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EVERY SUFFICIENTLY COHESIVE TOPOS IS INFINITESIMALLY GENERATED

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Résumé. Un topos $\mathcal{E}$ est *faiblement généré* par une sous-catégorie $\mathcal{C} \to \mathcal{E}$ si le sous-topos extrême $\mathcal{E} \to \mathcal{E}$ est le plus petit sous-topos de $\mathcal{E}$ contenant $\mathcal{C} \to \mathcal{E}$. Si la sous-catégorie est constituée d'un seul objet, nous disons que $\mathcal{E}$ est faiblement généré par cet objet. Par exemple, il est bien connu que chaque topos est faiblement généré par son classificateur de sous-objets. Le présent article est motivé par l'observation que certains 'gros' topos sont faiblement générés par un objet qui a exactement un point. Afin de mieux comprendre ce phénomène, nous abordons d'abord un problème plus général. Nous considérons une topologie (Lawvere-Tierney) dans un topos $\mathcal{E}$ et prouvons une condition suffisante pour que le classificateur de sous-objets denses associé génère faiblement $\mathcal{E}$. Nous nous concentrons ensuite sur les morphismes géométriques pré-cohésifs $p : \mathcal{E} \to \mathcal{S}$ avec $\mathcal{S}$ Booléen. Nous montrons que si le classificateur de sous-objets de $\mathcal{E}$ est connexe (Sufficient Cohesion) alors $\mathcal{E}$ est faiblement généré par le classificateur de sous-objets $\neg\neg$-denses.
Abstract. A topos $\mathcal{E}$ is *weakly generated* by a full subcategory $\mathcal{C} \to \mathcal{E}$ if the extreme subtopos $\mathcal{E} \to \mathcal{E}$ is the smallest subtopos of $\mathcal{E}$ containing $\mathcal{C} \to \mathcal{E}$. If the full subcategory consists of only one object then we say that $\mathcal{E}$ is weakly generated by that object. For instance, it is well-known that every topos is weakly generated by its subobject classifier. The present paper is motivated by the observation that certain 'gros' toposes are weakly generated by an object that has exactly one point. In order to better understand this phenomenon we first address a more general problem. We consider a (Lawvere-Tierney)

topology in a topos $\mathcal{E}$ and prove a sufficient condition for the associated classifier of dense subobjects to weakly generate $\mathcal{E}$. We then concentrate on pre-cohesive geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ with Boolean $\mathcal{S}$. We show that if the subobject classifier of $\mathcal{E}$ is connected (Sufficient Cohesion) then $\mathcal{E}$ is weakly generated by the classifier of $\neg\neg$-dense subobjects.
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1. Weak generation

In [7], Lawvere recalls having read that "the basic program of infinitesimal calculus, continuum mechanics, and differential geometry is that all the world can be reconstructed from the infinitely small" and then proposes a mathematical formulation of the idea that a topos may be generated by a single object T "which in some of several senses is infinitely small. Of course T is not just a single point; but it may *have* only a single point, or more generally the set of components functor may agree with the functor represented by 1 on T and its products and sums". This proposal is refined in Section VII of [8] and we elaborate on that.

Recall that a geometric morphism $s : \mathcal{E} \to \mathcal{L}$ between toposes is *connected* if its inverse image $s^* : \mathcal{S} \to \mathcal{E}$ is full and faithful. In Section VII of [8] the following concept is introduced.

Definition 1.1. Given a connected morphism $s : \mathcal{E} \to \mathcal{L}$ of toposes, let j in $\mathcal{E}$ be the strongest localness operator for which every s^*Y (for Y in $\mathcal{L}$) is a j-sheaf. If j is actually the identity map on the truth-value space, then $\mathcal{E}$ is *weakly generated by* s.

In other words, $\mathcal{E}$ is weakly generated by a connected $s : \mathcal{E} \to \mathcal{S}$ if the smallest subtopos containing the full subcategory $s^* : \mathcal{L} \to \mathcal{E}$ is the whole of $\mathcal{E}$. The next example is Proposition VII.6 in [8].

Example 1.2 (The topos of reversible graphs is weakly generated by loops)**.** Let M be the four-element monoid of endofunctions of a two-element set. Collapsing the two constant maps in M determines a quotient morphism of monoids $M \to N$ to a three-element monoid. This quotient induces a(n hyper-)connected geometric morphism $s : \widehat{M} \to \widehat{N}$ between the associated toposes of presheaves (see Example A4.6.9 in [5]). To prove that $\widehat{M}$ is weakly generated by s it is useful to picture the objects of $\widehat{M}$ as 'reversible' reflexive graphs. Then $s^* : \widehat{N} \to \widehat{M}$ is the full subcategory determined by those graphs that only have loops. If we let A be the result of applying s^* to the standard generator of $\widehat{N}$ then the exponential A^A contains the standard generator I of $\widehat{M}$ as a retract. Since $\widehat{M}$ has no proper subtoposes containing I, $\widehat{M}$ is weakly generated by $s : \widehat{M} \to \widehat{N}$.

It seems clear that Section VII in [8] tacitly suggests the possibility of a more general result. The purpose of the present paper is to prove such a result. We will recall the relevant definitions but we will also assume that the reader is more or less familiar with [8, 11]. For the moment, though, it is convenient to generalize.

Definition 1.3. The topos $\mathcal{E}$ is *weakly generated* by a full subcategory $\mathcal{C} \to \mathcal{E}$ if the identity $\mathcal{E} \to \mathcal{E}$ is the smallest subtopos containing $\mathcal{C} \to \mathcal{E}$.

Of course, $\mathcal{E}$ is weakly generated by a connected geometric morphism $s : \mathcal{E} \to \mathcal{L}$ in the sense of Definition 1.1 if and only if $\mathcal{E}$ is weakly generated by the full subcategory $s^* : \mathcal{L} \to \mathcal{E}$. In particular, every topos is weakly generated by itself.

Lemma 1.4. *If $\mathcal{C}$ is a small category and J is a subcanonical Grothendieck topology on it then,* $\mathrm{Sh}(\mathcal{C}, J)$ *is weakly generated by the restricted Yoneda embedding* $\mathcal{C} \to \mathrm{Sh}(\mathcal{C}, J)$ *if and only if J is the canonical topology.*

Proof. Follows from the definition of canonical topology. □

On the other hand, the existence of non trivial subcanonical topologies implies that the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ need not weakly generate the topos $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$.

Any object X in the topos $\mathcal{E}$ determines a full subcategory of $\mathcal{E}$ with exactly one object. If $\mathcal{E}$ is weakly generated by this subcategory then we say that $\mathcal{E}$ is *weakly generated by X*.

Example 1.5 (Any topos is weakly generated by its subobject classifier)**.** See paragraph before Remark A4.3.10 in [5]. The argument suggested there is that, since subtopos inclusions are exponential ideals, a subtopos containing Ω must contain all power-objects. So the direct image of the subtopos is a logical functor, and hence an equivalence (by A2.3.9 loc. cit.).

The details of Example 1.2 show that the topos $\widehat{M}$ of reversible graphs is weakly generated by the object A which, incidentally, has exactly one point.

Example 1.6 (Johnstone's topological topos is weakly generated by 2)**.** Let Σ be the full subcategory of $\mathbf{Top}$ determined by two objects: the terminal and the one-point compactification $\mathbb{N}^+$ of $\mathbb{N}$. Let J be the canonical topology on Σ so that $\mathcal{J} = \mathrm{Sh}(\Sigma, J)$ is the *topological topos* introduced in [3]. We claim that $\mathcal{J}$ is weakly generated by the discrete space 2 with two points. To prove this let $i : \mathcal{F} \to \mathcal{J}$ be a subtopos containing 2. Then the Cantor-space $2^{\mathbb{N}}$ is also in the exponential ideal $i_* : \mathcal{F} \to \mathcal{J}$. It is not difficult to prove that $\mathbb{N}^+$ is a retract of $2^{\mathbb{N}}$ and, since retracts of sheaves are sheaves, we have that $\mathbb{N}^+$ is in $\mathcal{F}$, so $i : \mathcal{F} \to \mathcal{E}$ is an equivalence because J is the canonical topology. (Concerning the proof that $\mathbb{N}^+$ is a retract of $2^{\mathbb{N}}$, one may do it by exhibiting an explicit continuous retraction of the continuous injection $\mathbb{N}^+ \to 2^{\mathbb{N}}$ that sends $n \in \mathbb{N}$ to the sequence that starts with 0 in the first n positions and ends with an infinite sequence of 1's. Alternatively, one may invoke a more general result saying that every nonempty closed subset of $2^{\mathbb{N}}$ is a retract of $2^{\mathbb{N}}$. The original source of this result seems to be [16].)

