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# MOBI SPACES AND GEODESICS FOR THE N-SPHERE

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**Résumé.** Nous introduisons une structure algébrique, appelée *mobi space*, qui peut être utilisée comme modèle pour les espaces où il existe des chemins géodésiques entre deux points quelconques. Cette nouvelle structure est semblable aux modules sur un anneau. Nous présentons des exemples diversifiés et montrons que la formule d'interpolation linéaire sphérique, qui reproduit les géodésiques sur les $n$-sphères, est un exemple de *mobi space*.

**Abstract.** We introduce an algebraic system, called mobi space, which can be used as a model for spaces with geodesic paths between any two of their points. This new algebraic structure is similar to modules over a ring. We present various examples and show that the formula for spherical linear interpolation, which gives geodesics on the $n$-sphere, is an example of a mobi space.

**Keywords.** Mobility algebra, mobi algebra, mobi space, affine space, affine mobi space, unit interval, ternary operation, geodesic path, geodesics, sphere, n-sphere, Slerp, damped harmonic oscillator, projectiles.

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## 1. Introduction

The purpose of this work is to introduce an algebraic system which can be used to model spaces with geodesics. The main idea stems from the interplay between algebra and geometry. In affine geometry the notion of affine space is well suited for this purpose. Indeed, in an affine space we have scalar multiplication, addition and subtraction and so it is possible to parametrize, for any instant $t \in [0, 1]$, a straight line between points $x$ and $y$ with the formula $(1 - t)x + ty$. Such a line is clearly a geodesic path from $x$ to $y$. In general terms, we may use an operation $q = q(x, t, y)$ to indicate the position, at an instant $t$, of a particle moving in a space $X$ from a point $x$ to a point $y$. If the particle is moving along a geodesic path then this operation must certainly verify some conditions. The aim of this project is to present an algebraic structure, $(X, q)$, with axioms that are verified by any operation $q$ representing a geodesic path in a space between any two of its points.

First results concerning this investigation were presented in [5] where a binary operation, obtained by fixing $t$ to a value that positions the particle at half way from $x$ to $y$, is studied. The whole movement of a particle on a geodesic path is captured when the variable $t$ is allowed to range over a set of values, of which the unit interval is the most natural choice. The investigation of those structures and properties relevant to our study led us to the discovery of a new algebraic structure that was called mobi algebra [6]. A mobi algebra (or mobility algebra), besides being a suitable algebraic model for the unit interval, offers an interesting comparison with rings. A slogan may be used to illustrate that comparison: *a mobi algebra is to the unit interval in the same way as a ring is to the set of reals*. A mobi algebra is an algebraic system $(A, p, 0, ½, 1)$ consisting of a set together with a ternary operation $p$ and three constants (see Definition 2.1). Every ring in which 2 is invertible has a mobi algebra structure, while a mobi algebra in which the element ½ is invertible has a ring structure [6].

Following the analogy with rings, and extending it to modules over a ring, we have arrived at a new structure, called mobility space (mobi space for short). If $A$ is a mobi algebra then a mobi space, say $(X, q)$, is defined over a mobi algebra in the sense that $q = q(x, t, y)$ operates on $x, y \in X$ and $t \in A$. The axioms defining a mobi space (Definition 2.2) are similar to the ones defining a mobi algebra. Several significant examples illustrate the strength of these axioms.

Every space with unique geodesics can be given a mobi space structure [9]. However, when geodesics are not unique (for instance when connecting antipodal points on the sphere) it is still possible to define a mobi space structure on that space. This is done by making appropriate choices and it is illustrated in the last section of this paper. The example of the sphere is considered with spherical linear interpolation (Slerp) whose formula gives rise to a mobi space structure.

## 2. Mobi space

In this section we give the definition of a mobi space over a mobi algebra. Its main purpose is to serve as a model for spaces with a geodesic path connecting any two points. It is similar to a module over a ring in the sense that it has an associated mobi algebra which behaves as the set of scalars. In [8], we show that the particular case of an affine mobi space is indeed the same as a module over a ring when the mobi algebra is a ring. In the last section, we present examples of geodesics on the $n$-sphere and on an hyperbolic $n$-space as mobi spaces over the unit interval.

Let us begin by briefly recalling the notion of a mobi algebra, introduced in [6].

**Definition 2.1.** *A mobi algebra is a system* $(A, p, 0, ½, 1)$*, in which* $A$ *is a set,* $p$ *is a ternary operation and* $0$*,* $½$ *and* $1$ *are elements of A, that satisfies the following axioms:*

**(A1)** $p(1, ½, 0) = ½$

**(A2)** $p(0, a, 1) = a$

**(A3)** $p(a, b, a) = a$

**(A4)** $p(a, 0, b) = a$

**(A5)** $p(a, 1, b) = b$

**(A6)** $p(a, ½, b_1) = p(a, ½, b_2) \implies b_1 = b_2$

**(A7)** $p(a, p(c_1, c_2, c_3), b) = p(p(a, c_1, b), c_2, p(a, c_3, b))$

**(A8)** $p(p(a_1, c, b_1), ½, p(a_2, c, b_2)) = p(p(a_1, ½, a_2), c, p(b_1, ½, b_2))$.

In this paper, the structure $(A, p, 0, \frac{1}{2}, 1)$ with $A = [0, 1]$ and

$$p(a, b, c) = a + b(c - a), \tag{1}$$

is called the *canonical mobi algebra*. Note that ½ is used to denote an element in an arbitrary mobi algebra, whereas $\frac{1}{2}$ is the real number in $[0,1]$.

A mobi space is defined over a mobi algebra as follows.

**Definition 2.2.** *Let $(A, p, 0, ½, 1)$ be a mobi algebra. An $A$-mobi space $(X, q)$, consists of a set $X$ and a map $q\colon X \times A \times X \to X$ such that:*

**(X1)** $q(x, 0, y) = x$

**(X2)** $q(x, 1, y) = y$

**(X3)** $q(x, a, x) = x$

**(X4)** $q(x, ½, y_1) = q(x, ½, y_2) \implies y_1 = y_2$

**(X5)** $q(q(x, a, y), b, q(x, c, y)) = q(x, p(a, b, c), y)$

The axioms **(X1)** to **(X5)** are the natural generalizations of axioms **(A3)** to **(A7)** of a mobi algebra. A natural generalization of **(A8)** is

$$q(q(x_1, a, y_1), ½, q(x_2, a, y_2)) = q(q(x_1, ½, x_2), a, q(y_1, ½, y_2)). \tag{2}$$

This condition, however, is too restrictive and is not in general verified by geodesic paths. That is the reason why we do not include it. When condition (2) is satisfied for all $x_1, x_2, y_1, y_2 \in X$ and $a \in A$, we call the $A$-mobi space $(X, q)$ *affine* and speak of an $A$-mobi affine space (see Subsection 3.4 for examples and counterexamples, see also [8]).

If we write $x \oplus y$ instead of $q(x, ½, y)$ and consider the special case of equation (2) when $a = ½$ then we get the usual medial law

$$(x_1 \oplus y_1) \oplus (x_2 \oplus y_2) = (x_1 \oplus x_2) \oplus (y_1 \oplus y_2).$$

As an illustration of the fact that the medial law does not hold true in general for geodesic paths, let us consider the example of the unit sphere. The midpoint $a \oplus b$ of two points $a$ and $b$ on the equator is again on the equator. Midpoint of North Pole $n$ and any point $c$ on equator is on the $45^{\text{th}}$ parallel. But the geodesic midpoint of two points on the $45^{\text{th}}$ parallel does not live on the $45^{\text{th}}$ parallel, but somewhat to the North of it; the $45^{\text{th}}$ parallel is not a geodesic. So $(a \oplus b) \oplus (n \oplus n)$ is on the $45^{\text{th}}$ parallel, but $(a \oplus n) \oplus (b \oplus n)$

is not, but is north of it. This phenomenon is an aspect of the Gaussian curvature of the sphere.

