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THE SERRE AUTOMORPHISM VIA HOMOTOPY ACTIONS AND THE COBORDISM HYPOTHESIS FOR ORIENTED MANIFOLDS

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Résumé. Nous construisons explicitement une action de $SO(2)$ sur une version squelettique de la bicatégorie des cobordismes à bords à deux dimensions. Par l'hypothèse du cobordisme bidimensionnel pour les variétés à bords, nous obtenons une action de $SO(2)$ sur le noyau des objets complètement dualisables de la bicatégorie cible. Cette action coïncide avec celle donnée par l'automorphisme de Serre. Nous donnons une description explicite de la bicatégorie des points fixes homotopiques de cette action, et discutons de sa relation avec la classification des théories quantiques des champs topologiques en 2 dimensions.

Abstract. We explicitly construct an $SO(2)$-action on a skeletal version of the 2-dimensional framed bordism bicategory. By the 2-dimensional Cobordism Hypothesis for framed manifolds, we obtain an $SO(2)$-action on the core of fully-dualizable objects of the target bicategory. This action is shown to coincide with the one given by the Serre automorphism. We give an explicit description of the bicategory of homotopy fixed points of this action, and discuss its relation to the classification of oriented 2d topological quantum field theories.

Keywords. Serre Automorphism, Cobordism Hypothesis, Topological Quantum Field Theory, Group actions, Bicategory, Homotopy Fixed Point.
Mathematics Subject Classification (2010). 81T45

1. Introduction

As defined by Atiyah in [Ati88] and Segal in [Seg04], an n-dimensional Topological Quantum Field Theory (TQFT) consists of a functor between two symmetric monoidal categories, namely a category of n-cobordisms, and a category of algebraic objects. This definition was introduced to axiomatize the locality properties of the path integral, and has given rise to a fruitful interplay between mathematics and physics in the last 30 years. A prominent example is given by a quantum-field-theoretic interpretation of the Jones polynomial by Witten in [Wit89].

More recently, there has been a renewed interest in the study of TQFTs, due in great part to the Baez-Dolan Cobordism Hypothesis and its proof by Lurie, whose main objects of investigation are *fully extended* TQFTs. These are a generalization of the notion of n-dimensional TQFTs, where data is assigned to manifolds of codimension up to n. The Baez-Dolan Cobordism Hypothesis, originally stated in [BD95], and proved by Lurie in [Lur09] in an ∞-categorical version, can be stated as follows: fully extended *framed* TQFTs are classified by their value on a point, which must be a fully dualizable object in the target symmetric monoidal (∞, n)-category $\mathcal{C}$. Moreover, the ∞-groupoid $\mathscr{K}(\mathcal{C}^{\text{fd}})$ given by the core of fully dualizable objects of $\mathcal{C}$ carries a homotopy $O(n)$-action induced by the "rotation of the framing" on the framed (∞, n)-cobordism category [Lur09, Corollary 2.4.10]. The inclusion $SO(n) \hookrightarrow O(n)$ then induces an $SO(n)$-action on $\mathscr{K}(\mathcal{C}^{\text{fd}})$. By the Cobordism Hyothesis for manifolds whose tangent bundle is equipped with an additional G-structure, homotopy fixed-points for this action classify fully extended *oriented* TQFTs. It is relevant to notice that in [Lur09] the homotopy $O(n)$-action on the framed (∞, n)-category of cobordisms is not explicitly constructed, or even briefly sketched. For an extensive introduction to extended TQFTs and the Cobordism Hypothesis, we refer the reader to [Fre12].

Blurring the distinction between $(\infty, 2)$-categories and bicategories, in [FHLT10] it is argued that in the case where the target is given by the bicategory Alg_2 of algebras, bimodules, and intertwiners, the fully dualizable objects are semisimple finite-dimensional algebras, and that the additional $SO(2)$-fixed-points structure should correspond to the structure of a symmetric Frobenius algebra. Via a direct construction, in [SP09] it is showed that the bigroupoid Frob of Frobenius algebras, Morita contexts and intertwiners indeed classifies fully extended oriented 2-dimensional TQFTs valued in Alg_2. In [Dav11], it is observed that the $SO(2)$-action given by the Serre automorphism on the core of fully-dualizable objects of Alg_2 is trivializable. In a purely bicategorical setting, in [HSV17] the homotopy-fixed-point bigroupoid of the $SO(2)$-action on Alg_2 is computed, and it is shown that it coincides with Frob.

In the present paper we provide an explicit $SO(2)$-action on the framed bordism bicategory, and show that the $SO(2)$-action induced on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ for any symmetric monoidal bicategory $\mathcal{C}$ is given by the Serre automorphism, regarded as a pseudo-natural isomorphism of the identity functor. More precisely, we make use of a presentation of the framed bordism bicategory provided in [Pst14] to construct such an $SO(2)$-action.

By the Cobordism Hypothesis for framed manifolds, which has been proven in the setting of bicategories in [Pst14], there is an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}}). \tag{1}$$

This equivalence allows us to transport the $SO(2)$-action on the framed bordism bicategory to the core of fully-dualizable objects of $\mathcal{C}$. We then prove that this induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is given precisely by the Serre automorphism, showing that the Serre automorphism has indeed a geometric origin, as expected from [Lur09].

Along the way, we also provide results concerning monoidal homotopy actions which are useful in determining when such actions are trivializable. The relevance for TQFT is the following: in the case of a trivializable $SO(2)$-action, *any* framed fully extended 2d TQFT can be promoted to an oriented one by providing the appropriate structure of a homotopy fixed point. In particular, we apply these results to the case of invertible 2d TQFTs, which have recently attracted interest for their application to condensed matter physics, more specifically to the study of topological insulators [Fre14a, Fre14b, FH16]. Namely, fully extended invertible TQFTs have been proposed as the low energy limit of short-range entanglement systems; see [Fre14b] for a discussion of these topics.

First defintions of monoidal bicategories appear in [KV94], [BN96] and [DS97], with a first full definition of a symmetric monoidal bicategory in [McC00]. We will refer to [SP09] for technical details. In section 5, we use the wire-diagram calculus developed in [Bar14].

It is worth noticing that the study of actions of groups on higher categories and their homotopy fixed points is also of independent interest, see for instance [EGNO15, BGM17] for the case of finite groups.

The paper is organized as follows.
In Section 2 we recall the notion of a fully-dualizable object in a symmetric monoidal bicategory $\mathcal{C}$. For each such an object X, we define the Serre automorphism as a certain 1-endomorphism of X. We show that the Serre automorphism is a pseudo-natural transformation of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which is moreover monoidal. This suffices to define an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

Section 3 investigates when a group action on a bicategory $\mathcal{C}$ is equivalent to the trivial action. We obtain a general criterion for when such an action is trivializable.

In Section 4, we compute the bicategory of homotopy fixed points of an $SO(2)$-action coming from a pseudo-natural transformation of the identity functor of an arbitrary bicategory $\mathcal{C}$. This generalizes the main result in [HSV17], which computes homotopy fixed points of the trivial $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$. Our more general theorem allows us to give an explicit description of the bicategory of homotopy fixed points of the Serre automorphism.

In Section 5, we introduce a skeletal version of the framed bordism bicategory by generators and relations, and define a non-trivial $SO(2)$-action on this bicategory. By the framed Cobordism Hypothesis, as in Equation (1), we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we prove to coincide with the one given by the Serre automorphism.

In Section 6 we discuss invertible 2d TQFTs, providing a general criterion for the trivialization of the $SO(2)$-action in this case.

In Section 7, we give an outlook on *homotopy co-invariants* of the $SO(2)$-action, and argue about their relation to the Cobordism Hypothesis for oriented manifolds.

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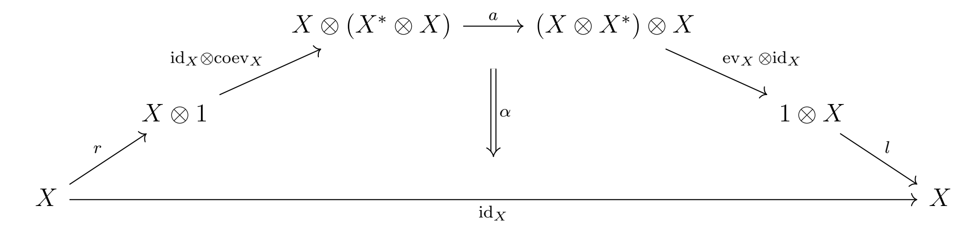
2. Fully-dualizable objects and the Serre automorphism

The aim of this section is to introduce the main objects of the present paper. On the algebraic side, these are fully-dualizable objects in a symmetric monoidal bicategory $\mathcal{C}$, and the Serre automorphism. Though some of the following material has already appeared in the literature, we recall the relevant definitions in order to fix notation. For details, we refer the reader to [Pst14].

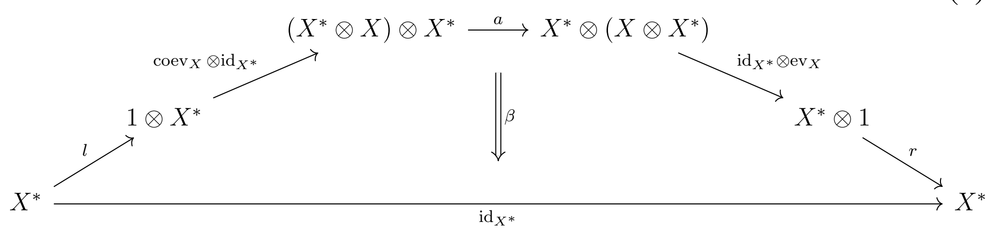
Definition 2.1. *A dual pair in a symmetric monoidal bicategory $\mathcal{C}$ consists of an object X, an object X^*, two 1-morphisms*

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to 1 \\ \mathrm{coev}_X &: 1 \to X^* \otimes X \end{aligned} \tag{2}$$

and two invertible 2-morphisms α and β in the diagrams below.



(3)



(4)

We call an object X of $\mathcal{C}$ dualizable if it can be completed to a dual pair. A dual pair is said to be coherent if the "swallowtail" equations are satisfied, as in [Pst14, Def. 2.6].

Remark 2.2. Given a dual pair, it is always possible to modify the 2-cell β in such a way that the swallowtail are fulfilled, cf. [Pst14, Theorem 2.7].

Dual pairs can be organized into a bicategory by defining appropriate 1- and 2-morphisms between them, cf. [Pst14, Section 2.1]. The bicategory of dual pairs turns out to be a 2-groupoid. Moreover, the bicategory of coherent dual pairs is equivalent to the core of dualizable objects in $\mathcal{C}$. In particular, this shows that any two coherent dual pairs over the same dualizable object are equivalent.

We now come to the stronger concept of fully-dualizability.

Definition 2.3. *An object X in a symmetric monoidal bicategory is called fully-dualizable if it can be completed into a dual pair and the evaluation and coevaluation maps admit both left- and right adjoints.*

Note that if left- and right adjoints exists, the adjoint maps will have adjoints themselves, since we work in a bicategorical setting, cf. [Pst14]. Note that if left- and right adjoints for the 1-morphisms ev and coev exist, these adjoint 1-morphisms will in turn have additional adjoints themselves. Thus, Definition 2.3 agrees with the definition of [Lur09] in the special case of bicategories.

