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A STUDY OF PENON WEAK n-CATEGORIES, PART 2: A MULTISIMPLICIAL NERVE CONSTRUCTION

Thomas COTTRELL

Résumé. Dans cet article nous faisons le premier pas vers une comparaison entre une définition algébique et une définition non-algébrique des n-catégories faibles. Cette comparaison prend la forme d'un foncteur 'nerf', selon la méthode établie pour passer du cadre algébrique au cadre non-algébrique. La définition algébrique que nous utilisons est due à Penon, et pour la définition non-algébrique nous utilisons une variante selon Simpson de la définition de Tamsamani. Comme prototype de notre construction du nerf, nous rappelons la construction du nerf pour les bicatégories proposée par Leinster et nous montrons que le nerf d'une bicatégorie ainsi obtenu est une 2-catégorie faible au sens de Tamsamani-Simpson. Nous définissons alors notre foncteur nerf pour les n-catégories faibles. Enfin nous prouvons que le nerf d'une 2-catégorie faible au sens de Penon est une 2-catégorie faible au sens de Tamsamani-Simpson, et nous faisons l'hypothèse que ce résultat s'étend aux niveaux n supérieurs.

Abstract. In this paper we take the first step towards a comparison between an algebraic and a non-algebraic definition of weak n-category. This comparison takes the form of a nerve functor, the established method of moving from the algebraic setting to the non-algebraic setting. The algebraic definition we use is that due to Penon, and the non-algebraic definition we use is Simpson's variant of Tamsamani's definition. As a prototype for our nerve construction, we recall a nerve construction for bicategories proposed by Leinster, and prove that the nerve of a bicategory given by this construction is a Tamsamani–Simpson weak 2-category. We then define our nerve functor for Penon weak n-categories. We prove that the nerve of a Penon weak 2-category is a Tamsamani–Simpson weak 2-category, and conjecture that this result holds for higher n.

Keywords. n-category, higher-dimensional category, nerve construction.
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1. Introduction

The aim of this paper, the second in a two-part series on Penon weak n-categories, is to make the first comparison between an algebraic definition and a non-algebraic definition of weak n-category. Many definitions of weak n-category have been proposed [32, 1, 3, 29, 33, 20, 26, 17, 18, 19], and it has been widely observed that each of these definitions is of one of two types: algebraic definitions, in which composites and coherence cells are explicitly specified, and non-algebraic definitions, in which a coherent choice of composites and constraint cells is merely required to exist [24, p. 5]. Although there is a large number of different definitions, relatively few comparisons have been made between them, and most of the comparisons that have been made are either exclusively between algebraic definitions, or exclusively between non-algebraic definitions [4, 10, 9, 21, 8, 11, 2]. Very little progress has been made in comparing algebraic and non-algebraic definitions, with the only existing comparisons being restricted to the case $n = 2$ (see [14, 24, 22, 16]). Moving between the algebraic and non-algebraic settings is difficult; it is not simply a case of taking a non-algebraic definition and making choices of composites and coherence cells, or of taking an algebraic definition and just asking for existence in place of specified structure.

One established method of moving between the algebraic and non-algebraic settings is the idea of a “nerve construction”. This idea arose from the well-known nerve construction for categories, which allows us to express a category as a simplicial set satisfying a “nerve condition”. Roughly speaking, a nerve construction takes an algebraic object, and produces from it a particular kind of presheaf, so a nerve construction can be seen as a way of passing from an algebraic setting to a non-algebraic setting. Various authors have given nerve constructions for algebraic definitions of weak n-category [34, 28, 7], but these have focussed on extracting a canonical nerve from a given algebraic notion of n-category, rather than making connections with existing non-algebraic definitions. This can be seen as creating a new non-algebraic definition corresponding to the given algebraic definition; the presheaves this approach gives are therefore specific to the chosen algebraic definition, and are unlikely to be presheaves on a category that arises naturally elsewhere. One exception to this is the case of strict ω-categories; Berger has shown that, in this case, the canonical nerve is a presheaf on a category that arises naturally as a wreath product of the simplex category Δ [5, 6].

In this paper we describe a nerve construction for weak n-categories. The algebraic definition this construction uses is that of Penon [29, 13], and it is designed to allow for comparison with the non-algebraic definition due to Tamsamani and Simpson [33, 31].

The reason for choosing to use Penon weak n-categories over another algebraic definition is that we are able to give an explicit description of Penon's monad (described in detail in Part 1 of this series), and thus of a free Penon weak n-category. This was very useful when devising the nerve constructions in this paper; these constructions involve algebras that are almost free, and the construction of Penon's monad in this chapter made it possible to modify the free algebra construction in a way that would not be possible with other algebraic definition, such as those of Batanin and Leinster. In spite of its unusual construction, Penon's monad is known to arise from an n-globular operad with contraction and system of compositions (see [4]), so this definition belongs to a commonly studied family of definitions of weak n-category.

There are two reasons for choosing to use Tamsamani–Simpson weak n-categories for the comparison. First, algebraic definitions such as Penon's are generally globular, with a set of cells for each dimension. The Tamsamani–Simpson definition also draws a clear distinction between different dimensions of cell; although this is universally true of algebraic definitions, it is not so commonly true of non-algebraic "nerve-like" definitions. Second, we are able to use an existing nerve construction – Leinster's nerve construction for bicategories, described in Section 3, which compares bicategories to Tamsamani–Simpson weak 2-categories – as a prototype for our construction.

The paper is structured as follows: in Section 2 we recall the definition of Tamsamani–Simpson weak n-category. In Section 3 we recall a nerve construction for bicategories given by Lack and Paoli [22], and adapt this into a form which we will use as a prototype for our nerve construction for Penon weak n-categories, following earlier work of Leinster [24]. We then prove that the nerve of a bicategory given by this nerve construction is a Tamsamani–Simpson weak 2-category. In Section 4 we recall the definition of Penon weak n-category. In Section 5 we give our nerve construction for Penon weak n-categories in the case $n = 2$. In Section 6 we prove that the nerve of a Penon weak 2-category satisfies the Segal condition, and is therefore a Tamsamani–Simpson weak 2-category. The proof is unavoidably technical, and is also in some parts elementary, and we apologise for this; both Penon weak n-categories and Tamsamani–Simpson weak n-categories are naturally arising in their own contexts, but these contexts are very different, and it is inevitable that any comparison will be technically complicated. In this proof we use the notation for the cells of a Penon weak n-category given by our construction of Penon's monad from Part 1 of this series. In Section 7, we give our nerve construction for general n. Finally, in Section 8, we conjecture that the nerve it gives is a Tamsamani–Simpson weak n-category, and discuss possible directions for further investigation.

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2. Tamsamani–Simpson weak n-categories

In this section we recall Simpson's variant of Tamsamani's definition of weak n-category [33, 31]. We begin by generalising the definition of simplicial set to that of an *n-simplicial set* (often known as a *multisimplicial set* when not specifying the value of n).

Definition 2.1. The *category of n-simplicial sets* n-$\mathbf{SSet}$ is defined inductively as follows:

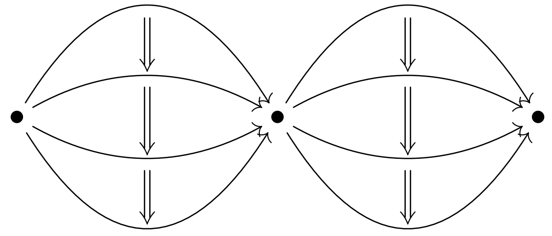
- 0-$\mathbf{SSet} := \mathbf{Set}$;
- for $n \geq 1$, n-$\mathbf{SSet} := [\Delta^{\mathrm{op}}, (n-1)\text{-}\mathbf{SSet}] \cong [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$, by cartesian closedness of $\mathbf{Cat}$.

We could have defined n-simplicial sets to be presheaves on Δ^n directly, but the form of the definition stated above highlights the fact that n-simplicial sets can be obtained by a process of repeated internalisation, which is a well-established method of adding extra dimensions; thus this illustrates why Δ^n is a reasonable category on which to take presheaves in a definition of weak n-category. Note that the inductive nature of this definition means that the definition of Tamsamani–Simpson weak n-category does not a priori allow for the case $n = \omega$. We write an object of Δ^n as an n-tuple

$$\mathbf{k} = (k_1, k_2, \ldots, k_n),$$

where, for all $1 \leq i \leq n$, $k_i \in \mathbb{N}$.

We now explain how we should think of the shapes of cells in an n-simplicial set for the purposes of the definition of Tamsamani–Simpson weak n-category. In Δ, the object $[k]$ can be thought of as a string of k composable morphisms. Similarly, in an n-simplicial set $A\colon (\Delta^n)^{\mathrm{op}} \to \mathbf{Set}$, the set $A(k_1, k_2, \ldots, k_n)$ can be thought of as the set of pasting diagrams called "cuboidal" by Leinster [25]. A cuboidal pasting diagram $(k_1, k_1, \ldots, k_n) \in \Delta^n$ consists of a grid of n-cells which is k_1 n-cells long, k_2 n-cells high, ..., and k_n n-cells wide; for example, the cuboidal pasting diagram $(2, 3) \in \Delta^2$ is shown in the diagram below.



For this to give a globular notion of weak n-category, we need to ensure that, if $g : x \to x'$ is a k-cell in a weak n-category, then the $(k-1)$-cells x

and x' have the same source and the same target. To do so we require that, for any j, if $k_j = 0$, i.e. the pasting diagram is 0 j-cells wide, then $j - 1$ should be the maximum dimension of cell in the diagram. In order to deal with this issue we use Simpson's approach, which is to use presheaves on a quotient of Δ^n, denoted Θ^n, rather than using presheaves on Δ^n itself. Note that if we do not ensure that our cells are globular, we obtain a definition of weak n-tuple category (also known as a weak n-fold category).

We define Θ^n as a coequaliser in $\mathbf{Cat}$. The idea is to identify objects in Δ^n if they are to be thought of as the same cuboidal pasting diagram. For example, in Θ^2, given an object (j, k), if $j = 0$ the pasting diagram has zero width, so the value of k should make no difference since the pasting diagram must also have zero height. Thus in Θ^2 we identify all objects of the form $(0, k)$, so Θ^2 looks like:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
 & (1,2) \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & (2,2) & \cdots \\
 & (1,1) \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[u, shift left=4] \arrow[u, leftarrow, shift left=2] \arrow[u] \arrow[u, leftarrow, shift right=2] \arrow[u, shift right=4] & (2,1) \arrow[u, shift left=4] \arrow[u, leftarrow, shift left=2] \arrow[u] \arrow[u, leftarrow, shift right=2] \arrow[u, shift right=4] & \cdots \\
(0,0) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[ur, bend left, shift left=2] \arrow[ur, bend left, leftarrow] \arrow[ur, bend left, shift right=2] \arrow[uur, bend left, shift left=2] \arrow[uur, bend left, leftarrow] \arrow[uur, bend left, shift right=2] & (1,0) \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] & (2,0) \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] & \cdots
\end{tikzcd}$$

Similarly, for higher values of n, objects of Δ^n are identified in Θ^n if they differ only after a 0.

Definition 2.2. We define a category Θ^n as a coequaliser in $\mathbf{Cat}$ as follows: first, let R be the subcategory of $\Delta^n \times \Delta^n$ with

- objects: for all objects $(k_1, k_2, \ldots, k_n)$ of Δ^n,

  $$((k_1, k_2, \ldots, k_n), (k_1, k_2, \ldots, k_n))$$

  is in R; also, for a fixed j with $1 \leq j < n$,

  $$((k_1, k_2, \ldots, k_j, \ldots, k_n), (k'_1, k'_2, \ldots, k'_j, \ldots, k'_n))$$

  is in R if $k_j = 0 = k'_j$ and $k_i = k'_i$ for all $i < j$;

- morphisms: let l, $m \leq n$, and let $(\mathbf{k}, 0) = (k_1, \ldots, k_l, 0, \ldots, 0)$ and $(\mathbf{k}', 0) = (k'_1, \ldots, k'_m, 0, \ldots, 0)$ be objects of Δ^n. Then the morphism

  $$(\phi, \psi)\colon ((\mathbf{k}, 0), (\mathbf{k}, 0)) \to ((\mathbf{k}', 0), (\mathbf{k}', 0)),$$

  where $\phi = (\phi_1, \ldots, \phi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$, is in R if
  - for all $i \leq j$, $\phi_i = \psi_i$;
  - $\phi_j : [k_j] \to [k'{}_j]$ factors through $[0]$ in Δ.

Since R is a subcategory of $\Delta^n \times \Delta^n$, it comes equipped with projection maps π_1, $\pi_2\colon R \to \Delta^n$. The category Θ^n is defined to be the coequaliser of the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \Delta^n \end{tikzcd}$$

in $\mathbf{Cat}$. A presheaf

$$A\colon (\Theta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is called an *n-precategory.*

Given an n-precategory

$$A\colon (\Theta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

and given an object $\mathbf{j}$ of Θ^n, we refer to an element of the set $A(\mathbf{j})$ as a "$\mathbf{j}$-cell".

Note that this is not the only way of ensuring that we have globular cells; in the original definition, Tamsamani takes presheaves on Δ^n, then includes an extra condition to ensure that the cells are globular. In their expositions of Simpson's definition, both Cheng and Lauda [12] and Leinster [24] also take this approach. Using Simpson's approach does make a difference, since it leads to a definition of a weak n-category as a presheaf satisfying the Segal condition, with no extra conditions; this allows us to work with a presheaf category, with all the usual desirable properties these have, such as completeness, cocompleteness, and the existence of the Yoneda embedding.

We now discuss the Segal condition, Tamsamani's n-dimensional generalisation of the nerve condition for categories originating in [30]. The Segal condition is a condition on a family of morphisms of n-precategories, called the *Segal maps*; these Segal maps are defined to be induced by wide pullbacks.

In the nerve condition for categories the Segal maps are required to be isomorphisms, to ensure that well-defined, associative, unital composition could be extracted from the nerve. In the Segal condition for weak n-categories, we wish to weaken this since we only want composition that is associative and unital up to coherent isomorphism. If the Segal maps were maps of n-categories we would instead require them to be equivalences. However, the Segal maps are merely maps of n-precategories, so we cannot use the same notion of equivalence. A functor is an equivalence if it is full, faithful, and essentially surjective on objects; for a map of n-precategories, we can still define fullness and faithfulness in the same way, but we cannot define what it means for a map to be essentially surjective since we do not have a composition structure, and thus no notion of isomorphism between cells.

It was Simpson's insight that, instead of asking for essential surjectivity, one can demand surjectivity on 0-cells. Simpson observed that the resulting notion, which we call *contractibility*, is enough for the purposes of the Segal condition, although it is not enough to define equivalences in general. (Note that Simpson uses the phrase "easy equivalence" where we use "contractible map".)

Before defining contractibility, we establish some notation used in the definition. Let $0 \leq p \leq n$, and write $\mathbf{1}_p$ for the equivalence class in Θ^n of the object

$$(\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{n-p})$$

of Δ^n, which should be thought of as a single globular p-cell.

Let $A \colon (\Theta^n)^{\mathrm{op}} \to \mathbf{Set}$ be an n-precategory. In Δ, we have maps σ, $\tau : [0] \to [1]$, with $\sigma(0) = 0$ and $\tau(0) = 1$. We define the source and target maps (denoted s and t respectively) in A, for each p, as follows:

$$s = A(\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p-1}, \sigma, \mathrm{id}, \ldots, \mathrm{id}) : A(\mathbf{1}_p) \to A(\mathbf{1}_{p-1});$$

$$t = A(\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p-1}, \tau, \mathrm{id}, \ldots, \mathrm{id}) : A(\mathbf{1}_p) \to A(\mathbf{1}_{p-1}).$$

Note that this defines the underlying n-globular set of the n-precategory A, with the set of p-cells for each $0 \leq p \leq n$ given by $A(\mathbf{1}_p)$.

We now give the definition of contractibility.

Definition 2.3. Let $m \geq 1$, let A, $B \colon (\Theta^m)^{\mathrm{op}} \to \mathbf{Set}$ be m-precategories, and let $\alpha \colon A \to B$ be a map of m-precategories. For each $0 \leq p \leq m-1$, we write $A(\mathbf{1}_p) \times_{B(\mathbf{1}_p)} B(\mathbf{1}_{p+1}) \times_{B(\mathbf{1}_p)} A(\mathbf{1}_p)$ for the limit of the diagram

$$\begin{tikzcd}
 & & A(\mathbf{1}_p) \arrow[d, "\alpha_{\mathbf{1}_p}"] \\
 & B(\mathbf{1}_{p+1}) \arrow[r, "s"'] \arrow[d, "t"] & B(\mathbf{1}_p) \\
A(\mathbf{1}_p) \arrow[r, "\alpha_{\mathbf{1}_p}"'] & B(\mathbf{1}_p) &
\end{tikzcd}$$

in $\mathbf{Set}$. We also have a cone over this diagram with vertex $A(\mathbf{1}_{p+1})$, as shown in the diagram below:

$$\begin{tikzcd}
A(\mathbf{1}_{p+1}) \arrow[rr, "s"] \arrow[dr, "\alpha_{\mathbf{1}_{p+1}}"'] \arrow[dd, "t"'] & & A(\mathbf{1}_p) \arrow[d, "\alpha_{\mathbf{1}_p}"] \\
 & B(\mathbf{1}_{p+1}) \arrow[r, "s"'] \arrow[d, "t"] & B(\mathbf{1}_p) \\
A(\mathbf{1}_p) \arrow[r, "\alpha_{\mathbf{1}_p}"'] & B(\mathbf{1}_p) &
\end{tikzcd}$$

The universal property of the limit induces a unique map

$$\tilde{\alpha}_{\mathbf{1}_{p+1}} \colon A(\mathbf{1}_{p+1}) \to A(\mathbf{1}_p) \times_{B(\mathbf{1}_p)} B(\mathbf{1}_{p+1}) \times_{B(\mathbf{1}_p)} A(\mathbf{1}_p)$$

such that

$$\begin{tikzcd}
A_{\mathbf{1}_{p+1}} \arrow[rrrd, bend left=15, "s"] \arrow[rd, dashed, "\tilde{\alpha}_{\mathbf{1}_{p+1}}"] \arrow[rrdd, bend right=40, "\alpha_{\mathbf{1}_{p+1}}"'] \arrow[rddd, bend right=50, "t"'] & & & \\
& A(\mathbf{1}_p) \times_{B(\mathbf{1}_p)} B(\mathbf{1}_{p+1}) \times_{B(\mathbf{1}_p)} A(\mathbf{1}_p) \arrow[rr] \arrow[rd] \arrow[dd] & & A(\mathbf{1}_p) \arrow[d, "\alpha_{\mathbf{1}_p}"] \\
& & B(\mathbf{1}_{p+1}) \arrow[r, "s"'] \arrow[d, "t"] & B(\mathbf{1}_p) \\
& A(\mathbf{1}_p) \arrow[r, "\alpha_{\mathbf{1}_p}"'] & B(\mathbf{1}_p) &
\end{tikzcd}$$

commutes.

The map $\alpha\colon A \to B$ is said to be contractible if:

- the map $\alpha_{\mathbf{1}_0}\colon A(\mathbf{1}_0) \to B(\mathbf{1}_0)$ is surjective (this is surjectivity of α on objects);
- for each $0 \leq p \leq m-1$, the map
$$\tilde{\alpha}_{\mathbf{1}_{p+1}}\colon A(\mathbf{1}_{p+1}) \to A(\mathbf{1}_p) \times_{B(\mathbf{1}_p)} B(\mathbf{1}_{p+1}) \times_{B(\mathbf{1}_p)} A(\mathbf{1}_p)$$
is surjective (this gives fullness at dimension $(p+1)$);
- for each $p = m-1$, the map
$$\tilde{\alpha}_{\mathbf{1}_{p+1}}\colon A(\mathbf{1}_{p+1}) \to A(\mathbf{1}_p) \times_{B(\mathbf{1}_p)} B(\mathbf{1}_{p+1}) \times_{B(\mathbf{1}_p)} A(\mathbf{1}_p)$$
is injective (this gives faithfulness at dimension m).

Note that the definition of contractibility above is only concerned with the effect of A and B on $\mathbf{1}_p$. The set $A(\mathbf{1}_p)$ is the set of "globular p-cells", i.e. p-cells in A that are one 1-cell long, one 2-cell high, etc.; there are no cells composed end-to-end (and similarly for B).

We now give the construction of the Segal maps. In the nerve condition for categories one considers composable strings of k morphisms for every $k \in \mathbb{N}$; here we consider, for every $0 \leq m \leq n$, the composable strings of k m-cells for every k and every composite of m-cells.

Let $A\colon (\Theta^n)^{\mathrm{op}} \to \mathbf{Set}$ be an n-precategory. Then, for all $1 \leq m \leq n$, and all $\mathbf{k} = (k_1, \dots, k_{m-1})$, we have a functor

$$\begin{aligned} A(\mathbf{k}, -, -) : \Delta^{\mathrm{op}} &\to [(\Theta^{n-m})^{\mathrm{op}}, \mathbf{Set}] \\ [k] &\mapsto A(\mathbf{k}, k, -), \end{aligned}$$

with the effect on morphisms given by composition.

Consider the following diagram in Δ:

$$\begin{tikzcd}
 & & & & & {[k]} & & & & & \\
{[1]} \arrow[urrrrr, "\iota_1"] & & {[1]} \arrow[urrr, "\iota_2"] & & {[1]} \arrow[ur, "\iota_3"] & & \dots & & {[1]} \arrow[ulll, "\iota_{k-1}"'] & & {[1]} \arrow[ulllll, "\iota_k"'] \\
 & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & & & & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Applying the functor $A(\mathbf{k}, -, -)$ to this diagram gives us the following diagram in $[(\Theta^{n-m})^{\mathrm{op}}, \mathbf{Set}]$:

$$\begin{tikzcd}
 & & & A(\mathbf{k}, k, -) \arrow[dlll, "i_1"'] \arrow[dl, "i_2"'] \arrow[dr, "i_{k-1}"] \arrow[drrr, "i_k"] & & & \\
A(\mathbf{k}, 1, -) \arrow[dr, "t"'] & & A(\mathbf{k}, 1, -) \arrow[dl, "s"] & \dots & A(\mathbf{k}, 1, -) \arrow[dr, "t"'] & & A(\mathbf{k}, 1, -) \arrow[dl, "s"] \\
 & A(\mathbf{k}, 0, -) & & & & A(\mathbf{k}, 0, -) &
\end{tikzcd}$$

and this is a cone over the diagram:

$$\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{k}$$

$$\begin{tikzcd}
A(\mathbf{k}, 1, -) \arrow[dr, "t"'] & & A(\mathbf{k}, 1, -) \arrow[dl, "s"] & \dots & A(\mathbf{k}, 1, -) \arrow[dr, "t"'] & & A(\mathbf{k}, 1, -) \arrow[dl, "s"] \\
 & A(\mathbf{k}, 0, -) & & & & A(\mathbf{k}, 0, -) &
\end{tikzcd}$$

Since $\mathbf{Set}$ is complete, $[(\Theta^{n-m})^{\mathrm{op}}, \mathbf{Set}]$ is complete, so we can take the limit of this diagram, denoted

$$A(\mathbf{k}, 1, -) \times_{A(\mathbf{k},0,-)} \cdots \times_{A(\mathbf{k},0,-)} A(\mathbf{k}, 1, -),$$

called a “wide pullback”. The universal property of this wide pullback induces a unique morphism such that the diagram

$$\begin{tikzcd}
 & & & & A(\mathbf{k},k,-) \arrow[ddllll, bend right, "i_1"'] \arrow[ddll, bend right, "i_2"'] \arrow[ddrr, bend left, "i_{k-1}"] \arrow[ddrrrr, bend left, "i_k"] \arrow[d, dashed, "S_{\mathbf{k},k}"] & & & & \\
 & & & & A(\mathbf{k},1,-) \times_{A(\mathbf{k},0,-)} \cdots \times_{A(\mathbf{k},0,-)} A(\mathbf{k},1,-) \arrow[dllll] \arrow[dll] \arrow[drr] \arrow[drrrr] & & & & \\
A(\mathbf{k},1,-) \arrow[dr, "t"'] & & A(\mathbf{k},1,-) \arrow[dl, "s"] & & & & A(\mathbf{k},1,-) \arrow[dr, "t"'] & & A(\mathbf{k},1,-) \arrow[dl, "s"] \\
 & A(\mathbf{k},0,-) & & & & & & A(\mathbf{k},0,-) &
\end{tikzcd}$$

commutes. The maps $S_{\mathbf{k},k}$, for all $\mathbf{k} = (k_1, \ldots, k_{m-1})$ and all $k \in \mathbb{N}$, are called the *Segal maps*.

