CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
VOLUME LIX-3 (2018)

A STUDY OF PENON WEAK n-CATEGORIES, PART 1: MONAD INTERLEAVING

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Résumé. L'article propose une construction alternative de la monade utilisée par Penon pour définir les n-catégories faibles. La monade de Penon ajoute deux éléments de structure supplémentaires à une structure d'ensemble n-globulaire : une structure de magma donnant une composition, et une structure de contraction donnant une cohérence. Ces deux structures sont ajoutées à l'aide d'une approche d'entrelacement, suivant la méthode utilisée par Cheng pour construire l'opérade de Leinster pour les ω-catégories faibles. Nous concluons en utilisant notre construction pour donner une description explicite de l'opérade n-globulaire pour les n-catégories faibles de Penon.

Abstract. We give an alternative construction of the monad used by Penon to define weak n-categories. Penon's monad adds two pieces of extra structure to an n-globular set: a magma structure, giving composition, and a contraction structure, giving coherence. We add these two structures using an interleaving approach, following the method used by Cheng to construct Leinster's operad for weak ω-categories. We conclude by using our construction to give an explicit description of the n-globular operad for Penon weak n-categories.

Keywords. n-category, higher-dimensional category, monad interleaving, operad.
Mathematics Subject Classification (2010). 18A40, 18C15, 18D05, 18D50.

1. Introduction

The main purpose of this paper is to give a new construction of the monad used by Penon to define weak n-categories. Penon weak n-categories, introduced in [15], are defined as the algebras for the monad induced by a certain non-monadic adjunction. The left adjoint of this adjunction, which Penon originally constructed using computads, freely adds two pieces of structure: a "magma structure", which gives composition in a weak n-category, and a "contraction structure", which gives coherence. In our construction we add these two structures by a process of monad interleaving. The use of this method to construct Penon's left adjoint was suggested by Cheng in [7], who used monad interleaving to construct Leinster's globular operad for weak ω-categories; thus we hope that this new construction should facilitate a comparison between the two definitions.

In Section 2 we recall the definition of Penon weak n-categories, along with the necessary preliminaries. In Section 3 we give the interleaving construction of the left adjoint in the definition of Penon weak n-categories. Finally, in Section 4 we show that our construction gives an explicit description of the n-globular operad whose algebras are Penon weak n-categories; the existence of this operad was proved by Batanin [3].

Throughout the paper we use a variant of Penon's definition of weak n-categories, given in [3, 9]. Penon defined weak n-categories as the algebras for a monad on the category of *reflexive* globular sets (globular sets in which each cell has a putative identity cell at the dimension above). In [9] Cheng and Makkai observed that, in the finite dimensional case, Penon's definition did not encompass certain well-understood examples of weak n-categories, such as braided monoidal categories, but that this could be remedied by using globular sets instead of reflexive globular sets. Note that Penon originally gave his definition in the case $n = \omega$, whereas we take n to be finite (this modification of the definition for finite n is standard, see [12, 9]). Throughout the paper, the letter n always denotes a fixed natural number. It is straightforward to adapt our construction to the case $n = \omega$; we explain how to do this at the end of Section 3.

Acknowledgements

This paper is adapted from material from my PhD thesis, and the research it contains was funded by a University of Sheffield studentship. I would like to thank my supervisor Eugenia Cheng for her invaluable guidance and support. I would also like to thank Nick Gurski, Roald Koudenburg, Jonathan Elliott, Tom Athorne, Alex Corner and Ben Fuller for many useful discussions.

2. Definition of Penon weak n-categories

In this section we recall the non-reflexive variant of Penon's definition of weak n-category [15, 3, 9]. The idea of Penon's definition is to weaken the well-understood notion of strict n-category by means of a "contraction". To do this Penon considers "n-magmas": n-globular sets equipped with binary composition operations that are not required to satisfy any axioms (apart from the usual source and target conditions). He then asks when an n-magma is "coherent enough" to be considered a weak n-category. To answer this question he uses the fact that every strict n-category has an underlying n-magma to compare n-magmas with strict n-categories by considering maps

$$X \xrightarrow{f} S,$$

where X is an n-magma, S is the underlying n-magma of a strict n-category, and f preserves the n-magma structure. Penon defines a notion of a contraction on such a map, which lifts identities in S to equivalences in X, ensuring that the axioms that hold in S hold up to equivalence in X; by analogy with contractions in the topological sense, we can think of the axioms as holding "up to homotopy" in X.

Penon then defines a category whose objects are maps $f \colon X \to S$ as above equipped with contractions; we denote this category by $\mathcal{Q}$, following the notation of Leinster [12]. An object of $\mathcal{Q}$ can be thought of as consisting of an n-magma X and a way of contracting it down to a strict n-category S. There is a forgetful functor $\mathcal{Q} \to n\text{-}\mathbf{GSet}$ sending an object of $\mathcal{Q}$ to the underlying n-globular set of its magma part. This functor has a left adjoint, which induces a monad on n-$\mathbf{GSet}$, and a Penon weak n-category is defined to be an algebra for this monad.

We begin by recalling the definition of an n-globular set, the underlying data for a Penon weak n-category.

Definition 2.1. The *n-globe category* $\mathbb{G}$ is defined as the category with

- objects: natural numbers $0, 1, \ldots, n-1, n$;
- morphisms generated by, for each $1 \leq m \leq n$, morphisms

$$\sigma_m, \tau_m \colon (m-1) \to m$$

  such that $\sigma_{m+1}\sigma_m = \tau_{m+1}\sigma_m$ and $\sigma_{m+1}\tau_m = \tau_{m+1}\tau_m$ for $m \geq 2$ (called the "globularity conditions").

An *n-globular set* is a presheaf on $\mathbb{G}$. We write n-$\mathbf{GSet}$ for the category of n-globular sets $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

For an n-globular set $X \colon \mathbb{G}^{\mathrm{op}} \to \mathbf{Set}$, we write s for $X(\sigma_m)$, and t for $X(\tau_m)$, regardless of the value of m, and refer to them as the source and target maps respectively. We denote the set $X(m)$ by X_m. We say that two m-cells x, $y \in X_m$ are *parallel* if $s(x) = s(y)$ and $t(x) = t(y)$; note that all 0-cells are considered to be parallel.

We now recall the definition of an n-magma, an n-globular set equipped with composition operations.

Definition 2.2. An *n-magma* (or simply *magma*, when n is fixed) consists of an n-globular set X equipped with, for each m, p, with $0 \leq p < m \leq n$, a binary composition function

$$\circ^m_p \colon X_m \times_{X_p} X_m \to X_m,$$

where $X_m \times_{X_p} X_m$ denotes the pullback

$$\begin{tikzcd}
X_m \times_{X_p} X_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "s"] \\
X_m \arrow[r, "t"'] & X_p
\end{tikzcd}$$

in $\mathbf{Set}$; these composition functions must satisfy the following source and target conditions:

- if $p = m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(b \circ_p^m a) = s(a),\ t(b \circ_p^m a) = t(b);$$

- if $p < m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(b \circ_p^m a) = s(b) \circ_p^{m-1} s(a),\ t(b \circ_p^m a) = t(b) \circ_p^{m-1} t(a).$$

A map of n-magmas $f \colon X \to Y$ is a map of the underlying n-globular sets such that, for all m, p, with $0 \leq p < m \leq n$, and for all $(a, b) \in X_m \times_{X_p} X_m$,

$$f(b \circ_p^m a) = f(b) \circ_p^m f(a).$$

We write n-$\mathbf{Mag}$ for the category whose objects are n-magmas and whose morphisms are maps of n-magmas.

Observe that every strict n-category has an underlying n-magma, and we have a forgetful functor

$$n\text{-}\mathbf{Cat} \longrightarrow n\text{-}\mathbf{Mag}.$$

We now recall the definition of a contraction on a map of n-globular sets $f \colon X \to S$, where S is the underlying n-globular set of a strict n-category. Note that this definition does not require a magma structure on X. We must treat dimension n slightly differently, since there is no dimension $n+1$; to do so, we define a notion of a "tame" map of n-globular sets (the terminology is due to Leinster [13, Definition 9.3.1]), which ensures that we have equalities between n-cells where we would normally expect contraction $(n+1)$-cells.

It is common to express the definition of contraction in terms of lifting conditions [3, 4, 10]; however, we express the definition using pullbacks of sets since this approach allows for a straightforward construction of free contractions, which we describe in the next section.

In the following definition, X^c_{m+1} is the set of all pairs of m-cells requiring a contraction $(m+1)$-cell, i.e. the set of all pairs of parallel m-cells on X_m which are mapped by f to the same m-cell in S_m. For any $(a, a) \in X^c_{m+1}$, we write $\gamma_m(a, a) = 1_a$, since it is these contraction cells that give us the identities in a Penon weak n-category.

Definition 2.3. Let $f\colon X \to S$ be a map of n-globular sets, where S is the underlying n-globular set of a strict n-category. The map f is said to be *tame* if, given a, $b \in X_n$, if $s(a) = s(b)$, $t(a) = t(b)$, and $f_n(a) = f_n(b)$, then $a = b$.

For each $0 \leq m < n$, define a set X^c_{m+1} by the following pullback:

$$\begin{tikzcd}
X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,f_m)}"] \\
X_m \arrow[r, "{(s,t,f_m)}"'] & X_{m-1} \times X_{m-1} \times S_m.
\end{tikzcd}$$

Note that when $m = 0$, we take X_{m-1} to be the terminal set.

A *contraction* γ on a tame map $f\colon X \to S$ consists of, for each $0 \leq m < n$, a map

$$\gamma_{m+1}\colon X^c_{m+1} \to X_{m+1}$$

such that, for all $(a, b) \in X^c_{m+1}$,

- $s(\gamma_{m+1}(a, b)) = a$;
- $t(\gamma_{m+1}(a, b)) = b$;
- $f_{m+1}(\gamma_{m+1}(a, b)) = 1_{f_m(a)} = 1_{f_m(b)}$.

Note that we only ever speak of a contraction on a tame map; thus, whenever we state that a map is equipped with a contraction, the map is automatically assumed to be tame. One way to think about this is to say that we do require a contraction $(n+1)$-cell for each pair of n-cells in X^c_n, and the only $(n+1)$-cells in X are equalities.

Penon does not use the term "contraction"; instead, he uses the word "stretching" ("étirement"). This may appear somewhat counterintuitive, as the two words seem antonymous. However, Penon's terminology comes from viewing the same situation from a different point of view; rather than seeing S as a contracted version of X, Penon sees X as a stretched-out version of S. In the case in which X has a magma structure, Penon refers to a such a map as a "categorical stretching" ("étirement catégorique"). Categorical stretchings form a category $\mathcal{Q}$, which we now define.

Definition 2.4. The *category of n-categorical stretchings* $\mathcal{Q}$ is the category with

- objects: an object of $\mathcal{Q}$ consists of an n-magma X, a strict n-category S, and a map of n-magmas

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  equipped with a contraction γ;

- morphisms: a morphism in $\mathcal{Q}$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{Mag}$ such that

  – v is a map of strict n-categories;

  – writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 \leq m < n$, and $(a, b) \in X^c_{m+1}$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

  We denote such a morphism by (u, v).

For an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

of $\mathcal{Q}$, we refer to X as its *magma part* and S as its *strict n-category part*. There is a forgetful functor

$$\begin{array}{ccc} U\colon \mathcal{Q} & \longrightarrow & n\text{-}\mathbf{GSet} \\ \begin{array}{c} X \\ {\scriptstyle f}\big\downarrow \\ S \end{array} & \longmapsto & X \end{array}$$

and this functor has a left adjoint $F\colon n\text{-}\mathbf{GSet} \to \mathcal{Q}$. Penon gives a construction of this left adjoint in the second part of [15].

Definition 2.5. Let P be the monad on $n\text{-}\mathbf{GSet}$ induced by the adjunction $F \dashv U$. A Penon weak n-category is defined to be an algebra for the monad P, and $P\text{-}\mathbf{Alg}$ is the category of Penon weak n-categories.

3. Construction of Penon's left adjoint

In [15] Penon gave a construction of the left adjoint F, mentioned above, using computads (which he called "polygraphs", terminology due to Burroni [6]). In this section we give a new, alternative construction of the functor F, using a monad interleaving construction similar to that used by Cheng to construct the operad for Leinster weak ω-categories [7] (see also [11], which describes a more general interleaving argument).

The first step of our construction is the same as that of Penon. There is a forgetful functor $U_T\colon n\text{-}\mathbf{Cat} \to n\text{-}\mathbf{GSet}$ (the notation U_T is used because $n\text{-}\mathbf{Cat} = T\text{-}\mathbf{Alg}$, where T is the free strict n-category monad on $n\text{-}\mathbf{GSet}$), and we write $\mathcal{R}$ for the comma category

$$n\text{-}\mathbf{GSet} \downarrow U_T.$$

Thus an object of $\mathcal{R}$ is a map of n-globular sets $f\colon X \to S$, where S is the underlying n-globular set of a strict n-category. Since an object of $\mathcal{Q}$ consists of an object $f\colon X \to S$ of $\mathcal{R}$ equipped with a magma structure on X and a contraction on f, we can factorise the forgetful functor $U\colon \mathcal{Q} \to n\text{-}\mathbf{GSet}$ as

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "U"] \arrow[dr, "W"'] & & n\text{-}\mathbf{GSet} \\ & \mathcal{R} \arrow[ur, "V"'] & \end{tikzcd}$$

where W forgets the magma and contraction structures, and V sends an object $f\colon X \to S$ of $\mathcal{R}$ to its n-globular set part X. To construct a left adjoint to U we construct left adjoints to V and W separately. Constructing a left adjoint to V is straightforward: it sends an n-globular set X to $\eta^T_X\colon\ X \to TX$.