As a corollary we may conclude that $\mathcal{J}$ is weakly generated by the full subcategory $p^* : \mathbf{Set} \to \mathcal{J}$ of discrete spaces. In other words, $\mathcal{J}$ is weakly generated (in the sense of Definition 1.1) by the (connected) canonical geometric morphism $p : \mathcal{J} \to \mathbf{Set}$. (One naturally wonders about geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ such that $\mathcal{E}$ is weakly generated by $p^*(\Omega_{\mathcal{S}})$.)

There is a very explicit construction of the smallest subtopos containing a fixed object (see, e.g., Proposition A4.5.15 in [5]). So it should be possible to characterize those objects that weakly generate but, for our main result, we are going to use a more direct strategy, suggested by the next observation combining Example 1.5 and the argument in Example 1.6.

Lemma 1.7. *Let $\mathcal{E}$ be a topos with subobject classifier Ω. If there are objects J and X, and a monomorphism $\Omega \to J^X$, then $\mathcal{E}$ is weakly generated by J.*

Proof. Let $\mathcal{F} \to \mathcal{E}$ be a subtopos containing J. Since subtoposes are exponential ideals, J^X is also in $\mathcal{F}$. As Ω is injective, it is a retract of J^X and hence Ω is also in $\mathcal{F}$. $\square$

In Section 2 we introduce the notion of *substantial* object and prove a sufficient condition for such an object to weakly generate. In Section 3 we restrict attention to the case where the substantial object is the classifier of dense monos determined by a subtopos. The sufficient condition proved in Section 2 naturally leads to the consideration of the double negation (Lawvere-Tierney) topology. The main result in this section shows that, for this topology, substantiality is enough to weakly generate. As a side remark motivated by the results in Section 3 we prove, in Section 4, what seems to be a folklore result characterizing the quasi-closed topologies whose associated sheafification functors preserve the subobject classifier. In Section 5 we incorporate, into the general context of a subtopos, a left adjoint to sheafification. In this case, we obtain a sufficient condition for substantiality and therefore a sufficient condition for weak generation in certain cases. In Section 6 we address the original motivating context and prove that every sufficiently cohesive topos over a Boolean base is weakly generated by its subcategory of 'Leibniz' objects. In Section 7 present a characterization (due to an anonymous referee) of substantial objects in toposes. This characterization may be applied to give simple characterizations of presheaf and spatial toposes whose classifiers of $\neg\neg$-dense subobjects are substantial. In the final section we briefly discuss some elementary remarks that may be relevant for future work.

2. Substantial objects

Let $\mathcal{E}$ be a category with finite products and initial object 0.

Definition 2.1. An object J in $\mathcal{E}$ is *substantial* if the following two conditions hold:

1. J is well-supported, in the sense that the unique $J \to 1$ is a regular epimorphism.

2. For every object Y in $\mathcal{E}$, if the projection $\pi_0 : Y \times J \to Y$ is an isomorphism then Y is initial.

Let us state the following simple fact as a proposition in order to emphasize that Definition 2.1 is consistent with the idea of a 'non subterminal' object.

Proposition 2.2. *If J is both substantial and subterminal then the unique $0 \to 1$ is an isomorphism.*

Proof. If J is well-supported and subterminal then the projection $1 \times J \to 1$ is an isomorphism. So, if J is also substantial, then 1 is initial. $\square$

Consider the following simple source of examples.

Lemma 2.3. *Assume that* 0 *is strict initial in* $\mathcal{E}$*. If* J *has two disjoint points then* J *is substantial.*

Proof. Since it has a point, J is certainly well-supported. Also, by hypothesis, there are points $\bot, \top : 1 \to J$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r, "!"] & 1 \arrow[r, shift left, "\top"] \arrow[r, shift right, "\bot"'] & J \end{tikzcd}$$

is an equalizer. If the projection $\pi_0 : Y \times J \to Y$ is an iso then, as the diagram on the left below commutes,

$$\begin{tikzcd} Y \arrow[r, shift left, "\langle id, \top ! \rangle"] \arrow[r, shift right, "\langle id, \bot ! \rangle"'] & Y \times J \arrow[r, "\pi_0"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "!"] & 1 \arrow[r, shift left, "\top"] \arrow[r, shift right, "\bot"'] & J \end{tikzcd}$$

it follows that the diagram on the right above commutes. So $! : Y \to 1$ factors through $! : 0 \to 1$. Since 0 is strict by hypothesis, the factorization $Y \to 0$ is an iso. $\square$

For instance, $2 = 1 + 1$ is substantial in any extensive category with finite products. Similarly, the subobject classifier in any topos is substantial. We will be mainly interested in pointed substantial objects.

Let us assume from now on that $\mathcal{E}$ is a topos and fix a pointed object $\top : 1 \to J$ therein. The rest of the section is devoted to prove a sufficient condition, involving substantiality, for J to weakly generate $\mathcal{E}$.

Any subobject $w : W \to X$ determines the following two subobjects

$$W \times 1 \xrightarrow{w \times \top} X \times J \qquad\qquad W \times J \xrightarrow{w \times J} X \times J$$

of $X \times J$. So, given two subobjects $u : U \to X$ and $v : V \to X$ of X, we may consider the subobjects

$$U \times 1 \xrightarrow{u \times \top} X \times J \qquad\qquad V \times J \xrightarrow{v \times J} X \times J$$

and also their join

$$(U \times 1) \cup (V \times J) \xrightarrow{(u\times\top)\cup(v\times J)} X \times J$$

as subobjects of $X \times J$. Of particular interest for us will be the case where $v = \neg u : \neg U \to X$. The resulting subobject

$$(U \times 1) \cup (\neg U \times J) \xrightarrow{(u\times\top)\cup((\neg u)\times J)} X \times J$$

will be denoted by $\Psi u : \Psi U \to X \times J$. (It seems worth observing that in this case the two relevant subobjects of $X \times J$ are disjoint so the subobject Ψu of $X \times J$ coincides with the unique map

$$[u \times \top, (\neg u) \times J] : (U \times 1) + (\neg U \times J) \to X \times J$$

from the coproduct $(U \times 1) + (\neg U \times J)$.)

