CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXI-4 (2020)

THE FULLNESS AXIOM AND EXACT COMPLETION OF HOMOTOPY CATEGORIES

Jacopo Emmenegger

Résumé. Nous adoptons une formulation catégorique du "Fullness Axiom" de Aczel de la Théorie Constructive des Ensembles, dans le but de dériver la propriété que la complétion exacte est localement cartésienne fermée. Nous montrons, en tant qu'application, que cette formulation est vérifiée dans la catégorie homotopique de toute catégorie de modèles satisfaisant des faibles conditions additionnelles, en obtenant ainsi en particulier que la complétion exacte de la catégorie des espaces topologiques et classes homotopiques des applications continues est localement cartésienne fermée. Dans la perspective de la théorie des types, ces résultats donnent une motivation générale pour la fermeture cartésienne locale de la catégorie des setoïdes. Pourtant, les résultats et les démonstrations sont formulés seulement dans le langage des catégories, et les lecteurs n'ont besoin d'aucune connaissance préalable de la théorie des types ou de la théorie constructive des ensembles.

Abstract. We use a category-theoretic formulation of Aczel's Fullness Axiom from Constructive Set Theory to derive the local cartesian closure of an exact completion. As an application, we prove that such a formulation is valid in the homotopy category of any model category satisfying mild requirements, thus obtaining in particular the local cartesian closure of the exact completion of topological spaces and homotopy classes of maps. Under a type-theoretic reading, these results provide a general motivation for the local cartesian closure of the category

of setoids. However, results and proofs are formulated solely in the language of categories, and no knowledge of type theory or constructive set theory is required on the reader's part.
Keywords. Exact completion, homotopy category, fullness axiom, local cartesian closure, weak limits.
Mathematics Subject Classification (2020). 18D15; 18A35; 18E08; 55U35; 18B15; 18A15; 18F60.

Introduction

In the paper that generalises the exact completion construction to an arbitrary category with *weak* finite limits, where a universal arrow is not required to be unique, Carboni and Vitale advocated a deeper study of that construction applied to homotopy categories [9]. These categories, indeed, form a large class of natural examples of categories with weak finite limits, in the sense that they do not arise as projective covers of finitely complete categories. A first step in this direction was made by Gran and Vitale in [13], where they provide a complete characterisation of those exact completions of categories with weak finite limits (henceforth ex/wlex completions) that produce a pretopos, and apply this result to show that the exact completion of the category of topological spaces and homotopy classes of maps is indeed a pretopos. However, the problem of determining whether it is also locally cartesian closed is explicitly left open.

The author has given a complete characterisation of locally cartesian closed ex/wlex completions in [10]. That characterisation is however not much suited to the study of the ex/wlex completion of a homotopy category $\mathrm{Ho}\,\mathbb{M}$, when instead a formulation in terms of the original Quillen model category $\mathbb{M}$ would be preferable. The present paper provides a condition ensuring the local cartesian closure of the ex/wlex completion $(\mathrm{Ho}\,\mathbb{M})_{\mathrm{ex}}$ for a large class of model categories. Somewhat surprisingly, this condition turns out to be what Carboni and Rosolini named weak local cartesian closure in [8], that is, simply existence of weak dependent products.

As we shall prove in the last section, the homotopy quotient of a weak dependent product in $\mathbb{M}$ is a *dependent full diagram* in $\mathrm{Ho}\,\mathbb{M}$. The latter is a generalisation to arbitrary categories with weak finite limits of a concept introduced in [11] to prove the local cartesian closure of the exact completion of a well-pointed category with finite products and weak equalisers. Anther precursor is the axiom F for a class of small maps in [5], which is proved to be stable under ex/reg completions. We shall comment on the (tight) relation between this axiom and dependent full diagrams in Remark 3.2. Indeed, both the universal property of dependent full diagrams and axiom F are inspired by Aczel's Fullness Axiom from the constructive set theory CZF [1, 2]. The Fullness Axiom is a collection principle asserting the existence of what Aczel calls *full sets*, that is, sets containing enough total relations (a.k.a. multi-valued functions). This axiom is strictly weaker than the Power Set Axiom (and it is regarded as a predicative principle) but strong enough, in particular, to entail the Exponentiation Axiom, which asserts that functions between two sets form a set.

We prove that also in the general case of an ex/wlex completion, existence of dependent full diagrams in $\mathbb{C}$ is enough to derive the local cartesian closure of $\mathbb{C}_{\mathrm{ex}}$. That this should be possible follows from the observation, due to Erik Palmgren, that arrows $V \to Y$ out of a weak product $Z \leftarrow V \to X$ may be understood as families indexed on Z of multi-valued functions from X to Y. A more robust formulation of this observation is in Remark 2.1. In addition, dependent full diagrams endow the internal logic with universal quantification and implication which, in turn, can be used to extract from multi-valued functions only the functional ones (cf. Lemma 2.6). At this point it is enough to construct a suitable equivalence relation to obtain an exponential in $\mathbb{C}_{\mathrm{ex}}$.

In order to prove that dependent full diagrams are homotopy quotients of weak dependent products, we exploit the concepts of *path category* and *weak homotopy* Π-*type*, recently introduced by van den Berg and Moerdijk in [6]. A path category is a slight strengthening of Brown's fibration category. In particular, the subcategory $\mathbb{M}_{\mathrm{f}}$ on the fibrant objects of a model category $\mathbb{M}$ is a path category as soon as all the objects of $\mathbb{M}$ are cofibrant. Weak homotopy Π-types in a path category $\mathbb{C}$ are what van den Berg and Moerdijk use to derive the local cartesian closure of $(\mathrm{Ho}\,\mathbb{C})_{\mathrm{ex}}$, the homotopy exact completion of $\mathbb{C}$. We show that if $\mathbb{M}$ is right proper, then weak homotopy Π-types arise as fibrant replacements of weak dependent products, so that a weak dependent product in $\mathbb{M}$ gives rise to a weak homotopy Π-type in $\mathbb{M}_\mathrm{f}$. Furthermore, in the same way as pullbacks along fibrations enjoy the additional universal property of homotopy pullbacks, also weak homotopy Π-types enjoy an additional universal property up to homotopy, which shows that the homotopy quotient maps weak homotopy Π-types to dependent full diagrams.

