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Q-VALUED SETS AND RELATIONAL-SHEAVES

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Abstract

We show that a sheaf for a *quantaloid* is an idempotent suprema-preserving lax-semifunctor (a *relational-sheaf*). This implies that for a Grothendieck topos $\mathcal{E}$ a sheaf is a relational-sheaf on the category of relations of $\mathcal{E}$ and thus $\mathcal{E}$ is equivalent to the category of relational-sheaves and *functional-transformations*. The theory is developed in the context of enriched taxons, which are enriched semicategories with an added structural requirement.

Nous montrons qu'un faisceau de *quantaloides* est un semifoncteur lax idempotent, qui préserve les supréma (un *faisceau relationnel*). Ceci implique que pour un topos de Grothendieck $\mathcal{E}$, un faisceau est un faisceau relationnel sur la catégorie des relations de $\mathcal{E}$ et donc $\mathcal{E}$ est équivalent à la catégorie des faisceaux relationnels et *transformations fonctionnelles*. Cette théorie est développée dans le cadre de "taxons" enrichis, c'est à dire des semicatégories enrichies avec une condition structurelle additionelle. [1]

1 Introduction

The main result of this paper is that a sheaf for an involutive quantaloid(Q), is an involution and infima preserving lax-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{Rel}$ that is also an idempotent. We call such a lax-semifunctor a *relational-sheaf*. It follows from this that if $\mathcal{E}$ is a Grothedieck topos,

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then the category of relational-sheaves and transformations for the involutive quantaloid of relations on $\mathcal{E}$ is equivalent to $\mathcal{E}$ itself.

From $\mathcal{H}$ a complete Heyting algebra, Higgs[14] constructed the category of $\mathcal{H}$-valued sets and showed that this category is equivalent to the category of sheaves on $\mathcal{H}$. Using this as a template others, including Canani, Borceux and Cruciani[3], Mulvey[21], Van den Bossche[28], Gylys[12], Garraway[9] et al, have explored $\mathcal{Q}$-valued sets for $\mathcal{Q}$ a *quantale* and more generally for supremum-enriched categories (*quantaloids*). The term quantale was derived from physics in the context of quantum logic and was introduced by Mulvey[20] to represent the lattice of open sets for a non-commutative topology. Boolean and Heyting algebras are commutative quantales, leading us to interpret a quantale as a model for non-commutive logic. A classic example is the lattice of closed right-ideals of a C^*-algebra which is a quantale that uses the $(-)^*$ operation as an involution on the ideals. This particular quantale has been studied in detail by Mulvey and Pelletier[22] in their work generalizing the Gelfand-Naimark theorem. Quantaloids arise naturally in many settings, for example both the category of sets and relations, and in general the category of relations for a Grothedieck topos are quantaloids. Pitts[24], bringing together the ideas of Carboni/Walters[6] and Freyd[8], looked in detail at the category of *bounded complete distributive categories of relations (bcDCR)* noting that it is equivalent to the category of Grothendieck toposes. Essentially a distributive category of relations is a quantaloid with added structure that among other things endows the quantaloid with an involution. The completion (with respect to copruducts and the splitting of symmetric idempotents) of a distributive category of relations results in the usual definition of $\mathcal{Q}$-valued sets. Building on this we will construct categories of $\mathcal{Q}$-taxons and $\mathcal{Q}$-categories for a quantaloid. A good source for the background of these notions can be found in a series of papers by Stubbe[26, 27, 13].

The main building blocks for this paper are involutive supremum-enriched semicategories and it is to these that we will apply the term quantaloid. Enriched category theory grew out of the work of Benabou[4], Kelly[15] and others. This is generalized to enriched *taxons* which are enriched semicategories with additional structure. The concept originates from an idea of Koslowski in [16]. A taxon to him is a *sem-*

icategorical structure with the added condition that the composition morphism is a particular coequalizer. Both Garraway[9] and Moens et al.[19] expanded on this with a notion of enriched taxons (which Moens referred to as regular categories). The focus of the first work was to use taxons as a tool towards understanding $\mathcal{Q}$-valued sets while the second is a more in-depth study of enriched taxons (regular categories) in general. The major difference in the two approaches is that the former defines a transformation to be a family of morphisms indexed by the arrows in the base taxon while the latter defines them in the traditional way using an objects-indexed family of morphisms. This is the setting Stubbe[26][13] used when he worked with taxons enriched in a supremum-enriched category and more generally a supremum-enriched semicategory. In the present work we will use both forms to define morphisms of relational-presheaves and relational-sheaves.

Rosenthal[25] defined a relational-presheaf on a supremum-enriched category, $\mathcal{Q}$, to be a *lax-functor* $F : \mathcal{Q}^{co} \to \mathbf{Rel}$, and a morphism of relational-presheaves is a lax-natural transformation in which each morphism is a function. A relational-presheaf is then said to be continuous if it preserves infima. Rosenthal then showed that this category of $\mathcal{Q}$-categories is equivalent to the category of continuous relational-sheaves. In the present work we will work with involutive supremum-enriched taxons and define a relational-presheaf to be an involution and infima-preserving lax-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{Rel}$.

The purpose of this paper is to relate the ideas of $\mathcal{Q}$-valued sets and relational-presheaves using as a guide the enriched taxon structure and thus creating an equivalence of categories that shows that all Grothedieck toposes can be thought of as categories of a particular type of relational-presheaf. In particular we will show that a sheaf is a symmetric idempotent relational-presheaf.

We begin with an exploration of enriched taxons and natural transformations with a focus on the implications these have in the supremum and infimum-enriched settings. We follow this with a short exploration of quantaloids and of the main structure and properties of distributive categories of relations (DCR). Using this as our template to build from we define the categories of relational-presheaves and relational-sheaves. Next is an examination of $\mathcal{Q}$-taxons (which Stubbe[27] calls $\mathcal{Q}$-regular categories) focusing on the category of $\mathcal{Q}$-valued sets. In particular we will construct an equivalence between subcategories of the categories of $\mathcal{Q}$-taxons and $\mathcal{Q}$-profunctors that are maps ($\mathcal{Q}$-**Set**) and $\mathcal{Q}$-taxons and $\mathcal{Q}$-semifunctors ($\mathcal{Q}$-**Tax**). The former is usually referred to as the category of $\mathcal{Q}$-valued sets. We finish by constructing an equivalence between the category $\mathcal{Q}$-**Set** and the category of relational-sheaves from which it will follow that if $\mathcal{Q}$ is a bounded distributive category of relations, then the category of relational-sheaves on $\mathcal{Q}$ is a Grothendieck topos.

2 Enriched Taxons

There is an exercise early in MacLane[17] that asks the student to show that the traditional definition of a natural transformation as a family of object indexed morphisms has an equivalent formulation in terms of an arrows indexed family of morphisms. The equivalence is obtained by focusing on the identity morphisms in the base category. This equivalence fails if some of the identity arrows are missing as maybe the case in a semicategory. In this more general setting the arrows based definition is problematical since there is no canonical way to define the composition of two transformations. This problem is circumvented when we require that the composition morphism of a semicategory be a particular coequalizer.

Definition 2.1 Let $\mathcal{V}$ be a monoidal category with all coproducts. A *$\mathcal{V}$-enriched semicategory* $\mathcal{C}$ consists of the following

- A set $|\mathcal{C}|$ called the objects of $\mathcal{C}$.
- For each pair of objects $A, B \in |\mathcal{C}|$ an object in $\mathcal{V}$ denoted $\mathcal{C}(A, B)$.
- For each triple of objects $A, B, C \in |\mathcal{C}|$ a $\mathcal{V}$-morphism, called composition,

$$C_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C),$$

  which is required to satisfy the usual associativity diagram.

A $\mathcal{V}$-enriched semicategory is a $\mathcal{V}$-*enriched taxon* if,

$$m : \coprod_X \mathcal{C}(X, C) \otimes \mathcal{C}(A, X) \to \mathcal{C}(A, C),$$

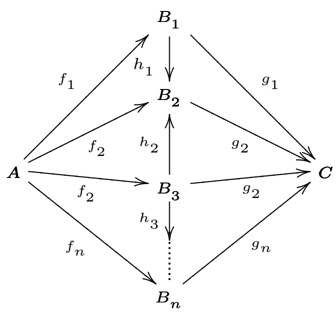
the canonical morphism obtained from the composition morphisms and the universal property of coproducts, is the coequalizer of

$$\coprod_{U,V} \mathcal{C}(U, C) \otimes \mathcal{C}(V, U) \otimes \mathcal{C}(A, V) \underset{m \circ 1}{\overset{1 \circ m}{\rightrightarrows}} \coprod_X \mathcal{C}(X, C) \otimes \mathcal{C}(A, X) \xrightarrow{m} \mathcal{C}(A, C)$$

where $1 \circ m$ and $m \circ 1$ are obtained from $\mathbf{1} \otimes m$ and $m \otimes \mathbf{1}$ respectively using the universal property of coproducts. ♢

We will denote the composite $\mathcal{C}_{ABC}(f, g)$ as fg or in certain instances for supremum-enriched semicategories by $f\&g$. The term taxon originated with unpublished work of Wood and Paré while they were exploring semicategories and Koslowski[16] used the term in this more specific setting. The enriched setting was studied in detail by Garraway[9] and Moens[19]. The latter refers to these as enriched regular-categories and defines them in terms of a coend instead of a coequalizer.

Example 2.2 When $\mathcal{V}$ is the monoidal category of sets and functions a **Set**-taxon can be thought of as a 'semicategory' with additional structure. When the composition arrow is a coequalizer of the appropriate type, then we have that composition is associative (so we need not actually require associativity for taxons). In addition we also have that if $g_1 f_1 = g_n f_n$ are equal composable arrows, then there is a zig-zag of arrows h_i and composable arrows g_i, f_i so that the following diagram (or with the h_i arrows reversed) commutes.



□

Definition 2.3 If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-semicategories, then a *semifunctor*, $F : \mathcal{C} \to \mathcal{D}$ consists of

- A function $F : |\mathcal{C}| \to |\mathcal{D}|$
- For each pair of objects $A, B \in |\mathcal{C}|$ a morphism

$$F_{AB} : \mathcal{C}(A,B) \to \mathcal{D}(FA, FB)$$

such that the following square commutes

$$\begin{array}{ccc}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) & \xrightarrow{F_{BC} \otimes F_{AB}} & \mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FB) \\
{\scriptstyle \mathcal{C}_{ABC}} \downarrow & = & \downarrow {\scriptstyle \mathcal{D}_{FAFBFC}} \\
\mathcal{C}(A,C) & \xrightarrow[F_{AC}]{} & \mathcal{D}(FA,FC)
\end{array}$$

♢

We will now use the arrows based definition of natural transformation as a template to define transformations of semifunctors between enriched taxons.

Definition 2.4 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two $\mathcal{V}$-semifunctors. A $\mathcal{V}$-*natural transformation* $\gamma : F \Rightarrow G$ consists of a $|\mathcal{C}| \times |\mathcal{C}|$ indexed family of $\mathcal{V}$-morphisms

$$\langle \gamma_{AB} : \mathcal{C}(A,B) \to \mathcal{D}(FA, GB) \rangle$$

with the property that for every triple of objects $A, B, C \in |\mathcal{C}|$ the following diagrams commute

$$\begin{array}{ccc}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) & \xrightarrow{G_{BC} \otimes \gamma_{AB}} & \mathcal{D}(GB,GC) \otimes \mathcal{D}(FA,GB) \\
{\scriptstyle \mathcal{C}_{ABC}} \downarrow & = & \downarrow {\scriptstyle \mathcal{D}_{FAGBGC}} \\
\mathcal{C}(A,C) & \xrightarrow[\gamma_{AC}]{} & \mathcal{D}(FA,GC)
\end{array}$$

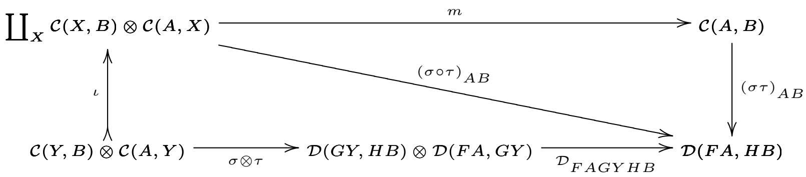
$$\begin{array}{ccc}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) & \xrightarrow{\gamma_{BC} \otimes F_{AB}} & \mathcal{D}(FB,GC) \otimes \mathcal{D}(FA,FB) \\
{\scriptstyle \mathcal{C}_{ABC}} \downarrow & = & \downarrow {\scriptstyle \mathcal{D}_{FAFBGC}} \\
\mathcal{C}(A,C) & \xrightarrow[\gamma_{AC}]{} & \mathcal{D}(FA,GC)
\end{array}$$

♢

The composite of two $\mathcal{V}$-natural transformations $F \overset{\tau}{\Longrightarrow} G \overset{\sigma}{\Longrightarrow} H$ is defined to be the family of unique $\mathcal{V}$-morphisms,

$$\langle (\sigma\tau)_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, HB) \rangle,$$

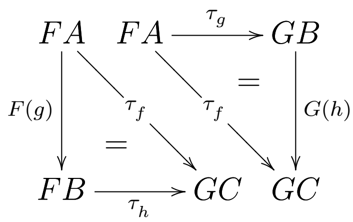
determined by the following diagram.



