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A simplified categorical approach to several Galois theories

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Résumé. Nous étudions le concept de structure de Galois et epimorphisme de Galois dans un contexte général. Notamment, une structure de Galois pour un épimorphisme $\pi\colon M \to B$ dans une catégorie $\mathcal{C}$ est l'action d'un groupe objet qui munit M d'une structure d'espace homogène dans la catégorie relative $\mathcal{C}_B$.

Abstract. We discuss the concept of Galois structure and Galois epimorphism in a general setting. Namely, a Galois structure for an epimorphism $\pi\colon M \to B$ in some category $\mathcal{C}$ is the action of a group object that gives to M the structure of principal homogeneous space in the relative category $\mathcal{C}_B$.

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1. Introduction

From its very starting point in the theory of polynomial equations with one variable [12], Galois theory proposes a systematic use of the principal homogeneous structure of the space of solutions of an equation. This idea was systematically applied by E. Vessiot [29] in his general approach to differential Galois theory. Today there are several Galois theories, with different domains of application.

It is clear that there is some common mathematical core within all these theories. This is usually explained through analogy. Most texts dedicated to several Galois theories develop them separately, establish some bridges, and point out these analogies between them, as in the book of R. and D. Douady [9].

There is a categorical approach to Galois theory initiated by Grothendieck ([14], see [10] for a more accessible exposition) and continued in [1] (see also [16]). This theory is further developed by Dubuc [11] and culminated by Joyal-Tierney [17]. A different approach to Galois theory is considered by G. Janelidze and F. Bourceux ([15], see also [2], chapter 5). This categorical Galois theory does not cover some natural incarnations of Galois theory, as differential Galois theory [25]. The main difference between Grothedieck approach and ours is the following: we do not see the Galois group as a set-theoretical group acting on an object but as a group object of the category. This line of thinking is inspired by some facts of differential Galois theory. For instance, the Galois group of a strongly normal extension [20] is an algebraic group defined over the constants, which can be seen as a particular kind a group object in the category of differential algebraic varieties. Some years ago A. Pillay generalized E. Kolchin's theory of strongly normal extensions [26]. A generalized strongly normal extension is a differential field extension whose group of automorphisms admits a natural structure of differential algebraic group, that is, a group object in the category of differential algebraic varieties.

Our framework also explains how some Galois theories are naturally extended. Most of them allow Galois structures (Definition 2.7) with Galois groups in some specific class of group objects in a category. By modifying the category, or by extending the class of possible Galois groups we obtain different extensions of Galois theory. For instance, classical Galois theory extends to Hopf-Galois theory by allowing a broader class of group objects.

We give some examples of how the proposed general definitions apply to the cases of classical Galois theory (algebraic and topological), and differential Galois theory. Then we explore the category of foliated smooth manifolds. Epimorphims in such category are partial Ehresmann conections. When examining Galois structures there naturally appear G-invariant connections. This is not surprising, G-invariant connections were in fact introduced in the context of Galois theory by E. Vessiot in the beggining to 20th century: they are the so-called automorphic systems appearing in [29].

We prove uniqueness of the Galois group for the irreducible case, Theorem 4.3. Finally we compare the real smooth and the complex algebraic cases.

2. General definitions

2.1 Split of groupoid actions

Let us consider $\mathcal{C}$ a category with binary products, kernels of pairs of morphisms, and a final object $\{\star\}$. Thus, there are also fibered products (pullbacks) as well as finite limits. We may define group objects and groupoid objects in $\mathcal{C}$.

Let G be a group object in $\mathcal{C}$. For each object X, the set $G(X) = \mathrm{Hom}(X, G)$ of X-elements of G is a group. An action of G in an object M is a morphism,

$$\alpha\colon G \times M \to M,$$

satisfying $\alpha \circ (\mu \times \mathrm{Id}_M) = \mu \circ (\mathrm{Id}_G \times \alpha)$ and $\alpha \circ ((e_G \circ \pi_M) \times \mathrm{Id}_M) = \mathrm{Id}_M$.[1] The action α induces a group morphism $\alpha\colon G(\{\star\}) \to \mathrm{Aut}(M)$, $g \mapsto \alpha \circ \langle g \circ \pi_M, \mathrm{Id}_M \rangle$.

From the action α we can form the *action groupoid* $G \ltimes M \rightrightarrows M$, with objects of objects M and object of arrows $G \ltimes M$. The source map is the projection π_2 onto the second factor M, and the target map is α. In terms of sets and elements, we have:

$$s(g, x) = x, \quad t(g, x) = \alpha(g, x), \quad (h, gx) \circ (g, x) = (hg, x).$$

Definition 2.1. *We say that a groupoid object $\mathcal{G} \rightrightarrows M$ splits in $\mathcal{C}$ if there is an action $\alpha\colon G \times M \to M$ an action of a group object and a groupoid isomorphism $\varphi\colon G \ltimes M \xrightarrow{\sim} \mathcal{G}$. In such a case, we say that G is a splitting group, α is a splitting action and φ is a splitting morphism for $\mathcal{G}$ in $\mathcal{C}$.*

Example 2.2. Let us remark that it is not in general possible to recover the group G from the action groupoid $G \ltimes M$. For instance, in the category of sets, let us consider two free and transitive actions of $\mathbb{Z}_4$ and $\mathbb{K}_4$ in a set $X = \{p_1, p_2, p_3, p_4\}$ of four elements. Since the actions are free and transitive we

[1] Where e_G represent the identity $e_G\colon \{\star\} \to G$ and π_M represents the unique morphism $\pi_M\colon M \to \{\star\}$.

have that the action groupoid is, in both cases, the total equivalence realtion $X \times X$. Therefore we have groupoid isomorphism:

$$\mathbb{K}_4 \ltimes X \xrightarrow{\sim} X \times X \xleftarrow{\sim} \mathbb{Z}_4 \ltimes X$$

$$(\sigma, p) \longrightarrow (p, \sigma \cdot p), (p, \tau \cdot p) \longleftarrow (\tau, p)$$

Thus, a split groupoid object may have different realizations as an action groupoid.

