CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXI-4 (2020)

PROJECTIVE COVERS OF 2-STAR-PERMUTABLE CATEGORIES

Vasileios Aravantinos-Sotiropoulos *

Résumé. Nous introduisons la notion de star-symétrie pour les relations dans une catégorie multipointée et nous l'utilisons pour obtenir une caractérisation des revêtements projectifs des catégories 2-star-permutables. Cela généralise les résultats de Rosicky-Vitale pour les catégories régulières de Mal'tsev [19] et aussi ceux de Gran-Rodelo pour les catégories régulières soustractives [13]. Nous appliquons la caractérisation en termes de star-symétrie pour retrouver les conditions syntaxiques définissant les variétés E-soustractives au sens de Ursini [20]

Abstract. We introduce the notion of star-symmetry for relations in a multi-pointed category and use it to obtain a characterization of the projective covers of 2-star-permutable categories. This generalizes the results of Rosický-Vitale for regular Mal'tsev categories [19], as well as those of Gran-Rodelo for regular subtractive categories [13]. We apply the characterization in terms of star-symmetry to recover the syntactic conditions defining E-subtractive varieties in the sense of Ursini [20].

Keywords. Multi-pointed category, star relation, Mal'tsev category, subtractive category, projective cover, variety of algebras.

Mathematics Subject Classification (2010). 18C05, 18A35, 08B05, 18G05.

---

*The author's research is funded by a doctoral grant Fonds Spéciaux de Recherche of the Université Catholique de Louvain.

1. Introduction

The notion of *multi-pointed category* has in recent years been introduced and studied as a setting where certain pointed and non-pointed contexts of interest in Categorical and Universal Algebra can be treated simultaneously. A multi-pointed category is simply a category $\mathcal{C}$ equipped with an *ideal* $\mathcal{N}$ of morphisms in the sense of Ehresmann [6], i.e. a collection of morphisms in $\mathcal{C}$ such that $fg \in \mathcal{N}$ whenever $f \in \mathcal{N}$ or $g \in \mathcal{N}$. The *pointed context* is captured by taking $\mathcal{N}$ to be the class of zero morphisms in a pointed category, while non-pointed settings, which are referred to as the *total context*, are captured by choosing $\mathcal{N}$ to be the class of all morphisms of a category. This has allowed the unification and extension of various results and characterizations known in pointed and non-pointed Categorical Algebra to the context of multi-pointed categories. First, in the article [12] the authors introduced the notion of a multi-pointed category with a *good theory of ideals* and unified results from the realm of *ideal determined* categories, on one hand, and *Barr-exact Goursat* categories, on the other. Next, in [11], notions of permutability of equivalence relations in multi-pointed categories were introduced and studied in connection with certain diagrammatic characterizations, known for regular *subtractive* categories and *Goursat* categories. Furthermore, in [10] the authors considered generalizations of homological lemmas, such as the *3 × 3 Lemma* and the *Short Five Lemma*. In non-pointed contexts the appropriate notion of exact sequence is that of *exact fork*, which is a sequence consisting of a kernel pair together with its coequalizer. Then, in a more general multi-pointed context, the pertinent notion becomes that of a *star-exact* sequence, which unifies the pointed and non-pointed versions, and allows for the aforementioned multi-pointed homological lemmas. Finally, in [14] the notion of 2-star-permutable category was studied as a common extension of both regular subtractive and regular Mal'tsev categories and characterizations of these categories via diagrams such as regular pushouts were generalized to a multi-pointed context. In the present note we want to add to this list a characterization of *projective covers* of regular 2-star-permutable multi-pointed categories.

There has been a lot of interest and work carried out in the literature on obtaining characterizations for the projective covers of various types of regular categories. The first result of this kind appears already in the work of Freyd in [7] in connection with his construction of the free abelian category on a given (pre-)additive one. About 3 decades later, Carboni and Vitale gave beautiful constructions for free *regular* and *exact* categories. Since abelian categories are in particular exact, these constructions can be used to recover in a nice conceptual manner the aforementioned one by Freyd, as well as other results on abelian categories (see [19]). One important feature of these *regular and exact completions* is that they apply to any category which is merely *weakly lex*, i.e. which is only required to have weak finite limits [5]. Then it turns out that any such category $\mathcal{C}$ appears as a projective cover inside both its regular completion and its exact completion and, furthermore, that a free exact category is the exact completion of any one of its projective covers. Such and other motivations have led various authors to establish characterizations for the projective covers of regular and exact categories that are extensive [15], Mal'tsev [19], protomodular, semi-abelian [8], unital, subtractive [13], Goursat [18] and others.

In this note we look at regular Mal'tsev and regular subtractive categories as special cases of the notion of 2-star-permutable category, following the line of research in [11], [14]. The aim here is to obtain a characterization of the projective covers of 2-star-permutable multi-pointed categories, thus unifying and subsuming the known characterizations in the Mal'tsev [19] and subtractive [13] settings. To accomplish this we first prove that 2-star-permutability is equivalent to a certain symmetry property of reflexive relations (3.2, 3.4), which specializes to known characterizations in both the total and pointed contexts. In the total context it becomes the well-known statement [4] that a regular category is Mal'tsev if and only if every reflexive relations in it is symmetric, while in the pointed context it says that a regular category is subtractive if and only if every reflexive relation in it is 0-symmetric [1, 16]. We then introduce the appropriate "weakening" of this symmetry property in the context of multi-pointed categories with only weak finite limits and weak kernels (3.8) and prove that this weakened property gives the desired characterization of projective covers (3.12). This result yields, in particular, a characterization of when the regular completion and the exact completion of a category with weak finite limits are 2-star-permutable. Finally, we apply the result to the case of varieties of universal algebras which have a non-empty set of constants, allowing us to recover the syntactic conditions defining *E-subtractive* varieties in the sense of Ursini [20].

Acknowledgments: The author would like to acknowledge with gratitude Professor Marino Gran for numerous helpful conversations and suggestions on the topic and presentation of this paper. He also thanks the referee for useful comments that improved the quality of the paper.

2. Preliminaries

2.1 Regular categories and relations

A finitely complete category $\mathcal{E}$ is called *regular* when every kernel pair in $\mathcal{E}$ has a coequalizer and moreover regular epimorphisms in $\mathcal{E}$ are stable under pullbacks. Equivalently, $\mathcal{E}$ is regular if it admits (regular epi, mono) factorizations of morphisms and these are stable under pullback.