Lemma 2.4. *If J is substantial then for any pair of subobjects $u : U \to X$ and $v : V \to X$, $u = v$ as subobjects of X if and only if $\Psi u = \Psi v$ as subobjects of $X \times J$.*

Proof. One direction is trivial (and does not need substantiality). For the other it is enough to prove that $\Psi u = \Psi v$ implies $v \leq u$. First let us pull back the subobject Ψu of $X \times J$ along $(V \cap \neg U) \times J \to X \times J$. The square below is easily seen to be a pullback

$$\begin{tikzcd} 0 = 0 \times 1 = (V \cap (\neg U) \cap U) \times 1 \arrow[r] \arrow[d, "! = (v\cap(\neg u)\cap u)\times\top"'] & U \times 1 \arrow[d, "u\times\top"] \\ (V \cap \neg U) \times J \arrow[r, "(v\cap(\neg u))\times J"'] & X \times J \end{tikzcd}$$

and, since $v \cap \neg u \leq \neg u$, the square below

$$\begin{tikzcd} (V \cap (\neg U)) \times J \arrow[r] \arrow[d, "id"'] & \neg U \times J \arrow[d, "(\neg u)\times J"] \\ (V \cap \neg U) \times J \arrow[r, "(v\cap(\neg u))\times J"'] & X \times J \end{tikzcd}$$

is also a pullback. As pulling back preserves joins we may conclude that the following square

$$\begin{tikzcd}
(V \cap (\neg U)) \times J \arrow[r] \arrow[d, "id"'] & \Psi U \arrow[d, "\Psi u"] \\
(V \cap \neg U) \times J \arrow[r, "(v \cap (\neg u)) \times J"'] & X \times J
\end{tikzcd}$$

is a pullback. A similar argument implies that the following diagram

$$\begin{tikzcd}
(V \cap (\neg U)) \times 1 \arrow[r] \arrow[d, "id \times \top"'] & \Psi V \arrow[d, "\Psi v"] \\
(V \cap \neg U) \times J \arrow[r, "(v \cap (\neg u)) \times J"'] & X \times J
\end{tikzcd}$$

is a pullback so, if $\Psi u = \Psi v$ then the object $(V \cap (\neg U)) \times J$ is isomorphic to $(V \cap (\neg U)) \times 1$ over $(V \cap \neg U) \times J$ which means that the projection $(V \cap (\neg U)) \times J \to V \cap (\neg U)$ is an isomorphism. Since J is substantial we may conclude that $V \cap (\neg U) = 0$.

Now we pullback Ψu along $v \times J : V \times J \to X \times J$. The following two squares are clearly pullbacks

$$\begin{tikzcd}
(U \cap V) \times 1 \arrow[r] \arrow[d, "(u \cap v) \times \top"'] & U \times 1 \arrow[d, "u \times \top"] \\
V \times J \arrow[r, "v \times J"'] & X \times J
\end{tikzcd}
\qquad
\begin{tikzcd}
(V \cap \neg U) \times J \arrow[r] \arrow[d, "(v \cap (\neg u)) \times J"'] & \neg U \times J \arrow[d, "\neg u \times J"] \\
V \times J \arrow[r, "v \times J"'] & X \times J
\end{tikzcd}$$

so, together with the fact that $V \cap (\neg U) = 0$, established in the previous paragraph, we obtain that the square on the left below

$$\begin{tikzcd}
(U \cap V) \times 1 \arrow[r] \arrow[d, "(u \cap v) \times \top"'] & \Psi U \arrow[d, "\Psi u"] \\
V \times J \arrow[r, "v \times J"'] & X \times J
\end{tikzcd}
\qquad
\begin{tikzcd}
V \times 1 \arrow[r] \arrow[d, "v \times \top"'] & \Psi V \arrow[d, "\Psi v"] \\
V \times J \arrow[r, "v \times J"'] & X \times J
\end{tikzcd}$$

is a pullback. A simpler calculation shows that the square on the right above is a pullback so, if $\Psi u = \Psi v$ then $(U \cap V) \times 1$ and $V \times 1$ are isomorphic over $V \times J$. Therefore, $u \cap v = v$ and hence, $v \leq u$ as subobjects of X. $\square$

The subobject $\top : 1 \to \Omega$ determines $\Psi\top : \Psi 1 \to \Omega \times J$ or, more explicitly,

$$(1 \times 1) \cup (1 \times J) \xrightarrow{(\top\times\top)\cup(\bot\times J)} X \times J$$

where, as usual, $\bot : 1 \to \Omega$ is the Heyting complement of $\top : 1 \to \Omega$.

Lemma 2.5. *If J is substantial then, for any subobject $u : U \to X$ there exists a unique map $\chi_u : X \to \Omega$ such that the following diagram*

$$\begin{tikzcd}
\Psi U \arrow[r] \arrow[d, "\Psi u"'] & \Psi 1 \arrow[d, "\Psi\top"] \\
X \times J \arrow[r, "\chi_u \times J"'] & \Omega \times J
\end{tikzcd}$$

is a pullback. Moreover, this $\chi_u : X \to \Omega$ is the characteristic map of the subobject u.

Proof. To prove existence consider the characteristic map $\chi_u : X \to \Omega$ of the subobject u of X. Since pulling back preserves unions, it is enough to check that the following two squares are pullbacks

$$\begin{tikzcd}
U \times 1 \arrow[r, "!\times!"] \arrow[d, "u\times\top"'] & 1 \times 1 \arrow[d, "\top\times\top"] \\
X \times J \arrow[r, "\chi_u \times J"'] & \Omega \times J
\end{tikzcd}
\qquad
\begin{tikzcd}
(\neg U) \times J \arrow[r, "!\times J"] \arrow[d, "(\neg u)\times J"'] & 1 \times J \arrow[d, "\bot\times J"] \\
X \times J \arrow[r, "\chi_u \times J"'] & \Omega \times J
\end{tikzcd}$$

but this follows because products of pullbacks are pullbacks. Notice that substantiality is not needed for this.

To prove uniqueness, let $\chi_u, \chi_v : X \to \Omega$ be two maps, say, characteristic of the subobjects $u : U \to X$ and $v : V \to X$ respectively. Assume that χ_u and χ_v pull $\Psi\top : \Psi 1 \to \Omega \times J$ to the same thing; that is, $\Psi u = \Psi v$. Then, as J is substantial, Lemma 2.4 allows us to conclude that $u = v$ as subobjects of X. Therefore, $\chi_u = \chi_v$. □

If J is not substantial then the uniqueness part of Lemma 2.5 does not hold. For example, if $\mathcal{E}$ is Boolean and $J = 1$ then any two maps $X \to \Omega$ induce (in the way described above) the same subobject of $X \times J \cong X$.

We now give a sufficient condition for the pointed object J to weakly generate the topos $\mathcal{E}$.

Proposition 2.6. *Let $\top : 1 \to J$ be a pointed object. If J is substantial and there is a map $\chi : \Omega \times J \to J$ such that the following diagram*

$$\begin{tikzcd}
\Psi 1 \arrow[r, "!"] \arrow[d, "\Psi\top"'] & 1 \arrow[d, "\top"] \\
\Omega \times J \arrow[r, "\chi"'] & J
\end{tikzcd}$$

is a pullback then $\mathcal{E}$ is weakly generated by J.

Proof. By Lemma 1.7 it is enough to prove that the transposition $\iota : \Omega \to J^J$ of χ is mono. Let $g, h : X \to \Omega$ be such that $\iota g = \iota h : X \to J^J$. Then

$$\begin{tikzcd}
X \times J \arrow[r, shift left, "g\times J"] \arrow[r, shift right, "h\times J"'] & \Omega \times J \arrow[r, "\iota\times J"] \arrow[rr, bend right, "\chi"'] & J^J \times J \arrow[r, "ev"] & J
\end{tikzcd}$$

commutes so that $\chi(g \times J) = \chi(h \times J) : X \times J \to J$. By hypothesis, the map χ pulls the point $\top : 1 \to J$ back to the subobject $\Psi\top : \Psi 1 \to \Omega \times J$, so $g \times J, h \times J : X \times J \to \Omega \times J$ pullback this subobject to the same subobject of $X \times J$. Lemma 2.5 implies that $g = h$. $\square$

In the next section we discuss a context where the hypotheses of Proposition 2.6 may naturally hold.

