CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LVI-3 (2015)

GENERALISED PUSHOUTS, CONNECTED COLIMITS AND CODISCRETE GROUPOIDS

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Résumé. Nous étudions brièvement une espèce de colimites, appellée ici 'pushout généralisé'. On prouve que, dans une catégorie quelconque, l'existence de ces colimites correspond à celle des colimites connexes; dans le cas fini on sait que ceci se reduit à l'existence de pushouts ordinaires et coégalisateurs (R. Paré, 1993). Cette étude a été motivée par la remarque que tout groupoïde est, *à équivalence près*, un pushout généralisé de groupoïdes codiscrets. Pour les groupoïdes fondamentaux d'espaces convenables nous donnons des résultats plus fins concernant des pushouts généralisés *finis*.

Abstract. This is a brief study of a particular kind of colimit, called a 'generalised pushout'. We prove that, in any category, generalised pushouts amount to connected colimits; in the finite case the latter are known to amount to ordinary pushouts and coequalisers (R. Paré, 1993). This study was motivated by remarking that every groupoid is, *up to equivalence*, a generalised pushout of codiscrete subgroupoids. For the fundamental groupoids of suitable spaces we get finer results concerning *finite* generalised pushouts.

Keywords. generalised pushout, connected colimit, fundamental groupoid.

Mathematics Subject Classification (2010). 18A30, 55A5.

1. Introduction

We are interested in colimits of a particular form, that will be called 'generalised pushouts' (see Section 2), following a line well represented in category theory: to study particular classes of (co)limits, like filtered colimits, flexible (co)limits, connected (co)limits, etc.

Generalised pushouts are *connected, non-simply-connected colimits* and therefore cannot be reduced to ordinary pushouts (see R. Paré [P1, P2]). We show in Sections 2 and 3 that generalised pushouts give all connected colimits and - in the finite case - amount to finite connected colimits; the latter can be reduced to pushouts and coequalisers, as proved in [P2].

The second part is about colimits of groupoids. In Section 4 we show that - up to categorical equivalence - *every* groupoid is a generalised pushout of codiscrete subgroupoids.

In the last Section 5 we consider cases where the fundamental groupoid $\pi_1 X$ of a space can be obtained as a *finite* colimit of this kind.

For instance, an obvious cover of the circle $\mathbf{S}^1$ with three open arcs X_1, X_2, X_3 shows that the fundamental groupoid $\pi_1(\mathbf{S}^1)$ is a *3-generalised pushout* of the codiscrete subgroupoids $\pi_1 X_i$ over codiscrete subgroupoids. Moreover if we form a subset A by choosing a point in each of the three intersections $X_i \cap X_j$, the (equivalent) restricted groupoid $\pi_1(\mathbf{S}^1)|A$ (with objects in A) is a 3-generalised pushout of *finite* codiscrete subgroupoids.

There are similar results for a compact differentiable manifold (Corollary 5.3), while a sphere with countably many handles would require a countable generalised pushout. More generally these facts hold for spaces having 'sufficiently good' covers, as we show in Theorem 5.2.

Let us stress the point that this article is not about the *concrete computation* of fundamental groupoids, which is already well covered in the literature (see [Bw2, BwS] and references therein): our goal is to isolate a kind of connected colimit motivated by the previous arguments and to study its categorical aspects.

As a related question, suggested by F.W. Lawvere, one might investigate 'codiscretely generated' toposes, like symmetric simplicial sets [Gr]. The van Kampen theorem for lextensive categories of [BwJ] might also be examined in the present line.

Here a connected category is assumed to be non-empty, as usual; but we do not follow this convention for spaces.

Acknowledgments. We gratefully acknowledge helpful information and suggestions from R. Brown, G. Janelidze, F.W. Lawvere and R. Paré. This work was supported by a PRIN Research Project and a Research Contract of Università di Genova.

2. Generalised pushouts

In an arbitrary category $\mathbf{C}$ we are interested in colimits of a particular form, that will be called 'generalised pushouts'.

We start from a non-empty set I and form the set $\hat{I}$ of its subsets $\{i, j\}$ having one or two elements, ordered by the relation $\{i, j\} \leq \{i\}$ and viewed as a connected category.