2.1 The Serre automorphism

Recall that by definition, the evaluation morphism for a fully dualizable object X admits both a right-adjoint ev_X^R and a left adjoint ev_X^L. We use these adjoints to define the Serre-automorphism of X:

Definition 2.4. *Let X be a fully-dualizable object in a symmetric monoidal bicategory. The Serre automorphism of X is the following composition of 1-morphisms:*

$$S_X : X \cong X \otimes 1 \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X^R} X \otimes X \otimes X^* \xrightarrow{\tau_{X,X} \otimes \mathrm{id}_{X^*}} X \otimes X \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes 1 \cong X. \tag{5}$$

Notice that the Serre automorphism is actually a 1-equivalence of X, since an inverse is given by the 1-morphism

$$S_X^{-1} = (\mathrm{id}_X \circ \mathrm{ev}_X) \circ (\tau_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes \mathrm{ev}_X^L), \tag{6}$$

cf. [Lur09, DSS13].

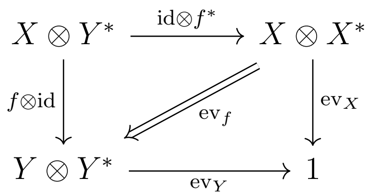
The next lemma is well-known [Lur09, Pst14], and is straightforward to show graphically.

Lemma 2.5. *Let X be fully-dualizable in $\mathcal{C}$. Then, there are 2-isomorphisms*

$$\begin{aligned} \mathrm{ev}_X^R &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X) \circ \mathrm{coev}_X \\ \mathrm{ev}_X^L &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X^{-1}) \circ \mathrm{coev}_X\,. \end{aligned} \tag{7}$$

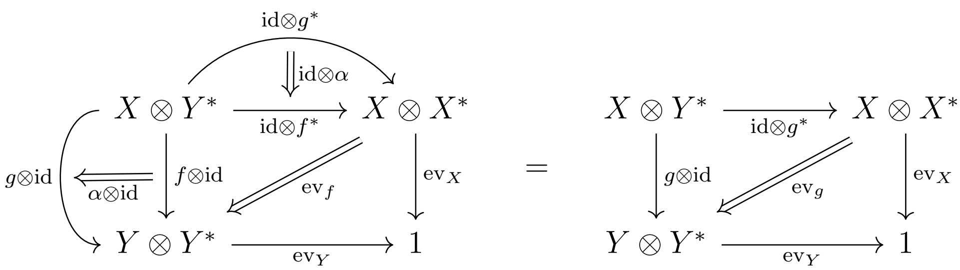
Next, we show that the Serre automorphism is actually a pseudo-natural transformation of the identity functor on the maximal subgroupoid of $\mathcal{C}$, as suggested in [Sch13]. To the best of our knowledge, a proof of this statement has not appeared in the literature so far, hence we illustrate the details in the following. We begin by showing that the evaluation 1-morphism is "dinatural".

Lemma 2.6. *Let X be dualizable in $\mathcal{C}$. The evaluation 1-morphism ev_X is "dinatural": for every 1-morphism $f : X \to Y$ between dualizable objects, there is a natural 2-isomorphism ev_f in the diagram below.*



(8)

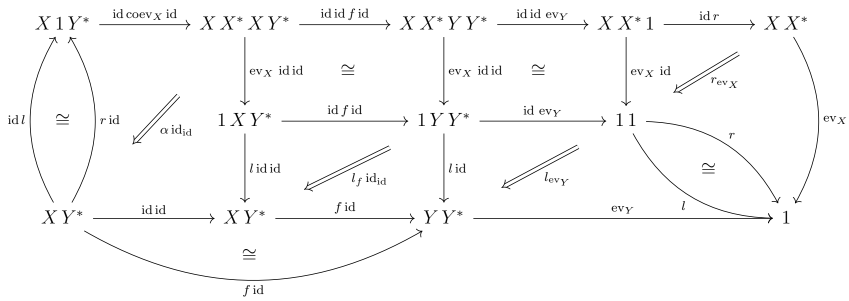
By "di-naturality", we explicitly mean that for every 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$, the following diagram commutes



(9)

Proof. We explicitly write out the definition of f^* and define ev_f to be the composition of the 2-morphisms in the diagram below.

$\mathrm{ev}_f :=$



(10)

Since the 2-morphism ev_f is given by the composition of associators and unitors which are natural 2-morphisms, it is natural itself, and thus the diagram in equation 9 commutes. □

In order to show that the Serre automorphism is pseudo-natural, we also need to show the dinaturality of the right adjoint of the evaluation.

Lemma 2.7. *For a fully-dualizable object X of $\mathcal{C}$, the right adjoint ev^R of the evaluation is "dinatural" with respect to 1-equivalences: for every 1-equivalence $f : X \to Y$ between fully-dualizable objects, there is a natural 2-isomorphism ev_f^R in the diagram below.*

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{ev}_X^R} & X \otimes X^* \\ {\scriptstyle \mathrm{ev}_Y^R}\big\downarrow & \swarrow\!\!\!\swarrow \ \mathrm{ev}_f^R & \big\downarrow{\scriptstyle f\otimes \mathrm{id}} \\ Y \otimes Y^* & \xrightarrow[\mathrm{id}\otimes f^*]{} & Y \otimes X^* \end{array} \tag{11}$$

Proof. In a first step, we show that $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is a right-adjoint to $\mathrm{ev}_X \circ (f^{-1} \otimes f^*)$. In formula:

$$(\mathrm{ev}_X \circ f^{-1} \otimes f^*)^R = f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R\,. \tag{12}$$

Indeed, let

$$\begin{aligned} \eta_X &: \mathrm{id}_{X\otimes X^*} \to \mathrm{ev}_X^R \circ \mathrm{ev}_X \\ \varepsilon_X &: \mathrm{ev}_X \circ \mathrm{ev}_X^R \to \mathrm{id}_1 \end{aligned} \tag{13}$$

be the unit and counit of the right-adjunction of ev_X and its right adjoint ev_X^R. We construct unit and counit for the adjunction in Equation (12). Let

$$\begin{aligned}
&\tilde{\varepsilon} : \mathrm{ev}_X \circ (f^{-1} \otimes f^*) \circ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \cong \mathrm{ev}_X \circ \mathrm{ev}_X^R \xrightarrow{\varepsilon_X} \mathrm{id}_1 \\
&\tilde{\eta} : \mathrm{id}_{Y \otimes Y^*} \cong (f \otimes (f^*)^{-1}) \circ (f^{-1} \otimes f^*) \\
&\xrightarrow{\mathrm{id} * \eta_X * \mathrm{id}} (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \circ \mathrm{ev}_X \circ (f^{-1} \otimes f^*).
\end{aligned} \tag{14}$$

Now, one checks that the quadruple

$$(\mathrm{ev}_X \circ (f^{-1} \otimes f^*),\ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R,\ \tilde{\varepsilon}, \tilde{\eta}) \tag{15}$$

fulfills indeed the axioms of an adjunction. This follows from the fact that the quadruple $(\mathrm{ev}_X, \mathrm{ev}_X^R, \varepsilon_X, \eta_X)$ is an adjunction. This shows Equation (12).

Now, notice that due to the dinaturality of the evaluation in Lemma 2.6, we have a natural 2-isomorphism

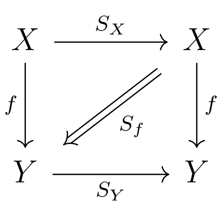
$$\mathrm{ev}_Y \cong \mathrm{ev}_X \circ (f^{-1} \otimes f^*). \tag{16}$$

Combining this 2-isomorphism with Equation (12) shows that the right adjoint of ev_Y is given by $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$. Since all right-adjoints are isomorphic the 1-morphism $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is isomorphic to ev_Y^R, as desired. □

We can now prove the following proposition.

Proposition 2.8. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Denote by $\mathscr{K}(\mathcal{C})$ the maximal sub-bigroupoid of $\mathcal{C}$. The Serre automorphism S is a pseudo-natural isomorphism of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.*

Proof. Let $f : X \to Y$ be a 1-morphism in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We need to provide a natural 2-isomorphism in the diagram



(17)

By spelling out the definition of the Serre automorphism, we see that this is equivalent to filling the following diagram with natural 2-cells:

$$\begin{array}{ccccccccccc} X & \longrightarrow & X\,1 & \xrightarrow{\mathrm{id}_X\ \mathrm{ev}_X^R} & X\,X\,X^* & \xrightarrow{\tau_{X,X}\ \mathrm{id}_{X^*}} & X\,X\,X^* & \xrightarrow{\mathrm{id}_X\ \mathrm{ev}_X} & X\,1 & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\,\mathrm{id}} & & \downarrow{\scriptstyle f\,f\,(f^*)^{-1}} & & \downarrow{\scriptstyle f\,f\,(f^*)^{-1}} & & \downarrow{\scriptstyle f\,\mathrm{id}} & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & Y\,1 & \xrightarrow[\mathrm{id}_Y\ \mathrm{ev}_Y^R]{} & Y\,Y\,Y^* & \xrightarrow[\tau_{Y,Y}\ \mathrm{id}_{Y^*}]{} & Y\,Y\,Y^* & \xrightarrow[\mathrm{id}_Y\ \mathrm{ev}_Y]{} & Y\,1 & \longrightarrow & Y \end{array} \tag{18}$$

The first, the last and the middle square can be filled with a natural 2-cell due to the fact that $\mathcal{C}$ is a symmetric monoidal bicategory. The square involving the evaluation commutes up to a 2-cell using the mate of the 2-cell of Lemma 2.6, while the square involving the right adjoint of the evaluation commutes up a 2-cell using the mate of the 2-cell of Lemma 2.7. □

2.2 Monoidality of the Serre automorphism

In this section we show that the Serre automorphism respects the monoidal structure. We will show that the Serre-automorphism is a *monoidal* pseudo-natural transformation of the identity functor. We begin with the following two lemmas:

Lemma 2.9. *Let $\mathcal{C}$ be a monoidal bicategory. Let X and Y be dualizable objects of $\mathcal{C}$. Then, there is a 1-equivalence $\xi_{X,Y} : (X \otimes Y)^* \cong Y^* \otimes X^*$. Furthermore, this 1-equivalence ξ is pseudo-natural: suppose that $f : X \to X'$ and $g : Y \to Y'$ are two 1-morphisms in $\mathcal{C}$. Then, there is a pseudo-natural 2-isomorphism in the diagram in equation* (19).

$$\begin{array}{ccc} (X \otimes Y)^* & \xrightarrow{\xi_{X,Y}} & Y^* \otimes X^* \\ {\scriptstyle (f\otimes g)^*}\uparrow & \swArrow{\scriptstyle \xi_{f,g}} & \uparrow{\scriptstyle g^*\otimes f^*} \\ (X' \otimes Y')^* & \xrightarrow[\xi_{X',Y'}]{} & Y'^* \otimes X'^* \end{array} \tag{19}$$

Proof. Define a 1-morphism $(X \otimes Y)^* \to Y^* \otimes X^*$ in $\mathcal{C}$ by the composition

$$(\mathrm{id}_{Y^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{ev}_{X\otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) \tag{20}$$

and define another 1-morphism $Y^* \otimes X^* \to (X \otimes Y)^*$ in $\mathcal{C}$ by the composition

$$(\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}^*_Y \otimes \mathrm{id}_{X^*}). \tag{21}$$

These two 1-morphisms are (up to invertible 2-cells) inverse to each other. This shows the first claim. The existence and the pseudo-naturality of the 2-isomorphism $\xi_{f,g}$ now follows from the definition of ξ and lemma 2.6. □

Now, we show that the evaluation 1-morphism respects the monoidal structure:

Lemma 2.10. *For a dualizable object X of a symmetric monoidal bicategory $\mathcal{C}$, the evaluation 1-morphism ev_X is a* monoidal *pseudo-dinatural transformation: namely, the following diagram commutes up to 2-isomorphism.*

$$\begin{array}{ccccc} (X\otimes Y)\otimes (X\otimes Y)^* & & \xrightarrow{\quad \mathrm{ev}_{X\otimes Y} \quad} & & 1 \\ \downarrow{\scriptstyle \mathrm{id}_{X\otimes Y}\otimes \xi} & & & & \downarrow \\ (X\otimes Y)\otimes Y^*\otimes X^* & \xrightarrow{\mathrm{id}_X\otimes \tau_{Y\otimes Y^*,X^*}} & X\otimes X^*\otimes Y\otimes Y^* & \xrightarrow{\mathrm{ev}_X\otimes \mathrm{ev}_Y} & 1\otimes 1 \end{array} \tag{22}$$

Here, the 1-equivalence ξ is due to Lemma 2.9.