3. Substantial classifiers of dense subobjects

The relation between subtoposes and universal closure operators is well-known. Let us briefly recall some relevant facts. Fix a subtopos $c : \mathcal{S} \to \mathcal{E}$ with unit $\eta : Id_{\mathcal{E}} \to c_* c^*$. For any subobject $u : U \to X$ in $\mathcal{E}$, its *closure* $\overline{u} : \overline{U} \to X$ is defined by declaring the left square below

$$\begin{tikzcd}
\overline{U} \arrow[r] \arrow[d, "\overline{u}"'] & c_*(c^* U) \arrow[d, "c_*(c^* u)"] \\
X \arrow[r, "\eta"'] & c_*(c^* X)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
J \arrow[r] \arrow[d, "\mathrm{j}"'] & c_*(c^* 1) \arrow[d, "c_*(c^* \top)"] \\
\Omega \arrow[r, "\eta"'] & c_*(c^* \Omega)
\end{tikzcd}$$

to be a pullback in $\mathcal{E}$. In particular, the closure of the subobject classifier $\top : 1 \to \Omega$ in $\mathcal{E}$ is denoted by $\mathrm{j} : J \to \Omega$ as on the right above. It follows that there exists a point $\top : 1 \to J$ such that the following diagram

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & J \arrow[d, "\mathrm{j}"] \\ & \Omega \end{tikzcd}$$

commutes. The resulting point $\top : 1 \to J$ is dense and it classifies dense subobjects (see Exercise V.1 in [12] or the paragraph following A4.4.2 in [5]).

Consider the subobject

$$\Psi 1 = (1 \times 1) + (1 \times J) \xrightarrow{\Psi\top = [\top\times\top, \bot\times J]} \Omega \times J$$

introduced before Lemma 2.5. At least part of the following result seems to be folklore.

Lemma 3.1. *With the notation above, the following are equivalent:*

1. *The subobject $\Psi\top : \Psi 1 \to \Omega \times J$ is dense in $\mathcal{E}$.*
2. *The map $[c^*\top, c^*\bot] : 1 + 1 \to c^*\Omega$ is an isomorphism in $\mathcal{S}$.*
3. *The reflection $c^* : \mathcal{E} \to \mathcal{S}$ preserves the subobject classifier and $\mathcal{S}$ is Boolean.*

Proof. The functor $c^* : \mathcal{E} \to \mathcal{S}$ applied to the map $\top \times \top : 1 \times 1 \to \Omega \times J$ results in the subobject $c^*\top \times c^*\top : 1 \times 1 \to (c^*\Omega) \times (c^*J)$ which is essentially just $c^*\top : c^*1 \to c^*\Omega$. On the other hand, $c^* : \mathcal{E} \to \mathcal{S}$ applied to the subobject $\bot \times J : 1 \times J \to \Omega \times J$ results in the subobject

$$(c^*\bot) \times (c^*J) : (c^*1) \times (c^*J) \to (c^*\Omega) \times (c^*J)$$

which is just $c^*\bot : c^*1 \to c^*\Omega$. Therefore, c^* applied to the whole subobject is just $[c^*\bot, c^*\top] : 1 + 1 \to c^*\Omega$. It follows that the first two items are equivalent.

The third item trivially implies the second. To complete the proof let j be the Lawvere-Tierney topology determined by the subtopos $\mathcal{S} \to \mathcal{E}$ and recall that the subobject classifier of $\mathcal{S}$ may be constructed in $\mathcal{E}$ as the equalizer $\Omega_j \to \Omega$ of $id, j : \Omega \to \Omega$ in $\mathcal{E}$. Clearly, the point $\top : 1 \to \Omega$ factors through $\Omega_j \to \Omega$ and it is well-known that the resulting point $\top : 1 \to \Omega_j$ classifies closed monos. In particular, let $\chi : 1 \to \Omega_j$ be the classifier of the closure $\overline{0} \to 1$ of $! : 0 \to 1$. Since $0 \to \overline{0}$ is dense, the following diagram commutes

$$\begin{tikzcd} c^*1 \arrow[r, "c^*\chi"] \arrow[dr, "c^*\bot"'] & c^*\Omega_j \arrow[d] \\ & c^*\Omega \end{tikzcd}$$

in $\mathcal{S} = \mathcal{E}_j$. Therefore, $[c^*\top, c^*\bot] : 1 + 1 \to c^*\Omega$ factors through the mono $\Omega_j = c^*\Omega_j \to c^*\Omega$ in $\mathcal{S}$. So, if the second item holds, then the monomorphism $\Omega_j \to c^*\Omega$ is an isomorphism, and $\mathcal{S}$ is Boolean. $\square$

Subtoposes $c : \mathcal{S} \to \mathcal{E}$ such that $c^* : \mathcal{E} \to \mathcal{S}$ preserves the subobject classifier are studied in Proposition A4.5.8 in [5]. Those such that $\mathcal{S}$ is Boolean are the *quasi-closed* ones (see Lemma A4.5.21 loc. cit.).

Proposition 3.2. *Let $c : \mathcal{S} \to \mathcal{E}$ be a subtopos such that $\mathcal{S}$ is Boolean and $c^* : \mathcal{E} \to \mathcal{S}$ preserves the subobject classifier. Let $\top : 1 \to J$ be the classifier of dense subobjects. If J is substantial then it weakly generates $\mathcal{E}$.*

Proof. Under the present hypotheses, Lemma 3.1 implies the existence of a unique morphism $\chi : \Omega \times J \to J$ such that the following diagram

$$\begin{tikzcd} \Psi 1 \arrow[r, "!"] \arrow[d, "\Psi\top"'] & 1 \arrow[d, "\top"] \\ \Omega \times J \arrow[r, "\chi"'] & J \end{tikzcd}$$

is a pullback, so we can apply Proposition 2.6. $\square$

For the reasons explained in Section 4 of [11] we are mainly interested in dense subtoposes (equivalently, those topologies that satisfy $j0 = 0$). The only dense quasi-closed topology is that determined by $\bot : 1 \to \Omega$. That is, the double-negation topology. Moreover, as observed in Example A4.5.9 in [5], the inverse image of $\mathcal{E}_{\neg\neg} \to \mathcal{E}$ preserves the subobject classifier. So we may conclude that, $\mathcal{E}_{\neg\neg} \to \mathcal{E}$ is the only dense subtopos satisfying the conditions in Lemma 3.1.

Definition 3.3. A topos $\mathcal{E}$ is *perfect* if its $\neg\neg$-dense subobject classifier is substantial.

The motivating case that led to Proposition 3.2 may now be stated as follows.

Corollary 3.4. *If the topos $\mathcal{E}$ is perfect then it is weakly generated by the classifier of $\neg\neg$-dense subobjects.*

It is possible to characterize perfect presheaf and spatial toposes directly, but the task is drastically simplified by a characterization of substantial objects in toposes suggested by an anonymous referee. We present this characterization and its applications in Section 7, which is fairly self contained, so the reader should find little trouble in reading it at his point if he wishes to do so. Incidentally, the 'perfect' terminology is justified by one of the applications. On the other hand, I don't know if the characterization by the referee may be applied to give a different proof of our main result. In any case, before continuing the path to the latter result, we briefly comment on a problem suggested by Lemma 3.1.

4. Regular elements

The third item of Lemma 3.1 suggests the problem of characterizing the quasi-closed topologies j such that the sheafification functor $\mathcal{E} \to \mathcal{E}_j$ preserves the subobject classifier. In this short section I give a solution that I learned from Rodolfo Ertola who provided a proof using Natural Deduction. Martin Hyland later informed me that a topos-theoretic argument (explained below) was known to Peter Johnstone already in 1973.

Let j be a topology in a topos $\mathcal{E}$ and let $\mathcal{E}_j \to \mathcal{E}$ be the associated subtopos. Proposition A4.5.8 in [5] shows that sheafification $\mathcal{E} \to \mathcal{E}_j$ preserves the subobject classifier if and only if

$$\begin{tikzcd} \Omega \arrow[r, "\langle j, id \rangle"] \arrow[d, "!"'] & \Omega \times \Omega \arrow[r, "\Rightarrow"] & \Omega \arrow[d, "j"] \\ 1 \arrow[rr, "\top"'] & & \Omega \end{tikzcd}$$

commutes. Also, recall that $j : \Omega \to \Omega$ is *quasi-closed* if it is a composite of the form

$$\Omega \xrightarrow{\langle id,!\rangle} \Omega \times 1 \xrightarrow{id\times\langle u,u\rangle} \Omega \times \Omega \times \Omega \xrightarrow{\Rightarrow\times id} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

for some $u : 1 \to \Omega$. If $U \to 1$ is the subterminal classified by u then the associated quasi-closed topology is denoted by $q(U)$ as in [5].

