CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
VOLUME LX-4 (2019)

GOURSAT COMPLETIONS

Diana RODELO and Idriss TCHOFFO NGUEFEU

Résumé. Nous caractérisons les catégories avec limites finies faibles dont les completions régulieres sont des catégories de Goursat.

Abstract. We characterize categories with weak finite limits whose regular completions give rise to Goursat categories.

Keywords. regular category, projective cover, Goursat category, 3-permuta–ble variety.

Mathematics Subject Classification (2010). 08C05, 18A35, 18B99,18E10.

1. Introduction

The construction of the free exact category over a category with finite limits was introduced in [3]. It was later improved to the construction of the free exact category over a category with finite weak limits (*weakly lex*) in [4]. This was possible because the uniqueness of the finite limits of the original category was never used in the construction; only the existence. In [4], the authors also considered the free regular category over a weakly lex one.

An important property of the free exact (or regular) construction is that such categories always have enough (regular) projectives. In fact, an exact category $\mathbb{A}$ may be seen as the exact completion of a weakly lex category if and only if it has enough projectives. If so, then $\mathbb{A}$ is the exact completion of any of its *projective covers*. Such a phenomenon is captured by varieties of universal algebras: they are the exact completions of their full subcategory of free algebras.

Having this link in mind, our main interest in studying this subject is to characterize projective covers of certain algebraic categories through simple properties involving projectives and to relate those properties to the known varietal characterizations in terms of the existence of operations of their varietal theories. Such kind of studies have been done for the projective covers of categories which are: Mal'tsev [11], protomodular and semi-abelian [5], (strongly) unital and subtractive [6].

The aim of this work is to obtain characterizations of the weakly lex categories whose regular completion is a Goursat (=3-permutable) category (Propositions 4.5 and 4.7). We then relate them to the existence of the quaternary operations which characterize the varieties of universal algebras which are 3-permutable (Remark 4.8).

2. Preliminaries

In this section, we briefly recall some elementary categorical notions needed in the following.

A category with finite limits is **regular** if regular epimorphisms are stable under pullback, and if kernel pairs have coequalizers. Equivalently, any arrow $f : A \longrightarrow B$ has a unique factorization $f = ir$ (up to isomorphism), where r is a regular epimorphism and i is a monomorphism and this factorization is pullback stable.

A **relation** R from X to Y is a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$. The opposite relation of R, denoted R^o, is the relation from Y to X given by the subobject $\langle r_2, r_1\rangle : R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. We shall identify a morphism $f : X \longrightarrow Y$ with the relation $\langle 1_X, f\rangle : X \rightarrowtail X \times Y$ and write f^o for its opposite relation. Given two relations $R \rightarrowtail X \times Y$ and $S \rightarrowtail Y \times Z$ in a regular category, we write $SR \rightarrowtail X \times Z$ for their relational composite. With the above notations, any relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ can be seen as the relational composite $r_2 r_1^o$. The properties collected in the following lemma are well known and easy to prove (see for instance [1]):

Lemma 2.1. *Let $f : X \longrightarrow Y$ be an arrow in a regular category $\mathbb{C}$, and let $f = ir$ be its (regular epimorphism, monomorphism) factorization. Then:*

1. *$f^o f$ is the kernel pair of f, thus $1_X \leqslant f^o f$; moreover, $1_X = f^o f$ if and only if f is a monomorphism;*

2. *ff^o is (i,i), thus $ff^o \leqslant 1_Y$; moreover, $ff^o = 1_Y$ if and only if f is a regular epimorphism;*

3. *$ff^of = f$ and $f^off^o = f^o$.*

A relation R on X is **reflexive** if $1_X \leqslant R$, **symmetric** if $R^o \leqslant R$, and **transitive** if $RR \leqslant R$. As usual, a relation R on X is an **equivalence relation** when it is reflexive, symmetric and transitive. In particular, a kernel pair $\langle f_1, f_2\rangle : \mathrm{Eq}(f) \rightarrowtail X \times X$ of a morphism $f : X \longrightarrow Y$ is an equivalence relation.

By dropping the assumption of uniqueness of the factorization in the definition of a limit, one obtains the definition of a weak limit. We call **weakly lex** any category with weak finite limits.

