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AFFINE COMBINATIONS IN AFFINE SCHEMES

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Résumé. Nous prouvons que la notion géométrique de *points voisins*, dérivée du “premier voisinage de la diagonale” en géométrie algébrique, a la propriété que toute combinaison affine d’un n-tuple quelconque de points mutuellement voisins a un sens invariant, dans tout schema affine. La preuve est obtenue par des considérations d’algèbre commutative élémentaire.

Abstract. The geometric notion of *neighbour points*, as derived from the “first neighbourhood of the diagonal” in algebraic geometry, is shown to have the property that affine combinations of any n-tuple of mutual neighbour points make invariant sense, in any affine scheme. The proof is a piece of elementary commutative algebra.

Keywords. First neighbourhood of the diagonal, neighbour points, affine schemes, affine combinations.

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Introduction

The notion of “neighbour points” in algebraic geometry is a geometric rendering of the notion of nilpotent elements in commutative rings, and was developed since the time of Study, Hjelmslev, later by Kähler, and notably, since the 1950s, by French algebraic geometry (Grothendieck, Weil et al.). The latter school introduced it via what they call *the first neighbourhood of the diagonal*.

In [4], [5] and [8] the neighbour notion was considered on an axiomatic basis, essentially for finite dimensional manifolds; one of the aims was to describe a combinatorial theory of differential forms.

In the specific context of algebraic geometry, such theory of differential forms was also developed in [2], where it applies not only to manifolds, but to arbitrary schemes.

One aspect, present in [5] and [8], but not in [2], is the possibility of forming *affine combinations* of finite sets of mutual (1st order) neighbour points. The present note completes this aspect, by giving the construction of such affine combinations, at least in the category of *affine* schemes[1] (the dual of the category of commutative rings or k-algebras).

The interest in having the possibility of such affine combinations is documented in several places in [8], and is in [5] the basis for constructing, for any manifold, a simplicial object, whose cochain complex is the deRham complex of the manifold.

One may say that the possibility of having affine combinations, for sets of mutual neighbour points, expresses in a concrete way the idea that spaces are "infinitesimally like affine spaces".

1. Neighbour maps between algebras

Let k be a commutative ring. Consider commutative k-algebras B and C and two algebra maps f and $g : B \to C$.[2] We say that they are *neighbours*, or more completely, *(first order) infinitesimal neighbours*, if

$$(f(a) - g(a)) \cdot (f(b) - g(b)) = 0 \text{ for all } a, b \in B, \tag{1}$$

or equivalently, if

$$f(a) \cdot g(b) + g(a) \cdot f(b) = f(a \cdot b) + g(a \cdot b) \text{ for all } a, b \in B. \tag{2}$$

(Note that this latter formulation makes no use of "minus".) When this holds, we write $f \sim g$ (or more completely, $f \sim_1 g$). The relation $\sim$ is a reflexive and symmetric relation (but not transitive). If the element $2 \in k$ is invertible, a third equivalent formulation of $f \sim g$ goes

$$(f(a) - g(a))^2 = 0 \text{ for all } a \in B. \tag{3}$$

[1]Added in proof: Since the construction is local in nature, it is not surprising that it may be extended to more general schemes, and also to the C^∞-context. These issues are dealt with in [1].

[2]Henceforth, "algebra" means throughout "commutative k-algebra", and "algebra map" (or just "map") means k-algebra homomorphism; and "linear" means k-linear. By $\otimes$, we mean $\otimes_k$.

For, it is clear that (1) implies (3). Conversely, assume (3), and let $a, b \in B$ be arbitrary, and apply (3) to the element $a + b$. Then by assumption, and using that f and g are algebra maps,

$$0 = (f(a+b) - g(a+b))^2 = [(f(a) - g(a)) + (f(b) - g(b))]^2$$
$$= (f(a) - g(a))^2 + (f(b) - g(b))^2 - 2(f(a) - g(a)) \cdot (f(b) - g(b)).$$

The two first terms are 0 by assumption, hence so is the third. Now divide by 2.

Note that if C has no zero-divisors, then $f \sim g$ is equivalent to $f = g$.