Proof. Let us construct a 2-isomorphism in the diagram in Equation (22). Consider the diagram in figure 1 on page 41: here, the composition of the horizontal arrows at the top, together with the two arrows on the vertical right are exactly the 1-morphism in Equation (22). The other arrow is given by $\mathrm{ev}_{X\otimes Y}$. We have not written down the tensor product, and left out isomorphisms of the form $1 \otimes X \cong X \cong X \otimes 1$ for readability. □

We can now establish the monoidality of the right adjoint of the evaluation via the following lemma.

Lemma 2.11. *Let $\mathcal{C}$ a symmetric monoidal bicategory, and let X and Y be fully-dualizable objects. Then, the right adjoint of the evaluation is monoidal. More precisely: if $\xi : (X \otimes Y)^* \to Y^* \otimes X^*$ is the 1-equivalence of Lemma*

2.9, the following diagram commutes up to 2-isomorphism.

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{ev}^R_{X\otimes Y}} & X\otimes Y\otimes (X\otimes Y)^* \\ {\scriptstyle \mathrm{ev}^R_X\otimes \mathrm{ev}^R_Y}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X\otimes Y}\otimes\xi} \\ X\otimes X^*\otimes Y\otimes Y^* & \xrightarrow[\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*}]{} & X\otimes Y\otimes Y^*\otimes X^* \end{array} \tag{23}$$

Proof. In a first step, we show that the right adjoint of the 1-morphism

$$(\mathrm{ev}_X\otimes \mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \tag{24}$$

is given by the 1-morphism

$$(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\circ\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y). \tag{25}$$

Indeed, if

$$\begin{aligned} \eta_X &: \mathrm{id}_{X\otimes X^*}\to \mathrm{ev}^R_X\circ\mathrm{ev}_X \\ \varepsilon_X &: \mathrm{ev}_X\circ\mathrm{ev}^R_X\to\mathrm{id}_1 \end{aligned} \tag{26}$$

are the unit and counit of the right-adjunction of ev_X and its right adjoint ev^R_X, we construct adjunction data for the adjunction in equations (24) and (25) as follows. Let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be the following 2-morphisms:

$$\begin{aligned} \tilde{\varepsilon} : &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi)\circ(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1}) \\ &\qquad\qquad\qquad\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ &\cong(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ &\cong(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y)\xrightarrow{\varepsilon_X\otimes\varepsilon_Y}\mathrm{id}_1 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tilde{\eta} :&\mathrm{id}_{X\otimes Y\otimes(X\otimes Y)^*}\cong(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \\ &\cong(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \\ &\xrightarrow{\mathrm{id}\otimes\eta_X\otimes\eta_Y\otimes\mathrm{id}}(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ &\qquad\qquad\circ(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \end{aligned} \tag{28}$$

One now shows that the two 1-morphisms in Equation (24) and (25), together with the two 2-morphisms $\tilde{\varepsilon}$ and $\tilde{\eta}$ form an adjunction. This gives that the two 1-morphisms in Equations (24) and (25) are adjoint.

Next, notice that the 1-morphism in Equation (24) is isomorphic to the 1-morphism $\mathrm{ev}_{X\otimes Y}$ by Lemma 2.10. Thus, the right adjoint of $\mathrm{ev}_{X\otimes Y}$ is given by the right adjoint of the 1-morphism in Equation (24), which is the 1-morphism in Equation (25) by the argument above. Since all adjoints are equivalent, this shows the lemma. □

We are now ready to prove that the Serre automorphism is a *monoidal* pseudo-natural transformation.

Proposition 2.12. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, the Serre automorphism is a* monoidal *pseudo-natural transformation of* $\mathrm{Id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$.

Proof. By definition (cf. [SP09, Definition 2.7]), we have to provide invertible 2-cells

$$\begin{aligned} \Pi_{X,Y} : S_{X\otimes Y} &\to S_X \otimes S_Y \\ M : S_1 &\to \mathrm{id}_1, \end{aligned} \tag{29}$$

satisfying suitable coherence equations. By the definition of the Serre automorphism in Definition 2.4, it suffices to show that the evaluation and its right adjoint are monoidal, since the braiding τ will be monoidal by definition. The monoidality of the evaluation is proven in Lemma 2.10, while the monoidality of its right adjoint follows from Lemma 2.11. These two lemmas thus provide an invertible 2-cell $S_{X\otimes Y} \cong S_X \otimes S_Y$. The second 2-cell $\mathrm{id}_1 \to S_1$ can be constructed in a similar way, by noticing that $1 \cong 1^*$.

The three coherence equations for a pseudo-natural transformation now read

$$\begin{aligned} \Pi_{X\otimes Y,Z} \circ (\Pi_{X\otimes Y} \otimes \mathrm{id}_{S_Z}) &= \Pi_{X,Y\otimes Z} \circ (\mathrm{id}_{S_X} \otimes \Pi_{Y,Z}) \\ \Pi_{1,X} &= M \otimes \mathrm{id}_{S_X} \\ \Pi_{X,1} &= \mathrm{id}_{S_X} \otimes M \end{aligned} \tag{30}$$

and can be checked directly by hand. □

3. Monoidal homotopy actions

In this section, we investigate homotopy actions on symmetric monoidal bicategories. In particular, we are interested in the case when the group

action is compatible with the monoidal structure. By a (homotopy) action of a topological group G on a bicategory $\mathcal{C}$, we mean a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, where $\Pi_2(G)$ is the fundamental 2-groupoid of G, and $\mathrm{Aut}(\mathcal{C})$ is the bicategory of auto-equivalences of $\mathcal{C}$. For details on homotopy actions of groups on bicategories, we refer the reader to [HSV17].
In order to simplify the exposition, we introduce the following

Definition 3.1. *Let G be a topological group. We will say that G is 2-truncated if $\pi_2(G, x)$ is trivial for every base point $x \in G$.*

Moreover, we will need also the following definition.

Definition 3.2. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. We will say that $\mathcal{C}$ is algebraically 1-connected if it is monoidally equivalent to B^2H, for some abelian group H.*

In the following, we denote by $\mathrm{Aut}_{\otimes}(\mathcal{C})$ the monoidal bicategory of invertible monoidal weak 2-functors of $\mathcal{C}$, invertible monoidal pseudo-natural transformations, and invertible monoidal modifications. Details of the construction can be found in [Hes17, Appendix A].

Definition 3.3. *Let $\mathcal{C}$ be a symmetric monoidal category and G be a topological group. A* monoidal *homotopy action of G on $\mathcal{C}$ is a monoidal morphism $\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$.*

We now prove a general criterion for when monoidal homotopy actions are trivializable.

Proposition 3.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let G be a path connected topological group. Assume that G is 2-truncated, and that $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is algebraically 1-connected, with abelian group H. If the second cohomology group $H^2_{grp}(\pi_1(G, e), H) \simeq 0$, then any monoidal homotopy action of G on $\mathcal{C}$ is pseudo-naturally isomorphic to the trivial action.*

Proof. Let $\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ be a weak monoidal 2-functor. Since $\mathrm{Aut}_{\otimes}(\mathcal{C})$ was assumed to be monoidally equivalent to B^2H for some abelian group H, the group action ρ is equivalent to a weak monoidal 2-functor $\rho : \Pi_2(G) \to B^2H$. Due to the fact that G is path connected and 2-truncated, we have that $\Pi_2(G) \simeq B\pi_1(G, e)$, where $\pi_1(G, e)$ is regarded as a discrete monoidal category. Thus, the monoidal homotopy action ρ is monoidally equivalent to a weak monoidal 2-functor $B\pi_1(G,e) \to B^2H$.

We claim that such functors are classified by $H^2_{grp}(\pi_1(G,e), H)$ up to monoidal pseudo-natural isomorphism. Indeed, let $F : B\pi_1(G,e) \to B^2H$ be a weak monoidal 2-functor. It is easy to see that F is trivial as a weak 2-functor, since we must have $F(*) = *$ on objects, $F(\gamma) = \mathrm{id}_*$ on 1-morphisms, and $B\pi_1(G)$ only has identity 2-morphisms. Thus, the only non-trivial data of F can come from the monoidal structure on F. The 1-dimensional components of the pseudo-natural transformations $\chi_{a,b} : F(a) \otimes F(b) \to F(a \otimes b)$ must be trivial since there are only identity 1-morphisms in B^2H. The 2-dimensional components of this pseudo-natural transformation consists of a 2-morphism $\chi_{\gamma,\gamma'}$ in B^2H for every pair of 1-morphisms $\gamma : a \to b$ and $\gamma' : a' \to b'$ in $B\pi_1(G)$ in the diagram in equation (31) below.

$$\begin{array}{ccc} F(a) \otimes F(a') & \xrightarrow{\chi_{a,a'}} & F(a \otimes a') \\ {\scriptstyle F(\gamma)\otimes F(\gamma')} \downarrow & \overset{\chi_{\gamma,\gamma'}}{\swarrow} & \downarrow {\scriptstyle F(\gamma\otimes\gamma')} \\ F(b) \otimes F(b') & \xrightarrow[\chi_{b,b'}]{} & F(b \otimes b') \end{array} \tag{31}$$

Hence, we obtain a 2-cochain $\pi_1(G) \times \pi_1(G) \to H$, which obeys the cocycle condition due to the coherence equations of a monoidal 2-functor, cf. [SP09, Definition 2.5].

One now checks that a monoidal pseudo-natural transformation between two such functors is exactly a 2-coboundary, which shows the claim. Since we assumed that $H^2_{grp}(\pi_1(G,e), H) \simeq 0$, the original action ρ must be trivializable. □

Next, we show that the bicategory $\mathrm{Alg}_2^{\mathrm{fd}}$ of finite-dimensional, semi-simple algebras, bimodules and intertwiners, equipped with the monoidal structure given by the *direct sum* fulfills the conditions of Proposition 3.4.