We now explain the interleaving argument, which is used to construct the left adjoint to W; this is where our construction differs from that of Penon. In an object of $\mathcal{Q}$ the magma structure and contraction structure exist independently of one another, and there are no axioms governing their interaction. Thus, we can define categories

- $\mathbf{Mag}_n$, whose objects are objects $f\colon X \to S$ of $\mathcal{R}$, together with a magma structure on X, which is respected by f;
- $\mathbf{Contr}_{n+1}$, whose objects are objects $f\colon X \to S$ of $\mathcal{R}$, together with a contraction. (Note that $n+1$ in the superscript here indicates that we have contraction equality $(n+1)$-cells ensuring tameness.)

The maps in these categories are required to respect the magma and contraction structures respectively. We can write the category $\mathcal{Q}$ as the pullback

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Mag}_n \arrow[d, "N"] \\
\mathbf{Contr}_{n+1} \arrow[r, "D"'] & \mathcal{R},
\end{tikzcd}$$

where N and D are the forgetful functors that forget the magma and contraction structures respectively. The functor N has a left adjoint M, which freely adds binary composites, and the functor D has a left adjoint C, which freely adds contraction cells. We wish to combine these left adjoints to obtain a left adjoint to $W\colon\ \mathcal{Q} \to \mathcal{R}$, which adds both the magma and contraction structures freely. However, we can't just add all of one structure, then all of the other, since with this approach we do not end up with enough cells. If we add a contraction structure first, followed by a magma structure, we do not get any contraction cells whose sources or targets are composites, such as unitors and associators. If we add a magma structure first, followed by a contraction structure, we do not get any composites involving contraction cells.

We therefore "interleave" the structures, one dimension at a time. To do so, we make the following observations:

- when we add contraction cells freely, the contraction m-cells depend only on cells at dimension $m-1$;
- when we add composites freely, the composites of m-cells depend only on cells at dimensions m and below.

This means that we can add the contraction cells and composites one dimension at a time; starting with dimension 1, we first add contraction cells freely, then add composites freely; we then move up to the next dimension and repeat the process.

To formalise this, we give separate dimension-by-dimension constructions of both the free contraction structure and the free magma structure, then interleave these constructions by lifting them to the case in which we have both a magma structure and a contraction structure. Thus we obtain a left adjoint to the forgetful functor $W\colon \mathcal{Q} \to \mathcal{R}$; by composing this with the left adjoint to the functor $V\colon \mathcal{R} \to n\text{-}\mathbf{GSet}$ we obtain the left adjoint F to $U\colon \mathcal{Q} \to n\text{-}\mathbf{GSet}$.

Owing to the length of this construction, this section is divided into four subsections. In Subsection 3.1 we construct the left adjoint to V. In Subsections 3.2 and 3.3 we give dimension-by-dimension constructions of the left adjoints to D and N respectively; these describe the free contraction structure and free magma structure. Finally, in Subsection 3.4 we then interleave these constructions to give a left adjoint to W.

3.1 Left adjoint to V

We begin by describing $\mathcal{R}$ explicitly, in order to establish some terminology, and to make clear its connection with $\mathcal{Q}$.

Definition 3.1. Write $\mathcal{R}$ to denote the comma category $n\text{-}\mathbf{GSet} \downarrow U_T$; explicitly, $\mathcal{R}$ is the category with

- objects: an object of $\mathcal{R}$ consists of an n-globular set X, a strict n-

category S, and a map of n-globular sets

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

- morphisms: a morphism in $\mathcal{R}$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in n-$\mathbf{GSet}$ such that v is a map of strict n-categories. We denote such a morphism by (u, v).

As in the case of $\mathcal{Q}$, for an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

we refer to X as its *n-globular set part* and S as its *strict n-category part*.

We have a forgetful functor $W\colon \mathcal{Q} \to \mathcal{R}$, which forgets the contraction and n-magma structures but leaves the underlying map of n-globular sets unchanged, and a forgetful functor $V\colon \mathcal{R} \to n$-$\mathbf{GSet}$, defined by

$$V(\, X \xrightarrow{f} S \,) = X;$$

these compose to give $V \circ W = U$. We construct left adjoints to V and W separately, then compose these to obtain the left adjoint to U. We begin with the construction of the left adjoint to V; we do this in more generality than we require, since this construction is valid for any monad T.

Definition 3.2. Let T be a monad on a category $\mathcal{C}$, and write $U_T\colon T$-$\mathbf{Alg} \to \mathcal{C}$ for the forgetful functor that sends a T-algebra to its underlying object in $\mathcal{C}$. Define a functor $H\colon \mathcal{C} \to \mathcal{C}/U_T$ as follows:

- on objects: for $X \in n$-$\mathbf{GSet}$,

$$H(X) = (\ X \xrightarrow{\eta^T_X} TX\ ),$$

  where TX has the structure of the free T-algebra on X;

- on morphisms: for $f\colon X \to Y$ in $\mathcal{C}$, $Hf = (f, Tf)$.

Proposition 3.3. *Write $V : \mathcal{C}/U_T \to \mathcal{C}$ for the forgetful functor defined by, for an object $f\colon X \to S$ of $\mathcal{C}/U_T$, where S has a T-algebra structure $\theta\colon TS \to S$,*

$$V(\ X \xrightarrow{f} S\ ) = X.$$

Then there is an adjunction $H \dashv V$.

Proof. First, we define the unit $\alpha\colon 1 \Rightarrow VH$ and the counit $\beta\colon HV \Rightarrow 1$. We have $VH = 1$, and we define $\alpha := \mathrm{id}$. To define β, let $f\colon X \to S$ in $\mathcal{C}/U_T$, where S has a T-algebra denoted by $\theta\colon TS \to S$. Observe that θ is a map of T-algebras since, by the algebra axioms, the diagram

$$\begin{tikzcd} T^2S \arrow[r, "T\theta"] \arrow[d, "\mu^T_S"'] & TS \arrow[d, "\theta"] \\ TS \arrow[r, "\theta"'] & S \end{tikzcd}$$

commutes. The component of β at $f\colon X \to S$, denoted β_f, is given by the commuting diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "f"] \arrow[dd, "\eta^T_X"'] & & X \arrow[dd, "f"] \\ & S \arrow[d, "\eta^T_S"] \arrow[dr, "\mathrm{id}_S"] & \\ TX \arrow[r, "Tf"'] & TS \arrow[r, "\theta"'] & S, \end{tikzcd}$$

as a map in $\mathcal{C}/U_T$. This diagram commutes since the left-hand square is a naturality square for η and the bottom-right triangle is the unit axiom for the algebra θ; the remaining square commutes trivially.

We now show that α and β satisfy the triangle identities. First, consider

$$\begin{tikzcd} V \arrow[r, "\alpha V"] \arrow[dr, "1_V"'] & VHV \arrow[d, "V\beta"] \\ & V. \end{tikzcd}$$

For $f\colon X \to S$ in $\mathcal{R}$,

$$V(\ X \xrightarrow{f} S\ ) = X = VHV(\ X \xrightarrow{f} S\ ),$$

$\alpha_X = 1_X$, and $U\beta_f = 1_X$, so this diagram commutes.

Now consider

$$\begin{tikzcd} H \arrow[r, "H\alpha"] \arrow[dr, "1_H"'] & HVH \arrow[d, "\beta H"] \\ & H. \end{tikzcd}$$

For $X \in \mathcal{C}$,

$$H(X) = (\ X \xrightarrow{\eta^T_X} TX\ ) = HVH(X),$$

$H\alpha_X = H\mathrm{id}_X = (\mathrm{id}_X, \mathrm{id}_{TX})$, and $\beta_{HX} = (\mathrm{id}, \mu_X \circ T\eta_X) = (\mathrm{id}_X, \mathrm{id}_{TX})$, so this diagram commutes. $\square$

This gives us the left adjoint to the functor $V\colon \mathcal{R} \to n\text{-}\mathbf{GSet}$.

3.2 Free contraction structure

We now construct the free contraction on an object of $\mathcal{R}$. In order to be able to use the construction in the interleaving argument in Section 3.4, we give the construction one dimension at a time. To do so, we define, for each $0 \le k \le n+1$, a category $\mathbf{Contr}_k$, an object of which consists of an object of $\mathcal{R}$ equipped with a contraction up to dimension k. (Observe that $\mathbf{Contr}_0 = \mathcal{R}$, and note that a "contraction at dimension $n+1$" refers to the tameness condition at dimension n.) We then have, for each $0 < k \le n+1$, a forgetful functor

$$D_k\colon \mathbf{Contr}_k \to \mathbf{Contr}_{k-1}.$$

We construct a left adjoint to each D_k, which freely adds a contraction structure at dimension k, leaving all other dimensions unchanged.

Definition 3.4. Let $f\colon X \to S$ be a map of n-globular sets, where S is the underlying n-globular set of a strict n-category, and let $0 \le k \le n$. Recall from Definition 2.3 that, for each $0 \le m < n$, the set X^c_{m+1} is defined by the pullback

$$\begin{tikzcd} X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,f_m)}"] \\ X_m \arrow[r, "{(s,t,f_m)}"'] & X_{m-1} \times X_{m-1} \times S_m. \end{tikzcd}$$

where, when $m = 0$, we take X_{m-1} to be the terminal set.

A k-contraction γ on the map f consists of, for each $0 \le m < k$, a map

$$\gamma_{m+1}\colon X^c_{m+1} \to X_{m+1}$$

such that, for $(a, b) \in X^c_m$,

$$s(\gamma_{m+1}(a, b)) = a,$$

$$t(\gamma_{m+1}(a, b)) = b,$$

$$f_{m+1}(\gamma_{m+1}(a, b)) = \mathrm{id}_{f(a)}.$$

Note that having an n-contraction on a map f is not the same as having contraction on f; for a contraction on f, we also require that f is tame (see Definition 2.3).

Definition 3.5. For each $0 \le k \le n$, define a category $\mathbf{Contr}_k$, with

- objects: an object of $\mathbf{Contr}_k$ consists of an n-globular set X, a strict n-category S, and a map of n-globular sets

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  equipped with a k-contraction γ;

- morphisms: a morphism in $\mathbf{Contr}_k$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{GSet}$ such that

  – v is a map of strict n-categories;

  – writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq k$, and $(a, b) \in X^c_m$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

Define a category $\mathbf{Contr}_{n+1}$, with

- objects: an object of $\mathbf{Contr}_{n+1}$ consists of an n-magma X, a strict n-category S, and a map of n-magmas

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  equipped with a contraction γ;

- morphisms: a morphism in $\mathbf{Contr}_{n+1}$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{GSet}$ such that

  – v is a map of strict n-categories;

– writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq n$, and $(a, b) \in X^c_m$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

For all $0 < k \leq n + 1$, we have a forgetful functor

$$D_k \colon \mathbf{Contr}_k \to \mathbf{Contr}_{k-1};$$

for $0 < k \leq n$, this functor forgets the contraction at dimension k, and for $k = n + 1$ it is the inclusion functor of the subcategory $\mathbf{Contr}_{n+1}$ into $\mathbf{Contr}_n$.

We now define a putative left adjoint C_k to the functor D_k; we will then prove that this functor is left adjoint to D_k in Proposition 3.7.

Definition 3.6. For each k, $0 < k \leq n$, we define a functor

$$C_k \colon \mathbf{Contr}_{k-1} \to \mathbf{Contr}_k.$$

We begin by giving the action of C_k on objects. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_{k-1}$, and write γ for its $(k-1)$-contraction (assuming $k > 1$; if $k = 1$, we have $\mathbf{Contr}_{k-1} = \mathbf{Contr}_0 = \mathcal{R}$, so there is no contraction on f). We define an object

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Contr}_k$, with k-contraction $\tilde{\gamma}$. The n-globular set $\tilde{X}$ is defined by:

- $\tilde{X}_j = X_j$ for all $j \neq k$;

- $\tilde{X}_k = X_k \amalg X_k^c$,
- for $(x, y) \in X_k^c \subseteq \tilde{X}_k$, $s(x, y) = x$, $t(x, y) = y$,
- for all other cells, sources and targets are inherited from X.

