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VARIETIES GENERATED BY COMPACT METRIC SPACES

by Ernie MANES

Abstract

Un ultrafiltre nonprincipal r sur ω choisit un point de convergence pour chaque séquence dans un espace métrique compact. La classe d'algèbres produites par cette opération est une sous-catégorie pleine d'espaces topologiques dénombrablement tendus et d'applications continues qui contient tous les espaces métriques compacts. Les équations qui déterminent cette classe sont précisément celles satisfaites par la fonction caractéristique $2^\omega \to 2$ de r.

Abstract

A nonprincipal ultrafilter r on ω chooses a convergent point for each sequence in a compact metric space. The class of algebras produced by this operation is a full subcategory of countably tight topological spaces and continuous maps which contains all compact metrizable spaces. The equations which determine this class are precisely those satisfied by the characteristic function $2^\omega \to 2$ of r.

Keywords ultrafilter, compact metric space, monad

AMS Classification 18C15, 03C05, 54E35

1 Introduction

In [2], Isbell showed that commutative real C^*-algebras can be presented as a variety with a few finitary operations and a single ω-ary operation. This paper presents results of this type for compact metrizable spaces.

Let βX be the Stone-Čech compactification of the discrete space X realized as the set of ultrafilters on X with basic open sets $\{\Box A : A \subset X\}$ where $\Box A = \{\mathcal{U} \in \beta X : A \in \mathcal{U}\}$.

In this paper, let Σ be the signature with a single operation of arity ω. The category of Σ-algebras will be denoted $\mathbf{Set}^{\Sigma}$. The unadorned symbol **Set** is the category of sets and functions.

For X a compact metrizable space and $r \in \beta\omega \backslash \omega$ a fixed nonprincipal ultrafilter, let $\delta_r : X^{\omega} \to X$ map the sequence $f : \omega \to X$ to the unique point to which the ultrafilter $fr \in \beta X$ converges. Let $\mathcal{V}_r$ be the variety of Σ-algebras generated by all such (X, δ_r) with X compact metrizable.

Recall that a topological space is **countably tight** if whenever $x \in \overline{A}$ there exists a countable subset $C \subset A$ with $x \in \overline{C}$. Let $\mathbf{Top}_{\omega}$ denote the category of countably tight topological spaces and continuous maps.

The main results to be proved are the following three theorems.

Theorem 1.1 *The functor $\mathcal{V}_r \to \mathbf{Top}_{\omega}$ mapping (X, δ) to the space whose closed sets are the δ-subalgebras of X is a well-defined full subcategory which contains all compact metrizable spaces.*

Theorem 1.2 *The equations defining $\mathcal{V}_r$ are precisely those satisfied by the Σ-algebra $(2, \chi_r)$ where $\chi_r : 2^{\omega} \to 2$ is the characteristic function of the ultrafilter r.*

Before stating the third theorem, three definitions are needed.

Definition 1.3 *For $x \in X$, let prin(x) be the principal ultrafilter $\{A \subset X : x \in A\}$. Let prin$_X : X \to \beta X$ denote the map $x \mapsto$ prin(x). For $\psi : X \to \beta Y$ let $\psi^{\#} : \beta X \to \beta Y$ be the Stone extension of ψ, the unique continuous extension of ψ, specifically*

$$\psi^{\#}\mathcal{U} \;=\; \{B \subset Y : \{x : B \in \psi x\} \in \mathcal{U}\}$$

For the special case $\psi = X \xrightarrow{f} Y \xrightarrow{\text{prin}_Y} \beta Y$, $\psi^{\#}$ is denoted βf. One checks

$$(\beta f)\mathcal{U} \;=\; \{B \subset Y : f^{-1}B \in \mathcal{U}\}$$

and that β is then an endofunctor of the category **Set** of sets and functions.

We follow topological convention and write $(\beta f)\mathcal{U}$ as $f\mathcal{U}$ when no confusion would result. This convention has already been used in the definition of δ_r above. As βf has form $\psi^{\#}$, it is continuous $\beta X \to \beta Y$.

T is a **subfunctor** of β if $TX \subset \beta X$ is such that for all functions $f : X \to Y$, βf maps TX into TY. This induces the map $Tf : TX \to TY$ rendering T a functor in its own right.

In a topological space, we use the notation $\mathcal{U} \rightharpoondown x$ to indicate that the ultrafilter $\mathcal{U}$ converges to x.

Definition 1.4 *Let T be a subfunctor of β and let X be a topological space. Say that $A \subset X$ is T**-closed** if given $A \in \mathcal{U} \rightharpoondown x$ with $\mathcal{U} \in TX$ then $x \in A$. X is a T**-space** if every T-closed set is closed. X is T-**compact** if each ultrafilter in TX converges in X. X is T**-Hausdorff** if each ultrafilter in TX converges to at most one point of X.*

The category of T-spaces will be denoted $\mathbf{Top}_T$ and the category of T-compact, T-Hausdorff T-spaces will be denoted $\mathbf{CT2}_T$.

