CALCULUS OF E-RELATIONS IN INCOMPLETE RELATIVELY REGULAR CATEGORIES

by Tamar JANELIDZE-GRAY

Résumé. Nous définissons une catégorie régulière relative incomplète comme une paire $(\mathbf{C}, \mathbf{E})$, où $\mathbf{C}$ est une catégorie arbitraire et $\mathbf{E}$ est une classe d'épimorphismes réguliers dans $\mathbf{C}$ satisfaisant certaines conditions. Nous développons ce que nous appelons un calcul relatif des relations dans ces catégories; on peut l'appliquer aux relations $(R, r_1, r_2) : A \to A$ dans $\mathbf{C}$ telles que les morphismes r_1 et r_2 sont dans $\mathbf{E}$. Cela généralise plusieurs résultats connus, y compris le travail récent avec J. Goedecke sur les catégories relatives de Goursat. Nous définissons les catégories régulières relatives incomplètes de Goursat et : (a) nous prouvons les versions *relatives incomplètes* des conditions équivalentes définissant les catégories régulières relatives de Goursat, (b): nous montrons que dans ce contexte l'axiome $\mathbf{E}$ -Goursat est équivalent à la version relative du Lemme 3×3.

Abstract. We define an incomplete relative regular category as a pair $(\mathbf{C}, \mathbf{E})$, where $\mathbf{C}$ is an arbitrary category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ satisfying certain conditions. We then develop what we call a relative calculus of relations in such categories; it applies to relations $(R, r_1, r_2) : A \to B$ in $\mathbf{C}$ having the morphisms r_1 and r_2 in $\mathbf{E}$. This generalizes previous results, including the recent work with J. Goedecke on relative Goursat categories. We define incomplete relative regular Goursat categories, and: (a) prove the *incomplete relative* versions of the equivalent conditions defining relative regular Goursat categories, (b): show that in this setting the $\mathbf{E}$-Goursat axiom is equivalent to the relative 3×3-Lemma.

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1. Introduction

An incomplete relative regular category is defined as a pair $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is a category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ satisfying suitable conditions. These conditions are such that:

(a) *Finitely complete relative case*: If $\mathbf{C}$ is a finitely complete category and $\mathbf{E}$ is a class of pullback stable regular epimorphisms in $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $(\mathbf{C}, \mathbf{E})$ is a relative regular category [4];

(b) *Absolute Case*: If $\mathbf{C}$ is finitely complete category with coequalizers of kernel pairs, and $\mathbf{E}$ is a class of all regular epimorphisms in $\mathbf{C}$, and pullback stable, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $\mathbf{C}$ is a regular category;

(c) *Trivial Case*: If $\mathbf{E}$ is the class of all isomorphisms in any category $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ always is an incomplete relative regular category.

Assuming that $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category, we define an $\mathbf{E}$-relation $(R, r_1, r_2) : A \to B$ in $\mathbf{C}$ as a relation R from A to B with r_1 and r_2 jointly monic morphisms in $\mathbf{E}$. The $\mathbf{E}$-relations have already been studied in the context of relative regular categories in [7] and [4], and also in a more general "*incomplete relative*" context in [8] and [9]. However, that incomplete relative context still assumed the existence of certain limits, as well as the pullbacks of morphisms in $\mathbf{E}$. In this paper we consider a more general setting, namely, we do not require the existence of those "special" limits, we only require the existence of pullbacks of morphisms in $\mathbf{E}$. It turns out that most of the results we had for $\mathbf{E}$-relations in [8] and [9] can be extended to this incomplete relative regular category setting.

Relative Mal'tsev and relative Goursat categories were introduced in [3] (see also [2]), and [4] respectively, and now we introduce the incomplete relative Mal'tsev and incomplete relative Goursat categories. Substantial part of this paper is devoted to incomplete relative regular Goursat categories, we show that the results about Goursat categories (see [1] and [5]), which have been extended to relative Goursat categories in [4], can also be extended to these incomplete relative regular Goursat categories.

The paper is organised as follows: In Section 2 we define incomplete relative regular categories and extend the notion of **E**-relations (see [9] and references therein) to this setting. In Section 3 we give some of the properties of **E**-relations, omitting most of the proofs since they are essentially the same as in the finitely complete relative case ([9], [7], and [4]). In Section 4 we define equivalence **E**-relations and state some of their properties, and then we define incomplete relative regular Mal'tsev categories. In Section 5 we define incomplete relative regular Goursat categories and we prove that the **E**-Goursat axiom, just like in the absolute and in the finitely complete relative cases ([1], [5], and [4]), is equivalent to several other equivalent conditions. Finally, in Section 6, we show that also in this incomplete relative context, the **E**-Goursat axiom is equivalent to the 3×3-Lemma (see [5] for the absolute case).