A functor $X \colon \hat{I} \to \mathbf{C}$ will be (partially) represented as the left diagram below

$$\begin{array}{ccc} & & X_i \\ & \overset{u_{ij}}{\nearrow} & \\ X_{ij} & & \\ & \underset{u_{ji}}{\searrow} & \\ & & X_j \end{array} \qquad (i, j \in I), \qquad \begin{array}{ccccc} & & X_i & & \\ & \nearrow & & \overset{f_i}{\searrow} & \\ X_{ij} & & & & Y \\ & \searrow & & \underset{f_j}{\nearrow} & \\ & & X_j & & \end{array} \tag{1}$$

A cocone of vertex Y amounts to family $f_i \colon X_i \to Y$ of morphisms of $\mathbf{C}$ such that all the right-hand squares above commute. The colimit will be called the *generalised pushout* of the objects X_i *over* the objects X_{ij}. We speak of a *finite* generalised pushout, or n*-generalised pushout* when the set I is finite, with n elements.

Plainly, 1-generalised pushouts are trivial and 2-generalised pushouts are ordinary pushouts. A 3-generalised pushout is the colimit of a diagram $X \colon \hat{I} \to \mathbf{C}$ with $I = \{1, 2, 3\}$

$$\begin{array}{ccccc} & & 3 & & \\ & \nearrow & & \nwarrow & \\ \{1,3\} & & & & \{2,3\} \\ \swarrow & & & & \searrow \\ 1 & \longleftarrow & \{1,2\} & \longrightarrow & 2 \end{array} \quad \hat{I} \qquad \begin{array}{ccccc} & & X_3 & & \\ & \nearrow & & \nwarrow & \\ X_{13} & & & & X_{23} \\ \swarrow & & & & \searrow \\ X_1 & \longleftarrow & X_{12} & \longrightarrow & X_2 \end{array} \quad X \tag{2}$$

as in the example of the Introduction for $\pi_1(\mathbf{S}^1)$. We prove below that they give all finite generalised pushouts and that their existence amounts to that of pushouts and coequalisers (Section 3).

The fact that pushouts are not sufficient to construct all finite generalised pushouts is already known from a paper of R. Paré [P1] (see Theorem 2) that characterises the limits that can be constructed with pullbacks. In fact the category $\hat{I}$ associated to $I = \{1, 2, 3\}$ (or any larger set) is not *simply connected*: the left figure above shows a non-trivial loop, that gives a non-trivial endomorphism in $\pi_1(\hat{I}) = \pi_1(\hat{I}^{op})$ (cf. [P1]).

3. Generalised pushouts and connected colimits

We shall now make use of a second paper of R. Paré on connected limits [P2], recalling a result from its Section 4 (written here in dual form).

Theorem 3.1 (R. Paré)**.** *The category* $\mathbf{C}$ *has arbitrary (resp. finite) connected colimits if and only if it has arbitrary (resp. finite) fibred coproducts and coequalisers.*

Here a *fibred coproduct* is the colimit of a family $h_i \colon X \to X_i$ of morphisms indexed by a non-empty set I. Of course the existence of finite fibred coproducts is equivalent to the existence of pushouts.

Lemma 3.2. *If the category* $\mathbf{C}$ *has arbitrary (resp. finite) generalised pushouts then it has arbitrary (resp. finite) fibred coproducts.*

Proof. Starting from a family $(h_i \colon X \to X_i)_{i \in I}$ $(I \neq \emptyset)$, we transform it into a diagram $X \colon \hat{I} \to \mathbf{C}$. After the given objects X_i, we let $X_{ij} = X$ for $i \neq j$ and

$$u_{ij} = h_i \colon X \to X_i, \text{ for } i \neq j, \qquad\qquad u_{ii} = \mathrm{id}X_i.$$

Now the given diagram $(h_i \colon X \to X_i)$ and the new one have the same cocones, namely the families of morphisms $f_i \colon X_i \to Y$ such that $f_i h_i = f_j h_j$ for all indices i, j. □

Lemma 3.3. *If the category* $\mathbf{C}$ *has 3-generalised pushouts then it has ordinary pushouts and coequalisers.*

Proof. Suppose that $\mathbf{C}$ has 3-generalised pushouts. To prove the existence of pushouts, starting from the left diagram below

$$X_{12} \begin{array}{l} \xrightarrow{u_{12}} X_1 \\ \xrightarrow[u_{21}]{} X_2 \end{array} \qquad X_{13} \begin{array}{l} \xrightarrow{u_{13}} X_1 \\ \xrightarrow[u_{31}]{} X_3 \end{array} \qquad X_{23} \begin{array}{l} \xrightarrow{u_{23}} X_2 \\ \xrightarrow[u_{32}]{} X_3 \end{array} \tag{3}$$

we extend it to a diagram on $I = \{1, 2, 3\}$, where $X_3 = X_2$, the second span above coincides with the first and the third consists of identities $u_{23} = u_{32} = \mathrm{id}X_2$. Plainly the extended diagram has the same cocones as the original one and its colimit is the pushout of the latter.

