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AUTOCATEGORIES: I. A COMMON SETTING FOR KNOTS AND 2-CATEGORIES

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Abstract. An *autograph* is an action of the free monoid with 2 generators; it could be drawn with no use of objects, by arrows drawn between arrows. As examples we get knot diagrams as well as 2-graphs. The notion of an *autocategory* is analoguous to the notion of category, by replacing the underlying graph by an autograph. Examples are knots or links diagrams (unstratified case), categories, 2-categories, double categories (stratified case), which so live in the same context, the category of autocategories.

Résumé. Un *autographe* est une action du monoïde libre à deux générateurs d et c, et peut être représenté en dessinant des flèches entre des flèches, sans utiliser d'objets. Par exemple nous avons les graphes et les 2-graphes. La notion d'*autocatégorie* est semblable à celle de catégorie, en remplaçant le graphe sous-jacent par un autographe. Les exemples sont les diagrammes de nœuds ou d'entrelacs (cas non-stratifiés), les catégories, 2-catégories et catégories doubles (cas stratifiés), qui ainsi résident dans la même catégorie des autocatégories.

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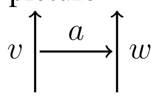
1. Autograph

Definition 1.1. *An* autograph $(A, (d, c))$ *is a set* A *of elements named* arrows, *equipped with two maps,* domain $d : A \to A$ *and* codomain $c : A \to A$, *i.e. a map* $\partial = (d, c) : A \to A \times A$. *Of course it is the same thing that an action on* A *of* $\mathbb{FM}(2) = \{d, c\}^*$, *the free monoïd on two generators* d *and* c.
We denote by Agraph *the category of autographs, with morphism maps* $f : A \to A'$ *with* $d'fa = fda$, $c'fa = fca$, *and* U : Agraph $\to$ Set *the forgetful functor given by* $\mathrm{U}((A, (d, c)) = A$.

Example 1.2. For $(G, (\delta, \gamma))$ a 2-generated group G (e.g. any finite simple group), with generators δ and γ, any G-set E is an autograph, with d and c given by the actions of δ and γ on E. In such a case d and c are invertible.

Remark 1.3. We represent the fact that the domain of an arrow $a \in A$ is the arrow v and its codomain is the arrow w, i.e. $da = v$ and $ca = w$, by: $a : v \to w$, or $v \overset{a}{\to} w$, or by the picture



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Definition 1.4. *1 .*[path] — *A (d, c)-path or a* path *of length k in an autograph $(A, (d, c))$ is a finite sequence of consecutive arrows $(z_n)_{0 \leq n \leq k-1}$ with*

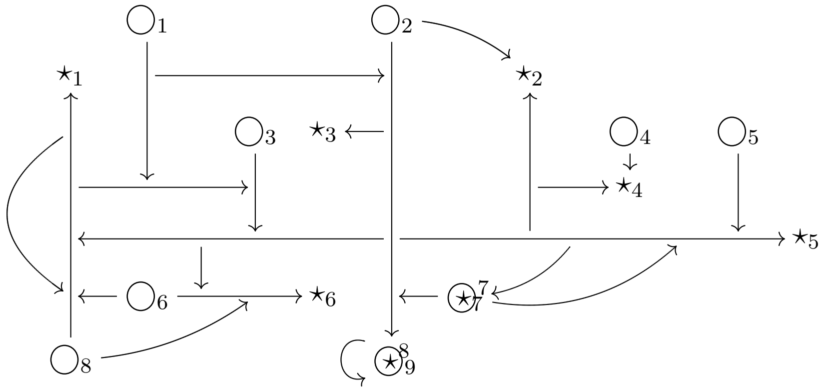
$$cz_0 = dz_1, \quad cz_1 = dz_2, \quad \ldots \quad cz_{k-2} = dz_{k-1}.$$

If there is no path of length > 1, then the autograph is U*-free. The set of paths in A is denoted* Path$(A, (d, c))$ *or shortly* Path(A).
2 .[descent] — *Given an autograph $(A, (d, c))$, a (d, c)-sequence or a downward sequence or a* descent *in $(A, (d, c))$ is a sequence $(x_n)_{n \geq 0}$ — finite or not — of elements of A with:*

$$\forall n \geq 0 \quad x_{n+1} \in \{dx_n, cx_n\}.$$

If there is no cyclic (resp. infinite) descent, then the autograph is stratifiable (resp.foundable). The set of (d, c)-sequences or descents in A is denoted Desc$(A, (d, c))$ *or shortly* Desc(A).