Under a type-theoretic reading, the results in the present paper provide a general motivation for the local cartesian closure of the category of setoids in Martin-Löf type theory. Indeed, the category of contexts of Martin-Löf type theory is a path category [4], see also [12], and ∏-types endow it with weak homotopy Π-types. More generally, we obtain a more elementary proof of the local cartesian closure of a homotopy exact completion. It should be noted that, under the reading of arrows out of a weak product as multi-valued functions, single-valued functions in a homotopy category $\mathrm{Ho}\,\mathbb{C}$ appear as "homotopy-irrelevant" arrows. Indeed, an arrow k out of a homotopy limit, say a homotopy pullback of $f\colon X \to Y$ and $g\colon Z \to Y$, induce an arrow in $(\mathrm{Ho}\,\mathbb{C})_{\mathrm{ex}}$ out of the actual pullback of f and g in if and only if values of k only depend on pairs (x, z) and not on the homotopy witnessing $f(x) \simeq g(z)$. The analogy with the role of homotopy-irrelevant fibrations in the argument for the local cartesian closure of $(\mathrm{Ho}\,\mathbb{C})_{\mathrm{ex}}$ from van den Berg and Moerdijk [6] may be worth further investigation. Indeed, in type-theoretic terminology, these are the proof-irrelevant setoid families whose importance has been stressed by Palmgren [16].

Furthermore, the observation that dependent full diagrams naturally arise as homotopy quotients of weak dependent products shows that existence of the former is not just a particular feature of the category of types in Martin-Löf type theory, the only example in [11]. On the contrary, it provides a large class of examples of categories with weak finite limits and dependent full diagrams. In particular, we obtain the local cartesian closure of the exact completion of the category of spaces and homotopy classes of maps, thus answering a question left open in [13].

The first half of the paper is devoted to the proof that existence of dependent full diagrams imply the local cartesian closure of the exact completion. In order to simplify the presentation, we split the argument in two steps. In Section 2, after a brief recap on ex/wlex completions, we define a non-indexed version of a full diagram in $\mathbb{C}$ and, assuming that $\mathbb{C}_{\mathrm{ex}}$ (or, equivalently, $\mathbb{C}$) has the needed structure for implication and universal quantification, we construct from it an exponential in $\mathbb{C}_{\mathrm{ex}}$. Section 3 contains the definition of the more general dependent full diagrams and the proof that their existence gives rise both to right adjoints to inverse images and to non-indexed full diagrams. Finally, Section 4 covers the case of homotopy categories.

Full diagrams

We briefly recall some basic facts about weak limits and the exact completion which are essential to our treatment. For additional background notions and notations we refer to [9, 8] and to section 1 of [10]. In this section and the next one $\mathbb{C}$ denotes a category with weak finite limits and $\mathbb{C}_{\mathrm{ex}}$ denotes its ex/wlex completion, that we shall refer to as the exact completion of $\mathbb{C}$. Regular epis are denoted with a triangle head, like in $A \twoheadrightarrow B$, while hook arrows $A \hookrightarrow B$ denote monos.

Recall that weak limits are defined as usual limits but without requiring uniqueness of a universal arrow. An arrow $f\colon V \to Y$ in $\mathbb{C}$ from a weak product $Z \leftarrow V \to X$ of Z and X is *determined by projections* [10] if it coequalises every pair of arrows jointly coequalised by the two weak product projections. A *weak exponential* [8, Definition 2.1] in $\mathbb{C}$ of two objects Y and X consists of an object W, a weak product $W \leftarrow V \to X$ and an arrow $V \to Y$ which is determined by projections, such that for any object W', weak product $W' \leftarrow V' \to X$ and arrow $V' \to Y$ determined by projections, there are (not necessarily unique) arrows $h\colon W' \to W$ and $k\colon V' \to V$ making the obvious diagram commute.

An object X in a category $\mathbb{E}$ is called *(regular) projective* if, for every regular epi $g\colon A \twoheadrightarrow B$ and arrow $f\colon X \to B$, there is a *lift* of f against g, *i.e.* an arrow $f'\colon X \to A$ such that $gf' = f$. A *projective cover* of an exact category $\mathbb{E}$ is a full subcategory $\mathbb{P}$ such that *(i)* every object in $\mathbb{P}$ is projective in $\mathbb{E}$, and *(ii)* every object in $\mathbb{E}$ is covered by an object in $\mathbb{P}$, *i.e.* for every A in $\mathbb{E}$ there are X in $\mathbb{P}$ and a regular epi $X \twoheadrightarrow A$. $\mathbb{E}$ has *enough projectives* if it has a projective cover. Whenever we are given a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$, we adopt the convention of using letters from P to Z for objects in $\mathbb{P}$. The reader should however keep in mind that $\mathbb{P}$ is not in general closed under limits that exist in $\mathbb{E}$.

Recall from [9] that, for a category $\mathbb{C}$ with weak finite limits, the exact completion $\mathbb{C}_{\text{ex}}$ can be described as the category whose objects are pseudo equivalence relations $R \rightrightarrows X$ in $\mathbb{C}$ and whose arrows from $R \rightrightarrows X$ to $S \rightrightarrows Y$ are equivalence classes of those arrows $X \to Y$ of $\mathbb{C}$ that map related elements to related elements, where $f, f' \colon X \to Y$ are equivalent if they are related in $S \rightrightarrows Y$. The full subcategory of $\mathbb{C}_{\text{ex}}$ on the free pseudo equivalence relations, *i.e.* those with equal legs, is a projective cover of $\mathbb{C}_{\text{ex}}$. It is equivalent to $\mathbb{C}$ via the embedding of $\mathbb{C}$ into $\mathbb{C}_{\text{ex}}$, that maps an object X to the pair of identities $\mathrm{id}_X, \mathrm{id}_X \colon X \rightrightarrows X$. Conversely, a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$ has weak finite limits and its exact competion $\mathbb{P}_{\text{ex}}$ is equivalent to $\mathbb{E}$ [9, Thm. 16]. A weak limit in $\mathbb{P}$ is obtained covering with a projective the corresponding limit in $\mathbb{E}$. More generally, a cone in $\mathbb{P}$ over a diagram $\mathcal{D}$ in $\mathbb{P}$ is a weak limit if and only if the unique arrow from the cone into the limit in $\mathbb{E}$ of $\mathcal{D}$ is a regular epi. see [10, Lemma 1.7].

Given a projective cover $\mathbb{P}$ of $\mathbb{E}$ exact and a weak product $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$ in $\mathbb{P}$, an arrow $f \colon V \to Y$ is determined by projections in $\mathbb{P}$ if and only if it factors in $\mathbb{E}$ through the regular epi $\langle p_1, p_2\rangle \colon V \twoheadrightarrow Z \times X$. It follows that $\mathbb{P}$ has weak exponentials if $\mathbb{E}$ is cartesian closed: given $Y, X \in \mathbb{P}$, one just need cover Y^X with a projective W and to do the same with $W \times X$. However, as argued in the last section of [10], the universal property of weak exponentials does not seem to be suited to prove the cartesian closure of $\mathbb{C}_{\text{ex}}$ when $\mathbb{C}$ only has weak finite limits. A complete characterisation in terms of what we called *extensional simple products* is presented in [10], but here we look at yet another (weakly) universal property.