The map $\tilde{f} \colon \tilde{X} \to S$ is defined by

- $\tilde{f}_j = f_j$ for all $j \neq k$;
- $\tilde{f}_k \colon \tilde{X}_k \to S_k$ is defined by
  - $\tilde{f}_k(\alpha) = f_k(\alpha)$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $\tilde{f}_k(x, y) = 1_{f_{k-1}(x)}$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

The k-contraction $\tilde{\gamma}$ on $\tilde{f}$ is defined by

- $\tilde{\gamma}_m = \gamma_m^{k-1}$ for all $m < k - 1$;
- $\tilde{\gamma}_{k-1} \colon X_k^c \to \tilde{X}_k$ is the inclusion into the coproduct $\tilde{X}_k = X_k \amalg X_k^c$.

This defines the action of C_k on objects.

We now give the action of C_k on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Contr}_{k-1}$. Define a morphism

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{f}"'] & \tilde{Y} \arrow[d, "\tilde{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Contr}_k$, where $\tilde{u}$ is defined by

- $\tilde{u}_j = u_j$ for all $j \neq k$;

- $\tilde{u}_k \colon \tilde{X}_k \to \tilde{Y}_k$ is given by
  - $\tilde{u}_k(\alpha) = u_k(\alpha)$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $\tilde{u}_k(x, y) = (u_{k-1}(x), u_{k-1}(y))$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

This defines the action of C_k on morphisms.

Proposition 3.7. *For all $0 < k \leq n$, there is an adjunction $C_k \dashv D_k$.*

Proof. We first define the unit $\eta \colon 1 \Rightarrow D_k C_k$, and counit $\epsilon : C_k D_k \Rightarrow 1$.

Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_{k-1}$, with $(k-1)$-contraction γ (assuming $k > 1$; if $k = 1$, we have $\mathbf{Contr}_{k-1} = \mathbf{Contr}_0 = \mathcal{R}$, so there is no contraction on f). Applying $D_k C_k$ gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{k-1}$ with the same $(k-1)$-contraction. The corresponding component of the unit η is given by the map

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_j = \begin{cases} 1_{X_j} & \text{if } j \neq k, \\ \text{the inclusion } X_j \hookrightarrow X_j \amalg X_j^c & \text{if } j = k. \end{cases}$$

Now let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_k$, with k-contraction γ. Applying C_kD_k gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_k$ with k-contraction $\tilde{\gamma}$, which is equal to γ at all dimensions except k. The corresponding component of the counit ϵ is given by the map

$$\begin{tikzcd} \tilde{X} \arrow[r, "\epsilon_X"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where ϵ_X is defined by

- $(\epsilon_X)_j = 1_{X_j}$ for all $j \neq k$;
- $(\epsilon_X)_k \colon \tilde{X}_k \to X_k$ is given by
  - $(\epsilon_X)_k(\alpha) = \alpha$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $(\epsilon_X)_k(x, y) = \tilde{\gamma}_k(x, y)$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

We now check that the triangle identities hold; consider the diagrams

$$\begin{tikzcd} D_k \arrow[r, "\eta D_k"] \arrow[dr, "1"'] & D_kC_kD_k \arrow[d, "D_k\epsilon"] \\ & D_k, \end{tikzcd} \qquad \begin{tikzcd} C_k \arrow[r, "C_k\eta"] \arrow[dr, "1"'] & C_kD_kC_k \arrow[d, "\epsilon C_k"] \\ & C_k. \end{tikzcd}$$

In all of the natural transformations in these diagrams, the components on the strict n-category parts are all identities, so to show that the diagrams commute we need only consider the components on the n-globular set parts. Since the components of the maps of n-globular sets are identities at every dimension except dimension k, we need only check that the corresponding diagrams of maps of sets of k-cells commute.

First, we must show that, given

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_k$, the diagram

$$\begin{tikzcd} X_k \arrow[r, "(\eta_X)_k"] \arrow[dr, "1_{X_k}"'] & \tilde{X}_k = X_k \amalg X^c_{k-1} \arrow[d, "(\epsilon_X)_k"] \\ & X_k \end{tikzcd}$$

commutes; this is true, since given $\alpha \in X_k$, we have

$$(\epsilon_X)_k \circ (\eta_X)_k(\alpha) = (\epsilon_X)_k(\alpha) = \alpha.$$

Secondly, we must show that, given

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{k-1}$ with $(k-1)$-contraction γ, the diagram

$$\begin{tikzcd} \tilde{X}_k \arrow[r, "(\tilde{\eta}_X)_k"] \arrow[dr, "1_{\tilde{X}_k}"'] & \tilde{X}_k \amalg \tilde{X}^c_k \arrow[d, "(\epsilon_{\tilde{X}})_k"] \\ & \tilde{X}_k \end{tikzcd}$$

commutes. We have two kinds of freely added contraction cells in $\tilde{X}_k \amalg \tilde{X}^c_k$; we write (x, y) for the contraction cells in X^c_k, and $[x, y]$ for those in $\tilde{X}^c_k$ (the latter being the specified contraction cells in this case). Given $\alpha \in X_k \subseteq \tilde{X}_k$,

$$(\epsilon_{\tilde{X}})_k \circ (\tilde{\eta}_X)_k(\alpha) = (\epsilon_{\tilde{X}})_k(\alpha) = \alpha;$$

given $(x, y) \in X^c_k \subseteq \tilde{X}_k$,

$$(\epsilon_{\tilde{X}})_k \circ (\tilde{\eta}_X)_k(x, y) = (\epsilon_{\tilde{X}})_k[x, y] = (x, y);$$

hence the diagram commutes.

Thus the triangle identities hold, and we have an adjunction $C_k \dashv D_k$, with unit η and counit ϵ. $\square$

We must also define C_{n+1} separately, since "adding contraction $(n+1)$-cells" consists of identifying certain n-cells rather than actually adding cells; we can think of this as adding equality $(n+1)$-cells between pairs of n-cells that would usually require a contraction cell between them.

Definition 3.8. We define a functor

$$C_{n+1} \colon \mathbf{Contr}_n \to \mathbf{Contr}_{n+1}.$$

We begin by giving the effect of C_{n+1} on objects. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_n$, with n-contraction γ. Define a set X^c_{n+1} and maps $\pi_1, \pi_2 \colon X^c_{n+1} \to X_n$ by the following pullback:

$$\begin{tikzcd} X^c_{n+1} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "{(s,t,f_n)}"] \\ X_n \arrow[r, "{(s,t,f_n)}"'] & X_{n-1} \times X_{n-1} \times S_n. \end{tikzcd}$$

The set X^c_{n+1} can be thought of as the set of pairs of n-cells to be identified, but note that there is some redundancy: for all $a \in X_n$, $(a, a) \in X^c_{n+1}$, and if we have $(a, b) \in X^c_{n+1}$ we also have $(b, a) \in X^c_{n+1}$.

We now define an object

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Contr}_{n+1}$ with contraction $\tilde{\gamma}$. For $0 \leq m < n$, define

$$\tilde{X}_m = X_m,$$

and define $\tilde{X}_n$ to be the coequaliser of the diagram

$$X^c_{n+1} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X_n.$$

For $0 \leq m < n$, define

$$\tilde{f}_m = f_m,$$

and define $\tilde{f}_n \colon \tilde{X}_n \to S_n$ to be the unique map such that

$$\begin{tikzcd} X^c_{n+1} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_n \arrow[r, "q"] \arrow[dr, "f_n"'] & \tilde{X}_n \arrow[d, dashed, "\tilde{f}_n"] \\ & & S_n \end{tikzcd}$$

commutes, where $q \colon X_n \to \tilde{X}_n$ is the coequaliser map. Finally, define $\tilde{\gamma}$ to be the n-contraction defined by

$$\tilde{\gamma}_m = \begin{cases} \gamma_m & \text{if } m < n, \\ q \circ \gamma_n & \text{if } m = n. \end{cases}$$

This defines the action of C_{n+1} on objects.

We now give the action of C_{n+1} on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Contr}_n$. Define a morphism

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{f}"'] & \tilde{Y} \arrow[d, "\tilde{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Contr}_{n+1}$, where, for $0 \leq m < n$,

$$\tilde{u}_m = u_m,$$

and $\tilde{u}_n \colon \tilde{X}_n \to \tilde{Y}_n$ is defined to be the unique map such that the diagram

$$\begin{tikzcd} X^c_{n+1} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_n \arrow[r, "q"] \arrow[d, "u_n"'] & \tilde{X}_n \arrow[d, dashed, "\tilde{u}_n"] \\ & Y_n \arrow[r, "p"'] & \tilde{Y}_n \end{tikzcd}$$

commutes, where p is the coequaliser map for $\tilde{Y}_n$. This defines the action of C_{n+1} on morphisms.

Proposition 3.9. *There is an adjunction $C_{n+1} \dashv D_{n+1}$.*

Proof. We first define the unit $\eta \colon 1 \Rightarrow D_{n+1}C_{n+1}$. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_n$. Applying $D_{n+1}C_{n+1}$ gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{n+1}$. The corresponding component of the unit η is given by the map

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_j = \begin{cases} 1_{X_j} & \text{if } j < n, \\ \text{the coequaliser map } q\colon X_n \to \tilde{X}_n & \text{if } j = n. \end{cases}$$

For the counit, observe that $C_{n+1}D_{n+1} = \mathrm{id}$, and if

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

is in the image of D_{n+1}, $\tilde{X} = X$ and $q = \mathrm{id}_X$, so $\eta = \mathrm{id}$. We define the counit

$$\epsilon : C_{n+1}D_{n+1} \Rightarrow 1$$

to be the identity. Thus all maps appearing in the diagrams for the triangle identities are identity maps, so both diagrams commute. Hence there is an adjunction $C_{n+1} \dashv D_{n+1}$. □

Thus Definitions 3.6 and 3.8 give us a dimension-by-dimension construction of the free contraction on an object of $\mathcal{R}$.

3.3 Free magma structure

We now construct the free n-magma on the source of an object of $\mathcal{R}$. As with the construction of the free contraction in the previous subsection, in order to be able to use the construction in the interleaving argument in Subsection 3.4, we give the construction one dimension at a time. To do so we define, for each $0 \leq j \leq n$, a category $\mathbf{Mag}_j$, an object of which consists of an object of $\mathcal{R}$ in which the source is equipped with a magma structure up to dimension j. (Observe that $\mathbf{Mag}_0 = \mathcal{R}$.) We then have, for each $0 < j \leq n$, a forgetful functor

$$N_j \colon \mathbf{Mag}_j \to \mathbf{Mag}_{j-1}.$$

We construct a left adjoint to each N_j, which freely adds a magma structure at dimension j, leaving all other dimensions unchanged.

In order to define what it means for an n-globular set to have a j-magma structure, we use the j-truncation functor

$$\mathrm{Tr}_j\colon n\text{-}\mathbf{GSet} \longrightarrow j\text{-}\mathbf{GSet},$$

which forgets the sets of m-cells for all $m > j$, and, for $m \leq j$, leaves the sets of m-cells and their source and target maps unchanged; the action on maps is defined similarly.

Definition 3.10. Define a category $\mathbf{Mag}_j$, with

- objects: an object of $\mathbf{Mag}_j$ consists of an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  in $\mathcal{R}$ such that $\mathrm{Tr}_j X$ is a j-magma, and $\mathrm{Tr}_j f$ is a map of j-magmas.

- morphisms: a morphism in $\mathbf{Mag}_j$ is a morphism

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in $\mathcal{R}$ such that $\mathrm{Tr}_j u$ is a map of j-magmas.

We can express the category $\mathbf{Mag}_j$ as a pullback. For any $j \in \mathbb{N}$ we have a commuting triangle of forgetful functors

$$\begin{tikzcd} j\text{-}\mathbf{Cat} \arrow[rr, "G"] \arrow[dr, "U_{T_j}"'] & & j\text{-}\mathbf{Mag} \arrow[dl, "E"] \\ & j\text{-}\mathbf{GSet} & \end{tikzcd}$$

in $\mathbf{CAT}$, where T_j is the free strict j-category monad (and thus $j\text{-}\mathbf{Cat} = T_J\text{-}\mathbf{Alg}$). We can then write $\mathbf{Mag}_j$ as the pullback

$$\begin{tikzcd} \mathbf{Mag}_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & n\text{-}\mathbf{GSet} \downarrow U_T \arrow[d, "\mathrm{Tr}_j"] \\ j\text{-}\mathbf{Mag} \downarrow G \arrow[r, "E"'] & j\text{-}\mathbf{GSet} \downarrow U_{T_j} \end{tikzcd}$$

For all $0 < j \leq n$, we have a forgetful functor

$$N_j \colon \mathbf{Mag}_j \to \mathbf{Mag}_{j-1},$$

which forgets the composition maps for j-cells. We will define, for each $0 < j \leq n$, a functor

$$M_j \colon \mathbf{Mag}_{j-1} \to \mathbf{Mag}_j$$

which freely adds binary composites at dimension j, taking an n-globular set equipped with a $(j-1)$-magma structure and adding a magma structure at dimension j to give an n-globular set equipped with a j-magma structure. We will then show that the functor M_j is left adjoint to the forgetful functor N_j.