Definition 1.5 *For $r \in \beta\omega\backslash\omega$, X a set, define $T_r X \subset \beta X$ transfinitely as follows.*

$$\begin{aligned} \mathcal{A}_0 &= \{\text{prin}(x) : x \in X\} \\ \mathcal{A}_\alpha &= \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma \cup \{\psi^{\#} r : \omega \xrightarrow{\psi} \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma\} \\ T_r X &= \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha \end{aligned}$$

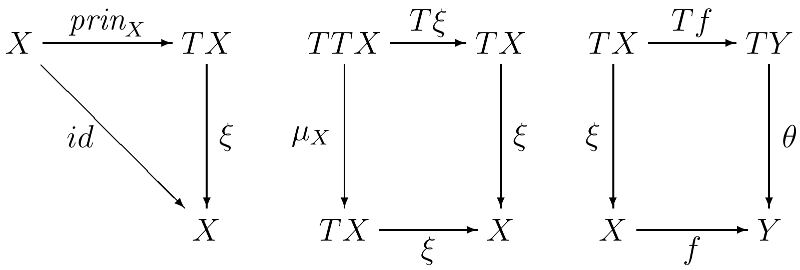
Theorem 1.6 *T_r is a subfunctor of β and $T_r X$ is the underlying set of the free algebra generated by X in $\mathcal{V}_r$. Further, as a category of countably tight spaces as in Theorem 1.1, $\mathcal{V}_r = \mathbf{CT2}_r$ is precisely the T_r-compact, T_r-Hausdorff T_r-spaces.*

2 Submonads of β

Definition 2.1 *Given $TX \subset \beta X$ for each set X, T is a* ***submonad*** *of β if some TW is non-empty and if T is closed under Stone extension, that is, given $\psi : X \to TY$, $(X \xrightarrow{\psi} TY \to \beta Y)^{\#}$ maps TX into TY.*

Notice that a submonad is a subfunctor since $\beta f = (X \xrightarrow{f} Y \xrightarrow{\text{prin}_Y} \beta Y)^{\#}$. Choosing W for which there exists $\mathcal{U} \in TW$, for $x \in X$ let $f : W \to X$ be constantly x so that $(Tf)\mathcal{U} = \text{prin}(x)$. This shows that TX contains all principal ultrafilters for all sets X.

Definition 2.2 *If T is a submonad of β, a T****-algebra*** *is a pair (X, ξ) where $\xi : TX \to X$ satisfies the leftmost two commutativities below.*



Here, μ_X is defined as $(id_{TX})^{\#}$.

If (X, ξ), (Y, θ) are T-algebras and $f : X \to Y$ is a function, f is a T-**homomorphism** $(X, \xi) \to (Y, \theta)$ if the square on the right above commutes. Since T is a functor, $id : (X, \xi) \to (X, \xi)$ is a T-homomorphism and T-homomorphisms compose, giving rise to the category $\mathbf{Set}^T$ of T-algebras.

Monads and their algebras are coextensive with (not necessarily finitary) universal algebra [3]. Heuristics: Think of (TX, μ_X) as the free T-algebra generated by the set X and think of $\xi : TX \to X$ as the unique T-homomorphism extending id_X. TX consists of the equivalence classes under the equations of the derived operations with variables in X and ξ interprets these operations in a particular algebra.

Rather than assume any of this background, we will give a self-contained treatment of all the properties we need based solely on facts that $\psi^{\#}$ is the unique continuous map $\beta X \to \beta Y$ with $\psi^{\#}\,\mathrm{prin}_X = \psi$, and that the principal ultrafilters are dense in βX.

For the balance of this section, T is a fixed submonad of β.

Proposition 2.3 *For $\phi : X \to TY$, $\psi : Y \to TZ$ the following* ***monad laws*** *hold.*

$$\begin{aligned} \phi^{\#} prin_X &= \phi \\ (prin_X)^{\#} &= id_{TX} \\ (\phi^{\#}\psi)^{\#} &= \phi^{\#}\psi^{\#} \end{aligned}$$

Proof *First let $T = \beta$. The first law states that $\phi^{\#}$ extends ϕ. For the second two, both sides are the unique continuous extension of the same map. For general T, as T is closed under prin and $(\cdot)^{\#}$, these laws continue to hold.* □

Lemma 2.4 *$prin : id \to T$ is a natural transformation.*

Proof We must show for all functions $f : X \to Y$ that the following square oommutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \mathrm{prin}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{prin}_Y} \\ TX & \xrightarrow[\ Tf\ ]{} & TY \end{array}$$

Indeed, using the definition of Tf and the first monad law,

$$(Tf)\,\mathrm{prin}_X = (\mathrm{prin}_Y f)^{\#}\,\mathrm{prin}_X = \mathrm{prin}_Y f \qquad \square$$