We now give Simpson’s variant of Tamsamani’s definition of weak n-category.

Definition 2.4. Let $n \in \mathbb{N}$. A Tamsamani–Simpson weak n-category is an n-precategory $A\colon (\Theta^n)^{\mathrm{op}} \to \mathbf{Set}$ such that, for all $1 \leq m \leq n$, $\mathbf{k} = (k_1, \ldots, k_{m-1}) \in \Delta^m$, and $[k] \in \Delta$, the Segal map

$$S_{\mathbf{k},k} : A(\mathbf{k},k,-) \to A(\mathbf{k},1,-) \times_{A(\mathbf{k},0,-)} \cdots \times_{A(\mathbf{k},0,-)} A(\mathbf{k},1,-)$$

is contractible.

3. A bisimplicial nerve construction for bicategories

In this section we describe a nerve construction for bicategories, due to Lack and Paoli [22], that will serve as a prototype for our nerve construction for Penon weak n-categories in Section 5. The description we give is an adaptation: the original definition given by Lack and Paoli depends on the use of both normal homomorphisms of bicategories and icons – concepts that we do not have in the case of Penon weak n-categories. Thus, we re-express this nerve in a form that uses only strict homomorphisms, so that it can be adapted to the context of Penon's definition.

The conceptual derivation of this nerve is as follows: first, consider the 2-functor J given by the composite of the canonical cosimplicial object $\Delta \to \mathbf{Cat}$ followed by the inclusion $\mathbf{Cat} \hookrightarrow \mathbf{Bicat}$ that realises each category as a bicategory with only identity 2-cells. This gives rise to a nerve functor

$$\begin{aligned} \mathbf{Bicat} &\to [\Delta^{\mathrm{op}}, \mathbf{Cat}] \\ \mathcal{B} &\mapsto \mathbf{Bicat}(J(-), \mathcal{B}). \end{aligned}$$

This is the method followed by Lack and Paoli. Note that one requires the 1-cells in $\mathbf{Bicat}$ to be normal homomorphisms and the 2-cells to be icons. Applying the standard nerve functor $N \colon \mathbf{Cat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ pointwise, one obtains

$$\mathbf{Bicat} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}] \xrightarrow{N \circ -} [\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathbf{Set}]] \cong [(\Delta^2)^{\mathrm{op}}, \mathbf{Set}].$$

In fact, the resulting nerve can be considered to be in $[(\Theta^2)^{\mathrm{op}}, \mathbf{Set}]$ without losing any information, since $\mathbf{Bicat}(J(0), \mathcal{B})$ is a discrete category, so this functor takes a bicategory and produces from it a 2-precategory as its nerve. This nerve matches an earlier nerve functor partially described by Leinster [24]; thus the description we give effectively completes Leinster's original definition. Leinster defined this nerve construction only on objects; we extend this to a nerve functor

$$\mathcal{N} \colon \mathbf{Bicat} \longrightarrow [(\Theta^2)^{\mathrm{op}}, \mathbf{Set}]$$

by describing the action on morphisms.

Before formally describing the nerve of a bicategory, we discuss the shapes of the simplicial cells in the nerve. The reason for giving this explanation is that the formal description is necessarily notation-heavy, as each (j, k)-cell of the nerve of a bicategory $\mathcal{B}$ is made up of multiple cells in $\mathcal{B}$. This explanation of shapes of cells also helps motivate the shapes of cells used in our nerve construction for Penon weak n-categories.

For all $k > 0$, $0 \leq i \leq k$, there is a map $d_i \colon [k-1] \to [k]$ in Δ given by

$$d_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i. \end{cases}$$

In the nerve of a category $\mathcal{N}\mathcal{C}$, a simplicial k-cell consists of a string of k composable morphisms, and the face maps $\mathcal{N}\mathcal{C}(d_i)$ are defined either to omit a single cell at one end of this string, or to compose a single pair of cells within the string. One would expect the definition of a $(k,0)$-cell in the nerve of a bicategory to be similar; however, one cannot define these face maps in exactly the same way, since composition of 1-cells in a bicategory is not associative. We now explain why this causes problems.

Suppose we define a $(k,0)$-cell in the nerve of a bicategory to consist just of a string of k composable morphisms, which we write as $(f_1, f_2, \dots, f_k)$, with the face maps defined using composition in the same way as in the nerve of a category. In Δ^2, the diagram

$$\begin{tikzcd}
(3,0) & (2,0) \arrow[l, "{(d_1,1)}"'] \\
(2,0) \arrow[u, "{(d_2,1)}"] & (1,0) \arrow[l, "{(d_1,1)}"] \arrow[u, "{(d_1,1)}"']
\end{tikzcd}$$

commutes. Write $\mathcal{N}\mathcal{B}$ for the nerve of $\mathcal{B}$; then, in order for $\mathcal{N}\mathcal{B}$ to be a bisimplicial set, the diagram

$$\begin{tikzcd}
\mathcal{N}\mathcal{B}(3,0) \arrow[r, "{\mathcal{N}\mathcal{B}(d_1,1)}"] \arrow[d, "{\mathcal{N}\mathcal{B}(d_2,1)}"'] & \mathcal{N}\mathcal{B}(2,0) \arrow[d, "{\mathcal{N}\mathcal{B}(d_1,1)}"] \\
\mathcal{N}\mathcal{B}(2,0) \arrow[r, "{\mathcal{N}\mathcal{B}(d_1,1)}"'] & \mathcal{N}\mathcal{B}(1,0)
\end{tikzcd}$$

must commute in $\mathbf{Set}$. However, consider a $(3,0)$-cell $(f,g,h) \in \mathcal{N}\mathcal{B}(3,0)$. Applying the maps along the top and right of the diagram above gives

$$(f,g,h) \xmapsto{\mathcal{N}\mathcal{B}(d_1,1)} (g \circ f, h) \xmapsto{\mathcal{N}\mathcal{B}(d_1,1)} (h \circ (g \circ f)),$$

whereas applying the maps along the left and bottom of the diagram gives

$$(f,g,h) \xmapsto{\mathcal{N}\mathcal{B}(d_2,1)} (f, h \circ g) \xmapsto{\mathcal{N}\mathcal{B}(d_1,1)} ((h \circ g) \circ f),$$

so the diagram does not commute.

Thus a $(k, 0)$-cell in the nerve of a bicategory consists not only of a string of k composable 1-cells, but of a whole k-simplex with 1-cells for its edges and isomorphism 2-cells for its faces; the data for each $(k, 0)$-cell includes all of its faces, not just those which make up the composable string of 1-cells. For example, a $(2, 0)$-cell looks like:

$$\begin{tikzcd}
 & a_1 \arrow[dr, "f_{12}"] & \\
a_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] \arrow[from=1-2, to=2-2, phantom, "\sim\Downarrow i_{012}"] & & a_2
\end{tikzcd}$$

This should be thought of as a pair of composable 1-cells, together with another 1-cell that would be a "valid choice" for their composite (but not necessarily their actual composite in the bicategory).

Similarly, a $(3, 0)$-cell looks like

$$\begin{tikzcd}
a_1 \arrow[r, "f_{12}"] \arrow[dr, phantom, near start, "\sim\Searrow\iota_{012}"] & a_2 \arrow[d, "f_{23}"] \arrow[dl, phantom, near end, "\iota_{023}\Swarrow\sim"] & & a_1 \arrow[r, "f_{12}"] \arrow[dr, "f_{13}"] & a_2 \arrow[d, "f_{23}"] \arrow[dl, phantom, near start, "\iota_{123}\Swarrow\sim"] \\
a_0 \arrow[u, "f_{01}"] \arrow[ur, "f_{02}"] \arrow[r, "f_{03}"'] & a_3 & = & a_0 \arrow[u, "f_{01}"] \arrow[r, "f_{03}"'] \arrow[ur, phantom, near start, "\sim\Downarrow\iota_{013}"] & a_3
\end{tikzcd}$$

i.e. a commuting tetrahedron whose faces are isomorphism 2-cells.

The (j, k)-cells in the nerve, for $k > 0$, are "simplicially weakened" versions cuboidal pasting diagrams. We usually draw these as grids of 2-cells; for example, we draw a $(3, 2)$-cell as:

$$\begin{tikzcd}
a_0 \arrow[r, bend left=60, "f^0_{01}"] \arrow[r, "f^1_{01}"] \arrow[r, bend right=60, "f^2_{01}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\alpha^1_{01}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha^2_{01}"] & a_1 \arrow[r, bend left=60, "f^0_{12}"] \arrow[r, "f^1_{12}"] \arrow[r, bend right=60, "f^2_{12}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\alpha^1_{12}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha^2_{12}"] & a_2 \arrow[r, bend left=60, "f^0_{23}"] \arrow[r, "f^1_{23}"] \arrow[r, bend right=60, "f^2_{23}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\alpha^1_{23}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha^2_{23}"] & a_3.
\end{tikzcd}$$

However, such diagrams are misleading since they do not capture the whole simplicial shape of the cell. In fact, each string of k composable 1-cells on the same "level" (i.e. with the same superscript) is a $(k, 0)$-cell, and all diagrams of 2-cells within each (j, k)-cell commute.

Note that the notation used in the diagrams above is the notation we use throughout this section. The subscripts and superscripts decorating each cell should be thought of as the coordinates of that cell, with the subscripts giving the horizontal coordinates, and superscripts giving the vertical coordinates.

We break the description of the nerve functor for bicategories into three parts. In Definition 3.1 we define, for a bicategory $\mathcal{B}$ and for each object (j, k) in Θ^2, a set $\mathcal{NB}(j, k)$, which is the set of (j, k)-cells in the nerve of $\mathcal{B}$. Then, in Definition 3.2, we extend this to a definition of a 2-precategory

$$\mathcal{NB}\colon (\Theta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

by describing the action of this presheaf on maps. This gives the action of the nerve functor

$$\mathcal{N}\colon \mathbf{Bicat} \longrightarrow [(\Theta^2)^{\mathrm{op}}, \mathbf{Set}].$$

on objects; in Definition 3.3 we give the action of this functor on maps.

Recall that an object of Θ^2 is an equivalence class of objects of Δ^2. An object of Δ^2 is in an equivalence class with more than one member if and only if it is of the form $(0, k)$. Thus, we treat the equivalence class of $(0, k)$ as the object $(0, 0)$ of Δ^2; all other equivalence classes are treated as their sole member. Note that the exact choice of representative does not make a difference to the definition.

Note that, ideally, we would give an abstract definition of the nerve of a bicategory by first defining a functor $i\colon \Theta^2 \to \mathbf{Bicat}$, then defining the nerve of a bicategory $\mathcal{B}$ to be given by $\mathbf{Bicat}(i(-), \mathcal{B})$, as one does when defining the nerve of a category. However, since we also want to avoid using normal homomorphisms or any kind of 2-cells, this is not practical as the bicategories in the image of the functor i are difficult to describe (in particular, they are not free, unlike in the case of the nerve of a category). We believe that describing these bicategories would require extra machinery (for example, we believe it could be done using computads) and is thus beyond the scope of this paper. Note that this is one of the reasons for using Penon weak n-categories in the remainder of the paper; in the case of Penon weak n-categories we are able to construct the nerve in this abstract way, by modifying the construction of a free Penon weak n-category, in a way that is not possible with bicategories. We do this in Sections 5 and 7.

Definition 3.1. Let $\mathcal{B}$ be a bicategory. We associate to $\mathcal{B}$ a 2-precategory $\mathcal{N}\mathcal{B}\colon (\Theta^2)^{\mathrm{op}} \to \mathbf{Set}$, called the *nerve of* $\mathcal{B}$, as follows:

Given $(j,k) \in \Theta^2$, $\mathcal{N}\mathcal{B}(j,k)$ is the set which has as its elements all quadruples

$$\left((a_u)_{0 \le u \le j}, (f^z_{uv})_{\substack{0 \le u < v \le j \\ 0 \le z \le k}}, (\alpha^z_{uv})_{\substack{0 \le u < v \le j \\ 1 \le z \le k}}, (\iota^z_{uvw})_{\substack{0 \le u < v < w \le j \\ 0 \le z \le k}} \right)$$

where

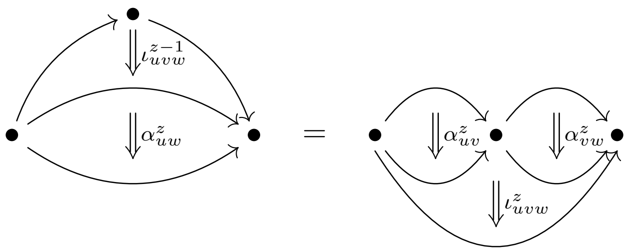
- each a_u is an object of $\mathcal{B}$;
- each $f^z_{uv} : a_u \to a_v$ is a 1-cell of $\mathcal{B}$;
- each $\alpha^z_{uv} : f^{z-1}_{uv} \to f^z_{uv}$ is a 2-cell of $\mathcal{B}$;
- each $\iota^z_{uvw} : f^z_{vw} \circ f^z_{uv} \to f^z_{uw}$ is an isomorphism 2-cell of $\mathcal{B}$, with inverse $(\iota^z_{uvw})^{-1}$;

and these cells satisfy the following axioms:

- for all $0 \le u < v < w \le j$, $1 \le z \le k$, the diagram

$$\begin{tikzcd}
f^{z-1}_{vw} \circ f^{z-1}_{uv} \arrow[r, "\iota^{z-1}_{uvw}"] \arrow[d, "\alpha^z_{vw} * \alpha^z_{uv}"'] & f^{z-1}_{uw} \arrow[d, "\alpha^z_{uw}"] \\
f^z_{vw} \circ f^z_{uv} \arrow[r, "\iota^z_{uvw}"'] & f^z_{uw}
\end{tikzcd}$$

  commutes; alternatively, we can draw this axiom as



- for all $0 \leq u < v < w < x \leq j$, $0 \leq z \leq k$, the diagram

$$\begin{tikzcd}
(f^z_{wx} \circ f^z_{vw}) \circ f^z_{uv} \arrow[rr, "s_{uvwx}"] \arrow[d, "\iota^z_{vwx} * 1_{f^z_{uv}}"'] & & f^z_{wx} \circ (f^z_{vw} \circ f^z_{uv}) \arrow[d, "1_{f^z_{wx}} * \iota^z_{uvw}"] \\
f^z_{vx} \circ f^z_{uv} \arrow[dr, "\iota^z_{uvx}"'] & & f^z_{wx} \circ f^z_{uw} \arrow[dl, "\iota^z_{uwx}"] \\
& f^z_{ux} &
\end{tikzcd}$$

  commutes, where

$$s_{uvwx} : (f^z_{wx} \circ f^z_{vw}) \circ f^z_{uv} \to f^z_{wx} \circ (f^z_{vw} \circ f^z_{uv})$$

  is the component of the appropriate associativity isomorphism for $\mathcal{B}$; alternatively, we can draw this axiom as

$$\begin{tikzcd}
& a_v \arrow[r, "f^z_{vw}"] \arrow[d, phantom, "\sim\Downarrow\iota_{uvw}"] & a_w \arrow[dr, "f^z_{wx}"] \arrow[d, phantom, "\iota_{uwx}\Downarrow\sim"] & & & & a_v \arrow[r, "f^z_{vw}"] \arrow[drr, "f^z_{vx}"'] \arrow[d, phantom, "\sim\Downarrow\iota_{uvx}"] & a_w \arrow[dr, "f^z_{wx}"] \arrow[phantom, d, "\iota_{vwx}\Downarrow\sim"] & \\
a_u \arrow[ur, "f^z_{uv}"] \arrow[urr, "f^z_{uw}"'] \arrow[rrr, "f^z_{ux}"'] & & & a_x & = & a_u \arrow[ur, "f^z_{uv}"] \arrow[rrr, "f^z_{ux}"'] & & & a_x
\end{tikzcd}$$

We now explain the action on maps in Θ^2, then make it precise in the next definition. Given a map $(p, q) \colon (l, m) \to (j, k)$ in Θ^2, we define a map

$$\mathcal{NB}(p, q) \colon \mathcal{NB}(j, k) \to \mathcal{NB}(l, m).$$

To understand what this map does, recall that an element of $\mathcal{NB}(j, k)$ consists of a collection of cells of $\mathcal{B}$ which form a (j, k)-cell, and that each of these cells has subscripts and (in some cases) superscripts which we think of as the coordinates of this cell within the (j, k)-cell. Given an element of $\mathcal{NB}(j, k)$, its image under $\mathcal{NB}(p, q)$ is the element of $\mathcal{NB}(l, m)$ made up of those cells whose horizontal coordinates are in the image of p and, where appropriate, whose vertical coordinate is in the image of q; any cells whose coordinates are not in the images of p and q are omitted, and cells with repeated coordinates are taken to be identities (or unitors in some cases).

Definition 3.2. Let $\mathcal{B}$ be a bicategory, and write l and r for its left and right unitors respectively. Let $(p,q)\colon (h,i) \to (j,k)$ be a map in Θ^2. We define a function of sets

$$\mathcal{N}\mathcal{B}(p,q)\colon \mathcal{N}\mathcal{B}(j,k) \to \mathcal{N}\mathcal{B}(h,i)$$

as follows:

$$\begin{aligned}\mathcal{N}\mathcal{B}(p,q)\colon &\left((a_u)_{0\le u\le j}, (f^z_{uv})_{\substack{0\le u<v\le j\\ 0\le z\le k}}, (\alpha^z_{uv})_{\substack{0\le u<v\le j\\ 1\le z\le k}}, (\iota^z_{uvw})_{\substack{0\le u<v<w\le j\\ 0\le z\le k}}\right)\\ &\longmapsto \left((b_u)_{0\le u\le h}, (g^z_{uv})_{\substack{0\le u<v\le h\\ 0\le z\le i}}, (\beta^z_{uv})_{\substack{0\le u<v\le h\\ 1\le z\le i}}, (\kappa^z_{uvw})_{\substack{0\le u<v<w\le h\\ 0\le z\le i}}\right)\end{aligned}$$

where

- $b_u = a_{p(u)}$
- $g^z_{uv} = \begin{cases} f^{q(z)}_{p(u)p(v)} & \text{if } p(u) \neq p(v),\\ \mathrm{id}_{a_{p(u)}} & \text{if } p(u) = p(v);\end{cases}$
- $\beta^z_{uv} = \begin{cases} \alpha^{q(z)}_{p(u)p(v)} & \text{if } p(u) \neq p(v), q(z-1) \neq q(z),\\ \mathrm{id}_{f^{q(z)}_{p(u)p(v)}} & \text{if } p(u) \neq p(v), q(z-1) = q(z),\\ \mathrm{id}_{\mathrm{id}_{a_{p(u)}}} & \text{if } p(u) = p(v);\end{cases}$
- $\kappa^z_{uvw} = \begin{cases} \iota^{q(z)}_{p(u)p(v)p(w)} & \text{if } p(u) \neq p(v) \neq p(w),\\ l_{f^{q(z)}_{p(u)p(v)}} & \text{if } p(u) \neq p(v) = p(w),\\ r_{f^{q(z)}_{p(u)p(v)}} & \text{if } p(u) = p(v) \neq p(w),\\ \mathrm{id}_{\mathrm{id}_{a_{p(u)}}} & \text{if } p(u) = p(v) = p(w).\end{cases}$

This defines the action of the nerve functor on objects; we now extend this to a definition of a nerve functor

$$\mathcal{N}\colon \mathbf{Bicat} \longrightarrow [(\Theta^2)^{\mathrm{op}}, \mathbf{Set}],$$

by describing the action of this functor on morphisms.

Definition 3.3. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a strict functor of bicategories. We define a map of bisimplicial sets $\mathcal{N}F\colon \mathcal{N}\mathcal{A} \to \mathcal{N}\mathcal{B}$ to be the map whose component $\mathcal{N}F_{(j,k)}\colon \mathcal{N}\mathcal{A}(j,k) \to \mathcal{N}\mathcal{B}(j,k)$, for each $(j,k) \in \Delta^2$, is given by

$$\begin{aligned}&\mathcal{N}F_{(j,k)}\left((a_u)_{0\le u\le j}, (f^z_{uv})_{\substack{0\le u<v\le j\\ 0\le z\le k}}, (\alpha^z_{uv})_{\substack{0\le u<v\le j\\ 1\le z\le k}}, (\iota^z_{uvw})_{\substack{0\le u<v<w\le j\\ 0\le z\le k}}\right)\\ &\qquad=\left((F(a_u))_{0\le u\le j}, (Ff^z_{uv})_{\substack{0\le u<v\le j\\ 0\le z\le k}}, (F\alpha^z_{uv})_{\substack{0\le u<v\le j\\ 1\le z\le k}}, (F\iota^z_{uvw})_{\substack{0\le u<v<w\le j\\ 0\le z\le k}}\right).\end{aligned}$$

The above defines a functor $\mathcal{N}\colon \mathbf{Bicat} \to [(\Theta^2)^{\mathrm{op}}, \mathbf{Set}]$, called the *nerve functor*.

The nerve of a bicategory satisfies the Segal condition, and is thus a Tamsamani–Simpson weak 2-category. Before giving the proof, we recall the definition of Tamsamani–Simpson weak n-category (Definition 2.4) in the case $n = 2$; the following is a slight unpacking of the definition, which treats Segal maps of the forms S_k and $S_{j,k}$ separately.

Definition 3.4. A *Tamsamani–Simpson weak* 2*-category* is a functor

$$A\colon (\Theta^2)^{\mathrm{op}} \to \mathbf{Set}$$

such that

(i) for each $k \ge 0$, the Segal map

$$S_k : A(k,-) \longrightarrow \underbrace{A(1,-) \times_{A(0,1)} \cdots \times_{A(0,1)} A(1,-)}_{k}$$

is contractible, i.e. it is surjective on objects, and full and faithful on 1-cells;

(ii) for each m, $k \ge 0$, the Segal map

$$S_{j,k} : A(j,k) \longrightarrow \underbrace{A(j,1) \times_{A(j,0)} \cdots \times_{A(j,0)} A(j,1)}_{k}$$

is a bijection.

