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COOPERATIVE PROPERTIES AND CONNECTED SUM

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Résumé. Un *cycle* est un graphe connexe 2-régulier. Une propriété relative aux cycles est coopérative si elle est valable pour tout cycle qui est la somme mod-2 de deux cycles se croisant dans un chemin nontrivial lorsque les deux sommands ont la propriété. Une telle propriété vaut pour tous les cycles si elle est valable pour les cycles dans une base à sommes connectées (CS), et tous les graphes ont des bases CS. Nous montrons que la commutativité "équivalence naturelle prés" est une propriété coopérative pour les cycles d'un diagramme dans un groupoide et que le critére du cycle de Kolmogorov est coopératif pour les cycles dans les chaînes de Markov.

Abstract. A *cycle* in a graph is a 2-regular connected subgraph. A property of cycles is *cooperative* if it holds for any cycle which is the mod-2 sum of two cycles intersecting in a nontrivial path when both summands have the property. Cooperative properties hold for all cycles when they hold for the cycles in a *connected sum* (CS) basis, and all graphs have CS bases. It is shown that cooperative properties include commutativity up to natural equivalence for cycles in a groupoid diagram and the Kolmogorov cycle criterion for reversibility of an irreducible, stationary, aperiodic Markov chain.

Keywords. Groupoid diagram, commutative up to a natural equivalence, Kolmogorov criterion, reversibility of a Markov chain, robust cycle basis.
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1. Introduction

We apply some concepts from graph theory to commutativity up to natural equivalence for diagrams in a groupoid category and to reversibility of Markov chains. See Harary [5] for undefined graph terminology below.

Properties for the cycles in a graph are exhibited which need only be checked *for the cycles in a basis* vs. *all* cycles in the graph. This avoids a combinatorial explosion. For instance, the 5-dimensional (binary) hypercube Q_5, with 32 vertices and 80 edges, contains more than 51 *billion* distinct cycle-subgraphs but has a basis with 49 elements.

While our program works for some interesting properties, not every cycle basis will do. One needs a *connected sum basis* (CS basis). This will enable construction of cycles in a system which involves topology, order, and hierarchy. Every graph has a CS basis; these bases are defined using the concept of *connected sum of cycles*.

The *connected sum of two cycles* Z_1 and Z_2 in a graph G is defined precisely when the intersection of the cycles is a nontrivial path, and, in that case, it is the symmetric difference of the edge sets (that is, the mod-2 sum).

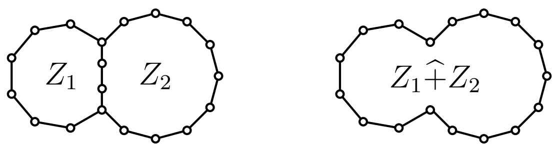


Figure 1: The connected sum of two cycles.

We write $Z_1 \widehat{+} Z_2$ for connected sum. The connected sum of two cycles is always a cycle, but their ordinary mod-2 sum is only guaranteed to have all vertices of even degree. Connected sum is commutative but not associative. The *connected sum of a sequence of cycles*, when it is defined, uses left-most parenthesization. So the sequence of cycles (Z_1, Z_2, Z_3) has a connected sum iff $Z_1 \cap Z_2$ and $(Z_1 \widehat{+} Z_2) \cap Z_3$ are nontrivial paths. Starting with a set $\mathcal{S}$ of cycles, one can form all possible connected sums for sequences from the set, and we call the resulting family of cycles the *robust closure* $\rho(S)$ of $\mathcal{S}$.

The edge sets of the even-degree subgraphs of G determine its *cycle space*, an $\mathbb{F}_2$-vector space, usually denoted $\mathbb{Z}(G)$, where addition mod-2 is symmetric difference. As every even degree graph has an edge-disjoint de-composition into cycles, there are bases for $\mathbb{Z}(G)$ consisting only of cycles. These are called *cycle bases*. See, e.g., [1, 6, 13, 17]. We introduced the following concepts in [7].

A cycle basis for a graph G is a *connected sum basis* if it can be used to construct every cycle in G by iteratively taking robust closure. Note that *topology* is involved in the definition of connected sum of two cycles, *order* in the choice of a sequence of cycles whose connected sum is defined giving the desired cycle, and *hierarchy* in the recursive construction of cycles.