Proposition 2.5 *For any set X, (TX, μ_X) is a T-algebra. Moreover, $\mu : TT \to T$ is a natural transformation.*

Proof Consider the diagram

$$\begin{array}{ccccc} TTX & \xrightarrow{\text{prin}_{TTX}} & TTTX & \xrightarrow{\mu_{TX}} & TTX \\ \mu_X \downarrow & \text{(A)} & \downarrow T\mu_X & \text{(B)} & \downarrow \mu_X \\ TX & \xrightarrow[\text{prin}_{TX}]{} & TTX & \xrightarrow[\mu_X]{} & TX \end{array}$$

Since $\mu_X = (id_{TX})^{\#}$, the horizontal rows are identity maps which gives one of the algebra laws as well as the fact that the perimeter (A,B) commutes. (A) commutes by Lemma 2.4. When $T = \beta$, all maps in (B) are continuous and the two paths $\beta\beta\beta X \to \beta X$ agree on the principal ultrafilters, so (B) commutes. Thus (B) commutes for T since T is a submonad. Similar principles show that μ is natural. Let $f : X \to Y$. We must show that (D) below commutes:

$$\begin{array}{ccccc} TX & \xrightarrow{\text{prin}_{TX}} & TTX & \xrightarrow{\mu_X} & TX \\ Tf \downarrow & \text{(C)} & \downarrow TTf & \text{(D)} & \downarrow Tf \\ TY & \xrightarrow[\text{prin}_{TY}]{} & TTY & \xrightarrow[\mu_Y]{} & TY \end{array}$$

As before, (C) and (C,D) commute and all maps in (D) are continuous when $T = \beta$ so (D) commutes. □

Theorem 2.6 (TX, μ_X) *is the free T-algebra generated by X. Specifically, if (Y, θ) is a T-algebra and $f : X \to Y$ is a function then*

$$f^{\#} \;=\; TX \xrightarrow{Tf} TY \xrightarrow{\theta} Y$$

is the unique T-homomorphism ψ with $\psi\, prin_X = f$.

Proof In retrospect, one of the algebra axioms on (Y, θ) asserts that $\theta : (TY, \mu_Y) \to (Y, \theta)$ is a T-homomorphism. Also, $Tf : (TX, \mu_X) \to (TY, \mu_Y)$ is a T-homomorphism precisely because μ is natural. Thus $f^{\#} : (TX, \mu_X) \to (Y, \theta)$ is a T-homomorphism. By Lemma 2.4 and the other algebra axiom, $f^{\#}\,\mathrm{prin}_X = \theta\,(Tf)\,\mathrm{prin}_X = \theta\,\mathrm{prin}_Y f = id_Y f = f$. Finally, suppose $\psi : (TX, \mu_Y) \to (Y, \theta)$ is a T-homomorphism with $\psi\,\mathrm{prin}_X = f$. Observe that the following triangle commutes:

$$\begin{array}{ccc} TX & \xrightarrow{T\mathrm{prin}_X} & TTX \\ & \searrow^{id} & \downarrow{\scriptstyle \mu_X} \\ & & TX \end{array}$$

As usual we need check this only for $T = \beta$. We have $\mu_X\,(\beta\,\mathrm{prin}_X)\,\mathrm{prin}_X = (id_{\beta X})^{\#}\,\mathrm{prin}_{\beta X}\,\mathrm{prin}_X = id_{\beta X}\,\mathrm{prin}_X = \mathrm{prin}_X$ so that the triangle commutes. We then have $\psi = \psi\, id_{\beta X} = \psi\,\mu_X\,(\beta \mathrm{prin}_X) = \theta\,(\beta\psi)\,(T\mathrm{prin}_X) = \theta\,\beta(\psi\,\mathrm{prin}_X) = \theta\,(\beta f) = f^{\#}$. □

We leave to the reader the routine verification that the two uses of $(\cdot)^{\#}$ agree on their overlap, that is, for $\psi : X \to TY$, $\psi^{\#} : TX \to TY$ is the unique T-homomorphism $(TX, \mu_X) \to (TY, \mu_Y)$ extending ψ.

Definition 2.7 *If (X_i, ξ_i) are T-algebras and $X = \prod X_i$ with projection maps $pr_i : X \to X_i$, there exists unique function ξ as shown.*

$$\begin{array}{ccc} TX & \xrightarrow{T pr_i} & TX_i \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi_i} \\ X & \xrightarrow[pr_i]{} & X_i \end{array} \qquad \begin{array}{ccc} TA & \xrightarrow{Ti} & TX \\ {\scriptstyle \xi_A}\downarrow & & \downarrow{\scriptstyle \xi} \\ A & \xrightarrow[i]{} & X \end{array}$$

*Then (X, ξ) is a T-algebra (the laws hold on each coordinate) and provides the product in the category $\mathbf{Set}^T$, so is called the **product** T-algebra. If (X, ξ) is a T-algebra, $A \subset X$ with inclusion $i : A \to X$ then*