An object P in a category is (regular) **projective** if, for any arrow $f : P \longrightarrow X$ and for any regular epimorphism $g : Y \twoheadrightarrow X$ there exists an arrow $h : P \longrightarrow Y$ such that $gh = f$. We say that a full subcategory $\mathbb{C}$ of $\mathbb{A}$ is a **projective cover** of $\mathbb{A}$ if two conditions are satisfied:

- any object of $\mathbb{C}$ is regular projective in $\mathbb{A}$;
- for any object X in $\mathbb{A}$, there exists a ($\mathbb{C}$-)cover of X, that is an object C in $\mathbb{C}$ and a regular epimorphism $C \twoheadrightarrow X$.

When $\mathbb{A}$ admits a projective cover, one says that $\mathbb{A}$ has *enough projectives*.

Remark 2.2. If $\mathbb{C}$ is a projective cover of a weakly lex category $\mathbb{A}$, then $\mathbb{C}$ is also weakly lex [4]. For example, let X and Y be objects in $\mathbb{C}$ and $X \longleftarrow W \longrightarrow Y$ a weak product of X and Y in $\mathbb{A}$. Then, for any cover $\bar{W} \twoheadrightarrow W$ of W, $X \longleftarrow \bar{W} \longrightarrow Y$ is a weak product of X and Y in $\mathbb{C}$. Furthermore, if $\mathbb{A}$ is a regular category and $X \longleftarrow P \longrightarrow Y$ a weak product of X and Y in $\mathbb{C}$, then the induced morphism $P \twoheadrightarrow X \times Y$ is a regular epimorphism. Similar remarks apply to all weak finite limits.

3. Goursat categories

In this section we review the notion of Goursat category and the characterizations of Goursat categories through regular images of equivalence relations and through Goursat pushouts.

Definition 3.1. [2, 1] *A regular category $\mathbb{C}$ is called a **Goursat category** when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.*

When $\mathbb{C}$ is a regular category, (R, r_1, r_2) is a relation on X and $f : X \twoheadrightarrow Y$ is a regular epimorphism, we define the **regular image of R along f** to be the relation $f(R)$ on Y induced by the (regular epimorphism, monomorphism) factorization $\langle s_1, s_2\rangle\psi$ of the composite $(f \times f)\langle r_1, r_2\rangle$:

$$\begin{array}{ccc} R & \xrightarrow{\psi} & f(R) \\ {\scriptstyle\langle r_1,r_2\rangle}\downarrow & & \downarrow{\scriptstyle\langle s_1,s_2\rangle} \\ X \times X & \xrightarrow[f\times f]{} & Y \times Y. \end{array}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2r_1^of^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric. In a general regular category $f(R)$ is not necessarily an equivalence relation. This is the case in a *Goursat category* according to the following theorem.

Theorem 3.2. [1] *A regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation R on X, the regular image $f(R) = fRf^o$ of R along f is an equivalence relation.*

If $\langle e_1, e_2\rangle : E \rightarrowtail X \times X$ is a reflexive relation, then the regular image of e_2 along the kernel pair of e_1 is given by $e_2(\mathrm{Eq}(e_1)) = e_2e_1^oe_1e_2^o = EE^o$. Goursat categories may also be characterized by such regular images:

Theorem 3.3. [1] *A regular category $\mathbb{C}$ is a Goursat category if and only if for any reflexive relation E, EE^o is an equivalence relation.*

Goursat categories are well known in Universal Algebra. In fact, by a classical theorem in [10], a variety of universal algebras is a Goursat category precisely when its theory has two quaternary operations p and q such that the identities $p(x, y, y, z) = x$, $q(x, y, y, z) = z$ and $p(x, x, y, y) = q(x, x, y, y)$ hold. Accordingly, the varieties of groups, Heyting algebras and implication algebras are Goursat categories. The category of topological groups, Hausdorff groups, right complemented semi-groups are also Goursat categories.