2. Incomplete relative regular categories and E-relations

Throughout the paper we assume that **C** is a category and **E** is a class of morphisms in **C** containing all isomorphisms. Consider the following conditions on $(\mathbf{C}, \mathbf{E})$:

Condition 2.1. (a) Every morphism in **E** is a regular epimorphism;

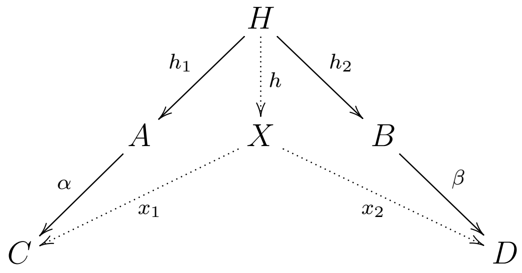
(b) The class **E** is closed under composition;

(c) If $f \in \mathbf{E}$ and $gf \in \mathbf{E}$, then $g \in \mathbf{E}$;

(d) If $f : A \to B$ and $f' : A' \to B$ are in **E**, then the pullback $(A \times_B A', \pi_1, \pi_2)$ of f and f' exists in **C** and the pullback projections π_1 and π_2 are in **E**;

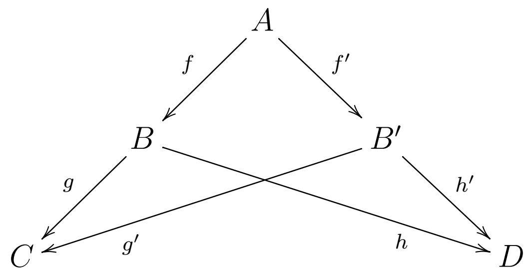
(e) If $h_1 : H \to A$ and $h_2 : H \to B$ are jointly monic morphisms in **C** and if $\alpha : A \to C$ and $\beta : B \to D$ are morphisms in **E**, then there exists a morphism $h : H \to X$ in **E** and jointly monic morphisms

$x_1 : X \to C$ and $x_2 : X \to D$ in $\mathbf{C}$ making the diagram



commutative.

Proposition 2.2. *Suppose* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(d) and 2.1(e), and let*



(2.1)

be a commutative diagram in $\mathbf{C}$. *If* f *and* f' *are in* $\mathbf{E}$ *and* (g, h) *and* (g', h') *are jointly monic pairs, then there exists a unique isomorphism* $\beta : B \to B'$ *with* $g'\beta = g$, $\beta f = f'$, *and* $h'\beta = h$.

Proof. Since f and f' are in $\mathbf{E}$, the kernel pairs of f and f' exist by Condition 2.1(d); moreover, they coincide since (g, h) and (g', h') are jointly monic pairs and the diagram (2.1) is commutative. Since every regular epimorphism is the coequalizer of its kernel pair (when the kernel pair exists), we conclude that there exists a unique isomorphism $\beta : B \to B'$ with $\beta f = f'$, and since f and f' are epimorphisms we obtain $g'\beta = g$ and $h'\beta = h$. $\square$

Remark 2.3. As follows from Proposition 2.2, under the assumptions of Conditions 2.1(a) and 2.1(d), the factorization in Condition 2.1(e) is unique up to an isomorphism.

Proposition 2.4. *Suppose* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(d), and 2.1(e). If a morphism* f *in* $\mathbf{C}$ *factors as* $f = em$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* $\mathbf{E}$.

Proof. Under the assumptions of Condition 2.1(e), take $h_1 = h_2 = m$ and $\alpha = \beta = e$. Then there exists a morphism $\bar{e}$ in $\mathbf{E}$ and jointly monic morphisms $\bar{m_1}$ and $\bar{m_2}$ in $\mathbf{C}$ such that $\bar{m_1}\bar{e} = \bar{m_2}\bar{e}$, and such factorization is unique by Remark 2.3. Since $\bar{e}$ is an epimorhism it follows that $\bar{m_1} = \bar{m_2}$, and therefore $em = \bar{m_1}\bar{e}$ is the desired factorization. □

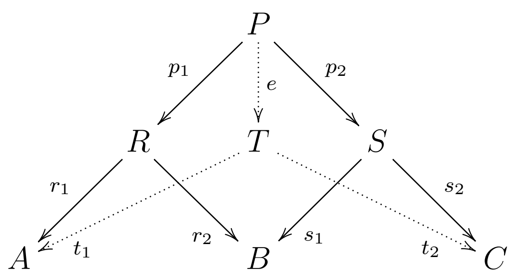
Definition 2.5. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular category if it satisfies Condition 2.1.

As follows from Proposition 2.4 and Definition 2.5, if $\mathbf{C}$ is a finitely complete category and $\mathbf{E}$ is pullback stable class of regular epimorphisms in $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $(\mathbf{C}, \mathbf{E})$ is a relative regular category [4] (see also [9]) (note that, obviously, every relative regular category is incomplete relative regular). In the *"absolute case"*, that is, when $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, if $\mathbf{C}$ has all finite limits and coequalizers of kernel pairs, and $\mathbf{E}$ is pullback stable, then the pair $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $\mathbf{C}$ is a regular category. On the other hand, if we take $\mathbf{E}$ to be the class of all isomorphisms in $\mathbf{C}$, which we call the *"trivial case"*, then any category $\mathbf{C}$ will satisfy Condition 2.1.

Throughout the rest of the paper we assume that $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category. We now extend the calculus of $\mathbf{E}$-relations [9] (see also [7], [8], [4]) to this *incomplete relative* context.