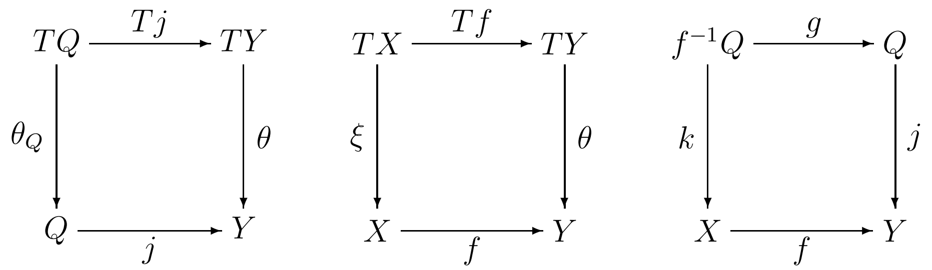
*A is a T-**subalgebra** of (X, ξ) if there exists ξ_A as shown, in which case, by similar reasoning, $i : (X, \xi_A) \to (X, \xi)$ is a T-homomorphism. T-algebra (Y, θ) is a **quotient algebra** of the T-algebra (X, ξ) if there exists a surjective T-homomorphism $(X, \xi) \to (Y, \theta)$.*

A class of T-algebras is a **variety** of T-algebras if it is closed under product algebras, subalgebras and quotient algebras. It is known [3] that such varieties are precisely those categories over **Set** which arise as an equationally definable class, so the terminology is justified even though this result is not needed in this paper.

If $\mathcal{V} = \mathbf{Set}^T$ or $\mathcal{V}$ is any variety of universal algebras and if $\mathcal{A} \subset \mathcal{V}$, the smallest variety containing $\mathcal{A}$ is $QSP(\mathcal{A})$ where $Q(\mathcal{B})$, $S(\mathcal{B})$, $P(\mathcal{B})$ are, respectively, the class of all quotient algebras, subalgebras, products of algebras in $\mathcal{B}$.

Lemma 2.8 *If $f : (X, \xi) \to (Y, \theta)$ is a T-homomorphism then the inverse image $f^{-1}Q$ of a subalgebra Q of (Y, θ) is a subalgebra of (X, ξ), and the direct image $fP \subset Y$ of a subalgebra P of (X, ξ) is a subalgebra of (Y, θ).*

Proof Consider



The first square shows how Q is a subalgebra and the second square shows that f is a homomorphism. The third square is the pullback square for $f^{-1}Q$. To show $f^{-1}Q$ is a subalgebra is to show that $\xi(Tk)$ factors through k. By the pullback property, it is equivalent to show that $f\xi(Tk)$ factors through j. To that end,

$$\begin{aligned} f\xi(Tk) &= \theta(Tf)(Tk) \text{ (T-homomorphism)} \\ &= \theta(Tj)(Tg) \\ &= j\theta_Q(Tg) \text{ (Q subalgebra)} \end{aligned}$$

Now let $i : P \to X$ be a subalgebra and factor $fi = P \xrightarrow{p} fP \xrightarrow{k} Y$ into its image, and show that fP is a subalgebra. As Tp is surjective, it suffices to show $\theta(Tk)(Tp)$ factors through k. Indeed,

$$\begin{aligned}\theta(Tk)(Tp) &= \theta(Tf)(Ti) \\ &= f\xi(Ti) \text{ (T-homomorphism)} \\ &= fi\xi_P \text{ (subalgebra)} \\ &= kp\xi_P \qquad \square\end{aligned}$$

3 T-algebras as spaces

We continue to fix a submonad T of β.

Lemma 3.1 *For (X, ξ) a T-algebra, the set of its subalgebras constitutes the closed sets of a topology $\mathcal{T}_\xi$ on X. The inclusion $(TX, \mathcal{T}_{\mu_X}) \to \beta X$ is continuous.*

Proof [4, Lemmas 3.1, 3.2]. □

Proposition 3.2 *If (X_i) is a family of T-spaces and $X = \prod X_i$ is the product set then a T-space topology exists on X rendering $pr_i : X \to X_i$ the product in $\mathbf{Top}_T$. Moreover, for $\mathcal{U} \in TX$, $\mathcal{U} \rightharpoonup x \Leftrightarrow \forall i \; pr_i\mathcal{U} \rightharpoonup x_i$.*

Proof [4, Theorem 5.4] □

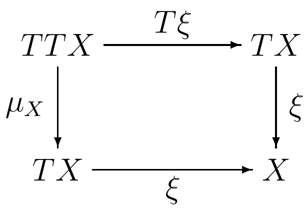
We denote the T-space product as $\bigotimes X_i$ to distinguish it from the Tychanoff topology $\prod X_i$. As $pr_j : \bigotimes X_i \to X_j$ is continuous, the T-space product topology is finer than pointwise convergence.