It is clear that the relation $\sim$ is stable under precomposition:

$$\text{if } h : B' \to B \text{ and } f \sim g : B \to C, \text{ then } f \circ h \sim g \circ h : B' \to C, \qquad (4)$$

and (by a small calculation), it is also stable under postcomposition:

$$\text{if } k : C \to C' \text{ and } f \sim g : B \to C, \text{ then } k \circ f \sim k \circ g : B \to C'. \qquad (5)$$

Also, if $h : B' \to B$ is a surjective algebra map, precomposition by h not only preserves the neighbour relation, it also reflects it, in the following sense

$$\text{if } f \circ h \sim g \circ h \text{ then } f \sim g. \qquad (6)$$

This is immediate from (1); the a and b occurring there is of the form $h(a')$ and $h(b')$ for suitable a' and b' in B', by surjectivity of h.

An alternative "element-free" formulation of the neighbour relation (Proposition 1.2 below) comes from a standard piece of commutative algebra. Recall that for commutative k-algebras A and B, the tensor product $A \otimes B$ carries structure of commutative k-algebra ($A \otimes B$ is in fact a coproduct of A and B); the multiplication map $m : B \otimes B \to B$ is a k-algebra homomorphism; so the kernel is an ideal $J \subseteq B \otimes B$.

The following is a classical description of the ideal $J \subseteq B \otimes B$; we include it for completeness.

Proposition 1.1. *The kernel J of $m : B \otimes B \to B$ is generated by the expressions $1 \otimes b - b \otimes 1$, for $b \in B$. Hence the ideal J^2 is generated by the expressions $(1 \otimes a - a \otimes 1) \cdot (1 \otimes b - b \otimes 1)$. Equivalently, J^2 is generated by the expressions*

$$1 \otimes ab \; + \; ab \otimes 1 \; - \; a \otimes b \; - \; b \otimes a.$$

Proof. It is clear that $1\otimes b - b\otimes 1$ is in J. Conversely, assume that $\sum_i a_i \otimes b_i$ is in J, i.e. that $\sum_i a_i \cdot b_i = 0$. Rewrite the ith term $a_i \otimes b_i$ as follows:

$$a_i \otimes b_i = a_i b_i \otimes 1 + (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$$

and sum over i; since $\sum_i a_i b_i = 0$, we are left with $\sum_i (a_i \otimes 1)\cdot(1\otimes b_i - b_i \otimes 1)$, which belongs to the $B \otimes B$-module generated by elements of the form $1\otimes b - b\otimes 1$. – The second assertion follows, since $ab\otimes 1 + 1\otimes ab - a\otimes b - b\otimes a$ is the product of the two generators $1 \otimes a - a \otimes 1$ and $1 \otimes b - b \otimes 1$, except for sign. (Note that the proof gave a slightly stronger result, namely that J is generated already as a B-module, by the elements $1 \otimes b - b \otimes 1$, via the algebra map $i_0 : B \to B \otimes B$, where $i_0(a) = a \otimes 1$). □

From the second assertion in this Proposition immediately follows that $f \sim g$ iff $\binom{f}{g} : B \otimes B \to C$ factors across the quotient map $B \otimes B \to (B \otimes B)/J^2$ (where $\binom{f}{g} : B \otimes B \to C$ denotes the map given by $a \otimes b \mapsto f(a) \cdot g(b)$); equivalently:

Proposition 1.2. *For $f, g : B \to C$, we have $f \sim g$ if and only if $\binom{f}{g} : B \otimes B \to C$ annihilates J^2.*

The two natural inclusion maps i_0 and $i_1 : B \to B \otimes B$ (given by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, respectively) are not in general neighbours, but when postcomposed with the quotient map $\pi : B \otimes B \to (B \otimes B)/J^2$, they are:

$$\pi \circ i_0 \sim \pi \circ i_1,$$

and this is in fact the universal pair of neighbour algebra maps with domain B.

2. Neighbours for polynomial algebras

We consider the polynomial algebra $B := k[X_1, \ldots, X_n]$. Identifying $B \otimes B$ with $k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$, the multiplication map m is the algebra map given by $Y_i \mapsto X_i$ and $Z_i \mapsto X_i$, so it is clear that the kernel J of m contains the n elements $Z_i - Y_i$. The following Proposition should be classical:

Proposition 2.1. *The ideal $J \subseteq B \otimes B$, for $B = k[X_1, \ldots, X_n]$, is generated (as a $B \otimes B$-module) by the n elements $Z_i - Y_i$.*