Definition 2.6. An $\mathbf{E}$-relation R from an object A to an object B in $\mathbf{C}$, written as $R : A \to B$, is a triple $R = (R, r_1, r_2)$ in which $r_1 : R \to A$ and $r_2 : R \to B$ are jointly monic morphisms in $\mathbf{E}$.

Let $(R, r_1, r_2) = R : A \to B$ and $(S, s_1, s_2) = S : B \to C$ be $\mathbf{E}$-relations in $\mathbf{C}$ and let (P, p_1, p_2) be the pullback of s_1 and r_2; by Condition 2.1(d) this pullback does exist and p_1 and p_2 are in $\mathbf{E}$. Since p_1 and p_2 are jointly monic and r_1 and s_2 are in $\mathbf{E}$, using Condition 2.1(e) we obtain the commutative diagram



(2.2)

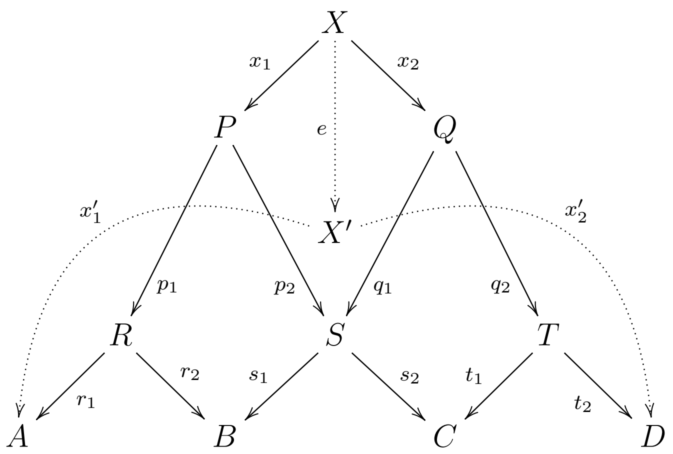
in which e is in $\mathbf{E}$, t_1 and t_2 are jointly monic, and such factorization ($t_1 e = r_1 p_1$ and $t_2 e = s_2 p_2$) is unique up to an isomorphism by Remark 2.3. Moreover, since r_1, p_1, s_2, and p_2 are in $\mathbf{E}$, the morphisms t_1 and t_2 are also in $\mathbf{E}$ by Conditions 2.1(b) and 2.1(c). Accordingly, we introduce:

Definition 2.7. If $R : A \to B$ and $S : B \to C$ are $\mathbf{E}$-relations in $\mathbf{C}$, then their composite $SR : A \to C$ is the $\mathbf{E}$-relation (T, t_1, t_2) in which T, t_1, and t_2 are defined as in the diagram (2.2) above.

It is well known that the composition of relations is associative in a regular category. The same is true for $\mathbf{E}$-relations in relative regular categories, and more generally in incomplete relative regular categories (the proof is essentially the same as in the finitely complete relative context, see Proposition 2.1.9 of [9]):

Proposition 2.8. *The composition of* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is associative (if we identify isomorphic relations).* □

As follows from the proof of Proposition 2.8 (see Proposition 2.1.9 of [9]), to construct the composite of $\mathbf{E}$-relations $(R, r_1, r_2) : A \to B$, $(S, s_1, s_2) : B \to C$, and $(T, t_1, t_2) : C \to D$, we first take the pullbacks (P, p_1, p_2) and (Q, q_1, q_2), of r_2 and s_1, and of s_2 and t_1 respectively, (which exist by Condition 2.1(d), and moreover, p_1, p_2, q_1, q_2 are in $\mathbf{E}$), then take the pullback (X, x_1, x_2) of p_2 and q_1 (which again exists by Condition 2.1), and then their composite $(X', x'_1, x'_2) : A \to D$ will be the $\mathbf{E}$-relation obtained from the following factorization:



(2.3)

In a similar way we can compose any finite number of $\mathbf{E}$-relations accordingly.

From now on, in the rest of the paper, we will identify the isomorphic relations. For each $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ there is an opposite $\mathbf{E}$-relation $R^\circ : B \to A$ given by the triple (R, r_2, r_1), and, just as in the absolute case, we have:

Proposition 2.9. *If* $(R, r_1, r_2) : A \to B$ *and* $(S, s_1, s_2) : B \to C$ *are* $\mathbf{E}$-*relations in* $\mathbf{C}$*, then:*

(i) $(R^\circ)^\circ = R$.

(ii) $(SR)^\circ = R^\circ S^\circ$. □

3. Properties of the E-relations

Most of the properties known for relations in a regular category have been extended to relative regular categories (see [7], [9], and [4]). In [8] we have proved that these properties also hold true when only some limits, namely the limits of some *special* diagrams (special case of which are pullbacks) existed. It turns out that the results can actually be proved in even more general setting, namely, when only the pullbacks of morphisms in $\mathbf{E}$ exist, i.e. in incomplete relative regular categories. We state some of these properties below, omitting the proofs since they are essentially the same as the proofs given in [9]:

Proposition 3.1. *Let* $(R, r_1, r_2) : A \to B$, $(R', r_1', r_2') : A \to B$, $(S, s_1, s_2) : B \to C$, *and* $(S', s_1', s_2') : B \to C$ *be* $\mathbf{E}$*-relations in* $\mathbf{C}$. *We have:*

(i) If $R \leq R'$ *then* $R^\circ \leq R'^\circ$.