In a mobi algebra $(A, p, 0, ½, 1)$, $\overline{a} \in A$ is defined for each element $a \in A$ as $\overline{a} = p(1, a, 0)$, which in the canonical case corresponds to $\overline{a} = 1-a$. Note that $½ \in A$ is the unique element with $\overline{½} = ½$. As an immediate consequence of the axioms of a mobi space, we get:

$$q(x, \overline{a}, y) = q(y, a, x). \tag{3}$$

Other properties of mobi spaces can be found in [7].

With the purpose of finding a general procedure to construct mobi spaces, let us consider a simple example with the variables $x, y \in \mathbb{R}$ and $t \in [0, 1]$. First, let us define a map

$$q(x, t, y) = x \cos(t) + y \sin(t)$$

and observe that it satisfies $q(x, 0, y) = x$ but not $q(x, 1, y) = y$. If we put

$$q(x, t, y) = x \cos\left(t\frac{\pi}{2}\right) + y \sin\left(t\frac{\pi}{2}\right)$$

then we have $q(x, 0, y) = x$ and $q(x, 1, y) = y$ but axiom **(X3)**, namely $q(x, t, x) = x$, fails for all values other than $x = 0$ or $t = 0, 1$.

If we change the map $q$ to be

$$q(x, t, y) = x \cos^2\left(t\frac{\pi}{2}\right) + y \sin^2\left(t\frac{\pi}{2}\right) \tag{4}$$

then we get axioms **(X3)** and **(X4)** but the axiom **(X5)** is not verified. Take for example, $x = 1$, $y = 0$, $r = t = \frac{1}{3}$ and $s = 1$ and observe that

$$q(x, r + t(s - r), y) = \cos^2\left(\frac{5\pi}{18}\right)$$

while

$$q(q(x, r, y), t, q(x, s, y)) = \cos^4\left(\frac{\pi}{6}\right)$$

and they are not equal.

One might expect that in order to fix this problem it would be sufficient to find a map $\theta$ such that

$$q(x, \theta(r + t(s - r)), y) = q(q(x, \theta(r), y), \theta(t), q(x, \theta(s), y)).$$

However, this is not so simple. Indeed, perhaps a first guess would be to consider the map $\theta(t) = \frac{2}{\pi}\arcsin(t)$. With this modification, (4) would become

$$q(x, \theta(t), y) = x + (y - x)t^2.$$

This new formula, however, still does not satisfy axiom **(X5)**.

There is, nevertheless, a general procedure that leads to a mobi space out of the formula $h(x, t, y) = x + (y - x)t^2$, but it involves some extra work. We have to introduce one extra dimension, while solving a certain system of equations (see Proposition 2.3 below).

First we solve the system of two equations

$$\begin{cases} A + (B - A)\, r^2 &= x \\ A + (B - A)\, s^2 &= y \end{cases}$$

which has a unique solution for every $x, y \in \mathbb{R}$, $r, s \in \mathbb{R}^+$ and $s \neq r$, namely

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{s^2 - r^2} \begin{pmatrix} s^2 & -r^2 \\ -(1 - s^2) & 1 - r^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

The mobi space on the set $\mathbb{R} \times \mathbb{R}^+$ (over the unit interval) is thus given by the formula

$$q((x, r), t, (y, s)) = (h(A, r + t(s - r), B), r + t(s - r))$$

with $s \neq r$ and $A$, $B$ obtained from equation $(5)$. When $s = r$ we put

$$q((x, r), t, (y, s)) = (x + t(y - x), r).$$

We end up with the operation $q : (\mathbb{R} \times \mathbb{R}^+) \times [0, 1] \times (\mathbb{R} \times \mathbb{R}^+) \to (\mathbb{R} \times \mathbb{R}^+)$ defined by

$$q\big((x, r), t, (y, s)\big) = \\ \begin{cases} \left(x + (y - x)\frac{2rt + (s-r)t^2}{r+s}, r + t(s - r)\right) & , if \quad s \neq r \\ \big(x + t(y - x), r\big) & , if \quad s = r \end{cases}, \tag{6}$$

turning $(\mathbb{R} \times \mathbb{R}^+, q)$ into a mobi space over the unit interval canonical mobi algebra. This procedure, which provides a way to construct examples of mobi spaces, is detailed in the next proposition and generalized in Section 4. Further examples are given in the next section. In particular, example 3.3(1) gives a physical intuition on this construction.

**Proposition 2.3.** *Let* $([0,1], p, 0, \frac{1}{2}, 0)$ *be the canonical mobi algebra. Consider two real valued functions* $f$ *and* $g$*, of one real variable, and let* $(I, \tau)$ *be a mobi space such that:*

1. $I \subseteq \mathbb{R}$ *is an interval of the real numbers;*
2. *the map* $\tau : I \times [0,1] \times I \to I$ *is defined as*
$$\tau(s, a, t) = s + a(t - s);$$
3. *for any* $s, t \in I$*, with* $t \neq s$*,*
$$f(s)\, g(t) \neq f(t)\, g(s). \tag{7}$$

*For any real vector space* $V$ *and any function* $K\colon I \to V$*, the structure* $(V \times I, q)$ *is a mobi space where* $q\colon (V \times I) \times [0,1] \times (V \times I) \to V \times I$ *is defined when* $t \neq s$ *by the formula (*$\tau_a = \tau(s, a, t)$*)*

$$q((x, s), a, (y, t)) = (f(\tau_a)A + g(\tau_a)B - K(\tau_a), \tau_a), \tag{8}$$

*with* $A, B \in V$ *the unique solutions of the system of equations*

$$\begin{cases} f(s)A + g(s)B &= x + K(s) \\ f(t)A + g(t)B &= y + K(t) \end{cases}, \tag{9}$$

*whereas*

$$q((x, s), a, (y, s)) = ((1 - a)x + ay, s). \tag{10}$$

*Proof.* This proposition is a particular case of Theorem 4.1 with $U = V = X$ a real vector space and $h(A, t, B) = Af(t) + Bg(t) - K(t)$. The unique solution (A,B) to the system

$$\begin{cases} h(A, s, B) &= x \\ h(A, t, B) &= y \end{cases},$$

for every $x, y \in V$ and $s, t \in I$, $s \neq t$, is guaranteed by condition (7). $\square$

Several applications of Proposition 2.3 are presented in Subsections 3.2 and 3.3.

## 3. Examples

In the following list of examples, the underlying mobi algebra structure is the canonical mobi algebra $([0,1], p, 0, \frac{1}{2}, 1)$ with $p$ as in (1). In each case, we present a set $X$ and a ternary operation $q(x,a,y) \in X$, for all $x, y \in X$, and $a \in [0,1]$, verifying the axioms of Definition 2.2. We also explain how to obtain some examples as an instance of a more general construction.

### 3.1 The canonical mobi space

1. Vector spaces provide examples. For instance:

$$X = \mathbb{R}^n \quad (n \in \mathbb{N})$$

and

$$q(x,a,y) = (1-a)x + a\,y.$$

2. The well known technique of transporting the structure provides us with other ways of presenting the canonical structure. For every bijective map $F\colon X \to X'$, with $X' \subseteq \mathbb{R}^n$ a convex set, we get a mobi space $(X,q)$ with

$$q(x,a,y) = F^{-1}((1-a)F(x) + a\,F(y)).$$

For instance, in the case of dimension one:

(a) If $F(x) = \log x$, $X = \mathbb{R}^+$ and $X' = \mathbb{R}$ then we get the mobi space $(X,q)$, with

$$q(x,a,y) = x^{1-a}y^a.$$

(b) If $F(x) = \frac{1}{x}$, then $(\mathbb{R}^+, q)$ is a mobi space with

$$q(x,a,y) = \frac{xy}{ax + (1-a)y}.$$

### 3.2 Examples obtained directly from Proposition 2.3

To apply Proposition 2.3, we need three functions $f$, $g$ and $K$ such that $f(s)g(t) \neq f(t)g(s)$ for all $s, t \in I, s \neq t$. If $g$ is a non-zero constant function, this condition just imply the injectivity of $f$. Let us begin with $K = 0$, $g = 1$ and a function $f$ injective in $I$.