Here $(\sigma \circ \tau)_{AB}$, which is derived from the universal property of coproducts, coequalizes the morphisms $1 \circ m$ and $m \circ 1$. Since m is a coequalizer this determines the unique morphism $(\sigma\tau)_{AB}$

We can interpret a $\mathcal{V}$-semifunctor F as a transformation τ_F which also happens to be the identify transformation for F since $\tau\tau_F = \tau = \tau_G\tau$. It is easy to see that this gives us a 2-category for which the interchange law holds[1]. We will denote this 2-category by $\mathcal{V}$-**Tax** and denote the associated hom categories by $\mathbf{Tax}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$. In the instance when $\mathcal{V}$ is the monoidal category of sets and functions then we simply call the 2-category **Tax** and the associated hom categories $\mathbf{Tax}(\mathcal{C}, \mathcal{D})$.

Example 2.5 In **Tax** the definition of a transformation $F \overset{\tau}{\Longrightarrow} G$ between two semifunctors $\mathcal{C}$ and $\mathcal{D}$, implies that for every morphism $f : A \to C$ in $\mathcal{C}$ there is an associated morphism $\tau_f : FA \to GC$ in $\mathcal{D}$ such that if $f = gh$, then the following diagrams commute.



If $\sigma : G \to H$ is a second transformation then the definition of the compostion of transformations implies that $(\sigma\tau)_f = \sigma_g\tau_h$ for some (and hence for all) composites $gh = f$.

Extending this example to the monoidal category of **Sup**-lattices (and by duality **Inf**-lattices) we find that the composition of transformations τ and σ is defined by setting $(\sigma\tau)_q$ equal to $\bigvee_i \{\sigma_{g_i}\tau_{h_i}\}$ for some, and hence for all, families of morphisms $\langle g_i h_i \rangle$ such that $\bigvee_i g_i h_i = f$. Note for later that if $\mathcal{Q}$ is a supremum-enriched taxon and $\mathcal{Q}_1$ is infimum-enriched then for $F, G, H : \mathcal{Q}^{co} \to \mathcal{Q}_1$ infima-preserving semifunctors and $\tau : F \Rightarrow G$, and $\sigma : G \Rightarrow H$ transformations, we have that $(\sigma\tau)_f = \bigwedge_i \sigma_{g_i}\tau_{h_i}$ for some family of morphisms $\langle g_i h_i \rangle$ such that $\bigvee_i g_i h_i = f$. Later we will use these ideas as a template to define the category of relational-presheaves. □

Henceforth assume that $\mathcal{V}$ is the monoidal category of supremum-enriched lattices (**Sup**). In this context we utilize the theory of lax-semifunctors and lax-transformations (bicategory morphisms in the sense of Benabou[4]) to generalize our notions leading to our definition of relational-presheaves and relational-sheaves.

Definition 2.6 Let $\mathcal{C}$ and $\mathcal{D}$ be two **Sup**-taxons. A *lax-semifunctor* consists of

- A function $F : |\mathcal{C}| \to |\mathcal{D}|$
- For each pair of objects $A, B \in |\mathcal{C}|$ a morphism

$$F_{AB} : \mathcal{C}(A,B) \to \mathcal{D}(FA, FB)$$

such that for every triple of objects $A, B, C \in |\mathcal{C}|$ we have the following inequality

$$\begin{array}{ccc}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) & \xrightarrow{F_{BC} \otimes F_{AB}} & \mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FB) \\
{\scriptstyle \mathcal{C}_{ABC}} \downarrow & \geq & \downarrow {\scriptstyle \mathcal{D}_{FAFBFC}} \\
\mathcal{C}(A,C) & \xrightarrow[F_{AB}]{} & \mathcal{D}(FA,FC)
\end{array}$$

♢

Definition 2.7 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two lax-semifunctors between **Sup**-taxons. A *pretransformation* $F \overset{\tau}{\Longrightarrow} G$ consists of an $|\mathcal{C}| \times |\mathcal{C}|$ indexed family of suprema-preserving morphisms

$$\left\langle \gamma_{AB} : \mathcal{C}(A,B) \to \mathcal{D}(FA, GB) \right\rangle$$

♢

We will define the composite of two pretransformations τ and σ to be the pretransformation obtained using the composition of transformations. That is $(\sigma\tau)_f$ equals $\bigvee_i\{\sigma_{g_i}\tau_{h_i}\}$ for some family of morphisms $\langle g_i, h_i\rangle$ such that $\bigvee_i g_i h_i = f$. It is easy to see that this is a quantaloid with the operations defined pointwise. Observe now that a lax-semifunctor F can be interpreted as a pretransformation $F \stackrel{\tau_F}{\Longrightarrow} F$ and by laxity that $\tau_F\tau_F \leq \tau_F$.

Definition 2.8 If a pretransformation $F \stackrel{\tau}{\Longrightarrow} G$ satisfies the inequalities $\tau\tau_F \leq \tau$ and $\tau_G\tau \leq \tau$, then we will call it a *modular-transformation.* Denote the semicategory of lax-semifunctors between **Sup**-taxons $\mathcal{C}$ and $\mathcal{D}$ and modular-transformations by $\mathbf{Lax}(\mathcal{C}, \mathcal{D})$ ♢

2.1 Modules and the Karoubian envelope

Definition 2.9 A *quantaloid* $\mathcal{Q}$ is a supremum-enriched semicategory together with an involution $()^* : \mathcal{Q}^{op} \to \mathcal{Q}$. The involution is a suprema-preserving semifunctor that is the identity function on objects and for which $()^* \circ ()^* = 1_{\mathcal{Q}}$.

♢

Example 2.10 The following are examples of quantaloids

- Boolean and Heyting algebras with involution the identity function.
- The power set of a group with the involution determined by the inverse operation. If A and B are subsets, then the composition is given by
$$AB = \{ab \mid a \in A \text{ and } b \in B\}$$
- The category of sets and relations is a quantaloid with involution given by the inverse relation.
- The category of relations of a Grothendieck topos is a quantaloid.
- The lattice of closed right-ideals of a C*-algebra with the involution determined by the inherent $(-)^*$ operation. If A and B

are closed right-ideals the composition is the closure of the set $AB = \{ab \mid a \in A \text{ and } b \in B\}$. For complex numbers the involution is conjugation.

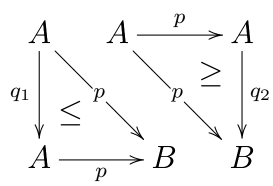
- The two object category constructed from a C*-algebra with objects 0 and 1 where the hom sets are respectively; closed two-sided ideals $Hom(0,0)$, closed right-ideals $Hom(1,0)$, closed left-ideals $Hom(0,1)$ and closed linear subspaces respectively $Hom(1,1)$. The composition of morphisms A and B is the closure of the set $AB = \{ab \mid a \in A \text{ and } b \in B\}$

□

At this point we will take a step back from our work and introduce the semicategory of modules for a quantaloid $\mathcal{Q}$. Recall that we require a quantaloid to come equipped with an involution and so any construct involving quantaloids that we make will incorporate some symmetry condition defined in terms of the involution

Definition 2.11 Let $\mathcal{Q}$ be a quantaloid. The semicategory of *modules* on $\mathcal{Q}$ has as its objects morphisms $q : A \to A$ in $\mathcal{Q}$, such that $qq \leq q$ (a module) and $q = q^*$ (symmetric).

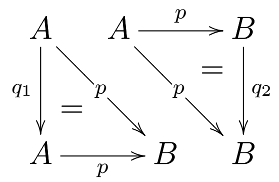
An arrow $q_1 \xrightarrow{p} q_2$ between modules $q_1 : A \to A$ and $q_2 : B \to B$ is a morphism $p : A \to B$ in $\mathcal{Q}$ that satisfies the inequalities $q_2 p \leq p$ and $pq_1 \leq p$. Pictorially we have



Denote this semicategory by $\mathbf{Mod}(\mathcal{Q})$. ♢

$\mathbf{Mod}(\mathcal{Q})$ is a quantaloid where the appropriate structure is defined pointwise. It is interesting to note that the semicategory $\mathbf{Mod}(\mathcal{Q})$ (temporarily suppressing the symmetry condition) is equivalent to the semicategory $\mathbf{Lax}(\mathbf{1}, \mathcal{Q})$. Also the semicategory of lax-semifunctors and modular-transformations is the semicategory of modules on the quantaloid of lax-semifunctors and pretransformations.

Definition 2.12 Let $\mathcal{Q}$ be a quantaloid. The *Karoubian envelope* is the category whose objects are symmetric idempotent arrows in $\mathcal{Q}$. if $q_1 : A \to A$ and $q_2 : B \to B$ are objects, then an arrow $p : q_1 \to q_2$ is a morphism $p : A \to B$ in $\mathcal{Q}$ that satisfies the following diagrams



Denote that Karoubian envelope of $\mathcal{Q}$ by $\mathbf{Kar}(\mathcal{Q})$. $\diamondsuit$

The Karoubian envelope construction will be of particular use to us later. We draw attention to the similarities in the definition of the Karoubian envelope and the arrows based definition of transformations. Building from this we will use the Karoubian envelope to define *relational-sheaves* and the transformations between them. In addition we will use the Karoubian envelope to motivate a definition of $\mathcal{Q}$-*categories* leading to an equivalence between the two structures.

Suppressing the symmetry condition one can argue that the Karoubian envelope is the preferable way to create a category out of a **Set**-taxon since it is a 2-semifunctor, is right-adjoint to the inclusion of **Cat** in **Tax**, and the functor $\mathbf{Kar}(-) : \mathbf{Tax} \to \mathbf{Cat}$ is the representable 2-semifunctor $\mathbf{Tax}(\mathbf{1}, -) : \mathbf{Tax} \to \mathbf{Cat}$, where $\mathbf{1}$ is the initial category (Garraway [9]).

It is easy to see that if $\mathcal{Q}$ is a quantaloid, then the category $\mathbf{Kar}(\mathcal{Q})$ is equivalent to the category $\mathbf{Kar}(\mathbf{Mod}(\mathcal{Q}))$. Thus, when $\mathcal{Q}$ is a **Sup**-taxon, $\mathbf{Kar}(\mathbf{Lax}(\mathbf{1}, \mathcal{Q}))$ is the category $\mathbf{Tax}_{\mathbf{Sup}}(\mathbf{1}, \mathcal{D})$ and in general:

Theorem 2.13 Let $\mathcal{C}$ and $\mathcal{D}$ be **Sup**-taxons, then the Karoubian envelope $\mathbf{Kar}(\mathbf{Lax}(\mathcal{C}, \mathcal{D}))$ is equivalent to the category $\mathbf{Tax}_{\mathbf{Sup}}(\mathcal{C}, \mathcal{D})$.

3 Distributive Categories of Relations

In this section we will do a quick review of distributive categories of relations (DCR), which are unital-quantaloids (supremum-enriched categories) with extra structure. Their study grew in particular out of the work of Carboni & Walters[6], Freyd[8] and others. First we begin by recalling some well know ideas.

Let $\mathcal{Q}$ be a unital-quantaloid. An arrow $A \xrightarrow{q} B$ in $\mathcal{Q}$ is a map, if there is a second arrow $B \xrightarrow{q^{\#}} A$, also in $\mathcal{Q}$, satisfying

$$1_A \leq q^{\#}q \text{ and } qq^{\#} \leq 1_B.$$

If q is a map then we represent its relationship to $q^{\#}$ by $q \dashv q^{\#}$. Denote the subcategory of $\mathcal{Q}$ whose arrows are maps by $\mathbf{Map}(\mathcal{Q})$. If $\mathcal{Q}$ is a unital-quantaloid and $A \xrightarrow{q} B$ is a map in $\mathcal{Q}$, then the following properties are well known.

