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TRIJUNCTIONS AND TRIADIC GALOIS CONNECTIONS

by René GUITART

Résumé. Dans cet article sont introduites les *trijonctions*, qui sont aux connexions galoisiennes triadiques ce que les adjonctions sont aux connexions galoisiennes. Nous décrivons le tripode trifibré associé à une trijonction, la trijonction entre topos de préfaisceaux associée a une trifibration discrète, et l'engendrement de toute trijonction par un bi-adjoint. À côté des exemples associés aux connexions galoisiennes triadiques, aux relations ternaires, d'autres le sont à des tenseurs symétriques, aux topos et univers algébriques.

Abstract. In this paper we introduce the notion of a *trijunction*, which is related to a triadic Galois connection just as an adjunction is to a Galois connection. We construct the trifibered tripod associated to a trijunction, the trijunction between toposes of presheaves associated to a discrete trifibration, and the generation of any trijunction by a bi-adjoint functor. While some examples are related to triadic Galois connections, to ternary relations, others are associated to some symmetric tensors, to toposes and algebraic universes.
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1. Introduction

A *trijunction* (definition 2.1) was introduced in [7] as a categorification of a triadic Galois connection [1], just as an adjunction [9] could be understood as a categorification of a Galois connection [13]: triadic Galois connections and Galois connections are trijunctions and adjunctions reduced to the case of posets (section 3). Any trijunction is generated by a bi-adjoint and determines a trifibration (section 2.1), and conversely a discrete trifibration determines a trijunction between toposes of presheaves. We give examples of trijunctions associated to adjunctions with parameters related to a symmetric tensor, and the constitutive auto-trijunctions of toposes or algebraic universes (section 4), which allow to reproduce internally triadic Galois connections.

2. Trijunctions, bi-adjunctions, discrete trifibrations

2.1 Trijunctions

Definition 2.1. *A* trijunction *between 3 categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, is the datum (γ, β, α) of 3 contravariant functors between any product of two of these categories and the third, i.e. 3 covariant functors as:*

$$\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}},\ \beta : \mathcal{C} \times \mathcal{A} \to \mathcal{B}^{\mathrm{op}},\ \alpha : \mathcal{B} \times \mathcal{C} \to \mathcal{A}^{\mathrm{op}}$$

and 3 natural equivalences with a circular condition

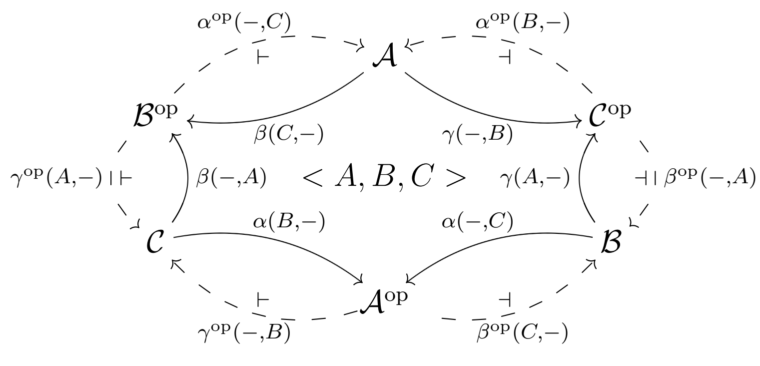
$$(-)^{\alpha,\gamma}(-)^{\gamma,\beta} = (-)^{\alpha,\beta} :$$

$$(-)^{\alpha,\gamma} : \mathrm{Hom}_{\mathcal{C}}(C, \gamma(A,B)) \simeq \mathrm{Hom}_{\mathcal{A}}(A, \alpha(B,C)) : (-)^{\gamma,\alpha} = ((-)^{\alpha,\gamma})^{-1},$$

$$(-)^{\gamma,\beta} : \mathrm{Hom}_{\mathcal{B}}(B, \beta(C,A)) \simeq \mathrm{Hom}_{\mathcal{C}}(C, \gamma(A,B)) : (-)^{\beta,\gamma} = ((-)^{\gamma,\alpha})^{-1},$$

$$(-)^{\beta,\alpha} : \mathrm{Hom}_{\mathcal{A}}(A, \alpha(B,C)) \simeq \mathrm{Hom}_{\mathcal{B}}(B, \beta(C,A)) : (-)^{\alpha,\beta} = ((-)^{\beta,\alpha})^{-1}.$$

Proposition 2.2. *Given a trijunction (γ, β, α) as in definition 2.1 and an object (A, B, C) of $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ we get 12 functors of one variable, in the bi-hexagon $< A, B, C >$, in which an exterior dotted line indicates a right adjoint to the corresponding internal unbroken line:*



Proof. Using known facts on adjunctions (recalled in section 2.5) the equivalences in definition 2.1 provide equivalences of adjunction when one argument is fixed, hence the adjunctions in the hexagon. □

Proposition 2.3. *Associated to adjunctions in the hexagon of proposition 2.2, there are 6 unit transformations which are* natural *in lower arguments and* dinatural *in upper arguments:*