Formally, we define a *property* of cycles to be a subset $\mathcal{P}$ of a graph's cycles. A *cooperative property* is one for which

$$Z_1, Z_2 \in \mathcal{P} \implies Z_1 \widehat{+} Z_2 \in \mathcal{P}. \tag{1}$$

Properties that hold for the cycles in a CS basis, and that are cooperative, will hold for all cycles. In contrast, for less carefully controlled cycle sums, where partial summands need not even be connected and where intersections of cycles can be arbitrary, properties holding for the cycles in a basis may not spread to the other cycles. Commutativity of cycles in a diagram turns out to be cooperative.

Cooperative properties involve additional structure superimposed on the graph. For diagram commutativity, this structure consists of a suitable diagram in a groupoid. Later we consider the structure of a Markov chain.

It may seem unnecessary to have a property for all cycles guaranteed by the members of a *special* cycle basis when it is almost a default assumption that properties of a graph related to cycles need only be checked for members of an *arbitrary* cycle basis. This belief could be due to two well-known examples:

Kirchoff's voltage law (*the sum of the voltages around any cycle is zero.*) By a linear-algebra argument, one need only check for the cycles in any basis.

A graph is bipartite if and only if each cycle has even length. Counting shows this holds for all cycles if it holds for the cycles in any basis.

A third example might come to mind. Many of the diagrams arising in elementary category theory and also in homological algebra are planar.

Plane diagrams commute iff the region boundaries commute. The region boundaries (of all the bounded regions) do constitute a cycle basis.

However, this last example, considered more carefully, shows that not all cycle bases will suffice. We provided a nonplanar diagram and a non-cs cycle basis where the cycles of the basis commute but some other cycles do *not* commute; see [4] and Figure 3 below. Further, the region-boundaries basis is a connected sum basis [10]. We showed in [4] that commutativity is a cooperative property so diagrams commute if (and only if) all cycles in a connected sum basis commute.

In this paper, the applicability of cooperative properties is demonstrated with two more examples: *commutativity up to a natural equivalence* is a cooperative property of cycles in a groupoid diagram, and the *Kolmogorov cycle criterion* (KCC) is a cooperative property of cycles in a Markov chain.

In §2 below, we review CS bases and §3 extends the machinery to directed graphs (digraphs). The results are applied in §4 to groupoid diagrams and in §5 to Markov chains; we conclude with a brief discussion.

2. Background on connected sum

For any graph H, we write $E(H)$ for the edge-set. The **connected sum** $Z_1 \widehat{+} Z_2$ of two cycle subgraphs of a graph G is just the usual mod-2 sum (i.e., symmetric difference of edge sets) but it is only defined when $Z_1 \cap Z_2$ is a path containing at least one edge.

Let $\mathrm{Cyc}(G)$ denote the set of all cycle-subgraphs of a graph G and let $\emptyset \neq \mathcal{S} \subseteq \mathrm{Cyc}(G)$. A sequence of not necessarily distinct cycles from $\mathcal{S}$

$$(Z_1, Z_2, \ldots, Z_k) \tag{2}$$

is called $\mathcal{S}$**-admissable** and its connected sum is defined by

$$\widehat{+}(Z_1, Z_2, \ldots, Z_k) := (\cdots((Z_1 \widehat{+} Z_2) \widehat{+} Z_3) \cdots) \widehat{+} Z_k. \tag{3}$$

provided that its members have pairwise intersections as specified so that all of the partial sums on the RHS of (3) are connected sums. Hence, the connected sum of a sequence is defined iff the sequence is $\mathcal{S}$-admissible.

The **robust closure** of $\mathcal{S}$ is the set of all cycles in G which are connected sums of $\mathcal{S}$-admissable sequences

$$\rho(\mathcal{S}) := \Big\{ Z : \exists\, \ell \geq 1, Z_i \in \mathcal{S},\ 1 \leq i \leq \ell,\ Z = \widehat{+}(Z_1, Z_2, \ldots, Z_\ell) \Big\}. \tag{4}$$

By definition, $\mathcal{S} \subseteq \rho(\mathcal{S}) \subseteq \mathrm{Cyc}(G)$ and ρ preserves inclusion. If $\mathcal{S} = \mathcal{B}$ is a cycle basis for which $\rho^k(\mathcal{B}) := \rho(\rho^{k-1}(\mathcal{B})) = \mathrm{Cyc}(G)$ for a positive integer k, then $\mathcal{B}$ is a **connected sum (CS) basis**; the **depth** of $\mathcal{B}$ is the least such k, the number of iterated robust closures needed to generate all cycles.