A *relation* R from X to Y in any finitely complete category is a subobject $\langle r_0, r_1\rangle : R \rightarrowtail X \times Y$. When $Y = X$ we will say that R is a relation on X and also denote this by a parallel pair $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$. The *opposite* relation R° is the relation given by $\langle r_1, r_0\rangle : R \rightarrowtail Y \times X$. Any morphism $f : X \to Y$ can be considered as a relation by identifying it with its graph $\langle 1_X, f\rangle : X \rightarrowtail X \times Y$. Then we will write f° to denote the opposite of the latter relation.

In the context of a regular category $\mathcal{E}$ [2] it is possible to define a composition of relations which, moreover, is associative. If R is a relation from X to Y and S is a relation from Y to Z, then we denote their composition by SR, which is a relation from X to Z. The diagonal relations $\Delta_X = \langle 1_X, 1_X\rangle : X \rightarrowtail X \times X$ act as identities for the composition of relations on either side. Furthermore, if a relation R is given by the subobject $\langle r_0, r_1\rangle : R \rightarrowtail X \times Y$, then we can write it as

$R = r_1 r_0^\circ$ in the above notation.

If $\mathrm{Eq}(f)$ denotes the kernel pair of a morphism $f : X \to Y$, then as a relation on X we have $\mathrm{Eq}(f) = f^\circ f$.

Let $f : X \to Y$ be a morphism and S be a relation on Y. We denote by $f^{-1}(S)$ the *inverse image* of the relation S along f, which is the relation on X defined as the pullback of the subobject $S \rightarrowtail Y \times Y$ along the morphism $f \times f : X \times X \to Y \times Y$. Then in the calculus of relations we have that $f^{-1}(S) = f^\circ S f$.

2.2 Projective covers

Let $\mathcal{E}$ be a category with a full subcategory $\mathcal{C}$. We say that $\mathcal{C}$ is a *projective cover* of $\mathcal{E}$ if the following two conditions hold:

- Every object of $\mathcal{C}$ is a regular projective in $\mathcal{E}$.
- For every object $E \in \mathcal{E}$ there exists a regular epimorphism $P \twoheadrightarrow E$ with $P \in \mathcal{C}$.

A regular epimorphism $P \twoheadrightarrow E$ with $P \in \mathcal{C}$ is called a $\mathcal{C}$-*cover* of E.

Even if $\mathcal{E}$ has limits of some type, $\mathcal{C}$ will in general only have *weak* limits of that type. So if $\mathcal{E}$ has finite limits (e.g. if it is regular), then $\mathcal{C}$ will be *weakly lex*, i.e. will have all weak finite limits. To construct the weak limit of a diagram in $\mathcal{C}$ one first constructs the actual limit in the ambient category $\mathcal{E}$ and then one takes a $\mathcal{C}$-cover of the latter limit.

Finally, every weakly lex category $\mathcal{C}$ appears as a projective cover inside both its regular completion $\mathcal{C}_{reg}$ and its exact completion $\mathcal{C}_{ex}$ in the sense of [5].

2.3 Multi-pointed categories and stars

We first recall here some basic notions introduced in [12].

A *multi-pointed category* is a pair $(\mathcal{C}, \mathcal{N})$ consisting of a category $\mathcal{C}$ and a distinguished class $\mathcal{N}$ of morphisms in $\mathcal{C}$ which is an *ideal*. The latter, as mentioned in the introduction, means that for any pair of arrows $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, either $f \in \mathcal{N}$ or $g \in \mathcal{N}$ implies that $gf \in \mathcal{N}$. The elements of $\mathcal{N}$ are usually referred to as *null* morphisms.

We will often by abuse say that $\mathcal{C}$ is a multi-pointed category and suppress the ideal $\mathcal{N}$ if there is no possibility of confusion. Before moving on, let us recall here the main examples of multi-pointed categories that we shall consider.

- A simple first example of multi-pointed category is obtained by taking any category $\mathcal{C}$ and defining $\mathcal{N}$ to be the collection of all morphisms in $\mathcal{C}$. This class of examples is known as the *total context*.
- A second example of importance arises when $\mathcal{C}$ is pointed (i.e. has a zero object) and $\mathcal{N}$ is defined as the collection of zero morphisms, i.e. the morphisms that factor through the zero object. This general class of examples is referred to as the *pointed context*.
- The previous example can in fact be seen as a special case of a more general class of multi-pointed categories, the so called *proto-pointed context* introduced in [12]. This refers to a category $\mathcal{C}$ in which every object has a smallest subobject and where a morphism $f : X \to Y$ is defined to be a null morphism precisely when it factors through the smallest subobject of Y. In the case of a variety $\mathbb{V}$ of universal algebras these morphisms are exactly those whose image is the subalgebra E_Y of Y generated by the constants. This latter situation has been called the *algebraic proto-pointed context* in [11] and is actually the motivation for the term "multi-pointed". Indeed, a proto-pointed category is the category-theoretic notion that corresponds to varieties with potentially more than one constant, such as unital rings and Heyting algebras, just as that of pointed category corresponds to varieties possessing a unique constant.

An $\mathcal{N}$-*kernel* of a morphism $f : X \to Y$ is a morphism $k : K \to X$ such that $fk \in \mathcal{N}$ and which is universal with this property, i.e. whenever $fg \in \mathcal{N}$ there is a unique morphism u such that $ku = g$. Note that k is then necessarily a monomorphism. Observe also that in the total context the $\mathcal{N}$-kernels are just identities, while in the pointed context we obtain the usual notion of kernel.

In the more general proto-pointed setting the kernel of f is generally the inverse image of the smallest subobject of Y, which in the algebraic case becomes precisely the subalgebra of X consisting of those elements that map to the subalgebra generated by the constants in Y. So for example, if $f : X \to Y$ is a morphism in the proto-pointed category **Heyt** of Heyting algebras, then the $\mathcal{N}$-kernel of f is the subalgebra $\{x \in X | f(x) = 0 \vee f(x) = 1\}$. Similarly, if f lives in the category **Ring** of unitary rings, then its kernel in the above sense is $\{x \in X | (\exists n \in \mathbb{Z}) f(x) = n \cdot 1\}$. Note how the latter is indeed a subring of X and hence defines a subobject in the category **Ring**, whereas the ordinary kernel does not.

Since we shall have occasion to deal with categories that only have weak finite limits, we will also correspondingly require the notion of *weak $\mathcal{N}$-kernel* of a morphism $f : X \to Y$. This is defined as $\mathcal{N}$-kernels above, but by only requiring existence of the factorization, not necessarily uniqueness.