Proposition 4.1. *For every subterminal $U \to 1$, the sheafification functor $\mathcal{E} \to \mathcal{E}_{q(U)}$ preserves the subobject classifier if and only if $\neg\neg U = U$.*

Proof. For brevity, let $j = q(U)$ and notice that $j\bot = u$ where $u : 1 \to \Omega$ is the classifying morphism of the subterminal U. If $\mathcal{E} \to \mathcal{E}_j$ preserves the subobject classifier then $\top \leq j(j\bot \Rightarrow \bot) = j(\neg u)$, which is equivalent to $\neg u \Rightarrow u \leq u$. Since $\neg\neg u \leq \neg u \Rightarrow u$, we may conclude that $\neg\neg u \leq u$.

On the other hand, the topos-theoretic proof of the converse goes as follows. First, recall that every geometric inclusion $\mathcal{E}_j \to \mathcal{E}$ has a unique dense/closed factorization that may be described as

$$\mathcal{E}_j \longrightarrow \mathcal{E}_{c(\mathrm{ext}(j))} \longrightarrow \mathcal{E}$$

where $\mathrm{ext}j$ is the *exterior* of j so that $c(\mathrm{ext}j)$ is the *closure* of j. See A4.5.19 and A4.5.20 in [5]. In particular, for $j = q(U)$, we have $c(\mathrm{ext}(j)) = c(U)$ and the factorization

$$\mathcal{E}_{q(U)} \longrightarrow \mathcal{E}_{c(U)} \longrightarrow \mathcal{E}$$

identifies $\mathcal{E}_{q(U)}$ with $(\mathcal{E}_{c(U)})_{\neg\neg}$. See the paragraph before A4.5.21 in [5].

Now, if $U = \neg V$ then the closure of $o(V)$ is

$$c(\mathrm{ext}(o(V))) = c(\neg V) = c(U)$$

so we have a dense inclusion $\mathcal{E}_{o(V)} \to \mathcal{E}_{c(U)}$ and then the composite

$$(\mathcal{E}_{o(V)})_{\neg\neg} \to \mathcal{E}_{o(V)} \to \mathcal{E}_{c(U)}$$

is a Boolean dense subtopos of $\mathcal{E}_{c(U)}$, so it must coincide with the subtopos $\mathcal{E}_{q(U)} = (\mathcal{E}_{c(U)})_{\neg\neg} \to \mathcal{E}_{c(U)}$. Therefore, the sheafification $\mathcal{E} \to \mathcal{E}_{q(U)}$ is the composite of two sheafifications (for $o(V)$ and $\neg\neg$) which are both known to preserve the subobject classifier. $\square$

In other words, $\mathcal{E} \to \mathcal{E}_{q(U)}$ preserves the subobject classifier if and only if U is *regular*.

5. A characterization of discrete objects

In this section we give a sufficient condition for the classifier of dense monos determined by an essential subtopos to be substantial. We state the results using the notation for centers of local morphisms in order to suggest the motivation. Let $c : \mathcal{S} \to \mathcal{E}$ be an essential subtopos and denote the counit of $c_! \dashv c^*$ by $\beta_X : c_!(c^* X) \to X$.

Lemma 5.1. *A subobject $u : U \to X$ in $\mathcal{E}$ is dense (w.r.t. the subtopos c) if and only if the counit $\beta : c_!(c^* X) \to X$ of $c_! \dashv c^*$ factors through u.*

Proof. If $c^* u : c^* U \to c^* X$ is an isomorphism in $\mathcal{S}$ then the transposition $c_!(c^* X) \to U$ of the inverse $(c^* u)^{-1} : c^* X \to x^* U$ shows that the counit $\beta : c_!(c^* X) \to X$ factors through $u : U \to X$. On the other hand, if there exists a morphism $v : c_!(c^* X) \to U$ such that $uv = \beta : c_!(c^* X) \to X$ then $(c^* u)(c^* v) = c^* \beta : c^*(c_!(c^* X)) \to c^* X$ and so the mono $c^* u$ is also split epic. $\square$

Recall that if we let the square on the left below be a pullback

$$\begin{tikzcd} J \arrow[r] \arrow[d, "j"'] & c_*(c^* 1) \arrow[d, "c_*(c^* \top)"] \\ \Omega \arrow[r, "\eta"'] & c_*(c^* \Omega) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & J \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

where η is the unit of $c^* \dashv c_*$ then the evident factorization $\top : 1 \to J$ on the right above is the classifier of dense monos.

Lemma 5.2. *If $c_! : \mathcal{S} \to \mathcal{E}$ preserves finite products and the counit β is monic then, for any object X in $\mathcal{E}$, the following are equivalent:*

1. *The counit $\beta : c_!(c^* X) \to X$ is an isomorphism.*
2. *There exists a unique map $X \to J$.*
3. *$c^*(J^X) = 1$.*

Proof. If $c^*(J^X) = 1$ then there exists a unique map $1 \to c^*(J^X)$ in $\mathcal{S}$. Since $c_! : \mathcal{S} \to \mathcal{E}$ preserves terminal object, there exists a unique map $X \to J$. If this is the case then, as β is monic, Lemma 5.1 implies that $\beta : c_!(c^*X) \to X$ and $id : X \to X$ must coincide. It remains to show that the first item implies the third. Since the functor $c_! : \mathcal{S} \to \mathcal{E}$ preserves finite products, the adjunction $c_! \dashv c^*$ is enriched. So, for any S in $\mathcal{S}$, $c^*(J^{c_!S}) = (c^*J)^S = 1^S = 1$. $\square$

An object X in $\mathcal{E}$ satisfying the equivalent conditions of Lemma 5.2 will be called *discrete*. Notice that the second and third items of Lemma 5.2 only involve the subtopos $c^* \dashv c_* : \mathcal{S} \to \mathcal{E}$ so the lemma suggests a definition of 'discrete object' in $\mathcal{E}$ relative to a subtopos. For example, the next lemma only needs the subtopos.

Lemma 5.3. *For every X in $\mathcal{E}$, if $\pi_0 : X \times J \to X$ is an isomorphism then, for every map $Y \to X$, Y is discrete.*

Proof. Since $\pi_0 : X \times J \to X$ is an isomorphism by hypothesis, for every object Y in $\mathcal{E}$ and every map $f : Y \to X$ there exists a unique $g : Y \to J$ such that $\pi_0 \langle f, g\rangle = f$. In other words, the existence a map $Y \to X$ implies the existence of a unique map $Y \to J$. By Lemma 5.2, such a Y is discrete. $\square$

So, assuming that the leftmost adjoint $c_!$ preserves finite products, if $\pi_0 : X \times J \to X$ is an iso then X is discrete and not only that but also every map with X as codomain has discrete domain.

Proposition 5.4. *Assume that $c_! : \mathcal{S} \to \mathcal{E}$ preserves finite products and that the counit β is monic. If, for every A in $\mathcal{S}$, $c_!A \times c_*2$ discrete implies A initial, then J is substantial.*

Proof. Assume that $\pi_0 : X \times J \to X$ is an iso. Lemma 5.3 implies that X is discrete, say $X = c_!A$. Also by Lemma 5.3, the projection $c_!A \times c_*2 \to c_!A$ has discrete domain. By hypothesis, A is initial and so, $X = c_!A = c_!0 = 0$. $\square$

For example, if we let $\mathcal{J}$ be the topological topos then the canonical geometric morphism $p : \mathcal{J} \to \mathbf{Set}$ is local so Proposition 5.4 applied to the center of p shows that $\mathcal{J}$ is perfect. (A different proof of this fact will be presented in Section 7.) By Corollary 3.4, the topological topos is weakly generated by an object with only one point.

Another application of Proposition 5.4 is discussed in the next section.

