Abstract. Continuing our first paper in this series, we study multiple limits in infinite-dimensional chiral multiple categories - a weak, partially lax form with directed interchangers.

After defining multiple limits, we prove that all of them can be constructed from (multiple) products, equalisers and tabulators - all of them assumed to be respected by faces and degeneracies. Tabulators appear thus to be the basic higher limits, as was already the case for double categories.

Intercategories, a laxer form of multiple category already studied in two previous papers, are also considered. In this more general setting the basic limits mentioned above can still be defined, but a general theory of multiple limits is not developed here.

Keywords. multiple category, double category, cubical set, limit.

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0. Introduction

Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical
categories can be seen as a particular case of multiple categories; their links with strict $\omega$-categories are made clear in the article [ABS].

The present series studies various ‘forms’ of weak or lax multiple categories, of finite or infinite dimension. They extend weak double categories [GP1 - GP4] and weak cubical categories [G1, G2, GP5]. More information on literature on higher dimensional category theory can be found in the Introduction of the first paper [GP8], here referred to as Part I.

Our main framework, a chiral multiple category, is briefly reviewed here, in Section 1; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the transversal direction), weak compositions $x +_iy$ in all positive (or geometric) directions $i$ and directed interchanges for the $i$- and $j$-compositions (for $0 < i < j$)

$$\chi_{ij} : (x +_i y) +_j (z +_i u) \to (x +_j z) +_i (y +_j u)$$

(1)

Part I also considers a laxer form already studied in two previous papers [GP6, GP7] for the 3-dimensional case, under the name of ‘intercategory’, that is particularly powerful: it covers duoidal categories, Gray categories, Verity double bicategories, monoidal double categories, etc. In this framework, extended in Part I to infinite dimension and recalled here in 1.9, there are also lower interchangers $(\tau_{ij}, \mu_{ij}, \delta_{ij})$ where positive degeneracies (i.e. weak identities) intervene; in particular degeneracies are no longer assumed to commute, but have a directed interchange for $0 < i < j$

$$\tau_{ij} : e_i e_j(x) \to e_j e_i(x)$$

(2)

Here we study multiple limits in the setting of chiral multiple categories. Part of the theory is briefly extended to intercategories, with the problems discussed below.

Our general definition of multiple limits (in 4.4) requires their preservation by faces and degeneracies (as in the cubical case [G2]). We prove that all of them can be constructed from (multiple) products, equalisers and tabulators. The latter appear thus to be the basic higher form of a limit, as was already the case for double and cubical categories. In particular this holds in a 2-category, where tabulators (of vertical identities) reduce to cotensors by the ordinal $2$; the previous result agrees thus with Theorem 10 of R. Street [St1], according to which all weighted limits in a 2-category can be constructed from such cotensors and ordinary limits.
More analytically, Section 1 contains a review of the basic notions of strict, weak and chiral multiple categories. We also introduce the ‘lift functors’ that will play a relevant role below.

Then, in Section 2, we begin our study of limits with the simple case of \textit{i-level limits}, for a \textit{positive} multi-index \( i = \{i_1, \ldots, i_n\} \). In a chiral multiple category \( A \), \( i \)-level limits are ordinary limits in the transversal category \( tv_i(A) \). When all these exist, \textit{and are preserved by faces and degeneracies} between transversal categories, we say that \( A \) has \textit{level multiple limits}. Of course, multiple products and multiple equalisers generate all of them.

Non-level limits, where the diagram and the limit object are not confined to a transversal category, are studied in the next two sections. The main theorems on the construction and preservation of multiple limits are stated in 3.6 and 4.5, and proved in Section 5.

The main example treated here is the chiral triple category \( SC(C) \) of \textit{spans and cospans} over a category \( C \) with pushouts and pullbacks (see 1.8, 2.1, 2.2, 3.7 and 4.6). One can similarly study multiple limits (and colimits) in other weak or chiral multiple categories of finite or infinite dimension, listed at the beginning of Section 2.

The relationship with the double limits of [GP1] are discussed in Sections 2 and 4. In the case of level limits (see 2.6) there are only some variations in terminology; for non-level limits there is a difference (see 4.7).

The general theory of multiple colimits is dual to that of multiple limits and is not written down explicitly. Showing this requires some technical expedient because - as we have seen in Part I - transversal duality turns a (right) chiral multiple category into a \textit{left-hand version} where all interchangers have the opposite direction. Thus, a multiple colimit in the chiral multiple category \( A \) is a multiple limit in a \textit{left} chiral multiple category \( A^{tv} \); but it can also be viewed as a \textit{multiple limit in a right chiral multiple category} \((A^{tv})^{-}\) indexed by the integers \( \leq 0 \) (reversing indices).

An extension of the general theory of multiple limits from the \textit{chiral} case to \textit{intercategories} presents serious problems, linked to the crucial fact that \textit{degeneracies no longer commute}. Yet, the basic limits can be easily extended.

To begin with, \textit{level limits} can be defined as here, in 2.2; one should nevertheless be aware that they do not behave so well as in the chiral case: see the end of Proposition 2.3. Tabulators can also be extended \textit{and even acquire}
richer forms: for instance, the total tabulator of a 12-cube gives now rise to
two distinct notions, the $e_1e_2$-tabulator and the $e_2e_1$-tabulator, as already
shown in Part I, Section 6. However it is not clear what a general definition
of limit should be: in a situation where degeneracies do not commute, even
defining the diagonal functor becomes complicated (see 3.1).

Notation. We follow the notation of Part I; the reference I.2.3 points to its
Subsection 2.3. The two-valued index $\alpha$ (or $\beta$) varies in the set $2 = \{0, 1\}$,
often written as $\{-, +\}$. The symbol $\subset$ denotes weak inclusion. Categories
and 2-categories are generally denoted as $A, B$; weak double categories
as $A, B$; weak or lax multiple categories as $A, B$.

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1. Multiple categories

After a review of the basic notions of strict multiple categories, taken from
Part I, we introduce the ‘lift functors’ that will play a relevant role in the
study of multiple limits. As it will be made clear later (in 4.8) they are a
surrogate for the path endofunctor of symmetric cubical categories. These
notions are then extended to chiral multiple categories, a weak and partially
lax version introduced in Part I.

1.1 Multiple sets

A multi-index $i$ is a finite set of natural numbers, possibly empty. Writing
$i \subset \mathbb{N}$ it will be understood that $i$ is finite; writing $i = \{i_1, \ldots, i_n\}$ we always
mean that $i$ has $n$ distinct elements, written in the natural order $i_1 < i_2 < \ldots < i_n$; the integer $n$ is called the dimension of $i$.

We use the following symbols

$$ij = ji = i \cup \{j\} \quad (\text{for } j \in \mathbb{N} \setminus i), \quad i|j = i \setminus \{j\} \quad (\text{for } j \in i). \quad (3)$$

A multiple set is a system of sets and mappings $X = ((X_i), (\partial_i^p), (e_i))$
under the following two assumptions.
For every multi-index $i = \{i_1, ..., i_n\}$, $X_i$ is a set whose elements are called $i$-cells of $X$ and said to be of dimension $n$. We write $X_s$, $X_i$, $X_{ij}$, etc., instead of $X_\emptyset$, $X_{\{i\}}$, $X_{\{i,j\}}$, etc.; thus $X_s$ is of dimension 0 while $X_0$, $X_1$, etc., are of dimension 1.

For $j \in i$ and $\alpha = \pm$ we have mappings, called faces and degeneracies of $X_i$

$$\partial^\alpha_j : X_i \rightarrow X_{i|j}, \quad e_j : X_{i|j} \rightarrow X_i$$

satisfying the multiple relations

$$\partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_i^\alpha \quad (i \neq j), \quad e_i e_j = e_j e_i \quad (i \neq j),$$

$$\partial_i^\alpha e_j = e_j \partial_i^\alpha \quad (i \neq j), \quad \partial_i^\alpha e_i = \text{id}.$$ (5)

Faces commute and degeneracies commute, but $\partial_i^\alpha$ and $e_i$ do not. These relations look similar to the cubical ones but much simpler, because here an index $i$ stands for a particular sort, instead of a mere position, and is never ‘renamed’. Note also that $\partial_i^\alpha$ acts on $X_j$ if $i$ belongs to the multi-index $i$ (and cancels it), while $e_i$ acts on $X_i$ if $i$ does not belong to $i$ (and inserts it); therefore $\partial_i^\alpha \partial_j^\beta$ and $e_i e_i$ make no sense, here: one cannot cancel or insert twice the same index.

If $i = j \cup k$ is a disjoint union and $\alpha = (\alpha_1, ..., \alpha_r)$ is a mapping $k = \{k_1, ..., k_r\} \rightarrow 2$, we have an iterated face and an iterated degeneracy (independent of the order of composition)

$$\partial_k^\alpha = \partial_{k_1}^{\alpha_1} ... \partial_{k_r}^{\alpha_r} : X_i \rightarrow X_j, \quad e_k = e_{k_1} ... e_{k_r} : X_j \rightarrow X_i.$$ (6)

In particular, the total $i$-degeneracy is the mapping

$$e_i = e_{i_1} ... e_{i_n} : X_s \rightarrow X_i.$$ (7)

1.2 Multiple categories

We recall the definition, from Part I.

A multiple category $A$ is, first of all, a multiple set of components $A_i$, whose elements are called $i$-cells. As above, $i$ is any multi-index, i.e. any finite subset of $\mathbb{N}$, and we write $A_s$, $A_i$, $A_{ij}$, etc., for $A_\emptyset$, $A_{\{i\}}$, $A_{\{i,j\}}$, etc.
Given two i-cells \(x, y\) which are \(i\)-consecutive (i.e. \(\partial_i^+(x) = \partial_i^-(y)\), with \(i \in \mathbb{N}\)), the \(i\)-composition \(x +_i y\) is defined and satisfies the following interactions with faces and degeneracies, for \(j \neq i\):

\[
\begin{align*}
\partial_i^-(x +_i y) &= \partial_i^-(x), \\
\partial_j^+(x +_i y) &= \partial_j^+(y), \\
\partial_j^+(x +_i y) &= \partial_j^+(x) +_i \partial_j^-(y), \\
e_j(x +_i y) &= e_j(x) +_i e_j(y).
\end{align*}
\]

For every multi-index \(i\) containing \(j\) we have a category \(\text{cat}_{i,j}(A)\) with objects in \(A_i\), arrows in \(A_{ij}\), faces \(\partial_j^n\), identities \(e_j\) and composition \(+_j\).

For \(i < j\) we have

\[
(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u) \quad \text{(binary ij-interchange), (9)}
\]

whenever these composites make sense. (Note that the lower interchanges are already expressed above.)