Proof. From Proposition 1.1, we know that J is generated by elements $P(\underline{Z})-P(\underline{Y})$, for $P \in k[\underline{X}]$ (where $\underline{X}$ denotes $X_1, \ldots, X_n$, and similarly for $\underline{Y}$ and $\underline{Z}$). So it suffices to prove that $P(\underline{Z}) - P(\underline{Y})$ is of the form

$$\sum_{i=1}^{n}(Z_i - Y_i)Q_i(\underline{Y}, \underline{Z}).$$

This is done by induction in n. For $n = 1$, it suffices, by linearity, to prove this fact for each monomial X^s. And this follows from the identity

$$Z^s - Y^s = (Z - Y) \cdot (Z^{s-1} + Z^{s-2}Y + \ldots + ZY^{s-2} + Y^{s-1}) \quad (7)$$

(for $s \geq 1$; for $s = 0$, we get 0). For the induction step: Write $P(\underline{X})$ as a sum of increasing powers of X_1,

$$P(X_1, X_2, \ldots) = P_0(X_2, \ldots) + X_1 P_1(X_2, \ldots) + X_1^2 P_2(X_2, \ldots) + \ldots.$$

Apply the induction hypothesis to the first term. The remaining terms are of the form $X_1^s P_s(X_2, \ldots)$ with $s \geq 1$; then for this term, the difference to be considered is

$$Y_1^s P_s(Y_2, \ldots) - Z_1^s P_s(Z_2, \ldots)$$

which we may write as

$$Y_1^s(P_s(Y_2, \ldots) - P_s(Z_2, \ldots)) + P_s(Z_2, \ldots)(Y_1^s - Z_1^s).$$

The first term in this sum is taken care of by the induction hypothesis, the second term uses the identity (7) which shows that this term is in the ideal generated by $(Z_1 - Y_1)$. $\square$

From this follows immediately

Proposition 2.2. *The ideal $J^2 \subseteq B \otimes B$, for $B = k[X_1, \ldots, X_n]$, is generated (as a $B \otimes B$-module) by the elements $(Z_i - Y_i)(Z_j - Y_j)$ (for $i, j = 1, \ldots, n$) (identifying $B \otimes B$ with $k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$).*

(The algebra $(B \otimes B)/J^2$ is the algebra representing the affine scheme “first neighbourhood of the diagonal” for the affine scheme represented by B, alluded to in the introduction.)

An algebra map $\underline{a} : k[X_1, \dots, X_n] \to C$ is completely given by an n-tuple of elements $a_i := \underline{a}(X_i) \in C$ $(i = 1, \dots, n)$. Let $\underline{b} : k[X_1, \dots, X_n] \to C$ be similarly given by the n-tuple $b_i \in C$. The decision when $\underline{a} \sim \underline{b}$ can be expressed equationally in terms of these two n-tuples of elements in C, i.e. as a purely equationally described condition on elements $(a_1, \dots, a_n, b_1, \dots, b_n) \in C^{2n}$:

Proposition 2.3. *Consider two algebra maps $\underline{a}$ and $\underline{b} : k[X_1, \dots, X_n] \to C$. Let $a_i := \underline{a}(X_i)$ and $b_i := \underline{b}(X_i)$. Then we have $\underline{a} \sim \underline{b}$ if and only if*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \tag{8}$$

for all $i, j = 1, \dots, n$.

Proof. We have that $\underline{a} \sim \underline{b}$ iff the algebra map $\binom{\underline{a}}{\underline{b}}$ annihilates the ideal J^2 for the algebra $k[X_1, \dots, X_n]$; and this in turn is equivalent to that it annihilates the set of generators for J^2 described in Proposition 2.2. But $\binom{\underline{a}}{\underline{b}}((Z_i - Y_i) \cdot (Z_j - Y_j)) = (b_i - a_i) \cdot (b_j - a_j)$, and then the result is immediate. $\square$

We therefore also say that the pair of n-tuples of elements in C

$$\begin{bmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{bmatrix}$$

are *neighbours* if (8) holds.

For brevity, we call an n-tuple $(c_1, \dots, c_n)$ of elements in C^n a *vector*, and denote it $\underline{c}$. Thus a vector $(c_1, \dots, c_n)$ is neighbour of the "zero" vector $\underline{0} = (0, \dots, 0)$ iff $c_i \cdot c_j = 0$ for all i and j.

