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SINGULARITIES AND REGULAR PATHS (AN ELEMENTARY INTRODUCTION TO SMOOTH HOMOTOPY)

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Résumé. Cet article est une introduction élémentaire à la topologie algébrique lisse, suivant une approche particulière: notre but est d'étudier des "espaces lisses avec singularités', par des méthodes d'homotopie adaptées à cette tâche. On explore ici des régions euclidiennes, moyennant des chemins de classe C^k, *en tenant compte du nombre de leurs arrêts* en fonction de k. Le groupoïde fondamental de l'espace acquiert ainsi une séquence de poids qui dépend d'un index de classe C^k et qui peut distinguer l'ordre des singularités "linéaires". On peut envisager d'appliquer ces méthodes à la théorie des réseaux.

Abstract. This article is a basic introduction to a particular approach within smooth algebraic topology: our aim is to study 'smooth spaces with singularities', by methods of homotopy theory adapted to this task. Here we explore euclidean regions by paths of (variable) class C^k, *counting their stops*. The fundamental groupoid of the space acquires thus a sequence of integral C^k-weights that depend on a smoothness index; a sequence that can distinguish 'linear' singularities and their order. These methods can be applied to the theory of networks.

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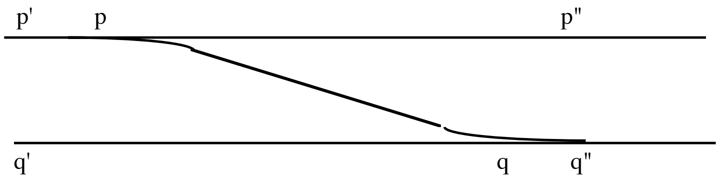
Introduction

We want to explore a 'smooth space with singularities', in order to distinguish its singularities and their order. Now, it is obvious and well-known that smooth paths can go through singularities, by braking at the crossing. Therefore, it will be important to count these stops, or - alternatively - to consider smooth paths that never stop.

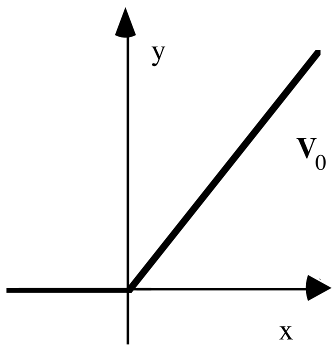
As a concrete situation which can be analysed in this way, consider a piece of railway network as represented below, with two main tracks linked by switches at p and q

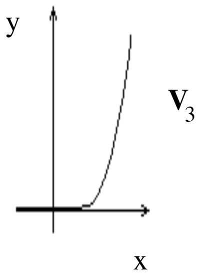
(1) 

Plainly, a train can move from p' to q" (or vice versa) without stopping, but cannot do the same from q' to p" (or vice versa). Moreover, a route through p' and q" should require a slowing down (with respect to a straight route), which can be expressed by letting the diverging junctions be (only) of class C^1. This example will be further analysed in 2.3(c).

Let us consider here a more basic space of this kind, the *standard deviation* $\mathbf{V}_k$, *of class* C^k $(0 \leq k < \infty)$, in the euclidean plane

(2) $\mathbf{V}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \leq 0, \; y = 0) \;\text{ or }\; (x \geq 0, \; y = x^{k+1})\},$





Take a curve $c \colon \mathbf{R} \to \mathbf{V}_k$ that crosses the singularity $(0, 0)$

(3) $c(t) = (x(t), 0)$ *or* $(x(t), x^{k+1}(t)),$ *for* $t \leq 0$ *or* $t \geq 0,$

where $x \colon \mathbf{R} \to \mathbf{R}$ is a strictly increasing C^∞-function that annihilates at 0. It is easy to see that $c \colon \mathbf{R} \to \mathbf{R}^2$ is always of class C^k; moreover the right (k+1)-derivative of a at 0 is the vector

$$(D_+^{k+1}c)(0) \; = \; (x^{(k+1)}(0), (k+1)!\, (x'(0))^{k+1}) \; \in \; \mathbf{R}^2,$$

so that c is of class C^{k+1} if and only if $x'(0) = 0$, and then $c'(0) = 0$. In other words, the class C^k of the singularity of $\mathbf{V}_k$ can be determined as the highest class of paths that go through the singularity without stopping.

In Section 1 we begin by considering a *euclidean space* X as a topological subspace of some standard euclidean space $\mathbf{R}^m$; we define a *C^k-path* $a: \mathbf{I} \to X$, in the usual way (1.2). For $k > 0$, a is said to be *C^k-regular* if it is constant or $a'(t) \in \mathbf{R}^m$ never vanishes on $\mathbf{I}$. More generally, for a continuous path a, we introduce a 'penalty' for each stop or breaking of C^k-smoothness, counted by an (extended) integral *C^k-weight* $w^k(a) \in \mathbf{N} \cup \{\infty\}$, so that a path a is C^k-regular if and only if $w^k(a) \leq 1$; the precise definition of the weight can be found in 1.3. This also defines a C^k-weight $w^k: \Pi_1(X) \to \mathbf{N} \cup \{\infty\}$ on the fundamental groupoid of the euclidean space X (1.6).

In the next section we analyse X at the basic level of the *existence* of paths, by a sequence of *tolerance relations* (reflexive and symmetric) on the set X itself, indexed by an *extended* natural number $k = 0, 1, \ldots, \infty$

(4) $x \,!_k\, y \quad \Leftrightarrow \quad$ (x and y are connected by a C^k-regular path in X).

For $k = 0$, this is the equivalence relation of path-connectedness and gives the classical quotient set $\Pi_0(X) = |X|/!_0$. More generally, we have a structure $R^k\Pi_0(X) = (X, !_k)$, consisting of a set equipped with a tolerance relation, which is better analysed in a *reduced form* $\mathrm{red}(R^k\Pi_0(X))$ (2.2).

For instance, in $\mathbf{V}_k$, the origin (0, 0) is in relation $!_\infty$ with any other point, but the points (– 1, 0) and (1, 1) are only in relation $!_h$ for $h \leq k$. For every $h > k$, the set $\mathrm{red}(R^h\Pi_0(X))$ consists of three equivalence classes: {(0, 0)}, the left open arm [(1, 0)] and the right open arm [(1, 0)]; the first is in relation with the other two, that are unrelated.

In Section 3 we consider the initial and terminal *k-jets* of a C^k-path in a euclidean space X, and define the *effective* k-jets at a point. Then, in Section 4, we extend the fundamental groupoid $\Pi_1(X)$ of the space (and its fundamental groups), introducing the *fundamental C^k-regular semicategory* $R^k\Pi_1(X)$ of X (see 4.4): its vertices are the 'regular' k-jets of X, its arrows are classes $[a]: j \to j'$ of C^k-regular paths; the homotopy relation used to define an arrow works at fixed initial and terminal k-jets. (A *semicategory* is the obvious generalisation of a category, without assumption of identities.)

In Section 5, we study the fundamental C^1-regular semicategory $R^1\Pi_1(X)$, and compare it with the fundamental groupoid $\Pi_1(T^*X)$ of the space of *non-zero* *tangent vectors*, isomorphic to the fundamental groupoid $\Pi_1(UTX)$ of the space of *unit tangent vectors* of X. We prove that this comparison is an isomorphism when X is a C^1-embedded manifold of dimension ≥ 2 (Theorem 5.4). Thus, the fundamental monoid $R^1\pi_1(\mathbf{R}^2, j)$ is isomorphic to the group of integers, by the winding number of a regular path, and expresses the possible 'shapes' of a planar loop realised with a smooth elastic wire. We also give examples where the comparison is not full (cf. 5.5).

We end with a more detailed study of tolerance sets, in Section 6.

This subject can be developed, working with 'convenient smooth structures', like C^∞-rings, Frölicher spaces, Chen spaces, diffeological spaces or other objects of synthetic differential geometry (cf. [MR, Fr, Ch, So, BH, Ko]). Smooth and directed algebraic topology [G3] can also be combined, to study 'directed smooth spaces'.

With respect to the existing literature about 'smooth paths' and 'smooth homotopy groups', our goals and results are completely different from those of Cherenack [Ck], or Caetano and Picken [CP], or Schreiber and Waldorf [SW1, SW2], where paths are allowed to stop or even required to be locally constant at the end-points. Such approaches have a smooth concatenation based on points (instead of jets), but do not distinguish what we want to explore. A recent paper by Sati, Schreiber and Stasheff [SSS] has applications of smooth cohomology to theoretical physics.

Jets of differentiable functions were introduced by C. Ehresmann, as equivalence classes of functions [E1, E2]. The notion of a tolerance set was introduced by E.C. Zeeman [Ze], in connection with mathematical models of the brain; the original name is 'tolerance space'. The interest of semicategories in category theory is recent: see [MBB].

1. Euclidean spaces and regular maps

We consider 'euclidean spaces' and maps of class C^k between them, where $k \leq \infty$ is an extended natural number, i.e. $k \in \bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. The standard interval is $\mathbf{I} = [0, 1]$. The usual concatenation of consecutive paths is written as $a*b$.

1.1. Euclidean spaces. A *euclidean space* will be just a topological subspace X of some standard euclidean space $\mathbf{R}^m$. (The standard euclidean spaces are viewed as naturally embedded, identifying $\mathbf{R}^m$ with $\mathbf{R}^m \times \{0\} \subset \mathbf{R}^{m+1}$; their union is a vector space of countable dimension, that can be equipped with the finest topology making the inclusions of all $\mathbf{R}^m$ continuous.)

Let us fix some examples, for future use.

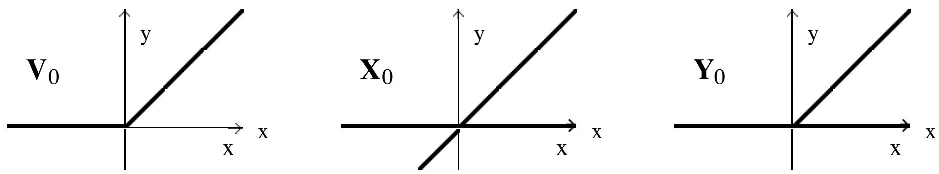
(a) Any real C^k-manifold M can be C^k-embedded in a suitable $\mathbf{R}^m$, and any subspace of M can be viewed as a subspace of $\mathbf{R}^m$.