1. With $f\colon \mathbb{R}^+ \to \mathbb{R};\ x \mapsto x^2$, we obtain

$$X = \mathbb{R} \times \mathbb{R}^+$$

with the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{2\,s\,a + (t-s)a^2}{t+s}, s + a(t-s)\right).$$

This is, in fact, the example displayed in equation (6). Note that, since $r, s \in \mathbb{R}^+$, the second branch in (6) is not necessary.

2. With $f\colon \mathbb{R}^+ \to \mathbb{R};\ x \mapsto \frac{1}{x}$, we get the set

$$X = \mathbb{R} \times \mathbb{R}^+$$

with the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{a\,t}{(1-a)s + a\,t}, s + a(t-s)\right).$$

3. With $f\colon \mathbb{R} \to \mathbb{R};\ x \mapsto x^3$, we can consider the set

$$X = \mathbb{R}^2$$

and the formula

$$q((x,s),a,(y,t)) =$$
$$\left(x + (y-x)\frac{3\,s^2\,a + 3\,s(t-s)a^2 + (t-s)^2\,a^3}{s^2 + s\,t + t^2}, s + a(t-s)\right),$$

if $(s,t) \neq (0,0)$, and

$$q((x,0),a,(y,0)) = (x + a\,(y-x), 0)\,.$$

4. In general, applying Proposition 2.3 with $g = 1$, $K = 0$ and $f$ an injective real function of one variable, we get a mobi space in any set $X \subseteq \mathbb{R}^2$ for which the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{f(s+(t-s)a) - f(s)}{f(t)-f(s)}, s + a(t-s)\right), \tag{11}$$

if $s \neq t$, and

$$q((x,s),a,(y,s)) = (x + a\,(y-x), s)\,,$$

defines a map $q\colon X \times [0,1] \times X \to X$, as it is the case when $X$ is a convex set.

Let us confirm, with a direct proof, that this operation $q$ verify property (3) of mobi spaces. For $t \neq s$, (11) implies

$$\begin{aligned} & q((x,s),1-a,(y,t)) \\ = & \left(x + (y-x)\frac{f(t+a(s-t)) - f(s)}{f(t)-f(s)}, t + a(s-t)\right) \\ = & \left(y + (x-y)\frac{f(t+a(s-t)) - f(t)}{f(s)-f(t)}, t + a(s-t)\right) \\ = & q((y,t),a,(x,s)). \end{aligned}$$

We will now see some examples obtained from physics.

### 3.3 Examples with physical interpretation

The following examples, from classical mechanics, can be viewed as an application of Proposition 2.3 with specific expressions for $f$, $g$ and $K$.

1. Consider a constant acceleration motion, with $x \in \mathbb{R}^n$, and the following position equation
$$x(t) = x_0 + v_0\,t - k\,t^2.$$
We can think, for instance, of a projectile motion in the plane $\mathbb{R}^2$ where $k$ would be $(0, \frac{g}{2})$ with $g$ being the gravitational acceleration near the

Earth's surface. The constants $x_0$ and $v_0$ correspond to the usual initial conditions $x(0) = x_0$ and $x'(0) = v_0$. Imposing boundary conditions like $x(0) = x_0$ and $x(1) = x_1$, lead to:

$$x(t) = x_0 + (x_1 - x_0)\,t + k\,t\,(1 - t).$$

Note that the operation $q$ defined as $q(x_0, t, x_1) = x(t)$ is not a mobi operation: in particular, idempotency $q(x, t, x) = x$ is not verified because a body could go up vertically and then down back to the same place; axiom **(X5)** is not verified either. The way to obtain a mobi space in this context is to let the variable $t$ flow freely in an extra dimension with boundary conditions like $x(t_0) = x_0$ and $x(t_1) = x_1$. These conditions lead to:

$$x(t_0 + a(t_1 - t_0)) = x_0 + a(x_1 - x_0) + k\,a\,(1 - a)(t_1 - t_0)^2.$$

In the scope of Proposition 2.3, we could say that $f(t) = t$, $g(t) = 1$ and $K(t) = k\,t^2$. For any $k \in \mathbb{R}^n$, we have then a mobi space $(X, q)$ over the canonical mobi algebra by taking the set

$$X = \mathbb{R}^{n+1}$$

with the formula

$$\begin{aligned} &q((x, s), a, (y, t)) = \\ &(x + a(y - x) + k\,a\,(1 - a)(t - s)^2, s + a\,(t - s))\,. \end{aligned} \tag{12}$$

Remarks:

(a) This example could be generalized to Special Relativity [4]. In this case, however, the operation $q$ is a partial operation because, in Minkowski spacetime, not every two points can be reached from one another if one point is not inside the *light cone* of the other.

(b) The result (12) may also be obtained from the geodesic equations in coordinates $(x, t)$, given by $\ddot{x} = -2k\,\dot{t}^2$ and $\ddot{t} = 0$, in a space where the invariant square of an infinitesimal line element is

$$dx^2 + 4\,k\,t\,dx\,dt + (4k^2t^2 - c^2)dt^2,$$

for any constant $c \neq 0$.

2. The solutions for the one-dimension motion of the well-known damped harmonic oscillator are of the form

$$Af(t) + Bg(t) - K(t),$$

where A and B are real parameters. If the oscillator is not driven, K(t)=0. Depending on the circumstances, we can have

(a) overdamping: $f(t) = e^{\alpha t},\ g(t) = e^{\beta t},\ \alpha \neq \beta$;
(b) critical damping: $f(t) = e^{\alpha t},\ g(t) = t\, e^{\alpha t}$;
(c) underdamping: $f(t) = e^{\alpha t}\sin(\beta t)$, $g(t) = e^{\alpha t}\cos(\beta t)$, $\beta \neq 0$;
(d) no damping: $f(t) = \sin(\beta t)$, $g(t) = \cos(\beta t)$, $\beta \neq 0$,

where $\alpha, \beta \in \mathbb{R}$ depend on the oscillatory system. For the first two cases, the determinant of the matrix (7) is non-zero for all $s, t \in \mathbb{R}$, with $t \neq s$ and therefore we can apply Proposition 2.3.

(a) In the overdamping case and for any $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$, we obtain the mobi space $(\mathbb{R}^2, q)$ over the canonical mobi algebra where $q$ is defined, for $t \neq s$, by

$$q((x,s),a,(y,t)) = \Big(\frac{e^{\alpha(1-a)(s-t)} - e^{\beta(1-a)(s-t)}}{e^{\alpha s+\beta t} - e^{\alpha t+\beta s}}\, e^{(\alpha+\beta)t}\, x + \frac{e^{\beta a(t-s)} - e^{\alpha a(t-s)}}{e^{\alpha s+\beta t} - e^{\alpha t+\beta s}}\, e^{(\alpha+\beta)s}\, y, s + a(t-s)\Big),$$

and, for $t = s$, by $q((x,s),a,(y,s)) = (x + a(y-x), s)$.

(b) In the critical damping case and for any $\alpha \in \mathbb{R}$, we obtain a mobi space $(\mathbb{R}^2, q)$ over the canonical mobi algebra with the formula

$$q((x,s),a,(y,t)) = \big((1-a)\, x\, e^{\alpha a(t-s)} + a\, y\, e^{\alpha(1-a)(s-t)}, s + a(t-s)\big)\,.$$

(c) For the case of underdamping, we can still apply Proposition 2.3 if we restrict the possible values of $s$ and $t$ to, for instance, $I = [0, \pi[$. The case $\alpha = 0$ and $\beta = 1$ is presented in the next item.

(d) Consider the case $f(t) = \sin(t)$ and $g(t) = \cos(t)$. Then, we obtain the mobi space $(\mathbb{R} \times [0, \pi[, q)$ with $q$ defined, for $t \neq s$, by

$$q((x, s), a, (y, t)) =$$
$$\left( \frac{\sin[(1-a)(t-s)]}{\sin(t-s)} x + \frac{\sin[a(t-s)]}{\sin(t-s)} y, s + a(t-s) \right).$$

The similarity between this formula and the mobi operation (31) of Section 5 will be analysed in a future work.