Thus to prove that the nerve of a bicategory is a Tamsamani–Simpson weak 2-category, we break this statement down into four propositions: one stating that each of the Segal maps $S_{j,k}$ is a bijection, and the other three stating the three conditions required for contractibility of the Segal maps S_k.

Proposition 3.5. *Let $\mathcal{B}$ be a bicategory. For all j, $k \geq 0$, the Segal map*

$$S_{j,k} : \mathcal{NB}(j,k) \longrightarrow \underbrace{\mathcal{NB}(j,1) \times_{\mathcal{NB}(j,0)} \cdots \times_{\mathcal{NB}(j,0)} \mathcal{NB}(j,1)}_{k}$$

is a bijection.

Proof. Let

$$\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 0 \leq z \leq k}}, (\alpha^z_{uv})_{\substack{0 \leq u < v \leq j \\ 1 \leq z \leq k}}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 0 \leq z \leq k}} \right)$$

be an element of $\mathcal{NB}(j,k)$. The function $S_{j,k}$ maps this to

$$\begin{aligned}
\Bigg(&\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 0 \leq z \leq 1}}, (\alpha^1_{uv})_{0 \leq u < v \leq j}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 0 \leq z \leq 1}} \right), \\
&\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 1 \leq z \leq 2}}, (\alpha^2_{uv})_{0 \leq u < v \leq j}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 1 \leq z \leq 2}} \right), \\
&\dots, \\
&\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ k-1 \leq z \leq k}}, (\alpha^k_{uv})_{0 \leq u < v \leq j}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ k-1 \leq z \leq k}} \right) \Bigg).
\end{aligned}$$

Every cell listed in the original element of $\mathcal{NB}(j,k)$ is listed in its image under $S_{j,k}$, so this function is injective. Furthermore, any element of the wide pullback

$$\underbrace{\mathcal{NB}(j,1) \times_{\mathcal{NB}(j,0)} \cdots \times_{\mathcal{NB}(j,0)} \mathcal{NB}(j,1)}_{k}$$

can be written in the form above. Thus $S_{j,k}$ is surjective.

Hence $S_{j,k}$ is a bijection. □

Proposition 3.6. *Let $\mathcal{B}$ be a bicategory. For all $k \geq 0$, the Segal map*

$$S_k : \mathcal{NB}(k,-) \longrightarrow \underbrace{\mathcal{NB}(1,-) \times_{\mathcal{NB}(0,0)} \cdots \times_{\mathcal{NB}(0,0)} \mathcal{NB}(1,-)}_{k}$$

is surjective on objects.

Proof. Let

$$\Big(\big(a_0 \xrightarrow{f^0_{01}} a_1 \big), \big(a_1 \xrightarrow{f^0_{12}} a_2 \big), \dots, \big(a_{k-1} \xrightarrow{f^0_{k-1,k}} a_k \big) \Big)$$

be an element of

$$\underbrace{A(1,0) \times_{A(0,0)} \cdots \times_{A(0,0)} A(1,0)}_{k}.$$

This is a string of k composable 1-cells in $\mathcal{B}$. We seek an element of $\mathcal{NB}(k,0)$ that maps to this under $(S_k)_0$. We define an element

$$\Big((a_u)_{0 \le u \le j}, (f^0_{uv})_{0 \le u < v \le j}, (\iota^0_{uvw})_{0 \le u < v < w \le j} \Big)$$

of $\mathcal{NB}(k,0)$; to do so we must define f^0_{uv} for every $v > u+1$, and we must define the ι^0_{uvw} for all $0 \le u < v < w \le k$. Our approach is to define each f^0_{uv} to be a composite of the cells of the form $f^0_{u,u+1}$, then define each ι^0_{uvw} to be a composite of constraint cells in $\mathcal{B}$ that mediate between the appropriate cells.

Let $0 \le u < u+1 < v \le j$, and define f^0_{uv} to be given by the composite

$$f^0_{uv} := (\cdots (f^0_{v-1,v} \circ f^0_{v-2,v-1}) \circ \cdots) \circ f^0_{u,u+1}.$$

Then, for all $0 \le u < v < w \le j$, there is a composite of constraint isomorphism 2-cells

$$\iota^0_{uvw} : f^0_{vw} \circ f^0_{uv} \to f^0_{uw}$$

in $\mathcal{B}$, which is unique by coherence for bicategories [15, 23].

This defines an element of $\mathcal{NB}(k,0)$; by construction we see that this element maps to

$$\Big(\big(a_0 \xrightarrow{f^0_{01}} a_1 \big), \big(a_1 \xrightarrow{f^0_{12}} a_2 \big), \dots, \big(a_{k-1} \xrightarrow{f^0_{k-1,k}} a_k \big) \Big)$$

under $(S_k)_0$, as required. Hence S_k is surjective on objects. □

To show that the Segal maps are full and faithful on 1-cells, we use the fact that there is some redundancy in the definition of $\mathcal{NB}(j,k)$. Specifically, to specify an element of $\mathcal{NB}(j,k)$ we only need to specify α^z_{uv} for $v = u+1$, rather than for all $u < v < j$ (note that we still have to specify every a_u, f^z_{uv} and ι^z_{uvw}). Since this fact is used in the proofs of both fullness and faithfulness, we state and prove it as a separate lemma:

Lemma 3.7. *Let $\mathcal{B}$ be a bicategory, let $j,\ k \in \mathbb{N}$, and suppose we have the following data:*

- *for all $0 \leq u \leq j$, an object a_u of $\mathcal{B}$;*
- *for all $0 \leq u < v \leq j$, $0 \leq z \leq k$, a 1-cell $f^z_{uv} : a_u \to a_v$ in $\mathcal{B}$;*
- *for all $0 \leq u < j$, $1 \leq z \leq k$, a 2-cell $\alpha^z_{u,u+1} : f^{z-1}_{u,u+1} \to f^z_{u,u+1}$ in $\mathcal{B}$;*
- *for all $0 \leq u < v < w \leq j$, $0 \leq z \leq k$, an isomorphism 2-cell $\iota^z_{uvw} : f^z_{vw} \circ f^z_{uv} \to f^z_{uw}$ in $\mathcal{B}$, with inverse $(\iota^z_{uvw})^{-1}$;*

such that the isomorphism 2-cells ι^z_{uvw} satisfy the pentagon axiom from the definition of $\mathcal{NB}$ on objects, Definition 3.1. Then this specifies a unique element

$$\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 0 \leq z \leq k}}, (\alpha^z_{uv})_{\substack{0 \leq u < v \leq j \\ 1 \leq z \leq k}}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 0 \leq z \leq k}} \right)$$

of $\mathcal{NB}(j,k)$.

Proof. We need to show that, for all $0 \leq u < u+1 < v \leq j$, $1 \leq z \leq k$, there is a unique choice of 2-cell α^z_{uv} in $\mathcal{B}$ such that the axioms for an element of $\mathcal{NB}(j,k)$ are satisfied. We do this by strong induction over v.

First, let $v = u+2$. For all $1 \leq z \leq k$, write $w := u+1$, and define $\alpha^z_{uv} = \alpha^z_{u,u+2}$ to be given by the composite

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha^z_{uw}"] \arrow[rr, bend left=70] \arrow[rr, bend right=70] \arrow[rr, phantom, bend left=45, "\Downarrow (\iota^{z-1}_{uwv})^{-1}"] \arrow[rr, phantom, bend right=45, "\Downarrow \iota^z_{uwv}"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha^z_{wv}"] & \bullet
\end{tikzcd}$$

in $\mathcal{B}$. By considering the composite $\alpha^z_{uv} \circ \iota^{z-1}_{uwv}$, we see that α^z_{uv} satisfies the square axiom from the definition of $\mathcal{NB}(j,k)$, Definition 3.1; furthermore, it is the only 2-cell of $\mathcal{B}$ satisfying these axioms, given that α^z_{uw}, α^z_{wv}, ι^{z-1}_{uwv} and ι^z_{uwv} are fixed.

Now let $m \geq 1$ and suppose we have defined α^z_{uv} for all $u+1 \leq v \leq u+m$. We define α^z_{uv} for $v = u+m+1$ as follows: let w be a natural number with $u < w < v$, and define α^z_{uv} to be given by the composite

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[rr, bend left=70] \arrow[rr, bend right=70] \arrow[r, phantom, "\Downarrow \alpha^z_{uw}"] \arrow[rr, phantom, bend left=50, "\Downarrow (\iota^{z-1}_{uwv})^{-1}"] \arrow[rr, phantom, bend right=50, "\Downarrow \iota^z_{uwv}"] & \bullet \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow \alpha^z_{wv}"] & \bullet
\end{tikzcd}$$

Note that the pentagon axiom from the definition of $\mathcal{NB}(j,k)$ ensures that this is independent of our choice of w. As before, by considering the composite $\alpha^z_{uv} \circ \iota^{z-1}_{uwv}$, we see that α^z_{uv} satisfies the square axiom from the definition of $\mathcal{NB}(j,k)$, Definition 3.1; furthermore, it is the only 2-cell of $\mathcal{B}$ satisfying these axioms, given that α^z_{uw}, α^z_{wv}, ι^{z-1}_{uwv} and ι^z_{uwv} are fixed.

This defines a unique element

$$\left((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 0 \leq z \leq k}}, (\alpha^z_{uv})_{\substack{0 \leq u < v \leq j \\ 1 \leq z \leq k}}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 0 \leq z \leq k}} \right)$$

of $\mathcal{NB}(j,k)$, as required. □

This now allows us to prove the Segal maps are full and faithful on 1-cells.

Proposition 3.8. *Let $\mathcal{B}$ be a bicategory. For all $k \geq 0$, the Segal map*

$$S_k : \mathcal{NB}(k,-) \longrightarrow \underbrace{\mathcal{NB}(1,-) \times_{\mathcal{NB}(0,0)} \cdots \times_{\mathcal{NB}(0,0)} \mathcal{NB}(1,-)}_{k}$$

is full on 1*-cells.*

Proof. Suppose we have two elements f, $g \in \mathcal{NB}(k,0)$, which we denote

$$f = \left((a_u)_{0 \leq u \leq k}, (f^0_{uv})_{0 \leq u < v \leq k}, (\iota^0_{uvw})_{0 \leq u < v < w \leq k} \right)$$

and

$$g = \left((b_u)_{0 \leq u \leq k}, (g^0_{uv})_{0 \leq u < v \leq k}, (\kappa^0_{uvw})_{0 \leq u < v < w \leq k} \right),$$

and suppose we have an element α of

$$\underbrace{\mathcal{NB}(1,1) \times_{\mathcal{NB}(0,0)} \cdots \times_{\mathcal{NB}(0,0)} \mathcal{NB}(1,1)}_{k},$$

with $s(\alpha) = S_k(f)$ and $t(\alpha) = S_k(g)$. Then, for all $0 \leq u \leq k$, $a_u = b_u$, and we can write α as

$$\alpha = \left(\left(\begin{tikzcd} a_0 \arrow[r, bend left, "f^0_{01}"{name=U}] \arrow[r, bend right, "g^0_{01}"'{name=D}] & a_1 \arrow[Rightarrow, from=U, to=D, "\alpha^1_{01}"] \end{tikzcd} \right), \left(\begin{tikzcd} a_1 \arrow[r, bend left, "f^0_{12}"{name=U}] \arrow[r, bend right, "g^0_{12}"'{name=D}] & a_2 \arrow[Rightarrow, from=U, to=D, "\alpha^1_{12}"] \end{tikzcd} \right), \ldots, \left(\begin{tikzcd} a_{k-1} \arrow[r, bend left, "f^0_{k-1,k}"{name=U}] \arrow[r, bend right, "g^0_{k-1,k}"'{name=D}] & a_k \arrow[Rightarrow, from=U, to=D, "\alpha^1_{k-1,k}"] \end{tikzcd} \right) \right).$$

By Lemma 3.7, α, combined with the isomorphism 2-cells ι^0_{uvw} and κ^0_{uvw}, defines a unique element

$$\left((a_u)_{0 \leq u \leq k}, (f^z_{uv})_{\substack{0 \leq u < v \leq k \\ 0 \leq z \leq 1}}, (\alpha^1_{uv})_{0 \leq u < v \leq k}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq k \\ 0 \leq z \leq 1}} \right)$$

of $\mathcal{NB}(k,1)$, where

- for all $0 \leq u < v \leq k$, $f^1_{uv} = g^0_{uv}$;
- for all $0 \leq u < v < w \leq k$, $\iota^1_{uvw} = \kappa^0_{uvw}$.

Denote this by $\hat{\alpha}$; then $s(\hat{\alpha}) = f$, $t(\hat{\alpha}) = g$, and $S_k(\hat{\alpha}) = \alpha$, so S_k is full on 1-cells. $\square$

Proposition 3.9. *Let $\mathcal{B}$ be a bicategory. For all $k \geq 0$, the Segal map*

$$S_k : \mathcal{NB}(k,-) \longrightarrow \underbrace{\mathcal{NB}(1,-) \times_{\mathcal{NB}(0,0)} \cdots \times_{\mathcal{NB}(0,0)} \mathcal{NB}(1,-)}_{k}$$

is faithful on 1*-cells.*

Proof. Suppose we have two parallel elements α, $\beta \in \mathcal{NB}(k,1)$ such that $(S_k)_1(\alpha) = (S_k)_1(\beta)$. We wish to show that $\alpha = \beta$. We can write f and g as

$$\alpha = \left((a_u)_{0 \leq u \leq k}, (f^z_{uv})_{\substack{0 \leq u < v \leq k \\ 0 \leq z \leq 1}}, (\alpha^1_{uv})_{0 \leq u < v \leq k}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq k \\ 0 \leq z \leq 1}} \right)$$

and

$$\beta = \left((a_u)_{0 \leq u \leq k}, (f^z_{uv})_{\substack{0 \leq u < v \leq k \\ 0 \leq z \leq 1}}, (\beta^1_{uv})_{0 \leq u < v \leq k}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq k \\ 0 \leq z \leq 1}} \right).$$

Note that the fact α and β are parallel tells us that they can only differ on their 2-cell parts. We write $(S_k)_1(\alpha) = (S_k)_1(\beta)$ as

$$\left(\left(\begin{tikzcd} a_0 \arrow[r, bend left, "f^0_{01}"] \arrow[r, bend right, "g^0_{01}"'] \arrow[r, phantom, "\Downarrow \gamma^1_{01}"] & a_1 \end{tikzcd} \right), \left(\begin{tikzcd} a_1 \arrow[r, bend left, "f^0_{12}"] \arrow[r, bend right, "g^0_{12}"'] \arrow[r, phantom, "\Downarrow \gamma^1_{12}"] & a_2 \end{tikzcd} \right), \dots, \left(\begin{tikzcd} a_{k-1} \arrow[r, bend left, "f^0_{k-1,k}"] \arrow[r, bend right, "g^0_{k-1,k}"'] \arrow[r, phantom, "\Downarrow \gamma^1_{k-1,k}"] & a_k \end{tikzcd} \right) \right),$$

which is an element of

$$\underbrace{\mathcal{N}\mathcal{B}(1,1) \times_{\mathcal{N}\mathcal{B}(0,0)} \cdots \times_{\mathcal{N}\mathcal{B}(0,0)} \mathcal{N}\mathcal{B}(1,1)}_{k}.$$

Furthermore, since $(S_k)_1(\alpha) = (S_k)_1(\beta)$, we have that, for all $0 \leq u < k$,

$$\alpha^1_{u,u+1} = \gamma^1_{u,u+1} = \beta^1_{u,u+1}.$$

Thus, by Lemma 3.7, for all $0 \leq u < v \leq k$, we have

$$\alpha^1_{uv} = \gamma^1_{uv} = \beta^1_{uv},$$

so $\alpha = \beta$, as required. □

We now have everything we need to prove that the nerve of a bicategory satisfies the Segal condition.

Theorem 3.10. *Let $\mathcal{B}$ be a bicategory. Then the nerve of $\mathcal{B}$, $\mathcal{N}\mathcal{B}$, satisfies the Segal condition, and is thus a Tamsamani–Simpson weak 2-category.*

Proof. For all j, $k \geq 0$, the Segal map

$$S_{j,k} : \mathcal{N}\mathcal{B}(j,k) \longrightarrow \underbrace{\mathcal{N}\mathcal{B}(j,1) \times_{\mathcal{N}\mathcal{B}(j,0)} \cdots \times_{\mathcal{N}\mathcal{B}(j,0)} \mathcal{N}\mathcal{B}(j,1)}_{k}$$

is a bijection by Proposition 3.5.

For all $k \geq 0$, the Segal map

$$S_k : \mathcal{NB}(k,-) \longrightarrow \underbrace{\mathcal{NB}(1,-) \times_{\mathcal{NB}(0,0)} \cdots \times_{\mathcal{NB}(0,0)} \mathcal{NB}(1,-)}_{k}$$

is surjective on 0-cells by Proposition 3.6, full on 1-cells by Proposition 3.8, and faithful on 1-cells by Proposition 3.9.

Thus $\mathcal{NB}$ satisfies the Segal condition, so it is a Tamsamani–Simpson weak 2-category. □

4. Penon weak n-categories

In this section we recall the non-reflexive variant of Penon's definition of weak n-category [29, 4, 13]. We refer the reader to Part 1 of this series for a more detailed description and an intuitive explanation; here we just give the formal definition.

We begin by recalling the definition of an n-globular set, the underlying data for a Penon weak n-category.

Definition 4.1. The *n-globe category* $\mathbb{G}$ is defined as the category with

- objects: natural numbers $0, 1, \ldots, n-1, n$;
- morphisms generated by, for each $1 \leq m \leq n$, morphisms

  $$\sigma_m, \tau_m \colon (m-1) \to m$$

  such that $\sigma_{m+1}\sigma_m = \tau_{m+1}\sigma_m$ and $\sigma_{m+1}\tau_m = \tau_{m+1}\tau_m$ for $m \geq 2$ (called the "globularity conditions").

An *n-globular set* is a presheaf on $\mathbb{G}$. We write n-$\mathbf{GSet}$ for the category of n-globular sets $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

For an n-globular set $X \colon \mathbb{G}^{\mathrm{op}} \to \mathbf{Set}$, we write s for $X(\sigma_m)$, and t for $X(\tau_m)$, regardless of the value of m, and refer to them as the source and target maps respectively. We denote the set $X(m)$ by X_m. We say that two m-cells $x, y \in X_m$ are *parallel* if $s(x) = s(y)$ and $t(x) = t(y)$; note that all 0-cells are considered to be parallel.

We now recall the definition of an n-magma, an n-globular set equipped with composition operations.

Definition 4.2. An *n-magma* (or simply *magma*, when n is fixed) consists of an n-globular set X equipped with, for each m, p, with $0 \leq p < m \leq n$, a binary composition function

$$\circ_p^m \colon X_m \times_{X_p} X_m \to X_m,$$

where $X_m \times_{X_p} X_m$ denotes the pullback

$$\begin{tikzcd} X_m \times_{X_p} X_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "s"] \\ X_m \arrow[r, "t"'] & X_p \end{tikzcd}$$

in $\mathbf{Set}$; these composition functions must satisfy the following source and target conditions:

- if $p = m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,
$$s(b \circ_p^m a) = s(a), \ t(b \circ_p^m a) = t(b);$$
- if $p < m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,
$$s(b \circ_p^m a) = s(b) \circ_p^{m-1} s(a), \ t(b \circ_p^m a) = t(b) \circ_p^{m-1} t(a).$$

A map of n-magmas $f \colon X \to Y$ is a map of the underlying n-globular sets such that, for all m, p, with $0 \leq p < m \leq n$, and for all $(a, b) \in X_m \times_{X_p} X_m$,

$$f(b \circ_p^m a) = f(b) \circ_p^m f(a).$$

We write n-$\mathbf{Mag}$ for the category whose objects are n-magmas and whose morphisms are maps of n-magmas.

Definition 4.3. Let $f \colon X \to S$ be a map of n-globular sets, where S is the underlying n-globular set of a strict n-category. The map f is said to be *tame* if, given a, $b \in X_n$, if $s(a) = s(b)$, $t(a) = t(b)$, and $f_n(a) = f_n(b)$, then $a = b$.

For each $0 \leq m < n$, define a set X^c_{m+1} by the following pullback:

$$\begin{tikzcd}
X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,f_m)}"] \\
X_m \arrow[r, "{(s,t,f_m)}"'] & X_{m-1} \times X_{m-1} \times S_m.
\end{tikzcd}$$

Note that when $m = 0$, we take X_{m-1} to be the terminal set.

A *contraction* γ on a tame map $f \colon X \to S$ consists of, for each $0 \leq m < n$, a map

$$\gamma_{m+1} \colon X^c_{m+1} \to X_{m+1}$$

such that, for all $(a, b) \in X^c_{m+1}$,

- $s(\gamma_{m+1}(a, b)) = a$;
- $t(\gamma_{m+1}(a, b)) = b$;
- $f_{m+1}(\gamma_{m+1}(a, b)) = 1_{f_m(a)} = 1_{f_m(b)}$.

Note that we only ever speak of a contraction on a tame map; thus, whenever we state that a map is equipped with a contraction, the map is automatically assumed to be tame. One way to think about this is to say that we do require a contraction $(n+1)$-cell for each pair of n-cells in X^c_n, and the only $(n+1)$-cells in X are equalities.

Definition 4.4. The *category of n-categorical stretchings $\mathcal{Q}$* is the category with

- objects: an object of $\mathcal{Q}$ consists of an n-magma X, a strict n-category S, and a map of n-magmas

$$\begin{tikzcd}
X \arrow[d, "f"'] \\
S
\end{tikzcd}$$

  equipped with a contraction γ;

- morphisms: a morphism in $\mathcal{Q}$ is a commuting square

  $$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{Mag}$ such that

  - v is a map of strict n-categories;
  - writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 \leq m < n$, and $(a, b) \in X^c_{m+1}$, we have
  $$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

  We denote such a morphism by (u, v).

There is a forgetful functor

$$\begin{tikzcd} U\colon \mathcal{Q} \arrow[r] & n\text{-}\mathbf{GSet} \\ X \arrow[d, "f"'] & \\ S & \end{tikzcd} \quad \longmapsto \quad X$$

and this functor has a left adjoint $F\colon n\text{-}\mathbf{GSet} \to \mathcal{Q}$.

Definition 4.5. Let P be the monad on n-$\mathbf{GSet}$ induced by the adjunction $F \dashv U$. A Penon weak n-category is defined to be an algebra for the monad P, and P-$\mathbf{Alg}$ is the category of Penon weak n-categories.