Examples of CS bases of depths 1 and 2 are given in [10] and [7]. The depth 1 case (called *robust* bases) includes plane graphs and complete graphs.

It was shown in [3] that for $n \geq 8$, the complete bipartite graph $K_{n,n}$ does not have *any* robust basis, but the basis consisting of all 4-cycles through a fixed edge from [7] is a CS basis. In [4], we gave a general method for constructing cs bases, involving *ear decompositions* [14], based on a theorem of Whitney [19], .

Recall that a **property** of cycles is a subset $\mathcal{P} \subseteq \mathrm{Cyc}(G)$. A property $\mathcal{P}$ is **cooperative** provided $Z_1, Z_2 \in \mathcal{P} \implies Z_1 \widehat{+} Z_2 \in \mathcal{P}$. The following is shown in [10], see also [7], [4].

Theorem 2.1. *If $\mathcal{P}$ is any cooperative property and $\mathcal{P}$ holds for all cycles in a connected sum basis for a graph G, then $\mathcal{P}$ holds for every cycle in G.*

3. Connected sum of directed cycles

It will be convenient to describe a directed versions of connected sum and cooperativity. We collect a few related definitions.

A *digraph* D is an ordered pair (V, A), where $V \neq \emptyset$ is a finite set of vertices and a set of arcs $A \subseteq V \times V$. The *underlying graph* $U(D)$ of D has the same vertex set with $vw \in E(U(D))$ iff $(v, w) \in A$ or $(w, v) \in A$. We also write $a \in A$ with $s(a) = v$ (v is the *source*) $t(a) = w$ (w is the *target*). Let $\deg_+(v)$ denote the *in-degree* of a vertex v which is the number of arcs a with $t(a) = v$ and let $\deg_-(v)$ denote *out-degree* of v, the number of arcs a with $s(a) = v$.

A *quiver* is an ordered pair (V, A), where $V \neq \emptyset$ is a set of vertices and A is a multiset, allowing for each $(v, w) \in V \times V$ a family $a_j, j \in J(v, w)$, of arcs, all with $s(a_j) = v, t(a_j) = w$. Quivers have an underlying multigraph and can be infinite. We write $D = (V, A)$ for both digraphs and quivers.

If D is a digraph (or quiver), then $U(D)$ will denote the *underlying* graph (or multigraph) obtained by discarding the direction of the arcs, replacing them by the corresponding edges.

A *directed walk (diwalk)* in D of length $\ell \geq 0$ is a sequence of vertices $(v_0, v_1, \ldots, v_\ell)$ and a sequence of arcs $(a_1, a_2, \ldots, a_\ell)$ such that

$$t(a_i) = v_i = s(a_{i+1}),\ 1 \leq i \leq \ell - 1,\ \ s(a_1) = v_0,\ \ t(a_\ell) = v_\ell.$$

If all vertices are distinct, the diwalk is called a *dipath*. If $v_0 = v_\ell$, the diwalk is called a *closed diwalk*. A *dicycle* is a closed diwalk where $v_i = v_j$ for $i < j$ implies $i = 0$, $j = \ell$, and $\ell \geq 1$. Loops are dicycles of length 1.

Given any graph G, one can form an *orientation digraph* by choosing for each edge of G exactly one of the two possible arcs.

Note that a digraph is a dicycle iff it is an orientation of a cycle such that

$$\deg_+(v) = 1 = \deg_-(v)$$

for all vertices v. Given a graph G, one forms the *symmetric digraph induced by* G, denoted $D(G)$, by replacing every edge of G by *both* possible arcs, so $D(G)$ is the union of the set of possible orientations. A digraph is *strongly connected* if every ordered pair of vertices is joined by a dipath from the first to the second. Hence, a dicycle is a minimal strongly connected digraph.

Two orientation digraphs that have underlying graphs sharing at least one edge will be in exactly one of the following relations with respect to their common edges: *consistently oriented* (in agreement on all); *oppositely oriented* (disagreeing on all); or *variably oriented*. We use this mostly for dicycles.