### 3.4 Affineness of the examples

We end this section with some comments on whether the examples presented verify the affine condition (2) or not. Examples like those corresponding to example 3.2.4 are, in general, not affine in the sense that they don't verify

$$q\left( q[(x_1, s_1), a, (y_1, t_1)], \frac{1}{2}, q[(x_2, s_2), a, (y_2, t_2)] \right)$$
$$= q\left( q[(x_1, s_1), \frac{1}{2}, (x_2, s_2)], a, q[(y_1, t_1), \frac{1}{2}, (y_2, t_2)] \right).$$

Indeed, in example 3.2.1 for instance, we have that

$$q\left( q[(0,0), \frac{1}{3}, (0,1)], \frac{1}{2}, q[(1,1), \frac{1}{3}, (0,0)] \right) = \left( \frac{5}{27}, \frac{1}{2} \right)$$

but

$$q\left( q[(0,0), \frac{1}{2}, (1,1)], \frac{1}{3}, q[(0,1), \frac{1}{2}, (0,0)] \right) = \left( \frac{1}{6}, \frac{1}{2} \right).$$

Similarly, in example 3.2.3, we have for instance:

$$q\left( q[(0,0), \frac{1}{3}, (0,1)], \frac{1}{2}, q[(1,1), \frac{1}{3}, (0,0)] \right) = \left( \frac{19}{189}, \frac{1}{2} \right)$$

while

$$q\left( q[(0,0), \frac{1}{2}, (1,1)], \frac{1}{3}, q[(0,1), \frac{1}{2}, (0,0)] \right) = \left( \frac{1}{12}, \frac{1}{2} \right).$$

Of course, the canonical mobi spaces 3.1 are affine. Examples 3.2.2, 3.3.1 and 3.3.2b correspond also to affine mobi spaces while 3.3.2a does not. Indeed, for 3.3.2a, we have for instance that

$$q\left(q[(0,0),\tfrac{1}{3},(0,1)],\tfrac{1}{6},q[(1,1),\tfrac{1}{3},(0,0)]\right)$$

$$=\left(\frac{(e^{\alpha/18}-e^{\beta/18})(e^{\alpha/3}+e^{\beta/3})}{e^{\alpha}-e^{\beta}},\frac{7}{18}\right)$$

but

$$q\left(q[(0,0),\tfrac{1}{6},(1,1)],\tfrac{1}{3},q[(0,1),\tfrac{1}{6},(0,0)]\right)$$

$$=\left(e^{2(\alpha+\beta)/3}\frac{(e^{-4\alpha/9}-e^{-4\beta/9})(e^{\beta/6}-e^{\alpha/6})}{(e^{2\alpha/3}-e^{2\beta/3})(e^{\alpha}-e^{\beta})},\frac{7}{18}\right).$$

The two results are different if $\alpha\neq\beta$. However, in the limit situation when $\beta\to\alpha$, the critical case is recovered and the two results are naturally equal. The example 3.3.2c is not affine either.

In the following section we will thoroughly analyse a procedure to construct examples of mobi spaces which in general are not affine mobi spaces. In a sequel to this work we will investigate the case of spaces with geodesics and how to construct mobi spaces out of them.

## 4. General construction for mobi spaces

We present here a general result from which Proposition 2.3 can be deduced. In general, the examples that are obtained in this way are not affine.

**Theorem 4.1.** *Let $(X,q_X)$ and $(I,q_I)$ be two mobi spaces over a mobi algebra $(A,p)$. Suppose the existence of two sets $U$, $V$ and a function $h\colon U\times I\times V\to X$ such that the system*

$$\begin{cases} h(\alpha,t_0,\beta) &= x_0 \\ h(\alpha,t_1,\beta) &= x_1 \end{cases} \tag{13}$$

*has a unique solution for every $x_0,x_1\in X$ and any $t_0,t_1\in I$ with $t_1\neq t_0$, namely*

$$\begin{cases} \alpha &= \alpha(x_0,t_0,x_1,t_1) \\ \beta &= \beta(x_0,t_0,x_1,t_1) \end{cases}. \tag{14}$$

*Then,* $(X \times I, q)$ *is a mobi space over the mobi algebra* $(A, p)$ *where*

$$q\colon (X \times I) \times A \times (X \times I) \to (X \times I)$$

*is defined using the map* $\chi$*, via (14),*

$$\chi(x_0, t_0, a, x_1, t_1) = \begin{cases} h[\alpha, q_I(t_0, a, t_1), \beta] & \textit{if} \quad t_1 \neq t_0 \\ q_X(x_0, a, x_1) & \textit{if} \quad t_1 = t_0 \end{cases} \tag{15}$$

*as*

$$q\left((x_0, t_0), a, (x_1, t_1)\right) = \left(\chi(x_0, t_0, a, x_1, t_1), q_I(t_0, a, t_1)\right).$$

*Proof.* The axioms **(X1)**, **(X2)** and **(X3)** are direct consequences of (13) and the fact that $q_X$ and $q_I$ are operations of mobi spaces. To prove **(X4)**, we first observe that

$$q\left[(x_0, t_0), ½, (x_1, t_1)\right] = q\left[(x_0, t_0), ½, (x_1', t_1')\right] \tag{16}$$

implies $q_I(t_0, ½, t_1) = q_I(t_0, ½, t_1')$ and hence $t_1' = t_1$. If $t_1 = t_0$, we also get $q_X(x_0, ½, x_1) = q_X(x_0, ½, x_1')$ and consequently $x_1' = x_1$. When $t_1 \neq t_0$, $t_1' = t_1$ and (16) imply

$$h[\alpha, q_I(t_0, ½, t_1), \beta] = h[\alpha', q_I(t_0, ½, t_1), \beta'] \equiv x_2,$$

where $\alpha' = \alpha(x_0, t_0, x_1', t_1)$ and $\beta' = \beta(x_0, t_0, x_1', t_1)$. Now, because $t_1 \neq t_0 \Rightarrow q_I(t_0, ½, t_1) \neq t_0$, the system

$$\begin{cases} h(\alpha, t_0, \beta) &= x_0 \\ h(\alpha, q_I(t_0, ½, t_1), \beta) &= x_2 \end{cases}$$

has a unique solution, we then conclude that $\alpha = \alpha'$ and $\beta = \beta'$ and consequently that

$$x_1' = h(\alpha', t_1, \beta') = h(\alpha, t_1, \beta) = x_1.$$

Let us now prove **(X5)**. We have to prove that $Q_1 = Q_2$ where:

$$Q_1 \equiv q\left[(x_0, t_0), p(a, b, c), (x_1, t_1)\right]$$

$$Q_2 \equiv q\Big(q\left[(x_0, t_0), a, (x_1, t_1)\right], b, q\left[(x_0, t_0), c, (x_1, t_1)\right]\Big).$$

To simplify the presentation of the proof, the following notations are used:

$$t_a = q_I(t_0, a, t_1), \quad t_c = q_I(t_0, c, t_1),$$

$$\chi_a = h(\alpha, t_a, \beta), \quad \chi_c = h(\alpha, t_c, \beta).$$

- Considering $t_0 \neq t_1$ and $t_a \neq t_c$, we have

$$\begin{aligned} Q_1 &= (h[\alpha, q_I(t_0, p(a,b,c), t_1), \beta], q_I[t_0, p(a,b,c), t_1]) \\ &= (h[\alpha, q_I(t_a, b, t_c), \beta], q_I[t_a, b, t_c]) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big[(\chi_a, t_a), b, (\chi_c, t_c)\Big] \\ &= (h[\tilde{\alpha}, q_I(t_a, b, t_c), \tilde{\beta}], q_I[t_a, b, t_c]) \end{aligned},$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the unique solutions of the system

$$\begin{cases} h(\tilde{\alpha}, t_a, \tilde{\beta}) &= \chi_a \\ h(\tilde{\alpha}, t_c, \tilde{\beta}) &= \chi_c \end{cases}$$

which imply that $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$, by definition of $\chi_a$ and $\chi_c$ and because $t_a \neq t_c$, therefore $Q_1 = Q_2$.