2.2 Normal epimorphisms

Let us recall that an action of a group (set) G in an object X is a group morphism $\phi\colon G \to \mathrm{Aut}(X)$. We say that $q\colon X \to Y$ is a *categorical quotient* of the action of G in X if:

1. For all $g \in G$, $q \circ \phi(g) = q$. In other words, q is G-invariant.
2. For all morphisms $f\colon X \to Z$ such that for all $g \in G$ $f \circ \phi(g) = f$ (i.e. f is G-invariant) there exists a unique $\bar{f}\colon Y \to Z$ such that $\bar{f} \circ q = f$.

Categorical quotients are epimorphisms and are unique up to isomorphims. Let us consider $\pi\colon M \to B$ an epimorphism in $\mathcal{C}$. The group group $\mathrm{Aut}_B(M)$ acts on M.

Definition 2.3. *We say that π is normal if it is the categorical quotient of M by the action of the group (set) $\mathrm{Aut}_B(M)$.*

Some categorical approaches to Galois theory rely in the notion of strict epimorphism ([1, I.10.2] see also [18, Def. 5.1.6]).

Definition 2.4. *Let $\pi\colon M \to B$ be an epimorphism.*

(a) A morphism $f\colon M \to Z$ is π-compatible if for any pair of morphisms $x, y\colon X \rightrightarrows M$ such that $\pi \circ x = \pi \circ y$ also $f \circ x = f \circ z$.

(b) π is a strict epimorphism if for any π-compatible f there is a unique $\bar{f}\colon B \to Z$ such that $f = \bar{f} \circ \pi$.

Proposition 2.5. *Let $\pi\colon M \to B$ be an epimorphism in a category $\mathcal{C}$.*

(a) *If π is normal then it is strict.*

(b) *Assume that any arrow with codomain M is invertible. Then, if π is strict π is normal.*

Proof. Let us consider an object Z an the composition map

$$\pi^*\colon \mathrm{Hom}(B, Z) \to \mathrm{Hom}(M, Z).$$

The image of π^* consists of π-compatible morphisms. Moreover, let us assume that $f\colon M \to Z$ is π-compatible. Then, for any $\sigma \in \mathrm{Aut}_B(M)$ we have $\pi \circ \sigma = \pi \circ \mathrm{Id}_M$ and therefore $f \circ \sigma = f$. It means that π-compatible morphisms are invariant under the action of $\mathrm{Aut}_B(M)$. In general we have a chain,[2]

$$\pi^*(\mathrm{Hom}(B, Z)) \subseteq \{\pi\text{-compatible morphisms}\} \subseteq \mathrm{Hom}(M, Z)^{\mathrm{Aut}_B(M)}.$$

Let us note the following:

(i) The epimorphism π is normal if and only if for any Z we have the equality between the first and third members of the chain.

(ii) The epimorphism π is strict if and only if for any Z we have the equality between the first and second members of the chain.

(a) Assume π normal. Then the three members of the above chain coincide. In particular, any π-compatible morphism factorizes.
(b) Assume that π is strict. We need to prove that any $\mathrm{Aut}_B(M)$ invariant morphism $f\colon M \to Z$ is π-compatible. Let $a, b\colon X \rightrightarrows M$ be a pair of morphisms such that $\pi \circ a = \pi \circ b$. Since f is $\mathrm{Aut}_B(M)$ invariant we have $f = f \circ (b \circ a^{-1})$ and from this $f \circ a = f \circ b$. Hence f is π-compatible. □

Remark 2.6. Let us recall that the notions of regular and effective epimorphism.

(a) An epimorphism $q\colon Y \to X$ is said to be regular if it is the coequalizer of a pair of morphisms $Z \rightrightarrows Y \to Z$.

[2] Here $\mathrm{Hom}(M, Z)^{\mathrm{Aut}_B(M)}$ stands for the set of $\mathrm{Aut}_B(M)$-invariant morphisms in $\mathrm{Hom}(M, Z)^{\mathrm{Aut}_B(M)}$.

(b) An epimorphism $q\colon Y \to X$ is said to be effective if it has a kernel pair and it is the coequalizer of a congruence of its kernel pair $\mathrm{KP}_q \rightrightarrows Y \to X$.

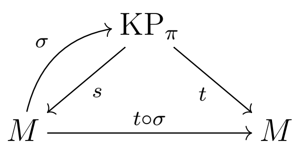
In a general category we have:

$$\text{effective} \implies \text{regular} \implies \text{strict}\,.$$

Moreover, in a category with pullbacks it is known that strict epimorphisms are effective. Therefore a *normal* epimorphism in a category with pullbacks is effective. If additionally, as stated in Proposition 2.5 (b), the epimorphism $\pi\colon M \to B$ satisfies that any arrow with codomain M is invertible, then π is normal if and only if it is effective. This equivalence between effectiveness and normality seems to be a key aspect in classical Galois theory.

2.3 Galois structures

The kernel pair of π, $\mathrm{KP}_\pi = M \times_B M \rightrightarrows M$, is a congruence (equivalence relation) in M, and therefore a grupoid object in $\mathcal{C}$. We set the source (s) and target (t) maps to be the first and second projection respectively. It represents the endomorphisms of M over B in the following sense: let $\mathrm{KP}_\pi(M)$ be the set of sections of the source map (s); the composition with the target map yields a bijection.



$$\mathrm{KP}_\pi(M) \xrightarrow{\sim} \mathrm{End}_B(M) \qquad \sigma \xrightarrow{\sim} t \circ \sigma$$

Let us consider an splitting action $\alpha\colon G \times M \to M$ of KP_π. The splitting isomorphism is necessarily

$$\langle \pi_2, \alpha \rangle \colon G \ltimes M \xrightarrow{\sim} \mathrm{KP}_\pi \quad (g, x) \mapsto (x, \alpha(g, x)),$$

which is completely determined by α. In other words, an splitting action of KP_π is an action that gives $\pi\colon M \to B$ the structure of *principal homogeneous space* modeled over $G \times B \to B$ in the relative category $\mathcal{C}_B$ of arrows over B.