- $q = qq^{\#}q$
- q is monomorphic if and only if $1_A = q^{\#}q$
- q is epimorphic if and only if $1_B = qq^{\#}$
- q is isomorphic if and only if it is both monomorphic and epimorphic.

A quantaloid $\mathcal{Q}$ satisfies Freyd's law of modularity if for every triple of arrows $A \xrightarrow{q} B$, $B \xrightarrow{p} C$ and $A \xrightarrow{r} C$ in $\mathcal{Q}$, then

$$pq \wedge r \leq p(q \wedge p^* r)$$

A particular consequence of Freyd's modularity law is that every map is defined in terms of its involute.

Theorem 3.1 If $\mathcal{Q}$ is a unital-quantaloid that satisfies Freyd's law of modularity and if $q \dashv q^{\#}$ (ie: q is a map), then $q^{\#} = q^*$.

Proof: First observe that

$$qq^{\#} = qq^{\#}q^{\#^*}q^{\#} \leq q^{\#^*}q^{\#} \quad \text{and} \quad q^{\#^*}q^* = q^{\#^*}q^*qq^* \leq qq^*$$

We now apply Freyd's law twice.

First we show that $q^{\#} \leq q^{*}$

$$\begin{aligned} q^{\#} &= q^{*}q^{\#^{*}}q^{\#} \wedge q^{\#} \\ &\leq q^{*}(q^{\#^{*}}q^{\#} \wedge qq^{\#}) \\ &= q^{*}qq^{\#} \\ &\leq q^{*} \end{aligned}$$

And now we show that $q^{*} \leq q^{\#}$ from which equality follows.

$$\begin{aligned} q^{*} &= q^{\#}qq^{*} \wedge q^{*} \\ &\leq q^{\#}(qq^{*} \wedge q^{\#^{*}}q^{*}) \\ &= q^{\#}q^{\#^{*}}q^{*} \\ &\leq q^{\#} \end{aligned}$$

∎

We now turn to a quick review of distributive categories of relations.

Definition 3.2 A unital-quantaloid is *cartesian* if there is a sup-functor $\times : \mathcal{Q} \otimes \mathcal{Q} \to \mathcal{Q}$ and a object I of $\mathcal{Q}$,

together with isomorphisms

- $a_{ABC} : A\times(B\times C) \sim (A\times B)\times C$
- $s_{AB} : A \times B \sim B \times A$
- $r_{A} : A \sim A \times I$

and morphisms

- $\Delta_{A} : A \to A \times A$
- $t_{A} : A \to I$

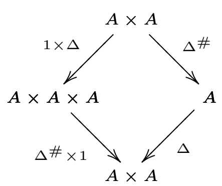
such that

i: The isomorphisms a, s, r are natural in $A, B, C \in |\mathcal{Q}|$ and satisfy the usual symmetric monoidal coherence conditions.

ii: The morphisms$\Delta$and t are maps and lax-natural in $A \in |\mathcal{Q}|$.

iii: The maps make $(\mathcal{Q}, A, I)$ into a commutative comonoid. ♢

If $\mathcal{Q}$ is a cartesian unital-quataloid, then an object $A \in \mathcal{Q}$ is *discrete* if the following square commutes.



A cartesian unital-quantaloid is called a *Distributive Category of relations* if every object is discrete. Denote the category of distributive categories of relations and suprema-preserving functors by DCR.

All distributive categories of relations are involutive, for if $A \xrightarrow{q} B$ is a morphism, then we define q^* to be the morphism

$$B \sim B \times I \xrightarrow{1\times\Delta t^\#} B \times A \times A \xrightarrow{1\times q\times 1} B \times B \times A \xrightarrow{t\Delta^\#\times 1} I \times A \sim A.$$

This gives an involution on $\mathcal{Q}$ which satisfies Freyd's modular laws.

Let $\mathcal{Q}$ be a unital-quantaloid, then a collection X of objects of $\mathcal{Q}$ is a *generating set* for $\mathcal{Q}$ if

$$1_A = \bigvee \{pq \mid cod(q) = dom(p) \in X \text{ and } pq \leq 1_A\}.$$

Say that $\mathcal{Q}$ is *bounded*, if it has a small set of generators. Of course, for any unital-quantaloid $\mathcal{Q}$, the set of objects automatically forms a generating set, so if it has a small set of objects it is bounded.

A Heyting algebra is a one object quantaloid where the composition is the meet operation and the involution is simply the identity functor. With this structure it is a distributive category of relations. More generally the category of relations for a Grothendieck topos is a bounded distributive category of relations. In addition this is complete in the sense that it has all coproducts and all symmetric idempotents split. This now leads us to the main result that the category of bounded complete distributive categories of relations (bcDCR) is equivalent to the category of Grothedieck toposes with arrows reversed(see for example Pitts[24]).

$$bcDCR \underset{Rel}{\overset{Map}{\rightleftarrows}} GTOP^{op} \quad (\sim)$$

The equivalence is given by sending a Grothendieck topos to its category of relations and in reverse a bounded complete distributive category of relations is sent to its subcategory of maps.

4 Relational-presheaves and Sheaves

One traditional way to define a presheaf on a site is as a covariant functor into the category of sets and functions. It is then a sheaf if it satisfies certain patching conditions. In this section we begin to develop an alternative formulation where a relational-presheaf on a quantaloid

$\mathcal{Q}$ is a particular lax-semifunctor with codomain the category of sets and relations. We then define a relational-sheaf to be an idempotent relational-presheaf. But first we need to define the semicategory of pretransformations from which we will construct the semicategories of relational-presheaves and relational-sheaves.

Recall that the category of relations and functions is supremum-enriched but not infimum-enriched. Unfortunately this tells us that we can not automatically apply the enriched taxon ideas directly. So we need to build the transformations from scratch using enriched taxon theory as a guide to the appropriate definitions.

Definition 4.1 Let $\mathcal{Q}$ be a quantaloid. The semicategory of *pretransformations* consist of

- **Objects** • A function $X : |\mathcal{Q}| \to |\ \mathbf{Rel}|$.
- **Arrows** • If X and Y are objects a *pretransformation* $X \stackrel{\tau}{\Longrightarrow} Y$ consists of a $|\mathcal{Q}|\times|\mathcal{Q}|$ indexed family of infima-preserving arrows

$$\left\langle \tau_{AB} : \mathcal{Q}^{co}(A,B) \to \mathbf{Rel}(X(A), Y(B)) \right\rangle$$

$\diamondsuit$

For simplicity sake, given a morphism $A \xrightarrow{f} B$, we will represent the relation $\tau_{AB}(f)$ by τ_f. If $a \in X(A)$ and $b \in Y(B)$, we say that a is τ_f related to b when $\tau_f(b,a) = 1$.

We now use the template of enriched taxons to define the composition of pretransformations. In this case though there is no guarantee that every arrow can be written as the supremum of the composition of some family of arrows, thus we must slightly relax things.

The composition of pretransformations $X \stackrel{\tau}{\Longrightarrow} Y \stackrel{\omega}{\Longrightarrow} Z$ is defined by setting $(\sigma\tau)_f(b,a) = 1$ if and only if there exists a family of composable morphisms $\langle g_i h_i \rangle$ such that

- $\bigvee g_i h_i \geq f$
- $\bigcap \sigma_{g_i} \tau_{h_i}(b,a) = 1.$

The semicategory of pretransformations is a quantaloid with the appropriate structure defined pointwise. In particular if there is a family of pretranformations $\tau_i : X \Rightarrow Y$, then $\left(\bigvee \tau_i\right)_q(b,a) = 1$ provided there exist an i such that $(\tau_i)_q(b,a) = 1$. Also define the involute of τ to be the pretransformation $\tau^\circ : Y \to X$ where, for each morphism q, $\tau^\circ_q = \tau^{-1}_q$ (the relation τ^{-1}_q is the inverse relation of τ_q). Denote the quantaloid of pretransformations by $PT(\mathcal{Q})$.

Definition 4.2 Let $\mathcal{Q}$ be a quantaloid, then

- The semicategory of *relational-presheaves* and *modular-transformations* on $\mathcal{Q}$ is the semicategory

$$RP(\mathcal{Q}) = \mathbf{Mod}\Big(PT(\mathcal{Q})\Big)$$

- The category of *relational-sheaves* and *transformations* is the category

$$RS(\mathcal{Q}) = \mathbf{Kar}\Big(PT(\mathcal{Q})\Big) \qquad \diamondsuit$$

Equivalently a relational-presheaf is a lax-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{Rel}$ that preserves infima and the involution. This is because a pretransformation that is a symmetric module can be thought of as a function $F : |\mathcal{Q}| \to |\mathbf{Rel}|$ (the associated object) together with the appropriate family of infima-preserving functions

$$\Big\langle F_{AB} : \mathcal{Q}^{co}(A,B) \to \mathbf{Rel}(X(A), Y(B)) \Big\rangle$$

with laxity a result of its being a module. Explicitly we have

$$F(q)F(p)(c,a) \leq F(qp)(c,a) \text{ and } F(q^*)(c,a) = F(q)(a,c).$$

We will use this notation henceforth for relational-presheaves and denote the associated modular-transformation by τ_F.

It follows that a relational-sheaf is a symmetric idempotent lax-semifunctor. Observe that a relational-presheaf F is an idempotent if and only if

$$\bigvee \Big\{ qp \;\Big|\; F(q)F(p)(c,a) = 1 \Big\} = \bigvee \Big\{ r \;\Big|\; F(r)(c,a) = 1 \Big\}.$$

There is a second type of transformation of relational-sheaves that we now wish to introduce, the functional-transformation. The definition is based on Rosenthal's[25] definition of a morphism as a lax-transformation in the traditional sense.

Definition 4.3 If F and G are relational-sheaves then a *functional-transformation* $F \overset{\tau}{\Rightarrow} G$ consists of a family of infima-preserving functions

$$\langle \tau_X : FX \to GX \rangle_{X \in |\mathcal{Q}|}$$

indexed by $|\mathcal{Q}|$, such that, for every morphism $q : X \to Y$ in $\mathcal{Q}$, the following square is an inequality

$$\begin{array}{ccc} FX & \xrightarrow{\tau_X} & GX \\ {\scriptstyle F(q)}\downarrow & \leq & \downarrow{\scriptstyle G(q)} \\ FY & \xrightarrow[\tau_Y]{} & GY \end{array}$$

$\diamondsuit$

Denote the category of relational-sheaves and functional-transformations by $RS_{fct}(\mathcal{Q})$. Later we will show that, under the right conditions, the category of maps in $RS(\mathcal{Q})$ is equivalent to the category $RS_{fct}(\mathcal{Q})$.

5 $\mathcal{Q}$-Taxons and $\mathcal{Q}$-Valued sets

In this section we will use the template of enriched taxon theory to explore what a taxon enriched in a quantaloid might look like. Observe that the composition morphism of a category can be thought of as a set-valued matrix. From this point of view we will build $\mathcal{Q}$-semicategories by starting with $\mathcal{Q}$-valued matrices. Then we construct $\mathcal{Q}$-semicategories, $\mathcal{Q}$-taxons and $\mathcal{Q}$-categories in an analogous way to our constructions of relational-presheaves and sheaves. The morphisms between these will come in essentially two flavours; the semifunctor and the profunctor/module.

Definition 5.1 Let $\mathcal{Q}$ be a quantaloid. The semicategory of matrices enriched in $\mathcal{Q}$ *($\mathcal{Q}$-matrices)* consists of

- Objects: Pairs (X,ρ_X), where X is a set and $\rho_X : X \to |\mathcal{Q}|$ is a function with codomain the objects of $\mathcal{Q}$.
- Arrows: A $\mathcal{Q}$-matrix, $(X,\rho_X) \xrightarrow{M} (Y,\rho_Y)$, is a binary function $M : Y \times X \to \mathcal{Q}$, such that $M(y,x) : \rho_Y(y) \to \rho_X(x)$ is a morphism in $\mathcal{Q}$.

The composite of two matrices $X \overset{M}{\to} Y \overset{N}{\to} Z$ is defined to be the matrix

$$\Big(N \circ M\Big)(z,x) = \bigvee_y \Big\{M(z,y) \ \&\ N(y,x)\Big\} \qquad \diamondsuit$$

If $(X,\rho) \xrightarrow{M} (Y,\rho)$ is a matrix, then there is a matrix $(Y,\rho) \xrightarrow{M^\circ} (X,\rho)$, called the involute of M, defined by setting for every $a \in X$ and $b \in Y$, $M^\circ(a,b) = M(b,a)^*$.