- Considering $t_0 = t_1$, and hence $t_a = t_c = t_0$, we have

$$\begin{aligned} Q_1 &= \Big(q_X[x_0, p(a,b,c), x_1], q_I[t_0, p(a,b,c), t_1]\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], t_0\Big) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big((q_X[x_0, a, x_1], t_a), b, (q_X[x_0, c, x_1], t_c)\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], q_I(t_a, b, t_c)\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], t_0\Big) \end{aligned}$$

implying that $Q_1 = Q_2$.

- Considering $t_0 \neq t_1$ and $t_a = t_c$, hence $\chi_a = \chi_c$, we have

$$\begin{aligned} Q_1 &= (h[\alpha, q_I(t_a, b, t_c), \beta], q_I[t_a, b, t_c]) \\ &= (\chi_a, t_a) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big[(\chi_a, t_a), b, (\chi_c, t_c)\Big] \\ &= \Big(q_X[\chi_a, b, \chi_c], q_I[t_a, b, t_c]\Big)\Big) \\ &= (\chi_a, t_a) \\ &= Q_1. \end{aligned}$$

□

As an example, consider $U = X = V = \mathbb{R}^+$, $I = \mathbb{R}_0^+$, $(A, p)$ the canonical mobi algebra and $h(\alpha, t, \beta) = \alpha\,\beta^t$. Then, for $t_0 \neq t_1$,

$$h[\alpha(x_0, t_0, x_1, t_1), t, \beta(x_0, t_0, x_1, t_1)] = x_0^{\frac{t-t_1}{t_0-t_1}}\, x_1^{\frac{t_0-t}{t_0-t_1}},$$

and if $t$ is $q_I(t_0, a, t_1) = t_0 + a\,(t_1 - t_0)$, we get:

$$q\,[(x_0, t_0), a, (x_1, t_1)] = (x_0^{1-a} x_1^a, t_0 + a\,(t_1 - t_0)).$$

This expression is well-defined even for $t_1 = t_0$. This leaves no option for $q_X$ if we want a continuous operation, as the only possibility is $q_X(x_0, a, x_1) = x_0^{1-a} x_1^a$. But any other mobi operation is allowed when $t_1 = t_0$ and we can write:

$$q\,[(x_0, t_0), a, (x_1, t_0)] = (q_X(x_0, a, x_1), t_0).$$

This example compares with Example 3.1.2a. Note that in Example 3.2.3, the branch corresponding to $(s, t) = (0, 0)$ cannot be obtained by continuity due to the fact that the limit $(s, t) \to (0, 0)$ does not exist. However, the *canonical* expression at $(0, 0)$ is the choice which corresponds to approaching the origin through the path $t = s$.

A useful particular case is when $X$ is a vector space and $h(\alpha, t, \beta) = \alpha f(t) + \beta g(t) - K(t)$ for some scalar maps $f$, $g$ and vector map $K$. The following proposition is a slight generalization of Proposition 2.3. Here, the canonical mobi algebra is replaced by an arbitrary one $(A, p)$, the real interval $I$ together with the map $\tau$ is replaced by a mobi space $(I, q_I)$ and the real vector space $V$ is replaced by the vector space $X$ over a field $F$. Moreover $q_X$ may be any mobi operation on $X$ rather than $q_X(x, a, y) = (1 - a)x + ay$ as considered in Proposition 2.3.

**Proposition 4.2.** *Let $(X, q_X)$ and $(I, q_I)$ be two mobi spaces over a mobi algebra $(A, p)$. Suppose moreover that $X$ is a vector space over a scalar field $F$ and let $f : I \to F$ and $g : I \to F$ be two functions such that, for any $t_0, t_1 \in I$ with $t_0 \neq t_1$, the following inequality holds*

$$f(t_0)\, g(t_1) \neq g(t_0)\, f(t_1). \tag{17}$$

*Furthermore, we consider a function $K : I \to X$. Then $(X \times I, q)$ is a mobi space over $(A, p)$ considering that*

$$q : (X \times I) \times A \times (X \times I) \to (X \times I)$$

*is defined as*

$$q\left[(x_0, t_0), a, (x_1, t_1)\right] = \Big(\chi_a(x_0, t_0, x_1, t_1), q_I(t_0, a, t_1)\Big)$$

*with*

$$\begin{aligned}
&\chi_a\left(x_0, t_0, x_1, t_1\right) \\
&= \frac{g(t_1)\ (x_0 + K(t_0)) - g(t_0)\ (x_1 + K(t_1))}{f(t_0)\, g(t_1) - f(t_1)\, g(t_0)}\, f[q_I(t_0, a, t_1)] \\
&- \frac{f(t_1)\ (x_0 + K(t_0)) - f(t_0)\ (x_1 + K(t_1))}{f(t_0)\, g(t_1) - f(t_1)\, g(t_0)}\, g[q_I(t_0, a, t_1)] \\
&- K[q_I(t_0, a, t_1)],
\end{aligned}$$

*when $t_1 \neq t_0$ and $\chi_a\left(x_0, t, x_1, t\right) = q_X(x_0, a, x_1)$ otherwise.*

*Proof.* This is just Theorem 4.1 for the case

$$h(\alpha, t, \beta) = \alpha\, f(t) + \beta\, g(t) - K(t).$$

With $U = V = X$, the system (13) simply reads

$$\begin{pmatrix} f(t_0) & g(t_0) \\ f(t_1) & g(t_1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 + K(t_0) \\ x_1 + K(t_1) \end{pmatrix}.$$

□

To illustrate this proposition, Example 3.2.4 can be generalized using an arbitrary mobi space $I$. Consider $h(\alpha, t, \beta) = \alpha f(t) + \beta$, in any set $X$ for which the next formula is well defined. Then, when $t_0 \neq t_1$,

$$q[(x_0, t_0), a, (x_1, t_1)] = \\ (x_0 + (x_1 - x_0)\frac{f(q_I(t_0, a, t_1)) - f(t_0)}{f(t_1) - f(t_0)}, q_I(t_0, a, t_1)).$$

When $t_0 = t_1$, $q[(x_0, t_0), t, (x_1, t_0)] = (q_X(x_0, a, x_1), t_0)$ for any mobi operation $q_X$.

Even when the system of equations (13) does not have a unique solution, then, in some cases, it is still possible to define a mobi-space. This will be illustrated with the formula for spherical linear interpolation giving geodesics on the $n$-sphere.

## 5. Geodesics on the $n$-sphere

The purpose of this section is to show that a mobi space can be obtained using the geodesic curves on the $n$-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_E = 1\}$$

and on one sheet of the two-sheeted hyperbolic $n$-space [15], as for instance

$$H^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_L = -1, x_1 > 0\}.$$

The notations $\langle\, ,\rangle_E$ and $\langle\, ,\rangle_L$ are used for the usual Euclidean and Lorentzian inner products, respectively. For the construction of the mobi operation for both cases at once, it is convenient to consider the family of functions

$$f(a) = \frac{e^{\alpha a} - e^{-\alpha a}}{2\alpha} \text{ and } g(a) = \frac{e^{\alpha a} + e^{-\alpha a}}{2}, \tag{18}$$

where $a \in \mathbb{R}$ and the parameter $\alpha$ is a non-zero complex number. These functions are real functions if and only if $\alpha$ is a real number or a pure imaginary number. In particular, we have that:

- $\alpha = 1 \Rightarrow f(a) = \sinh a$ and $g(a) = \cosh a$;

- $\alpha = i \Rightarrow f(a) = \sin a$ and $g(a) = \cos a$;
- $\alpha \to 0 \Rightarrow f(a) = a$ and $g(a) = 1$.