(ii) If $R \leq R'$ *then* $SR \leq SR'$.

(iii) If $R \leq R'$ *and* $S \leq S'$ *then* $SR \leq S'R'$.

□

Recall that, $R \leq R'$ means that there exists a morphism $t : R \to R'$ such that $r_1' t = r_1$ and $r_2' t = r_2$.

Remark 3.2. Any morphism $f : A \to B$ in $\mathbf{E}$ can be considered as an $\mathbf{E}$-relation $(A, 1_A, f)$ from A to B. The opposite $\mathbf{E}$-relation f° from B to A will then be the triple $(A, f, 1_A)$.

Proposition 3.3. *Let* $(R, r_1, r_2) : A \to B$ *be an* $\mathbf{E}$*-relation in* $\mathbf{C}$. *If* $RR^\circ \leq 1_B$ *then* $r_1 : R \to A$ *is an isomorphism.* □

Proposition 3.4. *If* $(R, r_1, r_2) : A \to B$ *is an* $\mathbf{E}$*-relation in* $\mathbf{C}$ *then* $R = r_2 {r_1}^\circ$. □

Proposition 3.5. *If* $f : A \to B$ *and* $g : C \to B$ *are the morphisms in* $\mathbf{E}$, *then the* $\mathbf{E}$*-relation* $g^\circ f$ *from* A *to* C *in* $\mathbf{C}$ *is given by the pullback* $(A \times_B C, p_1, p_2)$ *of* f *along* g.

□

Remark 3.6. As follows from Proposition 3.5, if $f : A \to B$ is a morphism in $\mathbf{E}$, then the $\mathbf{E}$-relation $f^\circ f : A \to A$ is given by the pullback $(A \times_B A, f_1, f_2)$ of f with itself. That is, $f^\circ f = (A \times_B A, f_1, f_2)$ is the kernel pair of f, and therefore $1_A \leq f^\circ f$.

Proposition 3.7. *If a morphism* $f : A \to B$ *is in* $\mathbf{E}$, *then* $ff^\circ = 1_B$.

□

Remark 3.8. It follows from Proposition 3.7 that for every morphism $f : A \to B$ in $\mathbf{E}$ the following equalities

$$ff^\circ f = f,$$

$$f^\circ f f^\circ = f^\circ$$

hold.

Theorem 3.9. *Let*

$$\begin{array}{ccc} D & \xrightarrow{k} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \tag{3.1}$$

be a diagram in $\mathbf{C}$. *If the morphisms* f, g, h, *and* k *are in* $\mathbf{E}$, *then:*

(i) $kh^\circ \leq g^\circ f$ *if and only if the diagram (3.1) commutes.*

(ii) $kh^\circ = g^\circ f$ *if and only if the diagram (3.1) commutes and the canonical morphism* $\langle h, k\rangle : D \to A \times_B C$ *is in* $\mathbf{E}$.

□

4. Equivalence E-relations

Just as in the absolute case, we can define equivalence $\mathbf{E}$-relations in an incomplete relative regular category $(\mathbf{C}, \mathbf{E})$ as follows:

Definition 4.1. An $\mathbf{E}$-relation $R : A \to A$ in $\mathbf{C}$ is said to be

(a) a reflexive $\mathbf{E}$-relation if $1_A \leq R$;

(b) a symmetric $\mathbf{E}$-relation if $R^\circ \leq R$ (so that $R^\circ = R$);

(c) a transitive $\mathbf{E}$-relation if $RR \leq R$;

(d) an equivalence $\mathbf{E}$-relation if it is reflexive, symmetric, and transitive.

As follows from Definition 4.1, if R is a reflexive and a transitive $\mathbf{E}$-relation then $RR = R$; indeed, since R is reflexive we have $R \leq RR$, which together with transitivity gives $RR = R$.

We now state some properties of equivalence $\mathbf{E}$-relations in incomplete relative regular categories, omitting the proofs again, since they are essentially the same as the proofs given in [9].

Proposition 4.2. *The composite of reflexive* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is a reflexive* $\mathbf{E}$*-relation.* □

Proposition 4.3. *Let* $R : A \to A$ *and* $S : A \to A$ *be equivalence* $\mathbf{E}$*-relations in* $\mathbf{C}$*. If the composite* SR *is an equivalence* $\mathbf{E}$*-relation, then* $SR = S \vee R$ *(i.e.* SR *is the smallest equivalence* $\mathbf{E}$*-relation containing both* S *and* R*).* □

Proposition 4.4. *If a morphism* $f : A \to B$ *is in* $\mathbf{E}$*, then the kernel pair* $(A \times_B A, f_1, f_2)$ *of* f *is an equivalence* $\mathbf{E}$*-relation in* $\mathbf{C}$*.* □

Definition 4.5. An $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ is said to be difunctional if $RR^{\circ}R = R$.