More generally, for an ordered pointed set \(N = (N, 0)\), an \(N\)-indexed multiple category \(A\) has components \(A_i\) indexed by (finite) multi-indices \(i \subseteq N\). If \(N\) is the ordinal set \(\mathbb{n} = \{0, ..., n - 1\}\) we obtain the \(n\)-dimensional version of a multiple category, called an \(n\)-tuple category. The 0-, 1- and 2-dimensional versions amount - respectively - to a set, a category or a double category.

### 1.3 Transversal categories

The transversal direction, corresponding to the index \(i = 0\), is treated differently in the theory: we think of it as the ‘dynamic’ direction, along which ‘transformation occurs’, while the positive directions \(i > 0\) are viewed as the ‘static’ or ‘geometric’ ones.

A positive multi-index \(\mathbf{i} = \{i_1, ..., i_n\}\) (with \(n \geq 0\) positive elements) has an ‘augmented’ multi-index \(0\mathbf{i} = \{0, i_1, ..., i_n\}\). The transversal category of \(i\)-cubes of \(A\)

\[
\text{tv}_i(A) = \text{cat}_{i,0}(A),
\]
has identities $1_x = \text{id}(x) = e_0(x) : x \to_0 x$ and composition $gf = f +_0 g$.

All these items are said to be of degree $n$ (though their dimension may be $n$ or $n + 1$): the degree always refers to the number of positive indices. In all of our examples, 0-composition is realised by the usual composition of mappings, while the ‘positive’ compositions (also called concatenations) are often obtained by operations (products, sums, tensor products, pullbacks, pushouts...) where reversing the order of the operands would only be confusing.

Faces and degeneracies give (ordinary) functors

$$\partial^a_i : tv_i(A) \to tv_i(A), \quad e_j : tv_i(A) \to tv_{ij}(A) \quad (j \notin i, \alpha = \pm).$$ (11)

In particular, the unique positive multi-index of degree 0, namely $\emptyset$, gives the category $tv_\ast(A)$ of objects of $A$ (i.e. $\ast$-cells) and their transversal maps (i.e. 0-cells).

An i-map $f : x \to_0 y$ is said to be i-special, or special in direction $i \in i$, if its $i$-faces are transversal identities (of $i|i$-cubes)

$$\partial^a_i f = e_0 \partial^a_i x = e_0 \partial^a_i y.$$ (12)

This, of course, implies that the i-cubes $x, y$ have the same $i$-faces. We say that $f$ is ij-special if it is special in both directions $i, j$.

### 1.4 Multiple functors and transversal transformations

A multiple functor $F : A \to B$ between multiple categories is a morphism of multiple sets $F = (F_i)$ that preserves all the composition laws. For an i-map $f : x \to_0 y$, we use one of the following forms

$$F(f) : F(x) \to_0 F(y), \quad F_{\partial^a}(f) : F_i(x) \to_0 F_i(y),$$

as may be convenient.

A transversal transformation $h : F \to G : A \to B$ between multiple functors consists of a face-consistent family of i-maps in $B$ (its components), for every positive multi-index $i$ and every $i$-cube $x$ in $A$

$$hx : F(x) \to_0 G(x) \quad (h_i x : F_i(x) \to_0 G_i(x)), \quad (j \in i).$$ (13)
The following axioms of naturality and coherence are required:

(trt.1) \( Gf.hx = hy.Ff \) (for \( f : x \to y \) in \( A \)),

(trt.2) \( h \) commutes with positive degeneracies and compositions:

\[
\begin{align*}
    h(e_j z) &= e_j(hz), \\
    h(x + j y) &= hx + j hy.
\end{align*}
\]

where \( i \) is a positive multi-index, \( j \in i \), \( x \) and \( y \) are \( j \)-consecutive \( i \)-cubes, \( z \) is an \( i|j \)-cube.

Given two multiple categories \( A, B \) we have thus the category \( \text{Mlc}(A, B) \) of their multiple functors and transversal transformations. All these form the 2-category \( \text{Mlc} \), in an obvious way.

More generally for any ordered pointed set \( N = (N, 0) \) we have the 2-category \( \text{Mlc}_N \) of \( N \)-indexed multiple categories, formed of ordinary categories \( \text{Mlc}_N(A, B) \).

1.5 Lift functors

For a positive integer \( j \) there is a \( j \)-directed lift functor with values in the 2-category of multiple categories indexed by the pointed set \( \mathbb{N}|j = \mathbb{N} \setminus \{j\} \)

\[
Q_j : \text{Mlc} \to \text{Mlc}_{\mathbb{N}|j}.
\]  

For a multiple category \( A \), the multiple category \( Q_j A \) is - loosely speaking - that part of \( A \) that contains the index \( j \), reindexed without it:

\[
\begin{align*}
    (Q_j A)_i &= A_{ij}, \\
    (\partial^p_i : (Q_j A)_i + (Q_j A)_{ij}) &= (\partial^p_i : A_{ij} + A_{ijj}), \\
    (e_i : (Q_j A)_{ii} + (Q_j A)_i) &= (e_i : A_{ij} + A_{ij}) \quad (i \in i \subset \mathbb{N}|j),
\end{align*}
\]

and similarly for compositions. In the same way for multiple functors \( F, G : A \to B \) and a transversal transformation \( h : F \to G : A \to B \) we let

\[
\begin{align*}
    (Q_j F)_i &= F_{ij}, \\
    (Q_j h)_i &= h_{ij} \quad (i \subset \mathbb{N}|j).
\end{align*}
\]

There is also an obvious restriction 2-functor \( R_j : \text{Mlc} \to \text{Mlc}_{\mathbb{N}|j} \), where the multiple category \( R_j A \) is that part of \( A \) that does not contain the index \( j \). The \( j \)-directed faces and degeneracies of \( A \) are not used in \( Q_j A \), but yield
three natural transformations, also called *faces* and *degeneracy*, with the following components for $i \subset N|j$

\[
D^α_j : Q_j \to R_j : \text{Mlc} \to \text{Mlc}_{N|j}, \quad (D^α_j)_i = ∂^α_j : A_{ij} \to A_i, \\
E_j : R_j \to Q_j : \text{Mlc} \to \text{Mlc}_{N|j}, \quad (E_j)_i = e_j : A_1 \to A_{ij}, \\
D^α_j E_j = \text{id}.
\] (17)

In particular, the objects and $*$-maps of $Q_j(A)$ are the $j$-cubes and $j$-maps of $A$, so that

\[
tv^*(Q_j(A)) = tv_j A, \quad tv^*(R_j(A)) = tv_A, \\
 tv^*(D^α_j) = ∂^α_j : tv_j A \to tv_A, \quad tv^*(E_j) = e_j : tv_A \to tv_j A.
\] (18)

Plainly all the functors $Q_j$ commute. By composing $n$ of them in any order we get an *iterated lift functor* of degree $n$, in a positive direction $i = \{i_1, \ldots, i_n\}$

\[
Q_1 : \text{Mlc} \to \text{Mlc}_{N|i}, \quad Q_1(A) = Q_{i_1} \ldots Q_{i_n}(A), \quad tv^*(Q_1(A)) = tv_{i_1}(A).
\] (19)

Again, there are faces and degeneracies (where $j \notin i$, $h \subset N|i_j$ and $hi = h \cup i$)

\[
D^α_j : Q_{ij} \to R_j Q_1 : \text{Mlc} \to \text{Mlc}_{N|ij}, \quad (D^α_j)_h = ∂^α_j : A_{hi} \to A_{hi}, \\
E_j : R_j Q_1 \to Q_{ij} : \text{Mlc} \to \text{Mlc}_{N|ij}, \quad (E_j)_h = e_j : A_{hi} \to A_{hi}, \\
 tv^*(D^α_j) = ∂^α_j : tv_{ij} A \to tv_A, \quad tv^*(E_j) = e_j : tv_A \to tv_{ij} A.
\] (20)

### 1.6 Transversal invariance

We now extend the notion of ‘horizontal invariance’ of double categories [GP1], obtaining a property that will be of use for multiple limits and should be expected of every ‘well formed’ multiple category.

We say that the multiple category $A$ is *transversally invariant* if its cubes are ‘transportable’ along transversally invertible maps. Precisely:
(i) given an i-cube $x$ of degree $n$ and a family of $2n$ invertible transversal maps $f^\alpha_i : y^\alpha_i \to \partial^\alpha_i x$ ($i \in \mathfrak{i}, \alpha = \pm$) with consistent positive faces (and otherwise arbitrary domains $y^\alpha_i$)

$$\partial^\alpha_i (f^\beta_j) = \partial^\beta_j (f^\alpha_i) \quad \text{for } i \neq j \text{ in } \mathfrak{i}, \quad (22)$$

there exists an invertible i-map $f : y \to_0 x$ (a ‘filler’, as in the Kan extension property) with positive faces $\partial^\alpha_i f = f^\alpha_i$ (and therefore $\partial^\alpha_i y = y^\alpha_i$).

Of course this property can be equivalently stated for a family of invertible maps $g^\alpha_i : \partial^\alpha_i x \to_0 y^\alpha_i$.

1.7 Weak multiple categories

*Weak multiple categories* have been introduced in Part I, Section 3.

Extending weak double categories [GP1 - GP4] and weak triple categories [GP6, GP7], the basic structure of a weak multiple category $\mathcal{A}$ is a multiple set with compositions in all directions. The composition laws in direction 0 are categorical and have a strict interchange with the other compositions.

On the other hand, the ‘positive’ compositions have transversally invertible comparisons called left $i$-unitor, right $i$-unitor, $i$-associator and $ij$-interchanger, for $0 < i < j$

$$\lambda_i x : (e_i \partial_i^- x) +_i x \to_0 x,$$
$$\rho_i x : x +_i (e_i \partial_i^+ x) \to_0 x,$$
$$\kappa_i (x, y, z) : x +_i (y +_i z) \to_0 (x +_i y) +_i z,$$
$$\chi_{ij} (x, y, z, u) : (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u),$$

under coherence conditions listed in I.3.3 and I.3.4.
Our main infinite-dimensional examples are of cubical type (see I.3.5). Essentially, this means that components, faces and degeneracies are invariant under renaming positive indices, in the same order. An i-cube can thus be indexed by \([n] = \{1, \ldots, n\}\) and called an n-cube; an i-map can be indexed by \([0[n] = \{0, 1, \ldots, n\}\) and called an n-map; again, such items are of order \(n\) and dimension \(n\) or \(n + 1\), respectively. (The examples below are also symmetric, by a natural action of each symmetric group \(S_n\) on the sets of \(n\)-cubes and \(n\)-maps, permuting the positive directions; see Part I.)