1. $\beta(C,-) \dashv \alpha^{\mathrm{op}}(-C)$ *and* $\gamma(-,B) \dashv \alpha^{\mathrm{op}}(B,-)$ *give on* $\mathcal{A}$*:*

$$\alpha\beta CAC \xleftarrow{\alpha_A^C} A \xrightarrow{\alpha_A^B} \alpha B\gamma AB$$

2. $(\alpha(-,C) \dashv \beta^{\mathrm{op}}(C,-)$ *and* $\gamma(A,-) \dashv \beta^{\mathrm{op}}(-,A)$ *give on* $\mathcal{B}$*:*

$$\beta C\alpha BC \xleftarrow{\beta_B^C} B \xrightarrow{\beta_B^A} \beta\gamma ABA$$

3. $\beta(-,A) \dashv \gamma^{\mathrm{op}}(A,-)$ *and* $\alpha(B,-) \dashv \gamma^{\mathrm{op}}(-,B)$ *give on* $\mathcal{C}$*:*

$$\gamma A\beta CA \xleftarrow{\gamma_C^A} C \xrightarrow{\gamma_C^B} \gamma\alpha BCB$$

We recover the equivalences $(-)^{\alpha,\beta}$ *etc., by:*

$$a : A \to \alpha(B,C) = b^{\alpha,\beta} = \alpha(b,C)\alpha_A^C = c^{\alpha,\gamma} = \alpha(B,c)\alpha_A^B,$$
$$b : B \to \beta(C,A) = c^{\beta,\gamma} = \beta(c,A)\beta_B^A = a^{\beta,\alpha} = \beta(C,a)\beta_B^C,$$
$$c : C \to \gamma(A,B) = a^{\gamma,\alpha} = \gamma(a,B)\gamma_C^B = b^{\gamma,\beta} = \gamma(A,b)\gamma_C^A.$$

Proof. For $\beta(C,-) \dashv \alpha^{\mathrm{op}}(-C)$ the unit is

$$\alpha_A^C = (1_{\beta(C,A)})^{\alpha,\beta} : A \to \alpha\beta CAC := \alpha(\beta(C,A),C)$$

This α_A^C is *natural* in A, i.e. such that, for any $u : A \to A'$,

$$\alpha_{A'}^C u = \alpha(\beta(C,u),C)\alpha_A^C,$$

and is *dinatural* in C, i.e. such that, for any $w : C \to C'$, we have:

$$\alpha(\beta(w,A),C)\alpha_A^C = \alpha(\beta(C',A),w)\alpha_A^{C'}.$$

The situation here is an "adjunction with a parameter" (see [11, p. 100]) in $\mathcal{C}$ between α and β, and the naturality and dinaturality are proved in [11, p. 216]; in fact the converse holds: if α_A^C is natural in A and dinatural in C, then $(-)^{\beta,\alpha}$ (or its inverse $(-)^{\alpha,\beta}$) is natural in its three arguments. This is indicated in [11] (exercise 2 p. 100 and exercise 1 p. 218): the unit $\eta_A^B : A \to R(B, L(A,B))$ of an adjunction with parameter is dinatural in B, and this is equivalent to the naturality of the adjunction τ itself in B. □

Proposition 2.4. *With the hypothesis and notations of propositions 2.2 and 2.3 we have* 6 equations of adjunction*:*

$$\alpha(B,\gamma^B_C)\alpha^B_{\alpha(B,C)} = 1_{\alpha(B,C)} = \alpha(\beta^C_B, C)\alpha^C_{\alpha(B,C)},$$

$$\beta(C,\alpha^C_A)\beta^C_{\beta(C,A)} = 1_{\beta(C,A)} = \beta(\gamma^A_C, A)\beta^A_{\beta(C,A)},$$

$$\gamma(A,\beta^A_B)\gamma^A_{\gamma(A,B)} = 1_{\gamma(A,B)} = \gamma(\alpha^B_A, B)\gamma^B_{\gamma(A,B)};$$

and we have the condition of circularity*, expressible in 6 equivalent ways:*

$$\alpha^B_A = \alpha(\beta^A_B, \gamma(A,B))\alpha^{\gamma(A,B)}_A, \quad \alpha^C_A = \alpha(\beta(C,A), \gamma^A_C)\alpha^{\beta(C,A)}_A,$$

$$\beta^C_B = \beta(\gamma^B_C, \alpha(B,C))\beta^{\alpha(B,C)}_B, \quad \beta^A_B = \beta(\gamma(A,B), \alpha^B_A)\beta^{\gamma(A,B)}_B,$$

$$\gamma^A_C = \gamma(\alpha^C_A, \beta(C,A))\gamma^{\beta(C,A)}_C, \quad \gamma^B_C = \gamma(\alpha(B,C), \beta^C_B)\gamma^{\alpha(B,C)}_C.$$