Remark 2.1. Let $\mathbb{E}$ be exact with a projective cover $\mathbb{P}$ and let Z, X, Y be three objects in $\mathbb{P}$. There is an isomorphism between the poset of subobjects $\mathrm{Sub}_{\mathbb{E}}(Z \times X \times Y)$ and the poset reflection $(\mathbb{P}/(Z, X, Y))_{\text{po}}$ of the category of spans

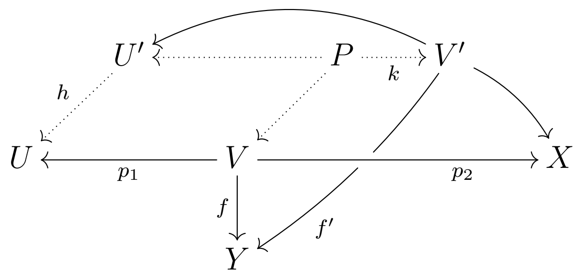
$$\begin{array}{ccccc} & & V & & \\ & \swarrow & \downarrow & \searrow & \\ Z & & X & & Y \end{array}$$

in $\mathbb{P}$ over Z, X and Y [9, Lemma 35]. This isomorphism restricts between those subobjects $R \hookrightarrow Z \times X \times Y$ such that $R \to Z \times X$ is regular epic, and those (equivalence classes of) spans such that $Z \leftarrow V \to X$ is a weak product. This restricts further between those subobjects such that $R \to Z \times X$ is iso, *i.e.* essentially graphs of arrows $Z \times X \to Y$, and those spans such that $Z \leftarrow V \to X$ is a weak product and $V \to Y$ is determined by projections.

The previous remark allows us to understand arrows $f \colon V \to Y$ in a category $\mathbb{C}$ with weak finite limits, where $Z \leftarrow V \to X$ is a weak product, as families, indexed by Z, of total relations (*i.e.* multi-valued functions) from X to Y. Such a total relation is functional (*i.e.* a single-valued function) precisely when f is determined by projections. This reading suggests that, in order to have a suitable universal property with respect to arbitrary arrows $V \to Y$ out of a weak product, we should look for some property of closure with respect to (families of) total relations from X to Y. A promising notion, that indeed proves to be useful, is that of a *full set* from the constructive set theory CZF.

A set f is full for two sets a and b if it consists of multi-valued functions from a to b and, for every multi-valued function r from a to b, there is $s \in f$ such that $s \subseteq r$. The *Fullness Axiom* states that, for any two sets a and b, there is a set which is full for a and b. This axiom was introduced in the context of Constructive Zermelo-Fraenkel set theory (CZF) by Peter Aczel in [1] in order to provide a simpler formulation of the axiom schema of Subset Collection. This axiom implies in particular the so-called Exponentiation Axiom, that the class of functions between two sets is a set. The next definition is inspired by the notion of full set.

Definition 2.2. *Let X and Y be two objects in a category $\mathbb{C}$ with weak finite limits. A* full diagram from X to Y *consists of a weak product $U \xleftarrow{p_1} V \xrightarrow{p_2} X$ and an arrow $f \colon V \to Y$ such that, for every object U', weak product $U' \leftarrow V' \to X$ and arrow $f \colon V' \to Y$, there are an arrow $h \colon U' \to U$, a weak pullback $U' \leftarrow P \to V$ of $U' \to U \leftarrow V$ and an arrow $k \colon P \to V'$ such that the diagram below commutes.*

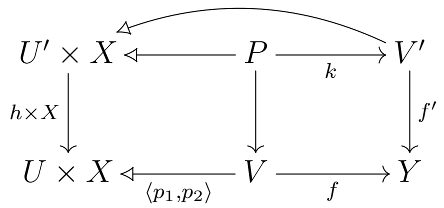


(1)

Remark 2.3.

1. The notion of full diagram is independent of the specific weak product in the following sense. If the pair $U \leftarrow V \to X$, $V \to Y$ is a full diagram from X to Y, then any other weak product $U \leftarrow W \to X$ together with the composite $W \to V \to Y$ is a full diagram from X to Y.

2. In the case of a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$, diagram (1) in Definition 2.2 can be written in $\mathbb{E}$ as



where the left-hand square is covering, *i.e.* the induced arrow from P to the pullback of $\langle p_1, p_2 \rangle$ and $h \times X$ is a regular epi.

Lemma 2.4. *Suppose that $\mathbb{C}$ has binary products. Then weak exponentials are full diagrams and any full diagram from X to Y gives rise to a weak exponential of Y and X.*

Proof. Since $\mathbb{C}$ has binary products, every weak product retracts onto the product of the same objects. Also, an arrow from a weak product is determined by projections if and only if it factors through the retraction onto the product. It follows that we may assume that the domain of a weak evaluation is a product, rather than just a weak product.

Let then W be a weak exponential of Y and X with weak evaluation $e\colon W \times X \to Y$. Given $f\colon V \to Y$ from a weak product $U' \leftarrow V \to X$

we can take $U \times X$ as P in (1): the arrow $k \colon U \times X \to V$ is a section of the retraction $V \to U \times X$ and the arrow $h \colon U \to W$ is obtained by the universal property of the weak exponential applied to the composite $fk \colon U \times X \to Y$.

For the converse, let $U \leftarrow V \to X$, $f \colon V \to Y$ form a full diagram and let $s \colon U \times X \hookrightarrow V$ be a section of the retraction $V \to U \times X$. We shall prove that $fs \colon U \times X \to Y$ exhibits U as a weak exponential of Y and X. Given $f' \colon U' \times X \to Y$, the universal property of full diagrams yields an arrow $h \colon U' \to U$ and a commutative diagram

$$\begin{array}{ccccc} U' \times X & \longleftarrow & P & \longrightarrow & U' \times X \\ {\scriptstyle h\times X}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} \\ U \times X & \longleftarrow & V & \xrightarrow{\ f\ } & Y \end{array}$$

where the left-hand square is a weak pullback. It follows that $P \to U' \times X$ has a section s' such that the diagram

$$\begin{array}{ccc} U' \times X & \overset{s'}{\hookrightarrow} & P \\ {\scriptstyle h\times X}\downarrow & & \downarrow \\ U \times X & \overset{s}{\hookrightarrow} & V \end{array}$$

commutes. The equation $fs(h \times X) = f'$ then follows immediately. □

In particular, a category with finite limits has full diagrams if and only if it has weak exponentials. Lemma 2.6 shows that, whenever the internal logic of $\mathbb{C}_{\text{ex}}$ (equivalently, of $\mathbb{C}$) supports implication and universal quantification, the left-to-right implication also holds when $\mathbb{C}$ only has weak finite limits.