(a) The strict symmetric cubical category \(\omega_{\text{Cub}}(C)\) of commutative cubes over a category \(C\). An \(n\)-cube is a functor \(x: 2^n \to C\) \((n \geq 0)\), where \(2\) is the ordinal category \(\bullet \to \bullet\); an \(n\)-map is a natural transformation of such functors. Applications of this multiple category (and its truncations) to algebraic K-theory can be found in [Sh].

(b) The weak symmetric cubical category \(\omega_{\text{Cosp}}(C)\) of cubical cospans over a category \(C\) with (a fixed choice) of pushouts has been constructed in [G1], to deal with higher-dimensional cobordism. An \(n\)-cube is a functor \(x: \wedge^n \to C\), where \(\wedge\) is the formal-cospan category \(\bullet \to \bullet \leftarrow \bullet\); again, an \(n\)-map is a natural transformation of such functors.

(c) The weak symmetric cubical category \(\omega_{\text{Span}}(C)\) of cubical span, over a category \(C\) with pullbacks, is similarly constructed over \(\vee = \wedge^{\text{op}}\), the formal-span category \(\bullet \leftarrow \bullet \to \bullet\) (see [G1]). It is transversally dual to \(\omega_{\text{Cosp}}(C^{\text{op}})\).

(d) The weak symmetric cubical category of cubical bispans, or cubical diamonds \(\omega_{\text{Bisp}}(C)\), over a category \(C\) with pullbacks and pushouts, is similarly constructed over a formal diamond category [G1].

1.8 Chiral multiple categories

Our main framework here is more general and partially lax. A chiral, or \(\chi\)-lax, multiple category \(A\) (see I.3.7) has the same data and axioms of a weak multiple category, except for the fact that the interchange comparisons \(\chi_{ij}\) \((0 < i < j)\) recalled above (in 1.7) are not supposed to be invertible.

Various examples are given in [GP7] and Part I, Section 4. For instance, if the category \(C\) has pullbacks and pushouts, the weak double cat-
Category \text{Span}(C)$, of arrows and spans of $C$, can be ‘amalgamated’ with the weak double category $\text{Cosp}(C)$, of arrows and cospans of $C$, to form a 3-dimensional structure: the chiral triple category $\text{SC}(C)$ whose 0-, 1- and 2-directed arrows are the arrows, spans and cospans of $C$, in this order (as required by the 12-interchanger). For higher dimensional examples, like $\text{Sp}_C^q(C)$, $\text{Sp}_C^\infty(C)$ and $\text{S}_{-\infty}^\infty(C)$ (and the corresponding left-chiral cases) see I.4.4; the latter structure is indexed by all integers, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

Chiral multiple categories, with their strict multiple functors and transversal transformations, form the 2-category $\text{StCmc}$.

As defined in I.3.9, a lax multiple functor $F: A \to B$ between chiral multiple categories, or lax functor for short, has components $F_i: A_i \to B_i$ for all multi-indices $i$ (often written as $F$) that agree with all faces, 0-degeneracies and 0-composition. Moreover, for every positive multi-index $i$ and $i \in i$, $F$ is equipped with $i$-special comparison $i$-maps $E_i$ that agree with faces

\begin{align}
E_i(x): & \ e_iF(x) \to 0 \ F(e_i x) \quad \text{(x in } A_i) \\
E_i(x, y): & \ F(x) +_i F(y) \to 0 \ F(z) \quad \text{ (z = x + i y in } A_i) \\
\partial^a_i E_i(x): & \ = E_i(\partial^a_i x) \quad \text{(j \neq i)} \\
\partial^a_j E_i(x, y): & \ = E_i(\partial^a_j x, \partial^a_j y) \quad \text{(j \neq i).}
\end{align}

These comparisons have to satisfy some axioms. We write down the naturality conditions (lmf.1-2), frequently used below, while the coherence conditions (lmf.3-5) can be found in I.3.9

(lmf.1) \textbf{(Naturality of unit comparisons)} For an $i|i$-map $f: x \to_0 y$ in $A$ we have:

$$F(e_i f).E_i(x) = E_i(g).e_i(F f): e_i F(x) \to_0 F(e_i y).$$

(lmf.2) \textbf{(Naturality of composition comparisons)} For two $i$-consecutive $i$-maps $f: x \to_0 x'$ and $g: y \to_0 y'$ in $A$ we have:

$$F(f +_i g).E_i(x, y) = E_i(x', y').(F f +_i F g): F x +_i F y \to_0 F(x' +_i y').$$

A transversal transformation $h: F \to G: A \to B$ between lax functors consists of a face-consistent family of i-maps in $B$ (its components), one for
every positive multi-index \( i \) and every \( i \)-cube \( x \) in \( A \)
\[
h x : F(x) \to_0 G(x), \quad h(\partial^a_i x) = \partial^a_i (hx), \tag{25}
\]
under the axioms (trt.1) and (trt.2L) of I.3.9

(trt.1) \( Gf.hx = hy.Ff \) (for \( f : x \to_0 y \) in \( A \)),

(trt.2L) for every positive multi-index \( i \) and \( i \in i \):
\[
h(e_i x) F_i (x) = G_i (x). e_i (hx) : e_i F(x) \to_0 G(e_i x),
\]
\[
h(x +_j y). F_i (x, y) = G_i (x, y). (hx +_i hy) : F(x) +_i F(y) \to_0 G(z). \tag{26}
\]

We have thus the 2-category \( L \times C \text{mc} \) of chiral multiple categories, lax functors and their transversal transformations. The lax multiple functor \( F \) is said to be \textit{unitary} if all its unit comparisons \( F_i (x) \) are transversal identities, so that \( F(e_i x) = e_i F(x) \) and \( F \) is a morphism of multiple sets.

The \textit{lift functor} and \textit{restriction functor} in direction \( j \) (see 1.5) are extended in the same form, for \( j > 0, j \notin i \):
\[
Q_j : L \times C \text{mc} \to L \times C \text{mc}_{Nj}, \quad (Q_j A)_1 = A_{ij},
\]
\[
R_j : L \times C \text{mc} \to L \times C \text{mc}_{Nj}, \quad (R_j A)_1 = A_i. \tag{26}
\]

Similarly one defines the 2-category \( C \times C \text{mc} \) for the colax case, where the comparisons of \textit{colax (multiple) functors} have the opposite direction. A \textit{pseudo (multiple) functor} is a lax functor whose comparisons are invertible (and is made colax by inverting its comparisons); such functors are the arrows of the 2-category \( Ps \times C \text{mc} \).

1.9 Intercategories

The more general case of \textit{intercategories}, studied in [GP6, GP7] and Part I (Sections 5 and 6), will only be considered here in a marginal way.

Let us recall that an intercategory \( A \) has, beside \( \chi_{ij} \), other three kinds of directed \( i-j \)-interchangers (for \( 0 < i < j \)), where identities intervene:

(a) \( \tau_{ij}(x) : e_je_i(x) \to_0 e_i e_j(x) \),

(b) \( \mu_{ij}(x, y) : e_i(x) +_j e_i(y) \to_0 e_i(x +_j y) \),

(c) \( \delta_{ij}(x, y) : e_j(x +_i y) \to_0 e_j(x) +_i e_j(y) \).
As proved in [GP7], three-dimensional intercategories comprise under a common form various structures previously studied, like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories. Literature on the these structures can be found in [GP7]; the inspiring case of duoidal (or 2-monoidal) categories can be found in [AM, BS, St2].

As already noted in Part I, various ‘anomalies’ appear with respect to the chiral case, that make problems for a theory of multiple limits in this setting. These will be briefly considered below (see 2.3 and 3.1), without further investigating a situation for which we do not yet have examples sufficiently rich to have good limits.

Some anomalies can already be remarked here. First, an intercategory $A$ is no longer a multiple set (unless each $\tau_{ij}$ is the identity). Second, a degeneracy $e_i$ ($i > 0$) is now lax with respect to every higher $j$-composition (for $j > i$, via $\tau_{ij}$ and $\mu_{ij}$) but colax with respect to every lower $j$-composition (for $0 < j < i$, via $\tau_{ji}$ and $\delta_{ji}$). Therefore, in the truncated $n$-dimensional case $e_1$ is lax with respect to all other compositions and $e_n$ is colax, but the other positive degeneracies (if any, i.e. for $n > 3$) are neither lax nor colax.

2. Multiple level limits

We begin our study of limits with the simple case of $i$-level limits, for a positive multi-index $i$.

In a chiral multiple category $A$, $i$-level limits are ordinary limits in the transversal category $tV_1(A)$ (as in the cubical case, see [G2]). When all these exist, and are preserved by faces and degeneracies, we say that $A$ has level multiple limits; of course they are ‘generated’ by multiple products and multiple equalisers.

Examples are given in the chiral triple category $SC(C)$ recalled in 1.8; they can be easily adapted to the weak multiple categories

$$\omega \text{Cub}(C), \quad \omega \text{Cosp}(C), \quad \omega \text{Span}(C), \quad \omega \text{Bisp}(C)$$

of 1.7, and to the chiral multiple categories

$$S_pC_q(C), \quad S_pC_{\infty}(C), \quad S_{-\infty}C_{\infty}(C)$$

recalled in 1.8.
Note that all of these are transversally invariant, a property of interest for limits as we show below, in 2.3 and 2.4.

Level limits can be extended to intercategories with the same definitions (see 1.9). But Proposition 2.3 and its consequences in 2.4 would partially fail.

Non-level limits, where the diagram and the limit cube are not confined to a transversal category, will be studied in the next two sections.

2.1 Products

Let us begin by examining various kinds of products in the chiral triple category \( A = SC(C) \).

Supposing that \( C \) has products, the same is true of its categories of diagrams, and (using the formal-span category \( \vee \) and the formal cospan \( \wedge \) recalled in 1.7) we have four types of products in \( A \) (indexed by a small set \( \Lambda \)):

- of objects (in \( C \)), with projections in \( A_0 \):
  \[ C = \prod_{\lambda \in \Lambda} C_{\lambda}, \quad p_{\lambda}: C \to C_{\lambda}, \]

- of 1-arrows (in \( C^\vee \)), with projections in \( A_{01} \):
  \[ f = \prod_{\lambda \in \Lambda} f_{\lambda}, \quad p_{\lambda}: f \to f_{\lambda}, \]

- of 2-arrows (in \( C^\wedge \)), with projections in \( A_{02} \):
  \[ u = \prod_{\lambda \in \Lambda} u_{\lambda}, \quad p_{\lambda}: u \to u_{\lambda}, \]

- of 12-cells (in \( C^{\vee \wedge} \)), with projections in \( A_{012} \):
  \[ \pi = \prod_{\lambda \in \Lambda} \pi_{\lambda}, \quad p_{\lambda}: \pi \to \pi_{\lambda}, \]

Faces and degeneracies preserve these products. Saying that the triple category \( SC(C) \) has triple products we mean all this. It is important to note that this is a global condition: we shall not define when, in a chiral triple category, a single product of objects \( \prod_{\lambda \in \Lambda} C_{\lambda} \) should be called ‘a triple product’.