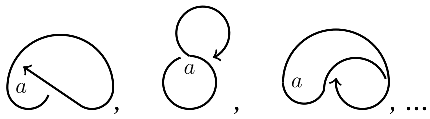
Example 1.5. A random example of a fragment of an *autograph* is:



where the $\bigcirc_j$ are sources, and the $\star_i$ are targets, with $\bigcirc_9 = \star_8$, in such a way that the picture looks like a kind of super-arrow from $(\bigcirc_j)_{j=1,\ldots,9}$ toward $(\star_i)_{i=1,\ldots,8}$; the sources and targets are 'open' or 'empty' places, in the sense that they must be filled by new arrows, for example by auto-arrows (cf. 2.1 and 2.2).

2. Auto-arrows and terminal autograph, self-reference

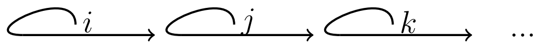
Definition 2.1. *In an autograph, an* 'auto'-arrow *is an auto-mapping from itself to itself, i.e. an arrow a with domain $da = a$ and codomain $ca = a$, i.e. a data which could be written as "$a : a \to a$", and be drawn as*



Remark 2.2. Of course very often — by way of *abbreviation* — we denote an 'auto'-arrow by a simple closed curve, as a circle $\bigcirc_a$, or a square $\square_a$, or even by a bullet $\bullet_a$, or a star $\star_a$, or a crossing $\times_a$, etc., and so we get a 'point' in our picture ; but such a 'point' is not at all considered as a static or stable 'object', rather it is an auto-modification.

Definition 2.3. *The* terminal autograph*, i.e. the terminal object in* Agraph*, consists of one letter named $*$; it will be denoted by $\mathrm{S}_* = \{* : * \to *\}$ (referencing to the shape 'S' of the second picture for an auto-arrow in 2.1). An auto-arrow a in A is equivalent to a morphism of autographs $a^\bullet : \mathrm{S}_* \to (A, (d, c))$, the* constant map *on a.*

Remark 2.4. The visualization of *self-reference* as an auto-arrow, or a 'partial auto'-arrow was introduced by Jean Schneider [7, 8], in order to modelize graphically the structure of time : an instant i is an operation i applied to itself i and producing a new thing j, which itself produces k, etc. and so the time is constructed:



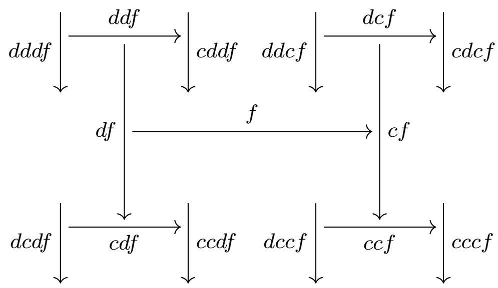
3. Free autographs on $\aleph_0$ generators

Proposition 3.1. *The* U*-free autograph on* $\aleph_0$ *generators* FA(3$\mathbb{N}$) *can be constructed as* $(\mathbb{N}, (t_1, t_2))$*, with* $\mathbb{N}$ *the set of natural numbers and* $d = t_1$*,* $c = t_2$*:*

$$t_1(m) = 3m + 1, \quad t_2(m) = 3m + 2 \qquad (\star)$$

Any finite or denumerable example of autograph is a quotient of this one. Furthermore, the set $\mathbb{R}$ *of real numbers is identified to a subset of the set of descents in* FA(3$\mathbb{N}$)*:* $\mathbb{R} \subset \mathrm{Desc}((\mathbb{N}, (t_1, t_2)))$.

Proof. 1 — The *free autograph* FA$(\{f\})$ *on one generator* f starts with:



According to a dyadic process, the beginning of FA$(\{f\})$ is pictured as this given H-binary tree. It could also be seen as a part of the Cayley graph for the free group on two generators s, t [2, Fig. 2.3., p.40] with $f \mapsto 1$, $df \mapsto s^{-1}$, $cf \mapsto s$, $ddf \mapsto ts^{-1}$, $cdf \mapsto t^{-1}s^{-1}$, $dcf \mapsto ts$, $ccf \mapsto t^{-1}s$, etc.