Finally, for the purpose of our nerve construction, it will necessary to use the fact that adjunction $F \dashv U$ can be factorised as:

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{R} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{Q} \arrow[l, shift left] \end{tikzcd}$$

where, writing $U_T\colon n\text{-}\mathbf{Cat} \to n\text{-}\mathbf{GSet}$ for the forgetful functor (the notation U_T is used because n-$\mathbf{Cat} = T$-$\mathbf{Alg}$, where T is the free strict n-category monad on n-$\mathbf{GSet}$), $\mathcal{R}$ is the comma category

$$n\text{-}\mathbf{GSet} \downarrow U_T.$$

5. The nerve construction for $n = 2$

In this section we construct a nerve functor for Penon weak 2-categories. The construction for the case of general n is given in Section 7; we present the 2-dimensional case separately since it is simpler, both conceptually and notationally, than the general case, but not too simple to exhibit all the key features of the n-dimensional construction. We are also able to prove that nerves satisfy the Segal condition in the case $n = 2$; we do this in Section 6. We use Leinster's nerve construction for bicategories as the prototype for our construction, and also use his notation. As in the previous section, we write P for the monad for Penon weak 2-categories, and T for the free strict 2-category monad.

When defining the nerve of a category, one common method is first to define a functor $I \colon \Delta \hookrightarrow \mathbf{Cat}$, and then define the nerve $\mathcal{NC}$ of a category $\mathcal{C}$ to be given by $\mathcal{NC} = \mathbf{Cat}(I(-), \mathcal{C})$. In analogy with this, to define our nerve functor for Penon weak 2-categories, we first define a functor

$$I_2 : \Theta^2 \longrightarrow P\text{-}\mathbf{Alg}.$$

This functor should give us, for each object of Θ^2, the corresponding cuboidal 2-pasting diagram, expressed as a freely generated Penon weak 2-category (recall that cuboidal pasting diagrams were discussed in Section 2, and again, in-depth, in Section 3). However, we have to be very careful about what we mean by "freely generated" in this context. Each cuboidal 2-pasting diagram has associated to it a 2-globular set whose cells are those which we draw in the pasting diagram. We could simply define I_2 to give us the free P-algebra on these 2-globular sets. Let $(j, k) \in \Theta^2$ and write $F_P(j, k)$ for the free P-algebra on the corresponding 2-globular set. We would then have, for a Penon weak 2-category $\mathcal{A}$, the nerve defined by

$$\mathcal{NA}(j, k) = P\text{-}\mathbf{Alg}(F_P(j, k), \mathcal{A}).$$

Consider the object $(2, 0)$ of Θ^2; writing f and g for the generating 1-cells, the free P-algebra on the corresponding 2-globular set looks like

$$\begin{tikzcd}
 & \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & \bullet,
\end{tikzcd}$$

(omitting identities and any composites involving identities). Thus, for $\mathcal{A} \in P$-**Alg**, the set $P\text{-}\mathbf{Alg}(F_P(2,0),\mathcal{A})$ is the set of all composable pairs of 1-cells in $\mathcal{A}$. However, we want an element of $\mathcal{NA}(2,0)$ to consist of a composable pair of 1-cells together with a choice of alternative composite, so we want $I_2(2,0)$ to look like

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "g\circ f"] \arrow[rr, bend right=50, "h"'] \arrow[rr, phantom, bend right=25, "\wr\|"] & & \bullet,
\end{tikzcd}$$

(once again omitting identities, etc.), where h is the choice of alternative composite. Note that these alternative composites are also required to allow us to define the face maps in our nerve; we cannot define the face maps using composition, as in the nerve of a category, because composition of 1-cells is not strictly associative in a Penon weak 2-category. We can think of this as weakening the maps in $\mathcal{NA}(2,0)$ on composites, but keeping them strict on identities. Thus, we may think we want to use a notion of normalised maps of Penon weak n-categories; that is, maps which preserve identities strictly but preserve composition only up to coherent isomorphism (note that there is no established definition of normalised maps of P-algebras, but for the purposes of this thought experiment this is not important). We would thus define

$$\mathcal{NA}(j,k) := P\text{-}\mathbf{Alg}_{\text{norm}}(F_P(j,k),\mathcal{A}),$$

where $P\text{-}\mathbf{Alg}_{\text{norm}}$ is the category of P-algebras and normalised maps. In fact, normalised maps turn out to be too weak, as we will now demonstrate. Consider the pasting diagram $(2,2)$ shown below:

$$\begin{tikzcd}
a_0 \arrow[r, bend left=60, "f_1"] \arrow[r, "g_1"'] \arrow[r, bend right=60, "h_1"'] \arrow[r, phantom, bend left=30, "\Downarrow"] \arrow[r, phantom, bend right=30, "\Downarrow"] & a_1 \arrow[r, bend left=60, "f_2"] \arrow[r, "g_2"'] \arrow[r, bend right=60, "h_2"'] \arrow[r, phantom, bend left=30, "\Downarrow"] \arrow[r, phantom, bend right=30, "\Downarrow"] & a_2
\end{tikzcd}$$

If we use normalised maps, each simplicial $(2,2)$-cell will include an extra 1-cell isomorphic to each of the binary composites of f's, g's and h's.

However, owing to the simplicial nature of Tamsamani–Simpson weak n-categories, we only wish to specify 1-cells in place of $f_2 \circ f_1$, $g_2 \circ g_1$, and $h_2 \circ h_1$. This is because we should have a 2-simplex of 1-cells at each "level" of the pasting diagram (here we have three such levels, one containing f_1 and f_2, one containing g_1 and g_2, and one containing h_1 and h_2) to allow us to define the face maps properly, but there should be no extra interaction between the levels. Recall from Definition 3.4 that the Segal map $S_{2,2}$ divides pasting diagrams of this shape along the 1-cells g_1 and g_2, and the Segal condition requires this map to be an isomorphism; if we add extra cells isomorphic to $h_2 \circ f_1$ and $h_1 \circ f_2$ to the diagram above, these cells are forgotten by $S_{2,2}$ so it is not an isomorphism.

We therefore want a method of weakening P-algebras that is biased towards specific choices of simplicial shapes. Such a method cannot be defined for a general P-algebra, since in general we have no notion of "level" like we do in a 2-pasting diagram. Thus, we define this weakening by explicitly stating which extra cells we are going to add. We do so by modifying the construction of the free Penon weak 2-category on a 2-globular set, using the construction of Penon's left adjoint from Part 1 of this series.

Recall from Section 4 that the adjunction inducing the monad P can be decomposed as

$$\begin{tikzcd}
n\text{-}\mathbf{GSet} \arrow[r, bend left, "H"] \arrow[r, phantom, "\perp"] & \mathcal{R} \arrow[l, bend left, "V"] \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q}, \arrow[l, bend left, "W"]
\end{tikzcd}$$

where $\mathcal{R}$ is the comma category $n\text{-}\mathbf{GSet} \downarrow U_T$, and $\mathcal{Q}$ is $\mathcal{R}$ with the added condition that the map part of each object is equipped with a contraction. Thus we can write the free P-algebra functor as the composite

$$\begin{tikzcd}
2\mathbf{GSet} \arrow[r, "H"] & \mathcal{R} \arrow[r, "J"] & \mathcal{Q} \arrow[r, "K"] & P\text{-}\mathbf{Alg},
\end{tikzcd}$$

where K is the Eilenberg–Moore comparison functor. Thus, instead of starting in $2\mathbf{GSet}$, we can start with an object of $\mathcal{R}$ and apply KJ to obtain a P-algebra that is "partially free" in the sense that the constraint cells and composites are still added freely (by the functor J), but the contraction is now taken over a different map, rather than a component of η^T. This allows us to add the isomorphism 2-cells we want using the contraction, thus avoiding the need to specify these cells individually.

Before defining the process in general we first describe a small example; specifically, we construct the P-algebra $I_2(2,1)$. Write $X(2,1)$ for the 2-globular set illustrated below:

$$\begin{tikzcd}
a_0 \arrow[r, bend left=50, "f^0_{01}"{name=U1}] \arrow[r, bend right=50, "f^1_{01}"'{name=D1}] & a_1 \arrow[r, bend left=50, "f^0_{12}"{name=U2}] \arrow[r, bend right=50, "f^1_{12}"'{name=D2}] & a_2
\arrow[Rightarrow, from=U1, to=D1, "\alpha^1_{01}"]
\arrow[Rightarrow, from=U2, to=D2, "\alpha^1_{12}"]
\end{tikzcd}$$

This is the associated 2-globular set of the pasting diagram, a concept introduced by Batanin [3, Proof of Proposition 4.2]. As explained earlier, we want $I_2(2,1)$ to be a "simplicially weakened" version of the free P-algebra on this 2-globular set, and to do so we construct an object of $\mathcal{R}$, then generate the "partially free" P-algebra on it. We take the strict 2-category part of this object of $\mathcal{R}$ to be the free strict 2-category on $X(2,1)$. To obtain the 2-globular set part of this object of $\mathcal{R}$ we add extra cells to $X(2,1)$ in the places where we want to weaken the diagram. Specifically, we add 1-cells

$$a_0 \xrightarrow{f^0_{02}} a_2 \quad \text{and} \quad a_0 \xrightarrow{f^1_{02}} a_2.$$

Based on Leinster's nerve construction for bicategories, we might also expect that we need to add a 2-cell

$$\begin{tikzcd}
a_0 \arrow[r, bend left=50, "f^0_{02}"{name=U}] \arrow[r, bend right=50, "f^1_{02}"'{name=D}] & a_2,
\arrow[Rightarrow, from=U, to=D, "\alpha^1_{02}"]
\end{tikzcd}$$

but this will be added automatically as a composite of other 2-cells, as we shall see later. We write $R(2,1)$ for the resulting 2-globular set; it can be drawn as:

$$\begin{tikzcd}
a_0 \arrow[rr, bend left=80, "f^0_{02}"] \arrow[rr, bend right=80, "f^1_{02}"'] \arrow[r, bend left=50, "f^0_{01}"{name=U1}] \arrow[r, bend right=50, "f^1_{01}"'{name=D1}] & a_1 \arrow[r, bend left=50, "f^0_{12}"{name=U2}] \arrow[r, bend right=50, "f^1_{12}"'{name=D2}] & a_2
\arrow[Rightarrow, from=U1, to=D1, "\alpha^1_{01}"]
\arrow[Rightarrow, from=U2, to=D2, "\alpha^1_{12}"]
\end{tikzcd}$$

To get an object of $\mathcal{R}$, we define a map

$$\theta_{(2,1)} \colon R(2,1) \longrightarrow TX(2,1)$$

as follows: $\theta_{(2,1)}$ leaves cells in $R(2,1)$ that are also in $X(2,1)$ unchanged; on the extra cells, we have

- $\theta_{(2,1)}(f^0_{02}) = f^0_{12} \circ f^0_{01}$;
- $\theta_{(2,1)}(f^1_{02}) = f^1_{12} \circ f^1_{01}$.

We now explain what happens when we apply the functor

$$J \colon \mathcal{R} \longrightarrow \mathcal{Q}$$

to

$$R(2,1) \xrightarrow{\theta_{(2,1)}} TX(2,1),$$

using the interleaving construction from Part 1 of this series. First we add contraction 1-cells; since $R(2,1)$ and $TX(2,1)$ have the same 0-cells, this just adds identities. We then generate composites of 1-cells freely; this adds $f^0_{12} \circ f^0_{01}$, $f^1_{12} \circ f^1_{01}$, $f^1_{12} \circ f^0_{01}$ and $f^0_{12} \circ f^1_{01}$, as well as composites involving identities. Next we add contraction 2-cells; this is where the "simplicial weakening" manifests itself. Observe that, after having generated 1-cell composites, we have pairs of 1-cells:

- f^0_{02} and $f^0_{12} \circ f^0_{01}$, which are parallel and are mapped to the same cell in $TX(2,1)$;
- f^1_{02} and $f^1_{12} \circ f^1_{01}$, which are parallel and are mapped to the same cell in $TX(2,1)$.

Thus, as well as the usual identities, associators, and unitors, generating contraction 2-cells freely adds the following cells:

$$\begin{tikzcd} & a_1 \arrow[dr, "f^0_{12}"] \arrow[d, phantom, "\Downarrow"] & \\ a_0 \arrow[ur, "f^0_{01}"] \arrow[rr, "f^0_{02}"'] & {} & a_2 \end{tikzcd} \qquad \begin{tikzcd} & a_1 \arrow[dr, "f^0_{12}"] \arrow[d, phantom, "\Uparrow"] & \\ a_0 \arrow[ur, "f^0_{01}"] \arrow[rr, "f^0_{02}"'] & {} & a_2 \end{tikzcd}$$

$$\begin{tikzcd}
& a_1 \arrow[dr, "f^1_{12}"] \arrow[d, phantom, "\Downarrow"] & & & a_1 \arrow[dr, "f^1_{12}"] \arrow[d, phantom, "\Uparrow"] & \\
a_0 \arrow[ur, "f^1_{01}"] \arrow[rr, "f^1_{02}"'] & {} & a_2 & a_0 \arrow[ur, "f^1_{01}"] \arrow[rr, "f^1_{02}"'] & {} & a_2
\end{tikzcd}$$

We generate composites of 2-cells, then "add contraction 3-cells", which forces all diagrams of 2-cells to commute. In particular, this forces the pairs of triangular cells shown above to be inverses of one another (and thus isomorphisms), and also gives us a 2-cell

$$\begin{tikzcd}
& & & & {} & \\
a_0 \arrow[rr, bend left, "f^0_{02}"] \arrow[rr, bend right, "f^1_{02}"'] \arrow[rr, phantom, "\Downarrow \alpha^1_{02}"] & & a_2 \arrow[r, phantom, "="] & a_0 \arrow[r, bend left, "f^0_{01}"] \arrow[r, bend right, "f^1_{01}"'] \arrow[r, phantom, "\Downarrow \alpha^1_{01}"] \arrow[rr, bend left=80, "f^0_{02}"] \arrow[rr, bend right=80, "f^1_{02}"'] & a_1 \arrow[r, bend left, "f^0_{12}"] \arrow[r, bend right, "f^1_{12}"'] \arrow[r, phantom, "\Downarrow \alpha^1_{12}"] \arrow[u, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & a_2 \\
& & & & {} &
\end{tikzcd}$$

Observe that this corresponds to the first axiom from Leinster's nerve construction (see Definition 3.1); adding "contraction 3-cells" also ensures that the second axiom holds when we perform this construction for longer cuboidal pasting diagrams.

This whole process gives an object of $\mathcal{Q}$, denoted

$$Q(j,k) \xrightarrow{\phi(j,k)} TX(j,k).$$

We obtain the P-algebra $I_2(2,1)$ by applying the Eilenberg–Moore comparison functor; the resulting P-algebra has as its underlying magma the magma part of the object of $\mathcal{Q}$ above.

Note that the triangular cells added by the free contraction are considered contraction cells in the object of $\mathcal{Q}$, but when we apply the Eilenberg–Moore comparison functor they are not contraction cells from the point of view of the P-algebra action. They retain their commutativity properties, however, so given any other P-algebra $\mathcal{A}$, a map of P-algebras

$$I_2(2,1) \longrightarrow \mathcal{A}$$

can map these cells to any suitably coherent choice of cells in $\mathcal{A}$; their images need not be contraction cells.

We now describe this construction for a general object of Θ^2. As above, we use Leinster's notation from his nerve construction for bicategories (Section 3). Recall that the subscripts and superscripts adorning each cell should be thought of as being the "coordinates" of that cell within the pasting diagram; the subscripts are the horizontal coordinates, and the superscripts are the vertical coordinates.

Note that an object of Θ^2 is an equivalence class of objects of Δ^2. An object of Δ^2 is in an equivalence class with more than one member if and only if it has a 0 in the first position. Thus, for the purposes of the following definition we represent the equivalence class of $(0, k)$ for all $k \in \mathbb{N}$ by the object $(0, 0)$ of Δ^2; all other equivalence classes are represented by their sole member.

Let (j, k) be an object of Θ^2; we first define the 2-globular set $X(j, k)$, the associated 2-globular set of the cuboidal pasting diagram (j, k), as follows:

- $X(j,k)_0 = \{a_u \mid u \in \mathbb{N}, 0 \leq u \leq j\}$;
- $X(j,k)_1 = \{f^z_{u,u+1} \mid u, z \in \mathbb{N}, 0 \leq u < j, 0 \leq z \leq k\}$;
- $X(j,k)_2 = \{\alpha^z_{u,u+1} \mid u, z \in \mathbb{N}, 0 \leq u < j, 1 \leq z \leq k\}$,

with source and target maps given by

$$s(f^z_{u,u+1}) = a_u,\ t(f^z_{u,u+1}) = a_{u+1},$$

$$s(\alpha^z_{u,u+1}) = f^{z-1}_{u,u+1},\ t(\alpha^z_{u,u+1}) = f^z_{u,u+1}.$$

We then add extra 1- and 2-cells to this to obtain a 2-globular set $R(j, k)$, defined as follows:

- $R(j,k)_0 = \{a_u \mid u \in \mathbb{N}, 0 \leq u \leq j\}$;
- $R(j,k)_1 = \{f^z_{uv} \mid u, v, z \in \mathbb{N}, 0 \leq u < v \leq j, 0 \leq z \leq k\}$;
- $R(j,k)_2 = \{\alpha^z_{u,u+1} \mid u, z \in \mathbb{N}, 0 \leq u < j, 1 \leq z \leq k\}$,

with source and target maps given by

$$s(f^z_{uv}) = a_u,\ t(f^z_{uv}) = a_v,$$

$$s(\alpha^z_{u,u+1}) = f^{z-1}_{u,u+1},\ t(\alpha^z_{u,u+1}) = f^z_{u,u+1}.$$

It is important to note that, in spite of the notation, this does not define functors X and R into 2-$\mathbf{GSet}$. This is because, at this stage of the construction, there is no way to define the action on maps in Θ^2, since we cannot map cells to identities as we do not have these in the 2-globular sets.

We now construct, for each $(j,k) \in \Theta^2$, an object

$$R(j,k) \xrightarrow{\theta_{(j,k)}} TX(j,k)$$

of $\mathcal{R}$. We define the map $\theta_{(j,k)}$ as follows:

- on 0-cells, $\theta_{(j,k)0}(a_u) = a_u$;
- on 1-cells, $\theta_{(j,k)1}(f^z_{uv}) = f^z_{v-1,v} \circ f^z_{v-2,v-1} \circ \cdots \circ f^z_{u,u+1}$;
- on 2-cells, $\theta_{(j,k)2}(\alpha^z_{u,u+1}) = \alpha^z_{u,u+1}$.

This map coincides with $\eta^T_{X(j,k)}$, the unit for the monad T, for all cells in $X(j,k)$; the extra cells in $R(j,k)$ can be thought of as weakenings of the composites at each level of the cuboidal pasting diagram, and $\theta_{(j,k)}$ maps each of these cells to the corresponding freely generated strict composite in $TX(j,k)$.

We now apply the functor $J\colon \mathcal{R} \to \mathcal{Q}$ to the object of $\mathcal{R}$ described above; this adds to $R(j,k)$ all the required composites and contraction cells. As demonstrated in the example above, this includes contraction cells in both directions between each of the extra 1-cells (those in $R(j,k)_1$ but not in $X(j,k)_1$) and the corresponding freely generated composites at the same level of the pasting diagram (i.e. of cells with the same z-coordinate). The tameness condition in the contraction ensures that these contraction 2-cells are isomorphisms. The extra 1-cells will give the necessary 1-dimensional faces in the nerve, and the contraction cells ensure that these are coherently isomorphic to the composites we originally had in the Penon weak

2-category whose nerve we are taking. We denote the resulting object of $\mathcal{Q}$ by

$$Q(j,k) \xrightarrow{\phi_{(j,k)}} TX(j,k).$$

We now extend this to a definition of a functor $E_2\colon \Theta^2 \to \mathcal{Q}$, with the action on objects as described above. To describe the action on a morphism in Θ^2, we first define a morphism in $\mathcal{R}$, and then take its transpose under the adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q} \arrow[l, shift left, "W"] \end{tikzcd}$$

to obtain a morphism in $\mathcal{Q}$.

Let $(p,q)\colon (j,k) \to (l,m)$ be a morphism in Θ^2. We define the strict 2-category part of the morphism of $\mathcal{R}$ first. Define a map of 2-globular sets $x(p,q)\colon X(j,k) \to TX(l,m)$ as follows:

- on 0-cells, $x(p,q)_0(a_u) = a_{p(u)}$;
- on 1-cells, $x(p,q)_1(f^z_{u,u+1}) =$

$$\begin{cases} f^{q(z)}_{p(u+1)-1,p(u+1)} \circ \cdots \circ f^{q(z)}_{p(u),p(u)+1} & \text{if } p(u) < p(u+1), \\ 1_{a_{p(u)}} & \text{if } p(u) = p(u+1); \end{cases}$$

- on 2-cells, $x(p,q)_2(\alpha^z_{u,u+1}) =$

$$\begin{cases} \alpha^{q(z)}_{p(u+1)-1,p(u+1)} * \cdots * \alpha^{q(z)}_{p(u),p(u)+1} & \text{if } p(u) < p(v), q(z-1) < q(z), \\ 1_{TX(p,q)_1(f^z_{u,u+1})} & \text{if } q(z-1) = q(z). \end{cases}$$

To obtain a map $TX(j,k) \to TX(l,m)$ we apply T and compose this with the multiplication for T, giving

$$TX(j,k) \xrightarrow{Tx(p,q)} T^2X(l,m) \xrightarrow{\mu^T_{X(l,m)}} TX(l,m)$$

We now define a map

$$\begin{tikzcd}
R(j,k) \arrow[rr, "r(p{,}q)"] \arrow[d, "\theta_{(j,k)}"'] & & Q(l,m) \arrow[d, "\phi_{(l,m)}"] \\
TX(j,k) \arrow[r, "Tx(p{,}q)"'] & T^2X(l,m) \arrow[r, "\mu^T_{X(l,m)}"'] & TX(l,m),
\end{tikzcd}$$

where the map $r(p,q)$ is defined as follows:

- on 0-cells, $R(p,q)_0(a_u) = a_{p(u)}$;
- on 1-cells,

$$R(p,q)_1(f^z_{uv}) = \begin{cases} f^{q(z)}_{p(u)p(v)} & \text{if } p(u) < p(v), \\ 1_{a_{p(u)}} & \text{if } p(u) = p(v); \end{cases}$$

- on 2-cells,

$$R(p,q)_2(\alpha^z_{uv}) = \begin{cases} \alpha^{q(z)}_{p(u)p(v)} & \text{if } p(u) < p(v), q(z-1) < q(z), \\ 1_{f^{q(z)}_{p(u)p(v)}} & \text{if } p(u) < p(v), q(z-1) = q(z), \\ 1_{1_{a_{p(u)}}} & \text{if } p(u) = p(v). \end{cases}$$

Finally, we take the transpose of this map under the adjunction

$$\begin{tikzcd}
\mathcal{R} \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "W"]
\end{tikzcd}$$

We write $\epsilon\colon JW \Rightarrow 1$ for the counit of this adjunction, and $\epsilon_{\phi_{(l,m)}}$ for the component corresponding to

$$Q(l,m) \xrightarrow{\phi_{(l,m)}} TX(l,m).$$

Then the transpose is given by the composite

$$\epsilon_{\phi_{(l,m)}} \circ J\big(r(p,q), \mu^T_{X(l,m)} \circ Tx(p,q)\big).$$

This allows us to define the functors $E_2\colon \Theta^2 \to \mathcal{Q}$ and $I_2\colon \Theta^2 \to P$-**Alg**.