It is now simpler and clearer to work in a chiral multiple category \( A \), rather than in a truncated case, as above.

Let \( n \geq 0 \) and let \( i \) be a positive multi-index (possibly empty). An \( i \)-product \( a = \prod_{\lambda \in \Lambda} a_{\lambda} \) will be an ordinary product in the transversal category \( tv_i(A) \) of \( i \)-cubes of \( A \) (recalled in Section 1). It comes with a family
\( p_\lambda: a \to_{0\lambda} a_\lambda \) of \( i \)-maps (i.e. cells of \( A_{0\lambda} \)) that satisfies the obvious universal property.

We say that \( A \):

(i) has \( i \)-products, or products of type \( i \), if all these products (indexed by an arbitrary small set \( \Lambda \)) exist,

(ii) has products if it has \( i \)-products for all positive multi-indices \( i \),

(iii) has multiple products if it has all products, and these are preserved by faces and degeneracies, viewed as (ordinary) functors (see (11))

\[
\partial^\alpha_i: \text{tv}_1(A) \to \text{tv}_{ij}(A), \quad e_j: \text{tv}_{ij}(A) \to \text{tv}_i(A) \quad (j \in i, \alpha = \pm). \quad (27)
\]

Of course this preservation is meant in the usual sense, up to isomorphism (i.e. invertible transversal maps); however, if this holds and \( A \) is transversally invariant (see 1.6), one can construct a choice of products that is strictly preserved by faces and degeneracies, starting from \( * \)-products and going up. This will be proved, more generally, in Proposition 2.3.

A \( * \)-product is also called a product of degree 0.

### 2.2 Level limits

We now let \( \Lambda \) be a small category.

There is an obvious chiral multiple category \( A^\Lambda \) whose \( i \)-cubes are the functors \( F: \Lambda \to \text{tv}_1(A) \) and whose \( i \)-maps are their natural transformations, composed as such. The positive faces, degeneracies and compositions are pointwise (as well as their comparisons):

\[
(\partial^\alpha_i F)(\lambda) = \partial^\alpha_i (F(\lambda)), \quad (e_j F)(\lambda) = e_j (F(\lambda)),
\]

\[
(F +_i G)(\lambda) = F(\lambda) +_i G(\lambda).
\]

The diagonal functor \( D: A \to A^\Lambda \) takes each \( i \)-cube \( a \) to the constant \( a \)-valued functor \( Da: \Lambda \to \text{tv}_1(A) \), and each \( i \)-map \( h: a \to_{0\lambda} b \) to the constant \( h \)-valued natural transformation \( Dh: Da \to Db: \Lambda \to \text{tv}_1(A) \).

The limit of the functor \( F \), called an \( i \)-level limit in \( A \), is an \( i \)-cube \( L \in A_i \) equipped with a universal natural transformation \( t: DL \to F: \Lambda \to \text{tv}_1(A) \), where \( DL: \Lambda \to \text{tv}_1(A) \) is the constant functor at \( L \). It is an \( i \)-product if \( \Lambda \) is discrete and an \( i \)-equaliser if \( \Lambda \) is the category \( 0 \to 1 \).
We say that $A$:

(i) **has $i$-level limits on** $\Lambda$ if all the functors $\Lambda \to tv_1(A)$ have a limit,

(ii) **has level limits on** $\Lambda$ if it has such limits for all positive multi-indices $i$,

(iii) **has level multiple limits on** $\Lambda$ if it has such level limits, and these are preserved by faces and degeneracies (as specified in (27)),

(iv) **has level multiple limits** if the previous property holds for every small category $\Lambda$.

Obviously, the multiple category $A$ has level multiple limits if and only if it has multiple products and multiple equalisers. **Finite level limits** work in the same way, with finite multiple products.

In particular, a $\star$-level limit is a limit in the transversal category $tv_\star(A)$, associated to the multi-index $\varnothing$, of degree 0; it will also be called a **level limit of degree 0**.

Extending the case of multiple products considered in 2.1, if the category $C$ is complete (or finitely complete) so are its categories of diagrams, and the chiral triple category $SC(C)$ has level triple limits (or the finite ones).

### 2.3 Proposition (Level limits and invariance)

Let $\Lambda$ be a category and $A$ a transversally invariant chiral multiple category (see 1.6). If $A$ has level multiple limits on $\Lambda$, one can find a consistent choice of such limits. More precisely, one can fix for every positive multi-index $i$ and every functor $F : \Lambda \to tv_1(A)$ a limit of $F$

\[ L(F) \in A_i, \quad t(F) : DL(F) \to F : \Lambda \to tv_1(A), \]  

so that these choices are strictly preserved by faces and degeneracies:

\[ \partial_i^n(L(F)) = L(\partial_i^n F), \quad \partial_i^n(t(F)) = t(\partial_i^n F) \quad (i \in i), \]

\[ e_i(L(F)) = L(e_i F), \quad e_i(t(F)) = t(e_i F) \quad (i \notin i). \]

**Proof.** We proceed by induction on the degree $n$ of positive multi-indices. For $n = 0$ we just fix a choice $(L(F), t(F))$ of $\star$-level limits on $\Lambda$, for all $F : \Lambda \to tv_\star(A)$. Then, for $n \geq 1$, we suppose to have a consistent choice for all positive multi-indices of degree up to $n - 1$ and extend this choice to degree $n$, as follows.
For a functor $F : \Lambda \to tv_1(A)$ of degree $n$, we already have a choice $(L(\partial^\alpha_i F), t(\partial^\alpha_i F))$ of the limit of each of its faces. Let $(L, t)$ be an arbitrary limit of $F$; since faces preserve limits (in the usual, non-strict sense), there is a unique family of transversal isomorphisms $h_\alpha^\iota$ coherent with the limit cones (of degree $n - 1$)

$$h_\iota^\alpha : L(\partial^\iota_i F) \to \partial^\alpha_i L, \quad t(\partial^\iota_i F) = (\partial^\alpha_i t).h_\iota^\alpha \quad (i \in \iota, \alpha = \pm),$$

and this family has consistent faces (see (22)), as follows easily from their coherence with the limit cones of a lower degree (when $n \geq 2$, otherwise the consistency condition is void).

Now, because of the hypothesis of transversal invariance, this family can be filled with a transversal isomorphism $h$, yielding a choice for $L(F)$ and $t(F)$

$$h : L(F) \to L, \quad t(F) = t.Dh : DL(F) \to F.$$  

By construction this extension of $L$ is strictly preserved by all faces. To make it also consistent with degeneracies, we assume that - in the previous construction - the following constraint has been followed: for an $i$-degenerate functor $F = e_i G : \Lambda \to tv_1(A)$ we always choose the pair $(e_i L(G), e_i t(G))$ as its limit $(L, t)$. This allows us to take $h_\iota^\alpha = \text{id}(L(G))$ (for all $i \in \iota$ and $\alpha = \pm$), and finally $h = \text{id}(L)$, that is

$$L(F) = e_i L(G), \quad t(F) = e_i t(G) : DL(F) \to F.$$  

If $F$ is also $j$-degenerate, then $F = e_i e_j H = e_j e_i H$; therefore, by the inductive assumption, both procedures give the same result: $e_i L(G) = e_i e_j L(H) = e_j e_i L(e_i H)$.

Note that this point would fail in an intercategory with $e_i e_j \neq e_j e_i$. □

### 2.4 Level limits as unitary lax functors

The previous proposition shows that, if the chiral multiple category $A$ is transversally invariant and has level multiple limits on the small category $\Lambda$, we can form a unitary lax functor $L$ and a transversal transformation $t$

$$L : A^\Lambda \to A, \quad t : DL \to 1 : A^\Lambda \to A^\Lambda,$$

such that, on every $i$-cube $F$, the pair $(L(F), t(F))$ is the level limit of the functor $F$, as in (28).
Indeed, after defining $L$ and $t$ on all i-cubes $F$, by a consistent choice (which is possible by the proposition itself), we define $L(h)$ for every natural transformation $h : F \to G : \Lambda \to tv_1(A)$. By the universal property of limits, there is precisely one i-map $L(h)$ such that

$$L(h) : L(F) \to_0 L(G), \quad h.t(F) = t(G).DL(h),$$

and this extension on i-maps is obviously the only one that makes the family $t(F) : DL(F) \to F$ into a transversal transformation $DL \to 1$. The lax comparison for i-composition (with $i \in i$)

$$L_i(F, G) : L(F) +_i L(G) \to_0 L(F +_i G),
\quad t(F +_i G).DL_i(F, G) = t(F) +_i t(G),$$

comes from the universal property of $L(F +_i G)$ as a limit.

In the hypotheses above we say that $A$ has lax functorial limits on $\Lambda$. We say that $A$ has pseudo (resp. strict) functorial limits on $\Lambda$ if $L$ is a pseudo functor (resp. can be chosen as a strict functor).

### 2.5 Level limits and liftings

Let us recall (from (19) and 1.8) that, for a positive multi-index $i$, the chiral multiple category $A$ has a lifting $Q_i(A)$ such that

$$tv_*(Q_i(A)) = tv_1(A).$$

Therefore, an i-level limit in $A$ is the same as a $*$-level limit in $Q_i(A)$. The chiral multiple category $A$

(i) has i-level limits on $\Lambda$ if and only if its lifting $Q_i(A)$ has $*$-level limits on $\Lambda$,

(ii) has level limits on $\Lambda$ if and only if all its liftings $Q_i(A)$ have $*$-level limits,

(iii) has level multiple limits on $\Lambda$ if and only if all its liftings $Q_i(A)$ have $*$-level limits and these are preserved by faces and degeneracies, namely the multiple functors $D^\alpha_j = D^\alpha_j(A)$ and $E_j = E_j(A)$ for $j \notin i$ and $\alpha = \pm$ (see 1.5)

$$D^\alpha_j : Q_{ij}(A) \to R_j Q_i(A),
\quad E_j : R_j Q_i(A) \to Q_{ij}(A),
\quad tv_*(D^\alpha_j) = \partial^\alpha_j : tv_1 A \to tv_1 A,
\quad tv_*(E_j) = e_j : tv_1 A \to tv_1 A,$$  

(37)
(iv) has level multiple limits if the previous property holds for every small category \( \Lambda \).

2.6 Level limits in weak double categories

Let \( \mathbb{A} \) be a weak double category, viewed as the weak multiple category \( \text{sk}_2(\mathbb{A}) \), by adding degenerate items of all the missing types (cf. I.2.7).