First, recall that descent in exact categories allows us to prove the following, where $\mathbb{X}_{\text{po}}$ denotes the poset reflection of the category $\mathbb{X}$. See also [10, Remark 1.9].

Lemma 2.5. $\mathbb{C}_{\text{ex}}$ *has right adjoints to inverse images if and only if* $\mathbb{C}$ *has right adjoints to weak pullback functors, i.e. the functors* $(\mathbb{C}/X)_{\text{po}} \to (\mathbb{C}/Y)_{\text{po}}$ *induced by weak pullback along arrows* $f \colon Y \to X$.

Proof. One direction follows by the existence of natural isomorphisms $\mathrm{Sub}_{\mathbb{C}_{\mathrm{ex}}}(\Gamma_{\mathrm{ex}} X) \cong (\mathbb{C}/X)_{\mathrm{po}}$. For the other direction apply Theorem 2 in Section 3.7 of [3]. □

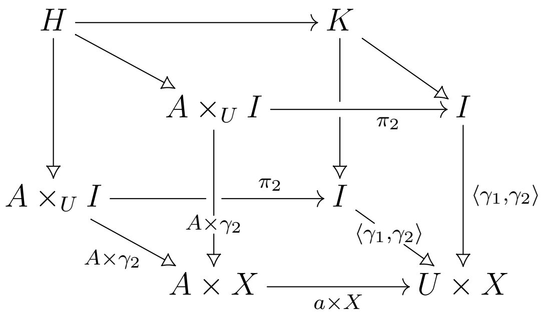
We now need a lemma which, for convenience, we formulate using an exact category $\mathbb{E}$ with a fixed projective cover $\mathbb{P}$.

Lemma 2.6. *Let $\mathbb{E}$ be an exact category with a projective cover $\mathbb{P}$ and suppose that $\mathbb{E}$ has right adjoints to inverse images along any arrow. If $\mathbb{P}$ has full diagrams, then for every X in $\mathbb{P}$ and B in $\mathbb{E}$ there are an object W in $\mathbb{P}$ and an arrow $W \times X \to B$ in $\mathbb{E}$ which are weakly terminal with respect to objects Z in $\mathbb{P}$ and arrows $Z \times X \to B$ in $\mathbb{E}$.*

Proof. Let $b\colon Y \twoheadrightarrow B$ be a cover of B with Y in $\mathbb{P}$, and take a full diagram $U \xleftarrow{p_1} V \xrightarrow{p_2} X$, $f\colon V \to Y$. The idea is to extract from U (codes of) functional relations. Let $\gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle\colon I \hookrightarrow U \times X \times B$ be the image factorisation of $\langle p_1, p_2, bf \rangle\colon V \to U \times X \times B$ and denote with $\phi\colon F \hookrightarrow U$ the subobject defined by the formula in context

$$u : U \mid (\forall x : X)(\forall y, y' : B)\, \gamma(u, x, y) \wedge \gamma(u, x, y') \Rightarrow y = y'.$$

In other words, given an arrow $a\colon A \to U$ in $\mathbb{E}$, consider the diagram

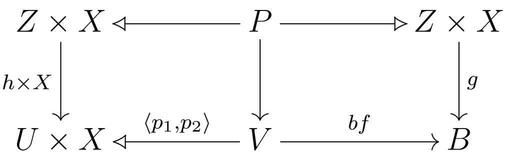


where all the squares are pullback. Then

$$a \le \phi \;\Leftrightarrow\; A \times_U I \xrightarrow{\pi_2} I \xrightarrow{\gamma_3} B \text{ coequalises } H \rightrightarrows A \times_U I. \qquad (2)$$

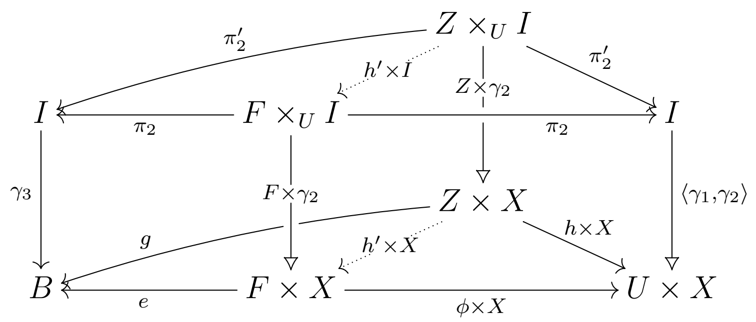
The existence of an arrow $e\colon F \times X \to B$ follows from (2) taking $a = \phi$. We shall show that e satisfies the required universal property. It then follows easily that $W \times X \to B$ satisfies it as well for any cover $W \twoheadrightarrow F$. In particular, $\mathbb{P}$ will have weak exponentials.

Given $g \colon Z \times X \to B$ with $Z \in \mathbb{P}$, let $g' \colon V' \to Y$ be a cover of g, *i.e.* be such that the right-hand square in diagram (3) is covering. By the universal property of a full diagram, we have, in particular, an arrow $h \colon Z \to U$ and a commutative diagram



(3)

where the left-hand square is covering. It follows that the induced arrow $q \colon P \to Z \times_U I$ is a regular epi. Consider now the solid arrows in diagram (4) below. The two right-hand squares are pullback and the front left-hand square commutes by definition of e.



(4)

In order to obtain an arrow $h' \colon Z \to F$ such that $e(h' \times X) = g$, it is enough to show that the square with side g commutes. Indeed, in this case, there is $h' \colon Z \to F$ such that $\phi h' = h$ by (2). It follows that the upper triangle(s) and the square with dotted sides in diagram (4) commute. Since $Z \times \gamma_2$ is (regular) epic, the lower left-hand triangle commutes as well.

To see that the square with side g in diagram (4) commutes note that, precomposing its two sides with $q \colon P \twoheadrightarrow Z \times_U I$ yields the right-hand square in diagram (3). The claim follows from the fact that q is (regular) epic. □

Theorem 2.7. *Suppose that $\mathbb{C}$ has right adjoints to weak pullback functors. If $\mathbb{C}$ has full diagrams, then $\mathbb{C}_{\mathrm{ex}}$ is cartesian closed.*

Proof. In the terminology of [10], Lemmas 2.5 and 2.6 prove that if a category has weak finite limits, right adjoints to weak pullback functors and full diagrams, then it has extensional exponentials too. The statement follows from Lemma 2.13 in [10]. □

Below we collect together the results in this section.