To prove that $\mathbf{C}$ has coequalisers, we start from two arrows $u, v \colon X \to Y$ and construct a new diagram on $I = \{1, 2, 3\}$, where $X_1 = X_2 = X_3 = Y$ and the three spans are specified below

$$X \begin{array}{l} \xrightarrow{u} X_1 \\ \xrightarrow[v]{} X_2 \end{array} \qquad Y \begin{array}{l} \xrightarrow{\mathrm{id}} X_1 \\ \xrightarrow[\mathrm{id}]{} X_3 \end{array} \qquad Y \begin{array}{l} \xrightarrow{\mathrm{id}} X_2 \\ \xrightarrow[\mathrm{id}]{} X_3 \end{array} \tag{4}$$

For the new diagram, a cocone $(f_i \colon X_i \to Z)$ is a map $f = f_1 = f_2 = f_3 \colon Y \to Z$ such that $fu = fv$; the 3-generalised pushout is thus the coequaliser of u, v. □

Corollary 3.4. *The following conditions on a category* **C** *are equivalent:*

(a) **C** *has generalised pushouts,*

(b) **C** *has fibred coproducts and coequalisers,*

(c) **C** *has connected colimits.*

Proof. (a) ⇒ (b) From the previous lemmas. (b) ⇒ (c) From Theorem 3.1. (c) ⇒ (a) Obvious. □

Corollary 3.5. *The following conditions on a category* **C** *are equivalent:*

(a) **C** *has finite generalised pushouts,*

(b) **C** *has 3-generalised pushouts,*

(c) **C** *has pushouts and coequalisers,*

(d) **C** *has finite connected colimits.*

Proof. (d) ⇒ (a) ⇒ (b) Obvious. (b) ⇒ (c) From 3.3. (c) ⇒ (d) From 3.1. □

We end by remarking that the 'weight' of generalised pushouts within finite colimits can be measured noting that each of the conditions below implies the following one

(a) **C** has finite colimits,

(b) **C** has finite generalised pushouts, or (equivalently) finite connected colimits, or pushouts and coequalisers, or 3-generalised pushouts,

(c) **C** has pushouts.

These implications cannot be reversed. In fact the category of non-empty sets has all colimits of non-empty diagrams but lacks an initial object. Secondly, every groupoid has (trivial) pushouts, but it has coequalisers if and only if it is an equivalence relation.

4. Generalised pushouts of groupoids

We prove now that all groupoids can be obtained as generalised pushouts of codiscrete groupoids. Of course a groupoid G is said to be *codiscrete*, or chaotic, if it has precisely one arrow $x \to y$ for any two vertices x, y; this means that G is either empty or equivalent to the singleton groupoid.

Proposition 4.1. *Every groupoid is categorically equivalent to a groupoid G that is a generalised pushout of finite codiscrete groupoids and inclusions. These groupoids can be assumed to be subgroupoids of G, and non-empty if G is connected.*

Proof. As a motivation of making appeal to categorical equivalence, note that a group has only one codiscrete subgroupoid, the trivial one, and cannot be a generalised pushout of codiscrete *sub*groupoids - unless it is trivial. (But one can prove, by an argument similar to the following one, that every connected groupoid on at least three vertices *is a generalised pushout of codiscrete subgroupoids and inclusions*.)

We can suppose that our groupoid is connected (non-empty); then it is equivalent to its skeleton, which is a group G_0, and we replace the latter with an equivalent groupoid G on three vertices, say 1, 2, 3. Let I be the set of commutative diagrams in G of the following form

$$\begin{array}{ccccc} & & 2 & & \\ & {\scriptstyle x}\nearrow & & \searrow{\scriptstyle y} & \\ 1 & & \xrightarrow[z]{\hspace{3em}} & & 3 \end{array} \qquad\qquad z = yx. \tag{5}$$

It will be convenient to denote this diagram by the triple (x, y, z), even though each pair of these arrows determines the third; we write as $F(x, y, z)$ the subgroupoid of G generated by these arrows: it is formed by all the objects, their identities, the three given maps and their inverses. I can be identified with the set of all the wide codiscrete subgroupoids of G. It is also easy to see that the set I is in bijective (non-canonical) correspondence with $G_0{\times}G_0$: after fixing a diagram (5) and identifying $G(1,1)$ with G_0, each pair $(g, h) \in G_0 \times G_0$ determines a triple (xg, y', zh), with $y' = zhg^{-1}x^{-1}$.