If $\mathcal{N}$-kernels exist for all morphisms in $\mathcal{C}$, then we shall say that $\mathcal{C}$ is a *multi-pointed category with kernels.* Similarly for weak $\mathcal{N}$-kernels.

We also record here for future use the following basic observation on the behavior of $\mathcal{N}$-kernels under pullback. For the sake of completeness, we also give the easy proof.

Lemma 2.1. *Consider the following pullback square in a multi-pointed category* $(\mathcal{C}, \mathcal{N})$.

$$\begin{array}{ccc} K' & \xrightarrow{k'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[k]{} & Y \end{array}$$

If k is the $\mathcal{N}$-kernel of some $f : Y \to Z$, then k' is the $\mathcal{N}$-kernel of $fg : X \to Z$.

Proof. Let $h : A \to X$ be such that $fgh \in \mathcal{N}$. Then, since k is the $\mathcal{N}$-kernel of f, there exists a unique $u : A \to K$ such that $ku = gh$. Now the universal property of the pullback gives a unique $v : A \to K'$ such that $g'v = u$ and $k'v = h$. Finally, note that k' is monomorphic because k is monomorphic. $\square$

A pair of morphisms $r = (r_0, r_1) : R \rightrightarrows X$ is called a *star* if $r_0 \in \mathcal{N}$. When it is moreover jointly monomorphic, we say that it is a *star relation*. In the total context this just defines a relation in the ordinary sense, whereas in the pointed case it is a relation whose first projection is zero. However, a more motivating example can be identified in the proto-pointed setting of the category **Ring** of unitary rings. Given any ideal $I \subseteq A$ inside the unitary ring A, we have an associated star relation R_I on A defined by $R_I := \bigcup_{n \in \mathbb{Z}} \{n\} \times (n + I)$. This star relation clearly uniquely determines the ideal I, but furthermore has the advantage that it is a subalgebra of $A \times A$ and hence lives in the category **Ring**, while the ideal I itself generally does not.

Given a relation R on an object X represented by the jointly monomorphic pair $r = (r_0, r_1) : R \rightrightarrows X$ and assuming $\mathcal{N}$-kernels exist, we define the *star* of R to be the relation R^* on X represented by the pair $(r_0 k_0, r_1 k_0)$ where $k_0 : K_0 \to R$ is the $\mathcal{N}$-kernel of r_0. Equivalently, one could say that R^* is the largest subrelation of R which is a star. In particular, when $R = \mathrm{Eq}(f)$ is the kernel pair of a morphism $f : X \to Y$, then $R^* = \mathrm{Eq}(f)^*$ is called the *star-kernel* of f.

In the context of a regular multi-pointed category it is possible to use the usual calculus of relations to develop a calculus of star relations, as is done in [11]. We shall not really need much of this though. We just record here the fact that, given relations R, S on an object X, we have that $(RS)^* = RS^*$.

Finally, we record an observation on how the star of a relation on an object X can be computed as a certain pullback involving the $\mathcal{N}$-kernel $\kappa_X : K_X \rightarrowtail X$ of the identity $1_X : X \to X$. Observe that this is just a generalization of the fact that the 0-class of a relation $R \rightarrowtail X \times X$ in a pointed category can be computed as the pullback of that relation along $\langle 0, 1 \rangle : X \to X \times X$. Since the more general statement does not appear in the literature, we also provide a proof.

Lemma 2.2. *Consider a relation* $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$ *in the multi-pointed category* $(\mathcal{E}, \mathcal{N})$ *with kernels. Then the* $\mathcal{N}$*-kernel* k_0 *of* r_0 *is obtained as the following pullback.*

$$\begin{array}{ccc} K_0 & \xrightarrow{\quad k_0 \quad} & R \\ {\scriptstyle \langle \overline{r_0}, r_1 k_0 \rangle} \downarrow & & \downarrow {\scriptstyle \langle r_0, r_1 \rangle} \\ K_X \times X & \xrightarrow[\kappa_X \times 1_X]{} & X \times X \end{array}$$

where $\kappa_X : K_X \rightarrowtail X$ *is the* $\mathcal{N}$*-kernel of the identity* $1_X : X \to X$.

Proof. We consider the $\mathcal{N}$-kernel $k_0 : K_0 \rightarrowtail R$ of r_0 and we will show that there is a pullback square as indicated.

First, observe that $r_0 k_0 \in \mathcal{N}$ implies that there is a $\overline{r_0} : K_0 \to K_X$ such that $\kappa_X \overline{r_0} = r_0 k_0$, giving the indicated morphism $K_0 \to K_X \times X$ in the above commutative diagram.

Now assume that $f = \langle f_0, f_1 \rangle : Z \to K_X \times X$ and $g : Z \to R$ are such that $(\kappa_X \times 1_X) f = \langle r_0, r_1 \rangle g$. Then $r_0 g = \kappa_X f_0$ and $r_1 g = f_1$. Since $\kappa_X \in \mathcal{N}$, the first of these implies that $r_0 g \in \mathcal{N}$ and hence there exists a unique $h : Z \to K_0$ such that $k_0 h = g$. Then also $\langle \overline{r_0}, r_1 k_0 \rangle h = \langle \overline{r_0} h, r_1 k_0 h \rangle = \langle \overline{r_0} h, r_1 g \rangle = \langle f_0, f_1 \rangle = f$, where $\overline{r_0} h = f_0$ follows because $\kappa_X \overline{r_0} h = r_0 k_0 h = r_0 g = \kappa_X f_0$ and κ_X is monomorphic. $\square$

3. 2-star-permutable categories

Let us recall the definition of 2-star-permutability from [11].

Definition 3.1. *Let* $\mathcal{C}$ *be a regular multi-pointed category with kernels. We say that* $\mathcal{C}$ *is* 2-star-permutable *if for any two effective equivalence relations* R, S *on an object* $X \in \mathcal{C}$ *we have* $RS^* = SR^*$.

In the total context, since the star of any relation is that relation itself, the definition says that effective equivalence relations are permutable, which yields precisely the regular Mal'tsev categories [4].

In the case of a pointed variety of universal algebras, the star of a relation R on X is the subrelation $R^* = \{(0, x) \in X \times X | (0, x) \in R\}$. More generally, in any pointed context, the star of the relation $\langle r_0, r_1 \rangle : R \rightarrowtail X \times X$ is the relation $\langle 0, c \rangle : C \rightarrowtail X \times X$ where $c : C \rightarrowtail X$ is the *0-class* of R, i.e. where the mono $c : C \rightarrowtail X$ is given by $c = r_1 \ker(r_0)$. Thus, the above definition says precisely that effective equivalence relations are *0-permutable* and this is known to characterize regular subtractive categories (see [16], [11] and [1] for the varietal case).