Corollary 2.8. *Suppose that $\mathbb{C}$ has right adjoints to weak pullback functors, and consider the following.*

1. *$\mathbb{C}$ has full diagrams.*
2. *$\mathbb{C}_{\mathrm{ex}}$ is cartesian closed.*
3. *$\mathbb{C}$ has weak exponentials.*

We have 1 ⇒ 2 ⇒ 3. If $\mathbb{C}$ has binary products, then 3 ⇒ 1.

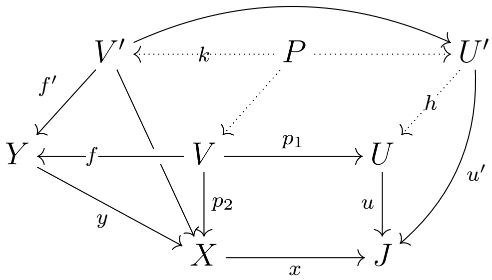
Dependent full diagrams

Recall that $\mathbb{C}$ denotes a category with weak finite limits and $\mathbb{C}_{\mathrm{ex}}$ its exact completion. In this section we define an indexed version of full diagrams, whose existence will endow the internal logic of $\mathbb{C}$ (hence of $\mathbb{C}_{\mathrm{ex}}$) with implication and universal quantification.

Definition 3.1. *Let $y\colon Y \to X$ and $x\colon X \to J$ be two arrows in a category with weak finite limits. A* dependent full diagram over *x, y is a commutative diagram*

$$\begin{array}{ccccc} Y & \xleftarrow{\;f\;} & V & \xrightarrow{\;p_1\;} & U \\ & \searrow^{y} & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle u} \\ & & X & \xrightarrow[\;x\;]{} & J \end{array} \tag{5}$$

such that the square is a weak pullback and, for every such diagram u', f' over y, x as below, there are an arrow $h\colon U' \to U$, a weak pullback $V \leftarrow P \to U'$ of $V \xrightarrow{p_1} U \xleftarrow{h} U'$ and an arrow $k\colon P \to V'$ making the diagram below commute.



The same observations as in Remark 2.3 apply, mutatis mutandis, to dependent full diagrams. Moreover, it is not difficult to see that dependent full diagrams generalise full families of pseudo-relations from [11], in the sense that the two notions coincide in well-pointed categories with finite products and weak equalisers.

Remark 3.2. Another category-theoretic version of Aczel's Fullness Axiom was introduced in [5] in the context of Algebraic Set Theory [15]. There one deals with classes of arrows, called *small maps*, in categories which are at least regular. According to the properties satisfied by the small maps, various set theories may be interpreted in this structure. In particular, in order to interpret the Fullness Axiom in a suitable category $\mathbb{E}$ equipped with a class of small maps $\mathcal{S}$, van den Berg and Moerdijk introduce a condition $\mathrm{F}(\mathcal{S})$, called **(F)** in [5, Section 3.7]. By taking the class $\mathcal{S}$ to consist of all arrows of $\mathbb{E}$, condition $\mathrm{F} = \mathrm{F}(\mathrm{Ar}\,\mathbb{E})$ makes sense for any regular category. If a regular category $\mathbb{E}$ has a projective cover $\mathbb{P}$, then a straightforward but lengthy computation shows that F holds in $\mathbb{E}$ if and only if $\mathbb{P}$ has dependent full diagrams. Moreover, Proposition 6.2.5 in [5] entails that condition F is stable under ex/reg completion. As a reg/wlex completion is in particular a regular category with enough projectives [9], it follows immediately that $\mathbb{C}_{\mathrm{ex}} \equiv (\mathbb{C}_{\mathrm{reg/wlex}})_{\mathrm{ex/reg}}$ satisfies F whenever $\mathbb{C}$ has dependent full diagrams.

Recall that a *weak dependent product* of two composable arrows $Y \xrightarrow{y} X \xrightarrow{x} J$ in a category with weak finite limits is a commutative diagram as (5) such that the square is a weak pullback, $f\colon V \to Y$ is determined by projections and, for every such diagram $u'\colon U' \to J, f'\colon V' \to Y$ over y and x, there are arrow $U' \to U$ and $V' \to V$ making the obvious diagram commute.

As for the non-indexed case, as soon as $\mathbb{C}$ has pullbacks, we may regard weak dependent products in $\mathbb{C}$ as those dependent full diagrams whose weak pullback is a pullback.

Lemma 3.3. *If $\mathbb{C}$ has pullbacks, weak dependent products are dependent full diagrams and any dependent full diagram over y, x gives rise to a weak dependent product of y, x.*

Proof. The lemma is proven similarly to Lemma 2.4. □

Lemma 3.4. *If $\mathbb{C}$ has dependent full diagrams, then it has full diagrams.*

Proof. A full diagram for two objects X and Y can be obtained as a dependent full diagram over $V \to U \to T$, where T is weakly terminal, U is a weak product of X and T, V is a weak product of U and Y and the arrows are the obvious projections. □

Lemma 3.5. *Let $\mathbb{C}$ be a category with weak finite limits. $\mathbb{C}$ has dependent full diagrams if and only if every slice of $\mathbb{C}$ has them.*

Proof. It follows from the fact that the forgetful functor $\mathbb{C}/J \to \mathbb{C}$ preserves and reflects weak pullbacks and dependent full diagrams. □

Lemma 3.6. *If $\mathbb{C}$ has dependent full diagrams, then $\mathbb{C}_{\mathrm{ex}}$ has right adjoints to inverse images.*

Proof. Using a choice of dependent full diagrams in $\mathbb{C}$ it is possible to define, for every $f\colon Y \to X$ in $\mathbb{C}$, functors $\forall^{\mathrm{w}}_f\colon (\mathbb{C}/Y)_{\mathrm{po}} \to (\mathbb{C}/X)_{\mathrm{po}}$. These functors are right adjoint to weak pullback functors by the universal property of dependent full diagrams. The statement follows from Lemma 2.5. □

Theorem 3.7. *If $\mathbb{C}$ has dependent full diagrams, then $\mathbb{C}_{\mathrm{ex}}$ is locally cartesian closed.*

Proof. It only remains to put together the previous results. Lemma 3.6 ensures that $\mathbb{C}_{\mathrm{ex}}$ has right adjoints to inverse images, whereas Lemmas 3.4 and 3.5 imply that $\mathbb{C}/X$ has full diagrams for every X in $\mathbb{C}$. Hence Theorem 2.7 yields the cartesian closure of $\mathbb{C}_{\mathrm{ex}}/(\Gamma_{\mathrm{ex}}X)$. The general statement follows now "descending" along a cover $\Gamma_{\mathrm{ex}}X \twoheadrightarrow A$ as in the proof of Theorem 3.6 in [10]. □

We again collect together the results of this section.