Theorem 4.6. *If* $(R, r_1, r_2) : A \to A$ *and* $(S, s_1, s_2) : A \to A$ *are equivalence* $\mathbf{E}$*-relations in* $\mathbf{C}$ *then the following conditions are equivalent:*

(a) $SR : A \to A$ *is an equivalence* $\mathbf{E}$*-relation.*

(b) $SR = RS$.

(c) Every $\mathbf{E}$*-relation is difunctional.*

(d) Every reflexive $\mathbf{E}$*-relation is an equivalence* $\mathbf{E}$*-relation.*

(e) Every reflexive $\mathbf{E}$*-relation is symmetric.*

(f) Every reflexive $\mathbf{E}$*-relation is transitive.*

□

Recall that a relative regular Mal'tsev category was defined in [3] as a relative regular category which satisfies any one of the conditions of Theorem 4.6 above (see also [2] and [6]). We now extend that definition to the "*incomplete relative*" context.

Definition 4.7. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular Mal'tsev category, if it is an incomplete relative regular category and satisfies any one of the conditions of Theorem 4.6 above.

In this paper we will emphasise on what we will define in the next section *incomplete relative regular Goursat category*. For, we will need the following

Proposition 4.8. *The following conditions are equivalent in* $(\mathbf{C}, \mathbf{E})$*:*

(a) for equivalence $\mathbf{E}$*-relations* R *and* S *on an object* A*, we have* $RSR = SRS$*;*

(b) this 3*-permutability* $RSR = SRS$ *holds when* R *and* S *are effective equivalence* $\mathbf{E}$*-relations;*

(c) every $\mathbf{E}$*-relation* P *satisfies* $PP^\circ PP^\circ = PP^\circ$*;*

(d) for every reflexive $\mathbf{E}$*-relation* E *on an object* A*, the* $\mathbf{E}$*-relation* EE° *is an equivalence* $\mathbf{E}$*-relation;*

(e) for every reflexive $\mathbf{E}$*-relation* E*, the* $\mathbf{E}$*-relation* EE° *is transitive;*

(f) for every reflexive $\mathbf{E}$*-relation* E *we have* $EE^\circ = E^\circ E$*.* □

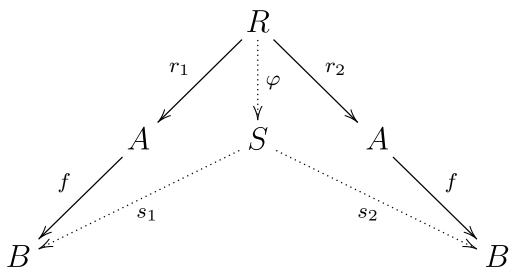
Again, we omit the proof since it follows the proof of Proposition 1.6 of [4].

5. Incomplete relative Goursat categories

Relative regular Goursat categories were introduce in [4], we now extend that definition to the "*incomplete relative*" context. First, let us define an $\mathbf{E}$-image of an endo-$\mathbf{E}$-relation in an incomplete relative regular category:

Definition 5.1. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative regular category. Given an $\mathbf{E}$-relation (R, r_1, r_2) on an object A in $\mathbf{C}$ and a morphism $f : A \to B$ in

E, we define the **E**-image of R along f to be the relation S on B which is obtained from the factorization



(5.1)

which exists by Condition 2.1(e). We write $f(R) = S$, which again is an **E**-relation by Conditions 2.1(b) and 2.1(c).

Note that if **C** has products then this definition is the same as Definition 1.7 of [4].

Proposition 5.2. *Let* $R = (R, r_1, r_2) : A \to A$ *be an* **E***-relation in* **C** *and let* $f : A \to B$ *be a morphism in* **E***. We have:*

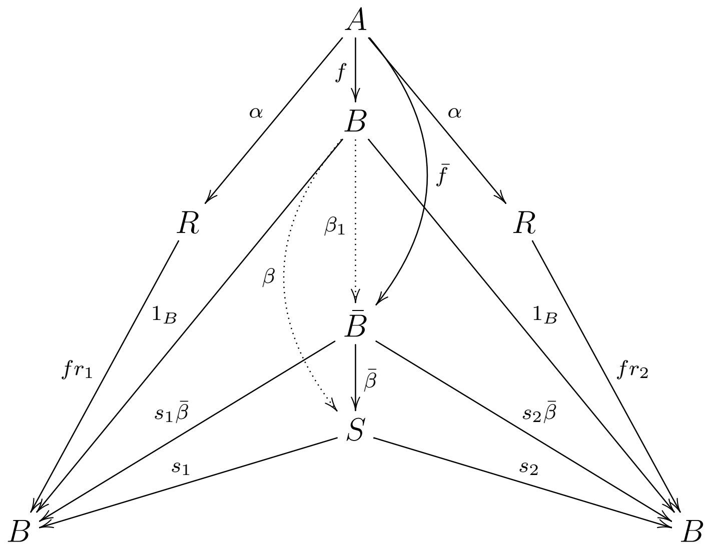
(i) If R *is a reflexive* **E***-relation then* $f(R)$ *is also a reflexive* **E***-relation.*