Define the **connected sum** $D_1 \widehat{+} D_2$ of two dicycles D_1, D_2 when both of the following hold: the underlying cycles $U(D_1)$ and $U(D_2)$ meet in a non-trivial path and the dicycles are oppositely oriented. In this case, $D_1 \widehat{+} D_2$ is the unique dicycle orientation of the cycle $U(D_1) \widehat{+} U(D_2)$ such that $D_1 \widehat{+} D_2$ is consistently oriented with D_1 and D_2. See Figure 2. We extend connected sum to sequences of dicycles, using leftmost parenthesization analogously with the undirected case.

Note that in any connected sum $\widehat{+}(D_1, D_2, D_3)$ of dicycles for which $U(D_3) \in \{U(D_1), U(D_2)\}$, the repeated cycle appears with both dicycle orientations.

A property of dicycles is *cooperative* if it holds for the connected sum of two dicycles whenever it holds for the summands.

For each cycle Z of G, we define

$$Z^{\pm} := \{Z^+, Z^-\},$$

where Z^+, Z^- are the oppositely-oriented dicycle orientations of Z.

If $\mathcal{S}$ is a set of cycles, we write the corresponding set of dicycles as

$$\mathcal{S}^\pm := \bigcup_{Z \in \mathcal{S}} Z^\pm$$

The orientation of a dicycle is determined on any nontrivial path subgraph. Because of the constrained intersections of connected sum sequences, one can choose the orientations so that each successive pairwise connected sum is the sum of two *oppositely oriented* dicycles. When this is done, the connected sum of the two underlying cycles is oriented consistently with the summands. The following theorem is sufficient to confirm this.

Theorem 3.1. *Let C be any cycle in a graph G with $C = \widehat{+}(Z_1, Z_2, \ldots, Z_k)$ and let C^ε be any orientation of C. Then there exists a unique sequence $(\varepsilon_1, \ldots, \varepsilon_k) \in \{+, -\}^k$ such that*

$$C^\varepsilon = \widehat{+}(Z_1^{\varepsilon_1}, Z_2^{\varepsilon_2}, \ldots, Z_k^{\varepsilon_k}). \tag{5}$$

Proof. By induction on k. The claim is trivial for $k = 1$ where $C = Z_1$ and ε_1 is determined by the orientation of C. Assume the result for $k - 1$ and suppose that C has a connected sum (3). Let $Z_k^{\varepsilon_k}$ be the unique orientation of Z_k consistent with C^ε. Put $C_k^\varepsilon := C^\varepsilon \widehat{+} Z_k^{-\varepsilon_k}$. Then

$$C_k = \widehat{+}(Z_1, \ldots, Z_{k-1}).$$

By the inductive hypothesis, we have unique $\varepsilon_1, \ldots, \varepsilon_{k-1}$ and (5) holds. □

The results here show that, as 1-chains over the integers, dicycles can be built so that all coefficients are 0, 1, or -1, using a connected sum basis. Further, a cooperative property holds for all dicycles in a digraph when it holds for all the dicycles in $\mathcal{B}^\pm$ where $\mathcal{B}$ is a CS basis for $U(D)$.

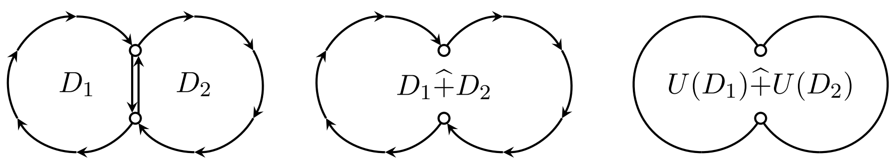


Figure 2: Two compatible dicycles (left), their connected sum (center) and the connected sum of their underlying graphs (right).

4. Diagrams in groupoids

In this section, we review our previous result on cooperativity of commutativity in groupoid diagrams and show how to extend this to groupoid diagrams which are only commutative up to a natural transformation.

Let D be a digraph and let $\mathcal{C}$ be a category. A *diagram* δ *of shape* D in $\mathcal{C}$ is a homomorphism from D to the underlying quiver of $\mathcal{C}$. For example, the digraph with vertices a, b, c, d and arcs $(a, b), (b, c), (a, d), (d, c)$ could be mapped by sending all vertices to a fixed object X in $\mathcal{C}$, with the arcs associated with various morphisms $X \to X$. See Mac Lane [15, p. 8].

