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# ORDER CONVERGENCE AND CONVERGENCE IN THE INTERVAL TOPOLOGY IN THE PRESENCE OF COMPACTOIDNESS

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**Résumé.** Nous utilisons le concept de filtre compactoid pour obtenir une généralisation avec une preuve simplifiée d'un résultat de van der Zypen qui établissait que si la topologie des intervaux est compacte, alors elle est plus grossière que la topologie de l'ordre. Notre version est localisée et s'applique non seulement aux topologies mais aussi aux convergences.

**Abstract.** Using the concept of compactoid filter, we obtain a generalized version with a simplified proof of a result of van der Zypen to the effect that whenever the interval topology is compact, it is coarser than the order topology. The present version is localized and applies not only to topologies but also to convergence structures.

**Keywords.** order topology; order convergence; interval topology; convergence space; pseudotopology; compactoid filter.

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Dominic van der Zypen has shown [14] that if a topology on a poset is finer than the interval topology and is compact, then it is coarser than the order topology. It is the aim of this short note to provide a stronger and "localized" version of this theorem, with a simpler proof. To localize the compactness hypothesis, I use *compactoid filters* (e.g., [5]). Recall that a filter is *compactoid* if each of its ultrafilters is convergent. Of course, every filter on a compact (topological or convergence) space is compactoid.

Moreover, every convergent filter is compactoid and a space $X$ is compact if and only if $\{X\}$ is compactoid. Hence compactoidness is a common generalization of compactness and convergence. More generally, a filter $\mathcal{F}$ is *compactoid at* $A \subset X$ if each of its ultrafilters has a limit point in $A$. Such generalization of compactness finds far reaching applications (e.g., [12, 13, 4, 6, 3, 8, 10, 11, 7, 1, 9, 2])

The result concerns various *convergence structures* on a poset $(P, \leq)$. Recall that a *convergence* $\xi$ on a set $X$ is a relation — denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \underset{\xi}{\to} x$ whenever $x$ and $\mathcal{F}$ are in relation — between $X$ and the set $\mathbb{F}X$ of filters on $X$, satisfying

1. $\{x\}^\uparrow \to x$ for every $x \in X$;
2. $\mathcal{F} \geq \mathcal{G} \Longrightarrow \lim \mathcal{F} \supset \lim \mathcal{G}$.

A map $f : (X, \xi) \to (Y, \tau)$ between two convergence spaces is *continuous* if

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}],$$

where $f[\mathcal{F}] = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$. We refer to [7] for a systematic study of convergence spaces.

The category **CONV** of convergence spaces and continuous maps is a topological cartesian closed and extensional category (hence a quasitopos). The set of convergence structures on a given set is a complete lattice for the order $\xi \leq \tau$ whenever $\mathrm{id} : (X, \tau) \to (X, \xi)$ is continuous. The category **TOP** of topological spaces and continuous maps is a concretely reflective subcategory. Indeed, calling $\xi$-closed a set containing the $\xi$-limit of each filter on it, the family of all $\xi$-closed sets for a convergence is the family of closed sets for a topology, denoted $\mathrm{T}\,\xi$ and called *topological reflection or topological modification of* $\xi$. It is the finest topology coarser than $\xi$. A convergence is a *pseudotopology* if $\mathcal{F}$ converges to $x$ whenever every ultrafilter finer than $\mathcal{F}$ does. The subcategory **PSTOP** of **CONV** formed by pseudotopological spaces and continuous maps is concretely reflective, and the *pseudotopological reflection* $\mathrm{S}\,\xi$ of a convergence $\xi$ is given by

$$\lim_{\mathrm{S}\,\xi} \mathcal{F} = \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \lim_\xi \mathcal{U},$$

where $\beta\mathcal{F}$ denotes the set of ultrafilters finer than $\mathcal{F}$.

If $A \subset (P, \leq)$ let $A^u = \bigcap_{a \in A} \uparrow a$ be the set of upper bounds of $A$ and $A^\ell = \bigcap_{a \in A} \downarrow a$ be the set of lower bounds. If now $\mathcal{A} \subset 2^P$, $\mathcal{A}^u = \bigcup_{A \in \mathcal{A}} A^u$ and $\mathcal{A}^\ell = \bigcup_{A \in \mathcal{A}} A^\ell$.

On a poset $(P, \leq)$, I consider the following convergence structures (it is well known and easy to verify that they satisfy the above axioms of convergence):

$$x \in \lim_- \mathcal{F} \text{ if } \bigvee \mathcal{F}^\ell \text{ exists and } x \leq \bigvee \mathcal{F}^\ell; \qquad \text{(lower convergence)}$$

$$x \in \lim_+ \mathcal{F} \text{ if } \bigwedge \mathcal{F}^u \text{ exists and } \bigwedge \mathcal{F}^u \leq x; \qquad \text{(upper convergence)}$$

$$\lim_o \mathcal{F} = \lim_- \mathcal{F} \cap \lim_+ \mathcal{F}. \qquad \text{(convergence in order)}$$

Since $\bigwedge \mathcal{F}^u \geq \bigvee \mathcal{F}^\ell$ ([1]), $x \in \lim_o \mathcal{F}$ if $\bigvee \mathcal{F}^\ell = x = \bigwedge \mathcal{F}^u$. Note also that in complete lattice, $\bigvee \mathcal{F}^\ell = \bigvee_{F \in \mathcal{F}} \bigwedge F$ and $\bigwedge \mathcal{F}^u = \bigwedge_{F \in \mathcal{F}} \bigvee F$.