Lemma 3.3 *For (X, ξ) a T-algebra, $(X, \mathcal{T}_\xi)$ is a T-compact T-space and, for $\mathcal{U} \in TX$, $\mathcal{U} \rightharpoonup \xi\mathcal{U}$.*

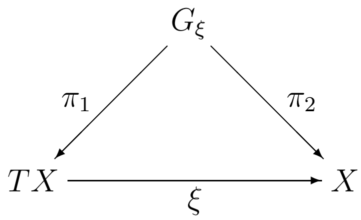
Proof The definition of a T-closed set and a T-subalgebra coincide so $(X, \mathcal{T}_\xi)$ is a T-space by Lemma 3.1. Let U be an open set with $\xi\mathcal{U} \in U$. As the complement U' is a subalgebra, we cannot have $\mathcal{U} \in TU'$ so $U' \notin \mathcal{U}$ and $U \in \mathcal{U}$. Thus $\mathcal{U} \rightharpoonup \xi\mathcal{U}$. □

Lemma 3.4 *Let X be a space in $\mathbf{CT2}_T$ and let $\xi : TX \to X$ be the T-restricted ultrafilter convergence function. Then (X, ξ) is a T-algebra.*

Proof $X \xrightarrow{\text{prin}_X} TX \xrightarrow{\xi} X = id_X$ since $\text{prin}(x)$ converges uniquely to x, and this is the first algebra axiom. We must show that the following square commutes.



We have from Proposition 2.5 and Lemma 3.3 that $(TX, \mathcal{T}_{\mu_X})$ is a T-space and that for $\mathcal{H} \in TTX$, $\mathcal{H} \rightharpoondown \mu_X(\mathcal{H})$. We first show that ξ is a closed subset of the T-space product $TX \otimes X$. Let $(\mathcal{U}_i, x_i)$ be a net in $TX \times X$ with $\xi \mathcal{U}_i = x_i$ and suppose $(\mathcal{U}_i, x_i) \rightharpoondown (\mathcal{U}, x)$ in $TX \otimes X$. Let U be open with $x \in U$. To show: $U \in \mathcal{U}$, since then $\xi \mathcal{U} = x$. Suppose not, so that $U' \in \mathcal{U}$. By Lemma 3.1, $\Box U' = \{\mathcal{V} \in TX : U' \in \mathcal{V}\}$ is a (not necessarily basic) open set in TX and, similarly, $\Box U' \times U$ is open in $TX \otimes X$. As $(\mathcal{U}_i, x_i)$ is eventually in $\Box U' \times U$ we have i exists with $\mathcal{U}_i \rightharpoondown x_i \in U$ but $U \notin \mathcal{U}_i$, the desired contradiction. Now let $\mathcal{H} \in TTX$. We must show $\xi(T\xi)\mathcal{H} = \xi \mu_X \mathcal{H}$. Let $G_\xi = \{(\mathcal{U}, x) : \xi \mathcal{U} = x\}$ be the graph of ξ. Observe that



commutes (where π_1, π_2 are the restricted projections) and that π_1 is bijective. There exists unique $\mathcal{H}^\star \in TG_\xi$ with $(T\pi_1)\mathcal{H}^\star = \mathcal{H}$. We also have $(T\pi_2)\mathcal{H}^\star = (T\xi)(T\pi_1)\mathcal{H}^\star = (T\xi)\mathcal{H}$. Thus $(T\pi_2)\mathcal{H}^\star \rightharpoondown \mu_X \mathcal{H}$ whereas $(T\xi)\mathcal{H} \rightharpoondown \xi(T\xi)\mathcal{H}$ so that, by Proposition 3.2, $\mathcal{H}^\star \rightharpoondown$

$(\mu_X\mathcal{H}, \xi(T\xi)\mathcal{H})$. But $G_\xi \in \mathcal{H}^\star$ with G_ξ closed, so we have $(\mu_X\mathcal{H}, \xi(T\xi)\mathcal{H}) \in G_\xi$ and $\xi\mu_X\mathcal{H} = \xi(T\xi)\mathcal{H}$ as desired. □

Lemma 3.5 *The T-space product of any family in $\mathbf{CT2}_T$ is in $\mathbf{CT2}_T$.*

Proof [4, Propositions 9.10, 10.10, Corollaries 9.11, 10.11]. □

Lemma 3.6 *For X a T-space, X is T-Hausdorff if and only if $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \otimes X$.*

Proof [4, Proposition 10.10]. □

Lemma 3.7 *A closed subspace of any space in $\mathbf{CT2}_T$ is in $\mathbf{CT2}_T$.*

Proof [4, Theorem 5.1, Propositions 9.6, 10.2]. □

We are now ready to identify the category of T-algebras as a full subcategory of topological spaces!

Theorem 3.8 $\mathbf{CT2}_T = \mathbf{Set}^T$. *This variety is generated by the two-element discrete space* 2.