Two dipaths in a digraph with a common source and a common target vertex are called *parallel*; in the extreme case, they are *internally disjoint* but this is not required. A diagram δ of shape D in $\mathcal{C}$ is *parallel-commutative* if for any two parallel dipaths in D, the corresponding dipaths in $\mathcal{C}$ give the same composite morphism. But any diagram parallel-commutes if its shape has no two distinct parallel paths (e.g., a cycle in which arcs alternate in direction).

Instead, we shall consider a stronger type of commutativity which, however, is only defined when the morphisms of the diagram are all invertible; that is, when the category $\mathcal{C}$ is a *groupoid* $\mathcal{G}$ [15, p. 20], [18, pp. 45, 134].

For a diagram

$$\delta : D \to \mathcal{G}$$

in a groupoid $\mathcal{G}$, we say that δ *groupoid-commutes* (g-commutes) if the composition around any cycle of the underlying graph of D induces an identity morphism in $\mathcal{G}$ [7], [4]. We assume that arc $x = (v, w)$ of D, traversed in proper order while going around the cycle, produces the morphism $\delta(x)$ but traversed in reverse, produces the morphism $\delta(x)^{-1}$ from $\delta(w)$ to $\delta(v)$.

It is easy to check that the groupoid commutativity of a cycle is independent of which traversal is chosen (i.e., of starting point and of clockwise vs counterclockwise orientation). But the particular identity morphism may depend on starting point.

Indeed, consider the following case, which is sufficient. Let the diagram have two objects X and Y with morphisms $a : X \to Y$ and $b : Y \to X$. Suppose that (i) $ba = 1_X$, where we write composition from right to left as usual. Hence, (ii) $ab = 1_Y$. Also, (i) implies (iii) $a^{-1}b^{-1} = 1_X$.

The following was shown in [10].

Theorem 4.1. *Groupoid commutativity is a cooperative property.*

The significance of the result follows from the fact that in [4], we exhibited a diagram $\delta : D \to \mathbb{C}^*$, where D is the orientation of the complete bipartite graph $K_{3,3}$ shown below, $\mathbb{C}^*$ is the group of nonzero complex numbers, and each morphism is rotation by $\frac{2\pi}{3}$, together with a particular basis $\mathcal{B}$ of $U(D)$, indicated on the right, such that δ commutes on the members of $\mathcal{B}$ (and indeed on any 6-cycle but does *not* commute on the 4-cycles. See Figure 3.

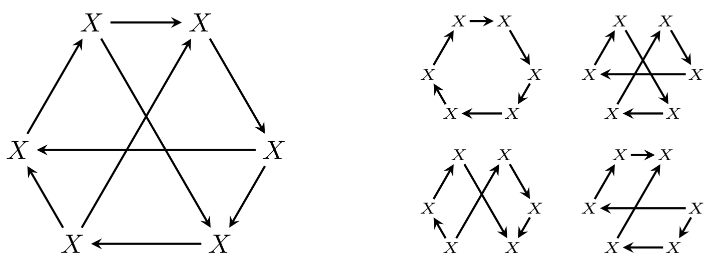


Figure 3: A noncommutative diagram (left) that commutes on a cycle basis (right). This basis is not a CS basis because no two of its members are compatible.

Assume where necessary that categories are *small* with only a *set* of objects. A *natural transformation* $\nu : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of $\mathcal{D}$-morphisms indexed by the objects of $\mathcal{C}$

$$\{\nu_x : Fx \to Gx\}_{x \in Obj(C)}$$

such that for every $\mathcal{C}$-morphism $\alpha : x \to y$, we have

$$\nu_y \circ F(\alpha) = G(\alpha) \circ \nu_x;$$

that is, all the associated squares commute. A *natural equivalence* is a natural transformation all of whose arrows are equivalences. A natural transformation is an equivalence iff it is invertible as a natural transformation.