There are many known characterizations of Goursat categories (see [1, 7, 8, 9] for instance). In particular the following characterization, through Goursat pushouts, will be useful:

Theorem 3.4. [7] *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Goursat category;*

(ii) *any commutative diagram of type* (I) *in $\mathbb{C}$, where α and β are regular epimorphisms and f and g are split epimorphisms*

$$
\begin{array}{ccc}
X & \overset{\alpha}{\twoheadrightarrow} & U \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & \text{(I)} & {\scriptstyle t}\uparrow\downarrow{\scriptstyle g} \\
Y & \underset{\beta}{\twoheadrightarrow} & W,
\end{array}
\qquad
\begin{array}{l}
g\alpha=\beta f \\
\alpha s=t\beta
\end{array}
$$

(which is necessarily a pushout) is a ***Goursat pushout****: the morphism $\lambda : \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$, induced by the universal property of kernel pair $\mathrm{Eq}(g)$ of g, is a regular epimorphism.*

Remark 3.5. Diagram (I) is a Goursat pushout precisely when the regular image of $\mathrm{Eq}(f)$ along α is (isomorphic to) $\mathrm{Eq}(g)$. From Theorem 3.4, it then follows that a regular category $\mathbb{C}$ is a Goursat category if and only if for any commutative diagram of type (I) one has $\alpha(\mathrm{Eq}(f)) = \mathrm{Eq}(g)$.

Note that Theorem 3.2 characterizes Goursat categories through the property that regular images of equivalence relations are equivalence relations, while Theorem 3.4 characterizes them through the property that regular images of certain kernel pairs are kernel pairs.

4. Projective covers of Goursat categories

In this section, we characterize the categories with weak finite limits whose regular completion are Goursat categories.

Definition 4.1. *Let $\mathbb{C}$ be a weakly lex category:*

1. *a **pseudo-relation** on an object X of $\mathbb{C}$ is a pair of parallel arrows* $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$; *a pseudo-relation is a relation if r_1 and r_2 are jointly monomorphic;*

2. *a pseudo-relation* $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ *on X is said to be:*

   - ***reflexive** when there is an arrow $r : X \longrightarrow R$ such that $r_1 r = 1_X = r_2 r$;*
   - ***symmetric** when there is an arrow $\sigma : R \longrightarrow R$ such that $r_2 = r_1\sigma$ and $r_1 = r_2\sigma$;*
   - ***transitive** if by considering a weak pullback*

$$\begin{array}{ccc} W & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X, \end{array}$$

   there is an arrow $t : W \longrightarrow R$ such that $r_1 t = r_1 p_1$ and $r_2 t = r_2 p_2$.
   - *a **pseudo-equivalence relation** if it is reflexive, symmetric and transitive.*

Remark that the transitivity of a pseudo-relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ does not depend on the choice of the weak pullback of r_1 and r_2; in fact, if

$$\begin{array}{ccc} \bar{W} & \xrightarrow{\bar{p_2}} & R \\ {\scriptstyle \bar{p_1}}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X, \end{array}$$

is another weak pullback, the factorization $\bar{W} \longrightarrow W$ composed with the transitivity arrow $t : W \longrightarrow R$ ensures that the pseudo-relation is transitive also with respect to the second weak pullback.

The following property from [12] (Proposition 1.1.9) will be useful in the sequel:

Proposition 4.2. [12] *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be a pseudo-relation in $\mathbb{C}$ and consider its (regular epimorphism, monomorphism) factorization in $\mathbb{A}$*

$$\begin{array}{ccc} R & \xrightarrow{(r_1,r_2)} & X \times X. \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle (e_1,e_2)} & \\ & E & \end{array}$$

Then, R is a pseudo-equivalence relation in $\mathbb{C}$ if and only if E is an equivalence relation in $\mathbb{A}$.

In order to characterize the projective covers $\mathbb{C}$ of Goursat categories $\mathbb{A}$, we should consider good properties characterizing Goursat categories which easily translate to the weakly lex context. A possible translation of the property in Theorem 3.2 should replace equivalence relations in $\mathbb{A}$ with pseudo-equivalence relations in $\mathbb{C}$ and regular epimorphisms in $\mathbb{A}$ with split epimorphisms in $\mathbb{C}$ (a regular epimorphism in $\mathbb{A}$ with a projective codomain is necessarily a split epimorphism). Thus, we introduce:

Definition 4.3. *Let $\mathbb{C}$ be a weakly lex category. We call $\mathbb{C}$ a **weak Goursat category** if, for any pseudo-equivalence relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ and any split epimorphism $X \overset{f}{\rightleftarrows} Y$, the composite $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation.*

Lemma 4.4. *If $\mathbb{C}$ is a regular weak Goursat category, then $\mathbb{C}$ is a Goursat category.*

Proof. We shall prove that for any reflexive relation $\langle e_1, e_2 \rangle : E \rightarrowtail X \times X$, EE^o is an equivalence relation (Theorem 3.3).