We first want to present an equivalent characterization of 2-star-permutability in terms of a symmetry property of reflexive relations. The symmetry property in question will be the following.

Definition 3.2. *Let $\mathcal{E}$ be a multi-pointed category with kernels and* $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$ *a relation in $\mathcal{E}$. We say that R is* left star-symmetric *if* $R^* \leq (R^\circ)^*$. *We say that it is* star-symmetric *if $R^* = (R^\circ)^*$, i.e. if both R and R° are left star-symmetric.*

Observe that in the pointed context left star-symmetry becomes the usual notion of *left 0-symmetry*, i.e. the statement that R satisfies the implication $(0,x) \in_A R \implies (x,0) \in_A R$ for any generalized element $x : A \to X$ of X. In an algebraic proto-pointed setting it is the implication $(e,x) \in_A R \implies (x,e) \in_A R$ *for every* $e \in E_X$, where E_X is the subalgebra generated by the constants. In the total context on the other hand, R being left star-symmetric just means that $R \leq R^\circ$, which is to say that R is a symmetric relation in the ordinary sense. In particular, in this case left star-symmetry and star-symmetry become equivalent.

Indeed, note more generally that for any generalized elements $x, y : A \to X$ in $\mathcal{E}$ we have that $(x,y) \in_A R^*$ precisely if $(x,y) \in_A R$ and $x \in \mathcal{N}$. Thus, R being left-star symmetric is saying that whenever $(n,y) \in_A R$ with $n \in \mathcal{N}$, then also $(y,n) \in_A R$.

We will need the following lemma, from [11], for the proof of our next proposition.

Lemma 3.3. *For any morphism $f : X \to Y$ and every relation S on Y in a multi-pointed category we have $(f^{-1}(S))^* = (f^{-1}(S^*))^*$.*

We can now present new equivalent characterizations of 2-star-permutability using the notion of star-symmetry. In fact, this allows us to also deduce that 2-star-permutability is equivalent to having the equality $RS^* = SR^*$ for any two equivalence relations R, S on the same object, not just effective ones. This does not seem to have appeared in the literature before.

Proposition 3.4. *For a regular multi-pointed category $\mathcal{C}$ with kernels the following are equivalent:*

1. *$\mathcal{C}$ is 2-star-permutable.*
2. *For any two equivalence relations R, S on an object $X \in \mathcal{C}$ we have $RS^* = SR^*$.*
3. *Every reflexive relation E in $\mathcal{C}$ is left star-symmetric, i.e. $E^* \leq (E^\circ)^*$.*
4. *Every reflexive relation E in $\mathcal{C}$ is star-symmetric, i.e. $E^* = (E^\circ)^*$.*

Proof. 1. $\implies$ 4. Let $E \overset{e_0}{\underset{e_1}{\rightrightarrows}} X$ be a reflexive relation with diagonal $\delta : X \to E$. Set $R \coloneqq \mathrm{Eq}(e_0) = e_0^\circ e_0$ and $S \coloneqq \mathrm{Eq}(e_1) = e_1^\circ e_1$, so that both R and S are effective equivalence relations on E. Observe that $\delta^{-1}(SR) = \delta^\circ e_1^\circ e_1 e_0^\circ e_0 \delta = e_1 e_0^\circ = E$ and $\delta^{-1}(RS) = \delta^\circ e_0^\circ e_0 e_1^\circ e_1 \delta = e_0 e_1^\circ = E^\circ$. Now using the assumption (1) and 3.3 we have

$$\begin{aligned} RS^* = SR^* &\implies (RS)^* = (SR)^* \\ &\implies \delta^{-1}((RS)^*) = \delta^{-1}((SR)^*) \\ &\implies \delta^{-1}((RS)^*)^* = \delta^{-1}((SR)^*)^* \\ &\implies \delta^{-1}(RS)^* = \delta^{-1}(SR)^* \\ &\implies (E^\circ)^* = E^*. \end{aligned}$$

4. $\implies$ 2. Consider the reflexive relation $E \coloneqq SR$ on X. Then we have $E^* = (E^\circ)^* \implies (SR)^* = (RS)^* \implies SR^* = RS^*$.

2. $\implies$ 1. Clear.

3. $\iff$ 4. Clear by considering both reflexive relations E and E°. □

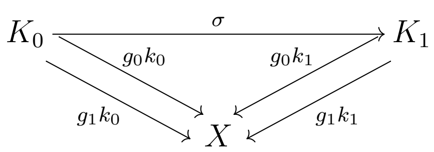
It should be observed that conditions (3) and (4) above can be formulated in any finitely complete multi-pointed category $(\mathcal{C}, \mathcal{N})$ with kernels, thus enlarging the class of categories for which the notion of 2-star-permutability can be considered to include non-regular ones. This generalizes the fact that the notion of Mal'tsev category can be formulated as a finitely complete category where every reflexive relation is symmetric [4], as well as the fact that subtractive categories can be defined as pointed finitely complete categories where every reflexive relation is 0-symmetric [16]. The following definition therefore appears pertinent.

Definition 3.5. *A multi-pointed category* $(\mathcal{C}, \mathcal{N})$ *is said to be* star-Mal'tsev *if every reflexive relation in* $\mathcal{C}$ *is left star-symmetric. Equivalently, if every reflexive relation in* $\mathcal{C}$ *is star-symmetric.*

With this terminology, 3.4 says that a regular multi-pointed category is 2-star-permutable if and only if it is star-Mal'tsev.

We now want to characterize the projective covers of 2-star-permutable regular multi-pointed categories, or, in other words, of regular star-Mal'tsev categories. In doing so, the following notion will play a key role. It is the appropriate adaptation of the notion of star-symmetry to the context of a multi-pointed category with only weak finite limits and weak kernels.