6. The case of pre-cohesive toposes

Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is called *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to a string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ such that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are full and faithful, $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products and the canonical natural transformation $\theta : p_* \to p_!$ is epic. This last condition is called the *Nullstellensatz*. (Alternatively, in the standard terminology, it is a local, hyperconnected and essential geometric morphism whose leftmost adjoint $p_!$ preserves finite products. In this form, the Nullstellensatz corresponds to hyperconnectedness, see [6].) We may also say that $\mathcal{E}$ is pre-cohesive over $\mathcal{S}$.

For example, let $\mathcal{C}$ be a small category with terminal object. Then, the canonical geometric morphism $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive if and only if every object of $\mathcal{C}$ has a point [6]. Further examples of pre-cohesive geometric morphisms may be found in [14, 13].

A pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is called a *quality type* if $\theta : p_* \to p_!$ is an iso. For instance, let $\mathcal{C}$ be a small category such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. Then p is a quality type if and only if every object has exactly one point [13].

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive. An object X in $\mathcal{E}$ is called *connected* if $p_! X = 1$. A pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ will be called *sufficiently cohesive* if the subobject classifier Ω of $\mathcal{S}$ is connected. We may also say that Sufficient Cohesion holds for p. For example, if $\mathcal{C}$ is small, has a terminal object and the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive then, p is sufficiently cohesive if and only if some object of $\mathcal{C}$ has two distinct points [14].

The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is said to satisfy *Connected Codiscreteness (CC)* if for every A in $\mathcal{S}$, the unique $p_!(p^! A) \to 1$ is mono. If $\mathcal{S}$ is De Morgan then CC is equivalent to Sufficient Cohesion (Corollary 6.6 in [11]). In general, CC implies Sufficient Cohesion but we don't know if the converse holds. Intuitively, these two conditions say that points and pieces are different concepts; in contrast to what happens in quality types. A precise statement is the following strengthening of Lemma 6.3 in [11].

Lemma 6.1. *Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive and satisfy CC. For every object A in $\mathcal{S}$, if $\theta_{p^! A} : p_*(p^! A) \to p_!(p^! A)$ is an isomorphism then A is subterminal. Therefore, if $p : \mathcal{E} \to \mathcal{S}$ is a quality type satisfying CC then $\mathcal{S}$ is inconsistent.*

Proof. Since $p^! : \mathcal{S} \to \mathcal{E}$ is full and faithful, the counit $\epsilon_A : p_*(p^! A) \to A$ is an iso. Then the composite

$$A \xrightarrow{\epsilon^{-1}} p_*(p^! A) \xrightarrow{\theta} p_!(p^! A) \xrightarrow{!} 1$$

is mono, because $\theta_{p^! A}$ is an isomorphism and $p_!(p^! A) \to 1$ is monic by CC.

If p is a quality type then θ is an iso so, in this case, for every A in $\mathcal{S}$, the unique map $A \to 1$ is mono. □

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive. The counit of the adjunction $p^* \dashv p_*$ will be denoted by $\beta : p^* p_* \to Id_{\mathcal{E}}$. As in Section 5, an object X in $\mathcal{E}$ is called *discrete* if the counit $\beta : p^*(p_* X) \to X$ is an iso. Also, let $\top : 1 \to J$ be the classifier of dense monos determined by the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$.

Theorem 6.2. *If the pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ satisfies CC then J is substantial.*

Proof. We apply Proposition 5.4 to the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$. So let A be an object in $\mathcal{S}$ and assume that $p^* A \times p^! 2$ is discrete in $\mathcal{E}$. That is, the counit $\beta : p^*(p_*(p^* A \times p^! 2)) \to p^* A \times p^! 2$ is an iso. Since the functors p_* and p^* preserve products, it follows that

$$p^*(p_*(p^* A)) \times p^*(p_*(p^! 2)) \xrightarrow{\beta \times \beta} p^* A \times p^! 2$$

is an iso. Since p^* and $p^!$ are fully faithful the unit $\alpha : Id_{\mathcal{S}} \to p_* p^*$ and counit $\epsilon : p_* p^* \to Id_{\mathcal{S}}$ are isos so the composite

$$p^* A \times p^* 2 \xrightarrow{p^* \alpha \times p^* \epsilon^{-1}} p^*(p_*(p^* A)) \times p^*(p_*(p^! 2)) \xrightarrow{\beta \times \beta} p^* A \times p^! 2$$

is also an iso. The composite $\beta(p^* \epsilon^{-1}) : p^* 2 \to p^! 2$ may be denoted by ϕ so the product $id \times \phi : p^* A \times p^* 2 \to p^* A \times p^! 2$ above is an iso. Since the leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products, the map

$$p_!(p^* A) \times p_!(p^* 2) \xrightarrow{id \times p_! \phi} p_!(p^* A) \times p_!(p^! 2)$$

is an iso. Since p^* is full and faithful the counit $\tau : p_!p^* \to Id_{\mathcal{S}}$ is an iso. Then, the composite below

$$\begin{tikzcd} A \times 2 \arrow[r, "\tau^{-1}\times\tau^{-1}"] \arrow[drr, "id\times((p_!\phi)\tau^{-1})"'] & p_!(p^*A) \times p_!(p^*2) \arrow[r, "id\times p_!\phi"] & p_!(p^*A) \times p_!(p^!2) \arrow[d, "\tau\times id"] \\ & & A \times p_!(p^!2) \end{tikzcd}$$

is an iso. Finally, since CC holds, $p^!2$ is connected (i.e. $p_!(p^!2) = 1$) so the projection $A \times 2 \to A$ is an iso. As 2 is substantial by Lemma 2.3, A is initial. $\square$

Notice that the converse does not hold as exemplified by the presheaf examples whose site satisfy that every object has exactly one point.

We can now prove one of the main results of the paper.

Corollary 6.3. *Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive and $\mathcal{S}$ be Boolean. If p is sufficiently cohesive then $\mathcal{E}$ is weakly generated by the classifier of $\neg\neg$-dense subobjects.*

Proof. By Corollary 4.5 in [11], the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ coincides with $\mathcal{E}_{\neg\neg} \to \mathcal{E}$. By Corollary 6.6 loc. cit., Sufficient Cohesion is equivalent to CC. Theorem 6.2 implies that the classifier of dense monos is substantial. (In other words, $\mathcal{E}$ is perfect.) So the result follows from Corollary 3.4. $\square$

Now that Corollary 6.3 is proved, it may be interesting to briefly recall the conversation that motivated it. During a meeting at Oaxaca in 2015 organized by F. Marmolejo, Lawvere envisaged a "new principle of logic" which is simply that the truth-value object would have the property that the only j-operator for which the top blob around true is a sheaf, is the identity. In other words, that the blob weakly generates the topos. In the context of a cohesive topos $p : \mathcal{E} \to \mathcal{S}$, the 'blob around true' is the vertex of the pullback

$$\begin{tikzcd} J \arrow[r] \arrow[d] & p^!(p_*1) \arrow[d, "p^!(p_*\top)"] \\ \Omega \arrow[r, "\eta"'] & p^!(p_*\Omega) \end{tikzcd}$$

because it is the largest subobject of Ω that collapses to the top point in the codiscretization $p^!(p_*\Omega)$ of Ω. Deprived of this specific geometric intuition, J may be identified with the codomain of the classifier of dense monos of the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$. Now, some non-triviality condition seemed needed in order to prove weak generation. Working out the details in the topos of reflexive graphs led to the idea of substantiality. In this way, the vibrant picture of an 'infinitesimal blob around the truth' generating the whole topos, became a cold proof that certain classifiers of dense monos are substantial and therefore weakly generate. The fact that Sufficient Cohesion implies substantiality of the relevant object connects the motivating idea and the end result. It is likely that the proof may be improved, but it is more tempting to pursue the new principle.

As suggested in the introduction we also want to show that every pre-cohesive topos as in the statement of Corollary 6.3 is weakly generated by a canonical quotient topos. We quickly discuss this quotient.