We now form a diagram $F \colon \hat{I} \to \mathbf{Gpd}$ of finite codiscrete groupoids and inclusions. F is already defined on the triples $(x, y, z) \in I$. For two *distinct* triples (x, y, z), (x', y', z') we distinguish two cases:

(i) if $x = x'$ or $y = y'$ or $z = z'$, we let

$$F(x, y, z; x', y', z') = (F(x, y, z) \cap F(x', y', z'))\hat{},$$

where $(-)\hat{}$ means taking out the isolated vertex (since the intersection itself is not codiscrete),
(ii) otherwise we let $F(x, y, z; x', y', z')$ be the (co)discrete groupoid on the vertex 1 (since we do not want to use the empty groupoid).

It is now evident that every cocone $f_{xyz}\colon F(x, y, z) \to H$ (for $(x, y, z) \in I$) of F has precisely one extension to a 'mapping' $f\colon G \to H$, which necessarily preserves identities and composition. The only point that is not completely trivial is showing that two arbitrary morphisms f_{xyz} and $f_{x'y'z'}$ of this cocone coincide on each vertex. Working for instance on the vertex 3, it is sufficient to consider the objects

$$F(x, y, z), \qquad F(x'', y', z), \qquad F(x', y', z') \qquad\qquad (x'' = z^{-1}y'),$$

so that the cocone condition gives: $f_{xyz}(3) = f_{x''y'z}(3) = f_{x'y'z'}(3)$. □

5. Good covers of spaces and manifolds

We end by investigating cases where the fundamental groupoid $\pi_1(X)$ of a space can be obtained as a finite generalised pushout of codiscrete groupoids.

First we need an extension of the van Kampen theorem for fundamental groupoids, as formulated by R. Brown in [Bw1, Bw2].

Theorem 5.1. *Let X be a space equipped with a family of subspaces $(X_i)_{i\in I}$ such that X is covered by their interior parts. Then the fundamental groupoid $\pi_1 X$ is (strictly) the generalised pushout of the groupoids $\pi_1 X_i$ over the groupoids $\pi_1(X_i \cap X_j)$.*

Proof. The binary case, concerning a pushout, is proved in [Bw2], Section 6.7.2. This extension can be proved by the same argument. (See also Exercise 6 of [Bw2], 6.7.) □

Secondly we recall (from [BT], Theorem 5.1) that every differentiable n-manifold X has a *good cover*, i.e. an open cover $(X_i)_{i\in I}$ such that all non-empty intersections $X_{i_1} \cap ... \cap X_{i_k}$ are diffeomorphic to $\mathbb{R}^n$. (Up to dimension 3, but not beyond, every topological manifold has a differentiable structure, so that this result can be extended [Sc, Mo].)

With this motivation we consider a topological space X having an open cover (X_i) such that all subspaces $X_i \cap X_j$ (including the original X_i, of course) are 1-connected: in other words we assume that the following fundamental groupoids are codiscrete

$$\pi_1(X_i) = C(X_i), \quad \pi_1(X_i \cap X_j) = C(X_i \cap X_j) = C(X_i) \cap C(X_j), \tag{6}$$

where $C(S)$ denotes the codiscrete groupoid on a set of objects S, possibly empty.

Theorem 5.2. *(a) In this hypothesis the fundamental groupoid $\pi_1 X$ is (in a strict sense) the generalised pushout of* a diagram $C\colon \hat{I} \to \mathbf{Gpd}$ of codiscrete groupoids and inclusions

$$C_{ij} \overset{u_{ij}}{\nearrow} C_i, \quad C_{ij} \underset{u_{ji}}{\searrow} C_j \qquad C_i = C(X_i), \qquad C_{ij} = C(X_i \cap X_j) = C_i \cap C_j. \tag{7}$$

All these objects are subgroupoids of $\pi_1 X$, the colimit.

(b) If X has a finite *cover (X_i) of the previous type (which is a consequence of the previous assumption, for a compact X), then $\pi_1 X$ is a* finite *generalised pushout of codiscrete groupoids and inclusions.*

(c) If, moreover, each subspace $X_{ijk} = X_i \cap X_j \cap X_k$ has a finite number of path components, we can choose a finite subset A of X which meets every such component so that the (equivalent) restricted groupoid $\pi_1 X|A$ is a finite *generalised pushout of* finite *codiscrete subgroupoids and inclusions.*

Proof. The first two points follow from Theorem 5.1. Point (c) follows from the Main Theorem of Brown - Salleh [BwS]. This article also shows that - here - it is not sufficient to consider the binary intersections $X_i \cap X_j$; the authors are indebted to R. Brown for helpful comments on this aspect. □

Corollary 5.3. *These results are automatically true for every differentiable manifold X. In the compact case, also the additional hypotheses of 5.2 (b), (c) automatically hold.*

Proof. Follows from the existence of good covers recalled above. □

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