Definition 3.6. *Let* $(\mathcal{C}, \mathcal{N})$ *be a weakly lex multi-pointed category with weak* $\mathcal{N}$*-kernels. A graph* $G \overset{g_0}{\underset{g_1}{\rightrightarrows}} X$ *in* $\mathcal{C}$ *is said to be* left star-symmetric *if , given weak* $\mathcal{N}$*-kernels* $k_0 : K_0 \to G$ *and* $k_1 : K_1 \to G$ *of* g_0 *and* g_1 *respectively, there exists a* $\sigma : K_0 \to K_1$ *such that the following diagram serially commutes*



i.e. such that $g_1k_1\sigma = g_0k_0$ *and* $g_0k_1\sigma = g_1k_0$ *both hold. We say that it is* star-symmetric *if both* G *and its opposite graph* $G \overset{g_1}{\underset{g_0}{\rightrightarrows}} X$ *are left star-symmetric.*

In other words, a graph G is left star-symmetric if a "weak star" of G factors through a weak star of the opposite graph. Note also that the definition does not depend on the chosen weak $\mathcal{N}$-kernels because any two weak $\mathcal{N}$-kernels of the same morphism factor through each other. Furthermore, it is clear that when G is a relation and $\mathcal{N}$-kernels exist the definition says precisely that $G^* \leq (G^\circ)^*$, i.e. that G is a left star-symmetric relation.

Remark 3.7. It is easy to see that in the total context we get the usual definition of a symmetric graph, since both $\mathcal{N}$-kernels are identities in this case. In the pointed context one of the two commutativities required above becomes trivial because $g_0k_0 = 0 = g_1k_1$ and we obtain the notion of a *left 0-symmetric* graph.

We now introduce the categories that will appear in our characterization of the projective covers of 2-star-permutable categories. These are the multi-pointed categories with weak finite limits and weak kernels which satisfy the appropriate "weakening" of the star-Mal'tsev property. Our terminology is inspired by that of Rosický-Vitale in [19] for the total context.

Definition 3.8. *We will say that a weakly lex multi-pointed category $\mathcal{C}$ with weak kernels is* star-G-Mal'tsev *if every reflexive graph in $\mathcal{C}$ is left star-symmetric. Equivalently, if every reflexive graph is star-symmetric.*

In what follows, we will be considering regular categories $\mathcal{E}$ together with a projective cover $\mathcal{C}$ of $\mathcal{E}$. We are thus interested in how ideals of morphisms in the projective cover are related to ideals of morphisms in the ambient regular category. A thorough analysis of this situation is contained in [9], from which we now borrow and record below the main points that will be of use in the remainder of this paper.

First, if $\mathcal{N}$ is an ideal of morphisms in the regular category $\mathcal{E}$, then we denote by $\mathcal{N}_\mathcal{C}$ the restriction of $\mathcal{N}$ to $\mathcal{C}$. It is then clear that $\mathcal{N}_\mathcal{C}$ is an ideal in $\mathcal{C}$.

Second, if we are given an ideal $\mathcal{N}$ in the projective cover $\mathcal{C}$, then we define $\mathcal{N}^\mathcal{E}$ to be the collection of morphisms $f : X \to Y$ in $\mathcal{E}$ for which there exists a commutative square

$$\begin{array}{ccc} P & \xrightarrow{n} & Q \\ {\scriptstyle p}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where p and q are regular epimorphisms and $n \in \mathcal{N}$. It is again not hard to check that $\mathcal{N}^{\mathcal{E}}$ is an ideal in $\mathcal{E}$.

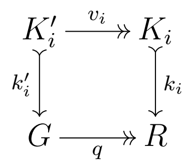
Lemma 3.9. *[9] Let $\mathcal{E}$ be a regular category having a projective cover $\mathcal{C}$.*

1. *For any ideal $\mathcal{N}$ in $\mathcal{E}$, if $\mathcal{E}$ has $\mathcal{N}$-kernels, then $\mathcal{C}$ has weak $\mathcal{N}_{\mathcal{C}}$-kernels, which can be computed by taking a projective cover of the domain of the $\mathcal{N}$-kernel in $\mathcal{E}$.*
2. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, the category $\mathcal{C}$ has weak $\mathcal{N}$-kernels if and only if $\mathcal{E}$ has $\mathcal{N}^{\mathcal{E}}$-kernels.*
3. *For any ideal $\mathcal{N}$ in $\mathcal{C}$ we have $(\mathcal{N}^{\mathcal{E}})_{\mathcal{C}} = \mathcal{N}$.*
4. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, regular epimorphisms are $\mathcal{N}^{\mathcal{E}}$-saturating in $\mathcal{E}$ (see 3.11).*

Before presenting our characterization, we find it useful to isolate the following fundamental observation.

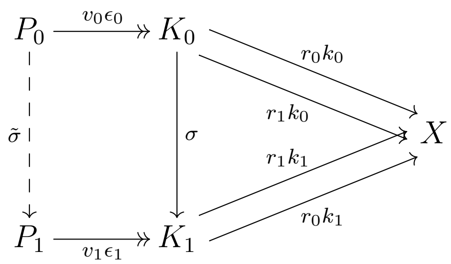
Lemma 3.10. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category $(\mathcal{E}, \mathcal{N})$ with kernels. Consider a graph $G \overset{g_0}{\underset{g_1}{\rightrightarrows}} X$ in $\mathcal{C}$ with its image factorization $\langle g_0, g_1 \rangle = G \overset{q}{\twoheadrightarrow} R \overset{\langle r_0, r_1 \rangle}{\rightarrowtail} X \times X$ in $\mathcal{E}$. Then G is a left star-symmetric graph if and only if R is a left star-symmetric relation.*

Proof. Consider $\mathcal{N}$-kernels $k_i : K_i \to R$ of r_i, for $i = 0, 1$. Form the pullbacks below for $i = 0, 1$ and then take $\mathcal{C}$-covers $\epsilon_i : P_i \twoheadrightarrow K_i'$.



By 2.1 we know that $k'_i : K'_i \rightarrowtail G$ is the $\mathcal{N}$-kernel of $r_i q = g_i$. Thus, we have that $u_i := k'_i \epsilon_i : P_i \to G$ is a weak $\mathcal{N}$-kernel of g_i in $\mathcal{C}$ for $i = 0, 1$ by 3.9.

Assume first that R is left star-symmetric, so that there exists a morphism $\sigma : K_0 \to K_1$ such that $r_1 k_1 \sigma = r_0 k_0$ and $r_0 k_1 \sigma = r_1 k_0$. By projectivity of P_0 and the fact that $v_1 \epsilon_1$ is a regular epimorphism, there exists a morphism $\tilde{\sigma} : P_0 \to P_1$ making the following diagram commute.