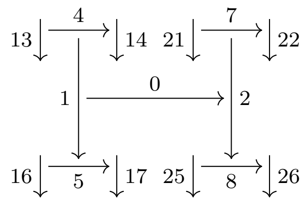
2 — With $d = 1$ and $c = 2$, and with $f = .$, to each element of $FA\{f\}$ is associated a triadic code with no 0 ; for example to $dcccdcf$ is associated the code .212221, and the associated rational number $\frac{2}{3} + \frac{1}{9} + \ldots + \frac{1}{729} = \frac{403}{729}$. At this level, the operations d and c are realized as $d = T_1$ and $c = T_2$:

$$T_1\Big(\frac{m}{n}\Big) = \frac{m}{n} + \frac{1}{3}\frac{1}{n}, \quad T_2\Big(\frac{m}{n}\Big) = \frac{m}{n} + \frac{1}{3}\frac{2}{n}. \qquad (\star\star)$$

Then FA$(\{f\})$ appears as a sub-autograph of the one consisting in the set $[0, 1[_{\text{rat}} = \{\frac{m}{n}; 0 \leq m < n\}$ equipped with d and c given by $(\star\star)$.

In fact, 403 determines completely the fraction $\frac{403}{729}$, the sequence 12221 being obtainable by successive divisions by 3, as the successive residues. So $FA\{f\}$ appears also as a sub-autograph of the one consisting in the set $\mathbb{N}$

equipped with $d = t_1$ and $c = t_2$ given by $(\star)$. We recover $\mathrm{FA}(\{f\})$ as the sub-structure $< 0 >$ generated by 0 :



$< 0 >$

3 — We get $\mathbb{N} = \sum_{n \geq 0} < 3n >$: for any $N \in \mathbb{N}$, successive divisions by 3 provide $N = 3q_1 + x_1, q_1 = 3q_2 + x_2, \ldots, q_{k-1} = 3q_k + x_k, q_k = 3q_{k+1}$, with $x_1, \ldots x_k \in \{1, 2\}$, $q_1, \ldots q_{k+1} \in \mathbb{N}$, and then $N \in < 3n >$ for the unique value $n = q_{k+1}$. So $(\mathbb{N}, (t_1, t_2))$, with t_1 and t_2 given by $(\star)$, appears as $\mathrm{FA}(3\mathbb{N})$, the free autograph on a denumerable set of generators $\{g_n\}$ with $g_n = 3n$, $3\mathbb{N} = \{3n; n \in \mathbb{N}\}$.
4 — Each $x \in \mathbb{R}$ is representable as $x = x_0 + \sum_{i \geq 1} \frac{x_i}{2^i}$, where, for every $i \geq 1$, $x_i \in \{0, 1\}$, and $x_0 \in \mathbb{N}$; if for one $s \geq 0$, $x_{s+j} = 1$ for all $j \geq 1$, then we replace x_s by $x_s + 1$ and all the x_{s+j}, $j \geq 1$, by 0: this new code determines the same x. After that, every x has a unique representation, with no infinite sequence of 1; then we associate to x the infinite sequence of elements of $< 3x_0 >$ with codes in d and c associated to $[x]_n = .x_1 x'_2 x'_3 \ldots x'_n$ with, for all i, $x'_i = x_i + 1$. For example to the real $\frac{5}{3}$ is associated the sequence $\ldots dcdcdc(3)$, or to the real π, of which the binary code is $11.001001000011111101\ldots$ is associated the sequence $\ldots cdcccccccddddcddcdd(3)$. So $\mathbb{R}$ appears as a completion of the autograph $\mathrm{FA}(3\mathbb{N})$, in terms of descents (definition 1.4): $\mathbb{R} \subset \mathrm{Desc}(\mathrm{FA}(3\mathbb{N}))$. □

4. Knots and links, surgery

Proposition 4.1. *Any oriented knot diagram* $\mathbb{K}$ *determines an associated autograph denoted by* $\mathrm{As}(\mathbb{K}) = (\mathrm{Arc}(\mathbb{K}), \alpha, \omega))$.

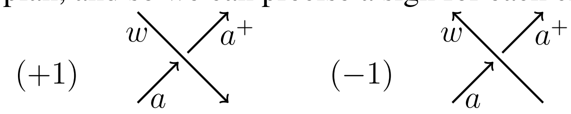
Proof. Given an oriented 'knot diagram' $\mathbb{K}$ [1, 5], i.e. a regular plan projection of a knot, with only isolated regular double points (the crossings of the diagram), following the orientation, from any crossing toward the next one, we get an oriented arc a, which is seen as an arrow from $v = \alpha(a)$ to

$w = \omega(a)$, if the first crossing is a crossing of v and a, and the second one is a crossing of w and a; so, on the set Arc($\mathbb{K}$) of arcs with these two maps $d = \alpha$ and $c = \omega$ we get a structure of autograph, denoted by As($\mathbb{K}$), in which for each a we have:

$$\xrightarrow{a^-} \alpha(a) \Big\uparrow \xrightarrow{a} \Big\uparrow \omega(a) \xrightarrow{a^+}$$