The present \( * \)-level limits in \( \mathbb{A} \), i.e. limits of ordinary functors \( \Lambda \to \text{tv}_* (\mathbb{A}) \), correspond to the ‘limits of horizontal functors’ in [GP1]. There are slight differences in terminology, essentially because the ‘2-dimensional universal property’ of double limits (see [GP1], 4.2) here is not required from the start but comes out of a condition of preservation by degeneracies.

As a particular case of the definitions in 2.2, we have the following cases.

(i) \( \mathbb{A} \) has \( * \)-level limits on a (small) category \( \Lambda \) if all the functors \( \Lambda \to \text{tv}_* (\mathbb{A}) \) have a limit. By the usual theorem on ordinary limits, all of them can be constructed from:

- small products \( \prod A_\Lambda \) of objects,
- equalisers of pairs \( f, g: A \to B \) of parallel horizontal arrows.

(i') \( \mathbb{A} \) has 1-level limits on \( \Lambda \) if all the functors \( \Lambda \to \text{tv}_1 (\mathbb{A}) \) have a limit. All of them can be constructed from:

- small products \( \prod u_\Lambda \) of vertical arrows,
- equalisers of pairs \( a, b: u \to v \) of double cells (between the same vertical arrows).

(ii) \( \mathbb{A} \) has level limits on \( \Lambda \) if it has \( * \)- and 1-level limits on \( \Lambda \).

(iii) \( \mathbb{A} \) has level double limits on \( \Lambda \) if it has such level limits, preserved by faces and degeneracies.

(iv) \( \mathbb{A} \) has level double limits if the previous property holds for every small category \( \Lambda \); this is equivalent to the existence of small double products and double equalisers.

Let us note again, as in 2.1, that the existence of (say) double products is now a global condition: it means the existence of products of objects and vertical arrows, consistently with faces and degeneracies. Here we are not defining when a single product \( \Pi A_\Lambda \) should be called a ‘double product’
(while in [GP1] this meant a product of objects preserved by vertical identities).

In [GP1] case (i) would be expressed saying that $A_\Lambda$ has 1-dimensional limits of horizontal functors on $\Lambda$. Case (iii) (resp. (iv)) would be expressed saying that $A_\Lambda$ can be given a lax choice of double limits for all horizontal functors defined on $\Lambda$ (resp. defined on some small category).

3. Multiple limits of degree zero

We now define ‘multiple limits’ of degree zero - namely those limits that produce objects. They extend the previous level limits of degree zero (or $\ast$-level limits), and are generated by the latter together with tabulators of degree zero (Theorem 3.6). The general case - limits that produce cubes of arbitrary dimension - will be treated in the next section.

3.1 The diagonal functor

Let $X$ and $A$ be chiral multiple categories, and let $X_i$ be small. Consider the diagonal functor (of ordinary categories)

$$D : tv, A \rightarrow PsCmc(X, A).$$

(38)

$D$ takes each object $A$ of $A$ to a unitary pseudo functor, ‘constant’ at $A$, via the family of the total $i$-degeneracies (see (7))

$$DA : X \rightarrow A,$$

$$DA(x) = e_i(A) \quad DA(f) = id(e_iA) \quad (\text{for } x \text{ and } f \text{ in } tv, X),$$

$$DA_i(x) = 1_{e_iA} : e_i(DA(x)) \rightarrow DA(e_ix) \quad (\text{for } x \in X_i),$$

$$DA_i(x, y) = \lambda_i : e_iA +_i e_iA \rightarrow e_iA \quad (\text{for } z = x +_i y \in X_i).$$

(39)

In fact, as required by axiom (lmf.3) of lax multiple functors (in I.3.9), the comparison $DA_i(x, y)$ above must be the unitor $\lambda_i(e_iA) = \rho_i(e_iA)$ of $A$, equivalently left or right (see I.3.3), that will generally be written as $\lambda_i$ for short.

Similarly, a $\ast$-map $h : A \rightarrow B$ in $A$ is sent to the constant transversal transformation

$$Dh : DA \rightarrow DB : X \rightarrow A, \quad (Dh)(x) = e_ih : e_iA \rightarrow e_iB \quad (x \in X_i).$$

(40)
DA is a strict multiple functor whenever A is pre-unitary (cf. I.3.2).

Note also that the definition of the diagonal functor D depends on the commutativity of degeneracies in A, which holds in the present chiral case. For a general 3-dimensional intercategory A one could define two functors

\[ D_{12} : tv_*A \to LxCmc(X,A), \quad D_{21} : tv_*A \to CxCmc(X,A) , \]  

(41)

where \( D_{ij}(A) \) sends a 12-cube \( x \) to \( D_{ij}(A)(x) = e_i e_j(A) \) (and any lower i-cube to \( e_i(A) \)). In higher dimension the situation is even more complex.

Still, in an intercategory we have level limits, defined as in Section 2, and some simple non-level limits that can be defined ad hoc, like the \( e_1 e_2\) -tabulator and the \( e_2 e_1\) -tabulator of a 12-cube considered in Part I, Section 6.

3.2 Cones

Let \( F : X \to A \) be a lax functor. A (transversal) cone of \( F \) will be a pair \((A, h : DA \to F)\) comprising an object \( A \) of \( A \) (the vertex of the cone) and a transversal transformation of lax functors \( h : DA \to F : X \to A \); in other words, it is an object of the ordinary comma category \((D \downarrow F)\), where \( F \) is viewed as an object of the category \( LxCmc(X,A) \).

By definition (see 1.8), the transversal transformation \( h \) amounts to assigning the following data:

- a transversal i-map \( hx : e_i(A) \to Fx \), for every i-cube \( x \) in \( X \), subject to the following axioms of naturality and coherence:

  (tc.1) \( Ff.hx = hy \) (for every i-map \( f : x \to y \) in \( X \)),

  (tc.2) \( h \) commutes with positive faces, and agrees with positive degeneracies and compositions:

  \[ h(\partial^a_i x) = \partial^a_i (hx), \quad (\text{for } x \text{ in } X_i), \]

  \[ h(e_i x) = F_i(x).e_i(hx) : e_i A \to F(e_i x) \quad (\text{for } x \text{ in } X_{i|}), \]

  \[ h(z) = F_i(x,y).(hx + hy).\lambda^{-1}_i : e_i A \to F(z) \quad (\text{for } z = x + y \text{ in } X_i), \]

where \( \lambda_i : e_i(A) + e_i(A) \to e_i(A) \) is a unitor of A (see (39)).

It is easy to see that a unitary lax functor \( S : A \to B \) preserves diagonalisation, in the sense that \( S.DA = D(SA) \). Therefore \( S \) takes a cone
(A, h: DA → F) of F: X → A to a cone (SA, Sh) of SF: X → B, and one can consider whether S preserves a limit. For a ‘general’ lax functor S one should transform cones using the comparison S(A), which will not be done here, for the sake of simplicity.

3.3 Definition (Limits of degree zero)

Given a lax functor F: X → A between chiral multiple categories, the (transversal) limit of degree zero \( \lim(F) = (L, t: DL → F) \) is a universal cone.

In other words:

(tl.0) \( L \) is an object of \( A \) and \( t: DL → F \) is a transversal transformation of lax functors,

(tl.1) for every cone \( (A, h: DA → F) \) there is precisely one \( \star \)-map \( f: A → L \) in \( A \) such that \( t.Df = h \).

We say that \( A \) has limits of degree zero on \( X \) if all these exist.

In particular, if \( X \) is the multiple category freely generated by a category \( \Lambda \), at \( \star \)-level, then \( A \) has 0-degree limits on \( X \) if and only if it has 0-degree level limits on \( \Lambda \) (see 2.2). Here freely generated at \( \star \)-level refers to a universal arrow from \( \Lambda \) to the functor \( tv_*: \text{Mlc} → \text{Cat} \).

3.4 Tabulators of degree zero

A is always a chiral multiple category. Let us recall that every positive multi-index \( i \) gives a ‘total’ degeneracy

\[ e_i = e_{i_1}...e_{i_n}: \text{tv}_*A → \text{tv}_1A. \]

An i-cube \( x \) of \( A \) can be viewed as a unitary pseudo functor \( x: u_1 → A \) where \( u_1 \) is the strict multiple category freely generated by one i-cube \( u_i \). The pseudo functor \( x \) sends \( u_1 \) to \( x \), and has comparisons \( x_i \) for \( i \)-composites that derive from the unitors of \( A \), as in the following cases

\[ x_i(e_i\partial^-_i u_1, u_1) = \lambda_i(x): e_i\partial^-_i x + i x → x, \]

\[ x_i(u_1, e_i\partial^+_i u_1) = \rho_i(x): x + i e_i\partial^+_i x → x. \]
Again, it is easy to see that this unitary pseudo functor $x: u_i \to A$ is preserved by a unitary lax functor $S: A \to B$, in the sense that the composite $S.x$ coincides with $S(x): u_i \to B$. All the pseudo functors $x: u_i \to A$ are strict precisely when $A$ is unitary.

The tabulator of degree zero of $x$ in $A$ will be the limit of this pseudo functor $x: u_i \to A$; we also speak of the total tabulator, or $i$-tabulator, of $x$.

The tabulator is thus an object $T = \top x = \top i x$ equipped with an $i$-map $t_x: e_i T \to 0 x$ such that the pair $(T, t_x: e_i T \to 0 x)$ is a universal arrow from the functor $e_i: \text{tv}_a A \to \text{tv}_l A$ to the object $x$ of $\text{tv}_l A$. Explicitly, this means that, for every object $A$ and every $i$-map $h: e_i A \to 0 x$ there is a unique ⋆-map $f$ such that

$$
\begin{array}{ccc}
e_i(A) & \xrightarrow{e_i(f)} & e_i(T) \\
h & \downarrow & \downarrow t_x \\
& & x \\
f: A \to 0 T,
\end{array}
$$

(43)

We say that $A$ has tabulators of degree zero if all these exist, for every positive multi-index $i$. Obviously, the tabulator of an object always exists and is the object itself.

When such tabulators exist, we can form for every positive multi-index $i$ a right adjoint functor

$$
\top_i: \text{tv}_l A \to \text{tv}_a A, \quad e_i \dashv \top_i,
$$

(44)

which is just the identity for $i = \emptyset$.