Now we have

$$\begin{aligned} g_1 u_1 \tilde{\sigma} &= r_1 q k'_1 \epsilon_1 \tilde{\sigma} \\ &= r_1 k_1 v_1 \epsilon_1 \tilde{\sigma} \\ &= r_1 k_1 \sigma v_0 \epsilon_0 \\ &= r_0 k_0 v_0 \epsilon_0 \\ &= r_0 q k'_0 \epsilon_0 \\ &= g_0 u_0 \end{aligned}$$

and similarly

$$\begin{aligned} g_0u_1\tilde{\sigma} &= r_0qk_1'\epsilon_1\tilde{\sigma} \\ &= r_0k_1v_1\epsilon_1\tilde{\sigma} \\ &= r_0k_1\sigma v_0\epsilon_0 \\ &= r_1k_0v_0\epsilon_0 \\ &= r_1qk_0'\epsilon_0 \\ &= g_1u_0 \end{aligned}$$

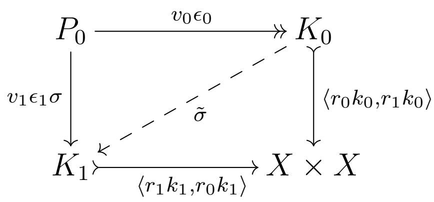
proving that G is left star-symmetric.

Conversely, assume that G is left star-symmetric. This means that there exists a $\sigma : P_0 \to P_1$ such that $g_1u_1\sigma = g_0u_0$ and $g_0u_1\sigma = g_1u_0$. We can then again calculate as follows:

$$\begin{aligned} r_1k_1v_1\epsilon_1\sigma &= r_1qk_1'\epsilon_1\sigma \\ &= g_1k_1'\epsilon_1\sigma \\ &= g_1u_1\sigma \\ &= g_0u_0 \\ &= r_0qk_0'\epsilon_0 \\ &= r_0k_0v_0\epsilon_0 \end{aligned}$$

$$\begin{aligned} r_0k_1v_1\epsilon_1\sigma &= r_0qk_1'\epsilon_1\sigma \\ &= g_0k_1'\epsilon_1\sigma \\ &= g_0u_1\sigma \\ &= g_1u_0 \\ &= r_1qk_0'\epsilon_0 \\ &= r_1k_0v_0\epsilon_0 \end{aligned}$$

This means that the square below commutes and so we obtain the indicated morphism $\tilde{\sigma}$ because $v_0\epsilon_0$ is a regular epimorphism and $\langle r_1k_1, r_0k_1\rangle$ is monomorphic, being the star of the relation R°.

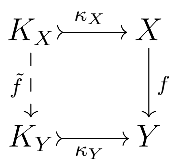


The commutation of the bottom triangle is precisely left star-symmetry of R. □

In order to prove our main result, we will need to impose an additional condition on the regular category $\mathcal{E}$ regarding the behavior of regular epimorphisms with respect to $\mathcal{N}$-kernels. This condition is familiar from the literature (see [9,11,14]) and is indeed mild enough that it includes all examples of interest. We now proceed to introduce the necessary notions.

Consider any object X in the multi-pointed category $(\mathcal{E}, \mathcal{N})$. We will denote by $\kappa_X : K_X \rightarrowtail X$ the $\mathcal{N}$-kernel of the identity morphism 1_X. Observe that by definition the generalized elements of K_X correspond precisely to the generalized elements of X that are in $\mathcal{N}$. Hence, K_X should be thought of as consisting of the "trivial elements" of the object X. Indeed, in the algebraic proto-pointed setting K_X is exactly what we have earlier in the text denoted by E_X, namely the subalgebra of X generated by the constants of the theory.

Now given any morphism $f : X \to Y$ in $\mathcal{E}$, we have a uniquely induced morphism $\tilde{f} : K_X \to K_Y$ making the following square commute.



Then we can introduce the following definition.

Definition 3.11. *A morphism $f : X \to Y$ in a multi-pointed category $(\mathcal{E}, \mathcal{N})$ is called* saturating *if the induced morphism $\tilde{f} : K_X \to K_Y$ is a regular epimorphism.*

Note that in the pointed context all morphisms are saturating, since $K_X = 0$ for any object X. The same holds in any algebraic proto-pointed setting, since every element $e \in X$ which is generated by constants is preserved under all homomorphisms $f : X \to Y$. Furthermore, it is not hard to see that regular epimorphisms are saturating in any proto-pointed context, not just the varietal one. In the total context, on the other hand, it is clear that the saturating morphisms are exactly the regular epimorphisms. In fact, that all regular epimorphisms are saturating is precisely what we shall require below.

Now we can present the main result of this note.

Theorem 3.12. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category with kernels $(\mathcal{E}, \mathcal{N})$ and assume regular epimorphisms in $\mathcal{E}$ are saturating. Then $(\mathcal{E}, \mathcal{N})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$ is star-G-Mal'tsev.*

Proof. Assume first that $\mathcal{E}$ is 2-star-permutable and consider any reflexive graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ with splitting $\delta : X \to G$. Consider its image factorization $\langle g_0, g_1 \rangle = G \overset{q}{\twoheadrightarrow} R \overset{\langle r_0, r_1 \rangle}{\rightarrowtail} X \times X$ in the regular category $\mathcal{E}$. Then the relation R on X is reflexive as well, since $r_i q \delta = g_i \delta = 1_X$ for $i = 0, 1$. Since $\mathcal{E}$ is 2-star-permutable, we know by 3.4 that R must be star-symmetric. Now 3.10 implies that G is (left) star-symmetric.

Conversely, assume that $(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$ is star-G-Mal'tsev. Consider any reflexive relation $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X$ in $\mathcal{E}$. We want to show that E is left star-symmetric.

Take a $\mathcal{C}$-cover $p : \tilde{X} \twoheadrightarrow X$ of X and consider the inverse image relation $E' \coloneqq p^{-1}(E)$. i.e. form the following pullback

$$\begin{array}{ccc} E' & \xrightarrow{\langle e_0', e_1' \rangle} & \tilde{X} \times \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p\times p} \\ E & \xrightarrow{\langle e_0, e_1 \rangle} & X \times X \end{array}$$

Now again take a $\mathcal{C}$-cover $\epsilon : G \twoheadrightarrow E'$ and set $g_0 := e_0'\epsilon$ and $g_1 := e_1'\epsilon$, so that we have a graph $G \overset{g_0}{\underset{g_1}{\rightrightarrows}} \tilde{X}$ in $\mathcal{C}$. Observe that the relation E' is reflexive, being the inverse image of a reflexive relation. It follows that the graph G is also reflexive. Indeed, if $\delta' : \tilde{X} \to E'$ is the diagonal of E', then by projectivity of $\tilde{X}$ we can lift to a $\tilde{\delta} : \tilde{X} \to G$ such that $\epsilon\tilde{\delta} = \delta'$ and then $g_i\tilde{\delta} = e_i'\epsilon\tilde{\delta} = e_i'\delta' = 1_{\tilde{X}}$ for $i = 0, 1$.