Lemma 3.5. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ be the bicategory where objects are given by finite-dimensional, semi-simple algebras, equipped with the monoidal structure given by the direct sum. By viewing $\mathcal{C}$ with the monoidal structure equipped by the direct sum, $\mathcal{C}$ turns into a* linear *bicategory. Then,* $\mathrm{Aut}_\otimes(\mathcal{C})$ *and* $B^2\mathbb{K}^*$ *are equivalent as symmetric monoidal bicategories.*

Proof. Let $F : \mathrm{Alg}_2^{\mathrm{fd}} \to \mathrm{Alg}_2^{\mathrm{fd}}$ be a weak monoidal 2-equivalence, and let A be a finite-dimensional, semi-simple algebra. Then A is isomorphic to a direct sum of matrix algebras. Calculating up to Morita equivalence and using that F has to preserve the single simple object $\mathbb{K}$ of Alg_2, we have

$$\begin{aligned} F(A) &\cong F\left(\bigoplus_i M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F\left(M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F(\mathbb{K}) \\ &\cong \bigoplus_i \mathbb{K} \cong \bigoplus_i M_{n_i}(\mathbb{K}) \cong A. \end{aligned} \tag{32}$$

A straightforward calculation using basic linear algebra confirms that these isomorphisms are even pseudo-natural. Thus, the functor F is pseudo-naturally isomorphic to the identity functor on $\mathrm{Alg}_2^{\mathrm{fd}}$.

Now, let $\eta : F \to G$ be a monoidal pseudo-natural isomorphism between two endofunctors of Alg_2. Since both F and G are pseudo-naturally isomorphic to the identity, we may consider instead a pseudo-natural isomorphism $\eta : \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}} \to \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}$. We claim that up to an invertible modification, the 1-equivalence $\eta_A : A \to A$ must be given by the bimodule ${}_AA_A$, which is the identity 1-morphism on A in Alg_2. Indeed, since η_A is assumed to be linear, it suffices to consider the case of $A = M_n(\mathbb{K})$ and to take direct sums. It is well-known that the only simple modules of A are given by $\mathbb{K}^n$. Thus,

$$\eta_A = (\mathbb{K}^n)^\alpha \otimes_{\mathbb{K}} (\mathbb{K}^n)^\beta, \tag{33}$$

where α and β are multiplicities. Now, [HSV17, Lemma 2.6] ensures that these multiplicities are trivial, and thus we have $\eta_A = {}_AA_A$ up to an invertible intertwiner. This shows that up to invertible modifications, all 1-morphisms in $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ must be identities.

Now, let m be a monoidal invertible endo-modification of the pseudo-natural transformation $\mathrm{id}_{\mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}}$. Then, the component $m_A : {}_AA_A \to {}_AA_A$ is an element of $\mathrm{End}_{(A,A)}(A) \cong \mathbb{K}$. As the modification square commutes automatically, this show that the 2-morphisms of $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ stand in bijection to $\mathbb{K}^*$. □

Remark 3.6. Notice that the symmetric monoidal structure on $\mathrm{Alg}_2^{\mathrm{fd}}$ considered above is *not* the standard one, which is instead the one induced by the tensor product of algebras, and which is the monoidal structure relevant for the remainder of the paper.

The last lemmas imply the following

Lemma 3.7. *Any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ equipped with the monoidal structure given by the direct sum is trivial.*

Proof. Since $\pi_1(SO(2), e) \simeq \mathbb{Z}$, and $H^2_{grp}(\mathbb{Z}, \mathbb{K}^*) \simeq H^2(S^1, \mathbb{K}^*) \simeq 0$, Proposition 3.4 and Lemma 3.5 ensure that any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ is trivializable. □

Recall that we regarded $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ as a monoidal bicategory with the monoidal structure given by direct sums.

Corollary 3.8. *Since $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$ are equivalent as additive categories, any $SO(2)$-action on $\mathrm{Vect}_2^{\mathrm{fd}}$ via linear morphisms is trivializable.*

Remark 3.9. The last two results rely on the fact that $\mathrm{Aut}_{\otimes}(\mathrm{Alg}_2^{\mathrm{fd}})$ and $\mathrm{Aut}_{\otimes}(\mathrm{Vect}_2^{\mathrm{fd}})$ are 1-connected as additive categories. This is due to the fact that fully-dualizable part of either Alg_2 or Vect_2 is semi-simple. An example in which the conditions in Proposition 3.4 do *not* hold is provided by the bicategory of Landau-Ginzburg models.

4. Computing homotopy fixed points

In this Section, we explicitly compute the bicategory of homotopy fixed points of an $SO(2)$-action which is induced by an arbitrary pseudo-natural equivalence of the identity functor of an arbitrary bicategory $\mathcal{C}$. Recall that a G-action on a bicategory $\mathcal{C}$ is a monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, or equivalently a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The bicategory of homotopy fixed points $\mathcal{C}^G$ is then given by the tri-limit of this trifunctor.

In Bicat, the tricategory of bicategory, this trilimit can be computed as follows: if $\Delta : B\Pi_2(G) \to \mathrm{Bicat}$ is the constant functor assigning to the one object $*$ the terminal bicategory with one object, the trilimit of the action functor ρ is given by

$$\mathcal{C}^G := \lim \rho = \mathrm{Nat}(\Delta, \rho), \tag{34}$$

the bicategory of tri-transformations between ρ and Δ. This definition is explicitly spelled out in [HSV17, Remark 3.11]. We begin by defining an

$SO(2)$-action on an arbitrary symmetric monoidal bicategory, starting from a pseudo-natural transformation of the identity functor on $\mathcal{C}$.

Definition 4.1. *Since $\Pi_2(SO(2))$ is equivalent to the bicategory with one object, $\mathbb{Z}$ worth of morphisms, and only identity 2-morphisms, we may define an $SO(2)$-action $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ by the following data:*

- *For every group element $g \in SO(2)$, we assign the identity functor of $\mathcal{C}$.*
- *For the generator $1 \in \mathbb{Z}$, we assign the pseudo-natural transformation of the identity functor given by α. Due to the monoidality, this determines the value of ρ on an arbitrary integer.*
- *Since there are only identity 2-morphisms in $\mathbb{Z}$, we have to assign these to identity 2-morphisms in $\mathcal{C}$.*
- *For composition of 1-morphisms, we assign the invertible modification $\rho(a + b) \cong \rho(a) \circ \rho(b)$ coming from the fact that α is a* monoidal *pseudo-natural transformation with respect to composition, which is the monoidal product in $\mathrm{Aut}_{\otimes}(\mathcal{C})$.*
- *In order to make ρ into a monoidal 2-functor, we have to assign additional data which we can choose to be trivial. In detail, we set $\rho(g \otimes h) := \rho(g) \otimes \rho(h)$, and $\rho(e) := \mathrm{id}_{\mathcal{C}}$. Finally, we choose ω, γ and δ as in [HSV17, Remark 3.8] to be identities.*

For a proof that this defines indeed a weak 2-functor, we refer to [Dav11, Lemma 3.2.3].

Our main example is the action of the Serre automorphism on the core of fully-dualizable objects:

Example 4.2. If $\mathcal{C}$ is a symmetric monoidal bicategory, consider $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the core of the fully-dualizable objects of $\mathcal{C}$. By Proposition 2.8, the Serre automorphism defines a pseudo-natural equivalence of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we denote by ρ^S.

The next theorem computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the action in Definition 4.1. This theorem generalizes [HSV17, Theorem 4.1], which only computes the bicategory of homotopy fixed points of the *trivial* $SO(2)$-action.

Theorem 4.3. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a monoidal pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Let ρ be the $SO(2)$-action on $\mathcal{C}$ as in Definition 4.1. Then, the bicategory of homotopy fixed points $\mathcal{C}^G$ is equivalent to the bicategory with*

- *objects: (c, λ) where c is an object of $\mathcal{C}$ and $\lambda : \alpha_c \to \mathrm{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \to (c', \lambda')$ in $\mathcal{C}^G$ are given by 1-morphisms $f : c \to c'$ in $\mathcal{C}$, so that the diagram*

$$\begin{array}{ccccc} \alpha_{c'} \circ f & \xleftarrow{\alpha_f} & f \circ \alpha_c & \xrightarrow{\mathrm{id}_f * \lambda} & f \circ \mathrm{id}_c \\ {\scriptstyle \lambda' * \mathrm{id}_f}\downarrow & & & & \downarrow \\ \mathrm{id}_c \circ f & & \longrightarrow & & f \end{array} \tag{35}$$

  commutes,
- *2-morphisms of $\mathcal{C}^G$ are given by 2-morphisms in $\mathcal{C}$.*

Proof. In order to prove the theorem, we need to explicitly unpack the definition of the bicategory of homotopy fixed points $\mathcal{C}^G$. This is done in [HSV17, Remark 3.11 - 3.14]. In the following, we will use the notation introduced in [HSV17].

The idea of the proof is to show that the forgetful functor which on objects of $\mathcal{C}^G$ forgets the data Θ, Π and M is an equivalence of bicategories. In order to show this, we need to analyze the bicategory of homotopy fixed points. We start with the objects of $\mathcal{C}^G$.

By definition, a homotopy fixed point of this action consists of

- An object $c \in \mathcal{C}$,
- A 1-equivalence $\Theta : c \to c$,

- For every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \alpha_c^n \circ \Theta \to \Theta \circ \mathrm{id}_c$ so that (Θ, Θ_n) fulfill the axioms of a pseudo-natural transformation,
- A 2-isomorphism $\Pi : \Theta \circ \Theta \to \Theta$ which obeys the modification square,
- Another 2-isomorphism $M : \Theta \to \mathrm{id}_c$

so that the following equations hold: Equation 3.18 of [HSV17] demands that

$$\Pi \circ (\mathrm{id}_\Theta * \Pi) = \Pi \circ (\Pi * \mathrm{id}_\Theta) \tag{36}$$

whereas Equation 3.19 of [HSV17] demands that Π equals the composition

$$\Theta \circ \Theta \xrightarrow{\mathrm{id}_\Theta * M} \Theta \circ \mathrm{id}_c \cong \Theta \tag{37}$$

and finally Equation 3.20 of [HSV17] tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \mathrm{id}_\Theta} \mathrm{id}_c \circ \Theta \cong \Theta. \tag{38}$$

Hence Π is fully specified by M. An explicit calculation using the two equations above then confirms that Equation (36) is automatically fulfilled. Indeed, by composing with Π^{-1} from the right, it suffices to show that $\mathrm{id}_\Theta * \Pi = \Pi * \mathrm{id}_\Theta$. Suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Then,

$$\begin{aligned} \mathrm{id}_\Theta * \Pi &= \mathrm{id}_\Theta * (M * \mathrm{id}_\Theta) && \text{by equation (38)} \\ &= (\mathrm{id}_\Theta * M) * \mathrm{id}_\Theta \\ &= \Pi * \mathrm{id}_\Theta && \text{by equation (37).} \end{aligned} \tag{39}$$

Adding appropriate associators shows that this is true in a general bicategory.

Note that by using the modification M, the 2-morphism $\Theta_n : \alpha_n^c \to \Theta \circ \mathrm{id}_c$ can be regarded as a 2-morphism $\lambda_n : \alpha_c \to \mathrm{id}_c$. Here, α_c^n is the n-times composition of 1-morphism α_c. Indeed, define λ_n by setting

$$\lambda_n := \left(\alpha_c \cong \alpha_c \circ \mathrm{id}_c \xrightarrow{\mathrm{id}_{\alpha_c} * M^{-1}} \alpha_c \circ \Theta \xrightarrow{\Theta_n} \Theta \circ \mathrm{id}_c \cong \Theta \xrightarrow{M} \mathrm{id}_c \right). \tag{40}$$

In a strict 2-category, the fact that Θ is a pseudo-natural transformation requires that $\lambda_0 = \mathrm{id}_c$ and that $\lambda_n = \lambda_1 * \cdots * \lambda_1$. In a bicategory, similar equations hold by adding coherence morphisms. Thus, λ_n is fully determined by λ_1. In order to simplify notation, we set $\lambda := \lambda_1 : \alpha_c \to \mathrm{id}_c$.