Proof. For example, between the unit α^C_A and the corresponding co-unit β^C_B we have the known equations of adjunctions recalled in proposition 2.11.
For example, as $\beta(C,A)$ is a functor in each variable, and as β^C_B is dinatural in C (proposition 2.3), the fourth circularity condition, expressing β^A_B, allows to deduce for any $c : C \to \gamma(A,B)$ that

$$\beta(c,A)\beta^A_B = \beta(C, \alpha(B,c)\alpha^B_A)\beta^C_B,$$

which (cf. proposition 2.3) is equivalent to $(-)^{\beta,\gamma} = (-)^{\beta,\alpha}(-)^{\alpha,\gamma}$. This implies conversely the fourth condition.
By the equations of adjunction, the six natural transformations $(-)^{\alpha,\beta}$ etc. are invertible (equivalence), and from the last equation we get the five analogs, and then any equation of circularity. □

2.2 Bi-adjunction

Definition 2.5. *A bi-functor* $\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ *is a* left bi-adjunction *if for every* A *in* $\mathcal{A}$ *the functor* $\gamma(A,-) : \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ *is a left adjoint, and for every* B *in* $\mathcal{B}$ *the functor* $\gamma(-,B) : \mathcal{A} \to \mathcal{C}^{\mathrm{op}}$ *is a left adjoint.*

Proposition 2.6. *1 — A bi-functor* $\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ *is a* left bi-adjunction *if and only if there is a trijunction* (γ, β, α)*, in the sense of definition 2.1. In this case,* β *and* γ *are unique up to natural isomorphisms.*

2 — A trijunction is completely determined up to isomorphisms by a functor $\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ with the datum for each object C of two objects αBC and βCA, with two morphisms

$$\gamma A \beta C A \xleftarrow{\gamma_C^A} C \xrightarrow{\gamma_C^B} \gamma \alpha B C B \ ,$$

such that for any $c : C \to \gamma AB$ there are two unique maps $a : A \to \alpha BC$ and $b : B \to \beta CA$ such that

$$c = \gamma(a, B)\gamma_C^B, \quad c = \gamma(A, b)\gamma_C^A.$$

Proof. 1 — The proposition is an application of known results (recalled in proposition 2.13 later). So we introduce β and α by $\gamma(A, -) \dashv \beta^{\mathrm{op}}(-, A)$ and $\gamma(-, B) \dashv \alpha^{\mathrm{op}}(B, -)$. With the formula $(\star)$ in the proof of proposition 2.13 we get bi-functors β and γ, with natural equivalences $(-)^{\gamma,\beta}$ and $(-)^{\alpha,\gamma}$, and we define $(-)^{\alpha,\beta}$ as the composition $(-)^{\alpha,\gamma}(-)^{\gamma,\beta}$.
2 — This results from the determination of adjoints by free objects.
So, all the data and equations in a trijunction (cf. propositions 2.2, 2.3 and 2.4) are consequences of these two "free object" properties. □

2.3 Discrete trifibration associated to a trijunction

A triadic Galois connection is known to be a generalization of a ternary relation (recalled in proposition 3.5 later); a similar understanding for a trijunction is in terms of trifibrations.

Definition 2.7. *Given a trijunction (γ, β, α) we construct its "trigraph", the category $\mathcal{G} = \mathcal{G}(\gamma, \beta, \alpha)$with objects $G = (a, b, c)$ as in*

$$a : A \to \alpha(B, C)$$

$$c : C \to \gamma(A, B) \qquad\qquad b : B \to \beta(C, A)$$

with

$$b = a^{\beta,\alpha},\ c = b^{\gamma,\beta},\ a = c^{\alpha,\gamma},$$

as in proposition 2.3; a morphism from (a, b, c) to (a', b', c') is a $g = (u, v, w) : (A, B, C) \to (A', B', C')$ with one of the equivalent conditions:

$$\alpha(v, w)a'u = a, \quad \beta(w, u)b'v = b, \quad \gamma(u, v)c'w = c.$$

Proposition 2.8. *We have a discrete fibration given by:*

$$\pi = \pi_{\gamma,\beta,\alpha} : \mathcal{G}(\gamma,\beta,\alpha) \to \mathcal{A}\times\mathcal{B}\times\mathcal{C} : (a,b,c) \mapsto (A,B,C)$$

$$\begin{array}{ccc} \mathcal{G} & \qquad & \begin{array}{ccc} & \mathcal{A} & \\ & \pi_{\mathcal{A}}\uparrow & \\ & \mathcal{G} & \\ \mathcal{C} \overset{\pi_{\mathcal{C}}}{\swarrow} & & \overset{\pi_{\mathcal{B}}}{\searrow} \mathcal{B} \end{array} \\ \downarrow \pi & & \\ \mathcal{A}\times\mathcal{B}\times\mathcal{C} & & \end{array}$$