Assuming that the tabulators of $x \in A_i$ and $z = \partial_j^a x$ exist (for $j \in i$), the projection $p_j^a x$ of $\top x = \top_i x$ will be the following ⋆-map of $A$

$$
\begin{array}{ccc}
e_{ij} \top x & \xrightarrow{e_{ij}(p_j^a x)} & e_{ij} \top (\partial_j^a x) \\
& \downarrow \partial_j^a(t_x) & \downarrow \partial_j^a x \\
& \quad z = \partial_j^a x & \quad t_z, e_{ij}(p_j^a x) = \partial_j^a(t_z).
\end{array}
$$

(45)

3.5 Tabulators and concatenation

We now examine the relationship between tabulators of $i$-cubes and (zero-ary or binary) $j$-concatenation, for $j \in i$.
(a) If the degenerate i-cube \( x = e_j z \) and the \( i|j \)-cube \( z \) have total tabulators in \( \mathcal{A} \), they are linked by a \textit{diagonal} transversal \( R \)-map \( d_j z \), defined as follows

\[
e_1(\top z) \xrightarrow{e_1(d_j z)} e_1(\top(e_j z)) \quad \xrightarrow{t_x} d_j z : \top z \to_0 \top(e_j z),
\]

(46)

This \( R \)-map \( d_j z \) is a section of both projections \( p_j^0 x \) (defined above) because

\[
t_x.e_{ij}(p_j^0 x.d_j z) = \partial_j^0(t_x).e_{ij}(d_j z) = \partial_j^0(t_x.e_1(d_j z)) = \partial_j^0(e_1 t_z) = t_z.
\]

(b) For a concatenation \( z = x +_j y \) of i-cubes, the three total tabulators of \( x, y, z \) are also related. The link goes through the ordinary pullback \( \top_j(x, y) \) of the objects \( \top x \) and \( \top y \), over the tabulator \( \top w \) of the \( i|j \)-cube \( w = \partial_j^+ x = \partial_j^- y \) \textit{(provided all these tabulators and such a pullback exist)}

\[
\top_j(x, y) \xrightarrow{p_j(x,y)} \top x \xrightarrow{p_j^+ x} t_w.e_{ij}(p_j^+ x) = \partial_j^+(t_x),
\]

\[
\top_j(x, y) \xrightarrow{q_j(x,y)} \top y \xrightarrow{p_j^- y} t_w.e_{ij}(p_j^- y) = \partial_j^-(t_y).
\]

(47)

We now have a \textit{diagonal} transversal \( R \)-map \( d_j(x, y) \) given by the universal property of \( \top z \)

\[
d_j(x, y) : \top_j(x, y) \to_0 \top z,
\]

\[
t_x.e_1(d_j(x, y)) = t_x.e_1 p_j(x, y) +_j t_y.e_1 q_j(x, y).
\]

(48)

The \( j \)-composition above is legitimate, by construction

\[
\partial_j^+(t_x.e_1 p_j(x, y)) = \partial_j^+(t_x).e_{ij}(p_j(x, y))
\]

\[
= t_w.e_{ij}(p_j^+ x).e_{ij}(p_j(x, y)) = t_w.e_{ij}(p_j^- y).e_{ij}(q_j(x, y))
\]

\[
= \partial_j^-(t_y).e_{ij}(q_j(x, y)) = \partial_j^-(t_y.e_1 q_j(x, y)).
\]

It is easy to show (and it also follows from the proof of the theorem below) that \( \top_j(x, y) \) is the transversal limit of the diagram \textit{formed} by \( z = x +_j y \) \textit{(based on the multiple category freely generated by two \( j \)-consecutive i-cubes)}.
3.6 Theorem (Construction and preservation of 0-degree limits)

Let $\mathcal{A}$ and $\mathcal{B}$ be chiral multiple categories.

(a) All limits of degree zero in $\mathcal{A}$ can be constructed from level limits of degree zero and tabulators of degree zero, or also from products, equalisers and tabulators - all of degree zero.

(b) If $\mathcal{A}$ has all limits of degree zero, a unitary lax multiple functor $\mathcal{A} \to \mathcal{B}$ preserves them if and only if it preserves products, equalisers and tabulators of degree zero.

Proof. See Section 5.

3.7 Examples

In the chiral triple category $\text{SC}(\mathcal{C})$ (over a category $\mathcal{C}$ with pullbacks and pushouts) we have the following three kinds of tabulators of degree zero (apart from the trivial tabulator of an object), already described in I.4.3.

(a) The tabulator of a 1-arrow $f$ (i.e. a span) is an object $\top_1 f$ with a universal 1-map $e_1(\top_1 f) \to 0 f$; the solution is the (trivial) limit of the span $f$, i.e. its middle object.

(b) The tabulator of a 2-arrow $u$ (a cospan) is an object $\top_2 u$ with a universal 2-map $e_2(\top_2 u) \to 0 u$; the solution is the pullback of $u$.

(c) The total tabulator of a 12-cell $\pi$ (a span of cospans) is an object $\top_{12} \pi$ with a universal 12-map $e_{12}(\top_{12} \pi) \to 0 \pi$; the solution is the limit of the diagram, i.e. the pullback of its middle cospan.

The two (non total) tabulators of degree 1 of the 12-cell $\pi$ will be reviewed below, in 4.6.

4. Multiple limits of arbitrary degree

We now introduce general limits in a chiral multiple category $\mathcal{A}$, taking advantage of the iterated lift functors $Q_i$ (see 1.5), where $i$ is a positive multi-index of degree $n \geq 0$. $\mathcal{X}$ is always a small chiral multiple category.

Let us recall that $u_i$ denotes the multiple category freely generated by one $i$-cube $u_i$ (as in 3.4).
4.1 A motivation

For a positive multi-index $i$ of degree $n \geq 0$, the limits (of degree 0) of multiple functors with values in the lifted chiral multiple category $Q_iA$ will be called limits of type $i$ (and degree $n$) in $A$; their results are thus $i$-cubes of $A$. They extend the limits of degree zero considered above, for $i = \emptyset$ and $Q_\emptyset A = A$.

Let us begin with some simple examples, based on a 2-dimensional cube $x \in A_{12}$, introducing definitions that will be made precise below.

(a) The cube $x \in A_{12}$ is the same as a unitary pseudo functor $x: u_{12} \to A$. We have already considered its tabulator of degree zero, namely an object $\top x = \top_{12}x$ with a universal 12-map $t_{12}(\top_{12}x) \to 0 x$ (where $e_{12} = e_1 e_2 = e_2 e_1 : A_s \to A_{12}$ is the composed degeneracy).

(b) But $x$ can also be viewed as a 1-arrow of $Q_2A$, i.e. a unitary pseudo functor $x: u_1 \to Q_2A$. Its $e_1$-tabulator (of degree 1) will be the total tabulator of $x$ as a 1-arrow of $Q_2A$; this amounts to a 2-arrow $\top_1 x$ of $A$ with a universal 12-map $t_1(\top_1 x) \to 0 x$ (where $e_1 : A_2 \to A_{12}$ is the degeneracy).

(c) Symmetrically, $x$ can be viewed as a 2-arrow of $Q_1A$, i.e. a unitary pseudo functor $x: u_2 \to Q_1A$. Its $e_2$-tabulator (of degree 1, again) will be the total tabulator of $x$ as a 2-arrow of $Q_1A$; this amounts to a 1-arrow $\top_2 x$ of $A$ with a universal 12-map $t_2(\top_2 x) \to 0 x$ (where $e_2 : A_1 \to A_{12}$ is the degeneracy).

(d) The 2-dimensional cube $x$ is also an object of $Q_{12}A$. Its tabulator of degree two is $x$ itself. This is a (trivial) level limit, while the previous limits are not level, i.e. are not limits in some transversal category of $A$.

4.2 General tabulators

An $i$-cube $x \in A_i$ is a unitary pseudo functor $x: u_i \to A$. For every $k \subset i$ we can also view $x$ as a pseudo functor $u_j \to Q_kA$ where $j = i \setminus k$, so that $x$ can have an $e_j$-tabulator, namely a $k$-cube $T = \top_j x \in A_k$ with a universal i-map $t_x : e_j(\top_j x) \to 0 x$. (Total tabulators correspond to $j = i$, while $j = \emptyset$ gives the trivial case $\top_\emptyset x = x$.)
The universal property says now that, for every $k$-cube $A$ and every $i$-map $h: e_j(A) \to_0 x$ there is a unique $k$-map $u$ such that

$$
\begin{array}{ccc}
e_j(A) & \xrightarrow{e_j(u)} & e_j(T) \\
\downarrow h & & \downarrow t_x \\
x & \rightarrow & t_x.e_j(u) = h.
\end{array}
$$

(49)

We say that the chiral multiple category $A$ has tabulators of all degrees if every $i$-cube $x \in A_i$ has all $j$-tabulators $\top_j x \in A_k$ (for $i = j \cup k$, disjoint union). We say that $A$ has multiple tabulators if it has tabulators of all degrees, preserved by faces and degeneracies.

In this case, if $A$ is transversally invariant, one can always make a choice of multiple tabulators such that this preservation is strict (as we have already seen in various examples of Part I):

$$
\partial^{p}_i(\top_j x) = \top_j(\partial^{p}_i x), \quad \top_j(e_i y) = e_i(\top_j y) \quad (j \subset i, \ i \in i \setminus j),
$$

(50)

for $x \in A_i$ and $y \in A_{i|j}$.

Note that these conditions are trivial if $j = \emptyset$ or $j = i$, whence for all weak double categories (where there is only one positive index). This remark will be important when reconsidering double limits, in 4.7.

### 4.3 Lemma (Basic properties of tabulators)

Let $A$ be a chiral multiple category.

(a) For an $i$-cube $x$ and a disjoint union $i = j \cup k$ we have

$$
\top_i x = \top_k \top_j x,
$$

(51)

provided that $\top_j x$ and $\top_k(\top_j x)$ exist.

(b) $A$ has tabulators of all degrees if and only if it has all elementary tabulators $\top_j x$ (for every positive multi-index $i$, every $j \in i$ and every $i$-cube $x$).

(c) If all $e_j$-tabulators of $i$-cubes exist in $A$ there is an ordinary adjunction

$$
e_j: tv_{i|j}(A) \rightleftarrows tv_i(A) : \top_j, \quad e_j \dashv \top_j \quad (j \in i),
$$

(52)
and \( e_j : tv_{i[j]}A \to tv_iA \) preserves colimits.

(d) If all \( e_j \)-cotabulators of \( i \)-cubes exist in \( A \), then \( e_j : tv_{i[j]}A \to tv_iA \) is a right adjoint and preserves the existing limits (so that a condition on multiple level limits in 2.2(iii) is automatically satisfied).

(e) In a weak double category \( A \) the existence of cotabulators of vertical arrows implies that all ordinary limits in \( tv_* (A) \) are preserved by vertical identities. (This has already been used in 1.5.5.)

Proof. (a) Composing universal arrows for 
\[
e_i = e_j e_k : tv_* A \to tv_k A \to tv_i A,
\]
one gets (a choice of) \( \top_i x \) from (a choice of) \( \top_j x \) and \( \top_k (\top_j x) \). The rest is obvious.