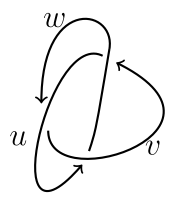
□

Remark 4.2. Elements of Arc($\mathbb{K}$) are not completely 'abstract', they are real arcs in the plan, and so we can precise a sign for each crossing :



So each arc a of the oriented knot diagram is equipped with a double sign (ϵ, η), where ϵ is the sign of its initial crossing and η is the sign of its final crossing; the data $(a, (\epsilon, \eta))$ is named a *doubly signed arc*, $(a, \epsilon\eta = \sigma)$ is named a *signed arc*, and a itself is an *unsigned arc* (unsigned, but oriented), with a predecessor a^- (crossing with v, sign ϵ) and a successor a^+ (crossing with w, sign η). Of course the values of ϵ and η determine the orientations on d and c.
Now, as the set of unsigned arcs, the set of signed arcs and the set of doubly signed arcs — in the given knot diagram $\mathbb{K}$ — are two other autographs associated to $\mathbb{K}$, denoted by As$_\sigma$($\mathbb{K}$) and As$_{\epsilon,\eta}$($\mathbb{K}$).
Of course the same constructions work for any oriented link diagram $\mathbb{L}$.

Example 4.3. The simplest example is the *trefoil knot*, with an oriented diagram $\mathbb{T}$, in which the double sign of each arc is $(+,+)$; the associated autograph As($\mathbb{T}$) is pictured and listed as follows:



$u : w \to v,$

$v : u \to w,$

$w : v \to u,$

or described by : $\alpha u = v, \omega u = w, \alpha v = w, \omega v = u, \alpha w = u, \omega w = v.$

Remark 4.4. In [8, p. 168-169], this diagram is commented almost as follows. With $R = {}^{v}{}_{w}$, $S = {}^{w}{}_{u}$, $I = {}^{u}{}_{v}$, we write, as for fractions (or matrices composition or tensors calculus), $S \star I = {}^{w}{}_{u} \star {}^{u}{}_{v} = {}^{w}{}_{v} = R^{-1}$; hence Schneider think of the trefoil as representing relations like :

$$S \star I = R^{-1}, \quad I \star R = S^{-1}, \quad R \star S = I^{-1},$$

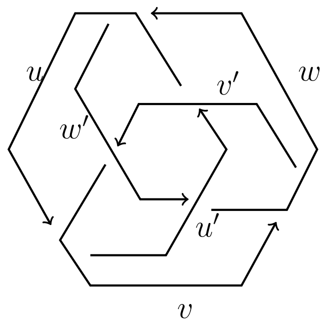
and the trefoil becomes a symbol of a borromean situation between R, S and I. Then the borromean schema is presented as an enrichment of the trefoil with three new crossing points r, s and i, with

$$R = I * i, S = R * r, I = S * s, r = i * I, s = r * R, i = s * S.$$

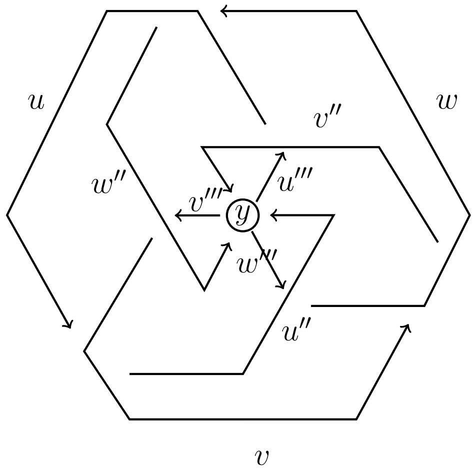
Here we proceed in a different way, trying to stay always at the level of autographs (arrows), i.e. with arcs rather than with crossing points (objects).

Example 4.5. An example is the *borromean* link, with an oriented diagram $\mathbb{B}$, and the associated autograph As($\mathbb{B}$) is listed and pictured as follows:

$$\begin{array}{ll} u : v' \to v, & u' : v \to v', \\ v : w' \to w, & v' : w \to w', \\ w : u' \to u, & w' : u \to u'. \end{array}$$