Proof. In fact $\mathcal{G}(\gamma,\beta,\alpha) = \mathcal{G}$ is isomorphic to the discrete fibration $\int\alpha$ associated to $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Id}_{\mathcal{A}^{\mathrm{op}}}\times\alpha^{\mathrm{op}}) : (\mathcal{A}\times\mathcal{B}\times\mathcal{C})^{\mathrm{op}} \to \mathrm{Ens}$, as well as the one $\int\beta$ associated to $\mathrm{Hom}_{\mathcal{B}}(\mathrm{Id}_{\mathcal{B}^{\mathrm{op}}}\times\beta^{\mathrm{op}}) : (\mathcal{B}\times\mathcal{C}\times\mathcal{A})^{\mathrm{op}} \to \mathrm{Ens}$ or the one $\int\gamma$ associated to $\mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}^{\mathrm{op}}}\times\gamma^{\mathrm{op}}) : (\mathcal{C}\times\mathcal{A}\times\mathcal{B})^{\mathrm{op}} \to \mathrm{Ens}$. So in the category of fibrations over $\mathcal{A}\times\mathcal{B}\times\mathcal{C}$ we have three isomorphisms

$$\begin{array}{ccc} & \int\alpha & \\ & \downarrow\simeq & \\ & \pi_{\gamma,\beta,\alpha} & \\ \int\gamma \overset{\simeq}{\nearrow} & & \overset{\simeq}{\nwarrow} \int\beta \end{array}$$

In fact the isomorphisms between these fibrations exactly correspond to equivalences in the definition (2.1) of the trijunction. □

2.4 From discrete trifibrations to trijunctions between presheaves

Proposition 2.9. *Given a functor* $R : (\mathcal{A}\times\mathcal{B}\times\mathcal{C})^{\mathrm{op}} \to \mathrm{Ens}$ *with* $\mathcal{A}$*,* $\mathcal{B}$*,* $\mathcal{C}$ *any small categories, or the associated discrete fibration* $\pi_R : \mathcal{G} \to \mathcal{A}\times\mathcal{B}\times\mathcal{C}$ *(called a discrete trifibration), there is an associated trijunction* $(\gamma_R,\beta_R,\alpha_R)$ *between toposes of presheaves* $\hat{\mathcal{A}} := \mathrm{Ens}^{\mathcal{A}^{\mathrm{op}}}$*,* $\hat{\mathcal{B}} := \mathrm{Ens}^{\mathcal{B}^{\mathrm{op}}}$*, and* $\hat{\mathcal{C}} := \mathrm{Ens}^{\mathcal{C}^{\mathrm{op}}}$*. Especially any bi-functor* $\mathcal{A}\times\mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ *determines such a trijunction.*

Proof. With $R_{\mathcal{C}}(C)(A,B) = R_{\mathcal{B}}(B)(C,A) = R_{\mathcal{A}}(A)(B,C) = R(A,B,C)$, $R_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{A}\times\mathcal{B})^{\mathrm{op}}}$, $R_{\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{C}\times\mathcal{A})^{\mathrm{op}}}$, $R_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{B}\times\mathcal{C})^{\mathrm{op}}}$. For F, G and H in $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, and $\hat{\mathcal{C}}$ we define $F\boxtimes G(A,B) = F(A)\times G(B)$, $H\boxtimes F(C,A) = H(C)\times F(A)$ and $G\boxtimes H(B,C) = G(B)\times H(C)$. Then

$$\gamma_R(F,G)(C) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}^{\mathrm{op}}}}(F\boxtimes G, R_{\mathcal{C}}(C)),$$
$$\beta_R(H,F)(B) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{C}^{\mathrm{op}}\times\mathcal{A}^{\mathrm{op}}}}(H\boxtimes F, R_{\mathcal{B}}(B)),$$
$$\alpha_R(G,H)(A) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{B}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}}}}(G\boxtimes H, R_{\mathcal{A}}(A)).$$

Then for example we associate to $\theta : F\boxtimes G\boxtimes H \to R$ a $\nu : H \to \gamma(F,G)$ by $(\nu_C(z))_{(A,B)}(x,y) = \theta_{(A,B,C)}(x,y,z)$. □

2.5 Annex 1: Classical facts on adjunctions

In 1958, Daniel Kan [9] introduced the notion of *adjoint functors*; then Peter Freyd (Princeton thesis, 1960) and William Lawvere (Columbia thesis, 1963) "emphasized the dominant position of adjunctions" [11, p. 103]:

Definition 2.10. *Let $\mathcal{A}$ and $\mathcal{C}$ be categories. Then a covariant functor $L : \mathcal{A} \to \mathcal{C}$ is called* left adjoint *to a covariant functor $R : \mathcal{C} \to \mathcal{A}$ (notation $\tau : L \dashv R$) if there exists a natural equivalence*

$$\tau : \mathrm{Hom}_{\mathcal{C}}(L(A), C) \simeq \mathrm{Hom}_{\mathcal{A}}(A, R(C)).$$

Proposition 2.11. *$\tau : L \dashv R$ is equivalent to $L \dashv R(\epsilon, \eta)$, with 2 natural transformations $\epsilon := \tau^{-1}(1_R) : LR \to \mathrm{Id}_{\mathcal{C}}$ and $\eta := \tau(1_L) : \mathrm{Id}_{\mathcal{A}} \to RL$ with the equations:*