Definition 5.1. Define a functor $E_2 \colon \Theta^2 \to \mathcal{Q}$ as follows:

- given an object $(j,k) \in \Theta^2$, $E_2(j,k)$ is defined to be the object

  $$Q(j,k) \xrightarrow{\phi_{(j,k)}} TX(j,k).$$

  of $\mathcal{Q}$;

- given a morphism $(p,q) : (j,k) \to (l,m)$ in Θ^2, $E_2(p,q)$ is defined to be the map

  $$\epsilon_{\phi_{(l,m)}} \circ J\big(r(p,q), \mu^T_{X(l,m)} \circ Tx(p,q)\big).$$

Write $K \colon \mathcal{Q} \to P\text{-}\mathbf{Alg}$ for the Eilenberg–Moore comparison functor for the adjunction

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "U"] \end{tikzcd}$$

We define a functor $I_2 := K \circ E_2 : \Theta^2 \to P\text{-}\mathbf{Alg}$.

We can now define the nerve functor for Penon weak 2-categories.

Definition 5.2. The *nerve functor* $\mathcal{N}$ for Penon weak 2-categories is defined by

$$\begin{tikzcd} \mathcal{N} \colon P\text{-}\mathbf{Alg} \arrow[r] & {[(\Theta^2)^{\mathrm{op}}, \mathbf{Set}]} \\ \mathcal{A} \arrow[d, "f"'] & P\text{-}\mathbf{Alg}(I_2(-), \mathcal{A}) \arrow[d, "f \circ -"] \\ \mathcal{B} \arrow[r, phantom, "\longmapsto" near start, shift left=20] & P\text{-}\mathbf{Alg}(I_2(-), \mathcal{B}). \end{tikzcd}$$

For a P-algebra $\mathcal{A}$, the presheaf $\mathcal{N}\mathcal{A} = P\text{-}\mathbf{Alg}(I_2(-), \mathcal{A})$ is called the *nerve of* $\mathcal{A}$.

6. The Segal condition

In this section we prove that the nerve of a Penon weak 2-category satisfies the Segal condition, and is therefore a Tamsamani–Simpson weak 2-category. Recall from Definition 3.4 that $\mathcal{N}\mathcal{A}$ satisfies the Segal condition if

(i) for all $j \geq 0$, the Segal map

$$S_j : \mathcal{N}\mathcal{A}(j,-) \longrightarrow \underbrace{\mathcal{N}\mathcal{A}(1,-) \times_{\mathcal{N}\mathcal{A}(0,1)} \cdots \times_{\mathcal{N}\mathcal{A}(0,1)} \mathcal{N}\mathcal{A}(1,-)}_{j}$$

is contractible, i.e. surjective on objects, full and faithful on 1-cells;

(ii) for all j, $k \geq 0$, the Segal map

$$S_{j,k} : \mathcal{N}\mathcal{A}(j,k) \longrightarrow \underbrace{\mathcal{N}\mathcal{A}(j,1) \times_{\mathcal{N}\mathcal{A}(j,0)} \cdots \times_{\mathcal{N}\mathcal{A}(j,0)} \mathcal{N}\mathcal{A}(j,1)}_{k}$$

is a bijection.

Our approach is to use the way in which nerve functor is defined to rewrite the Segal maps in terms of composition with certain maps of P-algebras; this then allows us to express most parts of the Segal condition (everything except surjectivity on objects) as statements describing certain P-algebras in the image of I_2 as colimits of diagrams in the image of I_2.

Before doing this, we establish some notation for certain free P-algebras in the image of I_2 that can be expressed as colimits of others; these P-algebras arise in the reformulation of the Segal condition described above. Observe that the free P-algebra functor F_P can be factorised as

$$\begin{tikzcd}
2\text{-}\mathbf{GSet} \arrow[rr, "F_P"] \arrow[dr, "F"'] & & P\text{-}\mathbf{Alg} \\
& \mathcal{Q} \arrow[ur, "K"'] &
\end{tikzcd}$$

Thus, we see from the construction of I_2 that, for (j,k) in Θ^2, if $R(j,k) = X(j,k)$, then $I_2(j,k) = F_P X(j,k)$. Since $R(j,k)$ and $X(j,k)$ differ only on 1-cells, this happens precisely when $R(j,k)_1 = X(j,k)_1$. This is true when $j = 0$ and $j = 1$, since

- for $j = 0$, $R(j,k)_1 = \emptyset = X(j,k)_1$;
- for $j = 1$, $R(j,k)_1 = \{f_{01}^z \mid 0 \leq z \leq k\} = X(j,k)_1$.

Thus $I_2(0,0) = F_P X(0,0)$, and $I_2(1,k) = F_P X(1,k)$ for all $k \in \mathbb{N}$. For $j \geq 2$, we have $f^0_{02} \in R(j,k)$, but $f^0_{02} \notin X(j,k)$, so this does not hold for $j \geq 2$.

Recall that, for all $k > 0$, $0 \leq i \leq k$, we have a map $d_i \colon [k-1] \to [k]$ in Δ given by

$$d_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

and consider the following diagram in P-$\mathbf{Alg}$:

$$\underbrace{\begin{tikzcd}
 & I_2(0,0) \arrow[dl, "I_2(d_0{,}1)"'] \arrow[dr, "I_2(d_1{,}1)"] & & & & I_2(0,0) \arrow[dl, "I_2(d_0{,}1)"'] \arrow[dr, "I_2(d_1{,}1)"] & \\
I_2(1,0) & & I_2(1,0) & \dots & I_2(1,0) & & I_2(1,0).
\end{tikzcd}}_{j \text{ copies of } I_2(1,0)}$$

Write $I_2(1,0)^{\amalg j}$ for the colimit of this diagram in P-$\mathbf{Alg}$. By the observations above, this diagram is the image under F_P of the diagram

$$\underbrace{\begin{tikzcd}
 & X(0,0) \arrow[dl, "a_1"'] \arrow[dr, "a_0"] & & & & X(0,0) \arrow[dl, "a_1"'] \arrow[dr, "a_0"] & \\
X(1,0) & & X(1,0) & \dots & X(1,0) & & X(1,0)
\end{tikzcd}}_{j \text{ copies of } X(1,0)}$$

in 2-$\mathbf{GSet}$, where $a_0 \colon X(0,0) \to X(1,0)$ maps the single 0-cell of $X(0,0)$ to a_0, and similarly for a_1. The colimit in 2-$\mathbf{GSet}$ of this diagram is $X(j,0)$, and thus

$$I_2(1,0)^{\amalg j} = F_P X(j,0),$$

the free P-algebra on a composable string of j 1-cells.

Similarly, write $I_2(1,1)^{\amalg j}$ for the colimit in P-$\mathbf{Alg}$ of the diagram

$$\underbrace{\begin{tikzcd}
 & I_2(0,1) \arrow[dl, "I_2(d_0{,}1)"'] \arrow[dr, "I_2(d_1{,}1)"] & & & & I_2(0,1) \arrow[dl, "I_2(d_0{,}1)"'] \arrow[dr, "I_2(d_1{,}1)"] & \\
I_2(1,1) & & I_2(1,1) & \dots & I_2(1,1) & & I_2(1,1),
\end{tikzcd}}_{j \text{ copies of } I_2(1,1)}$$

which is the image under F_P of the diagram

$$\underbrace{\begin{tikzcd} & X(0,1) \arrow[dl, "a_1"'] \arrow[dr, "a_0"] & & & & X(0,1) \arrow[dl, "a_1"'] \arrow[dr, "a_0"] & \\ X(1,1) & & X(1,1) & \dots & X(1,1) & & X(1,1) \end{tikzcd}}_{j \text{ copies of } X(1,1)}$$

in 2-$\mathbf{GSet}$. The colimit in 2-$\mathbf{GSet}$ of this diagram is $X(j,1)$, and thus

$$I_2(1,1)^{\amalg j} = F_P X(j,1),$$

the free P-algebra on a string of j 2-cells composable along boundary 0-cells.

We now rewrite the Segal maps of the form S_j in terms of composition with certain maps of P-algebras.

Lemma 6.1. *Let $\mathcal{A}$ be a Penon weak 2-category. For all $j > 0$, we have*

$$\underbrace{\mathcal{N}\mathcal{A}(1,-) \times_{\mathcal{N}\mathcal{A}(0,-)} \cdots \times_{\mathcal{N}\mathcal{A}(0,-)} \mathcal{N}\mathcal{A}(1,-)}_{j} \cong P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j}, \mathcal{A})$$

and the Segal map S_j is given by

$$S_j = \cdot \circ d^{\amalg j} \colon P\text{-}\mathbf{Alg}(I_2(j,-), \mathcal{A}) \longrightarrow P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j}, \mathcal{A}),$$

where $d^{\amalg j} \colon I_2(1,-)^{\amalg j} \to I_2(j,-)$ is a map in $[\Delta, P\text{-}\mathbf{Alg}]$ induced by the universal property of $I_2(1,-)^{\amalg j}$, defined in the proof.

Proof. We have the following functors:

$$\begin{tikzcd} \mathcal{N}^2\mathcal{A}(\cdot,-) \colon \Delta^{\mathrm{op}} \arrow[r] & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \\ k \arrow[d, "\alpha"'] & P\text{-}\mathbf{Alg}(I_2(k,-), \mathcal{A}) \arrow[d, "\cdot \circ I_2(\alpha,-)"] \\ j \arrow[r, phantom, "\longmapsto"] & P\text{-}\mathbf{Alg}(I_2(j,-), \mathcal{A}), \end{tikzcd}$$

$$\begin{tikzcd}
I_2(\cdot,-)\colon \Delta \arrow[r] & {[\Delta, P\text{-}\mathbf{Alg}]} \\
j \arrow[d, "\alpha"'] & I_2(j,-) \arrow[d, "I_2(\alpha,-)"] \\
k \arrow[r, phantom, "\longmapsto" near start, shift left=20pt] & I_2(k,-),
\end{tikzcd}$$

and

$$\begin{tikzcd}
P\text{-}\mathbf{Alg}(-,\mathcal{A})\colon [\Delta, P\text{-}\mathbf{Alg}]^{\mathrm{op}} \arrow[r] & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \\
X \arrow[d, "\delta"'] & P\text{-}\mathbf{Alg}(X(-),\mathcal{A}) \arrow[d, "-\circ\delta"] \\
Y & P\text{-}\mathbf{Alg}(Y(-),\mathcal{A}).
\end{tikzcd}$$

We can factorise $\mathcal{N}\mathcal{A}(\cdot,-)$ as follows:

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \arrow[rr, "\mathcal{N}\mathcal{A}(\cdot,-)"] \arrow[dr, "I_2(\cdot,-)"'] & & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \\
& {[\Delta, P\text{-}\mathbf{Alg}]^{\mathrm{op}}} \arrow[ur, "P\text{-}\mathbf{Alg}(-,\mathcal{A})"'] &
\end{tikzcd}$$

For each, $[j] \in \Delta$, we consider the actions of the functors $\mathcal{N}\mathcal{A}(\cdot,-)$ and $I_2(\cdot,-)$ on the diagram

$$\begin{tikzcd}
& & & & & {[j]} & & & \\
{[1]} \arrow[urrrrr, hook, "\iota_1"] & & {[1]} \arrow[urrr, hook, "\iota_2"] & & {[1]} \arrow[ur, hook, "\iota_3"] & \cdots & {[1]} \arrow[ul, hook', "\iota_{j-1}"'] & & {[1]} \arrow[ulll, hook', "\iota_j"'] \\
& {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

in Δ.

Applying $\mathcal{N}\mathcal{A}(\cdot,-)$ to this diagram gives

$$\begin{tikzcd}
 & & P\text{-}\mathbf{Alg}(I_2(j,-),\mathcal{A}) \arrow[dll] \arrow[ddl] \arrow[ddr] \arrow[drr] & & \\
P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dd, "t"'] & & & & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dd, "s"] \\
 & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dl, "s"] & \cdots & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dr, "t"'] & \\
P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A}) & & & & P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A})
\end{tikzcd}$$

which is a cone over the diagram

$$\begin{tikzcd}
P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dd, "t"'] & & & & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dd, "s"] \\
 & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dl, "s"] & \cdots & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dr, "t"'] & \\
P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A}) & & & & P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A})
\end{tikzcd}$$

Applying $I_2(\cdot,-)^{\mathrm{op}}$ to the original diagram gives

$$\begin{tikzcd}
 & & & I_2(j,-) \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] & \dots & I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] \\
 & I_2(0,-) & & & & I_2(0,-) &
\end{tikzcd}$$

in $[\Delta, P\text{-}\mathbf{Alg}]^{\mathrm{op}}$, which is a cone over the diagram

$$\overbrace{\begin{tikzcd}
I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] & \dots & I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] \\
 & I_2(0,-) & & & & I_2(0,-) &
\end{tikzcd}}^{j}$$

The limit of this diagram is $I_2(1,-)^{\amalg j}$, and this limit induces a unique map $d^{\amalg j}$ such that the diagram

$$\begin{tikzcd}
 & & & I_2(j,-) \arrow[d, dashed, "!d^{\amalg j}"] \arrow[dlll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[drrr, bend left] & & & \\
 & & & I_2(1,-)^{\amalg j} \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] & \dots & I_2(1,-) \arrow[dr, "t"'] & & I_2(1,-) \arrow[dl, "s"] \\
 & I_2(0,-) & & & & I_2(0,-) &
\end{tikzcd}$$

Applying $P\text{-}\mathbf{Alg}(-,\mathcal{A})$ to this diagram, we get:

$$\begin{tikzcd}
 & & & P\text{-}\mathbf{Alg}(I_2(j,-),\mathcal{A}) \arrow[d, dashed, "-\circ d^{\amalg j}"] \arrow[ddlll, bend right] \arrow[dddl, bend right] \arrow[dddr, bend left] \arrow[ddrrr, bend left] & & & \\
 & & & P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j},\mathcal{A}) \arrow[dlll] \arrow[ddl] \arrow[ddr] \arrow[drrr] & & & \\
P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[ddr, "t"'] & & & & & & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[ddl, "s"] \\
 & & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dl, "s"] & \cdots & P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \arrow[dr, "t"'] & & \\
 & P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A}) & & & & P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A}) &
\end{tikzcd}$$

Since $P\text{-}\mathbf{Alg}(-,\mathcal{A})$ is representable, it preserves limits [27, V.6 Theorem 3], so we have that

$$\underbrace{P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A}) \times_{P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A})} \cdots \times_{P\text{-}\mathbf{Alg}(I_2(0,-),\mathcal{A})} P\text{-}\mathbf{Alg}(I_2(1,-),\mathcal{A})}_{k}$$

$$\cong P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j},\mathcal{A})$$

and the Segal map S_j is given by composition with $d^{\amalg j}$, as required. □

Similarly, we now rewrite the Segal maps of the form $S_{j,k}$ in terms of composition with certain maps of P-algebras.

Lemma 6.2. *Let $\mathcal{A}$ be a Penon weak 2-category. For all j, $k > 0$, we have*

$$\underbrace{\mathcal{N}\mathcal{A}(j,1) \times_{\mathcal{N}\mathcal{A}(j,0)} \cdots \times_{\mathcal{N}\mathcal{A}(j,0)} \mathcal{N}\mathcal{A}(j,1)}_{k} \cong P\text{-}\mathbf{Alg}(I_2(j,1)^{\amalg k}, \mathcal{A})$$

and the Segal map $S_{j,k}$ is given by

$$S_{j,k} = \cdot \circ d^{\amalg k} \colon P\text{-}\mathbf{Alg}(I_2(j,k), \mathcal{A}) \longrightarrow P\text{-}\mathbf{Alg}(I_2(j,1)^{\amalg k}, \mathcal{A}),$$

where $d^{\amalg k} \colon I_2(j,1)^{\amalg k} \to I_2(j,k)$ is a map of P-algebras induced by the universal property of $I_2(j,1)^{\amalg k}$, defined in the proof.

Proof. We take a similar approach to that used in the proof of Lemma 6.1. For each $j > 0$, we have the following functors:

$$\begin{tikzcd} \mathcal{N}^2\mathcal{A}(j,\cdot) \colon \Delta^{\mathrm{op}} \arrow[r] & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \\ l \arrow[d, "\alpha"'] & P\text{-}\mathbf{Alg}(I_2(j,l), \mathcal{A}) \arrow[d, "\cdot \circ I_2(1_j,\alpha)"] \\ k \arrow[r, phantom, "\longmapsto" near start, shift left=8] & P\text{-}\mathbf{Alg}(I_2(j,k), \mathcal{A}), \end{tikzcd}$$

and

$$\begin{tikzcd} I_2(j,-) \colon \Delta \arrow[r] & {[\Delta, P\text{-}\mathbf{Alg}]} \\ k \arrow[d, "\alpha"'] & I_2(j,k) \arrow[d, "I_2(1_j,\alpha)"] \\ l \arrow[r, phantom, "\longmapsto" near start, shift left=8] & I_2(j,l), \end{tikzcd}$$

and we can factorise $\mathcal{N}\mathcal{A}(j,\cdot)$ as follows:

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[rr, "\mathcal{N}\mathcal{A}(j,\cdot)"] \arrow[dr, "I_2(j,\cdot)"'] & & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \\ & {[\Delta, P\text{-}\mathbf{Alg}]^{\mathrm{op}}} \arrow[ur, "P\text{-}\mathbf{Alg}(-,\mathcal{A})"'] & \end{tikzcd}$$

For each, $[k] \in \Delta$, we consider the effects of the functors $\mathcal{N}\mathcal{A}(j,\cdot)$ and $I_2(j,\cdot)$ on the diagram

$$\begin{tikzcd}
 & & & & & {[k]} & & & & & \\
{[1]} \arrow[urrrrr, "\iota_1"] & & {[1]} \arrow[urrr, "\iota_2"] & & {[1]} \arrow[ur, "\iota_3"] & & \cdots & & {[1]} \arrow[ulll, "\iota_{k-1}"] & & {[1]} \arrow[ulllll, "\iota_k"'] \\
 & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & & & & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

in Δ. By exactly the same argument as the case of S_j, we have a unique map $d^{\amalg k}$ such that

$$\begin{tikzcd}
 & & & I_2(j,k) \arrow[ddlll, bend right] \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[ddrrr, bend left] \arrow[d, dashed, "!d^{\amalg k}"] & & & \\
 & & & I_2(j,1)^{\amalg k} \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
I_2(j,1) \arrow[dr, "t"'] & & I_2(j,1) \arrow[dl, "s"] & \cdots & I_2(j,1) \arrow[dr, "t"'] & & I_2(j,1) \arrow[dl, "s"] \\
 & I_2(j,0) & & & & I_2(j,0) &
\end{tikzcd}$$

and applying the functor P-$\mathbf{Alg}(-,\mathcal{A})$ gives us the diagram

$$\begin{tikzcd}
 & & & P\text{-}\mathbf{Alg}(I_2(j,k),\mathcal{A}) \arrow[ddlll, bend right] \arrow[dddl, bend right] \arrow[dddr, bend left] \arrow[ddrrr, bend left] \arrow[d, dashed, "-\circ d^{\amalg k}"] & & & \\
 & & & P\text{-}\mathbf{Alg}(I_2(j,1)^{\amalg k},\mathcal{A}) \arrow[dlll] \arrow[ddl] \arrow[ddr] \arrow[drrr] & & & \\
P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \arrow[ddr, "t"'] & & & & & & P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \arrow[ddl, "s"] \\
 & & P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \arrow[dl, "s"] & \cdots & P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \arrow[dr, "t"'] & & \\
 & P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A}) & & & & P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A}) &
\end{tikzcd}$$

Thus we have that

$$\underbrace{P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \times_{P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A})} \cdots \times_{P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A})} P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A})}_{k}$$

$$\cong P\text{-}\mathbf{Alg}(I_2(j,1)^{\amalg k},\mathcal{A})$$

and the Segal map $S_{j,k}$ is given by composition with $d^{\amalg k}$, as required. □

We now use Lemmas 6.1 and 6.2 to prove that the nerve of a Penon weak 2-category satisfies the Segal condition. We begin with the Segal maps of the form S_j.

Proposition 6.3. *Let $\mathcal{A}$ be a Penon weak 2-category. For all $j > 0$, the Segal map*

$$S_j \colon \mathcal{NA}(j,-) \to \underbrace{\mathcal{NA}(1,-) \times_{\mathcal{NA}(0,-)} \cdots \times_{\mathcal{NA}(0,-)} \mathcal{NA}(1,-)}_{j}$$

is surjective on 0-cells, i.e. the map

$$(S_j)_0 \colon \mathcal{NA}(j,0) \to \underbrace{\mathcal{NA}(1,0) \times_{\mathcal{NA}(0,0)} \cdots \times_{\mathcal{NA}(0,0)} \mathcal{NA}(1,0)}_{j}$$

is surjective.

Proof. By Lemma 6.1, the Segal map S_j is given by

$$S_j = \cdot \circ d^{\amalg j} \colon P\text{-}\mathbf{Alg}(I_2(j,-),\mathcal{A}) \to P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j},\mathcal{A}),$$

so we need to show that

$$(S_j)_0 = \cdot \circ d^{\amalg j} \colon P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A}) \to P\text{-}\mathbf{Alg}(I_2(1,0)^{\amalg j},\mathcal{A})$$

is surjective. Let $\phi \colon I_2(1,0)^{\amalg j} \to \mathcal{A}$ be a map of Penon weak 2-categories. We must find a map $\psi \colon I_2(j,0) \to \mathcal{A}$ such that $(S_k)_0(\psi) = \phi$, i.e. such that the diagram

$$\begin{tikzcd}
I_2(1,0)^{\amalg j} \arrow[rr, "\phi"] \arrow[dr, hook, "d^{\amalg j}"'] & & \mathcal{A} \\
& I_2(j,0) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes.

Write the P-algebra $\mathcal{A}$ as

$$PA \xrightarrow{\theta} A$$

so $U_P\mathcal{A} = A$. We define ψ by first defining a map into the free algebra F_PA, then composing this with the algebra action θ. Define a map

$$\begin{tikzcd}
R(j,0) \arrow[rr, "g"] \arrow[d, "\theta_{(j,0)}"'] & & PA \arrow[d, "p_A"] \\
TX(j,0) \arrow[r, "Th"'] & T^2A \arrow[r, "\mu^T_A"'] & TA
\end{tikzcd}$$

in $\mathcal{R}$ as follows:

The map $g\colon R(j,0) \to PA$ is defined by:

- for all $a_u \in R(j,0)_0$, $g_0(a_u) = \phi_0(a_u)$;
- for $f^0_{uv} \in R(j,0)_1$ with $v = u+1$,
$$g_1(f^0_{uv}) = \phi_1(f^0_{uv});$$
- for $f^0_{uv} \in R(j,0)_1$ with $v > u+1$
$$g_1(f^0_{uv}) = \Big(\Big(\cdots \Big(\phi_1(f^0_{v-1,v}) \circ \phi_1(f^0_{v-2,v-1})\Big) \circ \cdots\Big) \circ \phi_1(f^0_{u,u+1})\Big).$$

Note that $R(j,0)_2 = \emptyset$, so we do not need to define g on 2-cells.