The *order topology* on $(P, \leq)$ is the topological reflection $T o$ of the convergence in order $o$. Let $\tau_i^-$ denote the *lower interval topology,* generated by the complements of lower rays $(x] = \{y : y \leq x\}$. Accordingly, $\tau_i^+$ denotes the *upper interval topology*, generated by complements of upper rays $[x) = \{y : x \leq y\}$, and $\tau_i = \tau_i^- \vee \tau_i^+$ is the *interval topology* on $(P, \leq)$. Note that ([2])

$$\tau_i \leq \mathrm{T}\, o \leq o = + \vee -.$$

**Example 1.** In the (spatial) frame $\mathcal{O}(X)$ of open subsets of a topological space $X$, a basic open set for $\tau_i^-$ is of the form

$$\mathcal{O}(X) \setminus \{V \in \mathcal{O}(X) : V \subset U_0\} = \{V \in \mathcal{O}(X) : V \cap (U_0)^c \neq \emptyset\},$$

while a basic open set for $\tau_i^+$ is of the form

$$\mathcal{O}(X) \setminus \{V \in \mathcal{O}(X) : U_0 \subset V\} = \{V \in \mathcal{O}(X) : V^c \cap U_0 \neq \emptyset\}.$$

[1] because if $x \in \mathcal{F}^u$ and $y \in \mathcal{F}^\ell$ there is $F_1, F_2 \in \mathcal{F}$ with $x \in F_1^u$ and $y \in F_2^\ell$ so that $y \leq z \leq x$ for every $z \in F_1 \cap F_2$

[2] In fact $+ \geq \tau_i^+$ and $- \geq \tau_i^-$. If $x \in \lim_o \mathcal{F}$, that is, $\bigvee \mathcal{F}^\ell = x = \bigwedge \mathcal{F}^u$, and $x \in P \setminus [t)$ then $P \setminus [t) \in \mathcal{F}$ for otherwise $[t) \in \mathcal{F}^\#$, that is, every $F \in \mathcal{F}$ has an element $t_F \geq t$, so that $t \leq \bigwedge \mathcal{F}^u \leq x$ and $x \in [t)$; a contradiction.

The lower convergence is the counterpart on $\mathcal{O}(X)$ of the upper Kuratowski convergence, that is,

$$\mathcal{F} \underset{-}{\to} U \iff U \subset \bigcup_{F\in\mathcal{F}} \operatorname{int}_X(\bigcap_{O\in F} O),$$

and the upper convergence

$$\mathcal{F} \underset{+}{\to} U \iff \operatorname{int}_X(\bigcap_{F\in\mathcal{F}} \bigcup_{O\in F} O) \subset U.$$

**Theorem 2.** *Let $\tau$ be a convergence on a poset $(P, \leq)$. Then*

1. *Assume that $\tau \geq \tau_i^-$. If a filter $\mathcal{F}$ on $P$ is $\tau$-compactoid at $\{x\} \cup (x]^c$ and $\mathcal{F} \underset{+}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$;*
2. *Assume that $\tau \geq \tau_i^+$. If a filter $\mathcal{F}$ on $P$ is $\tau$-compactoid at $\{x\} \cup [x)^c$ and $\mathcal{F} \underset{-}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$;*
3. *Assume that $\tau \geq \tau_i$. If a filter $\mathcal{F}$ on $P$ is $\tau$-compactoid and $\mathcal{F} \underset{o}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$.*

*Proof.* $(1)$. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}$. Since $\mathcal{F}$ is $\tau$-compactoid, $\mathcal{U}$ is $\tau$-convergent to some $y \in \{x\} \cup (x]^c$. Assume $y \neq x$. Then $y \nleq x$ and there exists $z_0 \in \mathcal{U}^u$ such that $y \nleq z_0$, for otherwise, $y \leq \bigwedge \mathcal{U}^u \leq x$ because $\mathcal{U} \underset{+}{\to} x$. Therefore, $(z_0] \in \mathcal{U}$ and $y \in (z_0]^c$. But $(z_0]^c$ is $\tau_i^-$-open, hence $\tau$-open because $\tau \geq \tau_i^-$, and therefore belongs to $\mathcal{U}$ because $\mathcal{U} \underset{\tau}{\to} y$; a contradiction. Thus $x = y$ and $\mathcal{U} \underset{\tau}{\to} x$ for every ultrafilter $\mathcal{U}$ of $\mathcal{F}$. In other words, $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$.

$(2)$ is proved in a similar way.

$(3)$. If $\mathcal{F} \underset{o}{\to} x$, then $\mathcal{F} \underset{+}{\to} x$ and $\mathcal{F} \underset{-}{\to} x$. Moreover, $\mathcal{F}$ is $\tau$-compactoid at $P = \{x\} \cup (x]^c \cup \{x\} \cup [x)^c$. Hence an ultrafilter $\mathcal{U}$ of $\mathcal{F}$ converges to $x$ for $\tau$ by either $(1)$ or $(2)$ depending on where the limit point obtained by compactoidness lies. ☐

The following particular case of $(3)$ extends the main theorem (i.e., Theorem 2.1) of [14] from topologies to convergences.

**Corollary 3.** *Let $\tau$ be a compact convergence on a poset $(P, \leq)$. If $\tau \geq \tau_i$, then $o \geq \mathrm{S}\,\tau$ and therefore $\mathrm{T}\,o \geq \mathrm{T}\,\tau$.*

*Proof.* If $\tau$ is compact, then every filter on $P$ is $\tau$-compactoid. Therefore,

$$\mathcal{U} \underset{o}{\to} x \Longrightarrow \mathcal{U} \underset{\tau}{\to} x$$

for every ultrafilter $\mathcal{U}$, that is, $o \geq \mathrm{S}\,\tau$. □

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