(b) The *standard C^k-deviation* $\mathbf{V}_k$, the *standard C^k-crossing* $\mathbf{X}_k$ and the *standard C^k-switch* $\mathbf{Y}_k$ will be the following subspaces of the real plane (for $k \in \mathbf{N}$):

(1) $\mathbf{V}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \leq 0,\ y = 0) \text{ or } (x \geq 0,\ y = x^{k+1})\}$,

(2) $\mathbf{X}_k = \{(x, y) \in \mathbf{R}^2 \mid y = 0 \text{ or } y = x^{k+1}\}$,

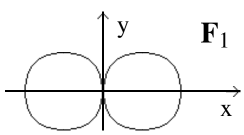
(3) $\mathbf{Y}_k = \{(x, y) \in \mathbf{R}^2 \mid y = 0 \text{ or } (x \geq 0,\ y = x^{k+1})\}$,



(c) We choose a lemniscate $\mathbf{E}_0$ as the *standard figure-eight curve of class C^0*. More generally, we denote by $\mathbf{E}_k$ a *standard figure-eight curve of class C^k*; it can be constructed in $\mathbf{R}^2$ from the *bounded C^k-crossing* $\mathbf{X}_k \cap \mathbf{B}^2$, linking smoothly its left arms together and its right ones as well. ($\mathbf{B}^2$ denotes the standard compact disc of $\mathbf{R}^2$.)

(d) We write $\mathbf{F}_k$ the *standard spectacles of class C^k*, that can be described as two smooth simple loops meeting at a point, with a contact of order k (precisely, i.e. not higher). Actually, the name of 'spectacles' is only appropriate for k *odd*, when there is a simple model in the plane, given by the union of two algebraic closed curves

(4) $\mathbf{F}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \pm 1)^{k+1} + y^{k+1} = 1\} \subset \mathbf{R}^2$,



For k *even*, $\mathbf{F}_k$ can be constructed in $\mathbf{R}^3$, starting from the bounded C^k-crossing $\mathbf{X}_k \cap \mathbf{B}^2$ and smoothly linking together its arms on $y = 0$, and the other arms (on $y = x^{k+1}$) as well. $\mathbf{F}_0$ can also be realised as two circles in $\mathbf{R}^3$, which meet in (only) one point, with different tangent lines.

It is easy to see that the even case has no model in the plane. Indeed, two smooth simple loops c, c' in the plane which meet at a point p, with a contact of even order k, must 'cross each other' at the meeting point (see below). Now, the complement of c in the plane has two connected components; if c, c' have no other meeting point, the complement of p in c' should stay in both components, which is absurd.

(If the loops c, c' have different tangent lines at p - only possible when $k = 0$ - they necessarily cross each other. Otherwise, in suitable cartesian coordinates, these curves can be locally represented around p as the graphs of two smooth functions $f, g\colon \mathbf{R} \to \mathbf{R}$ with $f(0) = g(0) = p$. Then the Taylor polynomial of degree $k+1$ of $h = f - g$, at 0, is a monomial of odd degree; therefore $h(x)$ changes of sign around the origin, and again our curves must cross each other.)

1.2. Smooth cubes. Smooth maps between euclidean spaces will be tested over smooth cubes. An n-*cube* of $X \subset \mathbf{R}^m$ is a continuous mapping

(1) $a\colon \mathbf{I}^n \to X$,

that will be viewed as a mapping $\mathbf{I}^n \to \mathbf{R}^m$ (with image in X) whenever useful.

This cube is said to be *of class* C^k, or a C^k-*cube*, if it has a C^k-extension $U \to \mathbf{R}^m$ over some open neighbourhood of $\mathbf{I}^n$ in $\mathbf{R}^n$ (or, equivalently, over $\mathbf{R}^n$); notice that this extension is *not* required to stay in X. For $t \in \mathbf{I}^n$, and a multi-index $i = (i_1,..., i_m) \in \mathbf{N}^m$ of height $|i| = i_1 +... + i_m$, the partial derivative of the component $a_j\colon \mathbf{I}^n \to \mathbf{R}$ is well defined (using any C^k-extension)

(2) $(\partial^{|i|}a_j / \partial t^i)(t) = (\partial^{|i|}a_j / \partial t_1^{i_1}... \partial t_n^{i_n})(t)$.

Equivalently, a C^k-cube can be defined as a mapping $a\colon \mathbf{I}^n \to X \subset \mathbf{R}^m$ that has continuous partial derivatives up to order k in the interior of $\mathbf{I}^n$, so that all such real functions have a continuous extension to $\mathbf{I}^n$. The equivalence can be proved using adequate extension theorems; for instance, Whitney's theorems as stated in Malgrange [Ma], Chapter 1.

1.3. Smooth weights and regular paths. Let $a\colon \mathbf{I} \to X$ be a continuous mapping with values in a euclidean space. It will also be called a C^0-*path*, or a C^0-*regular path*.

Let now $k \in \bar{\mathbf{N}}$ be positive. We let $St^k(a) \subset \mathbf{I}$ be the (possibly infinite) *set of* C^k-*stops* of the path a (including every breaking of C^k-smoothness):

(1) $\{0, 1\} \cup \{t \in\ ]0, 1[\ \mid$ either a is not C^k near t, or it is and $a'(t) = 0\}$,

where near t means on a convenient neighbourhood of t.

Then, we introduce a C^k-*weight* $w^k(a) \in \bar{\mathbf{N}}$, which - loosely speaking - counts each breaking of C^k-smoothness and each stop. Namely:

- $w^k(a) = \infty$ if a is not piecewise C^k;
- otherwise, $w^k(a) = (\#$ of connected components of $St^k(a)) - 1$.

We say that the path a is *C^k-regular* if $w^k(a) \leq 1$. This condition includes two cases:

- $w^k(a) = 0$ means that a is constant,
- $w^k(a) = 1$ means that a is of class C^k on **I** and $St^k(a)$ has precisely two connected components, those of 0 and 1; in other words the C^k-stops of a reduce to two disjoint closed intervals, possibly degenerate: $St^k(a) = [0, t_0] \cup [t_1, 1]$, where a is constant.

Any further unit in $w^k(a)$ means an internal C^k-stop point or an additional non-degenerate stop-interval. In brief, *a constant path costs nothing; otherwise, there is a fixed cost of 1/2 at departure and arrival, and a fixed cost of 1 at each C^k-stop* (independently of duration).

Notice that (always for $k > 0$) a path of class C^k is C^k-regular if and only if it is C^1-regular (i.e. it does not stop). Notice also that

(2) $w^k(a*b) \leq w^k(a) + w^k(b)$,

since there are two cases:

- $w^k(a) + w^k(b) = w^k(a*b) + 1$ if $c = a*b$ is C^k at a neighbourhood of $t = 1/2$ and $c'(1/2) \neq 0$,
- $w^k(a) + w^k(b) = w^k(a*b)$, otherwise.

1.4. Smooth maps. As an elementary way of introducing 'smooth spaces with singularities', let us introduce the category C^k**Euc** of subspaces of all the standard euclidean spaces $\mathbf{R}^m$, with *C^k-maps* $f: X \to Y$ between them; by this we mean a continuous mapping f that takes, by composition, the C^h-cubes of X into C^h-cubes of Y, for all $h \leq k$. Thus

(1) $C^\infty\mathbf{Euc} \subset C^k\mathbf{Euc} \subset C^{k'}\mathbf{Euc} \subset C^0\mathbf{Euc}$ $\qquad (0 \leq k' \leq k \leq \infty)$.

This definition agrees with the usual one when X and Y are C^k-manifolds. (Let us also recall that, by Boman's theorem [Bo], a map $f: X \to Y$ between C^∞-manifolds is C^∞ if and only if it preserves C^∞-paths.)

A C^k-map $\mathbf{I}^n \to Y$ is the same as a C^k-cube (according to the previous definition, 1.2). For general euclidean spaces ($X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$) we get a broader and (perhaps) better definition than by asking that f can be extended to a C^k-map

$U \to \mathbf{R}^m$ over some open neighbourhood of X in $\mathbf{R}^n$. For instance, if $X \subset \mathbf{R}^2$ is the union of three distinct (oriented) lines r_i through the origin, a C^1-map $f: X \to \mathbf{R}$ (as defined above) needs only to be separately C^1 on each line, whereas the other condition would also impose a relation on the directional derivatives $\partial f/\partial r_i$ at the origin.

We also notice that cubes can test smoothness in a 'finer' way than maps defined on euclidean *open* sets. For instance, if $X = \mathbf{V}_0$, a C^1-map $c: \mathbf{R} \to X$ with $c(0) = (0, 0)$ must have $c'(0) = 0$ and would test C^1-smoothness of functions defined over X in a less effective way than paths $a: \mathbf{I} \to X$ with initial or terminal point at the origin.

1.5. Pathwise regular maps. Let $f: X \to Y$ be a mapping between euclidean spaces. To say that it is *pathwise C^0-regular* will just mean that it is continuous.

For $k > 0$, we say that f is *pathwise C^k-regular* if it is a C^k-map and preserves, by composition, the *C^1-regular paths*. Then, it also preserves by composition the *C^h-regular paths*, for all $h \leq k$. If X and Y are C^k-embedded C^k-manifolds, a C^k-map is pathwise regular if and only if it is an immersion, i.e. the linear mapping $T_x f: T_x X \to T_{fx} Y$ is injective, for every $x \in X$.

These maps define the subcategory $C^k\mathbf{Reg} \subset C^k\mathbf{Euc}$ of *euclidean spaces and pathwise C^k-regular maps*

(1) $C^0\mathbf{Reg} = C^0\mathbf{Euc}$, $\qquad C^k\mathbf{Reg} = C^1\mathbf{Reg} \cap C^k\mathbf{Euc} \qquad (k > 0)$,

$C^\infty\mathbf{Reg} \subset C^k\mathbf{Reg} \subset C^{k'}\mathbf{Reg} \subset C^0\mathbf{Reg} = C^0\mathbf{Euc} \qquad (0 \leq k' \leq k \leq \infty)$.

Notice that $C^k\mathbf{Reg}$ lacks products, for $k > 0$; indeed, a cartesian projection $p_i: \mathbf{R}^2 \to \mathbf{R}$ is not immersion, and takes a regular, circular path to a path with stops.

1.6. The weighted fundamental groupoid. For a euclidean space X and an extended integer $k > 0$, the fundamental groupoid $\Pi_1(X)$ has a *C^k-weight* inherited from that of paths

(1) $w^k: \Pi_1(X) \to \bar{\mathbf{N}}$, $\qquad w^k[a] = \min \{w^k(b) \mid b \in [a]\}$.

Plainly, the identity at a point has C^k-weight 0; moreover, by1.3.2, a composed arrow $[a] + [b] = [a*b]$ gives:

(2) $w^k([a] + [b]) \leq w^k[a] + w^k[b]$.

If $f: X \to Y$ is a pathwise C^k-regular map (1.5) between euclidean spaces

(3) $w^k(fa) = w^k(a)$, $\qquad w^k[fa] \leq w^k[a]$.

2. Regular 0-homotopy

A euclidean space X is now equipped with an extended sequence of *k-regular* 0-*homotopy* objects $R^k\Pi_0(X)$. These are sets equipped with a tolerance relation, called *C^k-regular connectedness*.