The map $h\colon X(k,0) \to TA$ is defined by:

- for all $a_u \in X(j,0)_0$, $h_0(a_u) = \phi_0(a_u)$;
- for all $f^0_{u,u+1} \in X(j,1)_1$,
$$h_1(f^0_{u,u+1}) = p_A \circ \phi_1(f^0_{u,u+1})$$

Note that $X(j,0)_2 = \emptyset$, so we do not need to define h on 2-cells.

This defines a map in $\mathcal{R}$. We then take the transpose of this map under the the adjunction

$$\begin{tikzcd}
\mathcal{R} \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q} \arrow[l, shift left, "W"]
\end{tikzcd}$$

We write $\epsilon \colon JW \Rightarrow 1$ for the counit of this adjunction, and $\epsilon_{\phi_{\mathbf{k}}}$ for the component corresponding to

$$Q(j,0) \xrightarrow{\phi_{(j,0)}} TX(j,0).$$

Then the transpose is given by the composite

$$\epsilon_{\phi_{(j,0)}} \circ J(g, \mu^T_A \circ Th).$$

Finally, we apply the Eilenberg–Moore comparison functor $K \colon \mathcal{Q} \to P$-**Alg** to this; we write

$$\chi := K(\epsilon_{\phi_{(j,0)}} \circ J(g, \mu^T_A \circ Th)),$$

and define

$$\psi := \theta \circ \chi \colon I_2(j,0) \to \mathcal{A}.$$

We now check commutativity of the diagram

$$\begin{tikzcd}
I_2(1,0)^{\amalg j} \arrow[rr, "\phi"] \arrow[dr, hook, swap, "d^{\amalg j}"] & & \mathcal{A} \\
& I_2(j,0) \arrow[ur, swap, "\psi"] &
\end{tikzcd}$$

Since $I_2(1,0)^{\amalg j} \cong F_P X(j,0)$, this commutes if the diagram

$$\begin{tikzcd}
X(j,0) \arrow[r, "\eta^P_{X(j,0)}"] \arrow[dr, swap, "\eta^P_{X(j,0)}"] & U_P F_P X(j,0) \arrow[r, "U_P\phi"] & U_P\mathcal{A} \\
& U_P F_P X(j,0) \arrow[r, swap, "U_P d^{\amalg j}"] & U_P I_2(j,0) \arrow[u, swap, "U_P\psi"]
\end{tikzcd}$$

in 2-**GSet** commutes; we check this using an elementary approach. Since $X(j,0)_2 = \emptyset$, we do not have to check commutativity on 2-cells. We have

- for $a_u \in X(j,0)_0$,
$$U_P\psi_0 \circ U_P d_0^{\amalg j} \circ \eta^P_{X(j,0)}(a_u) = U_P\psi_0(a_u) = U_P\phi_0 \circ \eta^P_{X(j,0)}(a_u);$$
- for $f^z_{u,u+1} \in X(j,0)_1$,
$$U_P\psi_1 \circ U_P d_1^{\amalg j} \circ \eta^P_{X(j,0)}(f^z_{u,u+1}) = U_P\psi_1(f^z_{u,u+1}) = U_P\phi_1 \circ \eta^P_{X(j,0)}(f^z_{u,u+1});$$

hence the diagram commutes. Hence S_j is surjective on 0-cells. $\square$

We now use Lemma 6.1 to express the fullness and faithfulness part of the Segal condition in terms of colimits of P-algebras. Recall from Definition 2.3 that, given a map of simplicial sets $\alpha\colon A \to B$, we have an induced map $\tilde{\alpha}_1$ in **Set**, as shown in the diagram below:

$$\begin{tikzcd}
A_1 \arrow[dr, dashed, "\tilde{\alpha}_1"] \arrow[rrrd, bend left, "s"] \arrow[rrdd, bend right, "\alpha_1"'] \arrow[rddd, bend right=60, "t"'] & & & \\
& A_0 \times_{B_0} B_1 \times_{B_0} A_1 \arrow[rr] \arrow[rd] \arrow[dd] & & A_0 \arrow[d, "\alpha_0"] \\
& & B_1 \arrow[r, "s"'] \arrow[d, "t"] & B_0 \\
& A_0 \arrow[r, "\alpha_0"'] & B_0 &
\end{tikzcd}$$

and that α is full and faithful on 1-cells if the map $\tilde{\alpha}_1$ is an isomorphism. We wish to show that, for all $j \geq 0$, the Segal map

$$S_j\colon P\text{-}\mathbf{Alg}(I_2(j,-),\mathcal{A}) \longrightarrow P\text{-}\mathbf{Alg}(I_2(1,-)^{\amalg j},\mathcal{A})$$

is full and faithful on 1-cells. By the description of fullness and faithfulness above, this happens when the diagram

$$\begin{tikzcd}
P\text{-}\mathbf{Alg}(I_2(j,1),\mathcal{A}) \arrow[rr, "s"] \arrow[rd, "-\circ(d^{\amalg j})_1"] \arrow[dd, "t"'] & & P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A}) \arrow[d, "-\circ(d^{\amalg j})_0"] \\
& P\text{-}\mathbf{Alg}(I_2(1,1)^{\amalg j},\mathcal{A}) \arrow[r, "s"'] \arrow[d, "t"] & P\text{-}\mathbf{Alg}(I_2(1,0)^{\amalg j},\mathcal{A}) \\
P\text{-}\mathbf{Alg}(I_2(j,0),\mathcal{A}) \arrow[r, "-\circ(d^{\amalg j})_0"'] & P\text{-}\mathbf{Alg}(I_2(1,0)^{\amalg j},\mathcal{A}). &
\end{tikzcd}$$

is a limit cone in $\mathbf{Set}$. This cone lies in the image of the functor

$$P\text{-}\mathbf{Alg}(-,\mathcal{A})\colon P\text{-}\mathbf{Alg}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

and this functor is representable, so it preserves limits [27, V.6 Theorem 3]. Hence S_j is full and faithful on 1-cells if the diagram

$$\begin{tikzcd}
 & I_2(1,0)^{\amalg 3} \arrow[dl, "(d^{\amalg 3})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg 3}"] & & I_2(1,0)^{\amalg 3} \arrow[dl, "I_2(d_0,1)^{\amalg 3}"'] \arrow[dr, "(d^{\amalg 3})_0"] & \\
I_2(3,0) \arrow[drr, "I_2(1,d_1)"'] & & I_2(1,1)^{\amalg 3} \arrow[d, "(d^{\amalg 3})_1"] & & I_2(3,0) \arrow[dll, "I_2(1,d_0)"] \\
 & & I_2(3,1) & &
\end{tikzcd}$$

is a colimit cocone in P-$\mathbf{Alg}$.

Before proving this, we describe what this means in the case $j = 3$. The P-algebra $I_2(1,1)^{\amalg 3}$ consists of three 2-cells composed horizontally:

$$\begin{tikzcd}
a_0 \arrow[r, bend left=50, "f^0_{01}", ""{name=A, below}] \arrow[r, bend right=50, "f^1_{01}"', ""{name=B, above}] & a_1 \arrow[r, bend left=50, "f^0_{12}", ""{name=C, below}] \arrow[r, bend right=50, "f^1_{12}"', ""{name=D, above}] & a_2 \arrow[r, bend left=50, "f^0_{12}", ""{name=E, below}] \arrow[r, bend right=50, "f^1_{12}"', ""{name=F, above}] & a_3,
\arrow[Rightarrow, from=A, to=B, "\alpha^1_{01}"]
\arrow[Rightarrow, from=C, to=D, "\alpha^1_{12}"]
\arrow[Rightarrow, from=E, to=F, "\alpha^1_{23}"]
\end{tikzcd}$$

with the copies of $I_2(1,0)^{\amalg 3}$ in the diagram giving its source and target strings of 1-cells. The P-algebra $I_2(3,0)$ is a tetrahedron whose faces are isomorphism 2-cells:

$$\begin{tikzcd}
a_1 \arrow[r, "f_{12}"] \arrow[r, phantom, shift right=4ex, "\cong\Searrow"] & a_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, "\Swarrow\cong"] & & & a_1 \arrow[r, "f_{12}"] \arrow[drr, "f_{13}"] \arrow[d, phantom, shift left=4ex, "\cong\Downarrow"] & a_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, shift left=3ex, "\Swarrow\cong"] & \\
a_0 \arrow[u, "f_{01}"] \arrow[ur, "f_{02}"'] \arrow[rr, "f_{03}"'] & & a_3 & = & a_0 \arrow[u, "f_{01}"] \arrow[rr, "f_{03}"'] & & a_3.
\end{tikzcd}$$

Taking the colimit of the diagram glues one of these tetrahedra to the string of source 1-cells of $I_2(1,1)^{\amalg 3}$, and the other to the string of target 1-cells. Thus the fullness and faithfulness condition tells us that $I_2(3,1)$ can be obtained this way; it is a simplicially weakened version of the cuboidal pasting diagram $(3,1)$.

Lemma 6.4. *For all $j > 0$, the diagram*

$$\begin{tikzcd}
 & I_2(1,0)^{\amalg j} \arrow[dl, "(d^{\amalg j})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg j}"] & & I_2(1,0)^{\amalg j} \arrow[dl, "I_2(d_0,1)^{\amalg j}"'] \arrow[dr, "(d^{\amalg j})_0"] & \\
I_2(j,0) \arrow[drr, "I_2(1,d_1)"'] & & I_2(1,1)^{\amalg j} \arrow[d, "(d^{\amalg j})_1"] & & I_2(j,0) \arrow[dll, "I_2(1,d_0)"] \\
 & & I_2(j,1) & &
\end{tikzcd}$$

is a colimit cocone in P-$\mathbf{Alg}$.

To prove Lemma 6.4, we check directly that $I_2(j,1)$ satisfies the universal property for the colimit. In order to do this we must specify maps out of $I_2(j,1)$ and $I_2(j,k)$, which we define dimension by dimension, starting at dimension 0 and working up.

In this proof we write down the cells of $I_2(j,1)$ explicitly. We are able to do this using the description of the functor $J\colon \mathcal{R} \to \mathcal{Q}$ (which is used in the definition of I_2) given in Part 1 of this series.

Recall from the construction of $I_2(j,k)$ that at each dimension (excluding dimension 0), we have three types of cell: generating cells (those in $R(j,k)$), contraction cells, and composites. We use the following notation: for composites we write $\circ$ for composition of 1-cells and vertical composition of 2-cells, and $*$ for horizontal composition of 2-cells; for contraction cells, we write $[a,b]$ for the contraction cell from a to b. Since we are defining a map of P-algebras, once we have defined the effect of the map on generating cells and contraction cells, the effect on composites is determined by the fact that the map must preserve the P-algebra structure (in a way that we will make precise later). A similar statement is true for some of the contraction cells, but not all of them; due to the fact that (for $j > 1$) $I_2(j,k)$ is not a free P-algebra, only certain contraction cells are required to be preserved by the P-algebra structure. We refer to these cells as "algebraic contraction cells".

To see which contraction cells are algebraic contraction cells, suppose we are defining a map $\psi\colon I_2(j,k) \to \mathcal{A}$. This consists of a map of 2-globular sets $\psi\colon U_P I_2(j,k) = Q(j,k) \to A$ such that

$$\begin{tikzcd}
PQ(j,k) \arrow[r, "P\psi"] \arrow[d] & PA \arrow[d, "\theta"] \\
Q(j,k) \arrow[r, "\psi"'] & A
\end{tikzcd}$$

commutes, where the left-hand map is the algebra action for $I_2(j,k)$. The commutativity of this diagram is what ensures that the P-algebra structure is preserved. Thus, the contraction cells that must be preserved are precisely those which are recognised as contraction cells by the P-algebra structure, i.e. a contraction cell in $Q(j,k)$ is an algebraic contraction cell if it is the image under the algebra action $PQ(j,k) \to Q(j,k)$ of a contraction cell in $PQ(j,k)$. Since the only contraction 1-cells in $I_2(j,k)$ are the identities, all contraction 1-cells are algebraic. The algebraic contraction 2-cells in $I_2(j,k)$ consist of the identities, and any contraction cells that alter the bracketing of a composite, or alter the number of identities that appear in a composite, but do nothing else. In particular, the source and target of a non-identity algebraic contraction 2-cell in $I_2(j,k)$ are always composites of cells in $I_2(j,k)$, and these composites feature the same generating cells in the same order.

Another pivotal fact about $I_2(j,k)$ is that, in the construction, the functor $J\colon \mathcal{R} \to \mathcal{Q}$ "adds contraction 3-cells" (as well as adding other contraction cells and composites). This has the effect of identifying all parallel 2-cells, so in $I_2(j,k)$ there are no distinct parallel 2-cells. This allows us to write many of the contraction cells as composites of others.

Proof of Lemma 6.4. In this proof, we present the case $j = 3$, before moving on to the case of general j, since for a fixed value of j we are able to write down all of the cells in $I_2(j,1)$ (though note that we still omit certain composites). We use $j = 3$ rather than $j = 2$ (the simplest case of the lemma) because $I_2(2,1)$ is too small for this case to exhibit all the features of the general case.

Suppose we have a P-algebra $\mathcal{A}$ and a cocone

$$\begin{tikzcd}
 & I_2(1,0)^{\amalg 3} \arrow[dl, "(d^{\amalg 3})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg 3}"] & & I_2(1,0)^{\amalg 3} \arrow[dl, "I_2(d_0,1)^{\amalg 3}"'] \arrow[dr, "(d^{\amalg 3})_0"] & \\
I_2(3,0) \arrow[drr, "g"'] & & I_2(1,1)^{\amalg 3} \arrow[d, "\lambda"'] & & I_2(3,0) \arrow[dll, "h"] \\
 & & \mathcal{A} & &
\end{tikzcd}$$

in P-**Alg**. We define a map of P-algebras

$$\psi\colon I_2(3,1) \to \mathcal{A}$$

such that the diagram

$$\begin{tikzcd}
 & I_2(1,0)^{\amalg 3} \arrow[dl, "(d^{\amalg 3})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg 3}"] & & I_2(1,0)^{\amalg 3} \arrow[dl, "I_2(d_0,1)^{\amalg 3}"'] \arrow[dr, "(d^{\amalg 3})_0"] & \\
I_2(3,0) \arrow[drr, "I_2(1,d_1)"'] \arrow[ddrr, bend right, "g"'] & & I_2(1,1)^{\amalg 3} \arrow[d, "(d^{\amalg 3})_1"] \arrow[dd, bend right=60, "\lambda"'] & & I_2(3,0) \arrow[dll, "I_2(1,d_0)"] \arrow[ddll, bend left, "h"] \\
 & & I_2(3,1) \arrow[d, dashed, "\psi"] & & \\
 & & \mathcal{A} & &
\end{tikzcd}$$

commutes.

To define the map ψ, we first list the cells in $I_2(3,1)$. We list the cells by dimension, and for dimensions above 0, we break the list down further, into generating cells, contraction cells, and composites.

- 0-cells: a_u for all $0 \leq u \leq 3$;
- 1-cells:
  - Generating cells:

$$f^z_{uv} \text{ for all } 0 \leq u < v \leq 3,\ 0 \leq z \leq 1;$$

  - Contraction cells:

$$[a_u, a_u] = \mathrm{id}_{a_u} \text{ for all } 0 \leq u \leq 3;$$

  - Composites: Although we don't need to define the action of ψ on composites, since this is determined by the fact that ψ preserves the P-algebra structure, it is useful to list them here since we need to know what they are in order to write down the contraction 2-cells. Note that this list does not include composites involving identities.

$$f^z_{vw} \circ f^y_{uv} \text{ for all } 0 \leq u < v < w \leq 3,\ y, z \in \{0,1\};$$

$$(f^z_{23} \circ f^y_{12}) \circ f^x_{01},\ f^z_{23} \circ (f^y_{12} \circ f^x_{01}) \text{ for all } x, y, z \in \{0,1\}$$

- 2-cells:
  - Generating cells:

$$\alpha^1_{uv} \text{ for all } 0 \leq u < v \leq 3;$$

  - Contraction cells: There are three different types of contraction cell in $I_2(3,1)$ – the algebraic contraction cells, the triangular contraction cells corresponding to the cells denoted ι^z_{uvw} in Leinster nerve construction (see Section 3), and those which are composites of cells of the two other types.

    The algebraic contraction cells are those of the form:

$$[(f^z_{23} \circ f^y_{12}) \circ f^x_{01}, f^z_{23} \circ (f^y_{12} \circ f^x_{01})],$$

$$[f^z_{23} \circ (f^y_{12} \circ f^x_{01}), (f^z_{23} \circ f^y_{12}) \circ f^x_{01}],$$

    for all x, y, $z \in \{0,1\}$, as well as identities on all 1-cells. The triangular contraction cells, all of which lie in the image of either $I_2(1, d_1)$ or $I_2(1, d_0)$, are those of the form:

$$\begin{aligned} [f^0_{uw}, f^0_{vw} \circ f^0_{uv}] &= I_2(1, d_1)[f^0_{uw}, f^0_{vw} \circ f^0_{uv}], \\ [f^0_{vw} \circ f^0_{uv}, f^0_{uw}] &= I_2(1, d_1)[f^0_{vw} \circ f^0_{uv}, f^0_{uw}], \\ [f^1_{uw}, f^1_{vw} \circ f^1_{uv}] &= I_2(1, d_0)[f^0_{uw}, f^0_{vw} \circ f^0_{uv}], \end{aligned}$$

$$[f^1_{vw} \circ f^1_{uv}, f^1_{uw}] = I_2(1, d_0)[f^0_{vw} \circ f^0_{uv}, f^0_{uw}],$$

For all $0 \leq u < v < w \leq 3$. The remaining contraction cells are composites of those above:

$$\begin{aligned}
[f^0_{13} \circ f^1_{01}, (f^0_{23} \circ f^0_{12}) \circ f^1_{01}] &= [f^0_{13}, f^0_{23} \circ f^0_{12}] * [f^1_{01}, f^1_{01}],\\
[(f^0_{23} \circ f^0_{12}) \circ f^1_{01}, f^0_{13} \circ f^1_{01}] &= [f^0_{23} \circ f^0_{12}, f^0_{13}] * [f^1_{01}, f^1_{01}],\\
[f^1_{13} \circ f^0_{01}, (f^1_{23} \circ f^1_{12}) \circ f^0_{01}] &= [f^1_{13}, f^1_{23} \circ f^1_{12}] * [f^0_{01}, f^0_{01}],\\
[(f^1_{23} \circ f^1_{12}) \circ f^0_{01}, f^1_{13} \circ f^0_{01}] &= [f^1_{23} \circ f^1_{12}, f^1_{13}] * [f^0_{01}, f^0_{01}],\\
[f^0_{23} \circ f^1_{02}, f^0_{23} \circ (f^1_{12} \circ f^1_{01})] &= [f^0_{23}, f^0_{23}] * [f^1_{02}, f^1_{12} \circ f^1_{01}],\\
[f^0_{23} \circ (f^1_{12} \circ f^1_{01}), f^0_{23} \circ f^1_{02}] &= [f^0_{23}, f^0_{23}] * [f^1_{12} \circ f^1_{01}, f^1_{02}],\\
[f^1_{23} \circ f^0_{02}, f^1_{23} \circ (f^0_{12} \circ f^0_{01})] &= [f^1_{23}, f^1_{23}] * [f^0_{02}, f^0_{12} \circ f^0_{01}],\\
[f^1_{23} \circ (f^0_{12} \circ f^0_{01}), f^1_{23} \circ f^0_{02}] &= [f^1_{23}, f^1_{23}] * [f^0_{12} \circ f^0_{01}, f^0_{02}].
\end{aligned}$$

We now define the map $\psi\colon I_2(3, 1) \to \mathcal{A}$:

- On 0-cells:

$$\psi_0(a_u) := g_0(a_u) = h_0(a_u) = \lambda_0(a_u).$$

- On 1-cells:

$$\psi_1(f^z_{uv}) := \begin{cases} g_1(f^z_{uv}) & \text{if } z = 0,\\ h_1(f^{z-1}_{uv}) & \text{if } z = 1; \end{cases}$$

$$\psi_1[a_u, a_u] = \psi(\mathrm{id}_{a_u}) := \lambda_1(\mathrm{id}_{a_u}) = g_1(\mathrm{id}_{a_u}) = h_1(\mathrm{id}_{a_u}).$$

  We do not need to define the action of ψ_1 on composites explicitly; this is automatic since ψ must preserve the P-algebra structure.

- On 2-cells:

$$\begin{aligned}
&\psi_2(\alpha^1_{uv}) := \lambda(\alpha^1_{uv});\\
&\psi_2[(f^z_{23} \circ f^y_{12}) \circ f^x_{01}, f^z_{23} \circ (f^y_{12} \circ f^x_{01})]\\
&:=[\psi_1\big((f^z_{23} \circ f^y_{12}) \circ f^x_{01}\big), \psi_1\big(f^z_{23} \circ (f^y_{12} \circ f^x_{01})\big)];\\
&\psi_2[f^z_{23} \circ (f^y_{12} \circ f^x_{01}), (f^z_{23} \circ f^y_{12}) \circ f^x_{01}]\\
&:=[\psi_1\big(f^z_{23} \circ (f^y_{12} \circ f^x_{01})\big), \psi_1\big((f^z_{23} \circ f^y_{12}) \circ f^x_{01}\big)];
\end{aligned}$$

$$\begin{aligned}
\psi_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}] &:= g_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}];\\
\psi_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}] &:= g_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}];\\
\psi_2[f^1_{uw}, f^1_{vw} \circ f^1_{uv}] &:= h_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}];\\
\psi_2[f^1_{vw} \circ f^1_{uv}, f^1_{uw}] &:= h_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}].
\end{aligned}$$

> As with 1-cells, we do not need to define the action of ψ_2 on composites, including those contraction cells that are composites of others, since ψ must preserve the P-algebra structure.

We see by definition of ψ that it is a map of P-algebras, and that it makes the required diagram commute. It is clear that, at each stage of the construction of ψ, if we defined the map differently it would not have satisfied these conditions; in the case of the cells on which ψ is defined explicitly, any other definition would fail to make the diagram commute, and in the case of all other cells, any other definition would fail to give a map of P-algebras.