Before defining M_j, we first fix some notation that will be used in the construction of the free binary composites. Let X be an n-globular set equipped with a $(j-1)$-magma structure. For each $0 \leq p < j$, we can form the set of pairs of j-cells that are composable along p-cells using the following pullback:

$$\begin{tikzcd} X_j \times_{X_p} X_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_j \arrow[d, "s"] \\ X_j \arrow[r, "t"'] & X_p. \end{tikzcd}$$

We view $X_j \times_{X_p} X_j$ as the set of freely generated binary composites of j-cells along p-cells. We can form the set of freely generated binary composites of j-cells along boundaries of all dimensions by taking the coproduct of these sets over p. As the notation will become somewhat complicated in the definition of the left adjoint to N_j, we use the following shorthand:

$$X_j^2 := \coprod_{0 \leq p < j} X_j \times_{X_p} X_j.$$

This set comes equipped with source and target maps into X_{j-1}, which are defined in analogy with the sources and targets of composites in a magma structure from Definition 2.2, as follows:

- if $p = m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(a, b) = s(a),$$
$$t(a, b) = t(b)$$

- if $p < m-1$, given $(a,b) \in X_m \times_{X_p} X_m$,

$$s(a,b) = s(b) \circ_p^{m-1} s(a),$$

$$t(a,b) = t(b) \circ_p^{m-1} t(a).$$

The set X_j^2 contains only binary composites of "depth 1"; that is, it contains binary composites of pairs of j-cells in X, but it does not contain binary composites of binary composites, binary composites of binary composites of binary composites, etc. In order to obtain these composites of greater "depth", which we require in the free magma structure, we must iterate this process. To do so we define, for each $k \geq 0$, a set $X_j^{(k)}$ of composites of depth at most k. We have inclusion maps

$$X_j^{(k)} \hookrightarrow X_j^{(k+1)},$$

so this gives a sequence of sets; we take the colimit of this sequence to obtain the set of freely generated binary composites of all depths. We now describe and illustrate this iterative process for low depths of composite ($k \leq 2$).

When $k = 0$, we define

$$X_j^{(0)} = X_j,$$

with source and target maps $s, t \colon X_j^{(0)} \to X_{j-1}$ given by those in X.

When $k = 1$, we define

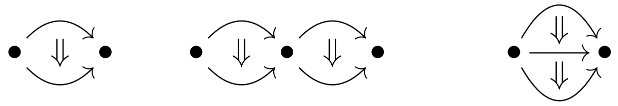
$$X_j^{(1)} := X_j + \left(X_j^{(0)}\right)^2 = X_j + X_j^2,$$

where the notation X_j^2 is shorthand, as described earlier. The set X_j^2 inherits source and target maps from $X_j^{(0)}$, so we have source and target maps

$$s, t \colon X_j^{(1)} \longrightarrow X_{j-1}$$

inherited from those for X_j and X_j^2. To see how this gives the set of composites of depth at most 1, we consider the case $j = 2$. By "expanding out" X_2^2, we see that $X_2^{(1)}$ contains the following shapes of composites:

$$X_2^{(1)} = \quad X_2 \quad + \quad X_2 \times_{X_0} X_2 \quad + \quad X_2 \times_{X_1} X_2$$



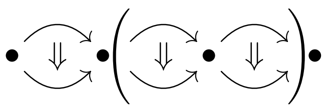
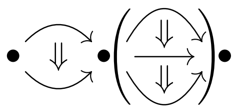
When $k = 2$, we define

$$X_j^{(2)} := X_j + \left(X_j^{(1)}\right)^2.$$

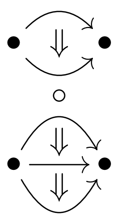
As in the case $k = 1$, this comes equipped with source and target maps. In the case $j = 2$, "expanding out" $\left(X_j^{(1)}\right)^2$ gives

$$\begin{aligned} X_2^{(2)} = \;& X_2 \;+\; X_2^2 \;+\; X_2 \times_{X_0} X_2^2 \;+\; X_2 \times_{X_1} X_2^2 \;+\; X_2^2 \times_{X_0} X_2 \\ &+\; X_2^2 \times_{X_1} X_2 \;+\; X_2^2 \times_{X_0} X_2^2 \;+\; X_2^2 \times_{X_1} X_2^2. \end{aligned}$$

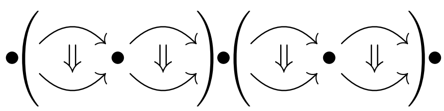
Thus $X_2^{(2)}$ contains the same shapes of composites that appear in $X_2^{(1)}$, as well as those composites of depth 2: in $X_2 \times_{X_0} X_2^2$ we have composites of the following shapes:

 and ;

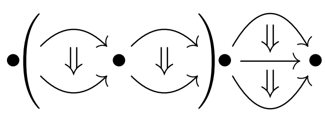
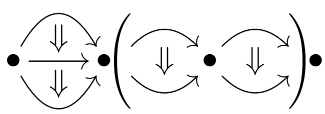
in $X_2 \times_{X_1} X_2^2$ we have composites of the following shape:

;

the shapes of composites in $X_2^2 \times_{X_0} X_2$ and $X_2^2 \times_{X_1} X_2$ are similar to those above; in $X_2^2 \times_{X_0} X_2^2$ we have composites of the following shapes:



and also

 and ;

and finally, in $X_2^2 \times_{X_1} X_2^2$ we have composites of the following shapes:

$$\begin{tikzcd}[row sep=small] \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet \\ \circ \\ \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[row sep=small] \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] & \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] & \bullet \\ & \circ & \\ \bullet \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet. \end{tikzcd}$$

Thus $X_2^{(2)}$ contains all binary composites of 2-cells of depth at most 2.

Since the construction of the the free j-magma structure consists of taking pullbacks and filtered colimits of sets, in order to define the composition maps at dimension j we require the following lemma due to Mac Lane [14, Theorem IX.2.1], which states that finite limits commute with filtered colimits in $\mathbf{Set}$. Note that this theorem still holds if $\mathbf{Set}$ is replaced by any locally finitely presentable category; see [1, Proposition 1.59].

Lemma 3.11 (Mac Lane)**.** *Let $\mathbb{I}$ be a finite category, and let $\mathbb{J}$ be a small, filtered category. Then for any bifunctor*

$$F\colon \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$$

the canonical arrow

$$\operatorname*{colim}_{j \in \mathbb{J}} \lim_{i \in \mathbb{I}} F(i, j) \longrightarrow \lim_{i \in \mathbb{I}} \operatorname*{colim}_{j \in \mathbb{J}} F(i, j)$$

is an isomorphism.

We now define a putative left adjoint M_j to the functor N_j; we will then prove that this functor is left adjoint to N_j in Proposition 3.13.

Definition 3.12. For each $0 < j \leq n$, we define a functor

$$M_j \colon \mathbf{Mag}_{j-1} \to \mathbf{Mag}_j.$$

We begin by giving the action of M_j on objects. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Mag}_{j-1}$. We will define an object

$$\begin{tikzcd} \widehat{X} \arrow[d, "\hat{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Mag}_j$, where $\widehat{X}$ differs from X only at dimension j. The set $\widehat{X}_j$ of j-cells of $\widehat{X}$ is the set of freely generated binary composites of j-cells of X. We define this as the colimit of a sequence of sets $X_j^{(k)}$, where $X_j^{(k)}$ is the set of freely generated binary composites of j-cells of X of depth at most k. We define $X_j^{(k)}$ by induction over k, as follows: when $k = 0$, define

$$X_j^{(0)} = X_j,$$

with source and target maps s, $t\colon X_j^{(0)} \to X_{j-1}$ given by those in X. Now suppose that $k > 0$ and we have defined $X_j^{(k-1)}$, equipped with source and target maps

$$s, t\colon X_j^{(k-1)} \longrightarrow X_{j-1}.$$

We define $X_j^{(k)}$ by

$$X_j^{(k)} := X_j + \left(X_j^{(k-1)}\right)^2.$$

Recall that the notation used above is shorthand, defined by

$$\left(X_j^{(k-1)}\right)^2 := \coprod_{0 \leq p < j} X_j^{(k-1)} \times_{X_p} X_j^{(k-1)},$$

and that this set inherits source and target maps from $X_j^{(k-1)}$. Thus we have source and target maps

$$s, t\colon X_j^{(k)} \longrightarrow X_{j-1}$$

inherited from those for X_j and $\left(X_j^{(k-1)}\right)^2$.

For each $k \geq 0$, we define a map

$$i^{(k)}\colon X_j^{(k)} \to X_j^{(k+1)},$$

which includes the freely generated composites in $X_j^{(k)}$ (those of depth at most k) into the set $X_j^{(k+1)}$ (which contains composites of depth at most $k+1$), and leaves the generating cells unchanged. The maps $i^{(k)}$ are defined by induction over k, as follows:

- for $k = 0$, $i^{(0)}$ is the coprojection map
$$i^{(0)} \colon X_j \to X_j + X_j^2;$$
- for $k \geq 1$, suppose we have defined $i^{(k-1)} \colon X_j^{(k-1)} \to X_j^{(k)}$. We define $i^{(k)}$ to be the map
$$i^{(k)} := 1_{X_j} + \coprod_{0 \leq p < j} \left(i^{(k-1)}, i^{(k-1)}\right) : X_j + \left(X_j^{(k-1)}\right)^2 \to X_j + \left(X_j^{(k)}\right)^2.$$

These sets and maps give us a diagram

$$X_j^{(0)} \xrightarrow{i^{(0)}} X_j^{(1)} \xrightarrow{i^{(1)}} X_j^{(2)} \xrightarrow{i^{(2)}} X_j^{(3)} \xrightarrow{i^{(3)}} \ldots$$

in $\mathbf{Set}$; we then define
$$\widehat{X}_j := \operatorname*{colim}_{k \geq 0} X_j^{(k)}.$$
For $m \neq j$, we define
$$\widehat{X}_m := X_m.$$
For $m \neq j, j+1$, the source and target maps
$$s, t \colon \widehat{X}_m \to \widehat{X}_{m-1}$$
are those inherited from X. Now write $c_j^{(k)} \colon X_j^{(k)} \to \widehat{X}_j$ for the coprojection maps. The source and target maps for $m = j+1$ are given by the composites

$$\widehat{X}_{j+1} = X_{j+1} \xrightarrow{s} X_j \xrightarrow{c_j^{(0)}} \widehat{X}_j$$

and

$$\widehat{X}_{j+1} = X_{j+1} \xrightarrow{t} X_j \xrightarrow{c_j^{(0)}} \widehat{X}_j$$

respectively. To define the source and target maps for $m = j$, recall that, for each k, we have source and target maps $s, t\colon X_j^{(k)} \to X_{j-1}$; we define $s, t\colon \widehat{X}_j \to X_{j-1}$ to be the unique maps induced by the colimit defining $\widehat{X}_j$ that make, for all $k \geq 1$, the diagrams

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "s"'] & \widehat{X}_j \arrow[d, dashed, "s"] \\ & X_{j-1} \end{tikzcd} \qquad \begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "t"'] & \widehat{X}_j \arrow[d, dashed, "t"] \\ & X_{j-1} \end{tikzcd}$$

commute respectively.

We now define the j-magma structure on $\widehat{X}$. For all $m < j$, and for all $0 \leq p < m$, the composition map

$$\circ_p^m\colon \widehat{X}_m \times_{\widehat{X}_p} \widehat{X}_m = X_m \times_{X_p} X_m \to X_m$$

is the corresponding composition map from the $(j-1)$-magma structure on X. To define the composition map $\circ_p^j$ for $0 \leq p < j$, we begin by observing that, by Lemma 3.11, we have an isomorphism

$$\operatorname*{colim}_{k,l\geq 0}\left(X_j^{(k)} \times_{X_p} X_j^{(l)}\right) \cong \left(\operatorname*{colim}_{k\geq 0} X_j^{(k)}\right) \times_{X_p} \left(\operatorname*{colim}_{l\geq 0} X_j^{(l)}\right) = \widehat{X}_j \times_{X_p} \widehat{X}_j.$$

Thus, to define the composition maps at dimension j, we define, for each $k, l > 0$, $0 \leq p < j$, a map

$$\circ_p^j\colon X_j^{(k)} \times_{X_p} X_j^{(l)} \to \widehat{X}_j.$$

To do so, observe that, in the case $k = l$, the source of the composition map above includes in $X_j^{(k+1)}$, which in turn includes in $\widehat{X}_j$; thus in this case we define the composition map to be the composite:

$$X_j^{(k)} \times_{X_p} X_j^{(k)} \hookrightarrow X_j^{(k+1)} \xrightarrow{c_j^{(k+1)}} \widehat{X}_j.$$

Now suppose that $k < l$; in this case we first include the source of the composition map in

$$X_j^{(l)} \times_{X_p} X_j^{(l)},$$

and we can then follow the same method as for $k = l$. Write

$$i^{(k,l)} := i^{(l)} \circ i^{(l-1)} \circ \cdots \circ i^{(k)} \colon X_j^{(k)} \longrightarrow X_j^{(l)},$$

and define $\circ_p^j$ to be the composite

$$X_j^{(k)} \times_{X_p} X_j^{(l)} \xrightarrow{(i^{(k,l)},\mathrm{id})} X_j^{(l)} \times_{X_p} X_j^{(l)} \hookrightarrow X_j^{(l+1)} \xrightarrow{c_j^{(l+1)}} \widehat{X}_j,$$

where the second map is the coprojection into the coproduct defining $X_j^{(l+1)}$. Similarly, for $l > k$, we define $\circ_p^j$ to be the composite

$$X_j^{(k)} \times_{X_p} X_j^{(l)} \xrightarrow{(\mathrm{id},i^{(k,l)})} X_j^{(k)} \times_{X_p} X_j^{(k)} \hookrightarrow X_j^{(k+1)} \xrightarrow{c_j^{(k+1)}} \widehat{X}_j,$$

Then $\circ_p^j \colon \widehat{X}_j \times_{X_p} \widehat{X}_j \to \widehat{X}_j$ is defined to be the unique map induced by universal property of

$$\widehat{X}_j \times_{X_p} \widehat{X}_j$$

as a colimit (using Lemma 3.11) such that, for all k, $l > 0$, the diagram

$$\begin{tikzcd}
X_j^{(k)} \times_{X_p} X_j^{(l)} \arrow[r, "\left(c_j^{(k)}, c_j^{(l)}\right)"] \arrow[dr, "\circ_p^j"'] & \widehat{X}_j \times_{X_p} \widehat{X}_j \arrow[d, dashed, "\circ_p^j"] \\
 & \widehat{X}_j
\end{tikzcd}$$

commutes. This defines a j-magma structure on $\widehat{X}$.