Let us note that the splitting action α induces a bijection between $G(M)$ and $\mathrm{KP}_\pi(M)$ and therefore a bijection,

$$G(M) \xrightarrow{\sim} \mathrm{End}_B(M), \quad g \mapsto \alpha_g = \alpha \circ \langle g, \mathrm{Id}_M\rangle.$$

However, such bijection is not compatible with the composition. We have

$$\alpha_g \circ \alpha_h = \alpha \circ \langle g, \mathrm{Id}_M\rangle \circ \alpha \circ \langle h, \mathrm{Id}_M\rangle = \alpha \circ \langle g \circ \alpha_h, \alpha_h\rangle$$

on the other hand,

$$\alpha_{gh} = \alpha \circ \langle g, \alpha_h\rangle.$$

It follows that, if $g = g \circ \alpha_h$ then, $\alpha_{gh} = \alpha_g \circ \alpha_h$. We see that this is satisfied if $g \in G(B)$, given that $\alpha_h \in \mathrm{End}_B(M)$ induces the identity in B. For *normal* epimorphisms this condition is optimal, as $G(M)^{\mathrm{Aut}_B(M)} = G(B)$. We have thus,

$$\begin{array}{ccccc} & & G(M) & \xrightarrow{\sim} & \mathrm{End}_B(M) \\ & & \uparrow & & \uparrow \\ G(\{\star\}) & \hookrightarrow & G(B) & \hookrightarrow & \mathrm{Aut}_B(M) \end{array}$$

where the maps in the lower row are injective group morphisms.

Definition 2.7. *Let $\pi\colon M \to B$ be an epimorphism in $\mathcal{C}$. A Galois structure for π is an splitting action $\alpha\colon G \times M \to M$ for KP_π such that the induced group morphism*

$$G(\{\star\}) \xrightarrow{\sim} \mathrm{Aut}_B(M), \quad g \mapsto \alpha_g$$

is an isomorphism.

Definition 2.8. *We say that an epimorphism $\pi\colon M \to B$ of $\mathcal{C}$ is Galois if satisfies the following conditions:*

(i) it is normal;

(ii) it admits a unique (up to isomorphism) Galois structure.

We call Galois group of π the group object Gal_π appearing in the unique Galois structure.

Note that if α is a Galois structure for π then we have isomorphisms,

$$G(\{\star\}) \xrightarrow{\sim} G(B) \xrightarrow{\sim} \mathrm{Aut}_B(M).$$

Given an splitting action α for KP_π as a groupoid object in $\mathcal{C}$ we may define an splitting action

$$\tilde{\alpha}\colon (G \times B) \times_B M \to M, \quad ((g, b), x) \mapsto \alpha(g, x),$$

for KP_π as a groupoid object in $\mathcal{C}_B$. In some cases, a splitting action may fail to be a Galois structure in the category $\mathcal{C}$ but be so in the relative category $\mathcal{C}_B$ of arrows over B.

Example 2.9. Let $\mathbf{Set}$ be the category of sets and $\pi\colon M \to B$ be a surjective map. In any case KP_π splits in $\mathbf{Set}_B$, and any splittig action is a Galois structure. The group object acting is a family of groups indexed by B and acting freely and transitively on the fibers of π. It is Galois if and only if the fibers have 1, 2 or 3 points.

However, KP_π splits in $\mathbf{Set}$ if and only if all fibers of π have exactly the same cardinal. Finally, π is Galois in $\mathbf{Set}$ if and only if it is a bijection, otherwise we may have the uniqueness for the Galois structure, but $G \subsetneq \mathrm{Aut}_B(M)$.

Example 2.10. Let $\mathbf{Mnf}$ be the category of smooth manifolds with smooth maps. By direct examination of the definition we have that a an splitting action for a submersion $\pi\colon M \to B$ is an structure of a principal bundle for some structure Lie group G. The splitting actions is far from being unique, moreover, G represents a very small part of $\mathrm{Aut}_B(M)$.

2.4 Galois correspondence

Let us recall that a congruence (internal equivalence relation) in M is a subobject of $M \times M$ having the reflexive, symmetric and transitive property. We say that a congruence $R \subseteq M \times M$ is *effective* if it is the kernel pair of an effective epimorphism. The class of an effective epimorphism up to isomorphisms of the codomain is called an effective quotient. We have then a diagram:

$$R \rightrightarrows M \to M/R.$$

The class $\mathrm{Rel}(M)$ of effective of congruences in M is partially ordered. For two congruences represented by monomorphisms $i\colon R \hookrightarrow M \times M$ and $i'\colon R' \hookrightarrow M \times M$ we say that $R \leq R'$ if there is $j\colon R \hookrightarrow R'$ such that $i' \circ j = i$. Analogously the class $\mathrm{Quot}(M)$ of effective quotients of M is ordered. For two effective quotients represented by effective epimorphisms $q\colon M \to Z$ and $q'\colon M \to Z'$ we say $q \geq q'$ if q' is q-compatible, so that there is $p\colon Z \to Z'$ such that $p \circ q = q'$. There is a natural bijective Galois connection between $\mathrm{Rel}(M)$ and $\mathrm{Quot}(M)$ of effective quotients of M given by the adjunctions:

$$\mathrm{KP}\colon \mathrm{Quot}(M) \to \mathrm{Rel}(M), \quad (q\colon M \to Z) \mapsto \mathrm{KP}_q = M \times_Z M,$$

$$\mathrm{coeq}\colon \mathrm{Rel}(M) \to \mathrm{Quot}(M), \quad R \mapsto (q\colon M \to M/R).$$

The quotient by a group action $\alpha\colon G \times M \to M$ is also understood in the above terms. We have $M/G = \mathrm{coeq}(\alpha, \pi_2)$ if such coequalizer exists in $\mathcal{C}$. Under suitable assumptions on the existence and nature of quotients by group actions, the general Galois connection gives rise to the classical Galois correspondence.