(ii) If R *is a symmetric* **E***-relation then* $f(R)$ *is also a symmetric* **E***-relation.*

Proof. (i): Suppose $R = (R, r_1, r_2) : A \to A$ is a reflexive **E**-relation in **C**. By the definition of a reflexive **E**-relation, there exists a morphism $\alpha : A \to R$ such that $r_1\alpha = 1_A = r_2\alpha$. Note here that α is a split monomorphism and therefore it is a monomorphism. Let $f : A \to B$ be a morphism in **E**; we have $(fr_1)\alpha = f = (fr_2)\alpha$, where fr_1 and fr_2 are in **E** since so are the morphisms f, r_1 and r_2. By Definition 5.1, the **E**-image of R along f is the **E**-relation (S, s_1, s_2) obtained from the factorization (5.1), therefore $fr_1 = s_1\varphi$ and $fr_2 = s_2\varphi$. Composing with α from the right on both sides of the last equality, we obtain $fr_1\alpha = s_1\varphi\alpha$ and $fr_2\alpha = s_2\varphi\alpha$.

On the other hand, since $\alpha : A \to R$ is a monomorphism and $\varphi : R \to S$ is in **E**, there exists a monomorphism $\bar{\beta} : \bar{B} \to S$ and a morphism $\bar{f} : A \to \bar{B}$ in **E** such that $\varphi\alpha = \bar{\beta}\bar{f}$.

We obtain the following diagram :



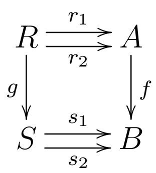
To prove that (S, s_1, s_2) is a reflexive $\mathbf{E}$-relation, we need to prove that there exists a morphism $\beta : B \to S$ such that $\beta s_1 = 1_B = \beta s_2$. Since $\bar{\beta}$ is a monomorphism, the morphisms $s_1\bar{\beta}$ and $s_2\bar{\beta}$ are jointly monic. Therefore, since f and $\bar{f}$ are in $\mathbf{E}$, and obvioulsy 1_B is jointly monic with itself, by Remark 2.3, the equalities $fr_1\alpha = f$, $fr_2\alpha = f$, $s_1\bar{\beta}\bar{f} = fr_1\alpha$, and $s_2\bar{\beta}\bar{f} = fr_2\alpha$ imply that there exists a unique morphism $\beta_1 : B \to \bar{B}$ such that $\beta_1 f = \bar{f}$. Now take $\beta = \bar{\beta}\beta_1$, then $s_1\beta = 1_B = s_2\beta$, as desired.

(ii): The proof easily follows from Remark 2.3. Indeed, if $R = (R, r_1, r_2) : A \to A$ is a symmetric $\mathbf{E}$-relation then there exists an isomorphism $r : R \to R$ such that $r_1 r = r_2$ and $r_2 r = r_1$. Letting $f(R) = (S, s_1, s_2)$, by Definition 5.1 we have that $s_1\varphi = fr_1$ and $s_2\varphi = fr_2$, yielding that $s_1\varphi r = fr_1 r = fr_2$ and $s_2\varphi r = fr_2 r = fr_1$. Therefore, by Remark 2.3 there exists a unique morphism $s : S \to S$ such that $s_2 s = s_1$ and $s_1 s = s_2$, i.e. $S^\circ \leq S$, proving that S is a symmetric $\mathbf{E}$-relation. □

The following Lemma and Corollary (Lemma 1.9 and Corollary 1.10 of [4]) also hold true in an incomplete relative regular category $(\mathbf{C}, \mathbf{E})$:

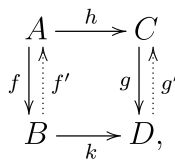
Lemma 5.3. *Given an* $\mathbf{E}$*-relation* (R, r_1, r_2) *on an object* A *in* $\mathbf{C}$ *and a morphism* $f : A \to B$ *in* $\mathbf{E}$*, the* $\mathbf{E}$*-image* $f(R)$ *can be formed as the composite* $f(R) = fRf^\circ = fr_2 r_1^\circ f^\circ$. □

Corollary 5.4. *Given a commutative diagram*



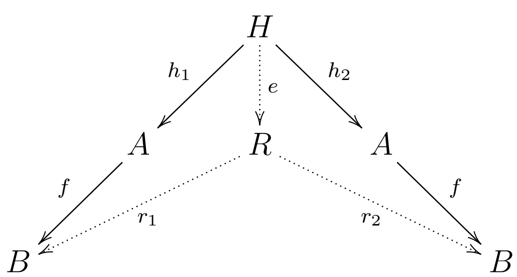
where R and S are $\mathbf{E}$-relations in $\mathbf{C}$ and f is in $\mathbf{E}$, the morphism g is in $\mathbf{E}$ if and only if $S = f(R)$, or equivalently if and only if $s_2s_1^\circ = fr_2r_1^\circ f^\circ$. If (R, r_1, r_2) and (S, s_1, s_2) are kernel pairs with coequalizers r and s in $\mathbf{E}$, then the latter is also equivalent to $s^\circ s = fr^\circ rf^\circ$. $\square$

Lemma 5.5. *Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative regular category. Given a morphism of (downward) split epimorphisms*



that is, f and g are split epimorphisms with splittings f' and g' respectively, and $kf = gh$ and $g'k = hf'$, if f, g, h, and k in are in $\mathbf{E}$, then the induced morphism between the kernel pairs of h and k is also in $\mathbf{E}$.