A 1-morphism of homotopy fixed points $(c, \Theta, \Theta_n, \Pi, M) \to (c', \Theta', \Theta'_n, \Pi', M')$ consists of:

- a 1-morphism $f : c \to c'$,
- an invertible 2-morphism $m : f \circ \Theta \to \Theta' \circ f$ which fulfills the modification square. Note that m is equivalent to a 2-isomorphism $m : f \to f'$ which can be seen by using the 2-morphism M.

The condition due to Equation 3.24 of [HSV17] demands that the following 2-isomorphism

$$f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \tag{41}$$

is equal to the 2-isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \mathrm{id}_f} \mathrm{id}_{c'} \circ f \cong f \tag{42}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \cong \mathrm{id}_{c'} \circ f \xrightarrow{M'^{-1} * \mathrm{id}_f} \Theta' \circ f \right) \tag{43}$$

Thus, m is fully determined by M and M'. The condition due to Equation 3.23 of [HSV17] reads

$$m \circ (\mathrm{id}_f * \Pi) = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \tag{44}$$

and is automatically satisfied, as an explicit calculation in [HSV17] confirms. Now, it suffices to look at the modification square of m, in Equation 3.25 of [HSV17]. This condition is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} \alpha_{c'} \circ f \circ \Theta & \xleftarrow{\alpha_f * \mathrm{id}_\Theta} & f \circ \alpha_c \circ \Theta & \xrightarrow{\mathrm{id}_f * \Theta_1} & f \circ \Theta \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\alpha_{c'}} * m} & & & & \Big\downarrow{\scriptstyle m} \\ \alpha_{c'} \circ \Theta' \circ f & & \xrightarrow[\Theta'_1 * \mathrm{id}_f]{} & & \Theta' \circ f \end{array} \tag{45}$$

Substituting m as in Equation (43) and Θ_1 for $\lambda := \lambda_1$ as defined in Equation (40), one confirms that this diagram commutes if and only if the diagram in Equation (35) commutes.

If (f, m) and (g, n) are 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-isomorphism $\beta : f \to g$ in $\mathcal{C}$. The condition coming from Equation 3.26 of [HSV17] then demands that the diagram

$$\begin{array}{ccc} f \circ \Theta & \xrightarrow{\quad m \quad} & \Theta' \circ f \\ {\scriptstyle \beta * \mathrm{id}_\Theta} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}_{\Theta'} * \beta} \\ g \circ \Theta & \xrightarrow[\quad n \quad]{} & \Theta' \circ g \end{array} \tag{46}$$

commutes. Using the fact that both m and n are uniquely specified by M and M', one quickly confirms that this diagram commutes automatically.

Our detailed analysis of the bicategory $\mathcal{C}^G$ shows that the forgetful functor which forgets the data Θ, M, and Π on objects and assigns Θ_1 to λ, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. □

Corollary 4.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action of the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\text{fd}})$ as in Example 4.2. Then, the bicategory of homotopy fixed points $\mathscr{K}(\mathcal{C}^{\text{fd}})^{SO(2)}$ is equivalent to a bicategory where*

- *objects are given by pairs (X, λ_X) with X a fully-dualizable object of $\mathcal{C}$ and $\lambda_X : S_X \to \mathrm{id}_X$ is a 2-isomorphism which trivializes the Serre automorphism,*
- *1-morphisms are given by 1-equivalences $f : X \to Y$ in $\mathcal{C}$, so that the diagram*

$$\begin{array}{ccccc} S_Y \circ f & \xleftarrow{\;S_f\;} & f \circ S_X & \xrightarrow{\mathrm{id}_f * \lambda_X} & f \circ \mathrm{id}_X \\ {\scriptstyle \lambda_Y * \mathrm{id}_f} \Big\downarrow & & & & \Big\downarrow \\ \mathrm{id}_X \circ f & & \xrightarrow{\qquad\qquad\qquad} & & f \end{array} \tag{47}$$

  commutes, and
- *2-morphisms are given by 2-isomorphisms in $\mathcal{C}$.*

Remark 4.5. Recall that we have defined the bicategory of homotopy fixed points $\mathcal{C}^G$ as the tri-limit of the action considered as a trifunctor $\rho : B\Pi_2(G) \to$

Bicat. Since we only consider symmetric monoidal bicategories, we actually obtain an action with values in $\mathrm{SymMonBicat}$, the tricategory of symmetric monoidal bicategories. It would be interesting to compute the limit of the action in this tricategory. We expect that this trilimit computed in $\mathrm{SymMonBicat}$ is given by $\mathcal{C}^G$ as a bicategory, with the symmetric monoidal structure induced by the symmetric monoidal structure of $\mathcal{C}$.

Remark 4.6. By [Dav11], the action via the Serre automorphism on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ is trivializable. The category of homotopy fixed points $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)}$ is then equivalent to the bigroupoid of symmetric, semi-simple Frobenius algebras.

Similarly, the action of the Serre automorphism on Vect_2 is trivializable. The bicategory of homotopy fixed points of this action is equivalent to the bicategory of finite Calabi-Yau categories, cf. [HSV17].

5. The 2-dimensional framed bordism bicategory

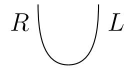
In this Section, we introduce a stricter version of the framed bordism bicategory $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$: this symmetric monoidal bicategory $\mathbb{F}_{cfd}$ is the free bicategory of a coherent fully-dual pair as introduced in [Pst14, Definition 3.13].

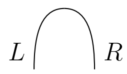
In order to efficiently work with this symmetric monoidal bicategory $\mathbb{F}_{cfd}$, we use a strictification result for symmetric monoidal bicategories as proven in [Bar14, Proposition 13]: any symmetric monoidal bicategory is equivalent to a *stringent* symmetric monoidal 2-category, which can be completely described in terms of a wire diagram calculus introduced in [Bar14]. In the following, we apply this strictification result to the symmetric monoidal bicategory $\mathbb{F}_{cfd}$, and provide a description using the wire diagram calculus developed in [Bar14], which we also refer to for the definition of a stringent symmetric monoidal 2-category.

Using this description, we define a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$. If $\mathcal{C}$ is an arbitrary symmetric monoidal bicategory, the action on $\mathbb{F}_{cfd}$ will induce an action on the functor bicategory $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ of symmetric monoidal functors. Using the Cobordism Hypothesis for framed manifolds, which has been proven in the bicategorical framework in [Pst14], we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We show that this induced action coming from the framed bordism bicategory is exactly the action given by the Serre automorphism.

We begin by recasting the definition of $\mathbb{F}_{cfd}$ in terms of the wire diagram calculus.

Definition 5.1. *The symmetric monoidal bicategory $\mathbb{F}_{cfd}$ consists of*

- *2 generating objects L and R,*
- *4 generating 1-morphisms, given by*
  - *a 1-morphism* $\mathrm{coev} : 1 \to R \otimes L$, *which we write as* 

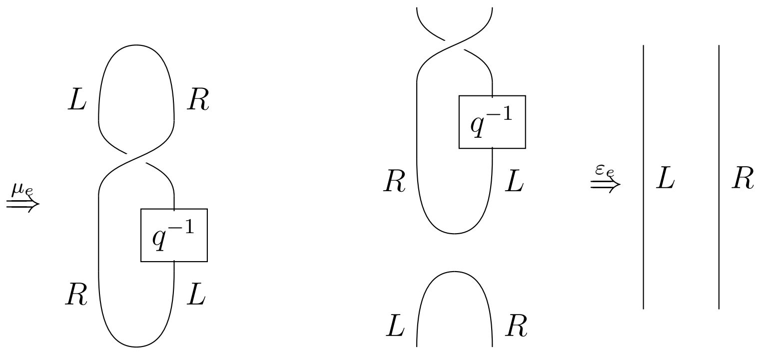
  - $\mathrm{ev} : L \otimes R \to 1$ *which we write as* 

  - *a 1-morphism $q : L \to L$,*
  - *another 1-morphism $q^{-1} : L \to L$,*
- *12 generating 2-cells given by*
  - *isomorphisms α, β, α^{-1} and β^{-1} as in Definition 2.1, which in pictorial form are given as follows:*

(48)

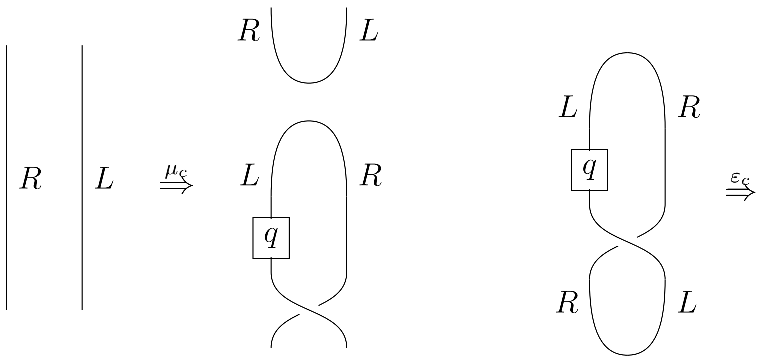
  - *isomorphisms $\psi : qq^{-1} \cong \mathrm{id}_L : \psi^{-1}$ and $\phi : q^{-1}q \cong \mathrm{id}_L : \phi^{-1}$*
  - *2-cells $\mu_e : \mathrm{id}_1 \to \mathrm{ev} \circ \mathrm{ev}^L$ and $\varepsilon_e : \mathrm{ev}^L \circ \mathrm{ev} \to \mathrm{id}_{L \otimes R}$, where* $\mathrm{ev}^L := \tau \circ (\mathrm{id}_R \otimes q^{-1}) \circ \mathrm{coev}$ *which in pictorial form are given*

as follows:



(49)

– *2-cells* $\mu_c : \mathrm{id}_{R\otimes L} \to \mathrm{coev} \circ \mathrm{coev}^L$ *and* $\varepsilon_c : \mathrm{coev}^L \circ \mathrm{coev} \to \mathrm{id}_1$, *where* $\mathrm{coev}^L := \mathrm{ev} \circ (q \otimes \mathrm{id}_R) \circ \tau$ *which in pictorial form are given as follows:*

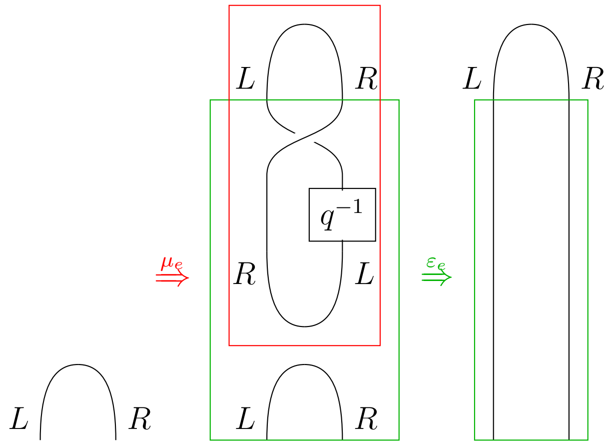


(50)

so that the following relations hold:

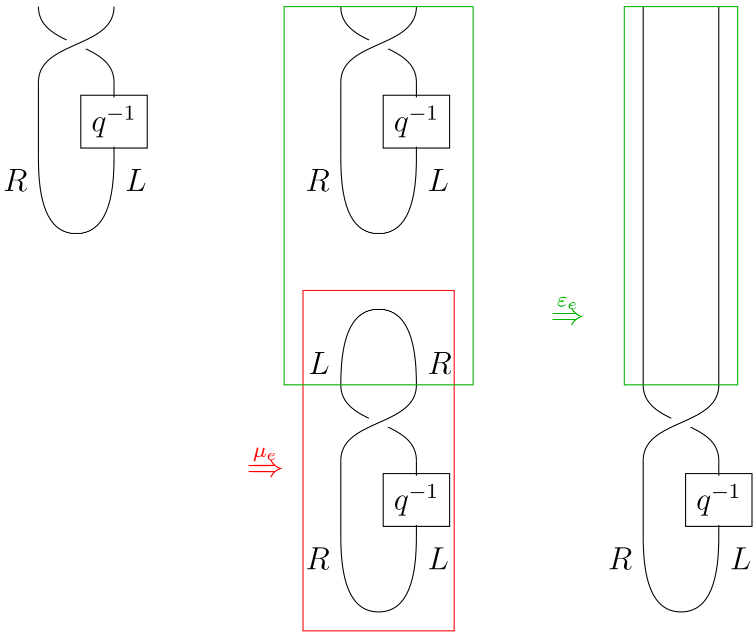
- α *and* α^{-1}, β *and* β^{-1}, ϕ *and* ϕ^{-1}, ψ *and* ψ^{-1} *are inverses to each other,*

- μ_e *and* ε_e *satisfy the two Zorro equations, which in pictorial form* *demand that the following composition of 2-morphisms*



(51)

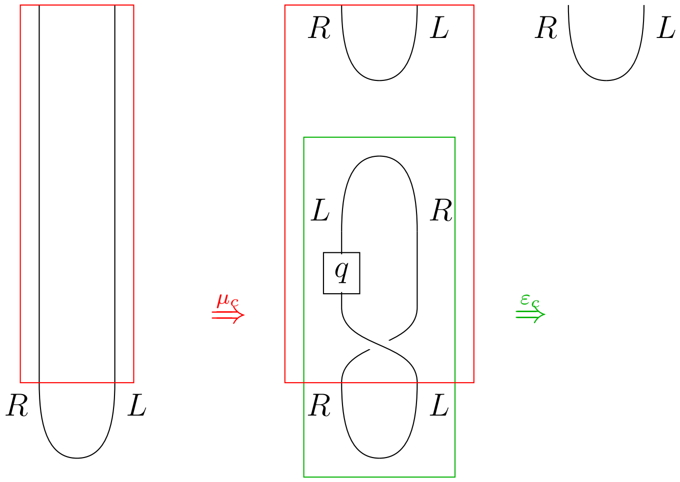
is equal to $\mathrm{id}_{\mathrm{ev}}$*, and that the following composition of 2-morphisms*



(52)

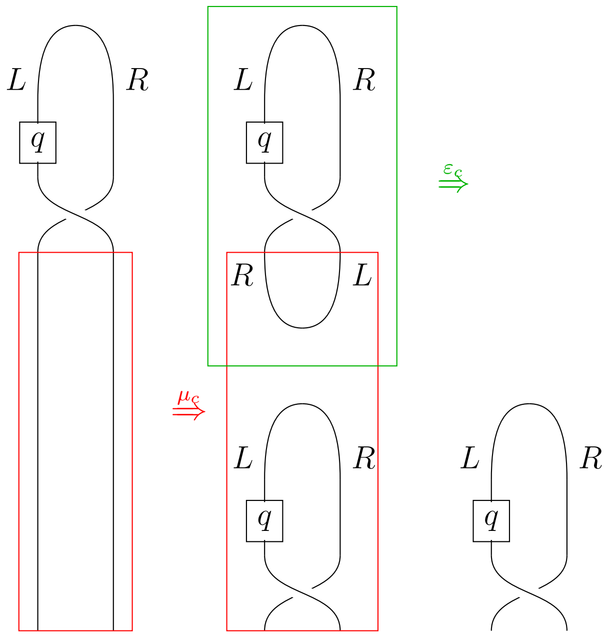
is equal to $\mathrm{id}_{\mathrm{ev}^L}$.

- μ_c *and* ε_c *satisfy the two Zorro equations, which in pictorial form* *demand that the composition*



(53)

is equal to $\mathrm{id}_{\mathrm{coev}}$*, and the composition of the following 2-morphisms*



(54)

is equal to $\mathrm{id}_{\mathrm{coev}^L}$.

- ϕ *and* ψ *satisfy triangle identities,*

- *the cusp-counit equations in figure 5 and 6 on p.33 of [Pst14] are satisfied,*
- *the swallowtail equations in figure 3 and 4 on p.15 of [Pst14] are satisfied.*

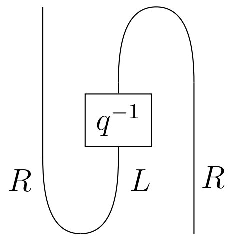
5.1 Action on the framed bordism bicategory

We can now proceed to construct an $SO(2)$-action on $\mathbb{F}_{cfd}$. This action will be vital for the remainder of the paper.

By Definition 4.1 it suffices to construct a pseudo-natural equivalence of the identity functor on $\mathbb{F}_{cfd}$ in order to construct an $SO(2)$-action. This pseudo-natural transformation is given as follows:

Definition 5.2. *Let $\mathbb{F}_{cfd}$ be the free symmetric monoidal bicategory on a coherent fully-dual object as in Definition 5.1. We construct a pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ of the identity functor on $\mathbb{F}_{cfd}$ as follows:*

- *For every object c of $\mathbb{F}_{cfd}$, we need to provide a 1-equivalence $\alpha_c : c \to c$.*
  - *For the object L of $\mathbb{F}_{cfd}$, we define $\alpha_L := q : L \to L$,*
  - *for the object R of $\mathbb{F}_{cfd}$, we set $\alpha_R := (q^{-1})^*$, which in pictorial form is given by*

$(q^{-1})^* :=$  (55)

- *for every 1-morphism $f : c \to d$ in $\mathbb{F}_{cfd}$, we need to provide a 2-isomorphism*

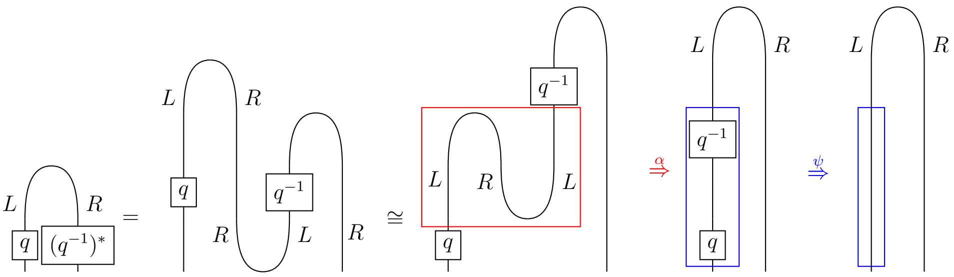
$$\alpha_f : f \circ \alpha_c \to \alpha_d \circ f. \tag{56}$$

  - *For the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ we define the 2-isomorphism $\alpha_q := \mathrm{id}_{q \circ q}$.*

– *For the 1-morphism $q^{-1} : L \to L$ we define the 2-isomorphism*

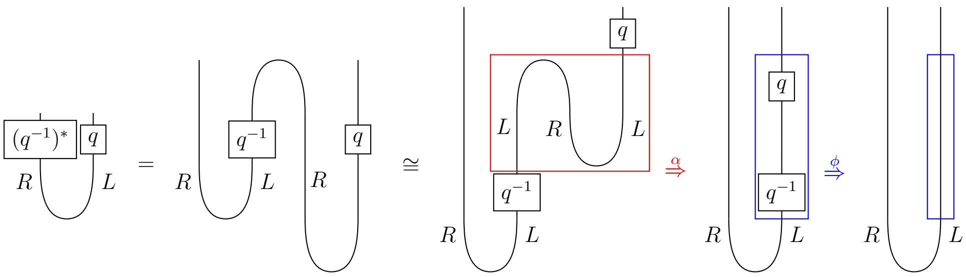
$$\alpha_{q^{-1}} := \left(q^{-1} \circ q \xrightarrow{\phi} \mathrm{id}_L \xrightarrow{\psi^{-1}} q \circ q^{-1} \right). \tag{57}$$

– *For the evaluation* $\mathrm{ev} : L \otimes R \to 1$, *we define the 2-isomorphism* α_{ev} *to be the following composition:*



(58)

– *For the coevaluation* $\mathrm{coev} : 1 \to R \otimes L$, *we define the 2-isomorphism* α_{coev} *to be the composition*



(59)

One now checks that this defines a pseudo-natural transformation of $\mathrm{id}_{\mathbb{F}_{cfd}}$. Using Definition 4.1 gives us a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$.

Remark 5.3. Note that the $SO(2)$-action on $\mathbb{F}_{cfd}$ does *not* send generators to generators: for instance, the 1-morphism $(q^{-1})^*$ in Equation (55) is not part of the generating data of $\mathbb{F}_{cfd}$.

Remark 5.4. Notice that the pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ constructed in Definition 5.2 is a *monoidal* pseudo-natural transformation.

This follows from the fact that we have defined α via generators and relations. In detail, we set

$$\begin{aligned} \alpha_X \otimes \alpha_Y &:= \alpha_{X \otimes Y} \\ \alpha_1 &:= \mathrm{id}_1. \end{aligned} \tag{60}$$

Thus, we can choose the additional data Π and M of a monoidal pseudo-natural transformation to be trivial, and we obtain an $SO(2)$-action on $\mathbb{F}_{cfd}$ via symmetric monoidal morphisms.

5.2 Induced action on functor categories

Starting from the action defined on $\mathbb{F}_{cfd}$, we induce an action on the bicategory of functors $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ for an arbitrary bicategory $\mathcal{C}$. The construction of the induced action on the bicategory of functors is a general construction. We provide details in the following.