Theorem 2.11. *Let $\pi\colon M \to B$ be a Galois epimorphism. Let us assume the following:*

(a) any subgrupoid object of the action groupoid $\mathrm{Gal}_\pi \ltimes M$ is of the form $H \ltimes M$ where H is a subgroup object of Gal_π;

(b) for any subgroup object $H \subseteq \mathrm{Gal}_\pi$ it does exists the effective quotient M/H.

Then the following sentences hold:

(i) The assignation:

$$H \subseteq G \ \rightsquigarrow \ q_H\colon M \to M/H,$$

establishes an order reversing bijective correspondence between the partially ordered class $\mathrm{Sub}(G)$ of subgroup objects of G and the partially ordered class $\mathrm{Quot}_{\geq\pi}(M)$ of intermediate effective quotients of M.

(ii) Let us consider any effective intermediate quotient $q\colon M \to Z$ *with corresponding subgroup* $H \subseteq \mathrm{Gal}_\pi$*. The restriction of the Galois structure* α *to* $H \times M$ *is a Galois structure for* q*.*

Proof. (i) It is clear that the assignation reverses order, for $H \subseteq H'$ we have $q_H \geq q'_H$. In order to see that it is bijective, let us construct its inverse correspondence. Let $q\colon M \to Z$ be a representative of an effective quotient with $q \geq \pi$. The kernel pair KP_q is an effective congruence in M and $\mathrm{KP}_q \leq \mathrm{KP}_\pi$. The splitting isomorphim establishes an isomorphism of KP_q with a subgroupoid object of $\mathrm{Gal}_\pi \ltimes M$ which, by condition (a), is of the form $H_q \ltimes M$ for a subgroup object H_q depending on q. We have that the effective epimorphism $q\colon M \to Z$ is equivalent to $q_{H_q}\colon M \to M/H_q$. Then we have:

$$H \rightsquigarrow q_H \rightsquigarrow H, \quad q \rightsquigarrow H_q \rightsquigarrow q.$$

(ii) It is enough to note that the splitting isomorphism $\langle \pi_2, \alpha \rangle$ maps $H \ltimes M$ onto KP_q. □

3. Classical Galois theory

3.1 Covering spaces

Let **Top** be the category of topological spaces. A covering map $\pi\colon M \to B$, with M and B connected, is a *Galois cover* if $\pi \times \mathrm{Id}_M\colon M \times_B M \to M$ is a trivial covering space. There is a Galois theory for covering spaces, analogous to classical Galois theory (see, for instance [19]).

Theorem 3.1. *Let* $\pi\colon M \to B$ *be a surjective local homeomorphism with* M *and* B *connected. The following are equivalent:*

(a) π *is a Galois cover.*

(b) π *is a Galois in* **Top**.

(c) KP_π *splits in* **Top**.

In any case, the Galois group object is $\mathrm{Gal}_\pi = \mathrm{Aut}_B(M)$ *with the discrete topology.*

Proof. $(c) \Rightarrow (a)$. Let us assume that there is an splitting isomorphism $\varphi\colon G \times M \xrightarrow{\sim} M \times_B M$. Then we have that the projection on the second factor $G \times M \to M$ is a local homeomorphism. Thus, G is discrete and $M \times_B M \to M$ is a trivial cover. It follows that π is a Galois cover. We also have $(b) \Rightarrow (c)$.

Let us see $(a) \Rightarrow (b)$. We assume that $M \times_B M \to M$ is a trivial cover, thus there is a trivialization,

$$\begin{array}{ccc} G \times M & \xrightarrow[\sim]{\varphi} & M \times_B M \\ & {\scriptstyle\bar{\pi}} \searrow \quad \swarrow {\scriptstyle\pi_1} & \\ & M & \end{array}$$

with G a discrete topological space. Let us check that there is a group structure on G such that it is isomorphic to $\mathrm{Aut}_B(M)$ and φ is the action of $\mathrm{Aut}_B(M)$ in M.

For each $g \in G$ let us consider the map $\sigma(g)\colon M \to M$ defined by the formula $\sigma(g)(x) = \pi_2(\varphi(g,x))$. It is a continuous map that induces the identity on B and thus, an automorphism of M over B. On the other hand, let σ be an automorphism of M over B. Then, the map $x \mapsto \varphi^{-1}(x, \sigma(x))$ is a section of $\bar{\pi}$. Since $\bar{\pi}$ is trivial, then there is a unique g in G such that $\varphi^{-1}(x,\sigma(x)) = (g,x)$. We define this g to be $g(\sigma)$. It is easy to check that those bijections inverse of each other. With the group operation in G induced by $\sigma_{gh} = \sigma_g \circ \sigma_h$ then we have that φ is a splitting morphism and thus π admits a Galois structure, where the action of G in M is isomorphic to that of $\mathrm{Aut}_B(M)$ endowed with the discrete topology, and thus unique.

Let us discuss the normality of π. In this context, it means that the action of $\mathrm{Aut}_B(M)$ is transitive on the fibers. Let m_1, m_2 be two points of M in the same fiber. Let g be the element of G such that $\varphi(g, m_1) = (m_1, m_2)$. Then, it is clear that $\sigma(g)(m_1) = m_2$. $\square$

Note that Galois covers are under the hypothesis of Theorem 2.11. The subgroupoids of $G \ltimes M$ are of the form $H \ltimes M$ with H a subgroup of G and the quotient M/H exists in **Top**. We obtain the well known correspondence between intermediate coverings and subgroups of G.

3.2 Algebraic Galois extensions

Let $\mathbf{Cmm}$ be the category of commutative rings with unit. The dual category $\mathbf{Cmm}^{\mathrm{op}}$ is the category of affine schemes.

Let us consider an extension of rings $i\colon K \hookrightarrow L$. The dual map $i^*\colon \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is an epimorphism in $\mathbf{Cmm}^{op}$. In this case the kernel pair is $\mathrm{Spec}(L \otimes_K L) \rightrightarrows \mathrm{Spec}(L)$ where the source and target maps are the dual of the canonical embeddings $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$ respectively.