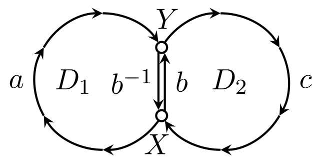
Let CAT be a small subcategory of the category of all small categories with functors as morphisms. We shall consider a fixed groupoid subcategory **G** of CAT. Let D be an orientation digraph and let $\delta : D \to$ **G** be any diagram. Then one can extend δ in a unique way to a diagram $\widehat{\delta}$ on the symmetric digraph which D induces

$$\widehat{\delta}: \widehat{D} := D(U(D)) \to \mathbf{G} \tag{6}$$

Commutativity up to natural equivalence for diagrams in $\mathbf{G}$ means that the composition functor around any dicycle in $\widehat{D}$ is naturally equivalent to the appropriate identity functor.

Theorem 4.2. *Commutativity up to natural equivalence is cooperative.*

Proof. Without loss of generality, we take two dicycles D_1 and D_2 which are oppositely oriented with $P := U(D_1) \cap U(D_2)$ a nontrivial path. Let P^+ be the orientation of P consistent with D_1 and let Y and X, resp., denote the first and last vertex of P^+. We write a for the composition of the morphisms along the path $D_1 - P^+$ from X to Y, and b for the composition along P^+ from Y to X. For the other dicycle, we do the same thing: let c denote the composition of the morphisms along $D_2 - P^-$ from Y to X. By definition, the composition of the morphisms in P^- from X to Y is b^{-1}.



Suppose now that both dicycles commute up to natural equivalences; let

$$\nu : b \circ a \Rightarrow 1_X \quad \text{and} \quad \mu : c \circ b^{-1} \Rightarrow 1_X. \tag{7}$$

where ν and μ are natural equivalences. We define a composition $\tau := \mu \Box \nu$ which is both a natural transformation and an equivalence; for every $x \in X$,

$$\tau_x := \mu_x \circ \Big((c \circ b^{-1})(\nu_x)\Big), \tag{8}$$

which is an X-morphism from $\Big((c \circ b^{-1}) \circ (b \circ a)\Big)(x) = (c \circ a)(x)$ to x. Hence, $D_1 \widehat{+} D_2$ commutes up to natural equivalence. This is illustrated in the following four commutative squares.

The first square expresses the fact that ν is a natural transformation; the second applies the functor cb^{-1}; the third expresses the naturality of μ; and the fourth is the (vertical) composition of the second and third squares.

$$\begin{array}{ccc} ba(x) & \xrightarrow{ba(\alpha)} & ba(x') \\ \nu_x \downarrow & & \downarrow \nu_{x'} \\ x & \xrightarrow[\alpha]{} & x' \end{array}$$

$$\begin{array}{ccc} ca(x) & \xrightarrow{ca(\alpha)} & ca(x') \\ \downarrow & & \downarrow \\ cb^{-1}(x) & \xrightarrow[cb^{-1}(\alpha)]{} & cb^{-1}(x') \end{array}$$

$$\begin{array}{ccc} cb^{-1}(x) & \xrightarrow{cb^{-1}(\alpha)} & cb^{-1}(x') \\ \mu_x \downarrow & & \downarrow \mu_{x'} \\ x & \xrightarrow[\alpha]{} & x' \end{array}$$

$$\begin{array}{ccc} ca(x) & \xrightarrow{ca(\alpha)} & ca(x') \\ \tau_x \downarrow & & \downarrow \tau_{x'} \\ x & \xrightarrow[\alpha]{} & x' \end{array}$$

This completes the proof. □

5. Application to Markov chains

In this section, we consider the *Kolmogorov cycle criterion* (KCC) for the reversibility of discrete Markov chains. See, e.g., Kelly [11, chap. 1].

Let $(X(t), t \in T)$ be a Markov chain with a finite or countable state-space $\mathcal{S} \subseteq \mathbb{N}_+$. A Markov chain is *stationary* if for all $n \in \mathbb{N}_+$ and all $\tau, t_1, \ldots, t_n \in T$,

$$\Big(X(t_1), X(t_2), \ldots, X(t_n)\Big) \sim \Big(X(\tau + t_1), X(\tau + t_2), \ldots, X(\tau + t_n)\Big),$$

where $A \sim B$ denotes the relation of equality of distributions.