Example 4.6. Starting with $\mathbb{B}$, in its 'hexagonal center' we can do a surgical procedure first consisting in the introduction of a Y-cut, cutting u' near the end in a point α, now separated into α^- and α^+, and similarly for v' and w', and then continued by the junctions of α^+ and β^-, β^+ and γ^-, γ^+ and α^-. Finally, after three Reidemeister moves of type I (twist)[1] to eliminate the three new loops, we get the picture of the trefoil $\mathbb{T}$. The data of this construction determine an autograph As$(\mathbb{B})_Y$, pictured and listed as:



$$\begin{array}{llll} u : v'' \to v, & v : w'' \to w, & w : u'' \to u, & \\ u'' : v \to y, & v'' : w \to y, & c'' : a \to y, & y : y \to y. \\ u''' : y \to v'', & v''' : y \to w'', & w''' : y \to u'', & \end{array}$$

In $\mathrm{As}(\mathbb{B})_Y$, the Y-cut is simulated by the introduction of an auto-arrow y and three arrows making the convenient junction: u''', v''', w'''.

Proposition 4.7. *At the level of autographs, the surgery procedure in 4.5 — explaining how to get $\mathbb{T}$ from $\mathbb{B}$ — could be translated by the construction of two maps between three free autocategories of paths on an autograph (see definition 6.1, 6.3):*

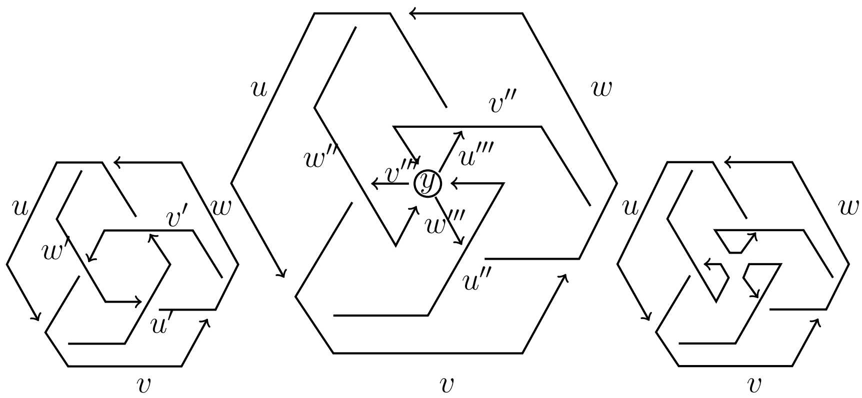
$$\mathrm{Path}(\mathrm{As}(\mathbb{B})) \xrightarrow{\bar{\beta}} \mathrm{Path}(\mathrm{As}(\mathbb{B})_Y) \xleftarrow{\bar{\tau}} \mathrm{Path}(\mathrm{As}(\mathbb{T})).$$

Proof. The reader is invited to follow the paths on the picture of $A(\mathbb{B})_Y$, in order to understand the transformations. The left mapping is determined by the map $\beta : \mathbb{B} \longrightarrow \mathrm{Path}(\mathrm{As}(\mathbb{B})_Y)$, and the right one is determined by $\tau : \mathbb{T} \longrightarrow \mathrm{Path}(\mathrm{As}(\mathbb{B})_Y)$, with

$$\beta(u) = u, \beta(v) = v, \beta(w) = w,$$

$$\beta(u') = u'''u'', \beta(v') = v'''v'', \beta(w') = w'''w'',$$

$$\tau(u) = uu'''v'', \tau(v) = vv'''w'', \tau(w) = ww'''u''.$$

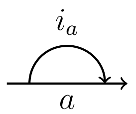


□

5. Identifiers, autographs and flexigraphs with identifiers

In the context of graphs, diagrams and categories, the notions of vertex, object, identity and unit, are almost identified. With the notion of autograph we clarify a distinction between them, at first by the elimination of the notion of object, and the introduction of the notion of auto-arrow. Secondly now we precise the notion of an *identifier*. In the next section we will introduce units and identities.

Remark 5.1. In presence of an object or a vertex X, in a category or a graph for example, we have to be careful and not to confuse the identity mapping $1_X : X \to X$ of the object X with an auto-arrow on X ; in such a situation, it is X itself which determines an auto-arrow $X : X \to X$, or better we have to consider that 1_X is an auto-arrow : $1_X : 1_X \to 1_X$ (see 6.2). But, in the general situation an identity mapping could exist on an arrow different from any object, even for an arrow which is not an auto-arrow, i.e. for such an arrow $\xrightarrow{a}$. In fact we could have a data $i_a : a \to a$ representing a so called 'identifier' on a, i.e. a selected endo-arrow of a, i_a, such that $di_a = a = ci_a$:



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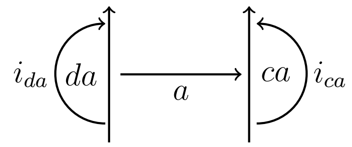
Such a data i_a determines a (by $di_a = a$), and so is really an 'identifier' of a (a 'name' of a), but is distinct from a, and is not an auto-arrow, and is not necessarily unique. Later we will see if a given identifier has to be an identity.