Group objects is $\mathbf{Cmm}^{\mathrm{op}}$ are commutative Hopf algebras. Thus, splitting actions in $\mathbf{Cmm}^{\mathrm{op}}$ are the already known Hopf-Galois structures, in the sense of Chase and Sweedler [7]. It is well known that Hopf-Galois structures are not unique in general.

Let us revisit classical Galois theory. Let us consider i to be a finite extension of fields. Classically, it is called a Galois extension if it satisfies one of the following equivalent conditions (see [27] pp. 140-141):

(a) L is separable and normal[3] over K.

(b) $|\mathrm{Aut}_K(L)| = \dim_K L$.

(c) $L \otimes_K L$ (with L-algebra structure given by the embedding $a \mapsto a \otimes 1$) is a finite trivial[4] L-algebra.

Let us consider $i\colon K \hookrightarrow L$ a Galois extension, and let G be $\mathrm{Aut}_K(L)$. Then, it is well known that the trivialization of $L \otimes_K L$ can be realized as a split. We have the trivial finite L-algebra $\mathrm{Maps}(G, L)$ and an isomorphism:

$$\varphi\colon L \otimes_K L \xrightarrow{\sim} \mathrm{Maps}(G, L) = \prod_{g \in G} L, \quad a \otimes b \mapsto f_{a\otimes b},$$

where $f_{a\otimes b}(g) = g(a)b$. Now we have that $\mathrm{Maps}(G, L) = \mathrm{Maps}(G, K) \otimes_K L$. Thus, in the dual category we have that the map,

$$\varphi^*\colon \mathrm{Spec}(\mathrm{Maps}(G, K)) \times_K \mathrm{Spec}(L) \xrightarrow{\sim} \mathrm{KP}_{i^*},$$

[3]It is clear that our categorical definition of normality coincides, in this context, with the classical definition $L^{\mathrm{Aut}_K(L)} = K$.

[4]A finite trivial L-algebra is an L-algebra isomorphic to a direct product of a finite number of copies of L, $\prod_{i \in I} L$.

is a splitting isomorphism of the groupoid KP_{i^*}. Noting that $\mathrm{Maps}(G,K) = \mathrm{Maps}(G,\mathbb{Z})\otimes_{\mathbb{Z}} K$ we see that the splitting isomorphism can be defined in the category $\mathbf{Cmm}^{op}$ and not only in the relative category $\mathbf{Cmm}_K^{op}$. We may state the following result.

Proposition 3.2. *Let us consider $i\colon K \hookrightarrow L$ a finite separable field extension, and $i^*\colon \mathrm{Spec}(L) \to \mathrm{Spec}(\mathrm{K})$ its dual morphism. The following are equivalent:*

(a) $i\colon K \hookrightarrow L$ is a Galois extension.

(b) i^* is Galois in $\mathbf{Cmm}^{\mathrm{op}}$.

(c) i^* is Galois in $\mathbf{Cmm}_K^{\mathrm{op}}$.

In such a case, if $G = \mathrm{Aut}_K(L)$, there is a natural action of G in $L\otimes_K L$ such that $(L\otimes_K L)^G$ is a Hopf K-algebra canonically isomorphic to $\mathrm{Maps}(G,K)$.

Let us fix a Galois extension $i\colon K \hookrightarrow L$ with group G. Let H be a subgroup of G. Then, we realize the field of invariants L^H as the equalizer, $L^H \to L \rightrightarrows L\otimes_{L^H} L$. Therefore, in the dual category $\mathrm{Spec}(L^H)$ appears as the effective quotient of $\mathrm{Spec}(L)$ by the action of the group object H. Moreover, since $G \ltimes \mathrm{Spec}(L)$ is the spectrum of a L-trivial algebra, we have that any subgroupoid is of the form $H \ltimes \mathrm{Spec}(L)$. We are under the hypothesis of Theorem 2.11, which in this particular case gives the classical Galois correspondence between intermediate field extensions and subgroups.

4. Foliated manifolds

4.1 Smooth foliated manifolds

Let $\mathbf{FMn}$ be the category of smooth manifolds endowed with regular foliations. Objects are pairs $(M,\mathcal{D})$ where M is a smooth manifold and $\mathcal{D}$ is an involutive linear subbundle of TM. Morphisms $f\colon (M,\mathcal{D}) \to (M',\mathcal{D}')$ are smooth maps $f\colon M \to M'$ such that for all $p \in M$ the differential d_pf induces a linear epimorphism from $\mathcal{D}_p$ to $\mathcal{D}'_p$. This implies that f maps leaves of $\mathcal{D}$ onto leaves of $\mathcal{D}'$ by local submersions. A manifold B admits two trivial structures of foliated manifold (B,TB), with only a leaf B and $(B,0_B)$ with point leaves.

Let $(G, \mathcal{D}_G)$ be a group object in **FMn**. It is clear that G is a Lie group. The existence of the identity element implies that the map,

$$(\{\star\}, 0_\star) \to (G, \mathcal{D}_G), \quad \star \mapsto e,$$

is a morphism of foliated manifolds, so that $\operatorname{rank}(\mathcal{D}_G) \leq \operatorname{rank}(0_\star) = 0$. It follows $\mathcal{D}_G = 0_G$. By abuse of notation we write G instead of $(G, 0_G)$. If is also clear that an action of G in $(M, \mathcal{D})$ to the category of foliated manifolds is an action of G in M by symmetries of $\mathcal{D}$. That is, for any $p \in M$ and $g \in G$ $d_pL_g(\mathcal{D}_p) = \mathcal{D}_{gp}$.

A *flat Ehresmann connection* in a submersion $\pi\colon M \to B$ is an involutive subbundle $\mathcal{F} \subset TM$ such that for each $p \in M$ the differential $d_p\pi$ is an isomorphism of $\mathcal{F}_p$ with $T_{\pi(p)}B$. We say that a foliated manifold $(M, \mathcal{F})$ is *irreducible* if it contains a dense leaf. Let us first analyze the case in which the basis M has a trivial structure of foliated manifold.