Definition 5.5. *Let $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ be a G-action on a bicategory $\mathcal{C}$, and let $\mathcal{D}$ be another bicategory. The G-action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ induced by ρ is defined as follows:*

- *On objects $g \in G$, we define an endofunctor $\tilde{\rho}(g)$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on objects F on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by $\tilde{\rho}(g)(F) := F \circ \rho(g^{-1})$. If $\alpha : F \to G$ is a 1-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we define*

$$\tilde{\rho}(g)(\alpha) := \qquad \begin{array}{ccc} F\rho(g^{-1})c & \xrightarrow{\alpha_{\rho(g^{-1})(c)}} & G\rho(g^{-1})c \\ {\scriptstyle F\rho(g^{-1})(f)}\big\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \alpha_{\rho(g^{-1})(f)}} & \big\downarrow{\scriptstyle G\rho(g^{-1})(f)} \\ F\rho(g^{-1})d & \xrightarrow[\alpha_{\rho(g^{-1})(d)}]{} & G\rho(g^{-1})d \end{array} \tag{61}$$

  If $m : \alpha \to \beta$ is a 2-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the value of $\tilde{\rho}(\gamma)$ is given by

$$\tilde{\rho}(\gamma)(m)_x := m_{\rho(g^{-1})(x)}. \tag{62}$$

- *on 1-morphisms $\gamma : g \to h$ of $\Pi_2(G)$, we define a 1-morphism $\tilde{\rho}(\gamma)$ in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ between the two endofunctors $F \mapsto F \circ \rho(g^{-1})$ and $F \mapsto F \circ \rho(h^{-1})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

  Explicitly, this means:

- *For each 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural transformation $\tilde{\rho}(\gamma)_F : F \circ \rho(g^{-1}) \to F \circ \rho(h^{-1})$ which we define via the diagram*

$$\begin{array}{ccc} F\rho(g^{-1})x & \xrightarrow{F(\rho(\gamma^{-1})_x)} & F\rho(h^{-1})x \\ {\scriptstyle F\rho(g^{-1})(f)}\downarrow & \swarrow\!\!\!\!\swarrow {\scriptstyle F(\rho(\gamma^{-1})_f)} & \downarrow{\scriptstyle F\rho(h^{-1})(f)} \\ F\rho(g^{-1})y & \xrightarrow[F(\rho(\gamma^{-1})_y)]{} & F\rho(h^{-1})y \end{array} \tag{63}$$

Here, γ^{-1} is the "inverse" path of γ given by $t \mapsto \gamma(t)^{-1}$, and $f : x \to y$ is a 1-morphism in $\mathcal{C}$.

- *For every pseudo-natural transformation $\alpha : F \to G$, we need to provide a modification $\tilde{\rho}(\gamma)_\alpha$ in the diagram*

$$\begin{array}{ccc} \tilde{\rho}(g)(F) & \xrightarrow{\tilde{\rho}(\gamma)_F} & \tilde{\rho}(h)(F) \\ {\scriptstyle \tilde{\rho}(g)(\alpha)}\downarrow & \swarrow\!\!\!\!\swarrow {\scriptstyle \tilde{\rho}(\gamma)_\alpha} & \downarrow{\scriptstyle \tilde{\rho}(h)(\alpha)} \\ \tilde{\rho}(g)(G) & \xrightarrow[\tilde{\rho}(\gamma)_G]{} & \tilde{\rho}(h)(G) \end{array} \tag{64}$$

which we define by

$$\tilde{\rho}(\gamma)_\alpha := \alpha^{-1}_{\rho(\gamma^{-1})_x}. \tag{65}$$

- *For the 2-morphisms in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ we proceed in a similar fashion: if $m : \gamma \to \gamma'$ is a 2-track, we have to provide a 2-morphism $\tilde{\rho}(m) : \tilde{\rho}(\gamma) \to \tilde{\rho}(\gamma')$ which can be done by explicitly writing down diagrams as above.*

The rest of the data of a monoidal functor $\tilde{\rho}$ is induced from the data of the monoidal functor ρ.

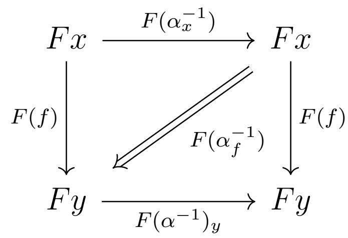
For $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal bicategories, the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})$ acquires a monoidal structure by "pointwise evaluation" of functors. Such a monoidal structure is also symmetric, see [SP09]. The following result is straightforward.

Lemma 5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal bicategories, and let ρ be a monoidal action of a group G on $\mathcal{C}$. Then ρ induces a monoidal action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}_\otimes(\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D}))$.*

Example 5.7. Our main example for induced actions is the $SO(2)$-action on $\mathbb{F}_{cfd}$ as in Definition 5.2. This action only depends on a pseudo-natural equivalence α of the identity functor on $\mathrm{id}_{\mathbb{F}_{cfd}}$. Consequently, the induced action on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ also only depends on a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. Using the definition above, we construct this induced pseudo-natural equivalence $\tilde{\alpha}$ as follows.

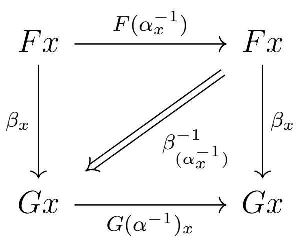
- For every 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural equivalence $\tilde{\alpha}_F : F \to F$, which is given by the diagram

$\tilde{\alpha}_F :=$



(66)

- for every pseudo-natural transformation $\beta : F \to G$, we need to give a modification $\tilde{\alpha}_\beta$, which we define by the diagram



(67)

This defines a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathrm{Fun}(\mathbb{F}_{cfd,}\mathcal{C})$. Note that $\mathbb{F}_{cfd}$ is even a *symmetric monoidal* bicategory. The $SO(2)$-action on $\mathbb{F}_{cfd}$ of Definition 5.2 is via symmetric monoidal homomorphisms by Remark 5.4. Hence, if $\mathcal{C}$ is also symmetric monoidal, then Lemma 5.6 provides a monoidal action on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$.

5.3 Induced action on the core of fully-dualizable objects

In this subsection, we compute the $SO(2)$-action on the core of fully-dualizable objects coming from the $SO(2)$-action on $\mathbb{F}_{cfd}$. Starting from the $SO(2)$-action on $\mathbb{F}_{cfd}$ as by Definition 5.2, we have shown in the previous subsection how to induce an $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ for $\mathcal{C}$ some symmetric monoidal bicategory. By the Cobordism Hypothesis for framed manifolds, we obtain an induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\text{fd}})$. More precisely, denote by

$$\begin{aligned} \mathrm{ev}_L : \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C}) &\to \mathscr{K}(\mathcal{C}^{\text{fd}}) \\ Z &\mapsto Z(L) \end{aligned} \tag{68}$$

the evaluation map. The Cobordism Hypothesis for framed manifolds in two dimensions [Pst14, Lur09] states that ev_L is an equivalence of symmetric monoidal bicategories. Hence, the composition of the $SO(2)$-action on $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ and (the inverse of) ev_L provides an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\text{fd}})$. The next proposition shows that this action is equivalent to the action ρ^S induced by the Serre automorphism which is illustrated in Example 4.2.

Proposition 5.8. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. By Definition 5.5, we obtain a monoidal $SO(2)$-action on* $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$*. Then, the monoidal $SO(2)$-action induced by the evaluation in Equation* (68) *on $\mathscr{K}(\mathcal{C}^{\text{fd}})$ is equivalent to ρ^S.*

Proof. Let

$$\rho : \Pi_2(SO(2)) \to \mathrm{Aut}_{\otimes}(\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})) \tag{69}$$

be the $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ as in Example 5.7. This action only depends on a monoidal pseudo-natural transformation α on the identity functor on $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$. By [Pst14], the 2-functor in Equation (68) which evaluates a framed field theory on the object L is an equivalence of bicategories. Thus, we obtain an $SO(2)$-action ρ' on $\mathscr{K}(\mathcal{C}^{\text{fd}})$. This action is given as follows. By Definition 4.1, we only need to provide a monoidal pseudo-natural transformation of the identity functor of $\mathscr{K}(\mathcal{C}^{\text{fd}})$. In order to write down this monoidal pseudo-natural transformation, note that the functor

$$\begin{aligned} \mathrm{Aut}_\otimes(\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})) &\to \mathrm{Aut}_\otimes(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \\ F &\mapsto \mathrm{ev}_L \circ F \circ \mathrm{ev}_L^{-1} \end{aligned} \tag{70}$$

is a monoidal equivalence. Hence, the induced pseudo-natural transformation of $\mathrm{id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$ is given as follows:

- For each fully-dualizable object c of $\mathcal{C}$, we assign the 1-morphism $\alpha'_c : c \to c$ defined by

$$\alpha'_c := \mathrm{ev}_L \left(\alpha_{\left(\mathrm{ev}_L^{-1}(c)\right)} \right) \tag{71}$$

- for each 1-equivalence $f : c \to d$ between fully-dualizable objects of $\mathcal{C}$, we define a 2-isomorphism $\alpha'_f : f \circ \alpha'_c \to \alpha'_d \circ f$ by the formula

$$\alpha'_f := \mathrm{ev}_L \left(\alpha_{\left(\mathrm{ev}_L^{-1}(f)\right)} \right). \tag{72}$$

Here, α is the pseudo-natural transformation as in Example 5.7. In order to see that α'_c is given by the Serre automorphism of the fully-dualizable object c, note that the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ is mapped to the Serre automorphism $S_{Z(L)}$ by the equivalence in Equation (68). □

Corollary 5.9. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. Consider the $SO(2)$-action ρ^S on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ induced by the Serre automorphism. Then the evaluation morphism ev_L induces an equivalence of bicategories*

$$\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}. \tag{73}$$

Proof. By Proposition 5.8, the equivalence of Equation (68) is $SO(2)$-equivariant. Thus, it induces an equivalence on homotopy fixed points, cf. [Hes16, Definition 5.3] for an explicit description. It is also possible to construct this equivalence directly: by theorem 4.3, the bicategory of homotopy fixed points $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)}$ is equivalent to the bicategory where

- objects are given by symmetric monoidal functors $Z : \mathbb{F}_{cfd} \to \mathcal{C}$, together with a modification $\lambda_Z : \tilde{\alpha}_Z \to \mathrm{id}_Z$. Explicitly, this means: if α is the endotransformation of the identity functor of $\mathbb{F}_{cfd}$ as in Definition 5.2, we obtain two 2-isomorphisms in $\mathcal{C}$:

$$\begin{aligned} \lambda_L &: Z(q^{-1}) \to \mathrm{id}_{Z(L)} \\ \lambda_R &: Z(((q^{-1})^*)^{-1}) \to \mathrm{id}_{Z(R)} \end{aligned} \tag{74}$$

  which are compatible with evaluation and coevaluation,

- 1-morphisms are given by symmetric monoidal pseudo-natural transformations $\mu : Z \to Z'$, so that the analogue of the diagram in Equation (35) commutes,

- 2-morphisms are given by symmetric monoidal modifications.

Now notice that $Z(q)$ is precisely the Serre automorphism of the object $Z(L)$. Thus, λ_L provides a trivialization of (the inverse of) the Serre automorphism. Applying theorem 4.3 again to the action of the Serre automorphism on the core of fully-dualizable objects shows that the functor $Z \mapsto (Z(L), \lambda_L)$ is an equivalence of homotopy fixed point bicategories. □

Remark 5.10. Note that in Corollary 5.9 we have proven that the evaluation induces an equivalence of bicategories $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$. We expect that this equivalence is an equivalence of *monoidal* bicategories. In order to prove this, one would have to explicitly work out the monoidal structure of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ which is induced from the monoidal structure of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

6. Invertible Field Theories

In the section, we consider the case of 2-dimensional oriented *invertible* topological field theories: such theories are in many ways easier to describe than arbitrary TQFTs, and play an important role in condensed matter physics and homotopy theory, as suggested in [Fre14a, Fre14b].

Denote with $\mathrm{Pic}(\mathcal{C})$ the *Picard groupoid* of a symmetric monoidal bicategory $\mathcal{C}$: it is defined as the maximal subgroupoid of $\mathcal{C}$ where the objects are invertible with respect to the monoidal structure of $\mathcal{C}$. Notice that $\mathrm{Pic}(\mathcal{C})$ inherits the symmetric monoidal structure from $\mathcal{C}$. Recall that $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ is equipped with a monoidal structure which is defined pointwise.