Thus, ψ is the unique map of P-algebras making the required diagram commute, so $I_2(3,1)$ is the colimit in P-**Alg** of the diagram

$$\begin{tikzcd}
& I_2(1,0)^{\amalg 3} \arrow[dl, "(d^{\amalg 3})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg 3}"] & & I_2(1,0)^{\amalg 3} \arrow[dl, "I_2(d_0,1)^{\amalg 3}"'] \arrow[dr, "(d^{\amalg 3})_0"] & \\
I_2(3,0) & & I_2(1,1)^{\amalg 3} & & I_2(3,0)
\end{tikzcd}$$

We now prove the lemma for a general value of j. Suppose we have a P-algebra $\mathcal{A}$ and a cocone

$$\begin{tikzcd}
& I_2(1,0)^{\amalg j} \arrow[dl, "(d^{\amalg j})_0"'] \arrow[dr, "I_2(d_1,1)^{\amalg j}"] & & I_2(1,0)^{\amalg j} \arrow[dl, "I_2(d_0,1)^{\amalg j}"'] \arrow[dr, "(d^{\amalg j})_0"] & \\
I_2(j,0) \arrow[drr, "g"'] & & I_2(1,1)^{\amalg j} \arrow[d, "\lambda"'] & & I_2(j,0) \arrow[dll, "h"] \\
& & \mathcal{A} & &
\end{tikzcd}$$

in P-**Alg**. We define a map of P-algebras

$$\psi\colon I_2(j,1) \to \mathcal{A}$$

such that the diagram

$$\begin{tikzcd}
 & I_2(1,0)^{\amalg j} \arrow[dl, "(d^{\amalg j})_0"'] \arrow[drr, "I_2(d_1,1)^{\amalg j}"] & & & & I_2(1,0)^{\amalg j} \arrow[dll, "I_2(d_0,1)^{\amalg j}"'] \arrow[dr, "(d^{\amalg j})_0"] & \\
I_2(j,0) \arrow[drrr, "I_2(1,d_1)"'] \arrow[ddrrr, bend right, "g"'] & & & I_2(1,1)^{\amalg j} \arrow[d, "(d^{\amalg j})_1"] \arrow[dd, bend right=60, "\lambda"'] & & & I_2(j,0) \arrow[dlll, "I_2(1,d_0)"] \arrow[ddlll, bend left, "h"] \\
 & & & I_2(j,1) \arrow[d, dashed, "\psi"] & & & \\
 & & & \mathcal{A} & & &
\end{tikzcd}$$

commutes.

To define the map ψ, we first list the cells in $I_2(j,1)$. As for the case $j = 3$, we list the cells by dimension, and for dimensions above 0, we list generating cells and contraction cells separately. Note that in this case we do not list the composites, since the notation would become very unwieldy; the action of ψ on composites is determined by the fact that it must preserve the P-algebra structure, so we do not need to list the composites explicitly.

- 0-cells: a_u for all $0 \leq u \leq j$;
- 1-cells:
  - Generating cells:

    $$f^z_{uv} \text{ for all } 0 \leq u < v \leq j,\ 0 \leq z \leq 1;$$

  - Contraction cells:

    $$[a_u, a_u] = \mathrm{id}_{a_u} \text{ for all } 0 \leq u \leq j;$$

- 2-cells:
  - Generating cells:

    $$\alpha^1_{uv} \text{ for all } 0 \leq u < v \leq j;$$

- Contraction cells: As in the case $j = 3$, we have algebraic contraction cells and triangular contraction cells corresponding to the cells ι^z_{uvw}; since all diagrams of contraction 2-cells commute in $I_2(j, 1)$, all other contraction cells can be expressed as composites of contraction cells of these two types.

  The algebraic contraction cells are those mediating between differently bracketed composites of the same 1-cells, and also identities on all 1-cells. The triangular contraction cells are those of the form:

$$\begin{aligned}
[f^0_{uw}, f^0_{vw} \circ f^0_{uv}] &= I_2(1, d_1)[f^0_{uw}, f^0_{vw} \circ f^0_{uv}],\\
[f^0_{vw} \circ f^0_{uv}, f^0_{uw}] &= I_2(1, d_1)[f^0_{vw} \circ f^0_{uv}, f^0_{uw}],\\
[f^1_{uw}, f^1_{vw} \circ f^1_{uv}] &= I_2(1, d_0)[f^0_{uw}, f^0_{vw} \circ f^0_{uv}],\\
[f^1_{vw} \circ f^1_{uv}, f^1_{uw}] &= I_2(1, d_0)[f^0_{vw} \circ f^0_{uv}, f^0_{uw}],
\end{aligned}$$

  for all $0 \leq u < v < w \leq j$. All remaining contraction cells are horizontal composites of those of the form

$$[f^z_{v_{m-1},v_m} \circ \cdots \circ f^z_{v_1,v_2} \circ f^z_{v_0,v_1}, f^z_{u_{l-1},u_l} \circ \cdots \circ f^z_{u_1,u_2} \circ f^z_{u_0,u_1}],$$

  for all l, $m \geq 2$, $0 \leq u_0 < u_1 < \cdots < u_l \leq j$, $u_0 = v_0 < v_1 < \cdots < v_m = u_l$, $0 \leq z \leq 1$. Note that we omit the choice of bracketing in the contraction cell above; there is one such cell for each choice of bracketing of the source and target. Each of these contraction cells can be written as a composite of algebraic contraction cells and the triangular contraction cells above.

We now define the map $\psi\colon I_2(j, 1) \to \mathcal{A}$:

- On 0-cells:

$$\psi_0(a_u) := g_0(a_u) = h_0(a_u) = \lambda_0(a_u).$$

- On 1-cells:

$$\psi_1(f^z_{uv}) := \begin{cases} g_1(f^z_{uv}) & \text{if } z = 0,\\ h_1(f^{z-1}_{uv}) & \text{if } z = 1;\end{cases}$$

$$\psi_1[a_u, a_u] = \psi(\mathrm{id}_{a_u}) := \lambda_1(\mathrm{id}_{a_u}) = g_1(\mathrm{id}_{a_u}) = h_1(\mathrm{id}_{a_u}).$$

As in the case $j = 3$, we do not need to describe the action of ψ on composites explicitly, since it must preserve the P-algebra structure.

- On 2-cells:

$$\begin{aligned}\psi_2(\alpha^1_{uv}) &:= \lambda(\alpha^1_{uv});\\ \psi_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}] &:= g_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}];\\ \psi_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}] &:= g_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}];\\ \psi_2[f^1_{uw}, f^1_{vw} \circ f^1_{uv}] &:= h_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}];\\ \psi_2[f^1_{vw} \circ f^1_{uv}, f^1_{uw}] &:= h_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}].\end{aligned}$$

  As in the case $j = 3$, we do not need to describe the action of ψ on the remaining 2-cells explicitly, since they are either algebraic contraction cells, or composites involving the algebraic contraction cells and those above.

We see by definition of ψ that it is a map of P-algebras, and that it makes the required diagram commute. It is clear that, at each stage of the construction of ψ, if we defined the map differently it would not have satisfied these conditions; in the case of the cells on which ψ is defined explicitly, any other definition would fail to make the diagram commute, and in the case of all other cells, any other definition would fail to give a map of P-algebras.

Thus, ψ is the unique map of P-algebras making the required diagram commute, so $I_2(j, 1)$ is the colimit in P-**Alg** of the diagram

$$\begin{tikzcd}[column sep=small] & I_2(1,0)^{\amalg j} \arrow[dl, "d_0^{\amalg j}"'] \arrow[dr, "I_2(d_1{,}1)^{\amalg j}"] & & I_2(1,0)^{\amalg j} \arrow[dl, "I_2(d_0{,}1)^{\amalg j}"'] \arrow[dr, "d_0^{\amalg j}"] & \\ I_2(j,0) & & I_2(1,1)^{\amalg j} & & I_2(j,0), \end{tikzcd}$$

as required. □

The following is now an immediate corollary of Lemma 6.4, via our characterisation of fullness and faithfulness of the Segal maps in terms of colimits in P-**Alg**.

Corollary 6.5. *Let $\mathcal{A}$ be a Penon weak 2 category. For all $j > 0$, the Segal map*

$$S_j \colon P\text{-}\mathbf{Alg}(I_2(j, -), \mathcal{A}) \longrightarrow P\text{-}\mathbf{Alg}(I_2(1, -)^{\amalg j}, \mathcal{A})$$

is full and faithful on 1-cells.

We now apply a similar argument to the Segal maps $S_{j,k}$, and reformulate the remaining part of the Segal condition in terms of colimits of P-algebras, as we did for S_j. By Lemma 6.2, $S_{j,k}$ is given by

$$S_{j,k} = \cdot \circ d^{\amalg k} \colon P\text{-}\mathbf{Alg}(I_2(j,k),\mathcal{A}) \longrightarrow P\text{-}\mathbf{Alg}(I_2(j,1)^{\amalg k},\mathcal{A}).$$

This is a bijection if $I_2(j,1)^{\amalg k} = I_2(j,k)$, and the map

$$d^{\amalg k} \colon I_2(j,1)^{\amalg k} \to I_2(j,k)$$

is the identity. This tells us that $I_2(j,k)$ can be obtained by gluing k copies of $I_2(j,1)$ along their boundary copies of $I_2(j,0)$. Thus, the Segal map $S_{j,k}$ is a bijection if the following lemma holds:

Lemma 6.6. *For all $j \geq 0$, $k > 0$, the diagram*

$$\begin{tikzcd}
& I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & & & & I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & \\
I_2(j,1) \arrow[drrr, bend right, "I_2(1{,}\iota_1)"'] & & I_2(j,1) \arrow[dr, "I_2(1{,}\iota_2)"'] & \cdots & I_2(j,1) \arrow[dl, "I_2(1{,}\iota_{k-1})"] & & I_2(j,1) \arrow[dlll, bend left, "I_2(1{,}\iota_k)"] \\
& & & I_2(j,k) & & &
\end{tikzcd}$$

is a colimit cocone in P-$\mathbf{Alg}$.

Proof. Let $\mathcal{A}$ be a Penon weak 2-category, and suppose we have a cocone

$$\begin{tikzcd}
& I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & & & & I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & \\
I_2(j,1) \arrow[drrr, bend right, "g^{(1)}"'] & & I_2(j,1) \arrow[dr, "g^{(2)}"'] & \cdots & I_2(j,1) \arrow[dl, "g^{(k-1)}"] & & I_2(j,1) \arrow[dlll, bend left, "g^{(k)}"] \\
& & & \mathcal{A} & & &
\end{tikzcd}$$

in P-$\mathbf{Alg}$. We define a map of P-algebras

$$\psi \colon I_2(j,k) \longrightarrow \mathcal{A}$$

such that the diagram

$$\begin{tikzcd}
& I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & & & & I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & \\
I_2(j,1) \arrow[drrr, bend right=15] \arrow[ddrrr, bend right=30, "g^{(1)}"'] & & I_2(j,1) \arrow[dr] \arrow[ddr, bend right=15, "g^{(2)}"'] & \cdots & I_2(j,1) \arrow[dl] \arrow[ddl, bend left=15, "g^{(k-1)}"] & & I_2(j,1) \arrow[dlll, bend left=15] \arrow[ddlll, bend left=30, "g^{(k)}"] \\
& & & I_2(j,k) \arrow[d, dashed, "\psi"] & & & \\
& & & \mathcal{A} & & &
\end{tikzcd}$$

commutes, and show that this is the unique such map of P-algebras. We take the same approach as in the proof of Lemma 6.4, defining the map by an elementary approach, and using the fact that it must preserve the P-algebra structure to avoid having to define it explicitly on every cell of $I_2(j,k)$. To do so we now list the cells of $I_2(j,k)$; we use the same notation as in Lemma 6.4, and note that, as before, we do not list composites or algebraic contraction cells.

- 0-cells: a_u for all $0 \leq u \leq j$;
- 1-cells:
  - Generating cells:

    $$f^z_{uv} \text{ for all } 0 \leq u < v \leq j,\ 0 \leq z \leq k;$$

  - Contraction cells:

    $$[a_u, a_u] = \mathrm{id}_{a_u} \text{ for all } 0 \leq u \leq j;$$

- 2-cells:
  - Generating cells:

    $$\alpha^z_{uv} \text{ for all } 0 \leq u < v \leq j,\ 1 \leq z \leq k;$$

- Contraction cells: As in Lemma 6.4, we have algebraic contraction cells and triangular contraction cells corresponding to the cells ι^z_{uvw} from Leinster's nerve construction for bicategories (Section 3); since all diagrams of contraction 2-cells commute in $I_2(j,1)$, all other contraction cells can be expressed as composites of contraction cells of these two types.

  The algebraic contraction cells are those mediating between differently bracketed composites of the same 1-cells, and also identities on all 1-cells. The triangular contraction cells are those of the form:
  $$[f^z_{uw}, f^z_{vw} \circ f^z_{uv}],$$
  and
  $$[f^z_{vw} \circ f^z_{uv}, f^z_{uw}],$$
  for all $0 \le u < v < w \le j$, $0 \le z \le k$. As in Lemma 6.4, all remaining contraction cells are composites of those above.

We now define the map $\psi\colon I_2(j,k) \to \mathcal{A}$:

- On 0-cells:
  $$\psi_0(a_u) := g^{(1)}_0(a_u).$$
- On 1-cells:
  $$\psi_1(f^z_{uv}) := \begin{cases} g^{(0)}_1(f^0_{uv}) & \text{if } z = 0, \\ g^{(z)}_1(f^1_{uv}) & \text{otherwise;} \end{cases}$$
  $$\psi_1[a_u, a_u] = \psi(\mathrm{id}_{a_u}) := g^{(1)}_1(\mathrm{id}_{a_u}).$$
  As in Lemma 6.4, we do not need to describe the action of ψ on composites explicitly, since it must preserve the P-algebra structure.
- On 2-cells:
  $$\psi_2(\alpha^z_{uv}) := g^{(z)}_2(\alpha^1_{uv});$$
  $$\psi_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}] := g^{(1)}_2[f^0_{uw}, f^0_{vw} \circ f^0_{uv}];$$
  $$\psi_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}] := g^{(1)}_2[f^0_{vw} \circ f^0_{uv}, f^0_{uw}];$$

> and for $1 \leq z \leq k$,
>
> $$\psi_2[f^z_{uw}, f^z_{vw} \circ f^z_{uv}] := g^{(z)}_2[f^1_{uw}, f^1_{vw} \circ f^1_{uv}];$$
> $$\psi_2[f^z_{vw} \circ f^z_{uv}, f^z_{uw}] := g^{(z)}_2[f^1_{vw} \circ f^1_{uv}, f^1_{uw}].$$
>
> As in Lemma 6.4, we do not need to describe the action of ψ on the remaining 2-cells explicitly, since they are either algebraic contraction cells, or composites involving the algebraic contraction cells and those above.

We see by definition of ψ that it is a map of P-algebras, and that it makes the required diagram commute. It is clear that, at each stage of the construction of ψ, if we defined the map differently it would not have satisfied these conditions; in the case of the cells on which ψ is defined explicitly, any other definition would fail to make the diagram commute, and in the case of all other cells, any other definition would fail to give a map of P-algebras.

Thus, ψ is the unique map of P-algebras making the required diagram commute, so $I_2(j,k)$ is the colimit in P-**Alg** of the diagram

$$\begin{tikzcd}
& I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & & & & I_2(j,0) \arrow[dl, "I_2(1{,}d_0)"'] \arrow[dr, "I_2(1{,}d_1)"] & \\
I_2(j,1) & & I_2(j,1) & \cdots & I_2(j,1) & & I_2(j,1)
\end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{k}$$

as required. ☐

The following is now an immediate corollary of Lemma 6.6:

Corollary 6.7. *Let $\mathcal{A}$ be a Penon weak 2-category. For each j, $k > 0$, the Segal map*

$$S_{j,k} \colon \mathcal{NA}(j,k) \to \underbrace{\mathcal{NA}(j,1) \times_{\mathcal{NA}(j,0)} \cdots \times_{\mathcal{NA}(j,0)} \mathcal{NA}(j,1)}_{k}$$

is a bijection.

We now have all the results we need to show that the nerve of a Penon weak 2-category is a Tamsamani–Simpson weak 2-category.

Theorem 6.8. *Let $\mathcal{A}$ be a Penon weak 2-category. Then the nerve $\mathcal{N}\mathcal{A}$ satisfies the Segal condition, and is thus a Tamsamani–Simpson weak 2-category.*

Proof. Let $\mathcal{A}$ be a Penon weak 2-category, and consider its nerve $\mathcal{N}\mathcal{A}$. For all $j \geq 0$, the Segal map

$$S_j : \mathcal{N}\mathcal{A}(j,-) \longrightarrow \underbrace{\mathcal{N}\mathcal{A}(1,-) \times_{\mathcal{N}\mathcal{A}(0,1)} \cdots \times_{\mathcal{N}\mathcal{A}(0,1)} \mathcal{N}\mathcal{A}(1,-)}_{j}$$

is surjective on objects by Proposition 6.3 and full and faithful on 1-cells by Corollary 6.5; hence S_j is contractible. Note that the proposition and corollary are valid only for $j > 0$, but for $j = 0$ the result holds trivially.

For all j, $k \geq 0$, the Segal map

$$S_{j,k} : \mathcal{N}\mathcal{A}(j,k) \longrightarrow \underbrace{\mathcal{N}\mathcal{A}(j,1) \times_{\mathcal{N}\mathcal{A}(j,0)} \cdots \times_{\mathcal{N}\mathcal{A}(j,0)} \mathcal{N}\mathcal{A}(j,1)}_{k}$$

is a bijection by Corollary 6.7. As above, this corollary is only valid for $k > 0$, but for $k = 0$ the result holds trivially.

Hence $\mathcal{N}\mathcal{A}$ satisfies the Segal condition, so it is a Tamsamani–Simpson weak 2-category. □

7. The nerve construction for general n

In this section we generalise the nerve construction for Penon weak 2-categories from Section 5 to a nerve construction for Penon weak n-categories for all $n \in \mathbb{N}$. As in Section 5, we write P for the monad for Penon weak n-categories, and T for the free strict n-category monad.

The construction proceeds analogously to that for $n = 2$. Since we are potentially working with a greater number of dimensions in the general case, we have to weaken composition in each cuboidal n-pasting diagram at every dimension (apart from dimensions 0 and n). The greater number of dimensions entails that the notation for the cells of the P-algebras we construct necessarily becomes more complicated and unwieldy.

In analogy with the case $n = 2$, when defining the nerve functor for Penon weak n-categories, we first define a functor $I_n \colon \Theta^n \to P$-**Alg** which gives us, for each object of Θ^n, the corresponding cuboidal n-pasting diagram expressed as a freely generated Penon weak n-category. We obtain the functor I_n by defining a functor $E_n\colon \Theta^n \to \mathcal{Q}$, then composing this with the Eilenberg–Moore comparison functor $K\colon \mathcal{Q} \to P$-**Alg** for the adjunction $F \dashv U$ defining the monad P.

As in the 2-dimensional case, for each object $\mathbf{j} = (j_1, j_2, \ldots, j_n)$ of Θ^n, we define two n-globular sets, $X(\mathbf{j})$ and $R(\mathbf{j})$; $X(\mathbf{j})$ is the associated n-globular set of the cuboidal pasting diagram $\mathbf{j}$, while $R(\mathbf{j})$ also contains extra cells to weaken the composition structure on certain simplicial shapes of composite. We then define an object of $\mathcal{R}$

$$R(\mathbf{j}) \xrightarrow{\theta_{\mathbf{j}}} TX(\mathbf{k}),$$

and define $E_n(\mathbf{j})$ to be the image of this under the functor $J\colon \mathcal{R} \to \mathcal{Q}$; that is, the left adjoint to the forgetful functor $W\colon \mathcal{Q} \to \mathcal{R}$.

Before giving the construction, once again we discuss the notation we will use. We will use a coordinate system similar to that used in the 2-dimensional construction. The difference is that, since higher dimensional cells require a greater number of coordinates, instead of using subscripts and superscripts, the coordinates of a cell will be written as a string in brackets. Thus, the m-cell

$$\alpha^m(u_0, v_0; u_1, v_1; \ldots ; u_{m-1}, v_{m-1}; z)$$

has source $(m-1)$-cell with coordinates $(u_0, v_0; \ldots ; u_{m-2}, v_{m-2}; u_{m-1})$ and target $(k-1)$-cell with coordinates $(u_0, v_0; \ldots ; u_{m-2}, v_{m-2}; v_{m-1})$. The z-coordinate indicates the position of this cell in relation to the other m-cells parallel to it, and the superscript m indicates the dimension of the cell. As in the 2-dimensional construction, each n-cell has the same coordinates as its target $(n-1)$-cell.

Recall that an object of Θ^n is an equivalence class of objects of Δ^n. An object of Δ^n is in an equivalence class with more than one member if and only if it has a 0 in the kth position for some $k < n$. Thus, for the purposes of the following definition we treat the equivalence class of $(l_1, \ldots, l_{m-1}, 0, l_{m+1}, \ldots, l_n)$, with $m < n$, as the object

$$(l_1, \ldots, l_{m-1}, 0, 0, \ldots, 0)$$

of Δ^n; all other equivalence classes are treated as their sole member.

Let $\mathbf{j} \in \Theta^n$ and define n-globular sets $X(\mathbf{j})$ and $R(\mathbf{j})$ as follows: the sets of cells of $X(\mathbf{j})$ are defined by

- $X(\mathbf{j})_0 = \{a_u \mid u \in \mathbb{N}, 0 \leq u \leq j_1\}$;
- for $0 < m < n$,
$$\begin{aligned} X(\mathbf{j})_m = \{&\alpha^m(u_1, u_1+1; u_2, u_2+1; \dots ; u_m, u_m+1; z) \\ &\mid 0 \leq u_l < j_l \text{ for all } 1 \leq l \leq m, 0 \leq z \leq j_{m+1}\}; \end{aligned}$$
- for $m = n$,
$$\begin{aligned} X(\mathbf{j})_n = \{&\alpha^n(u_1, u_1+1; u_2, u_2+1; \dots ; u_{n-1}, u_{n-1}+1; z) \\ &\mid 0 \leq u_l < j_l \text{ for all } 1 \leq l \leq n-1, 1 \leq z \leq j_n\}; \end{aligned}$$

and those for $R(\mathbf{j})$ are defined by

- $R(\mathbf{j})_0 = \{a_u \mid u \in \mathbb{N}, 0 \leq u \leq j_1\}$;
- for $0 < m < n$,
$$\begin{aligned} R(\mathbf{j})_m = \{&\alpha^m(u_1, v_1; u_2, v_2; \dots ; u_m, v_m; z) \\ &\mid 0 \leq u_l < v_l \leq j_l \text{ for all } 1 \leq l \leq m, 0 \leq z \leq j_{m+1}\}; \end{aligned}$$
- for $m = n$,
$$\begin{aligned} R(\mathbf{j})_n = \{&\alpha^n(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z) \\ &\mid 0 \leq u_l < v_l \leq j_l \text{ for all } 1 \leq l \leq n-1, 1 \leq z \leq j_n\}. \end{aligned}$$

For both $X(\mathbf{j})$ and $R(\mathbf{j})$, the source and target maps are defined by:

- for all 1-cells $\alpha^1(u_1, v_1; z)$,
$$s(\alpha^1(u_1, v_1; z)) = a_{u_1}, \ t(\alpha^1(u_1, v_1; z)) = a_{v_1};$$
- for all $1 < m < n$, and for all m-cells $\alpha^m(u_1, v_1; u_2, v_2; \dots ; u_m, v_m; z)$,
$$\begin{aligned} &s(\alpha^m(u_1, v_1; u_2, v_2; \dots ; u_m, v_m; z)) \\ &= \alpha^{m-1}(u_1, v_1; u_2, v_2; \dots ; u_{m-1}, v_{m-1}; u_m), \end{aligned}$$
  and
$$\begin{aligned} &t(\alpha^m(u_1, v_1; u_2, v_2; \dots ; u_m, v_m; z)) \\ &= \alpha^{m-1}(u_1, v_1; u_2, v_2; \dots ; u_{m-1}, v_{m-1}; v_m), \end{aligned}$$

- for all n-cells $\alpha^n(u_1, v_1; u_2, v_2; \dots ; u_{n-1},$

  $$\begin{aligned} &s(\alpha^n(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z)) \\ &= \alpha^{n-1}(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z-1), \end{aligned}$$

  and

  $$\begin{aligned} &t(\alpha^n(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z)) \\ &= \alpha^{n-1}(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z). \end{aligned}$$

Once again we note that, in spite of the notation, this does not define functors R and X into n-$\mathbf{GSet}$.