Remark. Even when $2 \in k$ is invertible, one cannot conclude that $(b_i - a_i)^2 = 0$ for all $i = 1, \dots, n$ implies $\underline{a} \sim \underline{b}$. For, consider $C := k[\epsilon_1, \epsilon_2] = k[\epsilon] \otimes k[\epsilon]$ (where $k[\epsilon]$ is the "ring of dual numbers over k", so $\epsilon^2 = 0$). Then the pair of n-tuples ($n = 2$ here) given by $(a_1, a_2) = (\epsilon_1, \epsilon_2)$ and $(b_1, b_2) := (0, 0)$ has $(a_i - b_i)^2 = \epsilon_i^2 = 0$ for $i = 1, 2$, but $(a_1 - b_1) \cdot (a_2 - b_2) = \epsilon_1 \cdot \epsilon_2$, which is not 0 in C.

We already have the notion of when two algebra maps f and $g : B \to C$ are neighbours. We also say that the pair (f, g) form an *infinitesimal 1-simplex* (with f and g as *vertices*). Also, we have with (8) the derived notion of when two vectors $\underline{a}$ and $\underline{b}$ in C^n are neighbours, or form an infinitesimal 1-simplex. This terminology is suited for being generalized to defining the notion of *infinitesimal p-simplex* of algebra maps $B \to C$, or of *infinitesimal p-simplex* of vectors in C^n (for $p = 1, 2, \ldots$), namely a $(p+1)$-tuple of mutual neighbouring algebra maps, resp. neighbouring vectors.

Proposition 2.3 generalizes immediately to infinitesimal p-simplices (where the Proposition is the special case of $p = 1$):

Proposition 2.4. *Consider $p+1$ algebra maps $\underline{a}_i : k[X_1, \ldots, X_n] \to C$ (for $i = 0, \ldots, p$), and let $a_{ij} \in C$ be $\underline{a}_i(X_j)$, for $j = 1, \ldots n$. Then the $\underline{a}_i$ form an infinitesimal p-simplex iff for all $i, i' = 0, \ldots p$ and $j, j' = 1, \ldots, n$*

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0. \tag{9}$$

3. Affine combinations of mutual neighbours

Let C be a k-algebra. An *affine* combination in a C-module means here a linear combination in the module, with coefficients from C, and where the sum of the coefficients is 1. We consider in particular the C-module $\mathrm{Lin}_k(B, C)$ of k-linear maps $B \to C$, where B is another k-algebra. Linear combinations of algebra maps are linear, but may fail to preserve the multiplicative structure (including 1). However

Theorem 3.1. *Let $f_0, \ldots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$, and let $t_0, \ldots, t_p$ be elements of C with $t_0 + \ldots + t_p = 1$. Then the affine combination*

$$\sum_{i=0}^{p} t_i \cdot f_i : B \to C$$

is an algebra map. The construction is natural in B and in C.

Proof. Since the sum is a k-linear map, it suffices to prove that it preserves the multiplicative structure. It clearly preserves 1. To prove that it preserves products $a \cdot b$, we should compare $\sum t_i f_i(a \cdot b)$. with

$$(\sum_i t_i f_i(a)) \cdot (\sum_j t_j f_j(b)) = \sum_{i,j} t_i t_j f_i(a) \cdot f_j(b).$$

Now use that $\sum_j t_j = 1$; then $\sum t_i f_i(a \cdot b)$ may be rewritten as

$$\sum_{ij} t_i t_j f_i(a \cdot b).$$

Compare the two displayed double sums: the terms with $i = j$ match since each f_i preserves multiplication. Consider a pair of indices $i \neq j$; the terms with index ij and ji from the first sum contribute $t_i t_j$ times

$$f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b), \tag{10}$$

and the terms terms with index ij and ji from the second sum contribute $t_i t_j$ times

$$f_i(a \cdot b) + f_j(a \cdot b), \tag{11}$$

and the two displayed contributions are equal, since $f_i \sim f_j$ (use the formulation (2)). The naturality assertion is clear. $\square$

Theorem 3.2. *Let Let $f_0, \ldots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$. Then any two affine combinations (with coefficients from C) of these maps are neighbours.*