For our purpose, we want to consider only real functions and therefore, for the rest of this section, it is understood that $\alpha$ is such that $f$ and $g$ are real. In any case, the functions (18) verify the following properties:

$$-\alpha^2 f^2(a) + g^2(a) = 1 \tag{19}$$
$$f(a)g(b) + f(b)g(a) = f(a+b) \tag{20}$$
$$\alpha^2 f(a)f(b) + g(a)g(b) = g(a+b) \tag{21}$$
$$f(-a) = -f(a) \quad , \quad g(-a) = g(a) \tag{22}$$
$$f(0) = 0 \quad , \quad g(0) = 1. \tag{23}$$

In general terms, let us consider an interval $I \in \mathbb{R}$ containing $0$ where g is injective. Let $V$ be an inner product space, with the inner product denoted by $\langle\,,\rangle$, and a subspace $X \subseteq \{x \in V \mid \langle x, x\rangle = -\alpha^2\}$. Inner product here means a nondegenerate symmetric bilinear form. We are going to show that when there exists a unique function

$$\theta : X \times X \to I$$

such that $-\alpha^2 g[\theta(x, y)] = \langle x, y\rangle$, $X$ can be given the structure of a mobi space. For instance, if $X$ is $S^n$, $\theta$ may be defined as

$$\theta(x, y) = \arccos(\langle x, y\rangle_E) \text{ with } I = [0, \pi]$$

and if $X = H^n$, $\theta$ may be defined as

$$\theta(x, y) = \operatorname{arccosh}(-\langle x, y\rangle_L) \text{ with } I = [0, +\infty[.$$

The expressions are similar for any pure imaginary or non-zero real number $\alpha$. The next two propositions show explicitly how to construct a mobi operation on $X$ using the functions $f$, $g$ and $\theta$. This construction is based on the spherical linear interpolation (Slerp) used in computer graphics [14]. The first proposition is for the cases where the geodesic between two points is unique which occur when the only zero of $f$, in $I$, is zero. This is what happens for $H^n$ but not for $S^n$ because $\sin(\pi) = 0$. Nevertheless, the Proposition 5.1 could still be applied to a portion of the $n$-sphere which does not contain antipodal points, such as for example $\{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_E = 1, x_1 > 0\}$.

**Proposition 5.1.** *Consider the* **real** *functions $f$ and $g$, of one real variable, verifying the properties* (19) *to* (23) *for some number $\alpha$. Suppose that $g$ is injective in an interval $I$ containing* 0 *and that, for $a \in I$, we have*

$$f(a) = 0 \Longleftrightarrow a = 0.$$

*Let $(V, \langle\ ,\ \rangle)$ be a real inner vector space and consider a subspace*

$$X \subseteq \{x \in V \mid \langle x, x\rangle = -\alpha^2\}.$$

*If there exists a unique function $\theta : X \times X \to I$ such that*

$$\langle x, y\rangle = -\alpha^2 g[\theta(x, y)] \tag{24}$$

*and $\theta(x, y) = 0 \Longleftrightarrow y = x$, then $(X, q)$ is a mobi space over the canonical mobi algebra where the ternary operation $q : X \times [0, 1] \times X \to X$ is defined, for $x \neq y$, by*

$$q(x, t, y) = \frac{f[\theta(x, y)\,(1 - t)]}{f[\theta(x, y)]}\, x + \frac{f[\theta(x, y)\, t]}{f[\theta(x, y)]}\, y, \tag{25}$$

*and, otherwise, by $q(x, t, x) = x$.*

*Proof.* To simplify the presentation, we use the notation $\Omega \equiv \theta(x, y)$. First, we have to prove that, when $x, y \in X$, $q(x, t, y)$ is still in $X$. The case $y = x$ is obvious. For $y \neq x$:

$$\begin{aligned}
\langle q(x, t, y), q(x, t, y)\rangle &= \frac{f^2(\Omega(1 - t))}{f^2(\Omega)}\langle x, x\rangle + \frac{f^2(\Omega t)}{f^2(\Omega)}\langle y, y\rangle \\
&+ 2\frac{f(\Omega(1 - t))f(\Omega t)}{f^2(\Omega)}\langle x, y\rangle
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\alpha^2}{f^2(\Omega)}(f^2(\Omega(1 - t)) + f^2(\Omega t) + 2f(\Omega(1 - t))f(\Omega t)g(\Omega)) \\
&= -\frac{\alpha^2}{f^2(\Omega)}(f(\Omega - \Omega t)(f(\Omega)g(\Omega t) + f(\Omega t)g(\Omega)) + f^2(\Omega t)) \\
&= -\frac{\alpha^2}{f^2(\Omega)}(f^2(\Omega)g^2(\Omega t) + f^2(\Omega t)(1 - g^2(\Omega)) \\
&= -\alpha^2(g^2(\Omega t) - \alpha^2 f^2(\Omega t)) = -\alpha^2.
\end{aligned}$$

Now, as $X$ is a subspace containing $x$ and $y$ and $q(x,t,y)$ is a linear combination of $x$ and $y$, we conclude that it is also in $X$. Axioms **(X1)**, **(X2)**, **(X3)** of a mobi space are a direct consequence of the definition of $q$. To prove **(X4)** we will use the notation $\Omega' \equiv \theta(x,y')$. If $x \neq y$ and $x \neq y'$, we have that $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ implies

$$\frac{f\left(\frac{\Omega}{2}\right)}{f(\Omega)}(x+y) = \frac{f\left(\frac{\Omega'}{2}\right)}{f(\Omega')}(x+y') \tag{26}$$

Applying the inner product with $x$ in both sides of this equation and using properties (19) to (21) in the form

$$f(\Omega) = 2f\left(\frac{\Omega}{2}\right) g\left(\frac{\Omega}{2}\right) \quad \text{and} \quad 1 + g(\Omega) = 2g^2\left(\frac{\Omega}{2}\right),$$

we get

$$\begin{aligned} & \frac{1}{2g\left(\frac{\Omega}{2}\right)}\langle x, x+y\rangle = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}\langle x, x+y'\rangle \\ \Rightarrow \quad & \frac{1}{2g\left(\frac{\Omega}{2}\right)}(-\alpha^2 - \alpha^2 g(\Omega)) = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}(-\alpha^2 - \alpha^2 g(\Omega')) \\ \Rightarrow \quad & g\left(\frac{\Omega}{2}\right) = g\left(\frac{\Omega'}{2}\right) \neq 0. \end{aligned}$$

Going back to (26) with this result, we conclude $y = y'$. If $x = y$ and $x \neq y'$, then $q\left(x, \frac{1}{2}, y\right) \neq q\left(x, \frac{1}{2}, y'\right)$. Indeed, $q\left(x, \frac{1}{2}, x\right) = q\left(x, \frac{1}{2}, y'\right)$ would imply

$$\begin{aligned} x = \frac{f\left(\frac{\Omega'}{2}\right)}{f(\Omega')}(x+y') \Rightarrow \langle x, x\rangle = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}\langle x, x+y'\rangle \\ \Rightarrow 1 = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}(1 + g(\Omega')) \Rightarrow g\left(\tfrac{\Omega'}{2}\right) = 1 \Rightarrow y' = x, \end{aligned}$$

in contradiction with the hypothesis. The case $x \neq y$ and $x = y'$ is similar. Obviously **(X4)** is also verified if $x = y$ and $x = y'$. The proof of **(X5)** begin with the observation that:

$$g[\theta(q(x,a,y), q(x,c,y))] = g[\theta(x,y)(c-a)]. \tag{27}$$

Indeed, beginning with the left-hand side of (27), if $y \neq x$:

$$\begin{aligned} & -\frac{1}{\alpha^2}\langle \frac{f(\Omega(1-a))}{f(\Omega)}x + \frac{f(\Omega a)}{f(\Omega)}y, \frac{f(\Omega(1-c))}{f(\Omega)}x + \frac{f(\Omega c)}{f(\Omega)}y \rangle \\ = & \frac{f(\Omega(1-a))f(\Omega(1-c))}{f^2(\Omega)} + \frac{f(\Omega a)f(\Omega c)}{f^2(\Omega)} \\ + & \left( \frac{f(\Omega(1-a))f(\Omega c) + f(\Omega(1-c))f(\Omega a)}{f^2(\Omega)} \right) g(\Omega) \\ = & \frac{f^2(\Omega)g(\Omega a)g(\Omega c) - g^2(\Omega)f(\Omega a)f(\Omega c) + f(\Omega a)f(\Omega c)}{f^2(\Omega)} \\ = & g(\Omega a)g(\Omega c) - \alpha^2 f(\Omega a)f(\Omega c) \\ = & g(\Omega a - \Omega c) = g(\Omega c - \Omega a) \end{aligned}$$