Proposition 4.1. *Let* $\pi\colon (M, \mathcal{F}) \to (B, TB)$ *be an epimorphism of foliated manifolds with* $\operatorname{rank}(\mathcal{F}) = \dim(B)$. *Then* π *is a submersion and* $\mathcal{F}$ *is a flat Ehresmann connection.*

Proof. For all $p \in M$ we have that $d_p\pi$ maps $\mathcal{F}_p$ onto T_pB. Therefore $d_p\pi$ is surjective for all $p \in M$ and π is a submersion. It is clear that $\mathcal{F}$ is a flat Ehresmann connection. □

Proposition 4.2. *Let* $\pi\colon (M, \mathcal{F}) \to (B, TB)$ *be a epimorphism of irreducible foliated manifolds with* $\operatorname{rank}\mathcal{F} = \dim B$. *The following are equivalent.*

- *(a)* KP_π *splits in* **FMn**.
- *(b)* π *is Galois in* **FMn**.
- *(c) There is a Lie group* G *acting on* M *such that* π *is a principal* G*-bundle and* $\mathcal{L}$ *is a* G*-invariant connection.*
- *(d) The above, with a unique* G.

In such a case G *is* $\operatorname{Aut}_{(B,TB)}(M, \mathcal{F})$.

Proof. Cases (a) and (c) are equivalent from the very definition of splitting action. It is also clear that (b) and (d) are equivalent. It remains to prove that (c) implies (d). Let us consider two principal structures $\beta\colon M \times H \to M$ and $\alpha\colon M \times G \to M$ such that $\mathcal{F}$ is simultaneously G and H-invariant. Let us see that these actions are conjugated by a Lie group isomorphism.

Let $\mathcal{L}$ be a dense leaf in M. We consider in $\mathcal{L}$ its intrinsic structure as smooth manifold, so that the projection $\mathcal{L} \to B$ is an étale map with arc-connected Hausdorff domain. Let us note that M and B are necessarily connected. Let x be any point of $\mathcal{L}$; there is a unique $h \in H$ such that $\alpha(x, g) = \beta(x, h)$. Let $\mathcal{L}'$ be the leaf of $\mathcal{F}$ passing through $\alpha(x, g) = \beta(x, h)$. Let us denote R_g^α and R_h^β the right translations by g and h respectively. Then, $R_g^\alpha|_{\mathcal{L}}$ and $R_h^\beta|_{\mathcal{L}}$ are homeomorphisms of $\mathcal{L}$ into $\mathcal{L}'$ that project onto the identity on B. They coincide on the point x, and thus they are the same, $R_g^\alpha|_{\mathcal{L}} = R_h^\beta|_{\mathcal{L}}$. Maps R_g^α and R_h^β are smooth and they coincide along the dense subset $\mathcal{L}$, thus they are equal. Finally, the map $G \to H$ that assigns to each g the only element h such that $\alpha(x, g) = \beta(x, h)$ is a group isomorphism. It is defined by composing and inverting smooth maps, so that, it is a Lie group isomorphism conjugating the actions α and β.

Moreover, the same argument proves that any automorphism $\varphi \in \mathrm{Aut}_{(B,TB)}(M, \mathcal{F})$ must be a translation by an element of G. □

The same idea can be generalized to the case in which the foliated structure of the basis is not trivial, but irreducible. Let $\pi\colon M \to B$ be a manifold submersion, and $\mathcal{D}$ a foliation in M. Let us recall that a flat $\mathcal{D}$*-connection* (or a flat partial connection in the direction of $\mathcal{D}$) is a foliation $\mathcal{F}$ in M that for all $p \in M$ the differential $d_p\pi$ maps $\mathcal{F}_p$ isomorphically onto $\mathcal{D}_p$. Note that a flat Ehresmann connection is the same that a flat TB-connection.

As in Proposition 4.1 if $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ is an submersion of foliated manifolds with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$ then $\mathcal{F}$ is a flat $\mathcal{D}$-connection.

Theorem 4.3. *Let $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ be a epimorphism of irreducible foliated manifolds with* $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$*. The following are equivalent.*

(a) KP_π *splits in* **FMn**.

(b) π *is Galois in* **FMn**.

(c) There is a Lie group G acting on M such that π is a principal G-bundle and $\mathcal{D}$ is a $\mathcal{D}$-partial G-invariant connection.

(d) The above, with a unique G.

In such a case G is $\mathrm{Aut}_{(B,\mathcal{D})}(M,\mathcal{F})$.

Proof. Let us consider $\mathcal{L}$ a dense leaf of $\mathcal{F}$. Then $\pi(\mathcal{L})$ is a dense leaf of $\mathcal{D}$. We may proceed as in the proof of Proposition 4.2 replacing the role of B by $\pi(\mathcal{F})$. □

4.2 Galois correspondence

From now on let $\pi\colon (M,\mathcal{F}) \to (B,\mathcal{D})$ be a Galois submersion of irreducible foliated manifolds with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$ with Galois group G. Let us check that we are under the hypothesis of Theorem 2.11.

Proposition 4.4. *Any object subgroupoid of the action groupoid $G \ltimes (M,\mathcal{F})$ is of the form $H \ltimes (M,\mathcal{F})$ with H a Lie subgroup of G.*

Proof. Let $(\mathcal{G},\mathcal{D}') \rightrightarrows (M,\mathcal{D})$ be a subgroupoid object of the action groupoid. Then $\mathcal{G}$ is a Lie subgroupoid of $G \ltimes M$ and if $(g,p) \in \mathcal{G}$ implies that the $\{g\} \times \mathcal{L}_p \subset \mathcal{G}$ where $\mathcal{L}_p$ is the leaf of $\mathcal{F}$ through p.