$(\epsilon L)(L\eta) = \mathrm{Id}_L, \quad (R\epsilon)(\eta R) = \mathrm{Id}_R$. *Furthermore we get τ and τ^{-1} by:*

$$\tau(c : LA \to C) = R(c)\eta_A, \quad \tau^{-1}(a : A \to RC) = \epsilon_C L(a).$$

Proof. This is coming from lemmas 6.2 p.306 and 6.2* p.307 in [9]. See also [11, chap. IV, p. 80-81]. □

Definition 2.12. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. Then a covariant functor $L : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called* left adjoint *— with a parameter in $\mathcal{B}$ — to a functor $R : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ contravariant in $\mathcal{B}$ and covariant in $\mathcal{C}$ if there exists a natural equivalence*

$$\tau : \mathrm{Hom}_{\mathcal{C}}(L(A, B), C) \simeq \mathrm{Hom}_{\mathcal{A}}(A, R(B, C)).$$

Proposition 2.13. *Given $L : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and for each object B in $\mathcal{B}$ a right adjoint R_B to $L(-, B)$, with $\tau_B : L(-, B) \dashv R_B$, then these functors determine a unique functor $R : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ with an equivalence τ as in definition 2.12, with for every $c : C \to C'$, $R(B, c) = R_B(c)$, and with $\tau(A, B, C) = \tau_B(A, C)$.*

Proof. This is proved as theorem 4.1 p. 300 in [9]. See also [11, p. 100]. With $\epsilon_B = \tau_B^{-1}(1_{R_B})$ and $\eta_{B'} = \tau_{B'}(1_{L(-,B')})$, an explicit formula for $R(b, c)$ with $b : B' \to B$ and $c : C \to C'$ is

$$R(b, c) = R_{B'}(c)R_{B'}(\epsilon_B(C))R_{B'}(L(R_B(C), b))\eta_{B'}(R_B(C)) \qquad (\star)$$

□

Kan was especially motivated by the case of $\otimes$:

Proposition 2.14. *The functor* $\otimes : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ *is left adjoint to* $\mathrm{Hom} : \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab}$*, in the sense of definition 2.12.*

3. Triadic Galois Connections and ternary relations

3.1 Triadic Galois connections and residuations

In relation with the calculus of ternary relations between sets and the "triadic concept analysis" as introduced in [10] and [14], and the notion of "*trilattice*", the notion of a *triadic Galois connection* has been introduced in 1997 by Klaus Biedermann [1], [2], [3]. We adapt his definition, without references to trilattices, and with a slightly different system of notations, in order to show that this notion is a particular case of a trijunction.
NB: In this section 3.1 we use and extend the classical properties of Galois connections (see 3.3) to triadic Galois connections. So we get a mini-model of the theory of trijunctions, namely its reduction to the case of posets.

Definition 3.1. *A* triadic Galois connection *between 3 posets* $\mathcal{A} = (A, \leq)$, $\mathcal{B} = (B, \leq)$ *and* $\mathcal{C} = (C, \leq)$ *is the datum* (γ, β, α) *of 3* decreasing *functions* $\gamma : A \times B \to C$, $\beta : C \times A \to B$, $\alpha : B \times C \to A$*, such that for all* $a \in A$, $b \in B$, $c \in C$*:*

$$c \leq \gamma(\alpha(b,c), b), \quad c \leq \gamma(a, \beta(c,a)),$$

$$b \leq \beta(\gamma(a,b), a), \quad b \leq \beta(c, \alpha(b,c)),$$

$$a \leq \alpha(\beta(c,a), c), \quad a \leq \alpha(b, \gamma(a,b)).$$

Proposition 3.2. *A triadic Galois connection is equivalent to the datum* (γ, β, α) *of 3* decreasing *functions* $\gamma : A \times B \to C$, $\beta : C \times A \to B$, $\alpha : B \times C \to A$*, such that*

$$\forall a \in A\, \forall b \in B\, \forall c \in C\, \Big[c \leq \gamma(a,b) \Leftrightarrow b \leq \beta(c,a) \Leftrightarrow a \leq \alpha(b,c)\Big].$$

Proposition 3.3. *A* triadic Galois connection *is exactly the special case of a* trijunction *according to definition 2.1 in which* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *are posets.*

Proposition 3.4. *1 — Let* $(M, \leq)$ *be a sup-lattice and let* $\otimes : M \times M \to M$ *a binary law compatible with* sup. *Then with* $\mathcal{A} = (M, \leq)$, $\mathcal{B} = (M, \leq)$, $\mathcal{C} = (M, \geq)$, *and with* $\gamma(a, b) = a \otimes b$, *we get a triadic Galois connection* (γ, β, α) *in the sense of (def. 3.1).*
2 — Let $(M, \leq)$ *a sup-lattice and a triadic Galois connection* (γ, β, α) *between* $\mathcal{A} = (M, \leq)$, $\mathcal{B} = (M, \leq)$, $\mathcal{C} = (M, \geq)$. *Then* γ *is a binary law compatible with* sup.