We now define the map $\hat{f} \colon \widehat{X} \to S$. At dimension j, $\hat{f}$ acts on a freely generated composite in $\widehat{X}_j$ by first applying f to each individual generating j-cell in the composite, then evaluating this composite via the magma structure on S; at all other dimensions it is the same as the map f.

For $m \neq j$, define

$$\hat{f}_m = f_m \colon \widehat{X}_m = X_m \to S_m.$$

To define $\hat{f}_m$ for $m = j$, we first define, for each $k \geq 0$, a map

$$f_j^{(k)} \colon X_j^{(k)} \to S_j.$$

When $k = 0$, define
$$f_j^{(k)} = f_j \colon X_j^{(0)} \to S_j.$$
Now let $k \geq 1$ and suppose we have defined the map
$$f_j^{(k-1)} \colon X_j^{(k-1)} \to S_j;$$
we define the map
$$f_j^{(k)} \colon X_j^{(k)} \to S_j$$
as follows: for each $0 \leq p < j$ there is a map
$$\left(f_j^{(k-1)}, f_j^{(k-1)}\right) : X_j^{(k-1)} \times_{X_p} X_j^{(k-1)} \to S_j \times_{S_p} S_j$$
induced by the universal property of $S_j \times_{S_p} S_j$. We compose each of these with the composition map $\circ_p^j$, and define $f_j^{(k)} \colon X_j^{(k)} \to S_j$ to be a coproduct of these composites, as follows:
$$\begin{aligned} f_j^{(k)} :=& f_j^{(0)} + \coprod_{0 \leq p < j} \left((\circ_p^j) \circ \left(f_j^{(k-1)}, f_j^{(k-1)}\right)\right) : \\ & X_j^{(k)} = X_j + \coprod_{0 \leq p < j} X_j^{(k-1)} \times_{X_p} X_j^{(k-1)} \to S_j. \end{aligned}$$

We then define $\hat{f}_j$ to be the unique map such that, for all $k \geq 1$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "f_j^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "\hat{f}_j"] \\ & S_j \end{tikzcd}$$

commutes.

Thus we have defined an object

$$\begin{tikzcd} \widehat{X} \arrow[d, "\hat{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Mag}_j$; this gives the action of M_j on objects.

We now give the action of M_j on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Mag}_{j-1}$. We define a morphism

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{u}"] \arrow[d, "\hat{f}"'] & \hat{Y} \arrow[d, "\hat{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Mag}_j$. At dimension j, the map $\hat{u}$ acts on a freely generated composite in $\hat{X}$ by applying u to each individual generating j-cell in the composite, thus giving a freely generated composite of j-cells in $\hat{Y}_j$; at all other dimensions it is the same as the map u. The construction of $\hat{u}$ is very similar to that of $\hat{f}$.

For $m \neq j$, we define $\hat{u}_m = u_m$. To define $\hat{u}_m$ for $m = j$, first we define, for each $k \geq 1$, a map

$$u_j^{(k)} \colon X_j^{(k)} \to Y_j^{(k)}.$$

When $k = 0$, define

$$u_j^{(1)} := u_j \colon X_j^{(1)} \to Y_j^{(1)}.$$

Now let $k \geq 1$ and suppose we have defined

$$u_j^{(k-1)} := u_j \colon X_j^{(k-1)} \to Y_j^{(k-1)};$$

we define $u_j^{(k)}$ as follows:

$$u_j^{(k)} := u_j^{(k-1)} + \coprod_{0 \leq p < j} \left(u_j^{(k-1)}, u_j^{(k-1)} \right) \colon X_j^{(k)} \to Y_j^{(k)}.$$

We then define $\hat{u}_j$ to be the unique map such that, for all $k \geq 1$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[d, "u_j^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "\hat{u}_j"] \\ Y_j^{(k)} \arrow[r, "c_j^{(k)}"'] & \hat{Y}_j \end{tikzcd}$$

commutes. This gives the action of the functor M_j on morphisms.

Proposition 3.13. *For all $0 < j \leq n$, there is an adjunction $M_j \dashv N_j$.*

Proof. We first define the unit $\eta\colon 1 \Rightarrow N_j M_j$ and counit $\epsilon\colon M_j N_j \Rightarrow 1$.

Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object in $\mathbf{Mag}_{j-1}$. Then the corresponding component of the unit map η is

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \widehat{X} \arrow[d, "\hat{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S, \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_k = \begin{cases} \mathrm{id}_{X_k} & \text{if } k \neq j, \\ \text{the coprojection map } c_j^{(0)}\colon X_j \to \widehat{X}_j & \text{if } k = j. \end{cases}$$

Naturality of η is immediate at dimensions $k \neq j$, and follows from the definition of the action of M_j on maps when $k = j$.

Now let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object in $\mathbf{Mag}_j$. The corresponding component of the counit map ϵ should be a map of the form

$$\begin{tikzcd} \widehat{X} \arrow[r, "\epsilon_X"] \arrow[d, "\hat{f}"'] & X \arrow[d, "f"] \\ S \arrow[r, "\mathrm{id}_S"'] & S. \end{tikzcd}$$

To define the map ϵ_X, recall that

$$\widehat{X}_j := \operatorname*{colim}_{k \geq 0} X_j^{(k)};$$

thus for each $k \geq 0$, we define a map

$$\epsilon_X^{(k)} \colon X_j^{(k)} \to X_j,$$

by induction over k.

When $k = 0$, $X_j^{(k)} = X_j$, and we define

$$\epsilon_X^{(0)} := \mathrm{id}_{X_j}.$$

Now suppose we have defined $\epsilon_X^{(k)}$ for some $k = l$. Recall that

$$X_j^{(l+1)} := X_j + \coprod_{0 \leq p < j} X_j^{(l)} \times_{X_p} X_j^{(l)}.$$

We define $\epsilon_X^{(l+1)}$ by

$$\epsilon_X^{(l+1)} := \mathrm{id}_{X_j} + \coprod_{0 \leq p < j} \left((\circ_p^j) \circ \left(\epsilon_X^{(l)}, \epsilon_X^{(l)} \right) \right) : X_j^{(l+1)} \longrightarrow X_j,$$

where $\circ_p^j$ is the composition map from the j-magma structure on X. We then define $(\epsilon_X)_j \colon \widehat{X}_j \to X_j$ to be the unique map such that, for all $k \geq 0$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "\epsilon_X^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "(\epsilon_X)_j"] \\ & X_j \end{tikzcd}$$

commutes. This defines the counit $\epsilon \colon M_j N_j \Rightarrow 1$. We now check naturality of ϵ. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Mag}_j$; since the components of ϵ are identities on strict n-category parts, and at all dimensions other than dimension j, to show that ϵ is natural we need only show that the diagram

$$\begin{tikzcd} \widehat{X}_j \arrow[r, "\hat{u}_j"] \arrow[d, "(\epsilon_X)_j"'] & \hat{Y}_j \arrow[d, "(\epsilon_Y)_j"] \\ X_j \arrow[r, "u_j"'] & Y_j \end{tikzcd}$$

commutes. By definition of $\widehat{X}$ as a colimit, this diagram commutes if, for each $k \geq 0$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "u_j^{(k)}"] \arrow[d, "\epsilon_X^{(k)}"'] & Y_j^{(k)} \arrow[d, "\epsilon_Y^{(k)}"] \\ X_j \arrow[r, "u_j"'] & Y_j \end{tikzcd}$$

commutes; we prove this by induction over k. It is immediate when $k = 0$, since $\epsilon_X^{(0)} = \mathrm{id}_{X_j}$ and $\epsilon_Y^{(0)} = \mathrm{id}_{Y_j}$. Now suppose we have shown that the diagram commutes for some $k = l$; then we have

$$\begin{aligned} u \circ \epsilon_X^{(l+1)} &= u_j + \coprod_{0 \leq p < j} \Big(u_j \circ (\circ_p^j) \circ \big(\epsilon_X^{(l)}, \epsilon_X^{(l)}\big) \Big) \\ &= u_j + \coprod_{0 \leq p < j} \Big((\circ_p^j) \circ \big(u_j \epsilon_X^{(l)}, u_j \epsilon_X^{(l)}\big) \Big) \\ &= u_j + \coprod_{0 \leq p < j} \Big((\circ_p^j) \circ \big(\epsilon_X^{(l)} u_j^{(l)}, \epsilon_X^{(l)} u_j^{(l)}\big) \Big) = \epsilon_j^{(l+1)} u_j^{(l+1)}, \end{aligned}$$

so the diagram commutes for $k = l + 1$. Thus, by induction, the diagram commutes for all $k \geq 0$. Hence ϵ is natural.

We now check that η and ϵ satisfy the triangle identities, i.e. that the diagrams

$$\begin{tikzcd} N_k \arrow[r, "\eta N_k"] \arrow[dr, "1"'] & N_k M_k N_k \arrow[d, "N_k \epsilon"] \\ & N_k, \end{tikzcd} \qquad \begin{tikzcd} M_k \arrow[r, "M_k \eta"] \arrow[dr, "1"'] & M_k N_k M_k \arrow[d, "\epsilon M_k"] \\ & M_k \end{tikzcd}$$

commute. In all of the natural transformations in these diagrams, the components on strict n-category parts are all identities, so to show that the diagrams commute we need only consider the components on n-globular set parts. Since the components of the maps of n-globular sets are identities at every dimension except dimension j, we need only check that the corresponding diagrams of maps of sets of j-cells commute.

For the first triangle identity, let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Mag}_j$. Then the diagram

$$\begin{tikzcd} X_j \arrow[r, "(\eta_X)_j = c_j^{(0)}"] \arrow[dr, "\epsilon_j^{(0)}"'] & \widehat{X}_j \arrow[d, "(\epsilon_X)_j"] \\ & X_j \end{tikzcd}$$

commutes by the universal property of $(\epsilon_X)_j$, so this triangle identity is satisfied.

Similarly, for the second triangle identity, let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Mag}_{j-1}$. Then the diagram

$$\begin{tikzcd}
\widehat{X}_j \arrow[r, "(\eta_{\widehat{X}})_j = c_j^{(0)}"] \arrow[dr, "\epsilon_j^{(0)}"'] & \widehat{\widehat{X}}_j \arrow[d, "(\epsilon_{\widehat{X}})_j"] \\
& \widehat{X}_j
\end{tikzcd}$$

commutes by the universal property of $(\epsilon_{\widehat{X}})_j$, so this triangle identity is satisfied.

Thus we have an adjunction $M_j \dashv N_j$, as required. $\square$

3.4 Interleaving the contraction and magma structures

We now explain the interleaving argument and show that we can interleave the constructions of Subsections 3.2 and 3.3 to give a construction of the left adjoint to the functor

$$W\colon \mathcal{Q} \to \mathcal{R}.$$

To do so we add the contraction and magma structures one dimension at a time, starting with dimension 1 and working upwards. At dimension m we first add free contraction cells using the functor C_m, then add free composites using the functor M_m, and then move up to the next dimension. Finally, we add "contraction $(n+1)$-cells" using the functor C_{n+1}, which identifies the appropriate cells at dimension n. Note that the method we use very closely follows the method used by Cheng in [7].

This construction is possible because of the dimensional dependencies of the functors C_k and M_j defined in Subsections 3.2 and 3.3; the contraction k-cells added by C_k only depend on the $(k-1)$-cells, and the composites added by the M_j only depend on the j-cells.

In order to describe this interleaving process formally, we define, for each $0 \leq j, k \leq n$, a category whose objects are objects of $\mathcal{R}$ equipped with both a j-magma structure and a k-contraction.