Proof. Let $\sum_i t_i f_i$ and $\sum_j s_j f_j$ be two such affine combinations. To prove that they are neighbours means (using (2)) to prove that for all a and b in B,

$$(\sum_i t_i f_i(a)) \cdot (\sum_j s_j f_j(b)) + (\sum_j s_j f_j(a)) \cdot (\sum_i t_i f_i(b)) \tag{12}$$

equals

$$\sum_i t_i f_i(a \cdot b) + \sum_j s_j f_j(a \cdot b). \tag{13}$$

Now (12) equals

$$\sum_{ij} t_i s_j f_i(a) \cdot f_j(b) + \sum_{ij} t_i s_j f_j(a) \cdot f_i(b) = \sum_{ij} t_i s_j [f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b)]$$

For (13), we use $\sum_j s_j = 1$ and $\sum_i t_i = 1$, to rewrite it as the left hand expression in

$$\sum_{ij} t_i s_j f_i(a \cdot b) + \sum_{ij} t_i s_j f_j(a \cdot b) = \sum_{ij} t_i s_j [f_i(a \cdot b) + f_j(a \cdot b)].$$

For each ij, the two square bracket expression match by (2), since $f_i \sim f_j$. $\square$

Combining these two results, we have

Theorem 3.3. *Let $f_0, \dots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$. Then in the C-module of k-linear maps $B \to C$, the affine subspace $\mathrm{Aff}_C(f_0, \dots, f_p)$ of affine combinations (with coefficients from C) of the f_is consists of algebra maps, and they are mutual neighbours.*

Note that these two Theorems are also vaild for commutative *rigs*, i.e. no negatives are needed for the notions or the theorems.

In [2], the authors describe an ideal $J_{0p}^{(2)}$. It is the sum of ideals J_{rs}^2 in the $p+1$-fold tensor product $B \otimes \dots \otimes B$, where J_{rs} is the ideal generated by $i_s(b) - i_r(b)$ for $b \in B$ and $r < s$ (with i_k the kth inclusion map). We shall here denote $J_{0p}^{(2)}$ just by $\overline{J}^{(2)}$ for brevity; it has the property that the $p+1$ inclusions $B \to B \otimes \dots \otimes B$ become mutual neighbours, when composed with the quotient map $\pi : B \otimes \dots \otimes B \to (B \otimes \dots \otimes B)/\overline{J}^{(2)}$, and this is in fact the universal $p+1$ tuple of mutual neighbour maps with domain B.

We may, for any given k-algebra B, encode the construction of Theorem 3.1 into one single canonical map which does not mention any individual $B \to C$. This we do by using the universal $p+1$-tuple of neighbour elements, and the generic $p+1$ tuple of elements (to be used as coefficients) with sum 1, meaning $(X_0, X_1, \dots, X_p) \in k[X_1, \dots, X_p]$ (where X_0 denotes $1-(X_1+ \dots + X_p)$). We shall construct a k-algebra map

$$B \to (B^{\otimes p+1}/\overline{J}^{(2)}) \otimes k[X_1, \dots, X_p]. \tag{14}$$

By the Yoneda Lemma, this is equivalent to giving a (set theoretical) map, natural in C,

$$\hom((B^{\otimes p+1}/\overline{J}^{(2)}) \otimes k[X_1, \dots, X_p], C) \to \hom(B, C),$$

(where $\hom$ denotes the set of k-algebra maps). An element on the left hand side is given by a $p+1$-tuple of mutual neighbouring algebra maps $f_i : B \to C$, together with a p-tuple $(t_1, \dots, t_p)$ of elements in C. With $t_0 := 1 - \sum_1^p t_i$, such data produce an element $\sum_0^p t_i \cdot f_i$ in $\hom(B, C)$, by Theorem 3.1, and the construction is natural in C by the last assertion in the Theorem.

The affine scheme defined by the algebra $B^{\otimes p+1}/\overline{J}^{(2)}$ is (essentially) called $\Delta_B^{(p)}$ in [2], and, (in axiomatic context, for manifolds, in a suitable sense), the corresponding object is called $M_{[p]}$ in [5] and $M_{(1,1,\dots,1)}$ in [4] I. 18 (for suitable M).