Denote the semicategory of matrices on $\mathcal{Q}$ by $\mathcal{Q}$-**Mat**. Notice that $\mathcal{Q}$-**Mat** is a quantaloid where the appropriate structure is defined point-wise. If $\mathcal{Q}$ is a unital-quantaloid, then $\mathcal{Q}$-**Mat** is the completion of $\mathcal{Q}$, as a unital-quantaloid, with respect to coproducts.

5.1 $\mathcal{Q}$-semicategories, $\mathcal{Q}$-Taxons and $\mathcal{Q}$-Categories

If $M : (X,\rho_X) \to (X,\rho_X)$ is an endomorphism, then we can interpret the value of M on a pair $a, b \in X$, $M(a,b)$, as a generalized notion of hom set. This now implies that we should require that for any triple $a, b, c \in X$ we have $M(a,b)\ \&\ M(b,c) \le M(a,c)$ Where & is the composition of arrows in $\mathcal{Q}$. Since $\mathcal{Q}$ is involutive it is natural (and helpful) to impose the symmetry condition that $M = M^\circ$.

Definition 5.2 Let $\mathcal{Q}$ be a quantaloid.

A $\mathcal{Q}$-*semicategory* is a matrix $(X,\rho) \xrightarrow{\delta_X} (X,\rho)$ such that

- $\delta_X \delta_X \le \delta_X$
- $\delta_X = \delta_X^\circ$. $\diamondsuit$

It is immediately clear that a $\mathcal{Q}$-semicategory is a $\mathcal{Q}$-matrix that is a symmetric module. We will denote a $\mathcal{Q}$-semicategory by (X,ρ_X,δ_X) and frequently, when the context is clear by (X,ρ,δ).

In a $\mathcal{Q}$-semicategory, the arrows $\delta(x,y)$ are elements of $\mathbf{Mod}(\mathcal{Q})$ since by construction

$$\begin{aligned} \delta(x,x) \mathbin{\&} \delta(x,x) &\leq \delta(x,x) \\ \delta(x,y) \mathbin{\&} \delta(y,y) &\leq \delta(x,y) \\ \delta(x,x) \mathbin{\&} \delta(x,y) &\leq \delta(x,y) \end{aligned}$$

Now we ask the question of when is a $\mathcal{Q}$-semicategory a $\mathcal{Q}$-taxon. Notice that in $\mathcal{Q}$-**Mat** the diagram, $\delta\delta\delta \underset{1\circ\leq}{\overset{\leq\circ 1}{\rightrightarrows}} \delta\delta \xrightarrow{\leq} \delta$, is a co-equalizer if and only if $\delta\delta = \delta$. Interpreting $\leq$ as a composition morphism leads naturally to our referring to such a $\mathcal{Q}$-semicategory as a $\mathcal{Q}$-taxon. Thus a $\mathcal{Q}$-taxon is a symmetric idempotent $\mathcal{Q}$-matrix.

When $\mathcal{Q}$ is a supremum-enriched category it is traditional to define a $\mathcal{Q}$-category as a $\mathcal{Q}$-semicategory together with 2-cell morphisms $\mathbf{1} \leq \delta(x,x)$ that satisfy the appropriate identity axioms. For $\mathcal{Q}$ a supremum-enriched semicategory we may lack the ability to construct such morphisms. Recall that we briefly argued that the Karoubian envelope was the preferable way to construct a category out of a taxon(see pg 10). So here we will use the Karoubian envelope as a guide to defining $\mathcal{Q}$-categories. With that in mind we require that each $\delta(x,x)$ behave like an identity morphism. In other words each arrow $\delta(x,x)$ is an object in $\mathbf{Kar}(\mathcal{Q})$ and each arrow $\delta(x,y)$ is a morphism between the objects in $\mathbf{Kar}(\mathcal{Q})$.

To recap we have

Definition 5.3 Let $(X,\rho,\delta\ )$ be a triple where (X,ρ_X) is a $\mathcal{Q}$-matrix object and $\delta : (X,\rho_X) \to (X,\rho_X)$ is an endomorphic $\mathcal{Q}$-matrix. Then

1. $(X,\rho,\delta\ )$ is a $\mathcal{Q}$-*semicategory*, if δ_X is a symmetric module $\mathcal{Q}$-matrix..
2. $(X,\rho,\delta\ )$ is a $\mathcal{Q}$-*Taxon*, if δ_X is a symmetric idempotent $\mathcal{Q}$-matrix.
3. $(X,\rho,\delta\ )$ is a $\mathcal{Q}$-*Category* if δ satisfies the following conditions.
   - For every $x \in X$, $\delta(x,x) \mathbin{\&} \delta(x,x) = \delta(x,x)$
   - For every pair $x, y \in X$
     $\delta(x,x) \mathbin{\&} \delta(x,y) = \delta(x,y)$
     $\delta(x,y) \mathbin{\&} \delta(y,y) = \delta(x,y)$ ♢

For any $\mathcal{Q}$-category we can interpret it as some construct utilizing the Karoubian envelope of $\mathcal{Q}$ in some way. We may use multiple copies of objects and arrows from **Kar**($\mathcal{Q}$) while not using others at all.

5.2 $\mathcal{Q}$-Morphisms

The morphisms of $\mathcal{Q}$-semicategories, taxons and categories come in two types. The $\mathcal{Q}$-enriched generalizations of profunctors (modules, distributors) and the generalizations of semifunctors. Those morphisms meant to represent profunctors we divide into two types, $\mathcal{Q}$-modules (morphisms from the semicategory **Mod**($\mathcal{Q}$-**Mat**)) and $\mathcal{Q}$-profunctors (morphisms from the category **Kar**($\mathcal{Q}$-**Mat**)).

Definition 5.4 Let (X,ρ,δ) and (Y,ρ,δ) be $\mathcal{Q}$-semicategories, then

1. A *$\mathcal{Q}$-Module* $(X,\rho,\delta) \xrightarrow{R} (Y,\rho,\delta)$ is a matrix $(X,\rho_X) \xrightarrow{R} (Y,\rho_Y)$ such that
   - $\delta_Y \circ R \leq R$
   - $R \circ \delta_X \leq R$.
2. A *$\mathcal{Q}$-Profunctor* $(X,\rho,\delta) \xrightarrow{R} (Y,\rho,\delta)$ is a matrix $(X,\rho_X) \xrightarrow{R} (Y,\rho_Y)$ such that
   - $\delta_Y \circ R = R$
   - $R \circ \delta_X = R$

The composition of $\mathcal{Q}$-modules and $\mathcal{Q}$-profunctors is the composition of $\mathcal{Q}$-Matrices. $\diamondsuit$

$\mathcal{Q}$-semicategories together with $\mathcal{Q}$-Modules form a supremum-enriched semicategory (denoted $\mathcal{Q}$-**Mod**) where the supremum is taken pointwise. This semicategory is the semicategory of modules of matrices **Mod**($\mathcal{Q}$-**Mat**). $\mathcal{Q}$-taxons together with $\mathcal{Q}$-profunctors for morphisms form a supremum-enriched category (denoted $\mathcal{Q}$-**Prof**). In this case $\mathcal{Q}$-**Prof** is the Karoubian envelope of $\mathcal{Q}$-**Mat** (**Kar**($\mathcal{Q}$-**Mat**)). Note that every $\mathcal{Q}$-taxon is a $\mathcal{Q}$-semicategory and that every $\mathcal{Q}$-profunctor is a $\mathcal{Q}$-module, thus $\mathcal{Q}$-**Mod** is a sub-semicategory of $\mathcal{Q}$-**Prof**. When $\mathcal{Q}$ is a unital-quantaloid, **Kar**($\mathcal{Q}$-**Mat**) is the completion of $\mathcal{Q}$ in the sense that it is a quantaloid with all coproducts and all symmetric idempotents split. So if $\mathcal{Q}$ is a bounded distributive category of relations, $\mathcal{Q}$-**Prof** is a bounded complete distributive category of relations.

Definition 5.5 Let $(X,\rho,\delta\ ) \xrightarrow{R} (Y,\rho,\delta\ )$ be a $\mathcal{Q}$-module between $\mathcal{Q}$-semicategories. Then R is a *symmetric-map (left-adjoint)* if

- $\delta_X \leq R^* \circ R$.
- $R \circ R^* \leq \delta_Y$. ♢

The category of $\mathcal{Q}$-taxons and $\mathcal{Q}$-profunctors that are symmetric-maps is traditionally referred to as the category of $\mathcal{Q}$-valued sets and is denoted by $\mathcal{Q}$-**Set**.

The semifunctors between $\mathcal{Q}$-semicategories is defined as one would expect in terms of a function on the objects and the appropriate morphism of hom sets.

Definition 5.6 Let $(X,\rho,\delta\ )$ and $(Y,\rho,\delta\ )$ be $\mathcal{Q}$-semicategories. A $\mathcal{Q}$-*semifunctor* $(X,\rho,\delta\ ) \xrightarrow{f} (Y,\rho,\delta\ )$ is a function $f : X \to Y$, such that

$$\delta_X(x_1, x_2) \leq \delta_Y(f(x_1), f(x_2)).$$

The composition of $\mathcal{Q}$-semifunctors is simply the composition of functions.

♢

Observe that the identity function is a $\mathcal{Q}$-semifunctor and is thus an identity morphism for the categories of $\mathcal{Q}$-semicategories, $\mathcal{Q}$-taxons or $\mathcal{Q}$-categories together with $\mathcal{Q}$-semifunctors. These are respectively denoted by $\mathcal{Q}$-**Scat**, $\mathcal{Q}$-**Tax** and $\mathcal{Q}$-**Cat**.

Example 5.7 Let $(X,\rho,\delta\ )$ be a $\mathcal{Q}$-semicategory and define a $\mathcal{Q}$-category

$$\mathbf{Kar}(X,\rho,\delta\ ) = (X_{\mathbf{Kar}}, \rho_{\mathbf{Kar}}, \delta_{\mathbf{Kar}})$$

as follows.

- $X_{\mathbf{Kar}} = X$.
- $\rho_{\mathbf{Kar}} = \rho_X$
- If $x, y \in X_{\mathbf{Kar}}$, then

  $$\delta_{\mathbf{Kar}}(x,y) = \bigvee \Big\{ \rho(y) \xrightarrow{p} \rho(x) \Big| p \le \delta(x,y) \text{ and } p\delta(x,x) = p = \delta(y,y)p \Big\}$$

To verify that this is a $\mathcal{Q}$-category we observe first that clearly

$$\delta_{\mathbf{Kar}}(x,z) \ \& \ \delta_{\mathbf{Kar}}(z,y) \le \delta_{\mathbf{Kar}}(x,y)$$

and so $\delta\delta \le \delta$. On the other hand

$$\begin{aligned} \delta_{\mathbf{Kar}}\delta_{\mathbf{Kar}}(x,y) &= \bigvee_z \{\delta_{\mathbf{Kar}}(x,z) \ \& \ \delta_{\mathbf{Kar}}(z,y) \ \} \\ &= \bigvee_z \Big\{ \vee \{p_1\} \ \& \ \vee \{p_2\} \ \Big\} \\ &\ge \bigvee \Big\{ p_1 \ \& \ \delta(y,y) \Big\} \quad \text{when } z = y \\ &= \bigvee \Big\{ p_1 \Big\} = \delta_{\mathbf{Kar}}(x,y) \end{aligned}$$

If $(X,\rho,\delta\ ) \xrightarrow{f} (Y,\rho,\delta\ )$ is a $\mathcal{Q}$-semifunctor between $\mathcal{Q}$-semicategories, then we let $\mathbf{Kar}(f) = f$. Clearly $\mathbf{Kar} : \mathcal{Q}\text{-}\mathbf{Tax} \to \mathcal{Q}\text{-}\mathbf{Cat}$ is a semifunctor. We claim that the inclusion of $\mathcal{Q}$-**Cat** into $\mathcal{Q}$-**Tax** is left-adjoint to **Kar**. The following two observations explicitly give us the unit and the counit of the adjunction.