Now consider $\mathcal{N}$-kernels $k_i : K_i \to E$ of e_i and $k_i' : K_i' \to E$ of e_i' in $\mathcal{E}$ for $i = 0, 1$. We then have induced morphisms $u_i : K_i' \to K_i$ such that $k_iu_i = qk_i'$. We claim that the u_i are regular epimorphisms.

$$\begin{array}{ccc} K_i' & \xrightarrow{k_i'} & E' \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle q} \\ K_i & \xrightarrow[k_i]{} & E \end{array}$$

To see this for u_0 we consider the following two commutative diagrams. In the first one the right-hand square is a pullback by construction, while the left-hand square is a pullback by 2.2. In the second diagram we know only that the right-hand square is a pullback, again by 2.2.

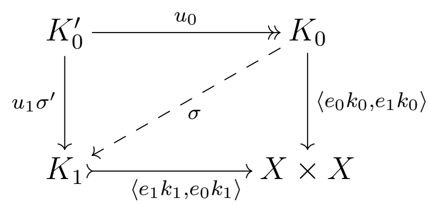
$$\begin{array}{ccccc} K_0' & \xrightarrow{k_0'} & E' & \xrightarrow{q} & E \\ {\scriptstyle \langle \overline{e_0'}, e_1'k_0' \rangle}\downarrow & & {\scriptstyle \langle e_0', e_1' \rangle}\downarrow & & \downarrow{\scriptstyle \langle e_0, e_1 \rangle} \\ K_{\tilde{X}} \times \tilde{X} & \xrightarrow[\kappa_{\tilde{X}} \times 1_{\tilde{X}}]{} & \tilde{X} \times \tilde{X} & \xrightarrow[p\times p]{} & X \times X \end{array}$$

$$\begin{array}{ccccc} K_0' & \xrightarrow{u_0} & K_0 & \xrightarrow{k_0} & E \\ {\scriptstyle \langle \overline{e_0'}, e_1'k_0' \rangle}\downarrow & & {\scriptstyle \langle \overline{e_0}, e_1k_0 \rangle}\downarrow & & \downarrow{\scriptstyle \langle e_0, e_1 \rangle} \\ K_{\tilde{X}} \times \tilde{X} & \xrightarrow[\tilde{p}\times p]{} & K_X \times X & \xrightarrow[\kappa_X \times 1_X]{} & X \times X \end{array}$$

Since $(p \times p)(\kappa_{\tilde{X}} \times 1_{\tilde{X}}) = (\kappa_X \times 1_X)(\tilde{p} \times p)$, we deduce that the outer rectangle in the second diagram is a pullback. Then by the usual pullback-cancellation property we have that the left-hand square is a pullback as well. But since both p and $\tilde{p}$ are regular epimorphisms, so is $\tilde{p} \times p$, since $\mathcal{E}$ is regular, and hence we deduce that the pullback u_0 is a regular epimorphism.

By the assumption that $\mathcal{C}$ is star-G-Mal'tsev, the reflexive graph G is left star-symmetric and so by 3.10 its image relation E' is also left star-symmetric. Thus, there exists a $\sigma' : K_0' \to K_1'$ such that $e_1' k_1' \sigma' = e_0' k_0'$ and $e_0' k_1' \sigma' = e_1' k_0'$.

Finally, consider the commutative square below.



Since u_0 is a regular epimorphism and $\langle e_1 k_1, e_0 k_1 \rangle$ is monomorphic (being the star of the relation E°), we get the indicated factorization $\sigma : K_0 \to K_1$, which shows that E is left star-symmetric. Thus, by 3.4 it follows that $\mathcal{E}$ is 2-star-permutable. □

The above result yields a characterization of when the regular and exact completion (in the sense of [5]) of a weakly lex multi-pointed category are 2-star-permutable.

Corollary 3.13. *Let $(\mathcal{C}, \mathcal{N})$ be a weakly lex multi-pointed category with weak kernels. Then $(\mathcal{C}_{reg}, \mathcal{N}^{\mathcal{C}_{reg}})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N})$ is star-G-Mal'tsev.*

Proof. $\mathcal{C}$ appears as a projective cover inside $\mathcal{C}_{reg}$. Then 3.12 indeed applies to give the result because by 3.9 we know that $\mathcal{C}_{reg}$ has $\mathcal{N}^{\mathcal{C}_{reg}}$-kernels, regular epimorphisms in $\mathcal{C}_{reg}$ are $\mathcal{N}^{\mathcal{C}_{reg}}$-saturating and $(\mathcal{N}^{\mathcal{C}_{reg}})_{\mathcal{C}} = \mathcal{N}$. □

In the exact same way we get the corresponding result about the exact completion $\mathcal{C}_{ex}$.

Corollary 3.14. *Let* $(\mathcal{C}, \mathcal{N})$ *be a weakly lex multi-pointed category with weak kernels. Then* $(\mathcal{C}_{ex}, \mathcal{N}^{\mathcal{C}_{ex}})$ *is 2-star-permutable if and only if* $(\mathcal{C}, \mathcal{N})$ *is star-G-Mal'tsev.*

Remark 3.15. We should comment here on how 3.12 extends the characterizations of projective covers for regular Mal'tsev categories, due to Rosicky-Vitale [19], and for regular subtractive categories, due to Gran-Rodelo [13].

For the Mal'tsev case, it is immediately clear from the definitions that we obtain exactly the same characterization as in [19], i.e. our star-G-Mal'tsev categories are exactly the G-Mal'tsev ones introduced therein. Note that in that paper G-Mal'tsev is initially defined by requiring that every reflexive graph be both symmetric and transitive, but this is equivalent to just requiring symmetry and that is in fact implicitly proved in [19].

In the pointed context, it is not immediate from the definitions that our star-G-Mal'tsev, which we should probably call *0-G-Mal'tsev* in this case, yields the *w-subtractive* categories of Gran-Rodelo [13]. Of course, since both characterize projective covers of the same class of regular categories, they turn out to be equivalent, since any weakly lex category can always be considered a projective cover of its regular completion. On the other hand, a direct proof of the equivalence of the two notions is also not too hard to construct.

Now suppose we are in an algebraic proto-pointed context and that the set of constants of the variety is nonempty. It was proved in [11] that in this case 2-star-permutability is equivalent to a priori more general properties such as *3-star-permutability* and the *symmetric saturation property*, but also to the syntactic condition defining *E-subtractive varieties* in the sense of [20]. We would like to conclude this note by showing how the equivalence with the latter notion can also directly be obtained from our characterization in terms of star-symmetry.