Proof. We follow the proof of Lemma 1.11 of [4]. Let (H, h_1, h_2) and (K, k_1, k_2) be the kernel pairs of h and k (they do exist since h and k are in $\mathbf{E}$), clearly the induced morphism $H \to K$ is again a split epimorphism. Since h_1 and h_2 are jointly monic and f is in $\mathbf{E}$, using Condition 2.1(e) we obtain the factorization



where e is in $\mathbf{E}$, and r_1 and r_2 are jointly monic morphisms in $\mathbf{E}$. Since e is in particular an epimorphism, the $\mathbf{E}$-relation R factors through the kernel pair K of k. But since $H \to K$ is a split epimorphism it follows that the induced morphism $R \to K$ is an isomorphism, therefore, $H \to K$ is in $\mathbf{E}$. □

We are now ready to prove the "*incomplete relative*" version of Theorem 2.1 of [4], which in the absolute case characterises regular Goursat categories (see [1] and [5]).

Theorem 5.6. *The following conditions are equivalent on* $(\mathbf{C}, \mathbf{E})$*:*

(a) the $\mathbf{E}$***-Goursat axiom*** *holds: given a morphism of (downward) split epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow\uparrow & & {\scriptstyle g}\downarrow\uparrow \\ B & \xrightarrow[k]{} & D \end{array} \qquad (5.2)$$

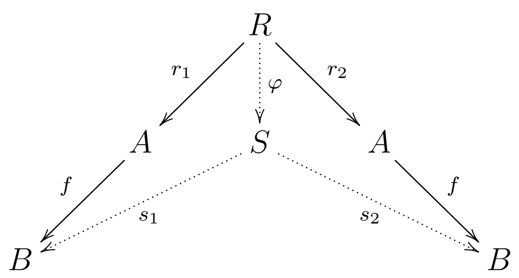
in $\mathbf{C}$ *with* f*,* g*,* h *and* k *in* $\mathbf{E}$*, the induced morphism between the kernel pairs of* f *and* g *is also in* $\mathbf{E}$*;*

(b) the $\mathbf{E}$*-image of an equivalence* $\mathbf{E}$*-relation is an equivalence* $\mathbf{E}$*-relation;*

(c) for every reflexive $\mathbf{E}$*-relation* E *on an object* A*, the* $\mathbf{E}$*-relation* EE° *is an equivalence* $\mathbf{E}$*-relation;*

(d) for equivalence $\mathbf{E}$*-relations* R *and* S *on an object* A*, we have* $RSR = SRS$*.*

Proof. Here again, we follow the proof of Theorem 2.1 from [4].
(a) ⇒ (b): Let (R, r_1, r_2) be an equivalence $\mathbf{E}$-relation on A and let $f : A \to B$ be in $\mathbf{E}$. We want to show that the $\mathbf{E}$-image $f(R) = (S, s_1, s_2)$ of R along f, obtained from the factorization



(5.3)

is again an equivalence $\mathbf{E}$-relation. Since S is a reflexive and a symmetric $\mathbf{E}$-relation, by Proposition 5.2 we only have to show that it is transitive, that is, $SS \leq S$. However, since S is a symmetric $\mathbf{E}$-relation, the transitivity of S will be proved if we show that $SS^{\circ} \leq S$. For, it is sufficient to show that there exists a morphism $t_S : S_1 \to S$, where (S_1, π_1, π_2) is the kernel pair of s_1, which makes the diagram

$$\begin{array}{ccc} S_1 & \overset{t_S}{\dashrightarrow} & S \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ S & \underset{s_2}{\longrightarrow} & B \end{array} \tag{5.4}$$

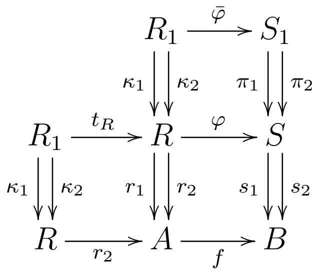
commutative. Since R is a (symmetric and) transitive $\mathbf{E}$-relation, there exists a morphism $t_R \colon R_1 \to R$, where $(R_1, \kappa_1, \kappa_2)$ is the kernel pair of r_1, making the corresponding diagram for R commutative:

$$\begin{array}{ccc} R_1 & \overset{t_R}{\longrightarrow} & R \\ {\scriptstyle \kappa_1}\downarrow\downarrow{\scriptstyle \kappa_1} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} \\ R & \underset{r_2}{\longrightarrow} & A \end{array}$$