Let $\mathcal{L}$ be a dense leaf of $\mathcal{F}$. Note that for any $g \in G$ and $p \in M$ the poinf (g,p) is an accumulation point of $\{g\} \times \mathcal{L}$. Therefore if $(g,p) \in \mathcal{G}$ implies $\{g\} \times \mathcal{L} \subseteq \mathcal{G}$ and therefore $\{g\} \times M \subseteq \mathcal{G}$. It follows that $\mathcal{G} = S \times M$ for some submanifold $S \subseteq G$. From the groupoid composition and inversion it follows that $S = H$ a Lie subgroup of G. □

By a G-manifold we mean a manifold X endowed with a left action of G. To any G-manifold X it corresponds an associated bundle with fiber X,

$$M \times_G X \to B$$

defined as the quotient of the direct product $M \times X$ by the equivalence relation $(pg,x) \sim (p,gx)$ for all $p \in M$, $g \in G$, $x \in X$. The G-invariant $\mathcal{D}$-connection induces an *associated $\mathcal{D}$-connection* $\mathcal{F} \times_G 0_X$ which is the projection on $M \times_G X$ of the direct product $\mathcal{F} \times_G 0_X$. We have that,

$$(M \times_G X, \mathcal{F} \times_G 0_X) \to (B,\mathcal{D})$$

is an epimorphism of foliated manifolds and $\mathcal{F} \times 0_X$ is a flat Ehresmann $\mathcal{D}$-connection. In particular if H is a Lie subgroup of G and $X = G/H$ is an homogeneous G-space we have,

$$M \times_G (G/H) = M/H$$

and the induced associated $\mathcal{D}$-connection is just the projection of $\mathcal{F}$ onto M/H. Therefore, in this case, Theorem 2.11 gives us a Galois correspondence between Lie subgroups of G and associated $\mathcal{D}$-connections in associated bundles whose fibers are homogeneous G-spaces.

4.3 Galois structures over $(B, \mathcal{D})$

Let us discuss Galois structures in the relative category $\mathbf{FMn}_{(B,\mathcal{D})}$ whose objects are smooth maps of foliated manifolds $(Z, \mathcal{D}_Z) \to (B, \mathcal{D})$. A *group bundle* $G \to B$ is a smooth bundle by Lie groups, where composition, inversion and identity depends smoothly on the base point. A *group $\mathcal{D}$-connection* in $G \to B$ is a $\mathcal{D}$-connection $\mathcal{D}$ in G such that leaves are compatible with composition. Linear bundles and linear $\mathcal{D}$-connections are the most usual examples of group bundles and group connections. Group bundles over B endowed with group $\mathcal{D}$-connections are group objects in $\mathbf{FMn}_{(B,\mathcal{D})}$. They are the smooth geometric counterpart of differential algebraic groups of finite dimension discussed by Buium in [5].

In the case of trivial foliated structure in the basis, group objects are locally Lie groups after change of basis, as the following result explains.

Proposition 4.5. *Let B be simply connected, and $q\colon (G, \mathcal{L}) \to B$ a group bundle with group connection (and therefore a group object in $\mathbf{FMn}_{(B,TB)}$). Let x be a point in M and G_x the fiber of G over x, then $(G, \mathcal{L}) \simeq (G_x, \{0\}) \times (B, TB)$.*

Proof. The argument is local, so we have to see that for each $x \in B$ there is a neighborhood U of x such that $(G|_U, \mathcal{L}|_U) \simeq (G_x, \{0\}) \times (U, TU)$. If this is the case, for each homotopy class of a path γ connecting x and y in B we have a group isomorphism $\gamma_*\colon G_x \to G_y$. If B is simply connected, those homotopy classes are unique for each y and the isomorphisms γ_* give us the trivialization of the group connection.

In fact, there are neighborhoods U of x in B, V_x of e_x (the identity element) in G_x, and V of e_x in G, and a decomposition $V \simeq U \times V_x$, such that the horizontal leaves of $\mathcal{L}$ in V have the form $\{g_x\} \times U$ for fixed $g \in V_x$.

Let us see that, for each $h_x \in G_x$ the leaf $\mathcal{F}$ of $\mathcal{L}$ that passes through h_x projects onto U. We may also assume that we take U small enough so that each connected component of $G|_U$ contains exactly one connected component of G_x. Let y be an accumulation point of $q(\mathcal{F})$ inside U. Let us consider h_y an element in G_y in the same connected component of $G|_U$ than h_x. Then there is a leaf $\mathcal{F}'$ of $\mathcal{L}|_U$ passing through h_x. Let U' be $q(\mathcal{F}')$ which is an open subset that intersects $q(\mathcal{F})$. By successive composition of $\mathcal{F}'$ with the leafs of $\mathcal{L}$ in $V|_{U'}$ we have that the connected component of $G|_U'$ containing h_y decomposes in leaves of $\mathcal{L}$. In particular, $\mathcal{F} \cap G|_{U'}$ is part of a leaf of such a decomposition. Finally, $y \in q(\mathcal{F})$. We have seen that $q(\mathcal{F})$ is an open subset that contains all its accumulation points inside U, so that $q(\mathcal{F}) = U$. Thus, $G|_U$ decomposes in leaves of $\mathcal{L}$. □

For the non-simply connected case, the classification of group connections may follow a similar path to the classification of linear connections. Classes of group connections may be given by classes of representations of the fundamental group $\Pi_1(x, B)$ into the group $\mathrm{Aut}(G_x)$ of automorphisms of the fiber. In the case of simply connected B there is no distinction between Galois structures in $\mathbf{FMn}$ or in $\mathbf{FMn}_{(B,TB)}$.

Corollary 4.6. *Let B be simply connected and let $\pi\colon (M, \mathcal{L}) \to (B, TB)$ be a submersion of foliated manifolds with* $\operatorname{rank}\mathcal{L} = \dim\, B$. *Then* KP_π *splits in* $\mathbf{FMn}$ *if and only if it splits in* $\mathbf{FMn}_{(B,TB)}$.

In the non-simply connected case, non trivial irreducible linear connections give us examples of splitting actions in the relative category. For instance, we may take, $B = S^1 \times S^1$. We take $G = \mathbb{R} \times B$ and $\mathcal{D} = \langle \partial_\theta + u\partial_u, \partial_\phi + \alpha u \partial u \rangle$ where u is the coordinate in $\mathbb{R}$ and α is an irrational number. Then, we have $(G, \mathcal{D}) \to (B, TB)$ is a group bundle with an irreducible group connection, locally isomorphic to the trivial additive bundle. The action of G on itself is an splitting action in $\mathbf{FMn}_{(B,TB)}$.