A Markov chain is *reversible* if for all $n \in \mathbb{N}_+$ and all $\tau, t_1, \ldots, t_n \in T$,

$$\Big(X(t_1), X(t_2), \ldots, X(t_n)\Big) \sim \Big(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n)\Big).$$

Reversibilty implies stationarity [11, p. 5]. Also, a stationary Markov chain $X(t)$ is *time-homogeneous*; i.e., for all $\tau, t_1, t_2 \in T$ and all $j, k \in \mathcal{S}$,

$$P(X(t_1 + \tau) = k | X(t_1) = j) = P(X(t_2 + \tau) = k | X(t_2) = j).$$

Write $p(j, k)$ for $P\Big(X(t + 1) = k \;\Big|\; X(t) = j\Big)$ as it is independent of t and for every state j the state transitions describe all events, so $\sum_{k \in \mathcal{S}} p(j, k) = 1$.

We define the *communications digraph* $D(X)$ of a Markov chain X to be the digraph with vertex set $\mathcal{S}$, where (j, k) is an arc iff $p(j, k) > 0$. A Markov chain is *irreducible* if and only if its communications digraph is *strongly connected* (there is a positive probability of a dipath joining each pair of states).

A Markov chain is *periodic* if there exists $d > 1$, $d \in \mathbb{N}_+$, such that

$$P\left(X(t + \tau) = j \;\Big|\; X(t) = j\right) > 0 \quad \Rightarrow \quad d | \tau$$

($d | \tau$ means d divides τ). If the chain is not periodic, it is called *aperiodic*.

The following result is well-known (e.g., [11, p. 6]).

Theorem 5.1. *A stationary, irreducible, and aperiodic Markov chain $X(t)$ is reversible if and only if there exists a function μ on $\mathcal{S}$ with $\mu(j) > 0$ for all j and $\sum_{j \in \mathcal{S}} \mu(j) = 1$ such that, for all $j, k \in \mathcal{S}$,* detailed balance *holds:*

$$\mu(j)p(j, k) = \mu(k)p(k, j). \tag{9}$$

As Kelly [11, p. 21] puts it, "... *it is natural to ask whether we can establish the reversibility of a process directly from the transition rates alone.*"

The *Kolmogorov Cycle Criterion* (KCC) makes this possible. (There are also extensions to continuous-time Markov processes which we omit here.) The KCC for a closed diwalk

$$\omega := (j_1, j_2, \ldots, j_n, j_1)$$

asserts that

$$P(\omega) = P(\omega^{op}), \tag{10}$$

where ω^{op} denotes the diwalk oppositely orientated to ω,

$$\omega^{op} = (j_1, j_n, \ldots, j_2, j_1)$$

and the probability of a diwalk is the product of the probabilities of its arcs,

$$P(\omega) := p(j_1, j_2) \ldots p(j_{n-1}, j_n)\, p(j_n, j_1).$$

We sketch Kelly's argument [11, p. 22] for the original result of [12].

Theorem 5.2. *A stationary, irreducible, and aperiodic Markov chain $X(t)$ is reversible if and only if the KCC* (10) *holds for all closed walks ω in $D(X)$.*

Proof. If X is reversible, then by the previous theorem, there exists a positive measure on $\mathcal{S}$ which satisfies detailed balance (9) for each oppositely oriented pair of arcs in $D(X)$. Take the product of the set of detailed-balance equations corresponding to the arcs in ω and divide by the product of the (positive!) measures of the states which occur (in reverse order) for the two opposing diwalk orientations. The result is equation (10).

Conversely, suppose that (10) holds for every closed walk. One defines a positive measure μ as follows. Select an arbitrary base-point j_0 in $\mathcal{S}$ and let $j \in \mathcal{S}$. As X is irreducible, there exists a diwalk (in fact, a dipath) ω from j to j_0 in the communications digraph $D(X)$. Define $\mu(j)$ by the following equation,

$$\mu(j) \;=\; B\,\frac{P(\omega^{op})}{P(\omega)}, \tag{11}$$

where B is an arbitrary positive constant that can later be adjusted to give a probability measure.

To see that $\mu(j)$ does not depend on the path from j to j_0, let ζ denote another j-j_0-dipath in $D(X)$. Using "$*$" to concatenate diwalks, the equation

$$\frac{P(\omega^{op})}{P(\omega)} = \frac{P(\zeta^{op})}{P(\zeta)} \tag{12}$$

holds by (10) as $P(\omega)P(\zeta^{op}) = P(\omega * \zeta^{op}) = P(\zeta * \omega^{op}) = P(\omega^{op})P(\zeta)$.