Proof. 1 — We take $\beta(c, a) = {}^a c := \sup_{a \otimes b \leq c} b$, $\alpha(b, c) = c^b := \sup_{a \otimes b \leq c} a$, i.e. (see [4, p. 325]) the right and left residuals c/a of c by a, $c \backslash b$ of c by b.
2 — $\gamma(a, -)$ is a left adjoint, and $\gamma(-, b)$ is a left adjoint too. □

3.2 Functional counterpart of a ternary relation

Proposition 3.5. *A triadic Galois connection* (γ, β, α) *between the posets* $(\mathcal{P}(A), \subseteq)$, $(\mathcal{P}(B), \subseteq)$ *and* $(\mathcal{P}(C), \subseteq)$ *is equivalent to the datum of a ternary relation* $R \subset A \times B \times C$, *according to the association:*

$$R = R_\gamma := \{(a, b, c); c \in \gamma(\{a\}, \{b\})\},$$

$$R = R_\beta := \{(a, b, c); b \in \beta(\{c\}, \{a\})\},$$

$$R = R_\alpha := \{(a, b, c); a \in \alpha(\{b\}, \{c\})\},$$

$$\gamma(A', B') = \gamma_R(A', B') := \{c; \forall a' \in A' \, \forall b' \in B' \ (a', b', c) \in R\},$$

$$\beta(C', A') = \beta_R(C', A') := \{b; \forall c' \in C' \, \forall a' \in A' \ (a', b, c') \in R\},$$

$$\alpha(B', C') = \alpha_R(B', C') := \{a; \forall b' \in B' \, \forall c' \in C' \ (a, b', c') \in R\}.$$

Furthermore

$$C' \leq \gamma(A', B') \Leftrightarrow B' \leq \beta(C', A') \Leftrightarrow A' \leq \alpha(B', C') \Leftrightarrow A' \times B' \times C' \subseteq R.$$

Proof. It is an immediate reformulation of Biedermann [1], [2], [3]. □

Proposition 3.6. *Given a ternary relation* $R \subseteq A \times B \times C$ *, and subsets* $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$, *we get, with the notations of 3.1 and with*

$$R^*_C(C') = \{(a, b); \forall c' \in C' \ (a, b, c') \in R\},$$

$$R^*_B(B') = \{(c, a); \forall b' \in B' \ (a, b', c) \in R\},$$

$$R^*_A(A') = \{(b,c); \forall a' \in A' \ (a',b,c) \in R\},$$

an hexagonal picture of seven equivalent conditions:

$$\begin{array}{ccccc}
 & & A' \subseteq \alpha_R(B',C') & & \\
 & \nearrow & & \nwarrow & \\
C' \times A' \subseteq R^*_B(B') & & & & A' \times B' \subseteq R^*_C(C') \\
\downarrow & & A' \times B' \times C' \subseteq R & & \downarrow \\
C' \subseteq \gamma_R(A',B') & & & & B' \subseteq \beta_R(C',A') \\
 & \nwarrow & & \nearrow & \\
 & & B' \times C' \subseteq R^*_A(A') & &
\end{array}$$

Furthermore each of the six operators α_R, R^*_A, β_R, R^*_B, γ_R, R^*_C, determines the five others, and the relation R itself.

Proof. It is a direct complement to proposition 3.5, in the style of [8]. For the last point starting for example from the datum of α_R, we get R^*_A by $R^*_A(A') = \cup_{A' \subseteq \alpha_R(B',C')} B' \times C'$, etc. □

Proposition 3.7. *A triadic Galois connection between $\mathcal{A} = (\mathcal{P}(E), \subseteq)$, $\mathcal{B} = (\mathcal{P}(E), \subseteq)$, $\mathcal{C} = (\mathcal{P}(E), \supseteq)$ is equivalent to the datum of a ternary relation $R \subset E^3$.*

Proof. A sup-compatible binary law $\gamma : \mathcal{P}(E)^2 \to \mathcal{P}(E)$ is equivalent to a map $r : E^2 \to \mathcal{P}(E)$, i.e. a ternary relation $R \subset E^3$. □

3.3 Annex 2: Classical facts on Galois connections

Clearly a posteriori an adjunction could be understood as a categorification of a Galois connection in the following sense of definition 3.8.

In his talk at the Summer Meeting of AMS at Chicago in 1941, Oystein Ore introduced — as a tool for the calculus of binary relations — the notion of a *Galois connexion* [13] (see also Garrett Birkhoff [4, p.124]) — or equivalently *Galois connection* (also named *Galois correspondence*) —, as follows.