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive. Denote by $s^* : \mathcal{L} \to \mathcal{E}$ the full subcategory consisting of those X in $\mathcal{E}$ such that $p_*X \to p_!X$ is an iso. Roughly speaking, the objects of $\mathcal{L}$ are those such that every piece has exactly one point. Objects in $\mathcal{L}$ are called *Leibniz spaces* in [10] where it is also suggested that these objects 'look like clouds of Leibnizian monads'. Roughly speaking each connected component in a Leibniz space consists of exactly one point together with some 'infinitesimals' around it. The following is a strengthening of Theorem 2 in [8].

Theorem 6.4. *The category $\mathcal{L}$ is a topos and the inclusion $s^* : \mathcal{L} \to \mathcal{E}$ is the inverse image of an essential geometric morphism.*

Proof. As suggested above, most of this result is proved in Theorem 2 in [8]. Observe that the Continuity condition is not required for the construction of the left adjoint $s_! : \mathcal{E} \to \mathcal{L}$ suggested in the last sentence of the proof there. On the other hand, the existence of the right adjoint to s^* rests on the assumption that $\mathcal{E}$ is has enough small limits. So, to complete the proof of the present result, it is enough to exhibit an elementary construction of the direct image $s_* : \mathcal{E} \to \mathcal{S}$. We leave it to the reader to prove that the top map in the following pullback

$$\begin{array}{ccc} s^*(s_*X) & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \eta} \\ p^*(p_*X) & \xrightarrow[\phi]{} & p^!(p_*X) \end{array}$$

is the counit of $s^* \dashv s_*$, where $\eta : X \to p^!(p_*X)$ is the unit of $p_* \dashv p^!$ and $\phi : p^* \to p^!$ is the canonical natural transformation from discrete to codiscrete. (Details and examples if this construction are the topic of joint work with F. Marmolejo to appear elsewhere.) □

Lawvere calls $s_* : \mathcal{E} \to \mathcal{L}$ the *canonical intensive quality* (of p). He says that $\mathcal{E}$ is *infinitesimally generated* if $\mathcal{E}$ is weakly generated by $s : \mathcal{E} \to \mathcal{L}$. See Proposition 6 in [8].

Corollary 6.5. *Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive and $\mathcal{S}$ be Boolean. If p is sufficiently cohesive then $\mathcal{E}$ is infinitesimally generated.*

Proof. As $\top : 1 \to J$ is dense, $p_*J = 1$ and, since $\theta_J : p_*J \to p_!J$ is epi, it is an iso. In other words, J is in $\mathcal{L}$. So the result follows from Corollary 6.3. □

Using the results in [11] we may conclude that if $p : \mathcal{E} \to \mathcal{S}$ is cohesive and sufficiently cohesive then $\mathcal{E}$ is infinitesimally generated. In contrast, notice that if $p : \mathcal{E} \to \mathcal{S}$ is a quality type then $s^* : \mathcal{L} \to \mathcal{E}$ is an equivalence so, in this case, $\mathcal{E}$ is trivially infinitesimally generated.

7. A characterization of substantial objects in toposes

In this section we present a characterization of substantial objects in toposes proposed by an anonymous referee, together with some applications. Let $\mathcal{E}$ be a topos.

Proposition 7.1. *If J is a well-supported object in $\mathcal{E}$ then the following are equivalent:*

1. *J is substantial.*

2. *For every subterminal U, if $\pi_0 : U \times J \to U$ is an isomorphism then U is initial.*

3. *The only open subtopos $f : \mathcal{F} \to \mathcal{E}$ such that $f^* J = 1$ is the degenerate one.*

Proof. To prove that the first two items are equivalent let the following squares be pullbacks in $\mathcal{E}$

$$\begin{tikzcd} Y \times J \arrow[r] \arrow[d, "\pi_0"'] & U \times J \arrow[r] \arrow[d, "\pi_0"'] & J \arrow[d, "!"] \\ Y \arrow[r] & U \arrow[r] & 1 \end{tikzcd}$$

where the bottom line is the epi/mono factorization of the unique $Y \to 1$. Since $\mathcal{E}$ is regular as a category, $\pi_0 : Y \times J \to Y$ is an isomorphism if and only if $\pi_0 : U \times J \to U$ is.

To prove that the second and third items are equivalent let $U \to 1$ be monic in $\mathcal{E}$ and let $f : \mathcal{E}/U \to \mathcal{E}/1 = \mathcal{E}$ be the induced open subtopos. Then $f^* J = 1$ if and only if $\pi_0 : U \times J \to U$ is an isomorphism. □

The following variant is also worth noting.

Corollary 7.2. *If $\top : 1 \to J$ is a pointed object in $\mathcal{E}$ then, J is substantial if and only if the only open subtopos $f : \mathcal{F} \to \mathcal{E}$ such that $f^*\top$ is an iso is the degenerate one.*

Proof. By Proposition 7.1 and the fact that $f^* J = 1$ if and only if $f^*\top$ is an iso. □

The referee also suggested that there may be a worthwhile connection with the following concept introduced in [2]: a monomorphism $m : U \to X$ in $\mathcal{E}$ is *strict* if the only subtopos $f : \mathcal{F} \to \mathcal{E}$ such that $f^* m$ is an isomorphism is the degenerate one. It is then obvious, by Corollary 7.2, that if $\top : 1 \to J$ is strict in the sense of Jibladze then J is substantial. Also, it is not difficult to prove that if J has a point disjoint from $\top : 1 \to J$ then $\top$ is strict; giving an alternative proof of Lemma 2.3 in the case that the underlying category is a topos.

The following is also due to the referee.

Corollary 7.3. *A topos $\mathcal{E}$ is perfect if and only if the only open Boolean subtopos of $\mathcal{E}$ is the degenerate one.*

Proof. Let $\top : 1 \to J$ be the classifier of $\neg\neg$-dense monos. By definition, $\mathcal{E}$ is perfect if and only if J is substantial. In turn, this is holds if and only if, the only open subtopos $\mathcal{E}/U \to \mathcal{E}$ such that $U^*J = 1$ is degenerate (Proposition 7.1). Since U^*J is the classifier of $\neg\neg$-dense monos in $\mathcal{E}/U$, $U^*J = 1$ implies that $\mathcal{E}/U$ is Boolean. □

Corollary 7.4. *If a topos $\mathcal{E}$ is 2-valued then, $\mathcal{E}$ is perfect if and only if it is not Boolean.*

Corollary 7.4 points at the following.

Corollary 7.5. *Let $\mathcal{S}$ be a 2-valued topos and let $p : \mathcal{E} \to \mathcal{S}$ be a hyperconnected geometric morphism. Then $\mathcal{E}$ is perfect if and only if it is not Boolean.*

Proof. Since p is hyperconnected the induced $p^* : \mathrm{Sub}_{\mathcal{S}}(1) \to \mathrm{Sub}_{\mathcal{E}}(1)$ is an isomorphism. □

In particular, Corollary 7.4 gives another proof that the topological topos [3] is perfect. Compare with the paragraph following Proposition 5.4.

I characterized the perfect presheaf toposes but we give here an improved statement and the proof suggested by the referee.

Proposition 7.6. *For any small category $\mathcal{C}$, $\widehat{\mathcal{C}}$ is perfect if and only if every object is the codomain of a non-invertible map.*

Proof. An object C in $\mathcal{C}$ will be called *strict* if every map with codomain C is an iso. Let $\mathcal{C}_0 \to \mathcal{C}$ be the full subcategory determined by the strict objects. It is clearly a sieve in $\mathcal{C}$ and so, by Example A4.5.2 in [5], the inclusion $\mathcal{C}_0 \to \mathcal{C}$ determines an open subtopos $\widehat{\mathcal{C}_0} \to \widehat{\mathcal{C}}$. Moreover, since $\mathcal{C}_0$ is a groupoid, $\widehat{\mathcal{C}_0}$ is Boolean by A1.4.2 in [5]. □

For instance, if $\mathcal{C}$ has a strict initial object then $\widehat{\mathcal{C}}$ is not perfect. On the other hand, notice that the hypotheses of the next result simply require that the canonical geometric morphism $\widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive [6, 13].