2.1. Tolerance sets. A *tolerance set* X will be a set equipped with a *tolerance relation* x!y, reflexive and symmetric. A *tolerance morphism* f: X → Y is a mapping between such sets which preserves the tolerance relation.

The category **Tol** of tolerance sets and morphisms is complete and cocomplete, with limits and colimits created by the forgetful functor U: **Tol** → **Set**. It will be analysed more deeply in the last section.

A tolerance set X has an *associated equivalence relation*

(1) $x \sim y \;\Leftrightarrow\; (\forall z \in X,\; z \,!\, x \Leftrightarrow z \,!\, y)$,

and we say that X (or its tolerance relation) is *reduced* if (1) is the identity relation.

The quotient $X/\sim$, equipped with the induced relation $\xi \,!\, \eta$ (denoted by the same symbol)

(2) $\mathrm{red}(X) = X/\sim$,

$$\xi \,!\, \eta \quad\Leftrightarrow\quad (\exists x \in \xi,\; \exists y \in \eta,\; x \,!\, y) \quad\Leftrightarrow\quad (\forall x \in \xi,\; \forall y \in \eta,\; x \,!\, y),$$

will be called the *reduced* tolerance set associated to X.

Indeed, it is easy to see that the induced relation $\xi \,!\, \eta$ on red(X) is necessarily reduced: $\xi \sim \eta$ implies $\xi = \eta$. (Let $[x] \sim [y]$; from $z \,!\, x$ it follows that $[z] \,!\, [x]$; then $[z] \,!\, [y]$ and $z \,!\, y$; the symmetric argument gives $x \sim y$.)

If X is *transitive*, i.e. its tolerance relation is an equivalence, then the associated equivalence relation coincides with !, and the quotient red(X) = X/**!** 'is a mere set' (in the sense that its induced tolerance relation is the identity).

The tolerance set red(X) is an effective description of X, which reduces its redundancy. However, *this reduction is not functorial*, and should be used with care: indeed, a tolerance morphism f: X → Y need not preserve the associated equivalence relation. (This is trivially true when X is transitive.)

2.2. Tolerance relations of regular connectedness. In the euclidean space X, every extended natural number $k \leq \infty$ defines a tolerance relation, called *C^k-regular connectedness* in X

(1) $x \,!_k\, y \;\Leftrightarrow\;$ (x and y are connected by a C^k-regular path in X).

This relation is preserved by pathwise C^k-regular maps. Generally, it is not transitive (for $k > 0$); but it is for a C^k-embedded C^k-manifold. Obviously, $x \,!_k\, y$ implies $x \,!_h\, y$, for $h \leq k$ in $\bar{\mathbf{N}}$.

By definition, the *k-regular* 0-*homotopy* object of X will be the tolerance set:

(2) $R^k\Pi_0(X) = (X, !_k)$ $(k \in \bar{\mathbf{N}})$.

We have thus an (extended) sequence of functors $R^k\Pi_0$: $C^k\mathbf{Reg} \to \mathbf{Tol}$, with values in the category of tolerance sets and tolerance maps. In particular, the tolerance set $R^0\Pi_0(X) = (X, !_0)$ is transitive and its reduction yields the usual set $\Pi_0(X)$ of path-components of X

(3) $\Pi_0(X) = \mathrm{red}(R^0\Pi_0(X))$.

We will often use the reduced tolerance set $\mathrm{red}(R^k\Pi_0(X))$ to describe $R^k\Pi_0(X)$, even though this quotient cannot be made into a functor on $C^k\mathbf{Reg}$, for $k > 0$.

2.3. Examples. (a) For the standard deviation $X = \mathbf{V}_k$ (1.1.1), $!_h$ is the chaotic relation (that links all pairs of points) when $h \leq k$. For $h > k$, we have $x \,!_h\, y$ if and only if x and y both belong to the left closed arm or the right closed arm of $\mathbf{V}_k$; these 'arms' meet at the origin, which is in relation $!_\infty$ with any other point.

Thus, for $h > k$, $\mathrm{red}(R^h\Pi_0(\mathbf{V}_k))$ has three elements, corresponding to the singularity and the two open arms ξ, η of $\mathbf{V}_k$; the singularity is $!_h$-related to the other two elements, which are not related

(1) $0 = [(0, 0)]$, $\xi = [(-1, 0)]$, $\eta = [(1, 1)]$,

(2) $0 \,!_h\, \xi$, $0 \,!_h\, \eta$ $(h > k)$.

The euclidean sets $\mathbf{X}_k$ and $\mathbf{Y}_k$ (1.1.2-3) yield similar results.

(b) For the C^k-figure eight $\mathbf{E}_k$ (1.1(c)), the set $\mathrm{red}(R^h\Pi_0(\mathbf{E}_k))$ has one element, for all h.

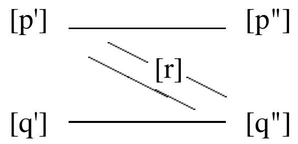
The space $\mathbf{F}_k$ (1.1(d)) gives a different result. The set $\mathrm{red}(R^h\Pi_0(\mathbf{F}_k))$ has three elements as soon as $h > k$ (and just one for $h \leq k$). These elements are the singularity at the origin and two 'punctured circles' ξ, η (i.e. circles without a point); the tolerance relation of $\mathrm{red}(R^k\Pi_0(\mathbf{F}_k))$ is described as above, in (2).

(c) Let us come back to the railway example, in figure (1) of the Introduction, where the route p', p, q, q" is assumed to be - precisely - of class C^1 at p and q. Then:

- the space is path-connected, and all pair of points are in relation $!_0$;
- p' and q', p" and q", q' and p" are not in relation $!_1$;
- no point of the upper line is in relation $!_2$ with any point of the lower line.

More precisely, the tolerance set $\mathrm{red}(R^1\Pi_0(X))$ consists of five equivalence classes, with the tolerance relation expressed by the following (non oriented) graph

(3)



For $k > 1$, $R^k\Pi_0(X)$ is transitive, and $red(R^k\Pi_0(X))$ has three *unrelated* classes: [p], [q], [r].

2.4. Remarks. A path in the euclidean space X is C^∞-regular if and only if it is C^k-regular, for all $k < \infty$. But the relation $a\ !_\infty\ b$ is strictly stronger than the conjunction of the relations $a\ !_k\ b$, for $k < \infty$; in other words, two points can be linked by suitable (different) paths of any possible C^k-class with $k < \infty$, without being linked by a regular C^∞-path.

For instance this happens in the euclidean space union of all deviations $\mathbf{V}_k$ (1.1.1)

(1) $X = \cup_{k\in\mathbf{N}} \mathbf{V}_k \subset \mathbf{R}^2$,

where $(-1, 0)\ !_k\ (1, 1)$ if and only if $k < \infty$.

3. Jets and paths

After a brief review of formal series, and their k-truncated versions, we consider the initial and terminal k-jets of a C^k-path in a euclidean space X, and define the *effective* k-jets at a point. Of course, jets can also be defined as equivalence classes of smooth functions, as in the original definition of C. Ehresmann [E1, E2].

3.1. Formal series and truncated polynomials. We begin by recalling the formalism of k-jets, as formal series, for $k = \infty$, or truncated series (i.e. truncated polynomials) for $k < \infty$.

Formal series $S = \Sigma\, a_i\tau^i$ in one variable τ, with coefficients in the real field, form a well-known $\mathbf{R}$-algebra $A_\infty = \mathbf{R}[[\tau]]$. They have a composition law (cf. Cartan [Ca])

(1) $S{\circ}T = \Sigma_i\, a_iT^i \qquad (S = \Sigma_i\, a_i\tau^i,\quad T = \Sigma_j\, b_j\tau^j)$,

provided the initial term b_0 *of* T *is zero*, so that the sum $\Sigma_i\, a_iT^i$ makes sense. (Indeed, the *order* $\omega(T^i)$, defined as the degree of the *lowest* non null coefficient of T^i, is $i.\omega(T) \geq i$; consequently, each coefficient of $S{\circ}T$ is computed by a finite sum of terms $a_i b_{j_1} ... b_{j_k}$.)

Notice that $S \circ 0 = a_0$ is the initial term of S, that is also written as $S(0)$. The algebraic properties of the composition law can be seen in [Ca].

For $k < \infty$, the algebra of k-*truncated series* (or k-truncated polynomials)

(2) $A_k = \mathbf{R}[[\tau]] / (\tau^{k+1}) = \mathbf{R}[\tau] / (\tau^{k+1})$,

has an induced composition law $[S] \circ [T] = [S \circ T]$ (when $T(0) = 0$).

A class $[S] = \Sigma_{i \leq k} a_i [\tau]^i$ will also be written as $\Sigma_{i \leq k} a_i \tau^i$, by abuse of notation. Operations in A_k are thus performed as with polynomials in an algebraic element τ such that $\tau^{k+1} = 0$: one omits all the terms of degree $> k$ ('the higher-order infinitesimals'), that come out of operations like product or composition of polynomials.

There are obvious *truncation epimorphisms*

(3) $tr_{kk'}: A_k \rightarrow A_{k'}$ $\qquad (0 \leq k' \leq k \leq \infty)$,

ending with $A_0 = \mathbf{R}$. We will refer to A_k as the algebra of *k-truncated series in one variable*, also when $k = \infty$ (and truncation is trivial).

We view A_k as the fibre bundle of k-jets of the real line

(4) $T_k\mathbf{R} = A_k$, $\qquad p: T_k\mathbf{R} \rightarrow \mathbf{R}, \quad p(S) = S(0)$.

Of course, it is a trivial bundle, and can be identified with the product of the line and its fibre at 0, the subalgebra $T_{k0}\mathbf{R}$ of k-truncated series with $S(0) = 0$

(5) $\mathbf{R} \times T_{k0}\mathbf{R} = T_k\mathbf{R}$, $\qquad (x, S) \mapsto x + S$.

Each fibre $\{x\} \times T_{k0}\mathbf{R}$ is a real vector space *at fixed* x, with the operations of $T_{k0}\mathbf{R}$:

(6) $(x + S) + (x + T) = x + S + T$, $\qquad \lambda.(x + S) = x + \lambda S$.

Composition is everywhere defined on the fibre $T_{k0}\mathbf{R}$. It *can* be extended, as in (1), letting:

(7) $(x + S) \circ T = x + S \circ T$.

3.2. Series in many variables. More generally, we will use formal series in the variables $\tau_1, ..., \tau_m$

(1) $S = \Sigma_i a_i \tau^i$, $\qquad i = (i_1, ..., i_m) \in \mathbf{N}^m$, $\qquad \tau^i = \tau_1^{i_1} \cdot ... \cdot \tau_m^{i_m}$,

and their $\mathbf{R}$-algebra $A_{m\infty} = \mathbf{R}[[\tau_1, ..., \tau_m]]$.