Definition 3.8. *A [dyadic]* Galois connection *between 2 posets $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ is the datum (β, α) of two* decreasing *functions $\beta : A \to B$ and $\alpha : B \to A$ such that*

$$\forall a \in A \left[a \leq \alpha(\beta(a))\right], \quad \forall b \in B \left[b \leq \beta(\alpha(b))\right].$$

Proposition 3.9. *It is equivalent for a Galois connection to assume that* α *and* β *are* ordinary *functions such that*

$$\forall a \in A \forall b \in B \Big[b \leq \beta(a) \Leftrightarrow a \leq \alpha(b) \Big].$$

Proposition 3.10. *A decreasing function* $\beta : A \to B$ *between two posets* $\mathcal{A}$ *and* $\mathcal{B}$ *determines two increasing functions* $\beta^l : \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ *and* $\beta^r : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$, *with* $\beta^l = \beta^{r\,\mathrm{op}}$ *and* $\beta^r = \beta^{l\,\mathrm{op}}$*; a* Galois connection *as* (β, α) *in 3.8 is exactly an* adjunction *in the sense of 2.10, namely* $\alpha^l \dashv \beta^r$*, or, equivalently,* $\beta^l \dashv \alpha^r$.

Proposition 3.11. *A Galois connection* (β, α) *between the posets* $(\mathcal{P}(A), \subseteq)$ *and* $(\mathcal{P}(B), \subseteq)$ *is equivalent to the datum of a binary relation* $R \subset A \times B$, *according to the association:*

$$R = \{(a, b); b \in \beta(\{a\})\} = \{(a, b); a \in \alpha(\{b\}),$$

$$\beta(A') = \{b; \forall a' \in A' \ (a', b) \in R\}, \quad \alpha(B') = \{a; \forall b' \in B' \ (a, b') \in R\}.$$

Furthermore

$$A' \subseteq \alpha(B') \Leftrightarrow B' \subseteq \beta(A') \Leftrightarrow A' \times B' \subseteq R.$$

Proof. See Ore [13, thm.10, p.503]. □

4. The auto-trijunction on a topos or an algebraic universe

4.1 Algebraic universe

We recall the definition of an *algebraic universe*, a notion we have developed in the 70's (see [5], [6]).
An *algebraic universe* is a category $\mathcal{X}$ with finite limits and colimits equipped with a contravariant functor $P : \mathcal{X} \to \mathcal{X}^{\mathrm{op}}$ such that $P \dashv P^{\mathrm{op}}$, this adjunction being monadic (analogous to *Stone duality*); we assume also that for any X in $\mathcal{X}$, the map $\eta_X : X \to PPX$ is factorized as $\psi_X a_X$ with $\psi_X : PX \to PPX$ (meeting map) and $a_X : X \to PX$ (atom map), and there are also $\pi_X : PX \to PPX$ (inclusion map), $\nu_X : PX \to PX$ (negation map) and $c_X : X^2 \to PX$ (pairing map); among these data a precise system of equations is assumed.

In any algebraic universe the construction P on objects is extended in two ways in a covariant functor: for $f : X \to Y$ we take:

$$\exists f = P(Pf.a_Y)\psi_X, \quad \forall f = P(Pf.a_Y)\pi_X,$$

$$\bigcup_X = P\eta_X\psi_{PX}, \quad \bigcap_X = P\eta_X P\pi_{PX}\psi_{PX}.$$

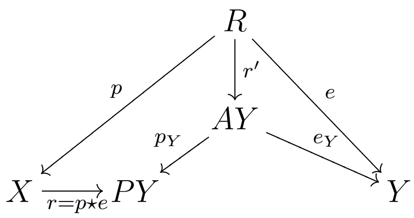
Given a relation $\rho = (p, e) : R \to A \times B$ we introduce its "characteristic map":

$$r = p \star e = (\exists e)(Pp)a_A : A \to PB.$$

Proposition 4.1. *(See [5]) Given a complete lattice equipped with a* sup-*compatible abelian monoid law* $\mathbb{L} = (L \leq, \otimes)$ *there is a structure of algebraic universe on* Ens *in which* $PX = L^X$*, and this generates the calculus of* $\mathbb{L}$*-fuzzy relations.*

4.2 Topos as an algebraic universe

An elementary *topos* (in the sense of Lawvere-Tierney, see [12]) is a category $\mathcal{E}$ with finite limits and colimits, with exponentials and subobject classifier. This is reducible to the conditions that $\mathcal{E}$ is with finite limits and colimits, and is such that for all object Y in $\mathcal{E}$ there is $(PY \xleftarrow{p_Y} AY \xrightarrow{e_Y} Y)$ such that for every $(X \xleftarrow{p} R \xrightarrow{e} Y)$ there is a unique $r = p \star e : X \longrightarrow PY$ and a unique $r' : R \longrightarrow AY$ with a pullback $(p, r'; r, p_Y)$ with $e = e_Y.r'$:



Proposition 4.2. *In a topos* $\mathcal{E}$ *the construction* P *is a contravariant functor which is its own adjoint:*

$$(P : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}) \dashv (P^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}),$$

and in fact with this P *we get a structure of an algebraic universe.*

Proof. It is well known. Given a morphism $f : Y \to X$, we get $P(f) : PX \to PY$ by $aX := 1_X \star 1_X$ and $Pf = ((a_X f) \star 1_Y$. Starting with $r : X \to PY$, $r = p{\star}e$, we get its "converse" $s : Y \to PX$ with $p = r^*(p_Y)$, $r' = p^*(r)$, $e = e_Y r'$, and $s = e{\star}p$. Then a structure of an algebraic universe is given by this P, with ψ and π in the internal language:

$$\psi_X(A) = \{B; \exists x(x \in A \,\&\, x \in B)\},\ \pi_X(A) = \{B; \forall x(x \in B \Rightarrow x \in A)\}.$$