Consider the (pseudo-)equivalence relation $\mathrm{Eq}(e_1) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} E$ and the split epimorphism e_2 (which is split by the reflexivity arrow). By assumption $\mathrm{Eq}(e_1) \underset{e_2\pi_2}{\overset{e_2\pi_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation. Its (regular epimorphism, monomorphism) factorization defines the regular image $e_2(\mathrm{Eq}(e_1)) = EE^o$

$$\begin{array}{ccc} \mathrm{Eq}(e_1) & \twoheadrightarrow & e_2(\mathrm{Eq}(e_1)) = EE^o \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & \downarrow\downarrow \\ E & \underset{e_2}{\overset{\leftarrow - - - - -}{\longrightarrow}} & X; \end{array}$$

thus EE^o is an equivalence relation. $\square$

We use Remark 2.2 repeatedly in the next results.

Proposition 4.5. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Then $\mathbb{A}$ is a Goursat category if and only if $\mathbb{C}$ is a weak Goursat category.*

Proof. Since $\mathbb{C}$ is a projective cover of a regular category $\mathbb{A}$, $\mathbb{C}$ is weakly lex.

Suppose that $\mathbb{A}$ is a Goursat category. Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and let $X \overset{f}{\underset{\leftarrow - -}{\longrightarrow}} Y$ be a split epimorphism in $\mathbb{C}$. For the (regular epimorphism, monomorphism) factorizations of $\langle r_1, r_2 \rangle$ and $\langle fr_1, fr_2 \rangle$ we get the following diagram

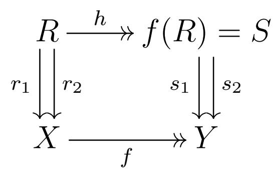
$$\begin{array}{ccccc} R & & \xrightarrow{\langle r_1, r_2 \rangle} & & X \times X \\ \| & {\scriptstyle p}\searrow & & \nearrow{\scriptstyle \langle e_1, e_2 \rangle} & \\ \| & & E & & \\ \| & & \vdots\, {\scriptstyle w} & & \downarrow {\scriptstyle f \times f} \\ \| & & S & & \\ \| & {\scriptstyle q}\nearrow & & \searrow{\scriptstyle \langle s_1, s_2 \rangle} & \\ R & & \xrightarrow[\langle fr_1, fr_2 \rangle]{} & & Y \times Y, \end{array} \tag{1}$$

where $w : E \longrightarrow S$ is induced by the strong epimorphism p

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & E \\ {\scriptstyle q}\downarrow & \overset{w}{\swarrow} & \downarrow {\scriptstyle (f \times f)\langle e_1, e_2 \rangle} \\ S & \underset{\langle s_1, s_2 \rangle}{\rightarrowtail} & Y \times Y. \end{array}$$

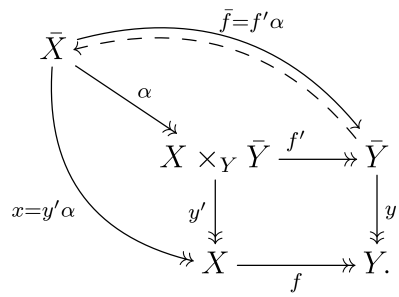
Then w is a regular epimorphism and by the commutativity of the right side of (1), one has $S = f(E)$. By Proposition 4.2, we know that E is an equivalence relation in $\mathbb{A}$. Since $\mathbb{A}$ is a Goursat category and f is a regular epimorphism (being a split one), then $S = f(E)$ is also an equivalence relation in $\mathbb{A}$ (Theorem 3.2) and by Proposition 4.2, we can conclude that $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation in $\mathbb{C}$.