4.4 Definition (Multiple limits)

We are now ready for a general definition of multiple limits in a chiral multiple category \( A \).

(a) For a positive multi-index \( i \subset \mathbb{N} \) and a chiral multiple category \( X \) we say that \( A \) has limits of type \( i \) on \( X \) if \( Q_i A \) has limits of degree zero on \( X \).

(b) We say that \( A \) has limits of type \( i \) if this happens for all small chiral multiple categories \( X \).

(c) We say that \( A \) has limits of all degrees (or all types) if this happens for all positive multi-indices \( i \).

(d) We say that \( A \) has multiple limits of all degrees if all the previous limits exist and are preserved by the multiple functors (see 1.5)
\[
D_j^a : Q_{ij} (A) \to R_j Q_i (A), \quad E_j : R_j Q_i (A) \to Q_{ij} (A) \quad (j \notin i). \quad (53)
\]

In this case, if \( A \) is transversally invariant, one can always operate a choice of multiple limits such that this preservation is strict (working as in Proposition 2.3).

We do not speak here of completeness: this notion should also involve the existence of 'companions' and 'adjoints' for all transversal maps, as shown by our study of Kan extensions in the domain of weak double categories [GP3, GP4].
4.5 Main Theorem (Construction and preservation of multiple limits)

Let \( A \) and \( B \) be chiral multiple categories.

(a) All multiple limits in \( A \) can be constructed from level multiple limits and multiple tabulators, or also from multiple products, multiple equalisers and multiple tabulators.

(b) If \( A \) has all multiple limits, a unitary lax multiple functor \( S : A \rightarrow B \) preserves them if and only if it preserves multiple products, multiple equalisers and multiple tabulators.

Similarly for finite limits and finite products.

Proof. Follows from Theorem 3.6, applied to the family of chiral multiple categories \( Q_i A \), together with the multiple functors of faces and degeneracies (see (53)) and the lax multiple functors \( Q_i S : Q_i A \rightarrow Q_i B \).

4.6 Examples

For a category \( C \) with pushouts and pullbacks we complete the discussion of tabulators in the chiral triple category \( SC(C) \), after the three types of tabulators of degree zero examined in 3.7. We start again from a 12-cube \( \pi : \vee \times \wedge \rightarrow C \) (a span of cospans in \( C \)).

(a) The \( e_1 \)-tabulator of \( \pi \) is a 2-arrow \( \top_1 \pi \) (a cospan) with a universal 12-map \( e_1 \top_1 \pi \rightarrow 0 \pi \); the solution is the middle cospan of \( \pi \).

(b) The \( e_2 \)-tabulator of \( \pi \) is a 1-arrow \( \top_2 \pi \) (a span) with a universal 12-map \( e_2 \top_2 \pi \rightarrow 0 \pi \); the solution is the obvious span whose objects are the pullbacks of the three cospans of \( \pi \).

These limits are preserved by faces and degeneracies. For instance:

- \( \partial_1^- (\top_2 \pi) = \top_2 (\partial_1^- \pi) \), which means that the domain of the span \( \top_2 \pi \) (described above) is the pullback of the cospan \( \partial_1^- \pi \),

- \( \top_2 (e_1 u) = e_1 (\top_2 u) \), i.e. the \( e_2 \)-tabulator of the 1-degenerate cell \( e_1 u \) (on the cospan \( u \)) is the degenerate span on the pullback of \( u \).

Finally, putting together the previous results (in 2.2 and 3.7): if \( C \) is a complete (or finitely complete) category with pushouts, then the chiral triple category \( SC(C) \) has multiple limits (or the finite ones).
4.7 Limits in weak double categories

We now complete the discussion of limits in a weak double category \( \mathbf{A} \), after the case of level limits examined in 2.6.

Here a consistent difference appears between the present analysis and that of [GP1]. In that paper all limits, including tabulators, were assumed to satisfy also a ‘two-dimensional universal property’ (namely condition (dl.2) in Definition 4.2). On the other hand multiple tabulators are here subject to preservation properties that only become non-trivial in dimension three or higher (as already remarked at the end of 4.2); the examples above (in 4.6) show that at least two positive indices are required to formulate non-trivial conditions of this type.

In other words, tabulators in a weak double category \( \mathbf{A} \) are here double tabulators, and the only limits that must be preserved by faces and degeneracies are the level ones, generated by products and equalisers of objects or vertical arrows of \( \mathbf{A} \).

The present terminology, a particular case of the definitions in 4.2 and 4.4, can thus be summarised as follows.

(a) \( \mathbf{A} \) has tabulators if every vertical arrow \( u \) (a 1-cube) has an object \( \top u = \top_1 u \) with a universal double cell \( e_1(\top_1 u) \to u \).

(b) \( \mathbf{A} \) has limits of degree zero (namely the limits that produce objects) if all the functors \( X \to \mathbf{A} \) (defined on a small weak double category) have a limit. Theorem 3.6 says that this condition amounts to the existence of:
- all products \( \prod \lambda A_\lambda \) of objects,
- all equalisers of pairs \( f, g : A \to B \) of parallel horizontal arrows,
- all tabulators \( \top u \) of vertical arrows.

(c) \( \mathbf{A} \) has limits of degree 1 (namely the limits that produce vertical arrows) if all the functors \( \Lambda \to tv_1(\mathbf{A}) = Q_1 \mathbf{A} \) (defined on a small category) have a limit. By the usual theorem on ordinary limits, this condition amounts to the existence of:
- products \( \prod u_\lambda \) of vertical arrows,
- equalisers of pairs \( a, b : u \to v \) of double cells (between the same vertical arrows).

(d) \( \mathbf{A} \) has limits of all degrees if both conditions (b) and (c) are satisfied.
(e) \( \mathcal{A} \) has double limits if all the previous limits exist and are preserved by the ordinary functors

\[
D_1^\alpha : \text{tv}_1 \mathcal{A} \to \text{tv}_* \mathcal{A}, \quad E_1 : \text{tv}_* \mathcal{A} \to \text{tv}_1 \mathcal{A},
\]

inasmuch as this makes sense (i.e. for ordinary limits in \( \text{tv}_* \mathcal{A} \) and \( \text{tv}_1 \mathcal{A} \), which amount to \( \star \)- and 1-level limits of \( \mathcal{A} \)).

Theorem 4.5 says that \( \mathcal{A} \) has double limits if and only if it has: double products, double equalisers and tabulators. Concretely, this amounts to the existence of the limits listed in (b) and (c), together with the conditions:

- products are preserved by domain, codomain and vertical identities,
- equalisers are preserved by domain, codomain and vertical identities.

If this holds and \( \mathcal{A} \) is transversally invariant (‘horizontally invariant’ in [GP1]), Proposition 2.3 says one can always choose double limits such that this preservation is strict. For products this means that:

- for a family of vertical arrows \( u_\lambda : A_\lambda \to B_\lambda \) we have \( \prod u_\lambda : \prod A_\lambda \to \prod B_\lambda \),
- for a family of objects \( A_\lambda \) the product of their vertical identities is the vertical identity of \( \prod A_\lambda \).

### 4.8 The symmetric cubical case

As analysed in [G1], weak symmetric cubical categories (with lax cubical functors) have a path endofunctor

\[
P : \text{LxWsc} \to \text{LxWsc},
\]

\[
P((\text{tv}_nA), (\partial_1^\alpha), (e_i), (+_i), (s_i), ...)
= ((\text{tv}_{n+1}A), (\partial_{i+1}^\alpha), (e_{i+1}), (+_{i+1}), (s_{i+1}), ...),
\]

which lifts all components of one degree and discards 1-indexed faces, degeneracies, transpositions and comparisons (the latter are omitted above). The discarded faces and degeneracy yield three natural transformations

\[
\partial_1^\alpha : P \xrightarrow{\text{deg}} 1 : e_1, \quad \partial_1^\alpha : e_1 = \text{id},
\]

which make \( P \) into a path endofunctor, from a structural point of view. The role of symmetries is crucial (without them we would have two non-isomorphic path-functors, and a plethora of higher path functors, their composites, see [G1]).
This situation cannot be extended to chiral multiple categories: the path endofunctor was replaced by the lift functors $Q_j : Lx Cmc \to Lx Cmc_{||j}$ and the restriction functors $R_j : Lx Cmc \to Lx Cmc_{||j}$ of 1.8, with faces and degeneracy

$$D_j^\alpha : Q_j \equiv R_j : E_j, \quad D_j^\alpha . E_j = \text{id.} \quad (57)$$

The whole system is consistent, by means of commutative squares

$$\begin{array}{ccc}
Lx Wse & \xrightarrow{P} & Lx Wse \\
\downarrow U & & \downarrow U_j \\
Lx Cmc & \xrightarrow{Q_j} & Lx Cmc_{||j}
\end{array} \quad \begin{array}{ccc}
Lx Wse & \xrightarrow{1} & Lx Wse \\
\downarrow U & & \downarrow U_j \\
Lx Cmc & \xrightarrow{R_j} & Lx Cmc_{||j}
\end{array} \quad (58)$$

where $U : Lx Wse \to Lx Cmc$ is the embedding described in I.2.8 (that gives rise to weak multiple categories of a symmetric cubical type) and $U_j = R_j U$.

In this way, cubical limits in weak symmetric cubical categories, dealt with in [G2], agree with multiple limits as presented here.

5. Proof of the theorem on multiple limits

We now prove Theorem 3.6. The argument is similar to the proof of the corresponding theorem for double limits [GP1], or its extension to cubical limits [G2].

5.1 Comments

Of course we only have to prove the ‘sufficiency’ part of the statement. We write down the argument for the construction of limits; the preservation property is proved in the same way.

The chiral multiple category $A$ is supposed to have all level limits of degree zero and all tabulators of degree zero (or total tabulators). The proof works by transforming a lax functor $F : X \to A$ of chiral multiple categories into a graph-morphism $G : X \to tv_A$ and taking the limit of the latter. The (directed) graph $X$ is a sort of ‘transversal subdivision’ of $X$, where every $i$-cube of $X$ is replaced with an object simulating its total tabulator.
The procedure is similar to computing the end of a functor $S: \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}$ as the limit of the associated functor $S^\#: \mathbf{C}^\# \to \mathbf{D}$ based on Kan’s subdivision category of $\mathbf{C}$ ([Ka], 1.10; [Ma], IX.5).

5.2 Transversal subdivision

The transversal subdivision $X$ of $X$ is a graph, formed by the following objects and arrows, for an arbitrary positive multi-index $i$ of degree $n \geq 0$, with arbitrary $j \in i$ and $\alpha = \pm$. (Note that this graph is finite whenever $X$ is.)