□

4.3 Symmetric tensors with right adjoints

Proposition 4.3. *With $\mathcal{A} = \mathrm{Ab}$, $\mathcal{B} = \mathrm{Ab}$, $\mathcal{C} = \mathrm{Ab}^{\mathrm{op}}$, we get a trijunction (def. 2.1) with $\gamma(A, B) = A \otimes B$, $\beta(C, A) = \mathrm{Hom}(A, C)$, and with $\alpha(B, C) = \mathrm{Hom}(B, C)$.*

Proof. This proposition results of proposition 2.14 , by imitation of proposition 3.4. Details of the proof arise also from proposition 2.6. □

Proposition 4.4. *In a symmetric monoidal closed category $\mathcal{E}$, there is a trijunction between $\mathcal{E}$, $\mathcal{E}$ and $\mathcal{E}^{\mathrm{op}}$ with*

$$\gamma(A, B) = A \otimes B, \quad \beta(C, A) = C^A, \quad \alpha(B, C) = C^B.$$

Proof. Analogous to the case in proposition 4.3. In a monoidal closed category, for any object B the functor $(-) \otimes B$ has a right adjoint $(-)^B$, and for any A the functor $A \otimes (-)$ has a right adjoint $(-)^A$. We conclude by proposition 2.6. □

Proposition 4.5. *In a symmetric monoidal closed category $\mathcal{E}$, with any object L, there is an associated (auto-)trijunction between $\mathcal{E}$, $\mathcal{E}$ and $\mathcal{E}$ with*

$$\gamma(A, B) = L^{A \otimes B}, \quad \beta(C, A) = L^{C \otimes A}, \quad \alpha(B, C) = L^{B \otimes C}.$$

Proof. $\mathrm{Hom}_{\mathcal{E}}(X, L^Y) \simeq \mathrm{hom}_{\mathcal{E}}(X \otimes Y, L) \simeq \mathrm{Hom}_{\mathcal{E}}(Y, L^X)$, so the functor $L^{(-)} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$ is left adjoint to L^{op}. One of the equivalences in a trijunction (definition 2.1) is given by: $\mathrm{Hom}_{\mathcal{E}}(A, L^{B \otimes C}) \simeq \mathrm{Hom}_{\mathcal{E}}(A \otimes (B \otimes C), L) \simeq \mathrm{Hom}_{\mathcal{E}}(B \otimes (A \otimes C), L) \simeq \mathrm{Hom}_{\mathcal{E}}(B, L^{A \otimes C})$. □

4.4 Canonical auto-trijunction on an algebraic universe

Proposition 4.6. *Given an algebraic universe $\mathcal{X}$ — for example a topos or a category of fuzzy sets (cf. propositions 4.1 and 4.2) — we get an auto-trijunction (γ, β, α) between $\mathcal{A} = \mathcal{X}$, $\mathcal{B} = \mathcal{X}$,$\mathcal{C} = \mathcal{X}$, with*

$$\gamma(A, B) = P(A \times B), \quad \beta(C, A) = P(C \times A), \quad \alpha(B, C) = P(B \times C).$$

Proof. An algebraic universe is a cartesian closed category, and we have $PX = P(1)^X$. So we have just to apply proposition 4.5. □

4.5 Toward a calculus of triadic Galois connections in a topos

In fact the auto-trijunction in proposition 4.6 does not depend on ψ, π, etc., but only on the composition $\psi.a = \eta$, the cartesian closed structure on the topos or the algebraic universe, and the object $P(1)$. Nevertheless:

Proposition 4.7. *In a topos $\mathcal{E}$, using the canonical auto-trijunction (proposition 4.6) and the data ψ, π, etc., we can internally recover a theory of Galois connections and triadic Galois connections.*

Proof. We indicate only the starting point. From a ternary relation $(p, q, r) : R \to A \times B \times C$, we can construct the different terms in the hexagon pictured in proposition 3.6 in the case of the category Ens.
We consider $c = r \star (p, q) : C \to P(A \times B)$, we know how to construct $\exists c : PC \to PP(A \times B)$, $\bigcap_{A \times B} : PP(A \times B) \to P(A \times B)$, and the composition $R^*_C = \bigcap_{A \times B} \exists c : PC \to P(A \times B)$.
We consider also $a' = (r, q) \star p : C \times B \to PA$, $\alpha_R = \bigcap_A \exists(a')$.
A calculus of ternary relations in terms of *internal triadic Galois connections* is available in any topos; this works also in any category of fuzzy sets.

□

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René Guitart,
Université Paris Diderot Paris 7. IMJ-PRG. UMR 7586
Bâtiment Sophie Germain.
Case 7012, 75205 Paris Cedex 13
rene.guitart@orange.fr