- If $(X,\rho,\delta\ )$ is a $\mathcal{Q}$-category, then $\mathbf{Kar}((X,\rho,\delta\ )) = (X,\rho,\delta\ )$ and $\eta_{(X,\rho,\delta)}$ is the identity $\mathcal{Q}$-semifunctor.
- Since $\delta_{\mathbf{Kar}} \le \delta_X$ the identity morphism on X is a $\mathcal{Q}$-semifunctor from $(X_{\mathbf{Kar}}, \rho_{\mathbf{Kar}}, \delta_{\mathbf{Kar}})$ to $(X,\rho,\delta\ )$. Thus setting $\varepsilon_{(X,\rho,\delta)}$ equal to the identity defines the counit $\varepsilon : inc \circ \mathbf{Kar} \Rightarrow \mathbf{1}_{\mathcal{Q}\text{-}\mathbf{Tax}}$. ∎

We will now proceed to construct functors between $\mathcal{Q}$-**Tax** and $\mathcal{Q}$-**Set** which will allow us to show that the category of $\mathcal{Q}$-valued sets is equivalent to the category of *complete* $\mathcal{Q}$-categories and *regular* $\mathcal{Q}$-semifunctors

5.3 The Functor $\Phi : \mathcal{Q}$-Tax$\longrightarrow\mathcal{Q}$ -Set

Let $(X,\rho,\delta\ )$ and $(Y,\rho,\delta\ )$ be $\mathcal{Q}$-taxons and define a functor

$$\Phi : \mathcal{Q}\textbf{-Tax} \longrightarrow \mathcal{Q}\ \textbf{-Set}$$

to be the identity on objects and if $(X,\rho,\delta\ ) \xrightarrow{f} (Y,\rho,\delta\ )$ is a $\mathcal{Q}$-semifunctor, then$\Phi(\ f) = R_f\ \left((X,\rho,\delta\ ) \xrightarrow{R_f} (Y,\rho,\delta\ )\right)$ is the $\mathcal{Q}$-module defined by setting.

$$R_f(y,x) = \delta_Y(y, f(x)).$$

Unfortunately at this point R_f need not be a $\mathcal{Q}$-profunctor since we only have $R_f\delta_X \leq R_f$ as seen below.

$$\begin{aligned}
\delta_Y R_f(y,x) &= \bigvee_{y'} \left\{\delta_Y(y,y')\ \&\ R_f(y',x)\right\} \\
&= \bigvee_{y'} \left\{\delta_Y\left(y,y'\right)\ \&\ \delta_Y\left(y', f(x)\right)\right\} \\
&= \delta_Y\left(y, f(x)\right) = R_f\left(y,x\right)
\end{aligned}$$

$$\begin{aligned}
R_f\delta_X(y,x) &= \bigvee_{x'} \left\{R_f\left(y,x'\right)\ \&\ \delta_X\left(x',x\right)\right\} \\
&= \bigvee_{x'} \left\{\delta_Y\left(y, f(x')\right)\ \&\ \delta_X\left(x',x\right)\right\} \\
&\leq \bigvee_{x'} \left\{\delta_Y\left(y, f(x')\right)\ \&\ \delta_Y\left(f(x'), f(x)\right)\right\} \\
&\leq \bigvee_{y'} \left\{\delta_Y\left(y, y'\right)\ \&\ \delta_Y\left(y', f(x)\right)\right\} \\
&\leq \delta_Y\left(y, f(x)\right) = R_f\left(y,x\right)
\end{aligned}$$

So for R_f to be a $\mathcal{Q}$-profunctor we need to require that $R_f\delta_X = R_f$.

Definition 5.8 A $\mathcal{Q}$-semifunctor $(X,\rho,\delta\ ) \xrightarrow{f} (Y,\rho,\delta\ )$ is *regular* if

$$R_f\delta_X = R_f.$$

◊

Let $(Y,\rho,\delta\ ) \xrightarrow{g} (Z,\rho,\delta\ )$ be second $\mathcal{Q}$-semifunctor, then $R_g R_f \leq R_{gf}$.

$$
\begin{aligned}
R_g R_f\big(z,x\big) &= \bigvee_y \Big\{R_g\big(z,y\big)\ \&\ R_f\big(y,x\big)\Big\} \\
&= \bigvee_y \Big\{\delta_Z\big(z,g(y)\big)\ \&\ \delta_Y\big(y,f(x)\big)\Big\} \quad \text{This row is } R_g\delta_Y\big(z,f(x)\big) \\
&\leq \bigvee_y \Big\{\delta_Z\big(z,g(y)\big)\ \&\ \delta_Z\big(g(y),gf(x)\big)\Big\} \\
&\leq \delta_Z\big(z,gf(x)\big) = R_{gf}\big(z,x\big)
\end{aligned}
$$

When g is a regular $\mathcal{Q}$-semifunctor $R_g R_f = R_{gf}$. We have equality since the second step above can be replaced by $R_g\delta_Y(z,f(x)\big) = R_g\big(z,f(x)\big)$, which is equal to $R_{gf}\big(z,x\big)$.

R_f is a symmetric-map.

$$
\begin{aligned}
R_f R_f^{\circ}\big(y,y''\big) &= \bigvee_x \Big\{R_f\big(y,x\big)\ \&\ R_f^{\circ}\big(x,y''\big)\Big\} \\
&= \bigvee_x \Big\{\Big\{\delta_Y\big(y,f(x)\big)\ \&\ \delta_Y\big(f(x),y''\big)\Big\} \\
&\leq \bigvee_{y'} \Big\{\delta_Y\big(y,y'\big)\ \&\ \delta_Y\big(y',y''\big)\Big\} \\
&\leq \delta_Y\big(y,y''\big)
\end{aligned}
$$

$$
\begin{aligned}
R_f^{\circ} R_f(x,x'') &= \bigvee_y \Big\{R_f^{\circ}\big(x,y\big)\ \&\ R_f\big(y,x''\big)\Big\} \\
&= \bigvee_y \Big\{\delta_Y^{\circ}\big(f(x),y\big)\ \&\ \delta_Y\big(y,f(x'')\big)\Big\} \\
&= \delta_Y\big(f(x),f(x'')\big) \\
&\geq \delta_X\big(x,x''\big)
\end{aligned}
$$

and thus $R_f \dashv R_f^{\circ}$, and hence when we restrict to regular $\mathcal{Q}$-semifunctors the image of$\Phi$is contained in the category of $\mathcal{Q}$-valued sets. Henceforth we will assume that all $\mathcal{Q}$-semifunctors are regular.

5.4 The Functor $\Psi : \mathcal{Q}$-Set$\longrightarrow\mathcal{Q}$ -Tax

Let $q\colon A\to A$ be an endomorphism in $\mathcal{Q}$ where $qq \le q$ and $q=q^*$, then there is a $\mathcal{Q}$-semicategory $[q] = (\{*\}, \rho_q, \delta_q)$, where $\rho_q(*) = domain(q)$ and $\delta_q(*,*) = q$. Observe that the full subcategory of $\mathcal{Q}$-**Mod** determined by the objects $[q]$ is equivalent to the semicategory **Mod**$(\mathcal{Q})$. If we restrict to objects $[q]$ which are symmetric idempotents, then the full subcategory of $\mathcal{Q}$-**Prof** determined by the objects $[q]$ is now equivalent to the category $Kar(\mathcal{Q})$.

When (X,ρ,δ) is a $\mathcal{Q}$-semicategory the $\mathcal{Q}$-morphisms

$$\alpha_x : [\delta(x,x)] \to (X,\rho,\delta)$$

defined by setting $\alpha_x(x',*) = \delta(x,x')$ are $\mathcal{Q}$-modules. We say that any morphism of this type is *representable*. For ease of notation we call the object $[\delta(x,x)] = [x]$ and the associated matrix δ_x. When (X,ρ,δ) is a $\mathcal{Q}$-taxon we have

$$\begin{aligned} \delta_X \alpha_x(x'',*) &= \bigvee_{x'} \left\{ \delta_X(x'',x') \;\&\; \delta_X(x',x) \right\} \\ &= \delta_X(x'',x) \\ &= \alpha_x(x'',*), \end{aligned}$$

So $\delta_X \alpha_x = \alpha_x$ (a left-module). Finally when (X,ρ,δ) is a $\mathcal{Q}$-category it is easy to show that we also have the equality $\alpha_x \delta_x = \alpha_x$, which tells us that α_x is a $\mathcal{Q}$-profunctor.

For the remainder of this section we will be using $\mathcal{Q}$-taxons exclusively and thus we will define *singletons* to be the appropriate left-module

Definition 5.9 Let $\mathcal{Q}$ be a quantaloid. If a $\mathcal{Q}$-module of the form

$$[q] \xrightarrow{\alpha} (X,\rho,\delta)$$

is a symmetric monomorphic map, and if $\delta_X \alpha = \alpha$ (α is a left-module), then α is called a *singleton* on (X,ρ,δ). ♢

The main tool in the construction of the functorΨ: $\mathcal{Q}$-**Set**$\to$ $\mathcal{Q}$-**Tax** is the use of *singleton* morphisms. When (X,ρ,δ) is a $\mathcal{Q}$-taxon then every representable morphism α_x is a singleton and our interest will be drawn towards the instance when all singletons are representable.

Example 5.10 Let $(X,\rho,\delta\ )$ be a $\mathcal{Q}$-taxon, then every representable morphism α_x is a singleton since

$$\begin{aligned}\alpha_x^\circ\alpha_x(*,*) &= \bigvee_{x'}\left\{\alpha_x^\circ(*,x') \;\&\; \alpha_x(x',*)\ \right\}\\ &= \bigvee_{x'}\left\{\delta_X(x,x') \;\&\; \delta_X(x',x)\right\}\\ &= \delta_X(x,x).\end{aligned}$$

So α is a monomorphism.

$$\begin{aligned}\alpha_x\alpha_x^\circ(x_1,x_2) &= \alpha_x(x_1,*) \;\&\; \alpha_x^\circ(*,x_2)\\ &= \delta_X(x_1,x) \;\&\; \delta_X(x,x_2)\\ &\leq \delta(x_1,x_2)\end{aligned}$$

Which shows that α is a map and thus a singleton. □

Singletons were the main tool used by Higgs[14] to show that his construction of $\mathcal{H}$-valued sets for a Heyting algebra $\mathcal{H}$ results in a category isomorphic to the category of sheaves for $\mathcal{H}$. His ideas were generalized in Garraway[10] to show that the category of $\mathcal{Q}$-valued sets can be thought of as sheaves on $\mathcal{Q}$. Stubbe[26][13], has explored in the non-involutive and involutive settings respectively showing how the construction can be interpreted as the Cauchy completion of $(X,\rho,\delta\ )$.

Let $(X,\rho,\delta\ )$ be a $\mathcal{Q}$-taxon and define $(\overline{X},\overline{\rho},\overline{\delta})$ to be the $\mathcal{Q}$-semicategory where

- $\overline{X}$ is the set of all singletons on $(X,\rho,\delta\ )$.
- If $[q]\xrightarrow{\alpha}(X,\rho,\delta\ )$ is an element of $\overline{X}$, then $\overline{\rho}(\alpha)$ is the domain of q.
- If $[q]\xrightarrow{\alpha}(X,\rho,\delta\ )$ and $[p]\xrightarrow{\beta}(Y,\rho,\delta\ )$ are elements of $\overline{X}$, then define $\overline{\delta}$ by setting $\overline{\delta}(\alpha,\beta\ ) = \alpha^\circ\beta(*,*)$. Unless needed, we will denote $\alpha^\circ\beta(*,*)$ by $\alpha^\circ\beta$.

It is easy to see that $\overline{\delta}$ is always symmetric since

$$\overline{\delta}^\circ(\alpha,\beta\ ) = \overline{\delta}(\beta,\alpha\ )^* = \left(\beta^\circ\alpha\right)^* = \alpha^\circ\beta = \overline{\delta}(\alpha,\beta\ )$$

To show that $(\overline{X}, \overline{\rho}, \overline{\delta})$ is a $\mathcal{Q}$-taxon we need to show that $\bigvee_\gamma \alpha^\circ\gamma\gamma^\circ\beta$ will equal $\alpha^\circ\beta$. In other words that $\overline{\delta\delta}$ equals $\overline{\delta}$.

$$\begin{aligned}
\delta_X\Big(x_1, x_2\Big) &\geq \bigvee_\gamma \gamma\gamma^\circ\Big(x_1, x_2\Big) && \text{each } \gamma \text{ is a map} \\
&\geq \bigvee_x \alpha_x\alpha_x^\circ\Big(x_1, x_2\Big) && \text{the representables} \\
&= \bigvee_x \delta\Big(x_1, x\Big) \ \&\ \delta^\circ\Big(x, x_2\Big) && \text{Which is } \delta\delta(x_1, x_2) \\
&= \delta_X\Big(x_1, x_2\Big) && \text{since } \delta \text{ is a } \mathcal{Q}\text{-taxon}
\end{aligned}$$

So now

$$\overline{\delta\delta}(\alpha,\beta) = \bigvee_\gamma \alpha^\circ\gamma\gamma^\circ\beta = \alpha^\circ\delta_X\beta = \alpha^\circ\beta = \overline{\delta}(\alpha,\beta)$$

And thus $\overline{\delta}$ is a $\mathcal{Q}$-taxon.