Now, there is a composition $S \circ T$, where T is a family of formal series $T_1, ...$,

T_m *with initial term zero*, on the same n variables, say $\vartheta_1,..., \vartheta_n$

(2) $S = \Sigma_i a_i\tau^i \in \mathbf{R}[[\tau_1,..., \tau_m]]$, $\quad T = (T_1,..., T_m) \in (\mathbf{R}[[\vartheta_1,..., \vartheta_n]])^m$,
$S \circ T = \Sigma_i a_i T^i$, $\quad T^i = T_1^{i_1} \cdot ... \cdot T_m^{i_m}$.

Again, the sum in $S \circ T$ makes sense because

$$\omega(T^i) = i_1\omega(T_1) + ... + i_m\omega(T_1) \geq i_1 + ... + i_m = |i|.$$

The k-truncated version

(3) $A_{mk} = \mathbf{R}[[\tau_1,..., \tau_m]] / I_k \quad (= \mathbf{R}[\tau_1,..., \tau_m] / J_k$, for $k < \infty$),

annihilates the ideal I_k (or J_k) of all series (or polynomials) whose order is $> k$, that is generated by the monomials $\tau^i = \tau_1^{i_1} \cdot ... \cdot \tau_m^{i_m}$ with $|i| = k+1$. Of course, $I_\infty = (0)$.

Again, we will refer to A_{mk} as the algebra of *k-truncated series* in m variables, even for $k = \infty$.

3.3. Jets of functions. Let A_{nk} be the algebra of k-truncated series in n variables $\vartheta_1,..., \vartheta_n$ (3.2), with $k \leq \infty$. For a C^k-mapping $f: U \to V$ between open euclidean spaces of dimensions n, m, we define the *k-jet* of f at a point $x \in U$

(1) $(j^k f)(x) \in (A_{nk})^m$, $\quad ((j^k f)(x))_i \in A_{nk} \quad (i = 1, ... m)$,
$((j^k f)(x))_i = \Sigma_{s \leq k} (s!)^{-1} D_s f_i(x) \cdot \vartheta^s$,
$D_s f_i(x) = (\partial^s f_i / \partial x_1^{h_1} ... \partial x_n^{h_n})_{|h|=s}$, $\quad \vartheta^s = (\vartheta_1^{h_1},..., \vartheta_n^{h_n})_{|h|=s}$,

where $\cdot$ is the scalar product of vectors indexed on all the n-tuples $h = (h_1,..., h_n)$ with a fixed sum $s = |h| = h_1 + ... + h_n$.

For instance, for $f: \mathbf{R}^2 \to \mathbf{R}$, and leaving the variable $x \in \mathbf{R}^2$ understood, $j^2 f$ yields the Taylor formula of f, of order 2, in the form:

$$j^2 f = f + \vartheta_1.\partial f/\partial x_1 + \vartheta_2.\partial f/\partial x_2 + 1/2(\vartheta_1^2.\partial^2 f/\partial x_1^2 + 2\vartheta_1\vartheta_2.\partial^2 f/\partial x_1 \partial x_2 + \vartheta_2^2.\partial^2 f/\partial x_2^2).$$

The jet of a composition can be expressed in a useful compact form, using the composition of formal series. Namely, if $f: U \to V$ and $g: V \to W$ are C^k-mappings between open euclidean spaces, so is gf and:

(2) $j^k(gf)(x) = ((j^k g)(f(x))) \circ ((j^k f)(x) - f(x))$.

Notice that $(j^k f)(x) - f(x)$ is a (possibly truncated) formal series with initial term 0, so that the composition makes sense. Concretely, we must replace the variable $\tau = (\tau_1,..., \tau_m)$ of the jet $(j^k g)(f(x))$ with the increment of $j^k f$.

Thus, for $k = 2$ and in one variable, the second term of (2) is a 2-truncated polynomial whose coefficients are (indeed) the derivatives at x of the composed function gf, up to the second:

$$(gf(x) + g'f(x).\tau + g''f(x).\tau^2)\circ(f'(x).\vartheta + f''(x).\vartheta^2)$$
$$= gf(x) + g'f(x).f'(x).\vartheta + (g'f(x).f''(x) + g''f(x).f'^2(x)).\vartheta^2.$$

We shall apply this formula also when f or g are C^k-cubes, taking into account the fact that they have C^k-extensions to open subsets containing their domain.

3.4. Initial and terminal jets of a path. Let us fix a euclidean space $X \subset \mathbf{R}^m$ and an extended natural number $k \leq \infty$.

Take a C^k-path $a: \mathbf{I} \to X \subset \mathbf{R}^m$ with $a(0) = x$ and $a(1) = x'$. Its *initial k-jet* (at $t = 0$) has components in the trivial fibre bundle $T_k\mathbf{R} = \mathbf{R} \times T_{k0}\mathbf{R}$ (3.1)

(1) $((j^ka)(0))_i = \Sigma_{h\leq k}\ (h!)^{-1}\ (a^{(h)}(0))_i\ .\ \tau^h \in T_k\mathbf{R} = \mathbf{R} \times T_{k0}\mathbf{R} \qquad (i = 1, ... m).$

It gives an element of the trivial fibre bundle $T_k\mathbf{R}^m = \mathbf{R}^m \times T_{k0}\mathbf{R}^m$

(2) $\partial_k^- a = (j^ka)(0) = x + \Sigma_{0<h\leq k}\ (h!)^{-1}\ a^{(h)}(0).\tau^h \in \{x\} \times T_{k0}\mathbf{R}^m,$

that will also be written in the form $x + (v^ka)(0)$.

Similarly, we have a *terminal k-jet* (at $t = 1$)

(3) $\partial_k^+ a = (j^ka)(1) = x' + (v^ka)(1)$
$\qquad = x' + \Sigma_{0<h\leq k}\ (h!)^{-1}\ a^{(h)}(1)\ .\ \tau^h \in \{x'\} \times T_{k0}\mathbf{R}^m.$

Initial and terminal jets determine each others. Indeed, let $b: \mathbf{I} \to X$ denote the reversed path of a, namely $b(t) = a(1 - t)$; then the initial k-jet of a and the terminal k-jet of b

(4) $\partial_k^- a = (j^ka)(0), \qquad\qquad \partial_k^+ b = (j^kb)(1) = \rho_k((j^ka)(0)),$

are linked by the involution ρ_k which changes sign to the derivatives of odd degree

(5) $\rho_k: T_k\mathbf{R}^m \to T_k\mathbf{R}^m, \qquad\qquad \rho_k(\Sigma_{h\leq k}\ a_h\tau^h) = \Sigma_{h\leq k}\ (-1)^h\ a_h\tau^h.$

3.5. Effective and virtual jets. We want now to define the set E_xT_kX of effective k-jets of X at x, as a subset of the vector space $\{x\} \times T_{k0}\mathbf{R}^m$.

In the same hypotheses as above (3.4), any initial k-jet of a C^k-path $a: \mathbf{I} \to X \subset \mathbf{R}^m$ with $a(0) = x$ will be called a *lower-effective k-jet of* X *at* x. Their set will be written as

(1) $E_x^-T_kX \subset \{x\} \times T_{k0}\mathbf{R}^m.$

The set of terminal k-jets of all paths ending at x will be written as

(2) $E_x^+T_kX = \rho_k(E_x^-T_kX)$,

(cf. 3.4.4) and called the set of *upper-effective k-jets of* X *at* x. Finally, we write

(3) $B_xT_kX = E_x^-T_kX \cap E_x^+T_kX$, $\quad\quad E_xT_kX = E_x^-T_kX \cup E_x^+T_kX$,

the sets of *bilateral* and *effective k-jets of* X *at* x. (The term 'effective' can be left understood.)

We will see (in 3.7) that the bilateral jets are precisely those that can be obtained as jets $(j^ka)(t)$ (of C^k-paths of X), at an *internal* point $t \in]0, 1[$. As a consequence, the effective jets are those that can be obtained at some $t \in \mathbf{I}$.

There are inclusions

(4) $B_xT_kX \subset E_x^\alpha T_kX \subset E_xT_kX \subset \{x\} \times T_{k0}\mathbf{R}^m \quad (\alpha = \pm)$.

Letting x vary, we get five 'fibred sets' on X

(5) $BT_kX \subset E^\alpha T_kX \subset ET_kX \subset T_k\mathbf{R}^m \quad\quad (\alpha = \pm)$.

For $k = 0$, we just have: $BT_0X = ET_0X = X \subset \mathbf{R}^m$.

For $k = 1$, the vector subspace spanned by $E_x^-T_1X$ (or $E_x^+T_1X$) in the real vector space of tangent vectors $\{x\} \times \mathbf{R}^m$ will be written as W_xT_1X and called the vector space of *virtual tangent vectors of* X *at* x. Notice that their collection WT_1X is not a fibre bundle, generally: the vector spaces W_xT_1X can have variable dimension, as is easy to see in the examples of 1.1.

The subset $E_x^\alpha T_1X$ inherits a *multiplication by real scalars* $\lambda \geq 0$, and will be viewed as a union of *semilinear subspaces* (semimodules on the semiring of weakly positive real numbers). Indeed, if $u: \mathbf{I} \to \mathbf{I}$ is any C^∞-function whose ∞-jet at 0 is $\lambda\tau$, the initial 1-jet of the reparametrised path au is

(6) $j^1(au)(0) = ((j^1a)(0) \circ ((j^1u)(0)) = a(0) + \lambda.(v^1a)(0)$.

The sets B_xT_1X and E_xT_1X (of bilateral and effective tangent vectors) inherit thus, from the vector space W_xT_1X, a *multiplication by real scalars*, and will be viewed as unions of linear subspaces of W_xT_1X.

Because of these multiplications, the following topological spaces

(7) $BT_1X \subset E^\alpha T_1X \subset ET_1X \subset WT_1X \subset X \times \mathbf{R}^m \quad\quad (\alpha = \pm)$,

admit $X \times \{0\}$ as a deformation retract, and are homotopically equivalent to X. *We also write* TX *for* ET_1X, *the fibred set of (effective) tangent vectors.*

Finally, we write E^*T_kX the set of the *regular* k-jets of X

(8) $E^*T_kX \subset ET_kX \subset T_k\mathbf{R}^m$,

i.e. the effective k-jets of X *with a non-zero term of degree 1*. These are also characterised below, in 3.7.