Definition 6.1. *An invertible framed TQFT with values in* $\mathcal{C}$ *is an invertible object in* $\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})$*. The space of invertible framed TQFTs with values in* $\mathcal{C}$ *is given by* $\mathrm{Pic}(\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C}))$.

Remark 6.2. Equivalently, an invertible TQFT assigns to the point in $\mathrm{Cob}_{2,1,0}$ an invertible object in $\mathcal{C}$, and to any 1- and 2-dimensional manifold it assigns invertible 1- and 2-morphisms.

Since the Cobordism Hypothesis provides a *monoidal* equivalence between $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ and $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the space of invertible framed TQFTs is given by $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))$, since taking the Picard groupoid behaves well with respect to monoidal equivalences.

We begin by proving the following:

Lemma 6.3. *Let* $\mathcal{C}$ *be a symmetric monoidal bicategory. Then, there is an equivalence of symmetric monoidal bicategories*

$$\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}). \tag{75}$$

Proof. First note that $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is a monoidal 2-groupoid, so there is an equivalence of bicategories $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}^{\mathrm{fd}})$. Now, it suffices to show that every object X in $\mathrm{Pic}(\mathcal{C})$ is already fully-dualizable. Indeed, denote the tensor-inverse of X by X^{-1}. By definition, we have 1-equivalences $X \otimes X^{-1} \cong 1$ and $1 \cong X^{-1} \otimes X$, which serve as evaluation and coevaluation. These maps may be promoted to adjoint 1-equivalences by [SP09, Proposition A.27]. Thus, the evaluation and coevaluation also admit adjoints, which suffices for fully-dualizability. $\square$

Notice that given a monoidal bicategory $\mathcal{C}$, any monoidal auto-equivalence of $\mathcal{C}$ preserves the Picard groupoid of $\mathcal{C}$, since it preserves invertibility of objects and (higher) morphisms. In particular, we have a monoidal 2-functor

$$\mathrm{Aut}_{\otimes}(\mathcal{C}) \to \mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \tag{76}$$

obtained by restriction. Since the $SO(2)$-action induced by the action on $\mathrm{Cob}_{2,1,0}$ is monoidal, it induces an action on $\mathrm{Pic}(\mathcal{C})$. To proceed, we need the following

Lemma 6.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that* $\mathrm{Pic}(\mathcal{C})$ *is monoidally equivalent to* $B^2\mathbb{K}^*$. *Then*

$$\mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*) \tag{77}$$

where the category on the right hand side is regarded as a discrete symmetric monoidal bicategory.

Proof. Since $\mathrm{Pic}(\mathcal{C}) \simeq B^2\mathbb{K}^*$ monoidally, we have to describe the Picard groupoid of the category of monoidal functors from $B^2\mathbb{K}^*$ to $B^2\mathbb{K}^*$. First, notice that the monoidal bicategory $B^2\mathbb{K}^*$ is the strict symmetric monoidal bicategory with a single object $\bullet$, and $B\mathbb{K}^*$ as the strict symmetric monoidal category of 1- and 2-morphisms. The bicategory of symmetric monoidal functors from $B^2\mathbb{K}^*$ to itself is then equivalent to the category $\mathrm{Fun}_{\otimes}(B\mathbb{K}^*, B\mathbb{K}^*)$ regarded as a bicategory with only identity 2-cells; see [CG07] for details.

By direct investigation, $\mathrm{Fun}_{\otimes}(B\mathbb{K}^*, B\mathbb{K}^*)$ is equivalent as a symmetric monoidal category to $\mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*)$ regarded as a discrete category. Indeed, any monoidal functor $F : B\mathbb{K}^* \to B\mathbb{K}^*$ is determined by a group homomorphism $\phi^F : \mathbb{K}^* \to \mathbb{K}^*$, and monoidality ensures that any natural transformation must correspond to the identity element in $\mathbb{K}^*$. Notice that the composition of monoidal functors $F' \circ F$ corresponds to $\phi^{F'} \circ \phi^F$. In follows then that the Picard groupoid of $\mathrm{Fun}_{\otimes}(B^2\mathbb{K}^*, B^2\mathbb{K}^*)$ is given by $\mathrm{Iso}(\mathbb{K}^*)$, which correspond to the invertible elements in the monoid $\mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*)$. □

Examples of symmetric monoidal bicategories satisfying the assumption of Lemma 6.4 are $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$. In general cases, we have the following

Lemma 6.5. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that* $\mathrm{Pic}(\mathcal{C})$ *is monoidally equivalent to* $B^2\mathbb{K}^*$. *Then any monoidal* $SO(2)$*-action on* $\mathrm{Pic}(\mathcal{C})$ *is trivializable.*

Proof. Since we have equivalences of monoidal bicategories $\Pi_2(SO(2)) \simeq B\mathbb{Z}$ and $\mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*)$, monoidal actions correspond to monoidal 2-functors $B\mathbb{Z} \to \mathrm{Iso}(\mathbb{K}^*)$: here we regard $B\mathbb{Z}$ as a symmetric monoidal bicategory with a single object, and the group $\mathrm{Iso}(\mathbb{K}^*)$ as a discrete symmetric monoidal bicategory, i.e. all 1- and 2-cells are identities. Monoidality implies that the single object of $B\mathbb{Z}$ is sent to the identity isomorphism of $\mathbb{K}^*$, which correspond to the identity functor on $\mathrm{Pic}(\mathcal{C})$. This forces the functor to be trivial on objects. It is clear that the action is also trivial on 1-

and 2-morphisms. Since there are no nontrivial morphisms in $\mathrm{Iso}(\mathbb{K}^*)$, the monoidal structure on the action ρ must also be trivial. □

Finally, we need the following

Lemma 6.6. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let ρ_S be the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism as in Example 4.2. Since this action is monoidal, it induces an action on $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C})$ by Lemma 6.3. We have then an equivalence of monoidal bicategories*

$$\mathrm{Pic}\left((\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}\right) \cong \mathrm{Pic}(\mathcal{C})^{SO(2)}. \tag{78}$$

Proof. Theorem 4.3 allows us to compute the two bicategories of homotopy fixed points explicitly: we see that both bicategories have invertible objects X of $\mathcal{C}$, together with the choice of a trivialization of the Serre automorphism as objects. The 1-morphisms of both bicategories are given by 1-equivalences between invertible objects of $\mathcal{C}$, so that the diagram in equation (47) commutes, while 2-morphisms are given by 2-isomorphisms in $\mathcal{C}$. □

The implication of the above lemmas is the following: when $\mathcal{C}$ is a symmetric monoidal bicategory with $\mathrm{Pic}(\mathcal{C}) \cong B^2\mathbb{K}^*$, the action of the Serre-automorphism on framed, invertible field theories with values in $\mathcal{C}$ is trivializable. Thus *all* framed invertible 2d TQFTs with values in $\mathcal{C}$ can be turned into orientable ones.

7. Comments on Homotopy Orbits

So far, we have constructed an $SO(2)$-action on the bicategory $\mathbb{F}_{cfd}$. We have shown how the action on $\mathbb{F}_{cfd}$ induces an action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$, and that via the (framed) Cobordism Hypothesis the induced action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ for framed manifolds agrees with the action of the Serre automorphism. As a consequence, we are able to provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{79}$$

in Corollary 5.9. We could then in principle deduce the Cobordism Hypothesis for oriented manifolds from 79, once we provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}). \tag{80}$$

The above equivalence can be proven directly by using a presentation of the oriented bordism bicategory via generators and relations, given in [SP09], and the notion of a Calabi-Yau object internal to a bicategory. The details appear in [Hes17].

Here, we want instead to comment on an alternative approach. Namely, in order to provide an equivalence as in (80), it suffices to identify the oriented bordism bicategory with the *colimit* of the $SO(2)$-action on $\mathbb{F}_{cfd}$. Indeed, recall that one may define a G-action on a bicategory $\mathcal{C}$ to be a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The tricategorical colimit of this functor will then be the bicategory of co-invariants or *homotopy orbits* of the G-action, denoted by $\mathcal{C}_G$. By Definition of the tricategorical colimit, and the fact that colimits are sent to limits by the Hom functor, we then obtain an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathcal{C}_G, \mathcal{D}) \cong \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})^{G}. \tag{81}$$

The following conjecture is then natural:

Conjecture 7.1. *The bicategory of co-invariants of the $SO(2)$-action on $\mathbb{F}_{cfd}$ is monoidally equivalent to the oriented bordism bicategory, i.e. we have a monoidal equivalence*

$$(\mathbb{F}_{cfd})_{SO(2)} \cong \mathrm{Cob}^{\mathrm{or}}_{2,1,0}\,. \tag{82}$$

Furthermore, the colimit is compatible with the monoidal structure.

Remark 7.2. We believe that this is not an isolated phenomenon, in the sense that any higher bordism category equipped with additional tangential structure should be obtained by taking an appropriate colimit of a G-action on the framed bordism category.

Given Conjecture 7.1 and Equation 81, we obtain the following sequence of monoidal equivalences

$$\begin{aligned}\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \\ &\cong \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}.\end{aligned} \tag{83}$$

Hence Conjecture 7.1 implies the Cobordism Hypothesis for oriented 2-manifolds. Notice that the chain of equivalences in 83 is natural in $\mathcal{C}$.
On the other hand, the Cobordism Hypothesis for oriented manifolds in 2-dimensions implies Conjecture 7.1 . Indeed, by using a tricategorical version of the Yoneda Lemma, as developed for instance in [Buh15], the chain of equivalences

$$\begin{aligned}\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C})\end{aligned} \tag{84}$$

implies that $\mathrm{Cob}^{\mathrm{or}}_{2,1,0}$ is equivalent to $(\mathbb{F}_{cfd})_{SO(2)}$, due to the uniqueness of representable objects.
We summarize the above arguments in the following

Lemma 7.3. *The Cobordism Hypothesis for oriented 2-dimensional manifolds is equivalent to Conjecture 7.1.*

It would then be of great interest to develop concrete constructions of homotopy co-invariants of actions of groups on bicategories, in the same spirit of [HSV17] and the present work, in order to verify directly the equivalence in Conjecture 7.1, and to extend the above arguments to general tangential G-structures.

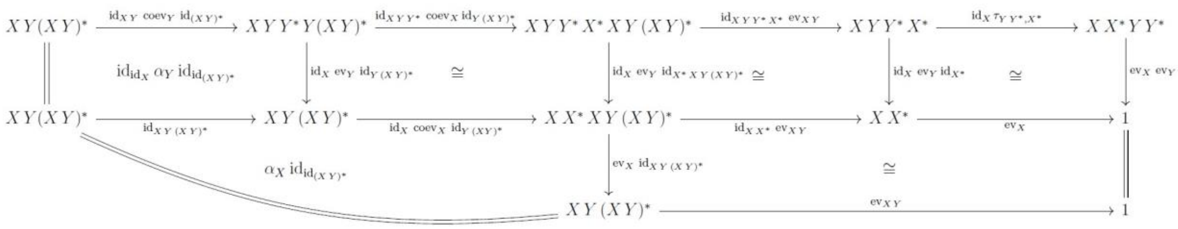


Figure 1: Diagram for the proof of Lemma 2.10

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