4. Affine combinations in a k-algebra C

The constructions and results of the previous Section concerning infinitesimal p-simplices of algebra maps $B \to C$, specialize (by taking $B = k[X_1, \dots, X_n]$, as in Section 2) to infinitesimal p-simplices of vectors in C^n; such a p-simplex is conveniently exhibited in a $(p+1) \times n$ matrix with entries a_{ij} from C:

$$\begin{bmatrix} a_{01} & \dots & a_{0n} \\ a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pn} \end{bmatrix}$$

in which the rows (the "vertices" of the simplex) are mutual neighbours. We may of course form affine (or even linear) combinations, with coefficients from C, of the rows of this matrix, whether or not the rows are mutual neighbours. But the same affine combination of the corresponding algebra maps is in general only a k-linear map, not an algebra map. However, if the rows are mutual neighbours in C^n, and hence the corresponding algebra maps are mutual neighbours $k[X_1, \dots, X_n] \to C$, we have, by Theorem 3.1 that the affine combinations of the rows of the matrix corresponds to the similar affine combination of the algebra maps. For, it suffices to check their equality on the X_is, since the X_is generate $k[X_1, \dots, X_n]$ as an algebra. Therefore, the Theorems 3.2 and 3.3 immediately translate into theorems about $p+1$-tuples of mutual neighbouring n-tuples of elements in the algebra C; recall that such a $p+1$-tuple may be identified with the rows of a $(p+1) \times n$ matrix with entries from C, satisfying the equations (9). We therefore have (cf. also [6])

Theorem 4.1. *Let the rows of a $(p+1) \times n$ matrix with entries from C be mutual neighbours. Then any two affine combinations (with coefficients from C) of these rows are neighbours. The set of all such affine combinations form an affine subspace of the C-module C^n.*

Let us consider in particular the case where the 0th row of a $(p+1) \times n$ matrix is the zero vector $(0, \dots, 0)$. Then the following is an elementary calculation:

Proposition 4.2. *Consider a $(p+1) \times n$ matrix $\{a_{ij}\}$ as above, but with $a_{0j} = 0$ for $j = 1, \dots n$. Then the rows form an infinitesimal p-simplex iff*

$$a_{ij} \cdot a_{i'j'} + a_{i'j} \cdot a_{ij'} = 0 \text{ for all } i, i' = 1, \dots p, j, j' = 1, \dots n \tag{15}$$

and

$$a_{ij} \cdot a_{ij'} = 0 \text{ for all } i = 1, \dots, p, j = 1, \dots n \tag{16}$$

hold. If 2 is invertible in C, the equations (16) follow from (15).

Proof. The last assertion follows by putting $i = i'$ in (15), and dividing by 2. Assume that the rows of the matrix form an infinitesimal p-simplex. Then (16) follows from $\underline{a}_i \sim \underline{0}$ by (8). The equation which asserts that $\underline{a}_i \sim \underline{a}_{i'}$ (for $i, i' = 1, \dots, p$) is

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0 \text{ for all } j, j' = 1, \dots n.$$

Multiplying out gives four terms, two of which vanish by virtue of (16), and the two remaining add up to (minus) the sum on the left of (15). For the converse implication, (16) give that the last p rows are $\sim \underline{0}$; and (16) and (15) jointly give that $\underline{a}_i \sim \underline{a}_{i'}$, by essentially the same calculation which we have already made. $\square$

When $\underline{0}$ is one of the vectors in a $p+1$-tuple, any linear combination of the remaining p vectors has the same value as a certain affine combination of all $p+1$ vectors, since the coefficient for $\underline{0}$ may be chosen arbitrarily without changing the value of the linear combination. Therefore the results on *affine* combinations of the rows in the $(p+1) \times n$ matrix with $\underline{0}$ as top row immediately translate to results about *linear* combinations of the remaining rows, i.e. they translate into results about $p \times n$ matrices, satisfying the equations (15) and (16); even the equations (15) suffice, if 2 is invertible. In this form, the results were obtained in the preprint [6], and are stated here for completeness. We assume that $2 \in k$ is invertible.

We use the notation from [4] I.16 and I. 18, where set of $p \times n$ matrices $\{a_{ij}\}$ satisfying (15) was denoted $\tilde{D}(p, n) \subseteq C^{p \cdot n}$ (we there consider algebras C over $k = \mathbb{Q}$, so (16) follows). In particular $\tilde{D}(2,2)$ consists of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{with} \quad a_{11} \cdot a_{22} + a_{12} \cdot a_{21} = 0.$$

Note that the determinant of such a matrix is 2 times the product of the diagonal entries. And also note that $\tilde{D}(2,2)$ is stable under transposition of matrices.