3.6. Theorem and Definition (Smooth concatenation of paths). *Let* $a, b\colon \mathbf{I} \to X$ *be two* C^k*-consecutive* C^k*-paths, which means that*

(1) $\partial_k^+ a = \partial_k^- b$,

i.e. $a^{(h)}(1) = b^{(h)}(0)$, *for all* $h \leq k$. *Then there is a* smoothly concatenated C^k*-path*

(2) $a + b\colon \mathbf{I} \to X$,

$(j^k(a + b))(0) = (j^k a)(0), \qquad (j^k(a + b))(1) = (j^k b)(1),$

$(a + b)(t) = a(\kappa(t))$ or $b(\kappa(t - {}^1/_2))$, for $0 \leq t \leq {}^1/_2$ or ${}^1/_2 \leq t \leq 1$.

We are using a concatenating C^∞-function $\kappa \in C^\infty(\mathbf{R}, \mathbf{R})$, *chosen once for all and satisfying*

(3) $(j^\infty\kappa)(0) = \tau, \qquad (j^\infty\kappa)({}^1/_4) = {}^1/_2 + \tau,$

$(j^\infty\kappa)({}^1/_2) = 1 + \tau, \qquad \kappa'(t) > 0.$

Moreover, if a *and* b *are* C^k*-regular, so is the concatenated path.*

Note. The function κ restricts to a strictly increasing diffeomorphism $[0, {}^1/_2] \to \mathbf{I}$, and replaces here the function 2t used for the usual concatenation $a*b$ (that is homotopic to the former, with fixed endpoints). *As an advantag*e, it has jet $2t + \tau$ (instead of $2t + 2\tau$) at the endpoints $(t = 0, {}^1/_2)$, *and leaves unchanged the initial jet of* a *and the terminal jet of* b. The similar condition at $t = {}^1/_4$ will be useful for associativity (in the proof of Theorem 4.3.)

Proof. The existence of a smooth function κ as above is obvious (or see Lemma 4.5, at the end of the next section).

Now, $a + b$ is of class C^k, because so is the 'pasting' of a, b at ${}^1/_2$. It suffices to apply the formula 3.3.2 for the jet of a composite (at the left and at the right of ${}^1/_2$)

$$j^k_-(a + b)({}^1/_2) = j^k(a\kappa)({}^1/_2) = (j^k a)(\kappa({}^1/_2))\circ((j^k\kappa)({}^1/_2)) - \kappa({}^1/_2))$$
$$= (j^k a)(1)\circ(\tau) = (j^k a)(1),$$
$$j^k_+(a + b)({}^1/_2) = j^k(b\kappa(\tau - {}^1/_2))({}^1/_2) = ((j^k b)(\kappa(0))\circ((j^k\kappa)(0)))\circ(\tau - {}^1/_2)$$
$$= (j^k b)(0)\circ(\tau) = (j^k b)(0).$$

Similarly, $a + b$ satisfies the initial and terminal conditions stated in (2).

Suppose now that a and b are C^k-regular, with $k > 0$. Then a and b never stop, and so does the concatenated path; indeed, $(a + b)'(\tau)$ is computed by one of the following formulas, and does not vanish at any $\tau \in \mathbf{I}$

$$a'(\kappa(\tau)).\kappa'(\tau), \qquad b'(\kappa(\tau - {}^1/_2)).\kappa'(\tau - {}^1/_2). \qquad \square$$

3.7. Corollary. *For a euclidean space X and $k \geq 0$:*

(a) the bilateral *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at an internal point $t_0 \in]0, 1[$, for some C^k-path a of X*;

(b) the effective *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at some point $t_0 \in [0, 1]$, for some C^k-path a of X*;

(c) the regular *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at some point $t_0 \in [0, 1]$, for some* C^k-regular *path a of X.*

Proof. Point (a) is obvious, using concatenation and smooth reparametrisation; then, (b) follows immediately. For (c), let a be a C^k-path of X , and suppose that $j = (j^k a)(t_0)$ has a non-zero term of degree 1. Then a satisfies the same property on a suitable neighbourhood of t_0 in $\mathbf{I}$; and we can restrict a to a suitable subinterval, and reparametrise it, so to obtain a C^k-regular path b which has the same k-jet at some point. $\square$

4. Fundamental smooth semicategories

After defining $R^k\Pi_0(X)$ in Section 2, we now want to analyse the fundamental groupoid $\Pi_1(X)$ of a euclidean space X. For $k > 0$, we use C^k-regular paths to get a *fundamental C^k-regular semicategory* $R^k\Pi_1(X)$: its vertices are the regular k-jets of X (3.5), and the homotopy relation used to define an arrow $[a]: j \to j'$ works at fixed initial and terminal k-jets.

4.1. Graphs of smooth paths. X is always a euclidean space. We now define its *graph of C^k-paths* C^kPX, and the *subgraph of C^k-regular paths* $R^kPX \subset C^kPX$.

For $k = 0$, $R^0PX = C^0PX = PX$ is just the graph of paths of X, with vertices the points of X and arrows $a: x \to x'$ the paths of X, from x to x'. It is a reflexive graph with composition; the latter is associative up to homotopy with fixed endpoints, and the quotient modulo this equivalence relation is the fundamental groupoid $\Pi_1(X)$ of the topological space X.

For $k > 0$, the vertices of C^kPX are the elements of ET_kX, i.e. the effective k-jets of X. An arrow $a: j \to j'$ is a C^k-path of X with the given initial and terminal jets

(1) $\partial_k^- a = (j^k a)(0) = j$, $\partial_k^+ a = (j^k a)(1) = j'$.

But we are more interested in the subgraph *of C^k-regular paths* $R^k PX \subset C^k PX$. Its vertices are the elements of $E^* T_k X$, i.e. the *regular* k-jets of X (with a non-zero term of degree 1, cf. 3.5 and 3.7). An arrow $a: j \to j'$, between two such jets $j, j' \in E^* T_k X$, is a *regular* C^k-path of X between the given end-jets.

The graph $R^k PX$ has the composition described above (Theorem 3.6), that we prove now to be associative up to the appropriate notion of homotopy.

4.2. Regular homotopy. Let two C^k-regular paths $a, b: \mathbf{I} \to X \subset \mathbf{R}^m$ be given. A *C^k-regular homotopy with fixed end jets*, from a to b, will be a C^k-cube $\varphi: \mathbf{I}^2 \to X$ (1.2) such that the C^k-paths $\varphi_t = \varphi(-, t): \mathbf{I} \to X$ *are regular* and satisfy the following conditions

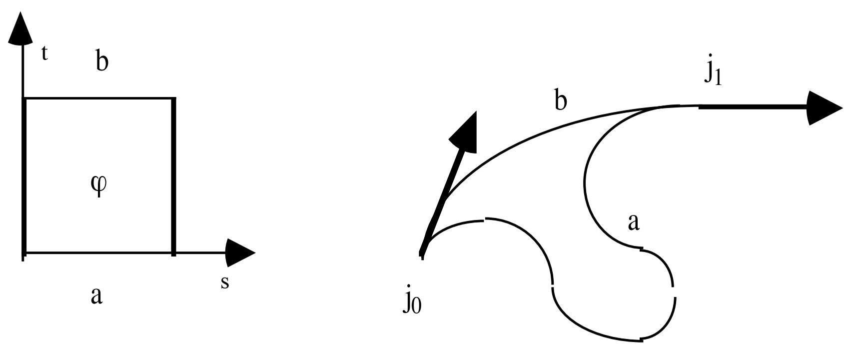
(i) $\varphi_0 = a$, $\varphi_1 = b$,

(ii) $(j^k \varphi_t)(0)$ and $(j^k \varphi_t)(1)$ are independent of $t \in \mathbf{I}$ (*fixed end jets*).

In the presence of (i), condition (ii) can be equivalently expressed as

(ii') $(j^k \varphi_t)(s) = (j^k a)(s) = (j^k b)(s)$, for $t \in \mathbf{I}$ and $s = 0, 1$.

The picture below represents the case $k = 1$, where the initial and terminal jets are (bound) vectors j_0, j_1; the homotopy φ is constant on the vertical edges of $\mathbf{I}^2$

(1) 

If such a 'distinguished' homotopy exists, we write $a \simeq_k b$. It is an equivalence relation for C^k-regular paths of X, because - plainly - these homotopies can be vertically reversed and pasted, and include the degenerate homotopy of a C^k-regular path: $0_a(s, t) = a(s)$.

Of course, $a \simeq_0 b$ is the ordinary relation of homotopy with fixed end-points. Moreover

(2) $a \simeq_k b \quad \Rightarrow \quad a \simeq_{k'} b \qquad (k' \leq k).$

4.3. Theorem. In the graph of C^k-regular paths R^kPX, the equivalence relation $\simeq_k$ agrees with concatenation and induces an associative operation on the quotient. This equivalence relation is preserved by pathwise C^k-regular maps.

Proof. (a) First we prove that, given four C^k-regular paths a, b, c, d, if

$$a, b: j' \to j, \qquad c, d: j \to j'', \qquad a \simeq_k b, \quad c \simeq_k d$$

then $a + b \simeq_k c + d$.

Let φ and ψ be our distinguished homotopies, that we want to paste horizontally. Since $\varphi(1, t) = a(1) = b(1)$ is constant, each partial derivative of a component φ_i annihilates at $\mathbf{I}\times\{1\}$, unless it only concerns derivation with respect to the first variable; the same holds for ψ at $\mathbf{I}\times\{0\}$; the relevant derivatives form the jet

$$(j^k\varphi_t)(1) = j = (j^k\psi_t)(0).$$

Therefore we can concatenate φ and ψ (horizontally), and obtain an (obviously) 'distinguished' homotopy; therefore $a + b \simeq_k c + d$.