Corollary 7.7. *Assume that $\mathcal{C}$ has a terminal object and that every object has a point. Then $\widehat{\mathcal{C}}$ is perfect if and only if $\mathcal{C}$ has a non terminal object.*

In other words, for p as in Corollary 7.7, the classifier of dense monos determined by the subtopos $p_* \dashv p^! : \mathbf{Set} \to \widehat{\mathcal{C}}$ is substantial if and only if p is not an equivalence.

Also, if $\mathcal{C}$ has a terminal object, every object of $\mathcal{C}$ has a point and some object of $\mathcal{C}$ has (at least) two points then $\widehat{\mathcal{C}}$ is perfect. So, if the pre-cohesive $\widehat{\mathcal{C}} \to \mathbf{Set}$ is Sufficiently Cohesive [14], then $\widehat{\mathcal{C}}$ is weakly generated by the classifier of $\neg\neg$-dense subobjects. This is the presheaf case of Theorem 6.2.

As a further by-product of the characterization of substantial objects in toposes, we characterize perfect spatial toposes.

Corollary 7.8. *For any spatial locale X, $\mathrm{Sh}(X)$ is perfect if and only if X has no isolated points.*

Proof. Let U be open in the space X and consider the associated open subtopos $\mathrm{Sh}(U) \cong \mathrm{Sh}(X)/U \to \mathrm{Sh}(X)$. It is well-known that $\mathrm{Sh}(U)$ is Boolean if and only if U is discrete (C3.5.3 in [5]). So $\mathrm{Sh}(U) \to \mathrm{Sh}(X)$ is degenerate if and only if U is empty. $\square$

Johnstone observes in Section 3.6 of [4] that there exists a largest open Boolean subtopos of $\mathcal{E}$. The associated subterminal may be defined as the interior of the $\neg\neg$-topology. He calls it the *Boolean core* of $\mathcal{E}$. It follows that $\mathcal{E}$ is perfect if and only if its Boolean core is degenerate. Explicit calculations of Boolean cores would make this observation immediately applicable.

It seems also relevant to recall that $\mathcal{E}$ is *scattered* if the subtopos $\mathcal{E}_{\neg\neg} \to \mathcal{E}$ is open (see [1]). Corollary 7.3 makes it clear that scattered and perfect are opposite concepts in the sense that: $\mathcal{E}$ is scattered and perfect if and only if $\mathcal{E}$ is degenerate.

The concepts of substantial object (in a topos), of strict mono, and of weak generation are all of the same form: a certain naturally defined class of subtoposes is actually trivial in some sense (i.e. collapses to the degenerate subtopos or collapses to the whole subtopos). Moreover, one of our main results says roughly that if an object is substantial then it weakly generates. In other words, if certain class of subtoposes is trivial in one sense then a different class of subtoposes is trivial in the opposite sense. Perhaps there is room for a more general treatment of these ideas by considering subobjects of Ω^Ω (alternatively, maps $\Omega^\Omega \to \Omega$) and studying what happens when certain such subobjects reduce to certain special points.

8. An explicit retraction $J^J \to \Omega$

Let $\mathcal{E}$ be a topos. Define $J \to \Omega$ by declaring that the square on the left below

$$\begin{tikzcd} J \arrow[r] \arrow[d, "\mathrm{j}"'] & 1 \arrow[d, "\top"] \\ \Omega \arrow[r, "\neg\neg"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & J \arrow[d, "\mathrm{j}"] \\ & \Omega \end{tikzcd}$$

is a pullback. As before we let $\top : 1 \to J$ be the unique map such that the triangle on the right above commutes. Also, let us denote the transposition of the identity by $\mathit{1} : 1 \to J^J$ and let $\rho : J^J \to \Omega$ be the unique map such that the following diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\mathit{1}"'] & 1 \arrow[d, "\top"] \\ J^J \arrow[r, "\rho"'] & \Omega \end{tikzcd}$$

is a pullback. We show below that, if J is substantial, then ρ is a retraction for the map $\iota : \Omega \to J^J$ defined in Proposition 3.2.

Lemma 8.1. *The diagram below*

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\mathit{1}"'] & \Omega \arrow[d, "\iota"] \\ & J^J \end{tikzcd}$$

commutes.

Proof. Simply transpose and calculate the associated dense mono. In more detail, the diagram below

$$\begin{tikzcd} 1 \times 1 \arrow[r, "in_0"] \arrow[d, "id \times \top"'] & (1 \times 1) + (1 \times J) \arrow[rr, "!"] \arrow[d, "{[\top \times \top, \bot \times J]}"] & & 1 \arrow[d, "\top"] \\ 1 \times J \arrow[r, "\top \times J"'] & \Omega \times J \arrow[r, "\iota \times J"] \arrow[rr, bend right, "\chi"'] & J^J \times J \arrow[r, "ev"] & J \end{tikzcd}$$

is a pullback. Indeed, the rectangle on the right is a pullback by the definition of ι (see Proposition 2.6). To show that the square on the left is a pullback it is enough to calculate the pullbacks below

$$\begin{tikzcd}
1\times 1 \arrow[r, "!"] \arrow[d, "id\times\top"'] & 1\times 1 \arrow[d, "\top\times\top"] \\
1\times J \arrow[r, "\top\times J"'] & \Omega\times J
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, "!"'] & 1\times J \arrow[d, "\bot\times J"] \\
1\times J \arrow[r, "\top\times J"'] & \Omega\times J
\end{tikzcd}$$

so the bottom composite of the rectangle in the beginning of the proof must be the projection $1 \times J \to J$. $\square$

We can now prove the promised result.

Proposition 8.2. *If J is substantial then $\rho : J^J \to \Omega$ is a retraction for $\iota : \Omega \to J^J$.*

Proof. If J is substantial then $\iota : \Omega \to J^J$ is mono. Consider now the proof of injectivity of Ω in Proposition IV.10.1 in [12]. In order to extend the horizontal map below

$$\begin{tikzcd}
\Omega \arrow[r, "id"] \arrow[d, "\iota"'] & \Omega \\
J^J &
\end{tikzcd}$$

along the vertical one, one must proceed as follows. Calculate the subobject of Ω classified by the top map (which is $\top : 1 \to \Omega$) and compose it with the vertical map to obtain the subobject $\iota\top : 1 \to J^J$. Its classifying map $J^J \to \Omega$ is the desired extension. By Lemma 8.1, the composite subobject is $\mathit{1} : 1 \to J^J$, so the extension is $\rho : J^J \to \Omega$. $\square$

By Lemma 2.5 the rectangle below

$$\begin{tikzcd}
(1\times 1)+(K\times J) \arrow[r] \arrow[d, "{[\mathit{1}\times\top, k\times J]}"'] & (1\times 1)+(1\times J) \arrow[r] \arrow[d, "{[\top\times\top, \bot\times J]}"] & 1 \arrow[d, "\top"] \\
J^J\times J \arrow[r, "\rho\times J"'] & \Omega\times J \arrow[r, "\chi"'] & J
\end{tikzcd}$$

is a pullback, where $k : K \to J^J$ is the Heyting complement of $\mathit{1} : 1 \to J^J$. So we could have defined the endomorphism $e = \iota\rho : J^J \to J^J$ directly as the transposition of the classifying morphism $J^J \times J \to J$ of the subobject $[\mathit{1} \times \top, k \times J] : (1 \times 1) + (K \times J) \to J^J \times J$. The discussion above implies that if J is substantial then e is idempotent. On the other hand, e may have some significance in a broader context.

The broader significance of the monoid J^J and its submonoid of Euler reals [9] will have to be studied elsewhere. It is suggestive that the object J is the $\neg\neg$-closure of a point in a rig, just as the object of 'infinitesimals' considered in [15].

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