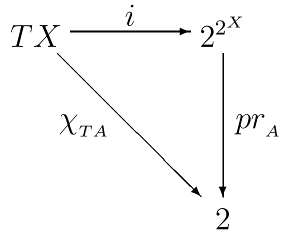
Proof We know that $\mathbf{CT2}_T \subset \mathbf{Set}^T$ at the level of objects from Lemma 3.4, and now observe that $\mathbf{CT2}_T$ is a full subcategory of $\mathbf{Set}^T$. Let X, Y be in $\mathbf{CT2}_T$ with T-restricted convergences $\xi : TX \to X$, $\theta : TY \to Y$ and let $f : X \to Y$ be continuous. Then $\mathcal{U} \in TX$, $\mathcal{U} \rightharpoondown x \Rightarrow f\mathcal{U} \rightharpoondown fx$, and this is just the commutative square that expresses that f is a T-homomorphism $(X, \xi) \to (Y, \theta)$. On the other hand, a homomorphism is continuous by Lemma 2.8.

We next show that $\mathbf{CT2}_T$ is a variety in $\mathbf{Set}^T$. Given a family (X_i) in $\mathbf{CT2}_T$ with T-restricted convergences ξ_i, the T-space product $X = \bigotimes X_i$ is itself in $\mathbf{CT2}_T$ by Lemma 3.5 and so has T-restricted convergence ξ. The squares

$$\begin{array}{ccc} T(\bigotimes X_i) & \xrightarrow{Tpr_j} & TX_j \\ \xi \downarrow & & \downarrow \xi_i \\ \bigotimes X_i & \xrightarrow[pr_j]{} & X_j \end{array}$$

commute because the projections are continuous, so indeed (X, ξ) is the product T-algebra. Now Lemma 2.8 gives that all T-homomorphisms are continuous and closed mappings. It follows immediately that all T-subalgebras are closed subspaces so by Lemma 3.7, $\mathbf{CT2}_T$ is closed under subalgebras. Now consider a surjective homomorphism $f : (X, \xi) \to (Y, \theta)$ of T-algebras with X in $\mathbf{CT2}_T$. As f is a closed mapping, Y has the quotient topology, so Y is a T-space by [4, Theorem 5.1]. Y is T-compact by [4, Proposition 9.2]. By Lemma 3.6, $\Delta_Y = (f \otimes f)\Delta_X$ is closed in $Y \otimes Y$, so Y is T-Hausdorff. This completes the proof that $\mathbf{CT2}_T$ is a variety in $\mathbf{Set}^T$.

Now let 2 be the two-element discrete space. It is obvious that all discrete spaces are T-spaces. As 2 is compact, Hausdorff and discrete, it is in $\mathbf{CT2}_T$ and hence is a T-algebra. Consider the inclusion $i : TX \to 2^{2^X}$. For $A \subset X$, the following triangle commutes.



This is because $pr_A \mathcal{U} = 1 \Leftrightarrow A \in \mathcal{U} \Leftrightarrow \mathcal{U} \in TA$. TA is a subalgebra of TX because μ is a natural transformation. $(TA)' = TA'$ because $\mathcal{U}$ contains exactly one of A, A', so TA is a clopen subset of TX. As χ_{TA} is continuous, it is a T-homomorphism. This proves that (TX, μ_X) is a subalgebra of a T-algebra power of 2. But every T-algebra (X, ξ) is a quotient (via ξ) of a TX. This shows that $\mathbf{Set}^T$ is the variety generated by 2. As 2 is in $\mathbf{CT2}_T$ and $\mathbf{CT2}_T$ is a variety, $\mathbf{CT2}_T = \mathbf{Set}^T$. □

Since every topological space is a β-space, we have the well-known

Corollary 3.9 $\mathbf{Set}^\beta$ *is the category of compact Hausdorff spaces.*

4 Proofs of the main theorems

Definition 4.1 *Define* $\beta_\omega X = \{\mathcal{U} \in \beta x :$ *there exists countable* $C \in \mathcal{U}\}$.

If $\psi : X \to \beta_\omega Y$ and $\mathcal{U} \in \beta_\omega X$ let $C \in \mathcal{U}$, $C_x \in \psi x$ $(x \in X)$ with C and all C_x countable. Then $D = \bigcup_{x \in C} C_x$ is countable. Recall $\psi^\# \mathcal{U} = \{B \subset Y : \{x : B \in \psi x\} \in \mathcal{U}\}$. For $x \in C$, $D \supset C_x \Rightarrow D \in \psi x$ so $\{x : D \in \psi x\} \supset C$. This shows, $D \in \psi^\# \mathcal{U}$, so β_ω is a submonad of β.

Proposition 4.2 *A topological space is countably tight if and only if it is a* β_ω*-space, that is,* $\mathbf{Top}_\omega = \mathbf{Top}_{\beta_\omega}$.

Proof [4, Theorem 4.9]. □

It is obvious from the definitions that if $S \subset T \subset \beta$ are subfunctors, every S-space is a T-space. Thus if $S \subset \beta_\omega$, all S-spaces are countably tight.