(b) We prove now that, for three consecutive C^k-regular paths a, b, c

$$(a + b) + c \simeq_k a + (b + c).$$

Recall that the diffeomorphism $\kappa: [0, 1/2] \to [0, 1]$ that defines concatenation (3.6) was chosen to satisfy

(1) $\kappa[0, 1/4] = [0, 1/2], \qquad \kappa[1/4, 1/2] = [1/2, 1],$

so that the two ternary composites are computed as

$$(2) \quad (a + b) + c = \begin{cases} a\kappa\kappa(t) & \textit{for } t \in [0, 1/4], \\ b\kappa(\kappa(t) - 1/2)) & \textit{for } t \in [1/4, 1/2], \\ c\kappa(t - 1/2) & \textit{for } t \in [1/2, 1], \end{cases}$$

$$(3) \quad a + (b + c) = \begin{cases} a\kappa(t) & \textit{for } t \in [0, 1/2], \\ b\kappa\kappa(t - 1/2) & \textit{for } t \in [1/2, 3/4], \\ c\kappa(\kappa(t - 1/2) - 1/2) & \textit{for } t \in [3/4, 1]. \end{cases}$$

This can be re-written using the Moore concatenation <abc> over [0, 3] and two C^∞-functions $\lambda, \mu: \mathbf{I} \to [0, 3]$ (reparametrisations)

$$(4) \quad <abc>(t) = \begin{cases} a(t) & \textit{for } t \in [0, 1], \\ b(t - 1) & \textit{for } t \in [1, 2], \\ c(t - 2) & \textit{for } t \in [2, 3], \end{cases}$$

(5) $(a + b) + c = \langle abc\rangle \circ \lambda$, $\qquad a + (b + c) = \langle abc\rangle \circ \mu$,

$$\lambda(t) = \begin{cases} \kappa\kappa(t) & \textit{for } t \in [0, {}^1/_4], \\ 1 + \kappa(\kappa(t) - {}^1/_2)) & \textit{for } t \in [{}^1/_4, {}^1/_2], \\ 2 + \kappa(t - {}^1/_2) & \textit{for } t \in [{}^1/_2, 1], \end{cases}$$

$$\mu(t) = \begin{cases} \kappa(t) & \textit{for } t \in [0, {}^1/_2], \\ 1 + \kappa\kappa(t - {}^1/_2) & \textit{for } t \in [{}^1/_2, {}^3/_4], \\ 2 + \kappa(\kappa(t - {}^1/_2) - {}^1/_2) & \textit{for } t \in [{}^3/_4, 1]. \end{cases}$$

Now, the function λ is C^∞, because at each pasting point $t = {}^1/_4$ or ${}^1/_2$ we get $(j^\infty\lambda)(t) = t + \tau$ (using the composition of jets and the hypotheses 3.6.3 on the jets of κ at these points). Similarly, μ is C^∞, and so is the affine homotopy ξ

(6) $\xi\colon \mathbf{I}^2 \to \mathbf{R}$, $\qquad \xi(s, t) = (1 - t).\lambda(s) + t.\mu(s)$,

with $\xi_0 = \xi(-, 0) = \lambda$ and $\xi_1 = \xi(-, 1) = \mu$. Its end-jets are fixed (independent of t)

$$(j^k\xi_t)(0) = (1 - t).j^k\lambda(0) + t.j^k\mu(0) = (1 - t).\tau + t.\tau = \tau,$$
$$(j^k\xi_t)(1) = (1 - t).j^k\lambda(1) + t.j^k\mu(1) = (1 - t).(3 + \tau) + t.(3 + \tau) = 3 + \tau.$$

At fixed t, $\xi_t = \xi(-, t)$ is an affine combination of λ, μ; since these are strictly increasing, so is ξ_t. Therefore $\xi\colon \mathbf{I}^2 \to \mathbf{R}$ is a C^k-regular homotopy from λ to μ, and $\langle abc\rangle \circ \xi$ is a regular homotopy from $(a + b) + c$ to $a + (b + c)$.

(c) Let $f\colon X \to Y$ be a pathwise C^k-regular map: by definition, it preserves C^k-regular paths and cubes. Let now $\varphi\colon \mathbf{I}^2 \to X$ be a C^k-regular homotopy satisfying the conditions (i), (ii) of 4.2. Then $f\varphi\colon \mathbf{I}^2 \to Y$ is a C^k-regular cube, with $(f\varphi)_0 = f\varphi_0 = fa$ and $(f\varphi)_1 = fb$. It has fixed end-jets, by applying the composition formula 3.3.2, for $s = 0, 1$

$$j^k(f\varphi_t)(s) = ((j^kf)(\varphi_t(s)) \circ ((j^k\varphi_t)(s) - \varphi(s)),$$

where $\varphi_t(s) = a(s)$ and $(j^k\varphi_t)(s) = (j^ka)(s)$ are both independent of t. □

4.4. The fundamental C^k-regular semicategory. For a euclidean space X, we will write

(1) $R^k\Pi_1(X) = R^kPX / \simeq_k$,

the quotient of the graph R^kPX (4.1) modulo the equivalence relation $\simeq_k$ (4.2), with the induced, associative concatenation.

$R^k\Pi_1(X)$ will be called the *fundamental C^k-regular semicategory of* X; where a *semicategory* is the obvious generalisation of a category, without assuming the existence of identities (cf. [MBB]).

We have thus defined a functor

(2) $R^k\Pi_1$: C^k**Reg** $\to$ s**Cat**,

with values in the category of small semicategories and *semifunctors* between them (preserving composition).

For any regular k-jet $j \in E^*T_kX$, there is a semigroup

(3) $R^k\pi_1(X, j) = R^k\Pi_1(X)(j, j)$.

Some computations of such semicategories and semigroups will be given in the next section.

If $k' \leq k$, let U: C^k**Reg** $\subset$ $C^{k'}$**Reg** be the inclusion (1.5). There are natural transformations

(4) $tr_{kk'}$: $R^k\Pi_1 \to (R^{k'}\Pi_1)\circ U$: C^k**Reg** $\to$ s**Cat** $\qquad (k \geq k')$,

whose component on the euclidean space X is the obvious functor

(5) $tr_{kk'}(X)$: $R^k\Pi_1(X) \to R^{k'}\Pi_1(X)$, $\qquad j \to tr_{kk'}(j)$,

that operates on k-jets by truncation (3.1.3), and on equivalence classes of path by 'inclusion' (taking 4.2.1 into account).

4.5. Lemma. *Let (a_n), (b_n) be two sequences of real numbers $(n \geq 0)$. Then there is a C^∞-function f:* **R** $\to$ **R** *whose sequences of derivatives at 0 and 1 are the given ones. Moreover*:

(a) if $a_0, b_0 > 0$, one can choose f so that $f(t) > 0$ over **R**;

(b) if $a_0 < b_0$ and $a_1, b_1 > 0$, one can choose f so that $f'(t) > 0$ over **R**.

Proof. By a well-known Borel's Lemma, there exist two C^∞-functions g, h: **R** $\to$ **R** whose sequences of derivatives at 0 and 1 are, respectively, the given sequences. Take a smooth 'bell' function β: **R** $\to$ **R** vanishing outside $]-\varepsilon, \varepsilon[$ $(\varepsilon < 1/3)$ and satisfying

$$\beta(0) = 1, \quad \beta^{(n)}(0) = 0, \qquad \beta(t) \geq 0 \qquad (n > 0,\ t \in \mathbf{R}),$$

then $f(t) = \beta(t).g(t) + \beta(t-1).h(t)$ satisfies our conditions.

In case (a), take $f(t) = c + \beta(t).(g(t) - c) + \beta(t-1).(h(t) - c)$, after choosing a positive $c < a_0, b_0$ and ε sufficiently small so that $g(t)$, $h(t+1) \geq c$ in $]-\varepsilon, \varepsilon[$.

In case (b), first consider the shifted sequences (a_{n+1}), (b_{n+1}) starting at $a_1, b_1 > 0$, and let u be a positive solution for them, as in the previous case. Now, the function

$$f(t) = a_0 + U(t), \qquad U(t) = \int_0^t u,$$

is a solution, provided that u is constructed so that the positive number $U(1)$ coincides with $b_0 - a_0 > 0$. This can always be done, either modifying β (to make $U(1)$ smaller) or adding to u a third bell-function with support contained in $[\varepsilon, 1 - \varepsilon]$ (to make $U(1)$ bigger). □

5. Comparison with the fundamental groupoid of tangent versors

We study in more detail the semicategory $R^1\Pi_1(X)$ of a euclidean space X, and compare it with the fundamental groupoid $\Pi_1(T^*X)$ of the space of *non-zero (effective) tangent vectors*, isomorphic to the fundamental groupoid $\Pi_1(UTX)$ of the space of *unit tangent vectors* of X. Under convenient hypotheses, this comparison is an isomorphism (Theorem 5.4).

5.1. The comparison. Let $X \subset \mathbf{R}^m$ be a euclidean space. Consider the obvious embedding

(1) $R^1PX \to C^0P(T^*X), \qquad a \mapsto \hat{a} = (a, a'),$

of the graph of regular C^1-paths into the graph of paths of the subspace of *non-zero* (effective) tangent vectors:

(2) $T^*X \subset TX = ET_1X, \qquad T^*X = TX \cap (X \times (\mathbf{R}^m \setminus \{0\}).$

Plainly, this embedding is the identity on vertices and consistent with concatenation. It is also consistent with the appropriate notions of homotopy: if $a \simeq_1 b$ in R^1PX (4.2), the C^1-regular homotopy $\varphi: \mathbf{I}^2 \to X$ consists of a family of regular C^1-paths $\varphi_t = \varphi(-, t): \mathbf{I} \to X$ $(t \in \mathbf{I})$, and can be lifted to a homotopy of paths in T^*X, with fixed endpoints

(3) $\hat{\varphi}: \mathbf{I}^2 \to T^*X, \qquad \hat{\varphi}(s, t) = \hat{\varphi}_t(s) = (\varphi(t, s), \partial\varphi/\partial s(t, s)),$
$\hat{\varphi}(0, -) = \hat{a}, \qquad \hat{\varphi}(1, -) = \hat{b},$

$\hat{\varphi}(s, t) = \hat{\varphi}_t(s) = (j^1\varphi_t)(s) \qquad$ (independent of $t \in \mathbf{I}$, for $s = 0, 1$).

Therefore, there is a canonical comparison semifunctor, that is the identity on the objects, the (bound) vectors of T^*X

(4) $\mathbf{u}: R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')].$

(Let us recall, from 3.5.7, that $TX \simeq X$ cannot give here a 'good' comparison.)

Now, the subspace $UTX \subset T^*X$ of *unit tangent vectors* is a strong deformation retract of T^*X. We identify their fundamental groupoids, by the canonical isomorphism

(5) $\Pi_1(UTX) \cong \Pi_1(T^*X)$,

induced by the embedding $UTX \subset T^*X$ and its retraction $p: T^*X \to UTX$ (the normalisation of non-zero vectors).

The comparison **u** need not be full (cf. 5.5). But we prove that it is an isomorphism when X is a C^1-embedded manifold of dimension ≥ 2 (Theorem 5.4, after the following two lemmas).

5.2. Lemma. *Let $f: S \to G$ be a semifunctor from a semicategory to a groupoid. Then f is full and faithful if and only if the following conditions hold:*

(a) for every object x of S, f restricts to an isomorphism of semigroups $S(x, x) \to G(f(x), f(x))$ (which is thus an isomorphism of groups);

(b) for every pair of objects x, y in S such that f(x), f(y) are connected in G, there is some arrow $a: x \to y$ in S.

Moreover, if all this holds true, S is a groupoid and f is actually a functor.

Proof. The necessity of these conditions is obvious, as well as the last remark. Conversely, let us suppose they hold and fix a pair of objects x, y of objects of S. Composition in S and G is written in additive notation.

If f(x), f(y) are connected in G, there is some arrow $a: x \to y$ in S. Since G is a groupoid, all the arrows of G(f(x), f(y)) can be expressed as $g = h + f(a)$, for some endomap $h \in G(f(x), f(x))$. Applying (a), we have that $g = f(b)$ for some $b: x \to y$ in S. Therefore, f is full.