Conversely, suppose that $\mathbb{C}$ is a weak Goursat category. Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be an equivalence relation in $\mathbb{A}$ and $f : X \twoheadrightarrow Y$ a regular epimorphism. We are going to show that $f(R) = S$



is an equivalence relation; it is obviously reflexive and symmetric. In order to conclude that $\mathbb{A}$ is a Goursat category, we must prove that S is transitive.

We begin by covering the regular epimorphism f in $\mathbb{A}$ with a split epimorphism $\bar{f}$ in $\mathbb{C}$. For that we take the cover $y : \bar{Y} \twoheadrightarrow Y$, consider the pullback of y and f in $\mathbb{A}$ and take its cover $\alpha : \bar{X} \twoheadrightarrow X \times_Y \bar{Y}$

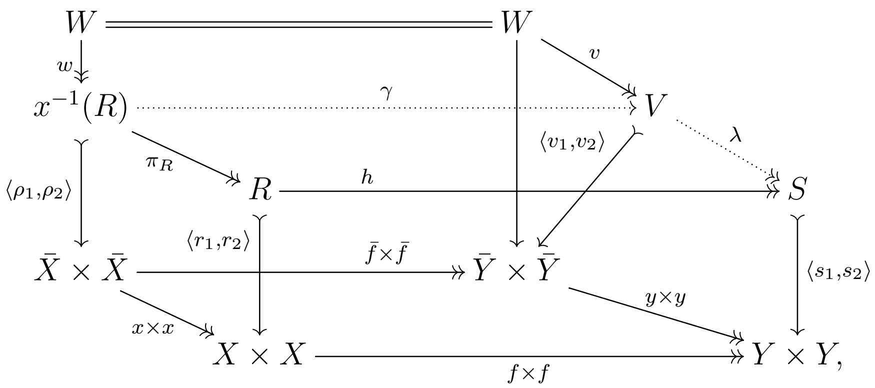


Since $\bar{f} = f'\alpha$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, it is a split epimorphism. Note that the above outer diagram is a *regular pushout*, so that

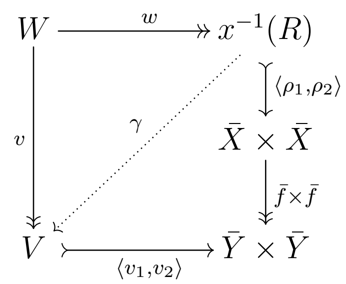
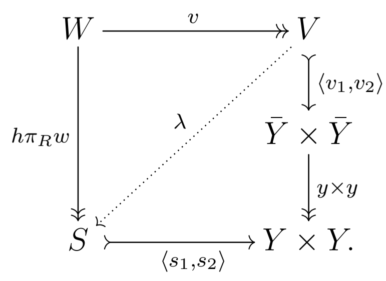
$$f^o y = x\bar{f}^o \quad \text{and} \quad y^o f = \bar{f} x^o \tag{2}$$

(Proposition 2.1 in [1]).

Next, we take the inverse image $x^{-1}(R)$ in $\mathbb{A}$, which is an equivalence relation since R is, and cover it to obtain a pseudo-equivalence $W \rightrightarrows \bar{X}$ in $\mathbb{C}$. By assumption $W \rightrightarrows \bar{X} \xrightarrow{\bar{f}} \bar{Y}$ is a pseudo-equivalence relation in $\mathbb{C}$ so it factors through an equivalence relation, say $V \underset{v_2}{\overset{v_1}{\rightrightarrows}} \bar{Y}$, in $\mathbb{A}$. We have



where γ and λ are induced by the strong epimorphisms w and v, respectively

 and 

Since γ is a regular epimorphism, we have $V = \bar{f}(x^{-1}(R))$. Since λ is a regular epimorphism, we have $S = y(V)$. One also has $V = y^{-1}(S)$ because

$$\begin{array}{rcll} y^{-1}(S) & = & y^o S y & \\ & = & y^o f(R) y & \\ & = & y^o f R f^o y & \\ & = & \bar{f} x^o R x \bar{f}^o & \text{(by (2))} \\ & = & \bar{f}(x^{-1}(R)) & \\ & = & V. & \end{array}$$