Also, $\mu(j) > 0$; indeed, as there is a diwalk η in $D(X)$ from j_0 to j, the concatenation $\omega * \eta$ is a closed diwalk of positive probability, so by (10), $P(\eta^{op}) > 0$; hence, with η^{op} instead of ω in (11), $\mu(j) > 0$.

It is routine to show that each arc satisfies (9). □

The following result applies our theory to obtain a more efficient characterization of reversibility.

Theorem 5.3. *A stationary, irreducible, and aperiodic Markov chain $X(t)$ is reversible if and only if the KCC* (10) *holds for all dicycles in $\mathcal{B}^{\pm}$, where $\mathcal{B}$ is any* CS *basis of $U(D(X))$*

Proof. One direction is trivial in view of Kolmogorov's theorem. In the opposite direction, suppose that his criteria hold for all the dicycles of a CS basis. Once we've established the next theorem, it follows that the KCC holds for all dicycles and hence X is reversible. □

Theorem 5.4. *The KCC is a cooperative property for dicycles in the communications digraph of a stationary, irreducible, and aperiodic Markov chain.*

Proof. We follow the same outline as in the proof of Theorem 4.2. Take oppositely oriented dicycles D_1 and D_2 with $P := U(D_1) \cap U(D_2)$ a nontrivial path. Let P^+ be the orientation of P consistent with D_1 and let Y and X, resp., denote the first and last vertex of P^+. We write α for the dipath $D_1 - P^+$ from X to Y, and β for the dipath P^+ from Y to X. Let γ denote the dipath $D_2 - P^-$ from Y to X. As D_1 and D_2 satisfy the KCC, we have the equations

$$P(\alpha)P(\beta) = P(\alpha^{op})P(\beta^{op})$$

$$P(\gamma)P(\beta^{op}) = P(\gamma^{op})P(\beta)$$

Multiplying the two equations and cancelling the positive term $P(\beta)P(\beta^{op})$ gives the KCC for $D_1 \widehat{+} D_2$, $P(\alpha)P(\gamma) = P(\alpha^{op})P(\gamma^{op})$. □

We now apply connected sum theory to an exercise [11, Ex. 1.5.2, p. 24].

Proposition 5.5 (Kelly). *Let $X(t)$ be a stationary, irreducible, and aperiodic Markov chain. If $j_0 \in \mathcal{S}$ is such that for every $j \in \mathcal{S}$, we have $p(j, j_0) > 0$, then X is reversible iff the KCC holds for all 3-cycles through j_0; that is, for all $j_1 \neq j_2 \in \mathcal{S} \setminus \{j_0\}$,*

$$p(j_0, j_1)\, p(j_1, j_2)\, p(j_2, j_0) \;=\; p(j_0, j_2)\, p(j_2, j_1)\, p(j_1, j_0).$$

Proof. The bouquet of triangles centered at j_0 is a CS basis for K_n for every $n \geq 3$ by [7, Proposition 1], and any dicycle in $D(X)$ is contained in some $D(K_n)$. □

6. Discussion

The implications of cooperativity for commutativity are interesting from the perspective of *the information theory of mathematics*. For example, one can define the structure of a group by a set of commutative diagrams and it is well-known that only a subset need to be checked. As with application of the Cube Lemma [16, p. 43], savings are modest. However, in a more complex situation, savings might be substantial, cf. [7], [9].

Perhaps the theory of diagrams which commute or commute up to natural equivalence could decrease the complexity of verifying commutativity for the groupoid diagrams involved in higher category theory and adjointness.

Indeed, the only result in the literature, of which we are aware, with a similar direction to ours is in Gray [2], who proved that a hypercube diagram in a 2-category is 2-commutative if and only if all its Q_3-subgraphs are 2-commutative.

Are there applications of connected sum theory to natural processes in biology and physics? The notion that cycles can be generated in a hierarchical fashion so that one must first prepare the ingredients in a previous stage before combining them in a connected sum could be a desirable feature.

Also, are there implications for the random spread of cooperative properties? Given a fixed probability that any one cycle will have the property, if the number of cs bases grows sufficiently rapidly as a function of the order of a graph family, then we might expect that there is a threshold number of vertices above which the property almost surely holds.

What other properties of cycles are cooperative?

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