By Theorem 3.8, the β_ω-algebras are the β_ω-compact, β_ω-Hausdorff countably tight spaces. β_ω-compact spaces were called *ultracompact* in [1]. It is worth noting that such spaces are easily characterized by an open covering property.

Proposition 4.3 *A topological space is* β_ω*-compact if and only if for each open cover, every countable subset has a finite subcover.*

Proof First assume X is β_ω-compact and suppose $\mathcal{A}$ is an open cover of X and C is a countable subset of X which has no finite subcover. Then $\{C\} \cup \{A' : A \in \mathcal{A}\}$ has the finite intersection property, so is contained in an ultrafilter $\mathcal{U}$. As $C \in \mathcal{U}$, $\mathcal{U} \in \beta_\omega X$. By hypothesis, x exists with $\mathcal{U} \rightharpoonup x$. Let $x \in A \in \mathcal{A}$. Then $A \in \mathcal{U}$, the desired contradiction. Conversely, suppose $\mathcal{U} \in \beta_\omega X$ does not converge so that each $x \in X$ is contained in an open set A_x with $A_x \notin \mathcal{U}$. Let C be countable, $C \in \mathcal{U}$. As $\{C\} \cup \{A'_x : x \in X\}$ has the finite intersection property, finitely many A_x cannot cover C. □

Since any intersection of submonads of β is again a submonad, every class of pairs $(\mathcal{U}, X)$ with $\mathcal{U} \in \beta X$ generates a submonad. We have

Proposition 4.4 *For $r \in \beta\omega\backslash\omega$, T_r as in Definition 1.5 is the submonad of β generated by r.*

Proof Let S be the submonad generated by r and let X be a set. That $T_rX \subset SX$ is obvious so it suffices to show T_r is a submonad. If $\psi : \omega \to T_rX = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, the image $\psi(\omega)$ is countable so lies in some $\mathcal{A}_\gamma$ with $\gamma < \omega_1$. Thus $\psi^\# r \in \mathcal{A}_{\gamma+1}$ by the induction hypothesis. □

Obviously $T_r \subset \beta_\omega$. Thus every T_r-space is countably tight.

Recall the signature Σ from the Introduction.

Theorem 4.5 *For $r \in \beta\omega\backslash\omega$, $\mathbf{Set}^{T_r} \to \mathbf{Set}^\Sigma$, $(X, \xi) \mapsto (X, \delta)$ where $\delta : X^\omega \to X$, $\delta(f) = f^\# r$ establishes an isomorphism of categories between $\mathbf{Set}^{T_r}$ and a variety of Σ-algebras and Σ-homomorphisms. Moreover the closed subsets of a T_r-algebra coincide with the Σ-subalgebras.*

Proof [4, Theorem 12.1]. □

Proposition 4.6 *Every compact metrizable space is a T_r-algebra.*

Proof Let X be a compact metrizable space. Clearly X is T_r-compact and T_r-Hausdorff so it suffices to show that every T_r-closed set $A \subset X$ is sequentially closed. If $x_n \in A$ and $x_n \rightharpoonup x$ and if $x \in U$ with U open, $\{n : x_n \in U\}$ is cofinite, hence in r, since r is non-principal. Thus $fr \rightharpoonup x$. As A is T_r-closed and $fr \in T_rX$, $x \in A$. □

Proofs of the main theorems

Let $\mathcal{W}$ be the variety of Σ-algebras isomorphic to $\mathbf{Set}^{T_r}$ as in Theorem 4.5. A compact metrizable space X lies in $\mathbf{Set}^{T_r}$ by Proposition 4.6 and is considered a Σ-algebra via $\delta : X^\omega \to X$ where $\delta(f) = f^\# r = \xi(fr)$ with ξ the T-restricted convergence function; in short, δ coincides with the map δ_r of the Introduction. As $\mathcal{V}_r$ is the variety of Σ-algebras generated by all (X, δ_r) we have $\mathcal{V}_r \subset \mathcal{W}$. By Theorem 3.8, $\mathcal{W}$ is the variety generated by the two-element compact metrizable space so that $\mathcal{W} \subset \mathcal{V}_r$. By Theorem 2.6, T_rX is the underlying set of the free T_r-algebra generated by X. This completes the proofs of Theorems 1.1, 1.6.

For Theorem 1.2, we must explore $\delta_r : 2^\omega \to 2$, $2 = \{0,1\}$. An element of 2^ω is the characteristic function χ_A for a subset $A \subset \omega$. The ultrafilter $\chi_A r$ is one of prin(0), prin(1). Now $\{1\} \in \chi_A r \Leftrightarrow A = \chi_A^{-1}(1) \in r$, so δ_r is the characteristic function of $r \subset 2^\omega$. As $(2, \chi_r)$ generates $\mathcal{W}$, Theorem 1.2 is proved. □

5 Exploring the equations

As we have seen, $\mathcal{V}_r$ is the class of those Σ-algebras which satisfy the equations which hold for $\chi_r : 2^\omega \to 2$ or, equivalently, which hold for arbitrary compact metrizable spaces under the operation $\delta_r : X^\omega \to X$ where $fr \rightharpoonup \delta_r(f)$. Ideally, one can discover a specific perspicuous list of such equations by which one can prove that all Σ-algebras which satisfy these equations are T_r-compact, T_r-Hausdorff T_r-spaces, so that the equations generate $\mathcal{V}_r$. Although some progress is reported in this final section, the question remains open.