Corollary 3.8. *Consider the following.*

1. $\mathbb{C}$ *has dependent full diagrams.*
2. $\mathbb{C}_{\mathrm{ex}}$ *is locally cartesian closed.*
3. $\mathbb{C}$ *has weak dependent products.*

We have 1 $\Rightarrow$ *2* $\Rightarrow$ *3. If* $\mathbb{C}$ *has pullbacks, then 3* $\Rightarrow$ *1.*

Full diagrams in homotopy categories

In this section we show that, under mild assumptions on a model category $\mathbb{M}$, the homotopy category $\mathrm{Ho}\,\mathbb{M}$ has dependent full diagrams if $\mathbb{M}$ has weak dependent products. Using well-known results, this implies that the exact completion of the homotopy categories on spaces and CW-complexes yields locally cartesian closed pretoposes.

$\xrightarrow{\sim}$ For basic notions on model categories and homotopical algebra we refer to [14]. Fibrations, cofibrations and weak equivalences are denoted as $\twoheadrightarrow$, $\rightarrowtail$ and $\xrightarrow{\sim}$, respectively. A path object factorisation for an object A is denoted as $A \overset{\sim}{\rightarrowtail} PA \twoheadrightarrow A \times A$, and a fibrewise path object factorisation for a fibration $p\colon A \twoheadrightarrow B$ as $A \overset{\sim}{\rightarrowtail} P_pA \twoheadrightarrow A \times_B A$. Since we shall not be concerned here with cylinder objects, we say that two arrows $f, g\colon C \to A$ in $\mathbb{M}$ are homotopic, written $f \simeq g$, if they are right homotopic (*i.e.* homotopic with respect to the path object PA). Similarly, we shall write $f \simeq_p g$ to mean that they are fibrewise right homotopic over the fibration p. Note that, since every fibration $p\colon A \twoheadrightarrow B$ is fibrant in the model category structure on $\mathbb{M}/B$ induced by that one on $\mathbb{M}$, fibrewise (right) homotopy is an equivalence relation on arrows $f, f'\colon X \to A$ such that $pf = pf'$. If every object in $\mathbb{M}$ is cofibrant, then it is also a congruence.

When every object in a model category $\mathbb{M}$ is cofibrant, the homotopy category $\mathrm{Ho}\,\mathbb{M}$ is equivalent to the category obtained quotienting the full subcategory $\mathbb{M}_f$ of $\mathbb{M}$ on fibrant objects by the homotopy relation [14]. Moreover, $\mathbb{M}_{\mathrm{f}}$ is a category of fibrant objects in the sense of Brown [7] where, in addition, every acyclic fibration has a section and where weak equivalences and homotopy equivalences coincide. A category of fibrant objects satisfying these additional properties is called a *path category* by van den Berg and Moerdijk [6]. More explicitly, a path category may be axiomatised as follows (cf. [6]). It has a terminal object and two classes of distinguished arrows closed under isomorphism and composition, called weak equivalences and fibrations, such that: *(i)* weak equivalences are closed under 2-out-of-6, *(ii)* terminal arrows are fibrations, *(iii)* pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback, and *(iv)* every acyclic fibration has a section.

Definition 4.1 ([6], Definition 5.2)**.** *Let $g\colon B \twoheadrightarrow A$ and $f\colon A \twoheadrightarrow I$ be two fibrations in a path category $\mathbb{C}$. A commuting diagram*

$$\begin{array}{ccccc} B & \xleftarrow{\;e\;} & U \times_I A & \twoheadrightarrow & U \\ & \searrow^{g} & \downarrow & & \downarrow u \\ & & A & \xrightarrow[f]{} & I \end{array}$$

is a homotopy weak dependent product of f and g *if for every such commutative diagram $u'\colon U' \twoheadrightarrow I$, $e'\colon U' \times_I A \to B$, there is $k\colon u' \to u$ over I such that $e(k \times A) \simeq_g e'$.*

Homotopy weak dependent products are called weak homotopy Π-types in [6]. As observed in [6], the (weak) universal property also holds when the arrow u' is not a fibration.

When the path category is $\mathbb{M}_{\mathrm{f}}$, a weak homotopy dependent product arises as fibrant replacement of a weak dependent product in $\mathbb{M}$. To prove this fact we need the following result, which is a reformulation for a model category of Theorem 2.38 from [6].

Theorem 4.2. *Let $\mathbb{M}$ be a model category and let A and B be cofibrant objects. Then every commutative square*

$$\begin{array}{ccc} A & \xrightarrow{\;k\;} & C \\ f \downarrow \wr & & \downarrow g \\ B & \xrightarrow[l]{} & D \end{array} \tag{6}$$

has a homotopy diagonal filler, i.e. an arrow $d\colon B \to C$ *such that* $gd = l$ *and* $df \simeq_g k$*. Moreover, such a filler is unique up to fibrewise homotopy over* g*.*

Proof. We shall first show that every commuting square (6) has a lower filler, *i.e.* an arrow d such that $gd = l$. This fact, in turn, allows us to obtain a homotopy witnessing the fact that the previously constructed lower filler is in fact a homotopy diagonal filler, and another homotopy witnessing its uniqueness.

Consider a factorisation of $\langle f, k\rangle\colon A \to B \times_D C$ into an acyclic cofibration $c\colon A \overset{\sim}{\rightarrowtail} E$ followed by a fibration $p\colon E \twoheadrightarrow B \times_D C$. In particular, E is cofibrant. From 2-out-of-3 we obtain that $\pi_1 p\colon E \twoheadrightarrow B$ is an acyclic fibration and, since B is cofibrant, it has a section $s\colon B \overset{\sim}{\to} E$. But then $d \coloneqq \pi_2 ps\colon B \to C$ is the required lower filler, as $gd = l\pi_1 ps = l$.

Therefore every commuting square (6) has a lower filler. In particular, we obtain a homotopy $s\pi_1 p \simeq_{(\pi_1 p)} \mathrm{id}_X$ as a lower filler in

$$\begin{array}{ccc} B & \longrightarrow & P_{\pi_1 p}E \\ {\scriptstyle s}\downarrow{\scriptstyle \wr} & & \downarrow \\ E & \xrightarrow[\langle s\pi_1 p, \mathrm{id}_X\rangle]{} & E \times_B E \end{array}$$

where the top horizontal arrow is s followed by reflexivity of $P_{\pi_1 p}E$. Since E is cofibrant, it is $df = \pi_2 p(s\pi_1 p)c \simeq_g \pi_2 pc = k$.