If $y = x$, then

$$\begin{aligned} g[\theta(q(x,a,y), q(x,c,y))] &= -\frac{1}{\alpha^2}\langle q(x,a,y), q(x,c,y) \rangle \\ &= -\frac{1}{\alpha^2}\langle x, x \rangle \\ &= 1 = g(0) = g[\theta(x,y)(c-a)]. \end{aligned}$$

Now, because $a, c \in [0,1]$ and $\Omega \in I$ imply $\Omega|c-a| \in I$, we can conclude, since $g$ is injective in I, that

$$|\theta(q(x,a,y), q(x,c,y))| = |\theta(x,y)(c-a)| = |\Omega(c-a)|. \tag{28}$$

By (22), this also imply that, for any $b \in [0,1]$, the following relation is true:

$$\frac{f[\theta(q(x,a,y), q(x,c,y))\, b]}{f[\theta(q(x,a,y), q(x,c,y))]} = \frac{f[\Omega(c-a)\, b]}{f[\Omega(c-a)]}.$$

With these results, we are able to prove **(X5)**. For simplification, we use the notation $\hat{c} \equiv c - a$. First, for $q(x,a,y) \neq q(x,c,y)$ and $x \neq y$:

$$
\begin{aligned}
& q[q(x,a,y),b,q(x,c,y)] \\
= \;& \frac{f[\Omega\,\hat{c}(1-b)]}{f[\Omega\,\hat{c}]}\,q(x,a,y)+\frac{f[\Omega\,\hat{c}\,b]}{f[\Omega\,\hat{c}]}\,q(x,\hat{c}+a,y) \\
= \;& \frac{g[\Omega\,\hat{c}\,b]f[\Omega(1-a)]-f[\Omega\,\hat{c}\,b]g[\Omega(1-a)]}{f(\Omega)}\,x \\
+ \;& \frac{g[\Omega\,\hat{c}\,b]f[\Omega\,a]+f[\Omega\,\hat{c}\,b]g[\Omega\,a]}{f(\Omega)}\,y \\
= \;& \frac{f[\Omega(1-a-\hat{c}\,b)]}{f(\Omega)}\,x+\frac{f[\Omega(a+\hat{c}\,b)]}{f(\Omega)}\,y \\
= \;& q[x,a+(c-a)b,y].
\end{aligned}
$$

If $q(x,a,y)=q(x,c,y)$, then $q(q(x,a,y),b,q(x,c,y))=q(x,a,y)$. On the other hand, from (27), we conclude that $x=y$ or $c=a$ and in both cases $q(x,a+b(c-a),y)=q(x,a,y)$. □

Before going to Proposition 5.2 that will explain how we can still get a mobi space out of a *Slerp type* formula on the $n$-sphere despite the fact that geodesics between antipodal points are not unique, let us take a closer look to the formula (25) in the case of $S^n$. This formula just gives the intersection between $S^n$ and a plane that contains the origin and the points $x$ and $y$, when $x$ and $y$ are not collinear. Starting at $x$ when $t=0$, a particle that goes to $y$ on that plane at constant speed will be, at an instant $t\in[0,1]$ at

$$q(x,t,y)=\cos(\Omega\,t)\,x+\sin(\Omega\,t)\,z \tag{29}$$

where

$$\cos(\Omega)\,x+\sin(\Omega)\,z=y. \tag{30}$$

Because $\langle x,x\rangle_E=\langle y,y\rangle_E=1$ and $\Omega\equiv\theta(x,y)=\arccos\langle x,y\rangle_E$, we have that $\langle x,z\rangle_E=0$ and $\langle z,z\rangle_E=1$. When $\sin(\Omega)\neq 0$, we can just solve (30) to obtain $z$ and then (29) reads as expected:

$$q(x,t,y)=\frac{\sin[\Omega\,(1-t)]}{\sin(\Omega)}\,x+\frac{\sin[\Omega\,t]}{\sin(\Omega)}\,y. \tag{31}$$

When $\Omega=0$, which means $x=y$, there is no journey to make: $q(x,t,x)=x$. When $\Omega=\pi$, which means $y=-x$, we have to choose the plane we wish to travel on. Equivalently, we have to chose the direction $v(x) \in \mathbb{R}^{n+1}$ we want to be playing the role of $z$. Of course, we still need $\langle x, v(x)\rangle_E = 0$ and $\langle v(x), v(x)\rangle_E = 1$. There is one more condition: to get a mobi space, we also need $v$ to be an even map because, from property (3), $q(x, t, -x) = q(-x, 1-t, x)$ which means that in a round trip, the going and the return must be done on the same path.

**Proposition 5.2.** *Consider the euclidean $n$-sphere*

$$X = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = 1\},$$

*a map $v : X \to X$ such that*

$$v(-x) = v(x) \text{ and } \langle x, v(x)\rangle = 0,$$

*and the map $\theta : X \times X \to [0, \pi]$ defined by*

$$\theta(x, y) = \arccos(\langle x, y\rangle).$$

*With $q : X \times [0, 1] \times X \to X$ defined by*

$$q(x,t,y) = \begin{cases} \dfrac{\sin[\theta(x,y)\,(1-t)]}{\sin[\theta(x,y)]}\,x + \dfrac{\sin[\theta(x,y)\,t]}{\sin[\theta(x,y)]}\,y & ,if \quad \theta(x,y) \in ]0,\pi[ \\ \cos[\theta(x,y)\,t]\,x + \sin[\theta(x,y)\,t]\,v(x) & ,if \quad \theta(x,y) \in \{0,\pi\} \end{cases},$$

*$(X, q)$ is a mobi space over the canonical mobi algebra.*

*Proof.* Most of the proof is the same as the proof of Proposition 5.1. We just have to consider the extra case $y = -x$ corresponding to $\Omega \equiv \theta(x, y) = \pi$. When $\Omega = \pi$, we have $q(x, 0, y) = \cos(0)x + \sin(0)v(x) = x$ and $q(x, 1, y) = \cos(\pi)x + \sin(\pi)v(x) = -x = y$, so Axioms **(X1)**, **(X2)** and **(X3)** of a mobi space are verified. Regarding **(X4)**, when $\Omega = \pi$ and $\Omega' \in ]0, \pi[$, we have that $q\left(x, \frac{1}{2}, y\right) \neq q\left(x, \frac{1}{2}, y'\right)$. Indeed $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ implies

$$v(x) = \frac{\sin\left(\frac{\Omega'}{2}\right)}{\sin(\Omega')}(x + y') \Rightarrow 0 = \cos\left(\frac{\Omega'}{2}\right) \Rightarrow \Omega' = \pi,$$

in contradiction with $\Omega' \in ]0, \pi[$. The case $\Omega = \pi$ and $\Omega' = 0$ is also incompatible with $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ because $v(x) \neq x$. Interchanging $y$ and $y'$ in the previous situations gives similar results. The case $\Omega = \pi$ and $\Omega' = \pi$ implies $y = -x = y'$, therefore **(X4)** is verified. Regarding **(X5)**, we first observe that (27) is valid for all $x, y \in X$. Indeed, if $y = -x$, then

$$\begin{aligned}
& \cos[\theta(q(x,a,y), q(x,c,y))] = \langle q(x,a,y), q(x,c,y)\rangle \\
= \quad & \langle \cos(\pi a)\, x + \sin(\pi a)\, v(x), \cos(\pi c)\, x + \sin(\pi c)\, v(x)\rangle \\
= \quad & \cos(\pi a)\cos(\pi c) + \sin(\pi a)\sin(\pi c) \\
= \quad & \cos[\pi(a-c)].
\end{aligned}$$