(a) For every $i$-cell $x$ of $X$ there is an object $x$ in $X$. For every $i$-map $f: x \to y$ of $X$ there is an arrow $f: x \to y$ in $X$.

(b) For every $i$-cell $x$ of $X$, we also add $2n$ arrows $p_j^\alpha x: x \to \partial_j^\alpha x$ (that simulate the projections (45) of the total tabulator of $x$, for $j \in i$ and $\alpha = \pm$).

(c) If $x = e_j z$ is degenerate (in direction $j$) we also add an arrow $d_j z: z \to e_j z$ (that simulates the diagonal map (46)).

(d) For every $j$-concatenation of $i$-cells $z = x +_j y$ in $X$, we also add an object $(x, y)_j$ in $X$ and three arrows

\[ p_j = p_j(x, y): (x, y)_j \to x, \quad q_j = q_j(x, y): (x, y)_j \to y, \quad d_j(x, y)_j: (x, y)_j \to z, \]

that simulate the pullback-object $\top_j(x, y)$ of (47), with its projections and the diagonal map (48).

5.3 The associated morphism of graphs

We now construct a graph-morphism $G: X \to \text{tv}_* A$ that naturally comes from $F$ and the existence of level limits and tabulators (of degree zero) in $A$.

(a) For every $i$-cell $x$ of $X$, we define $Gx$ as the following total tabulator (a $\star$-cube) of $A$

\[ G(x) = \top(Fx) \quad (t_{Fx}: e_0 G(x) \to_0 F(x)). \]
For every i-map \( f : x \to y \) of \( X \), we define \( Gf \) as the transversal map of \( A \) determined by the universal property of \( t_{Fy} \), as follows

\[
\begin{align*}
\begin{array}{c}
e_1 \top(Fx) \quad e_1(Gf) \\
t_{Fx}
\end{array}
\quad \begin{array}{c}
\top(Fy) \quad \top(Fy)
\end{array}
\quad \begin{array}{c}
gle f \quad t_{Fy}
\end{array}
\quad \begin{array}{c}
e_1(Gf) = Ff.t_{Fx}
\end{array}
\end{align*}
\]

(b) For \( z = \partial^\jmath x \) we define \( G(p^\jmath x) : Gx \to Gz \) as the following transversal map of \( A \)

\[
\begin{align*}
\begin{array}{c}
e_{ij} \top(Fx) \quad e_{ij}(G(p^\jmath x)) \\
t_{Fx}\partial^\jmath(t_{Fx})
\end{array}
\quad \begin{array}{c}
\top(Fz) \quad \top(Fz)
\end{array}
\quad \begin{array}{c}
gle(p^\jmath x) \quad t_{Fz}, e_{ij}(G(p^\jmath x)) = \partial^\jmath(t_{Fx}).
\end{array}
\end{align*}
\]

(c) For a degenerate i-cube \( x = e_jz \) (where \( z \) is an \( i|j \)-cube) the map \( G(d_jz) : Gz \to G(e_jz) \) is defined as follows

\[
\begin{align*}
\begin{array}{c}
e_4 \top(Fz) \quad e_4(G(d_jz)) \\
e_jt_{Fx}
\end{array}
\quad \begin{array}{c}
\top(Fe_jz) \quad \top(Fe_jz)
\end{array}
\quad \begin{array}{c}
gle(d_jz) \quad t_{Fx}, e_4(G(d_jz)) = F_jz.e_j(t_{Fx}).
\end{array}
\end{align*}
\]

(d) For a concatenation \( z = x + y \) of i-cubes, the object \( G(x, y)_j = \top_j(Fx, Fy) \) is the pullback of the objects \( \top_jFx \) and \( \top_jFy \), over the tabulator \( \top_jFw \) associated to the \( i|j \)-cube \( w = \partial^+_j x = \partial^-_j y \) (see (47)).

The arrows \( p_j(x, y) : (x, y)_j \to x \) and \( q_j(x, y) : (x, y)_j \to y \) of \( X \) are taken by \( G \) to the projections (47) of \( \top_j(Fx, Fy) \)

\[
\begin{align*}
\begin{array}{c}
c_j(x, y) \quad c_j(x, y)
\end{array}
\quad \begin{array}{c}
g_j(x, y)_j \quad g_j(x, y)_j
\end{array}
\quad \begin{array}{c}
G(p_j(x, y)) : G(x, y)_j \to Gx, \quad G(q_j(x, y)) : G(x, y)_j \to Gy,
\end{array}
\end{align*}
\]

so that \( (G(x, y)_j; Gp_j(x, y), Gq_j(x, y)) \) is the pullback of \( (p^+_j(Fx), p^-_j(Fy)) \) in \( tv_A \).
Finally, the arrow $d_j(x, y) : (x, y)_j \to z$ of $X$ is sent by $G$ to the diagonal (48) of $G(x, y)_i = \top_j(Fx, Fy)$, determined as follows

$$G(d_j(x, y)) : \top_j(Fx, Fy) \to_0 \top F(z),$$

$$t_{Fx}.e_i(G(d_j(x, y))) = e'_j(x, y).t_{Fy}.e_i(G(p_j(x, y)) + j t_{Fy}.e_i(G(q_j(x, y)))) \cdot \lambda_j^{-1},$$

$$e_i(G(x, y)_i) \xrightarrow{e'_j(G(d_j(x, y)))} e_i(\top(Fz)) \xrightarrow{t_{Fz}} Fz$$

$$e_i(G(x, y)_i) + j e_i(G(x, y)_i) \xrightarrow{t_{Fx}.e_iGp_j + j t_{Fy}.e_iGq_j} Fx + j Fy$$

The limit of this diagram $G : X \to tv_*A$ exists, by hypothesis.

5.4 From multiple cones to cones

In order to prove that the limit of $G$ gives the limit of degree 0 of $F$ we construct an isomorphism

$$(D \downarrow F) \to (D' \downarrow G),$$

from the comma category of transversal cones of the lax functor $F$ to the comma category of ordinary cones of the graph-morphism $G$. We proceed first in this direction, and then backwards.

Let $(A, h : DA \to F)$ be a cone of $F$. For every i-cube $x$ of $X$, we define $k(x) : A \to 0 Gx = \top(Fx)$ as the \textasteriskcentered-map of $A$ determined by the i-map $hx$, via the tabulator property

$$t_{Fx}.e_i(kx) = hx.$$  \hspace{1cm} (66)

Further, we define $k(x, y)_j : A \to_0 G(x, y)_j$ by means of the pullback-property of $G(x, y)_j$

$$p_j(x, y).k(x, y)_j = kx : A \to_0 Gx,$$

$$q_j(x, y).k(x, y)_j = ky : A \to_0 Gy.$$  \hspace{1cm} (67)

Let us verify that this family $k$ is indeed a cone of $G : X \to tv_*A$. 

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(a) Coherence with an $i$-map $f: x \to y$ (viewed as an arrow of $X$) means that $Gf.kx = ky$, which follows from the cancellation property of $t_{Fy}$

$$t_{Fy}.e_i(Gf.kx) = Ff.t_{Fx}.e_i(kx) = Ff.hx = hy = t_{Fy}.e_i(ky). \quad (68)$$

(b), (c) Coherence with the $X$-arrows $p_j(x): x \to \partial_j^a x$ and $d_j z: z \to e_j z = x$ follows from (62) and (63)

$$G(p_j^a(x)).kx = k(\partial_j^a x),$$

$$t_{Fx}.e_i(G(d_j z).kz) = E_j z.e_j(t_{Fx}).e_i(kz) = E_j z.e_j(t_{Fx}.e_i(kz)) \quad (69)$$

$$= E_j z.e_j(hz) = h(e_j z) = h(x) = t_{Fx}.e_i(kx).$$

(d) Coherence with the $X$-arrows $p_j = p_j(x, y)$ and $q_j = q_j(x, y)$ holds by construction (see (64)). For $d_j(x, y)$ and $z = x + y$ we have

$$t_{Fx}.e_i(G(d_j(x, y).k(x, y)_j))$$

$$= E_j(x, y). (t_{Fx}.e_ip_j + t_{Fy}.e_iq_j). \lambda_j^{-1}.e_i(k(x, y)_j)$$

$$= E_j(x, y). (t_{Fx}.e_ip_j + t_{Fy}.e_iq_j). (e_i(k(x, y)_j + e_i(k(x, y)_j). \lambda_j^{-1}$$

$$= E_j(x, y). (hx + hy). \lambda_j^{-1} = hz = t_{Fx}.e_i(kx). \quad (70)$$

Finally, a map of multiple cones

$$f: (A, h: DA \to F) \to (A', h': DA' \to F)$$

determines a map of $G$-cones $f: (A, k) \to (A', k')$, since

$$t_{Fx}.e_i(k'x.f) = h'x.e_i(f) = hx = t_{Fx}.e_i(kx). \quad (71)$$

5.5 From cones to multiple cones

In the reverse direction $(D' \downarrow G) \to (D \downarrow F)$ we just specify the procedure on cones. Given an ordinary cone $(A, k: DA \to G)$ of $G$, one forms a multiple cone $(A, h: DA \to F)$ by letting

$$hx = t_{Fx}.e_i(kx): e_i(A) \to x \quad (x \in A). \quad (72)$$
This satisfies (tc.1) (see 3.2) since, for \( f: x \rightarrow y \) in \( X \)

\[
F \cdot F \cdot x = F \cdot F \cdot t_{F \cdot x} \cdot e_1(kx) = t_{F \cdot y} \cdot e_1(G \cdot kx) = t_{F \cdot y} \cdot e_1(ky) = h_y.
\] (73)

Finally, to verify the condition (tc.2) for \( j \)-units and \( j \)-composition in \( X \) we operate much as above (with \( x = e_j z \) in the first case and \( z = x + j y \) in the second)

\[
E_j(z).e_j(hz) = E_j(z).e_j(t_{F \cdot z}.e_{i}(kz)) = E_j(z).e_j(t_{F \cdot z}.e_{i}(kz))
\]

\[
= t_{F \cdot x}.e_1(G(d_j z).k) = t_{F \cdot x}.e_1(kx) = h_x.
\] (74)

\[
hz = t_{F \cdot z}.e_1(kz) = t_{F \cdot z}.e_1(G(d_j (x, y)).k(x, y)_j) =
\]

\[
= E_j(x, y).(t_{F \cdot x}.e_{i}p_j + t_{F \cdot y}.e_{i}q_j).\lambda_j^{-1}.e_1k(x, y)_j
\]

\[
= E_j(x, y).(t_{F \cdot x}.e_{i}p_j + t_{F \cdot y}.e_{i}q_j).(e_1k(x, y)_j + e_1k(x, y)_j).\lambda_j^{-1}
\] (75)

\[
= E_j(x, y).(hx + hy).\lambda_j^{-1}.
\]

References


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