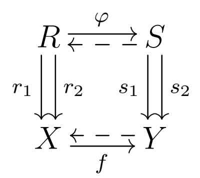
Finally, S is transitive since

$$\begin{array}{rcll} SS & = & yy^oSyy^oSyy^o & \text{(Lemma 2.1(2))} \\ & = & yy^{-1}(S)y^{-1}(S)y^o & \\ & = & yVVy^o & \\ & = & yVy^o & \text{(since V is an equivalence relation)} \\ & = & y(V) & \\ & = & S. & \end{array}$$

□

We may also consider weak Goursat categories through a property which is more similar to the one mentioned in Theorem 3.2:

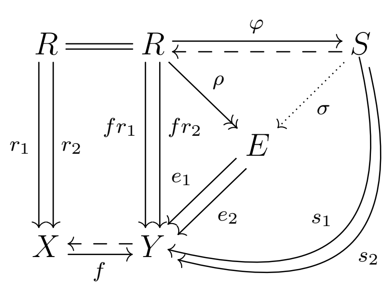
Lemma 4.6. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Then $\mathbb{C}$ is a weak Goursat category if and only if for any commutative diagram in $\mathbb{C}$*



(3)

such that f and φ are split epimorphism and R is a pseudo-equivalence relation, S is a pseudo-equivalence relation.

Proof. $(i) \Rightarrow (ii)$ Since $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation, by assumption $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} X$ is also a pseudo-equivalence relation and then its (regular epimorphism, monomorphism) factorization gives an equivalence relation $E \underset{e_2}{\overset{e_1}{\rightrightarrows}} Y$ in $\mathbb{A}$ (Proposition 4.2). We have the following commutative diagram



where $\sigma : S \longrightarrow E$ is induced by the strong (split) epimorphism φ

$$\begin{array}{ccc} R & \overset{\varphi}{\rightleftarrows} & S \\ {\scriptstyle\rho}\downarrow & \overset{\sigma}{\swarrow} & \downarrow{\scriptstyle\langle s_1,s_2\rangle} \\ E & \underset{\langle e_1,e_2\rangle}{\rightarrowtail} & Y \times Y. \end{array}$$

Then σ is a regular epimorphism and $S \underset{s_2}{\overset{s_1}{\rightrightarrows}} Y$ is a pseudo-equivalence relation (Proposition 4.2).

$(ii) \Rightarrow (i)$ Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and $X \overset{f}{\rightleftarrows} Y$ a split epimorphism. The following diagram is of the type (3)

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle fr_1}\downarrow\downarrow{\scriptstyle fr_2} \\ X & \underset{f}{\rightleftarrows} & Y. \end{array}$$

Since $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation, then by assumption $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation. □

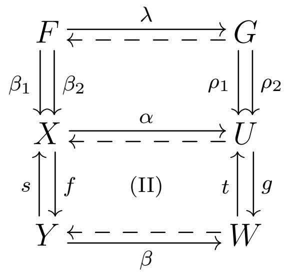
Alternatively, weak Goursat categories may be characterized through a property more similar to the one mentioned in Remark 3.5. A diagram of type (I) in a weakly lex context should have the regular epimorphisms α and β replaced by split epimorphisms; we call it of type (II). Note that such a diagram does not necessarily commute with the splittings of α and β.

Proposition 4.7. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. The following conditions are equivalent:*

(i) $\mathbb{A}$ is a Goursat category;

(ii) $\mathbb{C}$ is a weak Goursat category;

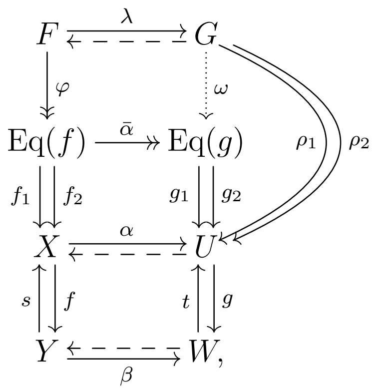
(iii) For any commutative diagram of type (II) *in* $\mathbb{C}$



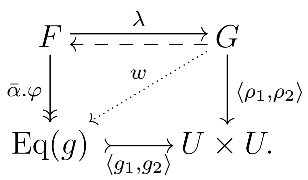
where F is a weak kernel pair of f and λ is a split epimorphism, G is a weak kernel pair of g.