The notation $\tilde{D}(p,n)$ may be consistently augmented to the case where $p = 1$; we say $(a_1, \ldots, a_n) \in \tilde{D}(1,n)$ if it is neighbour of $\underline{0} \in C^n$, i.e. if $a_j \cdot a_{j'} = 0$ for all $j, j' = 1, \ldots n$. (In [4], $\tilde{D}(1,n)$ is also denoted $D(n)$, and $D(1)$ is denoted D.)

It is clear that a $p \times n$ matrix belongs to $\tilde{D}(p,n)$ precisely when all its 2×2 sub-matrices do; this is just a reflection of the fact that the defining equations (15) only involve two row indices and two column indices at a time. From the transposition stability of $\tilde{D}(2,2)$ therefore follows that transposition $p \times n$ matrices takes $\tilde{D}(p,n)$ into $\tilde{D}(n,p)$.

Note that each of the rows of a matrix in $\tilde{D}(p,n)$ is a neighbour of $\underline{0} \in C^n$.

The results about affine combinations now has the following corollary in terms of linear combinations of the rows of matrices in $\tilde{D}(p,n)$:

Theorem 4.3. *Given a matrix $X \in \tilde{D}(p,n)$. Let a $(p+1) \times n$ matrix X' be obtained by adjoining to X a row which is a linear combination of the rows of X. Then X' is in $\tilde{D}(p+1,n)$.*

5. Geometric meaning

Commutative rings often come about as rings $O(M)$ of scalar valued functions on some space M, and this gives some geometric aspects (arising from the space M) into the algebra $O(M)$. Does *every* commutative ring (or k-algebra) come about this way? This depends of course what "space" is supposed to mean, and what the "scalars" and "functions" are. What could they be?

Algebraic geometry has over time developed a radical, almost self-referential answer. The first thing is to define the *category* $\mathcal{E}$ of spaces, and among these, a commutative ring object $R \in \mathcal{E}$ of *scalars*. The radical answer consists in taking the category $\mathcal{E}$ of spaces and functions to be the dual of the category $\mathcal{A}$ of commutative rings, and *any commutative ring B to be the ring $O(\overline{B})$ of scalar valued functions on the space $\overline{B}$ which it defines*. This will come about by letting the ring of scalars $R \in \mathcal{E}$ be the free ring in one generator: the polynomial ring in one variable, cf. (17) below.

To fix terminology, we elaborate this viewpoint. For flexibility and generality, we consider a commutative base ring k, and consider the category $\mathcal{A}$ of commutative k-algebras (the "absolute" case comes about by taking $k = \mathbb{Z}$).

So $\mathcal{E}$ is $\mathcal{A}^{op}$; for $B \in \mathcal{A}$, the corresponding object in $\mathcal{E}$ is denoted $\overline{B}$ or $\mathrm{Spec}(B)$; for $M \in \mathcal{E}$, the corresponding object in $\mathcal{A}$ is denoted $O(M)$. Thus, $B = O(\overline{B})$ and $M = \overline{O(M)}$.

We have in particular $k[X] \in \mathcal{A}$, the free k-algebra in one generator, and we put $R := \overline{k[X]}$. Then for any $M \in \mathcal{E}$,

$$\hom_{\mathcal{E}}(M, R) = \hom_{\mathcal{E}}(M, \overline{k[X]}) = \hom_{\mathcal{A}}(k[X], O(M)) \cong O(M), \quad (17)$$

(the last isomorphism because $k[X]$ is the free k-algebra in one generator X), and since the right hand side is a k-algebra, (naturally in M), we have that R is a k-algebra object in $\mathcal{E}$. And (17) documents that $O(M)$ is indeed canonically isomorphic as a k-algebra to the k-algebra of R-valued functions on M.

If $\mathcal{E}$ is a category with finite products, algebraic structure on an object R in $\mathcal{E}$ may be described in diagrammatic terms, but it is equivalent to description of the same kind of structure on the *sets* $\hom_{\mathcal{E}}(M, R)$ (naturally in M), thus is a description in terms of *elements*.

It is useful to think of, and speak of, such an element (map) $a : M \to R$ as an "element of R, defined at stage M", or just as a "generalized element (or generalized point[3]) of R defined at stage M". We may write $a \in R$, or $a \in_M R$, if we need to remember the "stage" at which the element a of R is defined; and we may drop the word "generalized".