We can take this one step further and show that $(\overline{X}, \overline{\rho}, \overline{\delta})$ is a $\mathcal{Q}$-category. To this end we first need to observe that since singletons are maps the following equalities must hold.

$$\begin{aligned}
\alpha^\circ\beta &= \alpha^\circ\beta\delta_q \\
&\leq \alpha^\circ\beta\beta^\circ\beta && \beta \text{ a map} \\
&\leq \alpha^\circ\delta_X\beta && \beta \text{ a map} \\
&= \alpha^\circ\beta
\end{aligned}
\qquad
\begin{aligned}
\alpha^\circ\beta &= \delta_q\alpha^\circ\beta \\
&\leq \alpha^\circ\alpha\alpha^\circ\beta && \alpha \text{ a map} \\
&\leq \alpha^\circ\delta_X\beta && \alpha \text{ a map} \\
&= \alpha^\circ\beta
\end{aligned}$$

From these we immediately obtain the desired equalities that illustrate that $(\overline{X}, \overline{\rho}, \overline{\delta})$ is a $\mathcal{Q}$-category.

$$\overline{\delta}\Big(\alpha,\beta\Big)\overline{\delta}\Big(\beta,\beta\Big) = \alpha^\circ\beta\beta^\circ\beta = \alpha^\circ\beta = \overline{\delta}\Big(\alpha,\beta\Big)$$
$$\overline{\delta}\Big(\alpha,\alpha\Big)\overline{\delta}\Big(\alpha,\beta\Big) = \alpha^\circ\alpha\alpha^\circ\beta = \alpha^\circ\beta = \overline{\delta}\Big(\alpha,\beta\Big)$$

Now we are set to define the functorΨ: $\mathcal{Q}$-**Set**$\longrightarrow\mathcal{Q}$ -**Tax**.

- On objects:$\Psi\ \Big((X,\rho,\delta\ )\Big) = (\overline{X}, \overline{\rho}, \overline{\delta})$.
- On arrows:$\Psi(\ R) = f_R$ where

$$f_R(\alpha) = [q] \xrightarrow{\alpha} (X,\rho,\delta\ ) \xrightarrow{R} (Y,\rho,\delta\ ) = R\alpha$$

To see that f_R is in fact a $\mathcal{Q}$-semifunctor observe that

$$\begin{aligned}
\overline{\delta}_X(\alpha,\beta) &= \alpha^\circ\beta \\
&= \alpha^\circ\delta\beta \qquad \beta \text{ is a singleton} \\
&\leq \alpha^\circ R^\circ R\beta R \qquad \text{a map} \\
&= (R\alpha)^\circ(R\beta) \\
&= \overline{\delta}_Y(f_R(\alpha), f_R(\beta))
\end{aligned}$$

If $(Y,\rho,\delta) \xrightarrow{S} (Z,\rho,\delta)$ is a second map, then

$$f_{SR}(\alpha) = SR\alpha = Sf_R(\alpha) = f_S\Big(f_R(\alpha)\Big) \qquad \text{Thus}\, \Psi(SR) = \Psi(S)\circ\Psi(R)$$

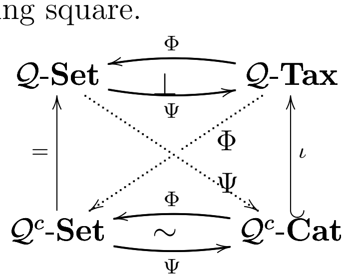
In addition, since (X,ρ,δ) is a $\mathcal{Q}$-taxon and $(X,\rho,\delta) \xrightarrow{R} (Y,\rho,\delta)$ is a $\mathcal{Q}$-profunctor, f_R is regular since

$$\begin{aligned}
\bigvee_\gamma \Big\{\delta_{\overline{Y}}(\alpha, R\gamma) \,\&\, \delta_{\overline{X}}(\gamma,\beta)\Big\} &= \bigvee_\gamma \alpha^\circ R\gamma\gamma^\circ\beta \\
&= \alpha^\circ R\delta_X\beta \\
&= \alpha^\circ R\beta \qquad \beta \text{ a singleton} \\
&= \delta_{\overline{Y}}(\alpha, R\beta)
\end{aligned}$$

So Ψ is a semifunctor whose image is contained in the category $\mathcal{Q}$-**Cat** and each $\mathcal{Q}$-semifunctor f_R is regular.

5.5 Transformations

Having constructed the needed functors we will now show that the functor $\mathcal{Q}$-**Tax** $\xrightarrow{\Phi}$ $\mathcal{Q}$-**Set** is left-adjoint to $\mathcal{Q}$-**Set** $\xrightarrow{\Psi}$ $\mathcal{Q}$-**Tax** and under a completeness condition the adjunction becomes an equivalence giving the following commuting square.



We start with the counit of the adjunction and show that it is a natural-isomorphism.

For each $\mathcal{Q}$-taxon $(X,\rho,\delta\ )$ define a $\mathcal{Q}$-profunctor $(\overline{X},\overline{\rho},\overline{\delta}) \xrightarrow{\varepsilon_X} (X,\rho,\delta\ )$ where $\varepsilon_X(x,\alpha\ ) = \alpha(x,*)$. These morphisms constitute the counit of the adjunction $(\varepsilon : \Phi\Psi \overset{\sim}{\Rightarrow} \mathbf{1})$.

To see that ε_X is a $\mathcal{Q}$-profunctor we have the two equalities,

$$\begin{aligned}
\varepsilon_X\overline{\delta}\big(x,\alpha\big) &= \bigvee_\beta \Big\{\varepsilon_X\big(x,\beta\big)\ \&\ \overline{\delta}\big(\beta,\alpha\big)\Big\} \\
&= \bigvee_\beta \Big\{\beta\beta^\circ\alpha\big(x,*\big)\Big\} \\
&= \delta_X\alpha(x,*) \\
&= \alpha(x,*) \\
&= \varepsilon_X(x,\alpha\ )
\end{aligned}$$

$$\begin{aligned}
\delta_X\varepsilon_X\big(x,\alpha\big) &= \delta_X\alpha\big(x,*\big) \\
&= \alpha\big(x,*\big) \\
&= \varepsilon_X\big(x,\alpha\big).
\end{aligned}$$

That ε_X is an isomorphism follows directly from the knowledge that each $(X,\rho,\delta\ )$ is $\mathcal{Q}$-taxon.

$$\begin{aligned}
\varepsilon_X\varepsilon_X^\circ(x,x') &= \bigvee_\alpha \Big\{\varepsilon_X(x,\alpha\ )\ \&\ \varepsilon_X^\circ(\alpha,x')\Big\} \\
&= \bigvee_\alpha \alpha\alpha^\circ(x,x') \\
&= \delta(x,x')
\end{aligned}$$

$$\begin{aligned}
\varepsilon_X^\circ\varepsilon_X(\alpha,\beta\ ) &= \bigvee_x \Big\{\varepsilon_X^\circ(\alpha,x)\ \&\ \varepsilon(x,\beta\ )\Big\} \\
&= \bigvee_x \Big\{\alpha^\circ(*,x)\ \&\ \beta(x,*)\Big\} \\
&= \alpha^\circ\beta \\
&= \overline{\delta}(\alpha,\beta\ )
\end{aligned}$$

To finish offwe need to show that ε is a natural, which means we wish to show that for any $\mathcal{Q}$-profunctor R that is a map the following square commutes.

$$\begin{array}{ccc} (\overline{X},\overline{\rho},\overline{\delta}) & \xrightarrow{\varepsilon_X} & (X,\rho,\delta\ ) \\ \overline{f_R}\big\downarrow & = & \big\downarrow R \\ (\overline{Y},\overline{\rho},\overline{\delta}) & \xrightarrow[\varepsilon_Y]{} & (Y,\rho,\delta\ ) \end{array}$$

where$\Phi\Psi(\ R)$ equals $\overline{f_R}$.

$$\begin{aligned} R\varepsilon_X\big(y,\alpha\big) &= \bigvee_x \Big\{R\big(y,x\big)\ \&\ \varepsilon_X\big(x,\alpha\big)\Big\} \\ &= \bigvee_x \Big\{R\big(y,x\big)\ \&\ \alpha\big(x,*\big)\Big\} \\ &= R\alpha\big(y,*\big) \end{aligned}$$

$$\begin{aligned} \varepsilon_Y\overline{f_R} &= \bigvee_\beta \Big\{\varepsilon_Y\big(y,\beta\big)\ \&\ \overline{f_R}\big(\beta,\alpha\big)\Big\} \\ &= \bigvee_\beta \Big\{\beta\big(y,*\big)\ \&\ \overline{\delta_Y}\big(\beta,R\alpha\big)\Big\} \\ &= \bigvee_\beta \Big\{\beta\beta^\circ R\alpha\big(y,*\big)\Big\} \\ &= R\alpha\big(y,*\big) \end{aligned}$$

Moving onto the unit of the adjunction, let $(X,\rho,\delta\ )$ be a $\mathcal{Q}$-taxon and define for each object $(X,\rho,\delta\ )$, a $\mathcal{Q}$-semifunctor $(X,\rho,\delta\ ) \xrightarrow{\eta_X} (\overline{X},\overline{\rho},\overline{\delta})$ by $\eta_X(x) = \alpha_x$. This is a natural-transformation $\eta\colon\mathbf{1}\Rightarrow\Psi\Phi$.

First we show that η_X is a $\mathcal{Q}$-semifunctor. For this we need to show that the inequality $\delta_X(x,x'') \leq \overline{\delta}_X(\alpha_x,\alpha_{x''})$ holds (in fact we obtain an equality).

$$\begin{aligned} \delta(x,x'') &= \bigvee_{x'} \Big\{\delta_X(x,x')\ \&\ \delta_X(x',x'')\Big\} \\ &= \bigvee_{x'} \Big\{\alpha_x^\circ(*,x')\ \&\ \alpha_{x''}(x',*)\Big\} \end{aligned}$$

$$\begin{aligned} &= \alpha_x^\circ \alpha_{x''}(*, *) \\ &= \overline{\delta}_X(\alpha_x, \alpha_{x''}) \end{aligned}$$

To see that η is a transformation observe that if $(X,\rho,\delta) \xrightarrow{f} (Y,\rho,\delta)$ is a regular $\mathcal{Q}$-semifunctor, then

$$\overline{R_f}\eta_X(x) = \overline{R_f}(\alpha_x) = R_f\alpha_x : [x] \to (X,\rho,\delta) \to (Y,\rho,\delta).$$

$$R_f\alpha_x\big(y,*\big) = R_f\delta_X\big(y,x\big) = R_f\big(y,x\big) = \delta_Y\big(y, f(x)\big) = \alpha_{f(x)}(y,*)$$

the second equality uses the fact that f is regular.

$$\eta_Y f(x) = \eta_Y\big(f(x)\big) = \alpha_{f(x)}$$

Thus the following is a commuting square

$$\begin{array}{ccc} (X,\rho,\delta) & \xrightarrow{\eta_X} & (\overline{X}, \overline{\rho}, \overline{\delta}) \\ {\scriptstyle \overline{R_f}}\downarrow & = & \downarrow{\scriptstyle f} \\ (Y,\rho,\delta) & \xrightarrow[\eta_Y]{} & (\overline{Y}, \overline{\rho}, \overline{\delta}) \end{array}$$

and hence η is a natural transformation. With similar computations we can see that the appropriate triangles commute telling us that$\Phi$is left-adjoint toΨ.

$$\mathcal{Q}\text{-}\mathbf{Set} \underset{\Psi}{\overset{\Phi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{Q}\text{-}\mathbf{Tax}$$

Definition 5.11 (X,ρ,δ) is *complete* if η_X is an isomorphism. $\diamondsuit$

Clearly (X,ρ,δ) is complete if and only if every singleton is representable. We will denote the full subcategories that consist of complete $\mathcal{Q}$-categories by $\mathcal{Q}^c$-**Set** and $\mathcal{Q}^c$-**Cat** respectively. An important result for us is that $(\overline{X}, \overline{\rho}, \overline{\delta})$ is complete and thus the image of Ψ will be contained in $\mathcal{Q}^c$-**Cat**.