4.4 Foliated complex algebraic varieties

Let $\mathbf{FVar}$ be the category of complex regular foliated varieties. Objects are $(M, \mathcal{D})$ where M is a complex variety and $\mathcal{D}$ is an involutive Zariski closed linear subbundle of TM. A foliated variety is called *irreducible* if it has a Zariski dense leaf, or equivalently, it does not have rational first integrals (except locally constant functions). Group objects in $\mathbf{FVar}$ are complex algebraic groups.

In this category, we can state Galois theory exactly in a way totally analogous to what has been done in $\mathbf{FMn}$.

Theorem 4.7. *Let $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ be a submersion of irreducible foliated varieties with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$. The following are equivalent.*

(a) KP_π *splits* $\mathbf{FVar}$.

(b) π *is Galois in* $\mathbf{FVar}$.

(c) There is an algebraic group G acting on M such that π is a principal G-bundle and $\mathcal{D}$ is a $\mathcal{D}$-partial G-invariant connection.

(d) The above, with a unique G.

In such a case G is $\operatorname{Aut}_{(B,\mathcal{D})}(M, \mathcal{L})$.

Proof. Totally analogous to the proofs given in Proposition 4.2 and Theorem 4.3. □

It is interesting to make the connection of this Galois theory with differential algebra. Let us fix $\pi\colon (M, \mathcal{L}) \to (B, \mathcal{D})$ a Galois submersion of irreducible foliated varieries with Galois group G and $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D} = r$. Let us note that, by elimination, it is always possible to find a system of commuting rational vector fields $\vec{D}_1, \ldots, \vec{D}_r$ that span $\mathcal{D}$ on the generic point of B. Let us fix $\Delta_B = (\vec{D}_1, \ldots, \vec{D}_r)$. We have that the field of rational functions $(\mathbb{C}(B), \Delta_B)$ is a differential field whose field of constants is $\mathbb{C}$.

The $\mathcal{D}$-connection $\mathcal{F}$ induce lifts of the rational vector fields $\vec{D}_j$ to $\mathcal{F}$-horizontal rational vector fields $\vec{F}_i$ in M that span $\mathcal{F}$ on the generic point of M. We set $\Delta_M = (\vec{F}_1, \ldots, \vec{F}_m)$ so that $(\mathbb{C}(M), \Delta_M)$ is also a differential field whose field of constants is $\mathbb{C}$. Since the projection of $\vec{F}_j$ is $\vec{D}_j$ we have that $\pi^*\colon (\mathbb{C}(B),\Delta_B)\hookrightarrow(\mathbb{C}(M),\Delta_M)$ is an differential field extension. We have the following geometric characterization of strongly normal extensions due to Bialynicki-Virula.

Proposition 4.8 ([3], in [4] p. 18)**.** *Let $(K,\Delta)\hookrightarrow(F,\Delta')$ be a differential field extension with K relatively algebraically closed in F and algebraically closed field of constants $C=K^{\Delta}=F^{\Delta'}$. The following are equivalent:*

1. *It is strongly normal in the sense of Kolchin.*
2. *There are a connected algebraic group G over C and a K-variety W such that:*
   (a) *W is a principal homogeneous space modeled over $G_K=G\times_C \operatorname{Spec}(K)$.*
   (b) *The field of rational functions in W is F.*
   (c) *The group G acts faithfully on F by differential automorphisms fixing K.*

Moreover the pair (G,W) is uniquely determined up to isomorphism and we have $G(C)=\operatorname{Aut}_{\Delta}(F/K)$.

This geometric characterization immediately yields the following.

Proposition 4.9. *Let $\pi\colon (M,\mathcal{L})\to(B,\mathcal{D})$ a Galois submersion of irreducible foliated varieries with Galois group G, and Δ_B, Δ_M as above. Assume any of the following equivalent hypothesis:*

1. *$\mathbb{C}(B)$ is relatively algebraically closed in $\mathbb{C}(M)$;*
2. *$\pi\colon M\to B$ has connected fibers;*
3. *G is connected.*

The differential field extension:

$$\pi^*\colon (\mathbb{C}(B),\Delta_B)\hookrightarrow(\mathbb{C}(M),\Delta_M)$$

is a strongly normal extension in the sense of Kolchin with Galois group G.

Proof. Let us consider: $M_B = M \times_B \operatorname{Spec} \mathbb{C}(B)$ and $G_B = M \times_C \operatorname{Spec} \mathbb{C}(B)$ as $\mathbb{C}(B)$-varieties. The splitting isomorhism:

$$M \times_{\mathbb{C}} G \xrightarrow{\sim} M \times_B M$$

changes of basis to an isomorphism of $\mathbb{C}(B)$-varieties,

$$M_B \times_{\mathbb{C}(B)} G_B \xrightarrow{\sim} M_B \times_{\mathbb{C}(B)} M_B$$

And therefore M_B is a principal homogenous space over G_B. The field of rational functions in M_B is also $\mathbb{C}(M)$. For any $g \in G$ we have a field automorphism,

$$R_g^* \colon \mathbb{C}(M) \to \mathbb{C}(M)$$

that fixes $\mathbb{C}(B)$ and the derivations $\vec{F}_j$ in Δ_M. This gives an inclusion,

$$G \to \operatorname{Aut}_\Delta(\mathbb{C}(M)/\mathbb{C}(B)), \quad g \mapsto R_g^*$$

and we conclude by Bialynicki-Virula's Proposition 4.8. □

Remark 4.10. The applications to differential algebra seem to go further. There have been several generalizations of differential Galois theory theory [26, 6] and a geometric characterization of strongly normal extensions [21, 22] which is very much in the flavour of Definition 2.7. We expect upcoming research clarifying how all those theories relate with the framework proposed here.

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