Suppose now that $a, b: x \to y$ in S are identified by f, and choose some $a': y \to x$ such that $f(a') = - f(a)$. Then $f(b + a') = f(a + a') = 0_{fx}$ and $b + a' = a + a' = 0_x$, by (a); similarly, $a' + b = a' + a = 0_y$, and finally $a = b$. □

5.3. Lemma. *For $X = \mathbf{R}^m$ and $m \geq 2$, the canonical comparison semifunctor*

(1) $\mathbf{u}: R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')],$

(cf. 5.1.4) is an isomorphism of groupoids. These are codiscrete for m > 2.

For m = 2, we have a connected groupoid whose groups of endoarrows are infinite cyclic

(2) $R^1\Pi_1(X)(j, j) \cong \Pi_1(T^*X)(j, j) \cong \Pi_1(\mathbf{S}^1)(j_0, j_0) \cong \mathbf{Z}.$

Proof. For m > 2,

$\Pi_1(UTX) = \Pi_1(\mathbf{R}^m \times \mathbf{S}^{m-1}) = \Pi_1(\mathbf{S}^{m-1})$,

is a codiscrete groupoid, i.e. between any two vertices there is precisely one arrow. The same is obviously true of $R^1\Pi_1(X)$, since any two C^1-regular paths $j \to j'$ can be deformed one into the other ($\mathbf{R}^3$ has 'sufficient room' to do that).

We now take $X = \mathbf{R}^2$ and apply the previous lemma. Its condition (b) is obviously satisfied: for any two vectors $j, j' \in T^*\mathbf{R}^2$ there exists a C^1-regular path a that gives an arrow $[a]: j \to j'$ in $R^1\Pi_1(X)$. We are left with considering the endoarrows of the semicategories in (1).

The space $X = \mathbf{R}^2$ will be given the usual orientation, by its embedding in $\mathbf{R}^3$ (with normal versor $(0, 0, 1)$).

If j_0 is the versor of the vector $j \in T^*X$, the canonical isomorphism

(3) $w\colon \Pi_1(T^*X)(j, j) \to \Pi_1(UTX)(j_0, j_0) = \Pi_1(\mathbf{R}^2 \times \mathbf{S}^1)(j_0, j_0) \to \mathbf{Z}$,

is computed as a winding number

(4) $w[a] = w(a_2)$,

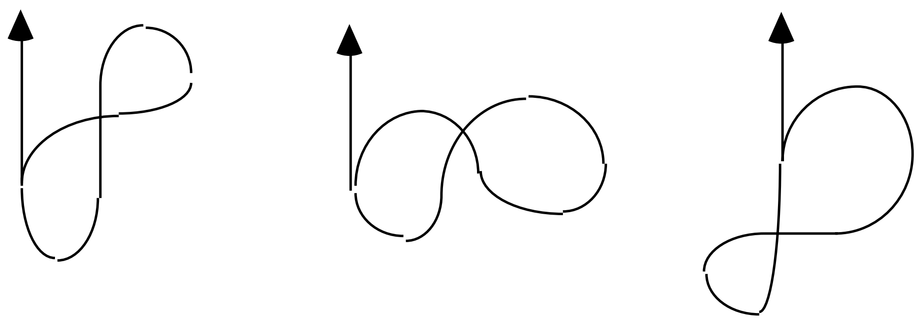
where $a = (a_1, a_2)$ and $a_2\colon \mathbf{I} \to \mathbf{R}^2 \setminus \{0\}$. It gives a winding-number homomorphism (of semigroups)

(5) $w\colon R^1\Pi_1(X)(j, j) \to \mathbf{Z}$, $\qquad [a] \mapsto w(a')$,

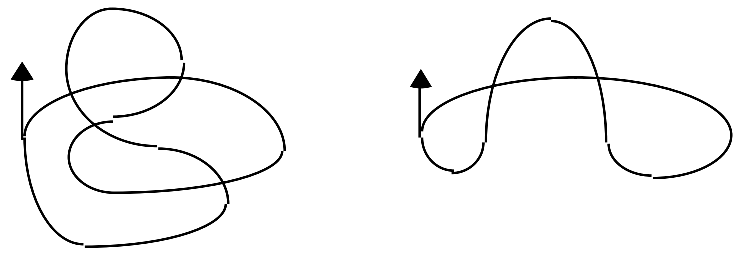
and it suffices to prove that this homomorphism is an isomorphism.

In fact, the semigroup $R^1\Pi_1(X)(j, j)$ is generated by two classes $[a], [b]$ with winding number 1 and -1, respectively. This proves that w is surjective.

But these two classes commute: $[a] + [b] = [b] + [a]$ (with winding number 0), as is shown by the following sequence of pictures



Moreover, $[b] + [a] + [b] = [a]$ (and, symmetrically, $[a] + [b] + [a] = [b]$), as proved by:



It follows that, for $n > 0$:

$$[b] + n[a] + [b] \;=\; (n - 1)[a] + [b] + [a] + [b] \;=\; n[a],$$

and every class is equivalent to $n[a]$, or $n[b]$, or $0_j = [a] + [b] = [b] + [a]$ $(n > 0)$. Therefore, the homomorphism w is bijective (and an isomorphism of groups). □

5.4. Theorem (Versors in a manifold). *Let $X \subset \mathbf{R}^m$ be a C^1-embedded manifold of dimension ≥ 2. Then $R^1\Pi_1(X)$ is a groupoid, and the canonical comparison semifunctor (5.1.4)*

(1) $\mathbf{u}\colon R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')],$

is an isomorphism of groupoids.

Proof. Again, since f is the identity on the objects, it is sufficient to prove that the canonical *semi*functor (1) is full and faithful. (But Lemma 5.2 would be of no real help here.)

(a) To prove that $\mathbf{u}$ is full, let us fix two vectors $j, j' \in T^*X$ and a path $b\colon j \to j'$ in the graph $P(T^*X)$, with projection $a\colon x \to x'$ in the graph PX; notice that a is just a continuous map $\mathbf{I} \to X$.

For $s \in]0, 1]$, we write $b_s\colon j \to b(s)$ the restriction of b to the interval $[0, s]$, reparametrised on $\mathbf{I}$, namely $b_s(t) = b(st)$ for $0 \leq t \leq 1$. Then we let

(2) $A = \{s \in]0, 1] \mid [b_s] \in \mathbf{u}(R^1\Pi_1(X)(j, b(s))\},$

and we have to prove that $1 \in A$.

First, the set A is not empty, because there exists a neighbourhood U of x in X that is C^1-diffeomorphic to a space $\mathbf{R}^n$ of dimension ≥ 2; if s is sufficiently small, b_s is a path in T^*U, and $\mathbf{u}\colon R^1\Pi_1(U) \to \Pi_1(T^*U)$ is full (and faithful), by Lemma 5.3.

Let $s_0 = \sup A \leq 1$, and let us prove that $s_0 \in A$. Choose a neighbourhood U

of $x_0 = a(s_0)$ with the same property as above, and some $s_1 \in A$ such that $a(s_1) \in U$. Then b_{s_1} ends at a vector $j_1 = b_{s_1}(1) = b(s_1) \in T^*U$ and $[b_{s_1}] \in \mathbf{u}(R^1\Pi_1(X)(j, j_1)$. But b stays in T^*U on some interval $[s_1, s_2]$ with $s_2 \geq s_0$, and this restriction can be replaced with a C^1-regular path in U, which can then be pasted to the one we already had on $[0, s_1]$, showing that $s_2 \in A$, and a fortiori $s_0 \in A$.

Moreover, $s_0 = 1$, otherwise in the previous argument we could take $s_2 > s_0$, and conclude $s_2 \in A$, a contradiction.

(b) Finally, to prove that f is faithful, let us take two paths $a, b: j \to j'$ in the graph R^1PX, such that $\mathbf{u}[a] = \mathbf{u}[b]$ in $\Pi_1(T^*X)$. This means that there exists a homotopy $\varphi: \mathbf{I}^2 \to T^*X$ such that

(i) $\varphi(-, 0) = (a, a')$, $\qquad \varphi(-, 1) = (b, b')$,

(ii) $(j^1\varphi(-, t))(0) = (a(0), a'(0)) = (b(0), b'(0))$, for all $t \in \mathbf{I}$,

(iii) $(j^1\varphi(-, t))(1) = (a(1), a'(1)) = (b(1), b'(1))$, for all $t \in \mathbf{I}$.

Notice that the intermediate paths $\varphi_\tau = \varphi(t, -): \mathbf{I} \to T^*X$, between $\varphi_0 = (a, a')$ and $\varphi_1 = (b, b')$, have a projection on X which need not even be of class C^1. We let A be the set of $s \in]0, 1]$ such that

- there exists a homotopy $\varphi: \mathbf{I}^2 \to T^*X$ with fixed end jets, whose projection $p\varphi: \mathbf{I}^2 \to X$ restricts to a C^1-regular homotopy $a_s \to b_s$,

where, without reparametrisation, a_s and b_s are the restrictions of a, b to $[0, s]$.

Again, it is sufficient to prove that $1 \in A$. The proof is similar to the previous one, and we only write down its beginning. The set A is not empty, because there exists a neighbourhood U of $x = a(0) = b(0)$ in X that is C^1-diffeomorphic to a space $\mathbf{R}^n$ of dimension ≥ 2; if s is sufficiently small, the paths a_s and b_s are in T^*U, and one can modify φ so that the restriction $(p\varphi)_s: [0, s] \times \mathbf{I} \to X$ is a C^1-regular homotopy. □

5.5. The circle and other curves. In dimension 1, this comparison need not be full, even for a manifold, namely the circle $\mathbf{S}^1 \subset \mathbf{R}^2$.

Let us fix the versor $j = (1, 0) + (0, 1).\tau$ of $UT\mathbf{S}^1 \subset T\mathbf{R}^2$. Then

(1) $R^1\pi_1(\mathbf{S}^1, j) = \mathbf{N}^*$,

is the additive semigroup of positive integers, properly contained in

(2) $\pi_1(UT\mathbf{S}^1, j) = \pi_1(\mathbf{S}^1 \times \mathbf{S}^0, j) = \mathbf{Z}$.

One can start from the standard path that turns around the circle n times (with λ

$= 2n\pi$)

(3) $a_n(t) = (\cos(\lambda t), \sin(\lambda t)), \qquad t \in \mathbf{I}.$

This has end-jets $(j_1a)(0) = (j_1a)(1) = (1, 0) + (0, \lambda).\tau$. Therefore, it is sufficient to reparametrise it as $a_n\varphi$, by a diffeomorphism $\varphi: \mathbf{R} \to \mathbf{R}$ with

$$\varphi(0) = 0, \qquad \varphi(1) = 1, \qquad \varphi'(t) > 0 \ (t \in \mathbf{R}), \qquad \varphi'(0) = \varphi'(1) = 1/\lambda.$$

It is not difficult to prove that $R^k\pi_1(\mathbf{S}^1, j) = \mathbf{N}^*$ holds for all $k \geq 1$ and $j \in T_k\mathbf{S}^1$, provided - obviously - that the coefficient of j of degree 1 is not null.