Definition 3.14. For each $0 \leq j \leq n$, $0 \leq k \leq n+1$, define a category $\mathcal{R}_{j,k}$ with

- objects: an object of $\mathcal{R}_{j,k}$ consists of an n-globular set X equipped with a j-magma structure, a strict n-category S, and a map of n-globular sets
$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$
that preserves the j-magma structure of X, equipped with a k-contraction γ;

- morphisms: a morphism in $\mathcal{R}_{j,k}$ is a commuting square
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$
in n-$\mathbf{GSet}$ such that
  - v is a map of strict n-categories;
  - u preserves the j-magma structure of X;
  - writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq n$, and $(a,b) \in X_m^c$, we have
$$u(\gamma_m(a,b)) = \delta_m(u(a), u(b)).$$

For $0 < j \leq n$, $0 < k \leq n+1$, we have forgetful functors
$$D_{j,k} \colon \mathcal{R}_{j,k} \to \mathcal{R}_{j,k-1},$$
which forgets the contraction structure at dimension k, and
$$N_{j,k} \colon \mathcal{R}_{j,k} \to \mathcal{R}_{j-1,k},$$
which forgets the magma structure at dimension j. Thus we can write the forgetful functor
$$W \colon \mathcal{Q} \to \mathcal{R}$$

as the composite

$$\mathcal{Q} = \mathcal{R}_{n,n+1} \xrightarrow{D_{n,n+1}} \mathcal{R}_{n,n} \xrightarrow{N_{n,n}} \mathcal{R}_{n-1,n} \xrightarrow{D_{n-1,n}} \cdots \xrightarrow{N_{1,1}} \mathcal{R}_{0,1} \xrightarrow{D_{0,1}} \mathcal{R}_{0,0} = \mathcal{R}.$$

In order to construct the left adjoint to W, we construct a left adjoint to each of the factors in the composite above, by lifting the constructions of C_k and M_j from Subsections 3.2 and 3.3 in a way that interacts properly with the forgetful functors

$$\mathcal{R}_{j,k} \to \mathbf{Mag}_j,$$

which forget the k-contraction structure entirely, and

$$\mathcal{R}_{j,k} \to \mathbf{Contr}_j,$$

which forget the j-magma structure entirely.

Lemma 3.15. *For all $0 < k \leq n+1$, the adjunction*

$$\begin{tikzcd} \mathbf{Contr}_{k-1} \arrow[r, shift left, "C_k"] \arrow[r, phantom, "\perp"] & \mathbf{Contr}_k \arrow[l, shift left, "D_k"] \end{tikzcd}$$

lifts to an adjunction

$$\begin{tikzcd} \mathcal{R}_{k-1,k-1} \arrow[r, shift left, "C_{k-1,k}"] \arrow[r, phantom, "\perp"] & \mathcal{R}_{k-1,k} \arrow[l, shift left, "D_{k-1,k}"] \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} \mathcal{R}_{k-1,k-1} \arrow[r, shift left, "C_{k-1,k}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{R}_{k-1,k} \arrow[l, shift left, "D_{k-1,k}"] \arrow[d] \\ \mathbf{Contr}_{k-1} \arrow[r, shift left, "C_k"] \arrow[r, phantom, "\perp"] & \mathbf{Contr}_k \arrow[l, shift left, "D_k"] \end{tikzcd}$$

commute serially.

Proof. We need to show that, given an object of $\mathbf{Contr}_{k-1}$ with n-globular set part X, if X is equipped with a $(k-1)$-magma structure, then this $(k-1)$-magma structure is "stable" under C_k; this is immediate since, by construction, C_k adds only k-cells to X, so the underlying $(k-1)$-globular set of X remains stable under C_k. □

Lemma 3.16. *For all $0 < j \leq n$, the adjunction*

$$\mathbf{Mag}_{j-1} \underset{N_j}{\overset{M_j}{\underset{\perp}{\rightleftarrows}}} \mathbf{Mag}_j$$

lifts to an adjunction

$$\mathcal{R}_{j-1,j} \underset{N_{j,j}}{\overset{M_{j,j}}{\underset{\perp}{\rightleftarrows}}} \mathcal{R}_{j,j}$$

making the diagram

$$\begin{tikzcd}
\mathcal{R}_{j-1,j} \arrow[r, shift left, "M_{j,j}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{R}_{j,j} \arrow[l, shift left, "N_{j,j}"] \arrow[d] \\
\mathbf{Mag}_{j-1} \arrow[r, shift left, "M_j"] \arrow[r, phantom, "\perp"] & \mathbf{Mag}_j \arrow[l, shift left, "N_j"]
\end{tikzcd}$$

commute serially.

Proof. We need to show that, given an object

$$\begin{tikzcd}
X \arrow[d, "f"'] \\
S
\end{tikzcd}$$

of $\mathbf{Mag}_{j-1}$, if f is equipped with a j-contraction γ, this j-contraction structure is "stable" under M_j. By construction, M_j adds only j-cells to X, so $\mathrm{Tr}_{j-1}X$ remains stable under M_j. The required contraction j-cells depend only on the $(j-1)$-cells of $\widehat{X}$, and we have $\widehat{X}^c_{j-1} = X^c_{j-1}$, so the contraction j-cells in $\widehat{X}$ are given by

$$X^c_{j-1} \xrightarrow{\gamma_{j-1}} X_j \xrightarrow{c_j} \widehat{X}_j.$$

For $m < j$, we have $\widehat{X}^c_{m-1} = X^c_{m-1}$, $\widehat{X}_m = X_m$, and the contraction m-cells are given by $\gamma_{m-1} \colon X^c_{m-1} \to X_m$. Hence the j-contraction structure is stable under M_j. $\square$

Combining Lemmas 3.15 and 3.16, we obtain a chain of adjunctions

$$\mathcal{R} = \mathcal{R}_{0,0} \underset{D_{0,1}}{\overset{C_{0,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{0,1} \underset{N_{1,1}}{\overset{M_{1,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \cdots \underset{D_{n-1,n}}{\overset{C_{n-1,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n-1,n} \underset{N_{n,n}}{\overset{M_{n,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n} \underset{D_{n,n+1}}{\overset{C_{n,n+1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n+1} = \mathcal{Q}.$$

Composing these, we obtain an adjunction

$$\mathcal{R} \underset{W}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q},$$

where $J = C_{n,n+1} \circ M_{n,n} \circ C_{n-1,n} \circ \cdots \circ M_{1,1} \circ C_{0,1}$. We then have

$$n\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q},$$

where $F = J \circ H$. Thus U has a left adjoint, so Penon weak n-categories are indeed well-defined, and moreover we have an explicit description of this left adjoint.

We now explain how to apply this construction in the case $n = \omega$. In this case, for each natural number k we have a composite adjunction

$$\mathcal{R} \underset{W_k}{\overset{J_k}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{k,k}$$

(i.e. $J_k = M_{k,k} \circ C_{k-1,k} \circ \cdots \circ M_{1,1} \circ C_{0,1}$). We define the functor $J \colon \mathcal{R} \longrightarrow \mathcal{Q}$ as follows: for an object A in $\mathcal{R}$, and for each natural number k,

$$(JA)_k := (J_k A)_k,$$

with magma structure, map of magmas, and contraction structure at dimension k given by those of $J_k A$. We then define $F := J \circ H$ as before, and we have

$$\omega\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

4. The operad for Penon weak n-categories

In [3], Batanin proved that there is an n-globular operad whose algebras are Penon weak n-categories, and that this operad can be equipped with a contraction and system of compositions. In this section we give a new, alternative proof of this fact using the construction of Penon's left adjoint from Section 3. Although it is not a new result, our proof is more direct than that of Batanin, offering an alternative point of view in a way that elucidates the structure of the operad, and makes clear the fact that the contraction and system of compositions arise naturally from the contraction and magma structure in the original definition of the monad P.

Throughout this section, we write T for the free strict n-category monad on n-$\mathbf{GSet}$. This is the monad induced by the adjunction

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & n\text{-}\mathbf{Cat}, \arrow[l, shift left] \end{tikzcd}$$

where n-$\mathbf{Cat}$ is the category of strict n-categories, and the right adjoint is the forgetful functor sending a strict n-category to its underlying n-globular set. We write $\eta^T\colon 1 \Rightarrow T$ for the unit of the monad T, and $\mu^T\colon T^2 \Rightarrow T$ for its multiplication. Similarly, for any monad P, we denote its unit by $\eta^P\colon 1 \Rightarrow P$ and its multiplication by $\mu^P\colon P^2 \Rightarrow P$.

We begin by recalling the definition of n-globular operad. These were introduced by Batanin [2]; as it is technically convenient for our purposes, we use a form of the definition that describes an n-globular operad as a cartesian map of monads (see [13, Corollary 6.2.4]).

Definition 4.1. An *n-globular operad* consists of a monad K on n-$\mathbf{GSet}$, and a cartesian map of monads $k\colon K \Rightarrow T$ (by which we mean a cartesian natural transformation $k\colon K \Rightarrow T$ respecting the monad structure). Given operads $k\colon K \Rightarrow T$, $k'\colon K' \Rightarrow T$, a *map of operads* $f\colon K \Rightarrow K'$ is a map of monads such that the diagram

$$\begin{tikzcd} K \arrow[rr, Rightarrow, "f"] \arrow[dr, Rightarrow, "k"'] & & K' \arrow[dl, Rightarrow, "k'"] \\ & T & \end{tikzcd}$$

commutes. The category of algebras for an operad $k\colon K \Rightarrow T$ is the category K-$\mathbf{Alg}$ of algebras for the monad K.

It is a straightforward and enlightening exercise to prove that the monad K is necessarily cartesian. We leave this to the reader.

In Definition 4.1, replacing n-$\mathbf{GSet}$ with $\mathbf{Set}$ and T with the free monoid monad yields a definition equivalent to that of classical non-symmetric operads. Both are examples of the more general notion of T-operads, introduced by Burroni [5] and described in detail by Leinster [13, Section 4.2]. Symmetric operads can then be obtained by equipping classical non-symmetric operads with a symmetric group action, but there is no such action in the case of n-globular operads. For the remainder of the paper "operad" is taken to mean "n-globular operad", since they are the only type of operads we use.

To prove that there is an operad whose algebras are Penon weak n-categories using Proposition 4.1 we must prove three facts: that there is a natural transformation

$$p\colon P \Longrightarrow T,$$

that this natural transformation is cartesian, and that it is a map of monads. Note that we know that the source of this natural transformation must be P to ensure that the algebras for the resulting operad are indeed P-algebras.

Proposition 4.2. *Recall from Definition 2.5 that $P\colon n\text{-}\mathbf{GSet} \to n\text{-}\mathbf{GSet}$ is the monad induced by the adjunction*

$$n\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

There is a natural transformation $p\colon P \Rightarrow T$ whose component p_X at an object X of n-$\mathbf{GSet}$ is given by

$$F(X) = (\ PX \xrightarrow{p_X} TX\ ),$$

an object of $\mathcal{Q}$.

Proof. Recall that there is a forgetful functor

$$U_T\colon n\text{-}\mathbf{Cat} \longrightarrow n\text{-}\mathbf{GSet}$$

that sends a strict n-category to its underlying n-globular set, and that the category $\mathcal{R}$ can be considered as the comma category

$$n\text{-}\mathbf{GSet} \downarrow U_T.$$

Write

$$\pi_1 \colon n\text{-}\mathbf{GSet} \downarrow U_T \to n\text{-}\mathbf{GSet}$$

and

$$\pi_2 \colon n\text{-}\mathbf{GSet} \downarrow U_T \to n\text{-}\mathbf{Cat}$$

for the projection maps, and consider the following diagram:

$$\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[d, "F"] & \\
 & \mathcal{Q} \arrow[d, "W"] & \\
 & n\text{-}\mathbf{GSet} \downarrow G \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] \arrow[d, phantom, "\Searrow"] & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat}. \arrow[ll, "G"]
\end{tikzcd}$$

Then the universal property of $n\text{-}\mathbf{GSet} \downarrow U_T$ as a 2-limit (see [16]) induces a unique natural transformation $p \colon P \Rightarrow T$ such that

$$\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[dl, bend right, "P"'] \arrow[dr, bend left, "F_T"] \arrow[d, phantom, "\Searrow p"] & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat} \arrow[ll, "G"]
\end{tikzcd}
=
\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[d, "F"] \arrow[ddl, bend right, "P"'] \arrow[ddr, bend left, "F_T"] & \\
 & \mathcal{Q} \arrow[d, "W"] & \\
 & n\text{-}\mathbf{GSet} \downarrow U_T \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] \arrow[d, phantom, "\Searrow"] & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat} \arrow[ll, "G"]
\end{tikzcd}$$

where F_T is the free strict n-category functor. □

Proposition 4.3. *The natural transformation $p \colon P \Rightarrow T$ is cartesian.*

To prove this, we must show that each naturality square for p is a pullback square. To do so, we use the construction of the adjunction

$$\begin{tikzcd}
n\text{-}\mathbf{GSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "U"]
\end{tikzcd}$$

from Section 2. Recall that this adjunction can be decomposed as

$$n\text{-}\mathbf{GSet} \underset{V}{\overset{H}{\rightleftarrows}} \mathcal{R} \underset{W}{\overset{J}{\rightleftarrows}} \mathcal{Q}.$$

Given a map $f\colon X \to Y$ in n-$\mathbf{GSet}$, the corresponding naturality square is obtained by applying the functor $J\colon \mathcal{R} \to \mathcal{Q}$ to the map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta^T_X"'] & Y \arrow[d, "\eta^T_Y"] \\ TX \arrow[r, "Tf"'] & TY. \end{tikzcd}$$

in $\mathcal{R}$, which is a pullback square in n-$\mathbf{GSet}$, since the free strict n-category monad T is cartesian [13, 4.1.18 and F.2.2]. Thus we prove that p is cartesian by proving that the functor J sends maps that are pullback squares to maps that are pullback squares (in fact, we do so only for a certain class of such maps). Recall that the adjunction $J \dashv W$ can be decomposed as the following chain of adjunctions:

$$\mathcal{R} = \mathcal{R}_{0,0} \underset{D_{0,1}}{\overset{C_{0,1}}{\rightleftarrows}} \mathcal{R}_{0,1} \underset{N_{1,1}}{\overset{M_{1,1}}{\rightleftarrows}} \dots \underset{D_{n-1,n}}{\overset{C_{n-1,n}}{\rightleftarrows}} \mathcal{R}_{n-1,n} \underset{N_{n,n}}{\overset{M_{n,n}}{\rightleftarrows}} \mathcal{R}_{n,n} \underset{D_{n,n+1}}{\overset{C_{n,n+1}}{\rightleftarrows}} \mathcal{R}_{n,n+1} = \mathcal{Q},$$

where the functor $C_{m,m+1}$ freely adds the contraction structure at dimension $m+1$, and the functor $M_{m,m}$ freely adds the magma structure at dimension m. We now prove three lemmas to show that each of these functors sends maps that are pullback squares to maps that are pullback squares, thus showing that their composite J does so as well. There are three lemmas since the functor $C_{n,n+1}$ must be treated separately from the functors $C_{m,m+1}$ for $0 \leq m \leq n-1$.