Definition 5.2. *An autograph A in which each arrow a is equipped with the data of a selected endo-arrow $i_{da} : da \to da$ named the* identifier *of da, and a selected endo-arrow $i_{ca} : ca \to ca$ named the* identifier *of ca, is called an* autograph with identifiers. *Then, with each arrow a, we have the picture*



Remark 5.3. Clearly i_{da} (or i_{ca}) does not depend really of a, but only of $v = da$ (or $w = ca$). Of course in an autograph with identifiers, if a is the domain (or the codomain) of something, i.e. $a = db$ or $a = cb$, then we have an identifier i_a on a. And if i_a itself is the domain (or codomain) of something, then we have an identifier i_{i_a} (or j_{i_a}), etc. But if a is not a domain or a codomain, then no identity on a is assumed to be specified.

Definition 5.4. *A* flexigraph *is the data of two sets G_0 and G_1, and three maps $\delta, \gamma : G_1 \to G_0$, $\phi : G_0 \to G_1$.*
The map ϕ is the flex*, and, as in the case of an autograph for d and c, δ and γ are thought as* 'domain' *and* 'codomain'*. The difference with an autograph is that there are two types of elements [0 (vertex), 1 (arrow)], and then domain and codomain look like 'objects' (or 'vertices'). For any $a \in G_1$ we get the picture :*

$$\phi X \Big\uparrow \qquad \delta a = X \xrightarrow{\ a\ } Y = \gamma a \qquad \Big\uparrow \phi Y \quad .$$

There are also 'isolated' ϕZ, for any Z which is not a δa or a γa.

Example 5.5. An autograph (A, d, c) is a special case of flexigraph, with $G_1 = A = G_0$, $\delta = d$, $\gamma = c$, and $\phi = 1_A$. Conversely a flexigraph $(G_1, G_0, \delta, \gamma, \phi)$ determines an autograph with $A = G_1$, $d = \phi\delta$, $c = \phi\gamma$. If the flexigraph is a *flexigraph with 'identities'*, i.e. is equipped with a map $\iota : G_0 \to G_1$ such that $\delta\iota = 1_{G_0} = \gamma\iota$ — that is to say that $(G_1, G_0, \delta, \gamma, \iota)$ is an oriented graph (usually named today a *graph*), and if ϕ in injective — then the associated autograph is with identifiers, with $i_{da} = \iota\delta a$, $i_{ca} = \iota\gamma a$. Of course this works in the special case of an oriented graph just seen as a flexigraph with $\phi := \iota$.

6. Autocategories

Definition 6.1. *An* autocategory $\mathcal{A}$ *is the data of an autograph* (A, d, c) *with identifiers (definitions 1.1 and 5.2), equipped with the data of a* composition law *for consecutive arrows, i.e. for any consecutive arrows*

$$f : p \to q, \quad g : q \to r \ \text{ (where } \ q = dg = cf),$$

we have a composed arrow denoted gf*, with* $d(gf) = df = p$ *and* $c(gf) = cg = r$*, i.e.*

$$\text{Position:} \quad gf : p \to r,$$

such that identities are units for composition:

$$\text{Unitarity:} \quad f i_{df} = f = i_{cf} f;$$

and such that the two compositions of three consecutive arrows are equal:

$$\text{Associativity:} \quad h(gf) = (hg)f, \ \text{ if } \ dh = cg \ \text{ and } \ dg = cf.$$

We denote by Acat *the category of autocategories, with morphism the maps* $F : A \to A'$ *with* $d'Fa = Fda$, $c'Fa = Fca$*, and* $F(ba) = F(b)F(a)$ *if* $db = ca$.
The forgetful functor V : Acat $\to$ Agraph *is given by* $\mathrm{V}(\mathcal{A}) = (A, d, c)$.

Proposition 6.2. *A category determines an autocategory* $\mathrm{Ass}(\mathcal{C})$*. Furthermore, any structure of flexigraph on its underlying graph, such that the flex* ϕ *in injective, determines another structure of autocategory* $\mathrm{Ass}(\mathcal{C}, \phi)$*, with the "same' arrows and the same composition law (but a very different underlying autograph); such a structure is named a* flexicategory *or* a category with a flexion.