Proof. $(i) \Leftrightarrow (ii)$ By Proposition 4.5.

$(i) \Rightarrow (iii)$ If we take the kernel pairs of f and g, then the induced morphism $\bar{\alpha} : \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ is a regular epimorphism by Theorem 3.4. Moreover, the induced morphism $\varphi : F \longrightarrow \mathrm{Eq}(f)$ is also a regular epimorphism. We get

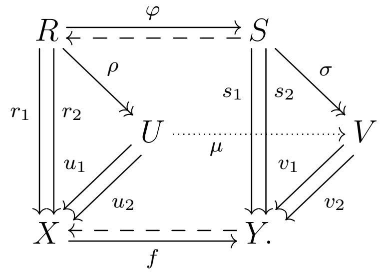


where $w : G \longrightarrow \mathrm{Eq}(g)$ is induced by the strong (split) epimorphism λ



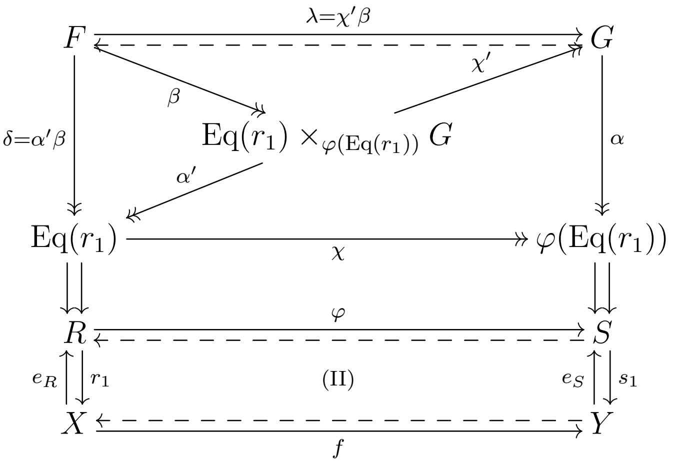
This implies that ω is a regular epimorphism and then $G \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} U$ is a weak kernel pair of g.

$(iii) \Rightarrow (ii)$ Consider diagram (3) in $\mathbb{C}$ where $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation. We want to prove that $S \underset{s_2}{\overset{s_1}{\rightrightarrows}} Y$ is also a pseudo-equivalence. Take the (regular epimorphism, monomorphism) factorization of R and S in $\mathbb{A}$ and the induced morphism μ making the following diagram commutative



Since μ is a regular epimorphism, $V = f(U)$ and consequently, V is reflexive and symmetric, as the regular image of the equivalence relation U (Theorem 3.2).

To conclude that S is a pseudo-equivalence relation, we just need to prove that V is transitive. We apply our assumption to the diagram

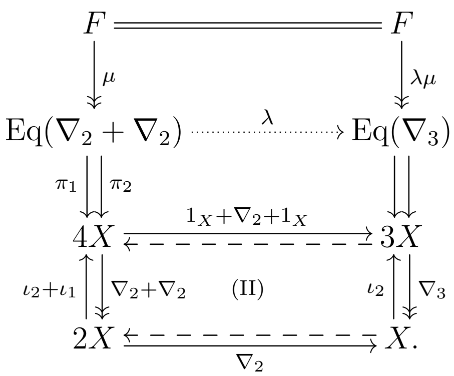


where G is a cover of the regular image $\varphi(\mathrm{Eq}(r_1))$ and F is a cover of the pullback $\mathrm{Eq}(r_1) \times_{\varphi(Eq(r_1))} G$. Note that $\lambda = \chi'\beta$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, so it is a split epimorphism. Since δ is a regular epimorphism, then $F \rightrightarrows R$ is a weak kernel pair of r_1. By assumption $G \rightrightarrows S$ is a weak kernel pair of s_1, thus $\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)$. We then have