Finally, given another homotopy diagonal filler d', the homotopy witnessing $d \simeq_g d'$ is obtained as a lower filler in

$$\begin{array}{ccc} A & \longrightarrow & P_g C \\ {\scriptstyle f}\downarrow{\scriptstyle \wr} & & \downarrow \\ B & \xrightarrow[\langle d, d'\rangle]{} & C \times_D C \end{array}$$

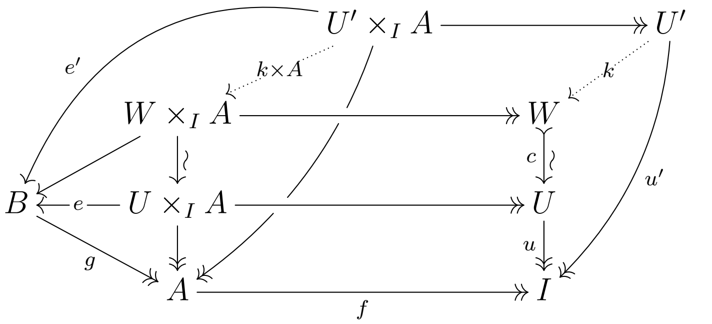
where the top horizontal arrow is the concatenation $df \simeq_g k \simeq_g d'f$. □

Remark 4.3. The argument used in the previous proof can be adapted to work in a path category, so to provide an alternative proof of Theorem 2.38 in [6]. To this aim it is enough to observe that, in a path category, the existence of lower fillers for commutative squares as in (6) is enough to derive that the homotopy relation is a congruence, as in the proof of Theorem 2.14 in [6].

Corollary 4.4. *Let* $\mathbb{M}$ *be a right proper model category where every object is cofibrant. If* $\mathbb{M}$ *has weak dependent products, then* $\mathbb{M}_\mathrm{f}$ *has homotopy weak dependent products for every pair of composable fibrations.*

Proof. Let $f\colon A \twoheadrightarrow I$ and $g\colon B \twoheadrightarrow A$ be two fibrations in $\mathbb{M}_\mathrm{f}$ and let $w\colon W \to I$, $d\colon W \times_I A \to B$ be a weak dependent product of them. Factor w as an acyclic cofibration $c\colon W \overset{\sim}{\rightarrowtail} U$ followed by a fibration $u\colon U \twoheadrightarrow I$. Since $\mathbb{M}$ is right proper, $W \times_I A \to U \times_I A$ is also a weak equivalence, hence we obtain $e\colon U \times_I A \to B$ as homotopy diagonal filler.

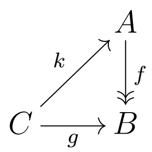
The required universal property is depicted in the diagram below



where $e(ck \times A) \simeq_g e'$ since e is just a homotopy diagonal filler. $\square$

Homotopy weak dependent products also enjoy another universal property with respect to certain homotopy diagrams. This is proved below in Lemma 4.9 and it is a consequence of the following result.

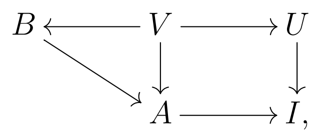
Proposition 4.5 ([6], Proposition 2.31)**.** *Let* $\mathbb{C}$ *be a path category and let*



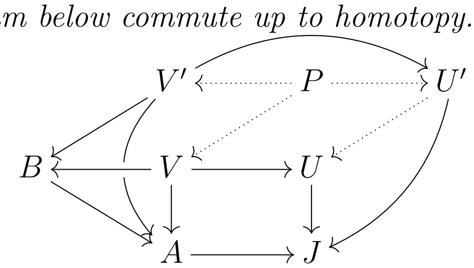
be a diagram that commutes up to homotopy. Then there is $k'\colon C \to A$ *such that* $k' \simeq k$ *and* $fk' = g$*.*

Remark 4.6. Proposition 4.5 has the important consequence that pullbacks in $\mathbb{M}_\mathrm{f}$ along fibrations are homotopy pullbacks and so are mapped to weak pullbacks in $\mathrm{Ho}\,\mathbb{M}$.

Definition 4.7. *Let $f\colon A \to I$ and $g\colon B \to A$ be two arrows in a path category $\mathbb{C}$. A diagram*



that commutes up to homotopy and where the square is a homotopy pullback, is homotopy full over f, g *if, for every such diagram over $B \to A \to I$ commuting up to homotopy, there are an arrow $U' \to U$, a homotopy pullback $V \leftarrow P \to U'$ of $V \to U \leftarrow U'$ and an arrow $P \to V'$ making the diagram below commute up to homotopy.*

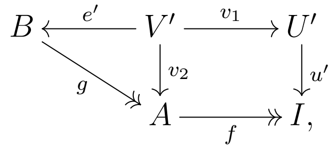


Remark 4.8. Let $\mathbb{C}$ be a path category. Since $\mathrm{Ho}\,\mathbb{C}$ is the quotient of $\mathbb{C}$ by the homotopy relation (cf. Theorem 2.16 in [6]), the image in $\mathrm{Ho}\,\mathbb{C}$ of a homotopy full diagram over f, g is a full diagram over $[f], [g]$.

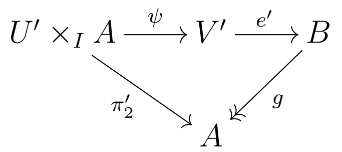
Lemma 4.9. *Let $\mathbb{C}$ be a path category and let $f\colon A \twoheadrightarrow I$ and $g\colon B \twoheadrightarrow A$ be two fibrations. A homotopy weak dependent product of f and g is a homotopy full diagram over f, g.*

Proof. Let $u\colon U \twoheadrightarrow I$, $e\colon U \times_I A \to B$ be a homotopy weak dependent product of f and g. Remark 4.6 implies that $U \times_I A$ is a homotopy pullback.