Proof. This definition is almost the same as the definition of a category [4], [6], excepted that now there are no objects. As in a category, the identifier i_a on a, if it exists, is unique, because it has to be a unit: in this case it is named an *identity*. This identity does exist when a is a domain or a codomain, but not only in the case where a is the auto-arrow associated to an object.
So, starting with a category $\mathcal{C}$, we get an autocategory $\mathrm{Ass}(\mathcal{C})$ by replacing each object $X \in \mathrm{Obj}(\mathcal{C})$ by an auto-arrow $i_X : i_X \to i_X$, where i_X 'is'

the arrow $1_X : X \to X$, identity on X in the category. If in $\mathcal{C}$ we have $f : X \to Y$, i.e. $X = \text{dom}(f)$ and $X = \text{cod}(f)$, then we consider that $i_X = d_{\text{Ass}(\mathcal{C})} f$, $i_X = c_{\text{Ass}(\mathcal{C})}(f)$. Composition is the same as in $\mathcal{C}$. Of course in this autocategory we recover the objects as being the auto-arrows which are units (as already it works in categories [4]).
If on $\mathcal{C}$ we have a flex $\phi : \text{Obj}(\mathcal{C}) \to \text{Arrow}(\mathcal{C})$ (see definition 5.4), we define d_ϕ and c_ϕ by $d_\phi(f) = \phi(d_{\mathcal{C}}(f))$ and $c_\phi(f) = \phi(c_{\mathcal{C}}(f))$, then for every object X, i_X become an arrow from $\phi(X)$ to $\phi(X)$, which is the identity on $\phi(X)$. □

Proposition 6.3. *To any autograph $(A, (d, c))$ there is associated a* V*-free autocategory* $\text{P}(A, (d, c)) = \Big(\text{Path}^\iota(A, (d, c)), D, C\Big)$, *which is the free autocategory on A.*

Proof. We consider paths $(z_n)_{0 \le n \le k-1}$ in A, shortly denoted by$(z_n)_k$ (definition 1.4), with $D((z_n)_k) := (dz_0)_0$ and $C((z_n)_k) = (cz_k)_0$, and so these paths are between paths of length 1 consisting in a domain or a codomain in A, i.e. an a of the form dx or cy; for any of these a we have to add an identity element I_a to $\text{Path}(A)$. So we get the set $\text{Path}^\iota(A)$. If a is an identifier i_b in A, and possibly an identity when A is an autocategory, then we should not confuse i_b, $(i_b) = (i_b)_0$, and I_{i_b}. In fact I_a plays the part of the empty sequence in the usual calculus of words; but here we need several empty words, one by domain or codomain a. Then the composition is given by concatenation of paths, and by the equations $(z_n)_k I_{z_0} = (z_n)_K$ and $I_{z_k}(z_n)_k = (z_n)_k$. □

7. Double categories and 2-categories as autocategories

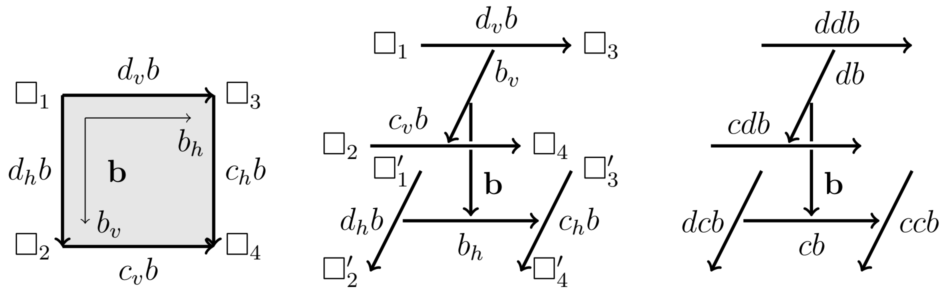
Proposition 7.1. *Any double category $\mathbb{C}$ (and especially any 2-category) is determined by an associated autocategory* $\text{Ass}(\mathbb{C})$.