Using the morphisms e_R and e_S which define the reflexivity of R and S respectively, we obtain a diagram

$$\begin{array}{ccc} R & \overset{\varphi}{\longrightarrow} & S \\ {\scriptstyle r_1}\downarrow\uparrow{\scriptstyle e_R} & & {\scriptstyle s_1}\downarrow\uparrow{\scriptstyle e_S} \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

where φ is the $\mathbf{E}$-part of the factorization in (5.3). By assumptions, the morphism $\overline{\varphi} \colon R_1 \to S_1$ between the kernel pairs of r_1 and s_1 is in $\mathbf{E}$. Combining the above two diagrams and adding the morphism $\bar{\varphi}$ to it, we obtain the diagram

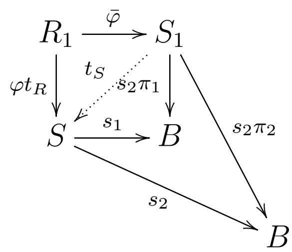


where, recall that, $(R_1, \kappa_1, \kappa_2)$ and (S_1, π_1, π_2) are the kernel pair of r_1 and s_1 respectively. We have:

$$s_1\varphi t_R = fr_1t_R = fr_2\kappa_1 = s_2\varphi\kappa_1 = s_2\pi_1\bar{\varphi}$$

$$s_2\varphi t_R = fr_2t_R = fr_2\kappa_2 = s_2\varphi\kappa_2 = s_2\pi_2\bar{\varphi}$$

Therefore, the following diagram



(5.5)

of solid arrows is commutative. We define the required morphism t_S : $S_1 \to S$ as follows. Since $\bar{\varphi}$ is in $\mathbf{E}$, the kernel pair (X, x_1, x_2) of $\bar{\varphi}$ exists. Moreover, since the above diagram is commutative and s_1 and s_2 are jointly monic, it follows that $\varphi t_R x_1 = \varphi t_R x_2$. Furthermore, since $\bar{\varphi}$ is a regular epimorphism, $\bar{\varphi}$ is the coequalizer of its kernel pair, and therefore there exists a unique morphism $t_S : S_1 \to S$ with $t_S\bar{\varphi} = \varphi t_R$. Now since $\bar{\varphi}$ is an epimorphism, the commutativity of the diagram (5.5) implies that $s_1t_S = s_2\pi_1$ and $s_2t_S = s_2\pi_2$, which gives us the commutativity of the desired diagram (5.4). This proves $(a) \Rightarrow (b)$.

The proofs for the remaining implications are the same as the proofs of the corresponding implications of Theorem 2.1 in [4]. □

We are now ready to give the following

Definition 5.7. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular Goursat category, if it is an incomplete relative regular category and satisfies any one of the conditions of Theorem 5.6 above.

6. The relative 3x3 Lemma

In this section we extend the results of Section 3 of [4] to the *"incomplete relative"* context. Just as in the absolute case, we have the following

Definition 6.1. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative regular category. We will say that the diagram

$$F \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \xrightarrow{f} B \tag{6.1}$$

is $\mathbf{E}$-exact when (f_1, f_2) is the kernel pair of f and f is in $\mathbf{E}$.

Notice that when (6.1) is $\mathbf{E}$-exact, the morphisms f_1 and f_2 are also in $\mathbf{E}$ by the pullback-stability if $\mathbf{E}$.

Since Theorems 3.9 and 5.6, Corollary 5.4, and Lemma 5.5, hold in incomplete relative regular categories, Theorem 3.3 and Theorem 3.4 of [4] also hold true in incomplete relative regular categories :

Theorem 6.2 (The relative 3×3-Lemma)**.** *Let $(\mathbf{C}, \mathbf{E})$ be a relative Goursat category. Given a commutative diagram*

$$\begin{array}{ccccc}
\overline{F} & \underset{\overline{h}_2}{\overset{\overline{h}_1}{\rightrightarrows}} & F & \xrightarrow{\overline{h}} & G \\
{\scriptstyle \overline{f}_2} \downdownarrows {\scriptstyle \overline{f}_1} & & {\scriptstyle f_2} \downdownarrows {\scriptstyle f_1} & & {\scriptstyle g_2} \downdownarrows {\scriptstyle g_1} \\
H & \underset{h_2}{\overset{h_1}{\rightrightarrows}} & A & \xrightarrow{h} & C \\
{\scriptstyle \overline{f}} \downarrow & & {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle g} \\
K & \underset{k_2}{\overset{k_1}{\rightrightarrows}} & B & \xrightarrow{k} & D
\end{array} \tag{6.2}$$

with $\mathbf{E}$-exact columns and middle row, the first row is $\mathbf{E}$-exact if and only if the third row is $\mathbf{E}$-exact.

Theorem 6.3. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative regular category. The following conditions are equivalent:*

(a) $(\mathbf{C}, \mathbf{E})$ *is an incomplete relative Goursat category;*

(b) the relative 3×3-Lemma holds in $(\mathbf{C}, \mathbf{E})$*;*

(c) in a diagram such as (6.2), if the first row is $\mathbf{E}$*-exact then the third row is also* $\mathbf{E}$*-exact;*

(d) in a diagram such as (6.2), if the third row is $\mathbf{E}$*-exact then the first row is also* $\mathbf{E}$*-exact.*

□

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