If $f : R \to S$ is a map in $\mathcal{E}$, then for an $a : M \to R$, we have the composite $f \circ a : M \to S$; viewing a and $f \circ a$ as generalized elements of

[3]Grothendieck called a map $M \to R$ "an M-valued point of R", extending the use in classical algebraic geometry, where one could talk about e.g. a complex-valued point, or point defined over $\mathbb{C}$, for R an arbitrary algebraic variety.

R and S, respectively, the latter is naturally written $f(a)$:

$$f(a) := f \circ a.$$

Subobjects of an arbitrary object $R \in \mathcal{E}$ may be characterized by which of generalized elements of R they contain. Maps $R \to S$ may be described by what they do to generalized elements of R (by post-composition of maps). This is essentially Yoneda's Lemma.

Consider in particular the theory of neighbour maps and their affine combinations, as developed in the previous sections. It deals with the category $\mathcal{A}$ of commutative k-algebras. We shall translate some of the notions and constructions into the dual category $\mathcal{E}$, i.e. into the category of affine schemes over k, using the terminology of generalized elements, or generalized points.

Thus (assuming for simplicity that 2 is invertible in k), we can consider the criterion (3) for the neighbour relation of algebra maps $f, g : B \to C$; it translates as follows. Let $\overline{f}$ and $\overline{g}$ be points of $\overline{B}$ (defined at stage $\overline{C}$). Then $\overline{f} \sim \overline{g}$ iff for all $a : \overline{B} \to R$, we have $(a(\overline{f}) - a(\overline{g}))^2 = 0$, or, changing the names of the objects and maps/elements in question, e.g. $X = \overline{B}$, and refraining from mentioning the common stage of definition of the elements x and y:

Two points x and y in X are neighbours iff for any scalar valued function α on X, $(\alpha(x) = \alpha(y))^2 = 0$.

Thus, the basic (first order) neighbour relation $\sim$ on any object M is determined by the set of scalar valued functions on it, and by which points in the ring object R of scalars have square 0. This implies that the neighbour relation is preserved by any map $\overline{B} \to \overline{B'}$ between affine schemes. The naturality of the construction of affine combinations of mutual neighbour points in $\overline{B}$ implies that the construction is preserved by any map $\overline{B} \to \overline{B'}$ between affine schemes.

The Proposition 2.3 gets the formulation:

Proposition 5.1. *Given two points $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n) \in R^n$. Then they are neighbours iff*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \tag{18}$$

for all $i, j = 1, \ldots, n$.

Here the (common) parameter space $\overline{C}$ of the $a_i s$ and b_is is not mentioned explicitly; it could be any affine scheme. Note that (18) is typographically the same as (8); in (18), the a_is etc. are (parametrized) points of R (parametrized by $\overline{C}$), in (8), they are elements in the algebra C; but these data correspond, by (17), and this correspondence preserves algebraic structure. – Similarly, Proposition 2.4 gets the reformulation:

Proposition 5.2. *A $p+1$-tuple $\{a_{ij}\}$ of points in R^n form an infinitesimal p-simplex iff the equations (9) hold.*

This formulation, as the other formulations in "synthetic" terms, are the ones that are suited to axiomatic treatment, as in Synthetic Differential Geometry, which almost exclusively[4] assumes a given commutative ring object R in a category $\mathcal{E}$, preferably a topos, as a basic ingredient in the axiomatics. (The category $\mathcal{E}$ of affine schemes is not a topos, but the category of presheaves on $\mathcal{E}$ is, and it, and some of its subtoposes, are the basic categories considered in modern algebraic geometry, like in [3].)

Proposition 5.3. *Given an affine scheme $\overline{B}$, with the k-algebra B finitely presentable. Then for any finite presentation (with n generators, say) of the algebra, the corresponding embedding $e : \overline{B} \to R^n$ preserves and reflects the relation $\sim$, and it preserves affine combinations of neighbour points.*

For, any map between affine schemes preserves the neighbour relation, and affine combinations of mutual neighbours. The argument for reflection is as for (6) since the presentation amounts to a surjective map of k-algebras $k[X_1, \dots, X_n] \to B$.

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[4]Exceptions are found in [10] (where R is *constructed* out of an assumed infinitesimal object T); and in [7] and [9], where part of the reasoning does not assume any algebraic notions.

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