Lemma 5.12 If (X,ρ,δ) is a $\mathcal{Q}$-taxon, then $(\overline{X}, \overline{\rho}, \overline{\delta})$ is complete.

Proof: Let $q : C \to C$ be a symmetric idempotent arrow and let $[q] \xrightarrow{A} (\overline{X}, \overline{\rho}, \overline{\delta})$ be a singleton. We want to show that A is equal to A_α for some singleton $\alpha : [q] \to (X,\rho,\delta\ )$. To this end define $[q] \xrightarrow{\alpha} (X,\rho,\delta\ )$ to be the needed singleton by setting

$$\alpha(x,*) = \bigvee_\gamma \Big\{\gamma(x,*)\ \&\ A(\gamma,*)\Big\}$$

$$\alpha^\circ(*,x) = \bigvee_\gamma \Big\{A^\circ(*,\gamma)\ \&\ \gamma^\circ(*,x)\Big\},$$

where the supremum is taken over all singletons $\gamma : [q] \to (X,\rho,\delta\ )$.

This is well defined since we are taking the supremum of morphisms of the form $C \xrightarrow{A(\gamma,*)} C \xrightarrow{\gamma(x,*)} \rho(x)$, which has the appropriate domain and codomain for $\alpha(x,*)$. First we show that α is a singleton.

$$\begin{aligned}
\delta_X\alpha(x,*) &= \bigvee_{x'} \Big\{\delta_X(x,x')\ \&\ \alpha(x',*)\Big\} \\
&= \bigvee_{x'} \Big\{\delta_X(x,x')\ \&\ \bigvee_\gamma \Big\{\gamma(x',*)\ \&\ A(\gamma,*)\Big\}\Big\} \\
&= \bigvee_{\gamma} \Big\{\bigvee_{x'} \Big\{\delta_X(x,x')\ \&\ \gamma(x',*)\Big\}\ \&\ A(\gamma,*)\Big\} \\
&= \bigvee_\gamma \{\gamma(x,*)\ \&\ A(x,*)\} = \alpha(x,*)
\end{aligned}$$

It is simpler to show that $\alpha\delta_q = \alpha$, and simillarly that α° is a morphism. We still need to show that α is monomorphic and that $\alpha \dashv \alpha^\circ$.

$$\begin{aligned}
\alpha\alpha^\circ(x,x') &= \alpha(x,*)\ \&\ \alpha^\circ(*,x') \\
&= \bigvee_\xi \Big\{\xi(x,*)\ \&\ A(\xi,*)\Big\}\ \&\ \bigvee_\gamma \Big\{A^\circ(*,\gamma)\ \&\ \gamma^\circ(*,x')\Big\} \\
&= \bigvee_{\xi,\gamma} \Big\{\xi(x,*)\ \&\ AA^\circ(\xi,\gamma\ )\ \&\ \gamma^\circ(*,x')\Big\} \\
&\le \bigvee_{\xi,\gamma} \Big\{\xi(x,*)\ \&\ \overline{\delta}(\xi,\gamma\ )\ \&\ \gamma^\circ(*,x')\Big\} \quad A \text{ is a map} \\
&= \bigvee_{\xi,\gamma} \Big\{\xi\xi^\circ\gamma\gamma^\circ(x,x')\Big\} = \delta_X(x,x')
\end{aligned}$$

and

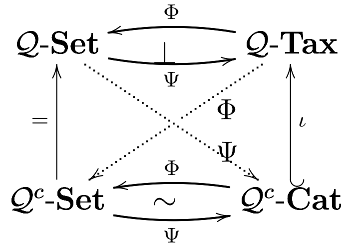
$$\begin{aligned}
\alpha^\circ\alpha(*,*) &= \bigvee_x \Big\{\alpha^\circ(*,x)\ \&\ \alpha(x,*)\Big\} \\
&= \bigvee_{x,\xi.\gamma} \Big\{A^\circ(*,\xi)\ \&\ \xi^\circ(*,x)\ \&\ \gamma(x,*)\ \&\ A(\gamma,*)\Big\} \\
&= \bigvee_{\xi,\gamma} \Big\{A^\circ(*,\xi)\ \&\ \xi^\circ\gamma\ \&\ A(\gamma,*)\} \\
&= \bigvee_{\xi,\gamma} \Big\{A^\circ(*,\xi)\ \&\ \overline{\delta}(\xi,\gamma)\ \&\ A(\gamma,\xi)\Big\} \\
&= A^\circ A(*,*) = q
\end{aligned}$$

Thus α is a singleton with domain the domain of A and codomain (X,ρ,δ). Observe that this implies that $\overline{\delta}(\alpha,\alpha) = q$. Finally we need to show that $A(\beta,*) = \overline{\delta}(\beta,\alpha)$.

$$\begin{aligned}
\overline{\delta}(\beta,\alpha) &= \bigvee_x \Big\{\beta^\circ(*,x)\ \&\ \alpha(x,*)\Big\} \\
&= \bigvee_x \left\{\beta^\circ(*,x)\ \&\ \bigvee_\gamma \Big\{\gamma(x,*)\ \&\ A(\gamma,*)\Big\}\right\} \\
&= \bigvee_{x,\gamma} \Big\{\beta^\circ(*,x)\ \&\ \gamma(x,*)\ \&\ A(\gamma,*)\Big\} \\
&= \bigvee_\gamma \Big\{\beta^\circ\gamma(*,*)\ \&\ A(\gamma,*)\Big\} \\
&= \bigvee_\gamma \Big\{\overline{\delta}(\beta,\gamma)\ \&\ A(\gamma,*)\Big\} = A(\beta,*)
\end{aligned}$$

Thus $(\overline{X},\overline{\rho},\overline{\delta})$ is complete. ∎

We now have an equivalence of the categories of $\mathcal{Q}$-valued sets and $\mathcal{Q}$-categories with the restriction to the case that every object is complete, and every $\mathcal{Q}$-semifunctor is regular.



Definition 5.13 A quantaloid is *strictly-Gelfand* if for all morphisms q, $qq^*q \le q$ implies that $qq^*q = q$. $\diamondsuit$

If $\mathcal{Q}$ is strictly-Gelfand, then every $\mathcal{Q}$-semicategory is a $\mathcal{Q}$-category, since the strictly-Gelfand condition forces equality in the last row below.

$$\begin{aligned} \delta(x,y)\delta(x,y)^*\delta(x,y) &= \delta(x,y)\delta^\circ(y,x)\delta(x,y) \\ &= \delta(x,y)\delta(y,x)\delta(x,y) \\ &\le \delta(x,x)\delta(x,y) \le \delta(x,y) \end{aligned}$$

Similarly we have $\delta(x,y)\delta(y,y) = \delta(x,y)$. Thus $\mathcal{Q}$-**Scat**, $\mathcal{Q}$-**Cat** and $\mathcal{Q}$-**Tax** are the same category when $\mathcal{Q}$ is strictly-Gelfand. It is easy to show that any quantaloid that satisfies Freyd's modular law is strictly-Gelfand and so every distributive category of relations is strictly-Gelfand. We can thus extrapolate these ideas to show that given a Grothendieck topos $\mathcal{E}$

$$\mathbf{Rel}(\mathcal{E})^c\text{-}\mathbf{Set} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Rel}(\mathcal{E})^c\text{-}\mathbf{Tax} \quad (\sim)$$

6 Relational-sheaves and $\mathcal{Q}$-Valued Sets

In this section we begin by constructing, for any quantaloid, an equivalence between the semicategory of $\mathcal{Q}$-semicategories and $\mathcal{Q}$-matrices ($\mathcal{Q}$-**Mat**) and the semicategory of relational-pretransformations (PT($\mathcal{Q}$)). Recall that when $\mathcal{Q}$ is a bounded distributive category of relations the category of $\mathcal{Q}$-valued sets is a Grothendieck topos, thus it will follow that the category of relational-sheaves on $\mathcal{Q}$ and the transformations that are maps is a Grothendieck topos. We start with a needed lemma that forms the essence of the equivalence.

Lemma 6.1 Let $\mathcal{Q}$ be a quantaloid and let $F, G : \mathcal{Q}^{co} \to \mathbf{Rel}$ be relational-presheaves. For each pair of objects A and B, a function $f : \mathcal{Q}^{co}(A,B) \to \mathbf{Rel}(FA, GB)$ is infima-preserving if and only if for every $b \in GB$ and $a \in FA$ the set $\{q \mid f(q)(b,a) = 1\}$ is a principal down-closed set.

Proof: If f preserves infima then we must have

$$f\Big(\bigvee\{q \mid f(q)(b,a)=1\ \}\Big)(b,a)=1,$$

thus $\{q \mid f(q)(b,a)=1\ \}$ is a principal down closed set. Now assume that for every $b \in GB$ and $a \in FA$ the set $\{q \mid f(q)(b,a)=1\ \}$ is a principal down closed set. Automatically we have $f(\vee q_i) \leq \wedge f(q_i)$. Let $\langle q_j \rangle$ be a family of morphisms and let $f(q_j)(b,a)=1$ for each j. Since $q_j \leq \vee q_j$ and since $\{q \mid f(q)(b,a)=1\ \}$ is a principal down closed set,

$$f\Big(\vee\, q_j\Big)(b,a)=1=\bigwedge\Big\{f(q_j)(b,a)\Big\}.$$

∎

The immediate implication of this lemma is that we can associate to each pair $a \in F(A)$, $b \in F(B)$ a particular $\mathcal{Q}$-morphism, which then enables us to construct the appropriate $\mathcal{Q}$-semicategory $(X,\rho,\delta\ )$. Similarly, we can apply this to transformations.

Recall that if $K, L : |\mathcal{Q}| \to |\ \mathbf{Rel}|$ are functions between the objects, then a pretransformation $K \overset{\tau}{\Longrightarrow} L$ consists of a $|\mathcal{Q}|\times|\mathcal{Q}|$ indexed family of infima-preserving arrows

$$\Big\langle \tau_{AB} : \mathcal{Q}^{co}(A,B) \to \mathbf{Rel}\Big(K(A), L(B)\Big)\Big\rangle.$$

These form a semicategory which is denoted $PT(\mathcal{Q})$. From this we defined the semicategory of relational-presheaves to be $\mathbf{Mod}(PT(\mathcal{Q}))$ and the category of relational-sheaves to be $\mathbf{Kar}(PT(\mathcal{Q}))$. These are denoted by $RP(\mathcal{Q})$ and $RS(\mathcal{Q})$ respectively.

Theorem 6.2 Let $\mathcal{Q}$ be a quantaloid, then the semicategory of $\mathcal{Q}$-matrices is equivalent to the semicategory of relational-pretransformations.

Proof: We first define Γ: $PT(\mathcal{Q}) \to \mathcal{Q}$-$\mathbf{Mat}$. Let $K, L : |\mathcal{Q}| \to |\ \mathbf{Rel}|$ be functions. In other words objects in the semicategory of pretransformations and let $K \overset{\tau}{\Longrightarrow} L$ be a relational-pretransformation, then

1. $\Gamma(\ K) = (X,\rho_X)$ where $X = \coprod_{A\in|\mathcal{Q}|} K(A)$, and $\rho_X(x)=A$ if $x \in K(A)$.

2. $\Gamma(\ \tau) = M_\tau$ where $M_\tau(y,x) = \bigvee\Big\{q \mid \tau_q(y,x)=1\Big\}$

To see that Γ preserves composition let σ be a second relational-pre-transformation

By definition $M_{\sigma\tau}(c,a) = \bigvee \Big\{ q \mid (\sigma\tau)_q(c,a) = 1 \Big\}$. But observe that $(\sigma\tau)_q(c,a) = 1$ if and only if there is a family of morphisms $\langle g_i, h_i \rangle$ in $\mathcal{Q}$, where $\bigvee(g_i h_i) \geq q$ and $\sigma_{g_i}\tau_{h_i}(c,a) = 1$ for all i. So it is evident that

$$q \leq \bigvee \Big\{ g_i h_i \mid \sigma_{g_i}\tau_{h_i}(c,a) = 1 \Big\}.$$

This is in turn less than or equal $M_\sigma M_\tau(c,a)$. Thus $M_{\sigma\tau}$ ls less than or equal $M_\sigma M_\tau$.