$$\begin{array}{rclr}
VV & = & v_2 v_1^o v_1 v_2^o & (\text{since } V \text{ is symmetric}) \\
 & = & v_2 \sigma \sigma^o v_1^o v_1 \sigma \sigma^o v_2^o & (\text{Lemma 2.1(2)}) \\
 & = & s_2 s_1^o s_1 s_2^o & (v_i \sigma = s_i) \\
 & = & s_2 \varphi r_1^o r_1 \varphi^o s_2^o & (\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)) \\
 & = & f r_2 r_1^o r_1 r_2^o f^o & (s_i \varphi = f r_i\ ) \\
 & = & f u_2 \rho \rho^o u_1^o u_1 \rho \rho^o u_2^o f^o & (u_i \rho = r_i) \\
 & = & f u_2 u_1^o u_1 u_2^o f^o & (\text{Lemma 2.1(2)}) \\
 & = & f U U f^o & (\text{since } U \text{ is an equivalence relation}) \\
 & = & f U f^o & (\text{since } U \text{ is an equivalence relation}) \\
 & = & V. & (\ f(U) = V)
\end{array}$$

□

Remark 4.8. When $\mathbb{A}$ is a 3-permutable variety and $\mathbb{C}$ its subcategory of free algebras, then the property stated in Proposition 4.7 (iii) is precisely what is needed to obtain the existence of the quaternary operations p and q which characterize 3-permutable varieties. Let X denote the free algebra on one element. Diagram (II) below belongs to $\mathbb{C}$



If F is a cover of $\mathrm{Eq}(\nabla_2 + \nabla_2)$), then $F \rightrightarrows 4X$ is a weak kernel pair of $\nabla_2 + \nabla_2$. By assumption $F \rightrightarrows 3X$ is a weak kernel pair of ∇_3, so that $\lambda\mu$ is surjective. We then conclude that λ is surjective and the existence of the quaternary operations p and q follows from Theorem 3 in [7].

Acknowledgements

The first author acknowledges partial financial assistance by Centro de Matemática da Universidade de Coimbra—UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.
The second author acknowledges financial assistance by Fonds de la Recherche Scientifique-FNRS Crédit Bref Séjour à l'étranger 2018/V 3/5/033 - IB/JN - 11440, which supported his stay at the University of Algarve, where this paper was partially written.

References

[1] A. Carboni, G.M. Kelly, M.C. Pedicchio, *Some remarks on Mal'tsev and Goursat categories*, Appl. Categ. Structures 1 (1993), no. 4, 385-421.

[2] A. Carboni, J. Lambek, M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra 69 (1991) 271–284.

[3] A. Carboni and R. Celia Magno, *The free exact category on a left exact one*, J. Austr. Math. Soc. Ser., A 33 (1982) 295-301.

[4] A. Carboni and E. Vitale, *Regular and exact completions*, J. Pure Appl. Algebra **125** (1998) 79-116.

[5] M. Gran, *Semi-abelian exact completions*, Homology, Hom. Appl. **4** (2002) 175-189.

[6] M. Gran and D. Rodelo, *On the characterization of Jónsson-Tarski and of subtractive varieties*, Diagrammes, Suppl. Vol. 67-68 (2012) 101-116.

[7] M. Gran and D. Rodelo, *A new characterisation of Goursat categories*, Appl. Categ. Structures 20 (2012), no. 3, 229-238.

[8] M. Gran and D. Rodelo, *Beck-Chevalley condition and Goursat categories*, J. Pure Appl. Algebra, 221 (2017) 2445-2457.

[9] M. Gran, D. Rodelo and I. Tchoffo Nguefeu, *Some remarks on connectors and groupoids in Goursat categories*, Logical Methods in Computer Science (2017) Vol. 13(3:14), 1-12.

[10] J. Hagemann and A. Mitschke, *On n-permutable congruences*, Algebra Universalis 3 (1973), 8-12.

[11] J. Rosicky and E.M. Vitale, *Exact completions and representations in abelian categories*, Homology, Hom. Appl. **3** (2001) 453-466.

[12] E. Vitale, *Left covering functors*, PhD thesis, Université catholique de Louvain (1994).

Diana Rodelo
CMUC, Department of Mathematics
University of Coimbra
3001–501 Coimbra, Portugal
Departamento de Matemática, Faculdade de Ciências e Tecnologia
Universidade do Algarve, Campus de Gambelas
8005–139 Faro, Portugal
drodelo@ualg.pt

Idriss Tchoffo Nguefeu
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
idriss.tchoffo@uclouvain.be