Corollary 3.16. *Let* $\mathbb{V}$ *be a variety of universal algebras and let* $E_{\mathbb{V}} \neq \emptyset$ *be its algebra of constants (i.e. the free* $\mathbb{V}$*-algebra on the empty set).*

Then $\mathbb{V}$ is 2-star-permutable if and only if the following syntactic condition holds:

For every $e \in E_{\mathbb{V}}$ there exists a binary term $s_e(x, y)$ such that $s_e(x, x) = e$ and $s_e(x, e) = x$.

Proof. Suppose 2-star-permutability holds and fix any $e \in E_{\mathbb{V}}$. We then consider a graph $F(x, y) \underset{g_1}{\overset{g_0}{\rightrightarrows}} F(x)$ between free algebras on 2 and 1 generator respectively, where g_0, g_1 are defined by setting $g_0(x) = x$, $g_0(y) = e$ and $g_1(x) = g_1(y) = x$. This graph is reflexive, since it is clearly split by the map $\delta : F(x) \to F(x, y)$ defined by $\delta(x) = x$. Since free algebras are projective, we can apply 3.12 (i.e. $\mathcal{C}$ here is the full subcategory of free algebras) to deduce that this graph must be star-symmetric.

Now we have $(e, x) = (g_0(y), g_1(y))$, so by the star-symmetry we must also have $(x, e) = (g_0(s_e), g_1(s_e))$ for some $s_e(x, y) \in F(x, y)$. Thus, $x = g_0(s_e(x, y)) = s_e(x, e)$ and $e = g_1(s_e(x, y)) = s_e(x, x)$.

Conversely, suppose we have binary terms $s_e(x, y)$ for all $e \in E_{\mathbb{V}}$ with the indicated properties. We will show that any reflexive relation $R \rightarrowtail X \times X$ in the variety $\mathbb{V}$ is left star-symmetric.

Indeed, assume that $(e, x) \in R$ for some $e \in E_X$ and $x \in X$. Since R is reflexive, we also have $(x, x) \in R$. By compatibility with the operations we then must have $(s_e(x, e), s_e(x, x)) \in R$, i.e. that $(x, e) \in R$. This concludes the proof. $\square$

As particular examples of E-subtractive varieties one has the categories **Ring** of unitary rings, as well as the categories **Heyt**, **Bool** of Heyting and Boolean algebras respectively. These are all in fact already Mal'tsev, but of course one also has examples of subtractive varieties which are not, such as that of *implication algebras* [17].

Remark 3.17. Since our main result is stated for any regular category, without requiring exactness, it can equally well be applied to *quasi-varieties* of universal algebras, since these are still regular categories. This encompasses further interesting examples, such as the category **RedRng** of *reduced rings*, i.e. unitary rings R satisfying $(\forall x \in R)(\forall n \geq$

$1)(x^n = 0 \implies x = 0)$. As a non-Mal'tsev example here one has the quasi-variety of *BCK algebras* (see [3], for example).

References

[1] P. Agliano, A. Ursini, *Ideals and other generalizations of congruence classes*, Journal of Australian Mathematical Society (Series A) 53 (1992) 103-115.

[2] M. Barr, *Exact Categories*, Lecture Notes in Mathematics, 236 Springer (1970) 1-120.

[3] W. J. Blok, J. G. Raftery, *On the quasivariety of BCK-algebras and its subvarieties*, Algebra Universalis, 33(1995) 68-90.

[4] A. Carboni, G. M. Kelly, M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Applied Categorical Structures, 1 (1993) 385-421.

[5] A. Carboni, E.M. Vitale, *Regular and Exact Completions*, Journal of Pure and Applied Algebra 125 (1998) 79-l 16

[6] C. Ehresmann, *Sur une notion générale de cohomologie*, C. R. Acad. Sci. Paris 259 (1964) 2050-2053.

[7] P. Freyd, *Representations in abelian categories*, Proc. Conference on Categorical Algebra, La Jolla, 1965 (Springer, Berlin, 1966)

[8] M. Gran, *Semi-abelian exact completions*, Homology, Homotopy and Applications, 4 (2002) 175-189.

[9] M. Gran, Z. Janelidze, *Star-regularity and regular completions*, Journal of Pure and Applied Algebra 218 (2014) 1771-1782.

[10] M. Gran, Z. Janelidze, D. Rodelo, *3 × 3 Lemma for Star-Exact Sequences*, Homology, Homotopy and Applications, vol. 14(2), 2012, pp. 1-22

[11] M. Gran, Z. Janelidze, D. Rodelo, A. Ursini, *Symmetry of Regular Diamonds, the Goursat property and Subtractivity*, Theory and Applications of Categories, Vol. 27, No. 6, 2012, pp. 80-96.

[12] M. Gran, Z. Janelidze, A. Ursini, *A good theory of ideals in regular multi-pointed categories*, Journal of Pure and Applied Algebra, 216 (2012) 1905-1919.

[13] M. Gran, D. Rodelo, *On the characterization of Jonsson-Tarski and of subtractive varieties*, Diagrammes, Volume S67-68 (2012), p. 101-115

[14] M. Gran, D. Rodelo, *On 2-star-permutability in regular multi-pointed categories*, Journal of Algebra and its Applications, Vol. 13, No. 8, 2014, 1450068.

[15] M. Gran, E. Vitale, *On the exact completion of the homotopy category*, Cah. Top. Geom. Diff Categoriques XXXIX (1998) 287-297.

[16] Z. Janelidze, *Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories*, Theory and Applications of Categories 16, 2006, 236-261.

[17] A. Mitschke, *Implication algebras are 3-permutable and 3-distributive*, Algebra Universalis, 182-186 (1971).

[18] D. Rodelo, I. Tchoffo Nguefeu, *Goursat Completions*, Cah. Top. Geom. Diff Categoriques LX (2019) 433-449.

[19] J. Rosický, E.M. Vitale, *Exact Completion and Representations in Abelian Categories*, Homology, Homotopy and Applications, vol.3(3), 2001, pp.453-466.

[20] A. Ursini, *Normal Subalgebras I*, Applied Categorical Structures, (2013) 21:209-236

Vasileios Aravantinos-Sotiropoulos
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
Chemin du Cyclotron 2
1348 Louvain-la-Neuve (Belgium)
vasileios.aravantinos@uclouvain.be