Proof. 1 — Let $\mathbb{C}$ be a double category [3], where d_h, d_v, c_h, c_v are horizontal and vertical domains and codomains, where ∞ and 8 are horizontal and vertical compositions. Let $\underline{\mathbb{C}}$ be the set of all elements in $\mathbb{C}$ (2-block, horizontal arrow, vertical arrow, or object).
The underlying set of $\text{Ass}(\mathbb{C})$ will be $\underline{\mathbb{C}}_v + \underline{\mathbb{C}}_u + \underline{\mathbb{C}}_h$, a sum of three copies of $\underline{\mathbb{C}}$, and an arbitrary element of $\mathbb{C}$ will have three avatars: if x is a designation

of any of these avatars, we get designations for all the avatars: x_u is the unoriented version, x_h is the horizontally oriented version, x_v is the vertically oriented version. So $(x_h)_h = x_h$, $(x_h)_v = x_v$, $(x_h)_u = x_u$, etc. Each avatar determines the others. In Ass($\mathbb{C}$) we consider that x_u is an arrow from x_v to x_h, i.e. $x_u : x_v \to x_h$, and so we define:

$$d(x_u) = x_v, \quad c(x_u) = x_h.$$

Let $\mathbf{b}$ be a not degenerated element of the double category, an unoriented 2-block, a 2-dim data as in picture [1]:



[1] [2] [3]

To get a determined operation (of type ∞ or 8) with $\mathbf{b} = b_u$ we need an orientation: to operate horizontally we use of b_h, and to operate vertically we use of b_v. We introduce, successively

$$d(b) = b_v, \quad c(b) = b_h$$

$$ddb = d(b_v) = (d_v b)_h, \quad cdb = c(b_v) = ((c_v b))_h,$$

$$dcb = d(b_h) = (d_h b)_v, \quad cdb = c(b_h) = (c_h b)_v,$$

$$\square_1 = dddb = (d_h d_v b)_v, \quad \square'_1 = ddcb = (d_v d_h b)_h.$$

We have

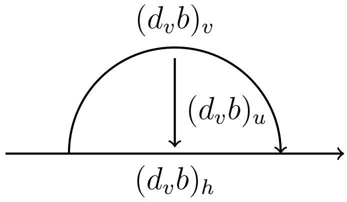
$$(\square_1)_h = \square'_1, \quad (\square'_1)_v = \square_1,$$

And the same facts for the three other corners.

Let us remark that, for clarity and simplicity, in picture [2] and [3] not all the existing arrows are drawn ; for example in fact $d_v b$ is for $(d_v b)_h$, but there are also $(d_v b)_v$, which has to be the vertical unity on $(d_v b)_h$ in $\mathbb{C}$, and here in Ass($\mathbb{C}$) its identity

$$i_{(d_v b)_h} = (d_v b)_v,$$

and then we have also $(d_v b)_u : (d_v b)_v \to (d_v b)_h$, etc.

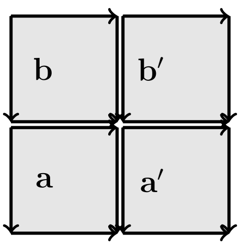


.

In Ass($\mathbb{C}$), if $dy = cx$, then x and y are both horizontal or both vertical. So a unique composition law "." in Ass($\mathbb{C}$) is defined as follows, if and only if $dy = cx$:

$$y.x = \begin{cases} y \infty x & : \text{if } y \text{ and } x \text{ are horizontal,} \\ y \, 8 \, x & : \text{if } y \text{ and } x \text{ are vertical.} \end{cases}$$

Now, to conclude, we have to consider the axiom which relates the two composition laws ∞ and 8 in $\mathbb{C}$, for a 'square of squares':



and to translate it in Ass($\mathbb{C}$).
This compatibility (distributivity) in $\mathbb{C}$ is

$$(a' \infty a) 8 (b' \infty b) = (a' 8 b') \infty (a 8 b),$$

when

$$d_v a = c_v b, d_v a' = c_v b', d_h a' = c_h a, d_h b' = c_h b.$$

The translation in Ass($\mathbb{C}$) is:

$$[(a'_h.a_h)_v.(b'_h.b_h)_v]_h = (a'_v.b'_v)_h.(a_v.b_v)_h,$$

as well as

$$(a'_h.a_h)_v.(b'_h.b_h)_v = [(a'_v.b'_v)_h.(a_v.b_v)_h]_v.$$

If we introduce on Ass($\mathbb{C}$) the involutive *transversal map* $(-)^\theta$ by

$$(x_h)^\theta = x_v, \quad (x_u)^\theta = x_u, \quad (x_v)^\theta = x_h,$$

such that

$$(x^\theta)^\theta = x, \qquad \text{(Inv)}$$

then we could recover the identities i_d and i_c on domains and codomains: $i_{dx} = (dx)^\theta, \quad i_{cx} = (cx)^\theta$, in such a way that for any x we have

$$(cx)^\theta x = x = x(dx)^\theta. \qquad \text{(Id)}$$

And with θ the compatibility becomes:

$$(a'.a)^\theta.(b'.b)^\theta = [(a'.b')^\theta.(a.b)^\theta]^\theta. \quad \text{(Comp)}$$

□

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