Note that we only consider maps whose the strict n-category part is a map in the image of T between free strict n-categories; this is as general as we need it to be to prove Proposition 4.3, and it allows us to use the fact that T is cartesian in the proofs of the lemmas.

Lemma 4.4. *Let $0 \leq m \leq n-1$ and suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{m,m}$ that is a pullback square in n-$\mathbf{GSet}$. Then its image under the functor

$$C_{m,m+1} \colon \mathcal{R}_{m,m} \longrightarrow \mathcal{R}_{m,m+1}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. The idea of the proof is as follows: the functor $C_{m,m+1}$ freely adds contraction $(m+1)$-cells to X and Y. These contraction cells are obtained by taking pullbacks in $\mathbf{Set}$, and then added to the sets of $(m+1)$-cells X_{m+1} and Y_{m+1} by taking coproducts in $\mathbf{Set}$. The action of $C_{m,m+1}$ on the map itself is then induced by the universal properties of these pullbacks and coproducts. Thus the image of this map under the functor $C_{m,m+1}$ is a coproduct of pullback squares (with some adjustments at the bottom to ensure that the strict n-category parts TX and TY remain unchanged). Since pullbacks commute with coproducts in $\mathbf{Set}$ [14, IX.2 Exercise 3], this coproduct of pullback squares is itself a pullback square.

Recall from Definition 2.3 that we have

$$\begin{tikzcd} X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,x_m)}"] \\ X_m \arrow[r, "{(s,t,x_m)}"'] & X_{m-1} \times X_{m-1} \times TA_m \end{tikzcd}$$

For $k \neq m+1$, we have $C_{m,m}(u, Tf)_k = (u, Tf)_k$, and since pullbacks in n-$\mathbf{GSet}$ are computed pointwise, we need only check that $C_{m,m}(u, Tf)_{m+1}$ is a pullback square, i.e. that

$$\begin{tikzcd} X_{m+1} \amalg X^c_{m+1} \arrow[r, "u_{m+1} \amalg u^c_{m+1}"] \arrow[d, "x_{m+1} \amalg x^c_{m+1}"'] & Y_{m+1} \amalg X^c_{m+1} \arrow[d, "y_{m+1} \amalg y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

is a pullback square. Since coproducts commute with pullbacks in $\mathbf{Set}$ [14, IX.2, exercise 3], this is true if the squares

$$
\begin{tikzcd}
X_{m+1} \arrow[r, "u_{m+1}"] \arrow[d, "x_{m+1}"'] & Y_{m+1} \arrow[d, "y_{m+1}"] \\
TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"'] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\
TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1}
\end{tikzcd}
$$

are both pullback squares. The left-hand square is a pullback square by hypothesis. For the right-hand square, suppose we have a cone

$$
\begin{tikzcd}
V \arrow[r, "v_1"] \arrow[d, "v_2"'] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\
TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1}
\end{tikzcd}
$$

in $\mathbf{Set}$. Recall that we have source and target maps $s, t \colon Y^c_{m+1} \to Y_m$ given by the projections from the pullback defining Y^c_{m+1}. Composing with these, and source and target maps for TA and TB, induces maps

$$
\begin{tikzcd}
V \arrow[dr, "!\sigma"] \arrow[drr, bend left, "sv_1"] \arrow[ddr, bend right, "sv_2"'] & & \\
& X_m \arrow[r, "u_m"] \arrow[d, "x_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_m \arrow[d, "y_m"] \\
& TA_m \arrow[r, "Tf_m"'] & TB_m,
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[dr, "!\tau"] \arrow[drr, bend left, "tv_1"] \arrow[ddr, bend right, "tv_2"'] & & \\
& X_m \arrow[r, "u_m"] \arrow[d, "x_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_m \arrow[d, "y_m"] \\
& TA_m \arrow[r, "Tf_m"'] & TB_m.
\end{tikzcd}
$$

The maps σ and τ give us a cone over the pullback square defining X^c_{m+1}; commutativity of this cone comes from the globularity conditions and the fact that every cell in the image of v_2 is an identity, so has the same source and target. Thus the universal property of X^c_{m+1} induces a unique map such that the diagram

$$\begin{tikzcd}
V \arrow[drr, bend left, "\sigma"] \arrow[dr, dashed, "v"] \arrow[ddr, bend right, "\tau"'] & & \\
& X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,x_m)}"] \\
& X_m \arrow[r, "{(s,t,x_m)}"'] & X_{m-1} \times X_{m-1} \times TA_m .
\end{tikzcd}$$

commutes.

We now check that v makes the diagram

$$\begin{tikzcd}
V \arrow[drr, bend left, "v_1"] \arrow[dr, "v"] \arrow[ddr, bend right, "v_2"'] & & \\
& X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\
& TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1}
\end{tikzcd}$$

commute. To show that the top triangle commutes, observe that the map $v_1 = u^c_{m+1} \circ v$ makes the following diagram commute:

$$\begin{tikzcd}
V \arrow[drrr, bend left, "\sigma"] \arrow[dr, "v"] \arrow[ddr, bend right, "\tau"'] & & & & \\
& X^c_{m+1} \arrow[rr, "s"] \arrow[dr, "u^c_{m+1}"'] \arrow[d, "t"'] & & X_m \arrow[dr, "u_m"] & \\
& X_m \arrow[dr, "u_m"'] & Y^c_{m+1} \arrow[rr, "s"] \arrow[d, "t"'] \arrow[drr, phantom, "\lrcorner", very near start] & & Y_m \arrow[d, "{(s,t,y_m)}"] \\
& & Y_m \arrow[rr, "{(s,t,y_m)}"'] & & Y_{m-1} \times Y_{m-1} \times TB_m .
\end{tikzcd}$$

Since $u_m\sigma = sv_1$ and $u_m\tau = tv_1$, by the universal property of Y^c_{m+1}, we have $u^c_{m+1} \circ v = v_1$.

To show that the left-hand triangle commutes, write $i\colon TA_m \to TA_{m+1}$ for the map that sends an m-cell to its identity $(m+1)$-cell, and consider that we can factorise $x^c_{m+1} \circ v$ as

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[rr, bend right, "\sigma"'] \arrow[rrr, bend right, "sv_2"'] & X^c_{m+1} \arrow[r, "s"] \arrow[rrr, bend left, "x^c_{m+1}"] & X_m \arrow[r, "x_m"] & TA_m \arrow[r, "i"] & TA_{m+1}. \end{tikzcd}$$

Thus we have $x^c_{m+1} \circ v = isv_2 = v_2$, since all cells in the image of v_2 are identities.

Finally, uniqueness of v comes from the universal property of X^c_{m+1}. Hence

$$\begin{tikzcd} X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

is a pullback square, so $C_{m,m+1}(u, Tf)$ is a pullback square. $\square$

We must treat the case $m = n$ separately.

Lemma 4.5. *Suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{n,n}$ that is a pullback square in n-$\mathbf{GSet}$. Then its image under the functor

$$C_{n,n+1} \colon \mathcal{R}_{n,n} \longrightarrow \mathcal{R}_{n,n+1} = \mathcal{Q}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. Recall from Definition 3.8 that we have

$$\begin{tikzcd} X^c_{n+1} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "{(s,t,x_n)}"] \\ X_n \arrow[r, "{(s,t,x_n)}"'] & X_{n-1} \times X_{n-1} \times TA_n, \end{tikzcd}$$

and that $\tilde{X}_n$ is defined to be the coequaliser of the diagram

$$X^c_{n+1} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X_n$$

in **Set**. We write $q\colon X_n \to \tilde{X}_n$ for the coprojection. The set $\tilde{Y}_n$ is defined similarly, and we write $r\colon Y_n \to \tilde{Y}_n$ for the coprojection. For all $0 \le m < n$ we have

$$C_{n,n+1}(u, Tf)_m = (u, Tf)_m,$$

and for $m = n$, we have that $C_{n,n+1}(u, Tf)_n$ is given by

$$\begin{tikzcd} \tilde{X}_n \arrow[r, "\tilde{u}_n"] \arrow[d, "\tilde{x}_n"'] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\ TA_n \arrow[r, "Tf_n"'] & TB_n, \end{tikzcd}$$

so we only need to check that this is a pullback square in **Set**.

Write w for the unique map making the diagram

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "x_n"'] & Y_n \arrow[dr, "r"] & \\ & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\ & TA_n \arrow[r, "Tf_n"'] & TB_n \end{tikzcd}$$

commute. We will show that, for a, $b \in X_n$, $w(a) = w(b)$ if and only if $(a, b) \in X^c_{n+1}$, and also that w is surjective; and thus $C_{n,n+1}(u, Tf)_n$ is a pullback square and $w = q$.

Let $(a, b) \in X^c_{n+1}$, so $x_n(a) = x_n(b)$, $s(a) = s(b)$, $t(a) = t(b)$. We have $(u_n(a), u_n(b)) \in Y^c_{n+1}$, so $ru_n(a) = ru_n(b)$. Thus

$$w(a) = (x_n(a), ru_n(a)) = (x_n(b), ru_n(b)) = w(b).$$

Now let a, $b \in X_n$ with $w(a) = w(b)$, so $x_n(a) = x_n(b)$, $ru_n(a) = ru_n(b)$. The source map $s\colon X_n \to X_{n-1}$ is the unique map making the diagram

$$\begin{tikzcd}
X_n \arrow[rr, "u_n"] \arrow[dr, "s"] \arrow[dd, "x_n"'] & & Y_n \arrow[dr, "s"] & \\
& X_{n-1} \arrow[rr, "u_{n-1}"] \arrow[dd, "x_{n-1}"] \arrow[drr, phantom, "\lrcorner", very near start] & & Y_{n-1} \arrow[dd, "y_{n-1}"] \\
TA_n \arrow[dr, "s"'] & & & \\
& TA_{n-1} \arrow[rr, "Tf_{n-1}"'] & & TB_{n-1}
\end{tikzcd}$$

commute. Thus, since $su_n(a) = su_n(b)$ and $sx_n(a) = sx_n(b)$, we have $s(a) = s(b)$. Similarly, $t(a) = t(b)$. Hence $(a, b) \in X^c_{n+1}$.

Now let $\pi \in TA_n$, $c \in \tilde{Y}_n$, with $Tf_n(\pi) = \tilde{y}_n(c)$. We wish to show that there is some $a \in X_n$ with $w(a) = (\pi, c)$, and thus that w is surjective. Since r is surjective, there exists $c' \in Y_n$ with $r(c') = c$. Since X_n is given by the pullback

$$\begin{tikzcd}
X_n \arrow[r, "u_n"] \arrow[d, "x_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_n \arrow[d, "y_n"] \\
TA_n \arrow[r, "Tf_n"'] & TB_n
\end{tikzcd}$$

and $yr(c') = Tf_n(\pi)$, we have $a \in X_n$ with $x_n(a) = \pi$, $u_n(a) = c'$. Thus $w(a) = (\pi, c)$, so w is surjective. Hence

$$\begin{tikzcd}
\tilde{X}_n \arrow[r, "\tilde{u}_n"] \arrow[d, "\tilde{x}_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\
TA_n \arrow[r, "Tf_n"'] & TB_n
\end{tikzcd}$$

is a pullback square. □

Thus we have shown that the functors adding the free contraction cells send maps that are pullback squares to maps that are pullback squares. We now do the same for the functors adding the free magma structure.