So, we have that, $\forall x, y \in X$:

$$\theta(q(x,a,y), q(x,c,y)) = \theta(x,y)|c-a|. \tag{32}$$

From equation (32), we conclude that $\theta(q(x,a,y), q(x,c,y)) = \pi$ if and only if $\Omega = \pi$ and $|c-a| = 1$ and that $\theta(q(x,a,y), q(x,c,y)) = 0$ if and only if $\Omega = 0$ or $c = a$. Therefore, besides the cases already proved in Proposition 5.1, we have to consider the following four situations:

1. $\theta(q(x,a,y), q(x,c,y)) = \pi$, $\Omega = \pi$ and
   (a) $c = 0$, $a = 1$
   (b) $c = 1$, $a = 0$
2. $\theta(q(x,a,y), q(x,c,y)) = 0$, $\Omega = \pi$ and $c = a$
3. $\theta(q(x,a,y), q(x,c,y)) \in ]0, \pi[$, $\Omega = \pi$, $c \neq a$ and $|c-a| \neq 1$.

For the situation (1a):

$$\begin{aligned}
q[q(x,a,y), b, q(x,c,y)] &= q(-x,b,x) = \cos(\pi b)(-x) + \sin(\pi b)\, v(-x) \\
q[x, a+b(c-a), y] &= q(x, 1-b, -x) \\
&= \cos(\pi - \pi b)\, x + \sin(\pi - \pi b)\, v(x) \\
&= -\cos(\pi b)\, x + \sin(\pi b)\, v(x).
\end{aligned}$$

The Axiom **(X5)** is ensured through the hypothesis $v(-x) = v(x)$. For the situation (1b):

$$\begin{aligned}
q[q(x,a,y), b, q(x,c,y)] = q(x,b,-x) &= \cos(\pi b)\, x + \sin(\pi b)\, v(x) \\
q[x, a+b(c-a), y] = q(x,b,-x) &= \cos(\pi b)\, x + \sin(\pi b)\, v(x).
\end{aligned}$$

In situation (2), $c = a$ and **(X3)** implies **(X5)**. Using $\hat{c} \equiv c - a$, we have for the situation (3):

$$\begin{aligned}
& q[q(x,a,y),b,q(x,c,y)] \\
= & \frac{\sin[\pi(c-a)(1-b)]}{\sin[\pi(c-a)]}(\cos(\pi a)\,x + \sin(\pi a)\,v(x)) \\
+ & \frac{\sin[\pi(c-a)b]}{\sin[\pi(c-a)]}(\cos(\pi c)\,x + \sin(\pi c)\,v(x)) \\
= & \frac{\sin[\pi\hat{c}(1-b)]\cos(\pi a) + \sin[\pi\hat{c}\,b]\cos(\pi(\hat{c}+a))}{\sin[\pi\,\hat{c}]}x \\
+ & \frac{\sin[\pi\hat{c}(1-b)]\sin(\pi a) + \sin[\pi\hat{c}\,b]\sin(\pi(\hat{c}+a))}{\sin[\pi\,\hat{c}]}v(x) \\
= & \cos[\pi\,\hat{c}\,b]\cos(\pi a) - \sin[\pi\,\hat{c}\,b]\sin[\pi a]\,x \\
+ & \cos[\pi\,\hat{c}\,b]\sin(\pi a) + \sin[\pi\,\hat{c}\,b]\cos[\pi a]\,v(x) \\
= & \cos[\pi(a+\hat{c}\,b)]\,x + \sin[\pi(a+\hat{c}\,b)]\,v(x) \\
= & q(x, a+b(c-a), y)
\end{aligned}$$

□

To finish this section, we present three examples of the map $v$ used in Proposition 5.2. First, consider the 1-sphere i.e. the circle. We can choose to move between antipodal points in the anticlockwise direction when starting somewhere at the top of the circle and in the clockwise direction when starting at the bottom. More specifically, if $x = (\cos\theta, \sin\theta), \theta \in [0, 2\pi[$, then $v$ is defined as

$$v(x) = \begin{cases} (-\sin\theta, \cos\theta) & if \quad \theta \in [0,\pi[ \\ (\sin\theta, -\cos\theta) & if \quad \theta \in [\pi, 2\pi[ \end{cases}.$$

Secondly, let us choose to connect two antipodal points on $S^2$, different from the poles, through the north pole and link the poles (on the z-axis) through the positive x-axis. This gives the following choice for $v$, considering $x_1^2 + x_2^2 + x_3^2 = 1$:

$$v(x_1,x_2,x_3) = \begin{cases} \dfrac{(-x_1x_3,\ -x_2x_3,\ 1-x_3^2)}{\sqrt{1-x_3^2}} & if \quad x_3 \neq \pm 1 \\ (1,0,0) & if \quad x_3 = \pm 1 \end{cases}.$$

As a third example, consider the 2-sphere parametrized in spherical coordinates as:

$$\{(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), (\theta,\varphi) \in ([0,2\pi[\times]0,\pi[) \cup (0,0) \cup (0,\pi)\}.$$

A possible map $v$ is the following:

$$v(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = \begin{cases} (-\sin\theta, \cos\theta, 0) & if \quad \varphi \in \left[0, \frac{\pi}{2}\right[ \text{ or } \left(\varphi = \frac{\pi}{2}, \theta \in [0,\pi[\right) \\ (\sin\theta, -\cos\theta, 0) & if \quad \varphi \in \left]\frac{\pi}{2}, \pi\right] \text{ or } \left(\varphi = \frac{\pi}{2}, \theta \in [\pi, 2\pi[\right) \end{cases}.$$

In this example, $v$ is on the equator in a plane rotated $\frac{\pi}{2}$ around the z-axis from the meridian of $x$. The choice $\theta = 0$ for the poles connects them through the positive y-axis. The other antipodal points are connected through a path that stays between the parallels of the two points, with an arbitrary choice for antipodal points on the equator.

## 6. Conclusion

We have introduced a new algebraic structure which captures some features of geodesic paths. Several examples were used to illustrate the difficulty in generating non-trivial examples (other than the affine case) and a general procedure was given in Theorem 4.1. This general construction, however, has the effect of raising an extra dimension. The construction commonly named as Slerp was used to show that mobi spaces include the example of geodesics on the $n$-sphere. It can be seen as a particular case of our general construction if we fix $t_0 = 0$ and let $t_1$ be a function of the end points $x_0$, $x_1$ as $t_1 = \theta(x_0, x_1)$. In that case there is no need to use the extra dimension, $I$, because $q$ can be defined through $q(x_0, a, x_1) = h(\alpha, q_I(0, a, \theta(x_0, x_1), \beta)$ with $\alpha$ and $\beta$ being the solutions to the system of equations (13). This, however, is done at the expense of imposing some tight conditions on the maps $h$ and $\theta$ and needs further investigations. Similarly, when $h$ is obtained as a geodesic flow (in a space with unique geodesics) we can take $t_0 = 0$ and $t_1 = 1$ in Theorem 4.1 and observe that $(X, q)$ is a mobi space with $q(x_0, a, x_1) = h(\alpha, a, \beta)$. Once again there is no need for the extra $I$-dimension but it comes with a cost of imposing extra conditions on the map $h$. It turns out that when $h$ is a geodesic flow in a space with unique geodesics then the required conditions are satisfied ([9]).

Some lines of future study include the connection with affine geometry [1, 10, 11] or the geometry of geodesics [2, 3]. As well as the study of affine mobi spaces *per se* [8]. The presence of an operation $x \oplus y = q(x, 1/2, y)$ admitting cancellation, together with the property $x \oplus y = y \oplus x$, tells us that the category of mobi spaces is a weakly Mal'tsev category [13, 12]. This was in fact the starting point that originated our investigation on mobi spaces.

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