We use ω as the set of variables in terms to specify equations. A particular example of a term is $\delta(t)$ for $t : \omega \to \omega$. For $I \in r$ with least element i_0 define $t_I : \omega \to \omega$ by $t_I n = n$ if $n \in I$, $t_I n = i_0$ otherwise. The first equation scheme we consider is

$$\delta(t_I) \quad = \quad \delta(id) \quad (\text{ for } I \in r) \tag{1}$$

For $f, g : \omega \to X$ let $f =_r g$ mean $\{n : fn = gn\} \in r$.

Lemma 5.1 *Equation (1) holds in a Σ-algebra (X, δ) if and only if for all $f, g : \omega \to X$, $f =_r g \Rightarrow \delta(f) = \delta(g)$.*

Proof $\Rightarrow$: If $f =_r g$ let $I = \{n : fn = gn\} \in r$. By (1), $\delta(f) = \delta(ft_I) = \delta(gt_I) = \delta(g)$.

$\Leftarrow$: The interpretation of $\delta(t_I)$ under $f : \omega \to X$ is $\delta(ft_I)$. Since $ft_I =_r f$ if $I \in r$, $\delta(ft_I) = \delta(f)$. □

Now $f =_r g \Rightarrow fr = gr$ since $A \in fr \Leftrightarrow f^{-1}A \in r \Rightarrow f^{-1}A \cap \{n : fn = gn\} \in r \Rightarrow g^{-1}A \in r$ (as $g^{-1}A \supset f^{-1}A \cap \{n : fn = gn\}$). Thus (1) holds in a compact metrizable space since if $f =_r g$, fr, gr converge to the same point since $fr = gr$.

The equation (1) for $(2, \chi_r)$ amounts to the following tautology for $A, B \subset \omega$.

$$(A \cap B) \cup (A' \cap B') \in r \quad \Rightarrow \quad (A \in r \Leftrightarrow B \in r)$$

Proposition 5.2 *If (X, δ) is a Σ-algebra satisfying (1), the subalgebras of (X, δ) form the closed sets of a T_r-compact T_r-space.*

Proof That $\emptyset$, X are subalgebras and that any non-empty intersection of subalgebras is a subalgebras is obvious. If A, B are subalgebras and $f : \omega \to A \cup B$ then $f^{-1}A \cup f^{-1}B = \omega$ so at least one of $f^{-1}A$, $f^{-1}B$ belongs to r. If $f^{-1}A \in r$ let $g : \omega \to A$ with $f =_r g$ so that $\delta(f) = \delta(g) \in A \subset A \cup B$. Similarly, $\delta(f) \in A \cup B$ if $f^{-1}B \in r$. This shows that the subalgebras form the closed sets of a topology. Let $f : \omega \to X$ and let $U \subset X$ be open with $\delta(f) \in U$. Set $I = \{n : fn \notin U\}$. If $I \in r$ then there exists $g : \omega \to U'$ with $f =_r g$ in which case, because U' is a subalgebra, $\delta(f) = \delta(g) \notin U$, a contradiction. Thus $I' \in r$ and $U \in fr$ which shows $fr \rightharpoonup \delta(f)$. That every ultrafilter in T_rX converges then follows from [4, Lemma 11.8]. Now A is closed if and only if for all $f : \omega \to A$, fr converges in A. As $fr \in T_rX$, it follows a fortiori that X is a T_r-space. □

A simple equation to add to the mix is

$$\delta(x, x, x, \ldots) \quad = \quad x \tag{2}$$

That this equation is satisfied by $\chi_r : 2^\omega \to 2$ is routine: $X \in r$, so $\chi_r(1, 1, 1, \ldots) = 1$; $\emptyset \notin r$, so $\chi_r(0, 0, 0, \ldots) = 0$.

Proposition 5.3 *If (X, δ) is a Σ-algebra satisfying equations (1,2), the subalgebras of (X, δ) form the closed sets of a T_r-compact, T1, T_r-space.*

Proof For $x \in X$, $\delta(x, x, x, \ldots) = x$, so $\{x\}$ is a subalgebra. Now use Proposition 5.2. □

The existence of further equations to force (X, δ) to be T_r-Hausdorff is guaranteed by Theorem 1.6. It remains open if a convenient set of such equations in the style of (1,2) can be found. With (1,2) alone, it is not clear that one can prove that continuous maps are homomorphisms.

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Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01002, USA
manes@math.umass.edu