Lemma 4.6. *Let $0 < m \leq n$ and suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{m-1,m}$ that is a pullback square in n-$\mathbf{GSet}$. Then its image under the functor

$$M_{m,m}\colon \mathcal{R}_{m-1,m} \longrightarrow \mathcal{R}_{m,m}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. The idea of this proof is similar to that of the proof of Lemma 4.4, but is slightly more complicated since the construction of $M_{m,m}$ uses filtered colimits as well as coproducts. The functor $M_{m,m}$ freely adds binary composites of m-cells to X and Y. These composites are added through a process of taking pullbacks, coproducts, and filtered colimits in $\mathbf{Set}$. The action of $M_{m,m}$ on the map itself is then induced by the universal properties of these pullbacks, coproducts, and filtered colimits. Thus the image of this map under the functor $M_{m,m}$ is a filtered colimit of coproducts of pullback squares (with some adjustments at the bottom to ensure that the strict n-category parts TX and TY remain unchanged). Since pullbacks commute with both coproducts and filtered colimits in $\mathbf{Set}$ [14, IX.2, Exercise 3 and Theorem 1], this filtered colimit of coproducts of pullback squares is itself a pullback square.

Recall the notation from Definition 3.12: we write

$$M_{m,m}(\, X \xrightarrow{x} TA \,) = \hat{X} \xrightarrow{\hat{x}} TA \,,$$

$$M_{m,m}(\, Y \xrightarrow{y} TB \,) = \hat{Y} \xrightarrow{\hat{y}} TB \,.$$

Since $M_{m,m}$ changes only dimension m, and since pullbacks in n-$\mathbf{GSet}$ are computed pointwise, we just need to check that

$$\begin{tikzcd} \hat{X}_m \arrow[r, "\hat{u}_m"] \arrow[d, "\hat{x}_m"'] & \hat{Y}_m \arrow[d, "\hat{y}_m"] \\ TA_m \arrow[r, "Tf_m"'] & TB_m \end{tikzcd}$$

is a pullback square in $\mathbf{Set}$. Recall that $\hat{X}_m$ and $\hat{Y}_m$ are defined as filtered colimits in $\mathbf{Set}$, with

$$\hat{X}_m := \operatorname*{colim}_{j \geq 1} X_m^{(j)}, \ \hat{Y}_m := \operatorname*{colim}_{j \geq 1} Y_m^{(j)}.$$

Since pullbacks commute with filtered colimits in $\mathbf{Set}$, we can prove that the above diagram is a pullback square by proving that, for each $j \geq 1$, the diagram

$$\begin{tikzcd}
X_m^{(j)} \arrow[r, "u_m^{(j)}"] \arrow[d, "x_m^{(j)}"'] & Y_m^{(j)} \arrow[d, "y_m^{(j)}"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square in $\mathbf{Set}$. We do this by induction. When $j = 1$, we have $X_m^{(j)} = X_m$, $Y_m^{(j)} = Y_m$, and the square above becomes is a pullback square by hypothesis.

Now suppose that $j > 1$, and we have shown that

$$\begin{tikzcd}
X_m^{(j-1)} \arrow[r, "u_m^{(j-1)}"] \arrow[d, "x_m^{(j-1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_m^{(j-1)} \arrow[d, "y_m^{(j-1)}"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square; we will show that

$$\begin{tikzcd}
X_m^{(j)} \arrow[r, "u_m^{(j)}"] \arrow[d, "x_m^{(j)}"'] & Y_m^{(j)} \arrow[d, "y_m^{(j)}"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square. Recall that $X_m^{(j)}$ is defined by

$$X_m^{(j)} := X_m \amalg \coprod_{0 \leq p < m} X_m^{(j-1)} \times_{X_p} X_m^{(j-1)},$$

and similarly for $Y_m^{(j)}$. Since pullbacks commute with coproducts in $\mathbf{Set}$, the above diagram is a pullback square if, for all $0 \leq p < m$, the diagram

$$\begin{tikzcd}
X_m^{(j-1)} \times_{X_p} X_m^{(j-1)} \arrow[r, "(u_m^{(j-1)}, u_m^{(j-1)})"] \arrow[d] & Y_m^{(j-1)} \times_{Y_p} Y_m^{(j-1)} \arrow[d] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square. We can write this as

$$\begin{tikzcd}
X_m^{(j-1)} \times_{X_p} X_m^{(j-1)} \arrow[r, "(u_m^{(j-1)}, u_m^{(j-1)})"] \arrow[d, "(x_m^{(j,1)}, x_m^{(j,1)})"'] & Y_m^{(j-1)} \times_{Y_p} Y_m^{(j-1)} \arrow[d, "(y_m^{(j,1)}, y_m^{(j,1)})"] \\
TA_m \times_{TA_p} TA_m \arrow[r, "(Tf_m, Tf_m)"'] \arrow[d, "\circ_p^m"'] & TB_m \times_{TB_p} TB_m \arrow[d, "\circ_p^m"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m.
\end{tikzcd}$$

The top square is a pullback of pullback squares, and hence is itself a pullback square. The fact that the bottom square is a pullback square is left as a straightforward exercise to the reader; it is an application of the fact that T is a cartesian monad [13, Example 4.1.18 and Theorem F.2.2], so the naturality squares for its multiplication μ^T are pullbacks squares, and the fact that T^2A and T^2B can be constructed via a series a pullbacks in n-$\mathbf{GSet}$ (see [13, F.1] and [8], which give constructions of T using this method).

Thus the diagram

$$\begin{tikzcd}
\hat{X}_m \arrow[r, "\hat{u}_m"] \arrow[d, "\hat{x}_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{Y}_m \arrow[d, "\hat{y}_m"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square in n-$\mathbf{GSet}$. Hence $M_{m,m}$ sends maps that are pullback squares to maps that are pullback squares, as required. $\square$

We now combine these results to prove that $p\colon P \Rightarrow T$ is cartesian.

Proof of Proposition 4.3. Combining the above results, and using the fact that $J\colon \mathcal{R} \to \mathcal{Q}$ is defined as the composite

$$J = C_{n,n+1} \circ M_{n,n} \circ C_{n-1,n} \circ \cdots \circ M_{1,1} \circ C_{0,1},$$

we see that, given a map (u, Tf) in $\mathcal{R}$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

is a pullback square in n-**GSet**, the map $J(u, Tf)$ in $\mathcal{Q}$ is also a pullback square in n-**GSet**. Take (u, Tf) to be

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta^T_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta^T_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

for any $f\colon A \to B$ in n-**GSet**, which is a pullback square since T is cartesian. Applying J gives us that

$$\begin{tikzcd} PA \arrow[r, "Pf"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & PB \arrow[d, "p_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

is a pullback square in n-**GSet**. Thus $p\colon P \Rightarrow T$ is a cartesian natural transformation. □

Thus the natural transformation $p\colon P \Rightarrow T$ satisfies one of the conditions in Proposition 4.1; to prove that it is an operad, we now only need to prove the following:

Proposition 4.7. *The natural transformation $p\colon P \Rightarrow T$ is a map of monads.*

Proof. We need to check that p satisfies the monad map axioms. To do so, recall that P is the monad induced by the adjunction

$$n\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}$$

defined in Section 3, and that this adjunction can be decomposed as

$$n\text{-}\mathbf{GSet} \underset{V}{\overset{H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R} \underset{W}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

Write α, β for the unit and counit of $H \dashv V$, and write κ, ζ for the unit and counit of $J \dashv W$. Then the unit $\eta = \eta^P$ of the adjunction $F \dashv U$ is given by the composite

$$1 \xrightarrow{\alpha} VH \xrightarrow{V\kappa H} VWJH = UF$$

and the counit ϵ of $F \dashv U$ is given by the composite

$$FU = JHVW \xrightarrow{J\beta W} JW \xrightarrow{\zeta} 1.$$

To show that p satisfies the axioms for a monad map we consider the unit η^P and counit ϵ for the adjunction $F \dashv U$. By Proposition 3.3, $\alpha = \mathrm{id}$, so $\eta^P = V\kappa H$. For all $X \in n$-$\mathbf{GSet}$, κ_{HX} is the map

$$\begin{tikzcd} X \arrow[r, "\eta^P_X"] \arrow[d, "\eta^T_X"'] & PX \arrow[d, "p_X"] \\ TX \arrow[r, "\mathrm{id}_{TX}"'] & TX \end{tikzcd}$$

in $\mathcal{R}$. Commutativity of this diagram shows that p satisfies the first axiom for a monad map.

For all $X \in n$-$\mathbf{GSet}$, ϵ_{FX} is the map

$$\begin{tikzcd} P^2X \arrow[rr, "\mu^P_X"] \arrow[d, "p_{PX}"'] & & PX \arrow[d, "p_X"] \\ TPX \arrow[r, "Tp_X"'] & T^2X \arrow[r, "\mu^T_X"'] & TX \end{tikzcd}$$

in Q. Commutativity of this diagram shows p satisfies the second axiom for a monad map.

Thus $p\colon P \Rightarrow T$ is a monad map. $\square$

Combining Propositions 4.3 and 4.7 gives us the following theorem:

Theorem 4.8. *There is an operad whose algebras are Penon weak n-categories, given by the cartesian map of monads $p\colon P \Rightarrow T$.*

Proof. The natural transformation $p\colon P \Rightarrow T$ is cartesian by Proposition 4.3, and is a monad map by Proposition 4.7. Thus it is an operad, and its category of algebras is P-**Alg**, the category of Penon weak n-categories. $\square$

In [2] Batanin uses two pieces of extra structure to identify which n-globular operads give sensible notions of weak n-category: a system of compositions, which gives the operad binary composition operations, and a contraction, which gives coherence. Both pieces of extra structure are defined on the "underlying collection" of an operad – its component on the terminal n-globular set. We now recall the necessary definitions, then show that the operad $p\colon P \Rightarrow T$ can be equipped with both structures in a way that arises naturally from the n-magma and contraction structures in the definition of P.

Definition 4.9. Given an n-globular operad $k\colon K \Rightarrow T$, its *underlying collection* is the component of k at the terminal n-globular set 1, that is $k_1\colon K1 \to T1$.

Definition 4.10. A *contraction* on an n-globular operad $k\colon K \Rightarrow T$ consists of a contraction (in the sense of Definition 2.3) on its underlying collection.

Definition 4.11. Let $0 \leq m \leq n$, and write $\eta_m := \eta^T_m(1)$, the single m-cell in the image of the unit map $\eta^T\colon 1 \to T1$. Define, for $0 \leq p \leq m \leq n$,

$$\beta^m_p = \begin{cases} \eta_m & \text{if } p = m, \\ \eta_m \circ^m_p \eta_m & \text{if } p < m. \end{cases}$$

Define an n-globular set S, in which

$$S_m := \{\beta^m_p \mid 0 \leq p \leq m \leq n\} \subseteq T1_m.$$

Write $s\colon S \to T1$ for the inclusion, and define the "unit" map $\eta^S\colon\ 1 \ \to\ S$ by $\eta^S_m(1) = \beta^m_m$.

A *system of compositions* on an n-globular operad $k\colon K \Rightarrow T$ consists of a map $\sigma\colon S \to K1$ in n-$\mathbf{GSet}$ such that the diagrams

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "s"'] & & K1 \arrow[dl, "k_1"] \\ & T1 & \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, "\eta^S"] \arrow[rr, bend right, "\eta^K_1"'] & S \arrow[r, "\sigma"] & K1 \end{tikzcd}$$

commute.

Proposition 4.12. *The operad P for Penon weak n-categories can be equipped with a contraction and system of compositions which arise naturally from the contraction on $p_1\colon\ P1\ \to\ T1$ and the magma structure on $P1$ respectively.*

Proof. The presence of the contraction is immediate, since

$$\begin{tikzcd} P1 \arrow[d, "p_1"] \\ T1 \end{tikzcd}$$

is an object of $\mathcal{Q}$, so is equipped with a contraction as constructed in Section 3. Similarly, $P1$ is equipped with a magma structure; we use this to define a system of compositions

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "s"'] & & P1 \arrow[dl, "p_1"] \\ & T1 & \end{tikzcd}$$

as follows: for all $0 \leq m \leq n$, writing 1 for the unique m-cell in the terminal n-globular set,

- $\sigma_m(\beta_m^m) := (\eta_1^P)_m(1) = 1$;
- for $0 \leq l \leq m$, $\sigma_m(\beta_l^m) := 1 \circ_l^m 1$.

From the definition of the magma structure on $P1$ given in Definition 3.12, this satisfies the source and target conditions for a map of n-globular sets, and the commutativity conditions required to be a map of collections. By definition of $\sigma_m(\beta_m^m)$,

$$\begin{tikzcd} 1 \arrow[r, "\epsilon^S"] \arrow[rr, bend right, "\eta_1^P"'] & S \arrow[r, "\sigma"] & P1 \end{tikzcd}$$

commutes. Thus, σ is a system of compositions on $p\colon P \Rightarrow T$. □

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