The spaces $\mathbf{E}_k, \mathbf{F}_k$ (1.1) also give free semigroups $R^k\pi_1(–, j)$, which is easy to compute.

5.6. The sphere. By theorem 5.4

(1) $R^1\pi_1(\mathbf{S}^2, j) = \pi_1(V\mathbf{S}^2, j)$.

This fundamental group can be easily computed with the van Kampen Theorem: $\pi_1(UT\mathbf{S}^2, j) = \mathbf{Z}_2$. But it is also easy to see directly that $R^1\pi_1(\mathbf{S}^2, j) = \mathbf{Z}_2$, since the stereographic embedding $f: \mathbf{R}^2 \to \mathbf{S}^2$ induces a surjective homomorphism of semigroups (hence of groups)

(2) $f_*: R^1\pi_1(\mathbf{R}^2, j) \to R^1\pi_1(\mathbf{S}^2, j)$,

which identifies the generator [a] with its opposite [b] (in the notation of the proof of Lemma 5.3).

6. Tolerance relations

We end with a more complete study of sets equipped with a tolerance relation, and their category.

6.1. Limits and colimits. Recall that a tolerance set X is a set equipped with a tolerance relation $x!y$, reflexive and symmetric. A tolerance morphism $f: X \to Y$ is a mapping between such sets which preserves the tolerance relation.

The category **Tol** of tolerance sets and morphisms is complete and cocomplete, with limits and colimits created by the forgetful functor U: **Tol** $\to$ **Set**. In particular, we have the following basic cases:

(a) the *product* $\prod X_i$ is the product of the underlying sets, with $(x_i)!(y_i)$ if and only if, for all indices i, $x_i ! y_i$ in X_i;

(b) the *equaliser* of $f, g: X \to Y$ is the equaliser $E = \{x \in X \mid fx = gx\}$ in **Set**,

with the restricted tolerance relation;
(c) the *sum* ΣX_i is the sum of the underlying sets, with $x!y$ if and only if this holds in one subset X_i;
(d) the *coequaliser* of $f, g: X \to Y$ is the coequaliser $E' = Y/R$ in **Set** (R is the equivalence relation of Y generated by $fx \sim gx$, for $x \in X$), equipped with the finest tolerance relation making the projection $Y \to Y/R$ a tolerance map; in other words, $[x]\,!\,[y]$ if and only if $x'!y'$ for some $x' \in [x]$ and $y' \in [y]$.

The following example will be referred to as the *test-case*: X is the union of the three coordinate planes of $\mathbf{R}^3$, and $x!y$ means that x and y are equal or lie in *one* such plane.

6.2. Tensor product and Hom. Our category **Tol** has a monoidal closed structure, with *tensor product* $X \otimes Y$ given by the cartesian product of the underlying sets, equipped with a tolerance which is *finer* than the cartesian one:

(1) $(x, y)\,!\,(x', y')$ if $(x!x'$ and $y = y')$ or $(x = x'$ and $y!y')$.

The identity is the terminal object $\top = \{*\}$, which acts under the tensor product as under product.

The internal hom-functor

(2) Hom: $\mathbf{Tol}^{op} \times \mathbf{Tol} \to \mathbf{Tol}$,

is obtained by equipping the set $\mathbf{Tol}(X, Y)$ with the *pointwise* tolerance relation:

(3) $f!g \Leftrightarrow$ (for all $x \in X$, $fx\,!\,gx$ in Y).

Now, it is trivial to verify that the exponential law in **Set** restricts to an isomorphism:

(4) $\mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)), \quad f \mapsto (x \mapsto f(x, -))$.

A *tolerance category* **A** will be a category enriched over the monoidal closed category **Tol**. This simply means that **A** is equipped with a binary relation ! between parallel maps, which is reflexive, symmetric and consistent with composition in the following weak sense

(5) if $g!g'$ then $hgf\,!\,hg'f$ (whenever the composition makes sense).

A will be said to be a *cartesian tolerance* category if the following stronger condition holds

(6) if $f!f'$ and $g!g'$ then $gf\,!\,g'f'$,

corresponding to enrichment with respect to the *cartesian* structure of **Tol**.

For instance, **Tol** itself is a tolerance category, but not a cartesian one. On the other hand, any cohesive category [G1, G2] satisfies the cartesian condition: for instance, the category of sets and partial mappings, where f!f' means that the partial mappings f, f': X → Y coincide on the elements of X on which they are both defined.

6.3. Club-structures. An equivalence relation over a set X can be equivalently assigned by means of a partition. Extending this well-known fact, a tolerance relation ! over X can be equivalently assigned by means of a *club-structure*, i.e. a set $\mathcal{A} \subset \mathcal{P}X$, whose elements will be called *clubs* of X (in the *test-case* 6.1, the clubs are the three coordinate planes).

Clubs must satisfy the following axioms:

(a) $\mathcal{A}$ is a covering of X (every point lies in a club);

(b) if A ⊂ X, and every pair a, a' ∈ A lies in a common club, then A is contained in a club;

(c) if A ⊂ B are clubs, then A = B.

No club can be empty (unless X = Ø). Moreover, if $A \in \mathcal{A}$, x ∈ X and every a ∈ A lies in a club containing x, then x ∈ A (because A∪{x} must be contained in some club, which has to coincide with A). More generally:

(d) if A ⊂ B ⊂ X, A is a club and all pairs of points of B lie in a common club, then A = B.

The bijective correspondence between our two notions is given by:

(1) $(X, !) \mapsto (X, \mathcal{A})$, the clubs being the maximal subsets of X which are pairwise !-linked,

(2) $(X, \mathcal{A}) \mapsto (X, !)$, where x!y if and only if x, y belong to a common club.

First, note that (1) is well defined by Zorn's lemma: every pairwise !-linked subset of X is contained in a maximal one. Now, it is obvious to verify that $(X, !) \mapsto (X, \mathcal{A}) \mapsto (X, !')$ produces a tolerance relation !' that coincides with !. On the other hand, consider the procedure $(X, \mathcal{A}) \mapsto (X, !) \mapsto (X, \mathcal{A}')$; if $A \in \mathcal{A}$, A is pairwise!-linked and therefore is contained in some maximal !-linked subset $A' \in \mathcal{A}'$, which has to coincide with A by (d). Conversely, if $A' \in \mathcal{A}'$, then A' is (maximal) !-linked and contained in some club $A \in \mathcal{A}$ (because of (b)); this also is !-linked, by definition of !, whence it coincides with A'.

In this correspondence, a *map* f: X → Y of club-sets is obviously a mapping of sets taking each club of X into some club of Y.

6.4. The associated equivalence relation. A tolerance set X usually contains a great redundancy, which can be cut out (as we have already seen in 2.1), much in the same way as in the procedure turning a preordered set into the associated ordered set.

For every point $x \in X$, the *star* of x will be

(1) $\mathrm{st}(x) = \{z \in X \mid x \,!\, z\}$ = union of the clubs containing x.

The equivalence relation *associated* to the link ! is produced by the mapping st: $X \to \mathcal{P}X$

(2) $x \sim y \;\Leftrightarrow\; \mathrm{st}(x) = \mathrm{st}(y)$,
$\Leftrightarrow$ for every $z \in X$, $z!x \Leftrightarrow z!y$,
$\Leftrightarrow$ the clubs containing x coincide with the ones containing y.

The quotient set $\mathrm{red}(X) = X/\!\sim$ corresponds thus, bijectively, to the set of stars of X, but should not be confused with the latter; the stars of X form a partition if and only if the link of X is an equivalence relation, in which case clubs and stars coincide. The set red(X) has an induced tolerance relation

(3) $[x] \,!\, [y] \;\Leftrightarrow\; x!y$ (independently of the choice of representatives),

that determines the one of X and is *reduced*, in the sense that its associated equivalence relation is the identity (cf. 2.1). Let us recall that the procedure of reduction is not functorial: a tolerance map $f\colon X \to Y$ need not preserve the equivalence relation associated to the tolerance relation.

In the test-case, the star of the origin is X itself; the star of each other point of an axis is the union of its two coordinate planes; the star of each other point is its coordinate plane. There are 7 equivalence classes: the origin [0], the three axes without the origin $[e_i]$, the three coordinate planes without their axes $[e_i + e_j]$ $(i \neq j)$.

6.5. The associated preorder. A tolerance set X has also an *associated preorder*

(1) $x \prec y \;\Leftrightarrow\; \mathrm{st}(x) \supset \mathrm{st}(y)$,
$\Leftrightarrow$ for every $z \in X$, $z!y \Rightarrow z!x$,
$\Leftrightarrow$ x belongs to each club containing y.

It determines the associated equivalence relation $x \sim y$ (as $x \prec y$ and $y \prec x$). Thus, the quotient $\mathrm{red}(X) = X/\!\sim$ is an ordered set (anti-isomorphic to the ordered set of stars)

(2) $[x] \leq [y]$ if $\mathrm{st}(x) \supset \mathrm{st}(y)$.

We say that x is a *maximal* element of X if st(x) is a club, if and only if x belongs to a unique club, if and only if (y!x!z ⇒ y!z), if and only if x is maximal in the associated preorder. In our test-case, $[0] < [e_i] < [e_i+e_j]$; the maximal elements of X are all the points which do not lie on some axis.

On the other hand, each preordered set $(X, \prec)$ has an associated tolerance

(3) x ! y if x, y have a common upper bound z $(x \prec z,\ y \prec z)$.

We say that a tolerance set (X, !) is *of preorder type* if these two procedures yield back the original tolerance relation; or equivalently if the link ! satisfies

(4) if x!y then there exists some z whose star is contained in $st(x) \cap st(y)$.

In fact, the converse implication always hold (if $st(z) \subset st(x) \cap st(y)$, then z!x, whence $x \in st(z) \subset st(y)$).

Our test case is of preorder type, whereas the tolerance $|x - y| < 1$ in **R** is not; its clubs are the open intervals of length 1, while $st(x) =]x - 1, x + 1[$.

6.6. Pointed tolerance relations. A *pointed tolerance set* is a pointed set $X = (X, 0_X)$ equipped with a tolerance relation such that $x \; ! \; 0_X$, for all $x \in X$.

Equivalently, all the clubs of X contain the base point. A morphism has to respect both structures. This defines the category **Tol.** of *pointed tolerance sets* (or pointed club-sets). Again, it is complete and cocomplete and has a canonical monoidal closed structure.

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