We now wish to construct, for each $\mathbf{j} \in \Theta^n$, an object of $\mathcal{R}$ which will consist of a map from $R(\mathbf{j})$ into the free strict n-category on $X(\mathbf{j})$. Before doing so, we must first establish notation for the freely generated composite cells in $TX(\mathbf{j})$. Following Penon's notation for composition in an n-magma (see Definition 4.2), given m-cells α_1, α_2 and $0 \leq p < m$, where the target p-cell of α_1 coincides with the source p-cell of α_2, we write $\alpha_2 \circ^m_p \alpha_1$ for their composite along boundary p-cells. For composites involving greater numbers of cells we extend this to summation-style notation; for m-cells α_i, $1 \leq i \leq k$ for some k, satisfying the appropriate source and target conditions to be composable, we write

$$\bigodot^{m,p}_{1 \leq i \leq k} \alpha_i := \alpha_k \circ^m_p \alpha_{k-1} \circ^m_p \cdots \circ^m_p \alpha_2 \circ^m_p \alpha_1.$$

We now define $\theta_{\mathbf{j}} \colon R(\mathbf{j}) \to TX(\mathbf{j})$ by:

- for $a_u \in R(\mathbf{j})_0$, $(\theta_{\mathbf{j}})_0(a_u) = a_u$;
- for $0 < m < n$, $(\theta_{\mathbf{j}})_m(\alpha^m(u_1, v_1; u_2, v_2; \dots ; u_m, v_m; z)) =$

  $$\bigodot^{m,m-1}_{u_m \leq w_m < v_m} \cdots \bigodot^{m,0}_{u_1 \leq w_1 < v_1} \alpha^m(w_1, w_1+1; w_2, w_2+1; \dots ; w_m, w_m+1; z)$$

- for $m = n$, $(\theta_{\mathbf{j}})_n(\alpha^n(u_1, v_1; u_2, v_2; \dots ; u_{n-1}, v_{n-1}; z)) =$

$$\underset{u_{n-1}\le w_{n-1}<v_{n-1}}{\overset{n,n-2}{\bigcirc}} \quad \cdots \quad \underset{u_1\le w_1<v_1}{\overset{n,0}{\bigcirc}} \; \alpha^m(w_1, w_1{+}1; w_2, w_2{+}1; \ldots ; w_{n-1}, w_{n-1}{+}1; z)$$

Similar to the 2-dimensional case, $\theta_{\mathbf{j}}$ coincides with $\eta^T_{X(\mathbf{j})}$ whenever $v_l = u_l + 1$ for all $0 \le l \le m - 1$.

To complete the construction of the action of the functor $E_n \colon \Theta^n \to \mathcal{Q}$ on objects, we apply the functor $J \colon \mathcal{R} \to \mathcal{Q}$ to $\theta_{\mathbf{j}} \colon R(\mathbf{j}) \to TX(\mathbf{j})$. This adds to $R(\mathbf{j})$ all the required composites and contraction cells, including those which ensure that the weakened composites (those cells in $R(\mathbf{j})$ but not in $X(\mathbf{j})$) are coherently equivalent to the corresponding freely generated composites at the same level in the pasting diagram. We denote the resulting object of $\mathcal{Q}$ by

$$Q(\mathbf{j}) \xrightarrow{\phi_{\mathbf{j}}} TX(\mathbf{j}).$$

We now define the action of the functor $E_n \colon \Theta^n \to \mathcal{Q}$ on morphisms. As in the 2-dimensional case, to do so we first define a morphism in $\mathcal{R}$, then take its transpose under the adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left=1.5, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q} \arrow[l, shift left=1.5, "W"] \end{tikzcd}$$

to obtain a morphism in $\mathcal{Q}$.

Let $\mathbf{p} : \mathbf{j} \to \mathbf{k}$ be a morphism in Θ^n. We define the strict n-category part of the morphism of $\mathcal{R}$ first. Define a map of 2-globular sets $x(\mathbf{p}) \colon X(\mathbf{j}) \to TX(\mathbf{k})$ as follows:

- for $a_u \in X(\mathbf{j})_0$, $x(\mathbf{p})_0(a_u) = a_{p_1(u)}$;
- for $0 < m < n$, $\alpha^m(u_1, u_1 + 1; \ldots ; u_m, u_m + 1; z) \in X(\mathbf{j})_m$, if for all $1 \le l \le m$ we have $p_l(u_l) < p_l(v_l)$, then

  $$x(\mathbf{p})_m(\alpha^m(u_1, u_1 + 1; \ldots ; u_m, u_m + 1; z)) = \\ \underset{\substack{p_m(u_m)\le w_m \\ <p_m(u_m+1)}}{\overset{m,m-1}{\bigcirc}} \quad \cdots \quad \underset{\substack{p_1(u_1)\le w_1 \\ <p_1(u_1+1)}}{\overset{m,0}{\bigcirc}} \; \alpha^m(w_1, w_1 + 1; \ldots ; w_m, w_m + 1; p_{m+1}(z));$$

  otherwise, for the smallest l such that $p_l(u_l) = p_l(v_l)$ we define

  $$x(\mathbf{p})_m(\alpha^m(u_1, u_1 + 1; \ldots ; u_m, u_m + 1; z))$$

to be the identity m-cell on the $(l-1)$-cell

$$\underset{\substack{p_{l-1}(u_{l-1})\leq w_{l-1}\\ <p_{l-1}(u_{l-1}+1)}}{\overset{l-1,l-2}{\bigcirc}} \cdots \underset{\substack{p_{1}(u_{1})\leq w_{1}\\ <p_{1}(u_{1}+1)}}{\overset{l-1,0}{\bigcirc}} \alpha^m(w_1, w_1+1; \dots ; w_{l-1}, w_{l-1}+1; p_l(u_l));$$

- for $\alpha^n(u_1, u_1+1; \dots ; u_{n-1}, u_{n-1}+1; z) \in X(\mathbf{j})_n$, if for all $1 \leq l \leq m$ we have $p_l(u_l) < p_l(v_l)$, and $p_n(z-1) < p_n(z)$, then

  $x(\mathbf{p})_n(\alpha^n(u_1, u_1+1; \dots ; u_{n-1}, u_{n-1}+1; z)) =$

  $$\underset{\substack{p_{n-1}(u_{n-1})\leq w_{n-1}\\ <p_{n-1}(u_{n-1}+1)}}{\overset{n,n-2}{\bigcirc}} \cdots \underset{\substack{p_{1}(u_{1})\leq w_{1}\\ <p_{1}(u_{1}+1)}}{\overset{n,0}{\bigcirc}} \alpha^m(w_1, w_1+1; \dots ; w_{n-1}, w_{n-1}+1; p_n(z));$$

  if for all $1 \leq l \leq m$ we have $p_l(u_l) < p_l(v_l)$, and $p_n(z-1) = p_n(z)$, then we define

  $$x(\mathbf{p})_n(\alpha^n(u_1, u_1+1; \dots ; u_{n-1}, u_{n-1}+1; z))$$

  to be the identity n-cell on the $(n-1)$-cell

  $$\underset{\substack{p_{n-1}(u_{n-1})\leq w_{n-1}\\ <p_{n-1}(u_{n-1}+1)}}{\overset{n-1,n-2}{\bigcirc}} \cdots \underset{\substack{p_{1}(u_{1})\leq w_{1}\\ <p_{1}(u_{1}+1)}}{\overset{n,0}{\bigcirc}} \alpha^m(w_1, w_1+1; \dots ; w_{n-1}, w_{n-1}+1; p_n(z));$$

  otherwise, for the smallest l such that $p_l(u_l) = p_l(v_l)$, we define

  $$x(\mathbf{p})_n(\alpha^n(u_1, u_1+1; \dots ; u_{n-1}, u_{n-1}+1; z))$$

  to be the identity m-cell on the $(l-1)$-cell

  $$\underset{\substack{p_{l-1}(u_{l-1})\leq w_{l-1}\\ <p_{l-1}(u_{l-1}+1)}}{\overset{l-1,l-2}{\bigcirc}} \cdots \underset{\substack{p_{1}(u_{1})\leq w_{1}\\ <p_{1}(u_{1}+1)}}{\overset{n,0}{\bigcirc}} \alpha^m(w_1, w_1+1; \dots ; w_{l-1}, w_{l-1}+1; p_l(u_l)).$$

To obtain a map $TX(\mathbf{j}) \to TX(\mathbf{k})$ we apply T and compose this with the multiplication for T, giving

$$TX(\mathbf{j}) \xrightarrow{Tx(\mathbf{p})} T^2X(\mathbf{k}) \xrightarrow{\mu^T_{X(\mathbf{k})}} TX(\mathbf{k})$$

We now define a map

$$\begin{tikzcd} R(\mathbf{j}) \arrow[rr, "r(\mathbf{p})"] \arrow[d, "\theta_{\mathbf{j}}"'] & & Q(\mathbf{k}) \arrow[d, "\phi_{\mathbf{k}}"] \\ TX(\mathbf{j}) \arrow[r, "Tx(\mathbf{p})"'] & T^2X(\mathbf{k}) \arrow[r, "\mu^T_{X(\mathbf{k})}"'] & TX(\mathbf{k}), \end{tikzcd}$$

where the map $r(\mathbf{p})$ is defined as follows:

- for $a_u \in R(\mathbf{j})_0$, $r(\mathbf{p})_0(a_u) = a_{p_1(u)}$;
- for $0 < m < n$, $\alpha^m(u_1, v_1; \dots ; u_m, v_m; z) \in R(\mathbf{j})_m$, if for all $1 \leq l \leq m$ we have $p_l(u_l) < p_l(v_l)$, then

  $$r(\mathbf{p})_m(\alpha^m(u_1, v_1; \dots ; u_m, v_m; z)) = \\ \alpha^m(p_1(u_1), p_1(v_1); \dots ; p_m(u_m), p_m(v_m); p_{m+1}(z));$$

  otherwise, for the smallest l such that $p_l(u_l) = p_l(v_l)$, we define

  $$r(\mathbf{p})_m(\alpha^m(u_1, v_1; \dots ; u_m, v_m; z))$$

  to be the identity m-cell on the $(l-1)$-cell

  $$\alpha^{l-1}(p_1(u_1), p_1(v_1); \dots ; p_{l-1}(u_{l-1}), p_{l-1}(v_{l-1}); p_l(u_l));$$

- for $\alpha^n(u_1, v_1; \dots ; u_{n-1}, v_{n-1}; z) \in R(\mathbf{j})_n$, if for all $1 \leq l \leq n-1$ we have $p_l(u_l) < p_l(v_l)$, and $p_n(z-1) < p_n(z)$, then

  $$r(\mathbf{p})_n(\alpha^m(u_1, v_1; \dots ; u_{n-1}, v_{n-1}; z)) = \\ \alpha^m(p_1(u_1), p_1(v_1); \dots ; p_{n-1}(u_{n-1}), p_{n-1}(v_{n-1}); p_n(z));$$

  if for all $1 \leq l \leq n-1$ we have $p_l(u_l) < p_l(v_l)$, and $p_n(z-1) = p_n(z)$, then we define

  $$r(\mathbf{p})_n(\alpha^m(u_1, v_1; \dots ; u_{n-1}, v_{n-1}; z))$$

  to be the identity n-cell on the $(n-1)$-cell

  $$\alpha^{n-1}(p_1(u_1), p_1(v_1); \dots ; p_{l-1}(u_{l-1}), p_{n-1}(v_{n-1}); p_l(z));$$

otherwise, for the smallest l such that $p_l(u_l) = p_l(v_l)$, we define

$$r(\mathbf{p})_n(\alpha^m(u_1, v_1; \dots ; u_{n-1}, v_{n-1}; z))$$

to be the identity m-cell on the $(l-1)$-cell

$$\alpha^{l-1}(p_1(u_1), p_1(v_1); \dots ; p_{l-1}(u_{l-1}), p_{l-1}(v_{l-1}); p_l(u_l)).$$

Finally, we take the transpose of this map under the adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "W"] \end{tikzcd}$$

We write $\epsilon \colon JW \Rightarrow 1$ for the counit of this adjunction, and $\epsilon_{\phi_{\mathbf{k}}}$ for the component corresponding to

$$Q(\mathbf{k}) \xrightarrow{\phi_{\mathbf{k}}} TX(\mathbf{k}).$$

Then the transpose is given by the composite

$$\epsilon_{\phi_{\mathbf{k}}} \circ J\big(r(\mathbf{p}), \mu^T_{X(\mathbf{k})} \circ Tx(\mathbf{p})\big).$$

This allows us to define the functors $E_n \colon \Theta^n \to \mathcal{Q}$ and $I_n \colon \Theta^n \to P$-**Alg**.

Definition 7.1. Define a functor $E_n \colon \Theta^n \to \mathcal{Q}$ as follows:

- given an object $\mathbf{j} \in \Theta^n$, $E_n(\mathbf{j})$ is defined to be the object

$$Q(\mathbf{j}) \xrightarrow{\phi_{(\mathbf{j})}} TX(\mathbf{j}).$$

  of $\mathcal{Q}$;

- given a morphism $\mathbf{p} \colon \mathbf{j} \to \mathbf{k}$ in Θ^n, $E_n(\mathbf{p})$ is defined to be the map

$$\epsilon_{\phi_{\mathbf{k}}} \circ J\big(r(\mathbf{p}), \mu^T_{X(\mathbf{k})} \circ Tx(\mathbf{p})\big).$$

Write $K \colon \mathcal{Q} \to P$-**Alg** for the Eilenberg–Moore comparison functor for the adjunction

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "U"] \end{tikzcd}$$

We define a functor $I_n := K \circ E_n : \Theta^n \to P$-**Alg**.

We can now define the nerve functor for Penon weak n-categories.

Definition 7.2. The *nerve functor* $\mathcal{N}$ for Penon weak n-categories is defined by

$$\begin{tikzcd}
\mathcal{N}\colon P\text{-}\mathbf{Alg} & \longrightarrow & {[(\Theta^n)^{\mathrm{op}}, \mathbf{Set}]} \\
\mathcal{A} \arrow[dd, "f"'] & & P\text{-}\mathbf{Alg}(I_n(-), \mathcal{A}) \arrow[dd, "f\circ -"] \\
& \longmapsto & \\
\mathcal{B} & & P\text{-}\mathbf{Alg}(I_n(-), \mathcal{B}).
\end{tikzcd}$$

For a P-algebra $\mathcal{A}$, the presheaf $\mathcal{N}\mathcal{A} = P\text{-}\mathbf{Alg}(I_n(-), \mathcal{A})$ is called the *nerve of* $\mathcal{A}$.

8. Directions for further investigation

In this section we discuss the questions that arise from this nerve construction, and what further results need to be proved in order to make a more complete comparison between Penon weak n-categories and Tamsamani–Simpson weak n-categories. The central question is whether the following conjecture holds:

Conjecture 8.1. *Let $\mathcal{A}$ be a Penon weak n-category. Then the nerve $\mathcal{N}\mathcal{A}$ satisfies the Segal condition, and is thus a Tamsamani–Simpson weak n-category.*

We have proved this only in the case $n = 2$ (Theorem 6.8). As in the 2-dimensional case, for general n we can express the Segal maps in terms of composition with wide pushouts of face maps, allowing us to rephrase some parts of the Segal condition in terms of colimits of P-algebras in the image of the functor $I_n\colon \Theta^n \to P\text{-}\mathbf{Alg}$ (for the 2-dimensional version, see Lemmas 6.1 and 6.2). However, it is not practical to generalise the proofs from the 2-dimensional case to the general case by hand, due to their elementary approach. The use of computers in mathematical proofs has become more prevalent in recent years, and it may be possible to generalise these elementary proofs for low values of n, by using a computer to perform the calculations of the cells in the P-algebras $I_n(\mathbf{j})$. To prove Conjecture 8.1 in general we would need a more abstract approach. The author believes that this would require a deeper understanding of the "partially free" P-algebras (those generated from an object of $\mathcal{R}$ rather than an n-globular set) used in the nerve construction; colimits of free P-algebras are easy to work with, since the free P-algebra functor preserves colimits, but this is not true for "partially free" P-algebras. Coherence for "partially free" Penon weak n-categories would likely play a key role in this, though we have not yet made this precise.

Another natural question to ask is whether the nerve functor for Penon weak n-categories is full and faithful. We now prove that it is faithful, then argue that it is not full and explain why this is the case.

Proposition 8.2. *The nerve functor* $\mathcal{N}\colon P\text{-}\mathbf{Alg} \to [(\Theta^n)^{\mathrm{op}}, \mathbf{Set}]$ *is faithful.*

Proof. The idea of the proof is as follows: every presheaf $(\Theta^n)^{\mathrm{op}} \to \mathbf{Set}$ has an underlying n-globular set, and in the case of the nerve of a Penon weak n-category, this is isomorphic to the underlying n-globular set of the original P-algebra. A map of P-algebras is a map of the underlying n-globular sets satisfying a certain commutativity condition, and when we apply the nerve functor to such a map the action on underlying n-globular sets remains unchanged.

For all $0 \leq k \leq n$, write

$$(\mathbf{1}_k, \mathbf{0}) := (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, 0, \ldots, 0) \in \Theta^n.$$

Observe that $R(\mathbf{1}_k, \mathbf{0}) = X(\mathbf{1}_k, \mathbf{0})$, so $I_n(\mathbf{1}_k, \mathbf{0}) = F_P X(\mathbf{1}_k, \mathbf{0})$, where

$$F_P\colon n\text{-}\mathbf{GSet} \longrightarrow P\text{-}\mathbf{Alg}$$

is the free P-algebra functor. Furthermore, for $k \in \mathbb{G}_n$,

$$X(\mathbf{1}_k, \mathbf{0}) \cong H_k = \mathbb{G}_n(-, k),$$

i.e. $X(\mathbf{1}_k, \mathbf{0})$ is a representable functor. Thus, by the Yoneda lemma, for any $A \in n\text{-}\mathbf{GSet}$,

$$A_k \cong n\text{-}\mathbf{GSet}(H_k, A) \cong n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}, A)),$$

naturally in A and k. Let $\mathcal{A} = (\theta_A\colon PA \to A)$ be a P-algebra. Then, by the adjunction $F_P \dashv U_P$,

$$n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}), A) \cong P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{A}),$$

naturally in $\mathcal{A}$.

Now suppose we have P-algebras $\mathcal{A} = (\theta_A\colon PA \to A), \mathcal{B} = (\theta_B\colon PB \to B)$, and maps of P-algebras u, $v\colon \mathcal{A} \to \mathcal{B}$ such that $\mathcal{N}u = \mathcal{N}v$. Thus, for each $0 \leq k \leq n$ we have

$$u \circ - = v \circ -\colon P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{A}) \to P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{B}).$$

We can write u_k as the composite shown in the diagram below:

$$\begin{tikzcd}
A_k \arrow[r, "u_k"] \arrow[d, "\cong"'] & B_k \\
n\text{-}\mathbf{GSet}(H_k, A) \arrow[r, "u\circ-"'] \arrow[d, "\cong"'] & n\text{-}\mathbf{GSet}(H_k, B) \arrow[u, "\cong"'] \\
n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}), A) \arrow[r, "u\circ-"'] \arrow[d, "\cong"'] & n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}), B) \arrow[u, "\cong"'] \\
P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{A}) \arrow[r, "u\circ-"'] & P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{B}) \arrow[u, "\cong"']
\end{tikzcd}$$

and similarly, we can write v_k as:

$$\begin{tikzcd}
A_k \arrow[r, "v_k"] \arrow[d, "\cong"'] & B_k \\
n\text{-}\mathbf{GSet}(H_k, A) \arrow[r, "v\circ-"'] \arrow[d, "\cong"'] & n\text{-}\mathbf{GSet}(H_k, B) \arrow[u, "\cong"'] \\
n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}), A) \arrow[r, "v\circ-"'] \arrow[d, "\cong"'] & n\text{-}\mathbf{GSet}(X(\mathbf{1}_k, \mathbf{0}), B) \arrow[u, "\cong"'] \\
P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{A}) \arrow[r, "v\circ-"'] & P\text{-}\mathbf{Alg}(I_n(\mathbf{1}_k, \mathbf{0}), \mathcal{B}). \arrow[u, "\cong"']
\end{tikzcd}$$

Since $u\circ- = v\circ-$, these diagrams give us that $u_k = v_k$ for all $0 \leq k \leq n$, so $u = v$. Hence the nerve functor $\mathcal{N}\colon P\text{-}\mathbf{Alg} \to [(\Theta^n)^{\mathrm{op}}, \mathbf{Set}]$ is faithful. $\square$

To see that the nerve functor is not full, consider the P-algebra illustrated below:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "h"] \arrow[rr, bend right=50, "k"'] \arrow[rr, phantom, bend right=25, "\cong"] & & \bullet
\end{tikzcd}$$

where $g \circ f = h$. Any endomorphism of this P-algebra that sends f to f and g to g must also send h to h, since maps of P-algebras preserve composition, and $h = g \circ f$. However, when we consider endomorphisms of the nerve of this P-algebra, we see that there are endomorphisms sending f to f and g to g that send h to k; such endomorphisms are not in the image of the nerve functor.

This illustrates a key difference between algebraic and non-algebraic definitions of weak n-category: in the algebraic case the natural notion of map preserves the composition structure, but in the non-algebraic case there is no specified composition structure to preserve. In the example above, once we have applied the nerve functor we no longer remember which cell was $g \circ f$, and morphisms can now map h to any legitimate choice of composite.

Note that maps of nerves are still required to preserve identities, however, since these are specified by degeneracy maps. This means that maps of Tamsamani–Simpson weak n-categories behave like normalised maps, i.e. those that preserve identities strictly, but are only required to preserve composition weakly. This has been formalised in the 2-dimensional case by Lack and Paoli [22]. There is currently no definition of normalised maps of Penon weak n-categories, and we believe that such a definition would be necessary to adapt our nerve construction to give a full nerve functor for Penon weak n-categories.

One final question raised by this work is whether every Tamsamani–Simpson weak n-category arises as the nerve of a Penon weak n-category. To answer this question we would need to construct a Penon weak n-category from a Tamsamani–Simpson weak n-category. Note that there will be no canonical way to do this, since it would involve making choices of composites.

This nerve construction is a first step towards understanding the relationships between algebraic and non-algebraic definitions of weak n-categories. We have made a connection between the algebraic definition of Penon weak n-categories and the non-algebraic setting in which Tamsamani–Simpson weak n-categories are defined, allowing for the relationship between these definitions to be studied. Our nerve construction is the first to allow for such a comparison, and we believe that it should pave the way for more connections to be made between algebraic and non-algebraic definitions of weak n-category.

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Thomas Cottrell
Department of Mathematical Sciences
University of Bath
Claverton Down
Bath
B2 7AY
UK
t.cottrell@bath.ac.uk