Let now



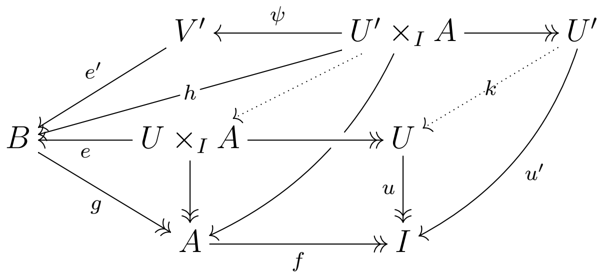
be commutative up to homotopy and such that the square is a homotopy pullback. Hence there is an arrow $\psi\colon U' \times_I A \to V'$ such that $v_1\psi \simeq \pi_1'$ and $v_2\psi \simeq \pi_2'$. In particular, the diagram below commutes up to homotopy



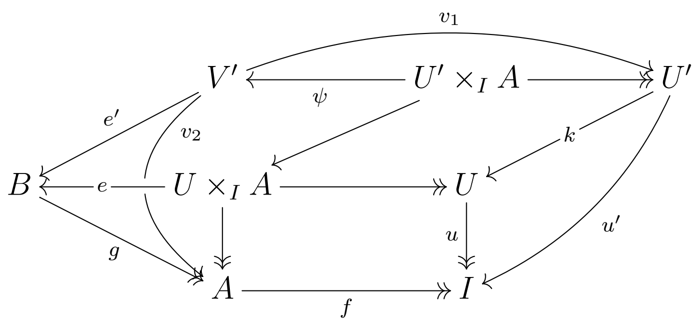
(7)

and Proposition 4.5 implies that there is $h\colon U' \times_I A \to B$ that makes the above triangle commute and which is homotopic to $e'\psi$.

The universal property of the homotopy weak dependent product then yields an arrow $k\colon U' \to U$ such that everything in the diagram below commutes strictly except for the two top-left triangles with common edge h, which only commute up to homotopy.



Hence, as required, the square with two dotted sides above is a homotopy pullback and the diagram below commutes up to homotopy.



□

Theorem 4.10. *Let* $\mathbb{M}$ *be a right proper model category where every object is cofibrant. If* $\mathbb{M}$ *has weak dependent products, then* $\mathrm{Ho}\,\mathbb{M}$ *has dependent full diagrams and, in turn,* $(\mathrm{Ho}\,\mathbb{M})_{\mathrm{ex}}$ *is locally cartesian closed.*

Proof. Lemma 4.9 and Remark 4.8 yield a full diagram over $[f], [g]$ whenever f and g are both fibrations. Since arrows in $\mathbb{M}$ factor as weak equivalences and fibrations and the former are isomorphisms in $\mathrm{Ho}\,\mathbb{M}$, this is enough to conclude that $\mathrm{Ho}\,\mathbb{M}$ has dependent full diagrams. The last statement is an application of Theorem 3.7. □

As an application of Theorem 4.10, consider the two standard model structures on the category of topological spaces by Quillen [17] and by Strøm [18], which we denote by Top_Q and Top_S, respectively. The latter is right proper and every space is cofibrant. Furthermore, Carboni and Rosolini showed that Top has weak dependent products [8]. Therefore $(\mathrm{Ho}\,\mathsf{Top}_S)_{\mathrm{ex}}$ not only is a pretopos, as proved in [13], but it is also locally cartesian closed. This answers a question left open by Gran and Vitale in [13]. In addition, although in Top_Q the cofibrant objects are just the CW-complexes, Top_Q is Quillen equivalent to simplicial sets with the Quillen model structure. This latter category does satisfy the hypothesis of our theorem, therefore $(\mathrm{Ho}\,\mathsf{Top}_Q)_{\mathrm{ex}}$ is locally cartesian closed too.

Acknowledgements

I am in debt with the late Erik Palmgren for sharing with me his idea of using the Fullness Axiom to obtain local cartesian closure. The results in this paper were presented at the 5th Workshop on Categorical Methods in Non-Abelian Algebra, Louvain-la-Neuve, June 1-3 2017, and at the International Category Theory Conference in Vancouver, July 16-22 2017. I gratefully thank the organisers of both events for giving me the opportunity to speak. This paper was partly written while I was visiting the Hausdorff Research Institute for Mathematics in Bonn during May 2018 in occasion of the trimester program Types, Sets and Constructions. I thank the Institute for providing an excellent working environment. Support from the Royal Swedish Academy of Sciences is also acknowledged. Finally, I would like to express my gratitude to the anonymous referee for useful comments on a previous draft of the paper.

References

[1] P. Aczel. The type theoretic interpretation of constructive set theory. In A. MacIntyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium '77*, volume 96 of *Studies in Logic and the Foundations of Mathematics*, pages 55–66. North-Holland, Amsterdam, 1978.

[2] P. Aczel and M. Rathjen. Notes on Constructive Set Theory. Technical Report 40, Mittag-Leffler Institute, The Swedish Royal Academy of Sciences, 2001.

[3] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer-Verlag, New York, 1985.

[4] B. van den Berg. Path categories and propositional identity types. *ACM Trans. Comput. Logic*, 19(2):15:1–15:32, 2018.

[5] B. van den Berg and I. Moerdijk. Aspects of predicative algebraic set theory I: Exact completion. *Ann. Pure Appl. Log.*, 156(1):123–159, 2008.

[6] B. van den Berg and I. Moerdijk. Exact completion of path categories and algebraic set theory. Part I: Exact completion of path categories. *J. Pure Appl. Algebra*, 222(10):3137–3181, 2018.

[7] K.S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973.

[8] A. Carboni and G. Rosolini. Locally cartesian closed exact completions. *J. Pure Appl. Algebra*, 154(1-3):103–116, 2000.

[9] A. Carboni and E.M. Vitale. Regular and exact completions. *J. Pure Appl. Algebra*, 125(1-3):79–116, 1998.

[10] J. Emmenegger. On the local cartesian closure of exact completions. To appear in *J. Pure Appl. Algebra.* Preprint available at arXiv:1804.08585.

[11] J. Emmenegger and E. Palmgren. Exact completion and constructive theories of sets. To appear in *J. Symb. Log.* Preprint available at arXiv:1710.10685.

[12] N. Gambino and R. Garner. The identity type weak factorisation system. *Theoret. Comput. Sci.*, 409(1):94–109, 2008.

[13] M. Gran and E. M. Vitale. On the exact completion of the homotopy category. *Cah. Topol. Géom. Différ. Catég.*, 39(4):287–297, 1998.

[14] M. Hovey. *Model categories.* Amer. Math. Soc., Providence, R.I., 1999.

[15] A. Joyal and I. Moerdijk. *Algebraic Set Theory.* London Mathematical Society Lecture Note Series. Cambridge University Press, 1995.

[16] E. Palmgren. Proof-relevance of families of setoids and identity in type theory. *Arch. Math. Logic*, 51(1):35–47, 2012.

[17] D.G. Quillen. Homotopical algebra. Lecture Notes in Mathematics. 43. Springer-Verlag, Berlin-Heidelberg-New York. VI, 1967.

[18] Arne Strøm. The homotopy category is a homotopy category. *Arch. Math.*, 23:435–441, 1972.

Jacopo Emmenegger
School of Computer Science
University of Birmingham
Birmingham B15 2TT, UK
op.emmen@gmail.com