For $M_\sigma M_\tau \leq M_{\sigma\tau}$ we have

$$\begin{aligned} M_\sigma M_\tau(c,a) &= \bigvee_b \Big\{ M_\sigma(c,a) \ \& \ M_\tau(b,a) \ \Big\} \\ &= \bigvee_b \Big\{ \vee \{q \mid \sigma_q(c,b) = 1\} \ \& \ \vee \{p \mid \tau_p(b,a) = 1\} \Big\} \\ &= \bigvee \Big\{ qp \mid \sigma_q \tau_p(c,a) = 1 \Big\} \\ &\leq \bigvee \Big\{ q \mid \Big(\sigma\tau\Big)_q(c,a) = 1 \Big\} \\ &= M_{\sigma\tau}(c,a) \end{aligned}$$

Let Λ: $\mathcal{Q}$-$Mat \to PT(\mathcal{Q})$ be defined on a matrix $(X,\rho_X) \xrightarrow{M} (Y,\rho_Y)$ by setting

1.$\Lambda \ \Big((X,\rho_X)\Big) = \overline{X}$ where $\overline{X}(A) = \Big\{ x \in X \mid \rho(x) = A \Big\}$

2.$\Lambda(\ M) = \tau_M$ where $\tau_{M_q}\Big(y,x\Big) = 1$ if and only if $q \leq M\Big(x,y\Big)$.

For two composible matrices M and N, $(\tau_M \tau_N)_q(c,a) = 1$ if and only if there exists a family of composible morphisms $\langle g_i, h_i \rangle$ such that $\vee g_i h_i \geq q$ and $\tau_{M_{g_i}} \tau_{N_{h_i}}(c,a) = 1$ for all i. This is if and only if there is a b_i for each i such that

$$\tau_{M_{g_i}}(c,b_i) = 1 = \tau_{N_{h_i}}(b_i,x)$$

If and only if

$$\begin{aligned} q &\leq \bigvee_{b_i} \Big\{ M(c,b_i) \ \& \ N(b_i,a) \Big\} \\ &\leq \bigvee_b \Big\{ M(c,b) \ \& \ N(b,a) \Big\} \end{aligned}$$

This implies that $q \leq MN(c,a)$, thus $\tau_M \tau_N \leq \tau_{MN}$.

Assume that $\tau_{MN_q}(c,a) = 1$, which implies that $q \leq MN(c,a)$. By definition $MN(c,a)$ is equal to $\vee_b\{M(c,b) \ \& \ N(b,a)\}$. We pick as our family of morphisms $\langle M(c,b), N(b,a)\rangle$. This gives us the other needed inequality $\tau_{MN} \leq \tau_M \tau_N$ and thus Λ is a semifunctor. Clearly we have an isomorphism of semicategories.

$$\Gamma\,\Lambda = \mathbf{1} \quad \text{and} \quad \Lambda\,\Gamma = \mathbf{1}.$$ ∎

A consequence of this is that we can interpret the semicategory of pretransformations as the completion of a quantaloid with respect to coproducts. An immediate corollary is that the following categories are equivalent.

$\mathcal{Q}$-**Mod** = **Mod**($\mathcal{Q}$-**Mat**) $\sim$ **Mod**$(PT(\mathcal{Q})) = RP(\mathcal{Q})$ (relational-presheaves)
$\mathcal{Q}$-**Prof** = **Kar**($\mathcal{Q}$-**Mat**) $\sim$ **Kar**$(PT(\mathcal{Q})) = RS(\mathcal{Q})$ (relational-sheaves)

Explicitly Γ and Λ can be extended to relational-sheaves and $\mathcal{Q}$-taxons as follows. For F a relational-sheaf, set $\Gamma(F) = (X_F, \rho_F, \delta_F)$ where (X_F, ρ_F) is the matrix object and $\delta_F = \Gamma(\tau_F) = M_\tau$.

Let (X,ρ,δ) be a $\mathcal{Q}$-semicategory, then $\Lambda\big((X,\rho,\delta)\big) = F_X$, where $F_X = \tau_\delta$. It is simple consequence of the constructions that F_X is a relational-presheaf. if F is a relational-sheaf then $\Gamma(F) = (X_F, \rho_F, \delta_F)$ where X_F and ρ_F are determined as above for $\Gamma\big((X,\rho_X)\big)$ and $\delta_F = M_{\tau_F}$. It is easy to see that both Γ and Λ preserve the associated identity morphisms when the functors are respectively restricted to $\mathcal{Q}$-taxons and relational-sheaves.

When $\mathcal{Q}$ is a bounded complete distributive category of relations the category **Map**$\big(RS(\mathcal{Q})\big)$ is a Grothendieck topos and a relational-sheaf (symmetric idempotent pretransformation) can then be interpreted as a sheaf.

Next we show that the categories of $\mathcal{Q}$-taxons and $\mathcal{Q}$-semifunctors is equivalent to the category of relational-sheaves and functional-transformations.

Theorem 6.3 For $\mathcal{Q}$ a quantaloid the category of $\mathcal{Q}$-taxons and $\mathcal{Q}$-semifunctors is equivalent to the category of relational-sheaves and functional-transformations.

Proof: On the objects the functors are described as above.

Let $(X,\rho,\delta\ ) \xrightarrow{f} (Y,\rho,\delta\ )$ be a $\mathcal{Q}$-semifunctor and define $\tau^f : F_X \Rightarrow F_Y$ by $\tau^f_A(x) = f(x)$. We need to show that for any morphism $q : A \to B$ the following square is an inequality.

$$\begin{array}{ccc} F_X(A) & \xrightarrow{\tau^f_A} & F_Y(A) \\ {\scriptstyle F_X(q)}\downarrow & \leq & \downarrow{\scriptstyle F_Y(q)} \\ F_X(B) & \xrightarrow[\tau^f_B]{} & F_Y(B) \end{array}$$

To this end observe that $F_Y(q)\tau^f_A(b,a) = 1$ if and only if there exists a' such that $F_Y(b,a') = 1 = \tau^f_A(a',a)$. Since τ^f_A is a function this is true if and only if $f(a) = a'$ and $q \leq \delta_Y(b, f(a))$.

On the other hand $\tau^f_B F_X(q)(b,a) = 1$ if and only if there is a b' such that $f(b') = b$ and $q \leq \delta_X(b', a)$. Since f is a $\mathcal{Q}$-semifunctor

$$q \leq \delta_X(b', a) \leq \delta_Y(b, f(a)).$$

Thus $\tau_B F_X \leq F_Y \tau_A$ and we see that τ is a functional-transformation. Simply checking the details we have a functor $\overline{\Lambda} : \mathcal{Q}\text{-}\mathbf{Tax} \to RS_{fct}(\mathcal{Q})$.

The construction of $\overline{\Gamma} : RS_{fct}(\mathcal{Q}) \to \mathcal{Q}\text{-}\mathbf{Tax}$ is identical toΓon the objects. If $F \overset{\tau}{\Rightarrow} G$ is a functional-transformation then define $\overline{\Gamma}(\tau)$ to be the $\mathcal{Q}$-semifunctor $(X_F, \rho_F, \delta_F) \xrightarrow{f_\tau} (X_G, \rho_G, \delta_G)$ given by setting for each $x \in F(A)$, $f_\tau(x) = \tau_A(x)$.

This is a $\mathcal{Q}$-semifunctor. Let $q : A \to B$, then $q \leq \delta_F(b,a)$ if and only if $F(q)(b,a) = 1$. Since τ is a $\mathcal{Q}$-semifunctor, $\tau_B(b) = f_\tau(b)$ which means that $\tau_B(f_\tau(b), a) = 1$. Since $\tau_B F(q) \leq G(q)\tau_A$ we must have that $G(q)\tau_A(f_\tau(b), a) = 1$. Because τ_A is a function $G(q)(f_\tau(b), f_\tau(a)) = 1$ and $\tau_A(f_\tau(a), a) = 1$. Thus $q \leq \delta_G(f_\tau(b), f_\tau(a))$ and we can conclude that $\delta_F(b,a) \leq \delta_G(f_\tau(b), f_\tau(a))$. Hence f_τ is a $\mathcal{Q}$-semifunctor. That $\overline{\Gamma}$ is a functor now follows easily. Again with simple computations we have an isomorphism of categories

$$\overline{\Gamma}\,\overline{\Lambda} = \mathbf{1} \quad \text{and} \quad \overline{\Lambda}\,\overline{\Gamma} = \mathbf{1}.$$

∎

For completeness we describe for relational-sheaves the properties that are equivalent to those needed to create the equivalence between $\mathcal{Q}^c$-**Set** and $\mathcal{Q}^c$-**Tax**.

Definition 6.4

- Let F be a relational-presheaf. A modular-transformation τ is a *singleton on* F if for some symmetric idempotent, $q : A \to A$, in $\mathcal{Q}$, there is a relational-sheaf F_q, where

  $$F_q(X) = \begin{cases} \{*\} & \text{if } X = A \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{and} \qquad F_q(p)(*,*) = 1 \quad \text{IFF} \quad p \leq q$$

  with $\tau : F_q \to F$ is a monomorphic map.

- A functional-transformation $F \overset{\tau}{\Rightarrow} G$ is *regular* if $M_\tau M_F = M_\tau$

- A singleton τ is *representable* if there exists x such that for every q we have $\tau_q(y,*) = F(q)(y,x)$.

- A relational-sheaf is *complete* if every singleton $\tau : F_q \to F$ is representable.

- A relational-presheaf is *Karoubian* if for every pair $a \in F(A)$ and $b \in F(B)$; $F(p)(b,a) = 1$ implies

  - there exists morphisms q_1, q_2 such that

    $$F(q_1)(b,b) = 1 = F(q_2)(b,a) \text{ and } q_1 q_2 \geq p$$

  - there exists morphisms q_3, q_4 such that

    $$F(q_3)(b,a) = 1 = F(q_4)(a,a) \text{ and } q_3 q_4 \geq p$$

  (The relational-presheaf equivalent to being a $\mathcal{Q}$-category) ♢

It is now easy to show that the category of relational-sheaves and regular functional-transformations is equivalent to the category of $\mathcal{Q}$-taxons and regular $\mathcal{Q}$-semifunctors and thus we have the following equivalences.

Corollary 6.5 If $\mathcal{Q}$ is a bounded distributive category of relations, then the following are equivalent categories.

1: $\mathcal{Q}^c$-**Tax**

2: $\mathcal{Q}^c$-**Set**

3: $RS(\mathcal{Q})^c_{fct}$

4: $\mathbf{Map}\Big(RS(\mathcal{Q})\Big)^c$

In addition each is a Grothendieck topos. ∎

Corollary 6.6 If $\mathcal{E}$ is a Grothendieck topos, then the following are equivalent categories.

1: $\mathcal{E}$

2: $\mathbf{Rel}(\mathcal{E})^c$-**Tax**

3: $\mathbf{Rel}(\mathcal{E})^c$-**Set**

4: $RS\Big(\mathbf{Rel}(\mathcal{E})\Big)^c_{fctn}$

5: $\mathbf{Map}\Big(RS\Big(\mathbf{Rel}(\mathcal{E})\Big)^c\Big)$ ∎

We thus have for $\mathcal{E}$ a Grothendieck topos that a sheaf on $\mathcal{E}$ in the traditional sense is an idempotent infima and involution preserving lax-semifunctor from the category $\mathbf{Rel}(\mathcal{E})^{co}$ to the category of sets and relations on $\mathcal{E}$ (ie: a relational-sheaf). In particular for $\mathcal{H}$ a Heyting algebra, interpreted as a one object quantaloid, a sheaf F is an idempotent symmetric lax-semifunctor $F : \mathcal{H}^{co} \to \mathbf{Rel}$. That is

$$\tau_F \tau_F = \tau_F$$

$$\tau_F(q)(y, x) = \tau_F(q)(x, y)$$

The focus on this paper has been to relate enriched taxon theory, $\mathcal{Q}$-valued set theory and relational-presheaves. There are many aspects that warrant further study, for example what can be said about the structures when we focus on order enriched semicategories. This may be of interest since a version of lemma 6.1 says that a function is order-preserving if the associated hom set is down-closed. Other directions include focusing on relational-presheaves and sheaves and develop in more detail their theory along the lines of sheaf theory. One could also expand on enriched-taxon theory with respect to enriched-category theory with an emphasis on where the existence of identity morphisms is indispensable.

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