
CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXIII-2

A FUNCTORIAL APPROACH TO DIFFERENTIAL CALCULUS

Wolfgang Bertram, Jérémy Haut

Résumé. Nous montrons que le calcul différentiel (sous sa forme usuelle, ou sous la forme du *calcul différentiel topologique*) admet un plongement plein et fidèle dans une catégorie de foncteurs (des foncteurs d'une petite catégorie dite *catégorie des algèbres tangentes ancrées* vers des ensembles ancrés). Pour préparer cette approche, nous définissons une nouvelle version, plus symétrique, du calcul différentiel, où l'application *ancre* joue un rôle central.

Abstract. We show that differential calculus (in its usual form, or in the general form of *topological differential calculus*) can be fully imdedded into a functor category (functors from a small category of *anchored tangent algebras* to anchored sets). To prepare this approach, we define a new, symmetric, presentation of differential calculus, whose main feature is the central rôle played by the *anchor map*, which we study in detail.

Keywords. differential calculus, functor category, anchor, tangent algebra.

Mathematics Subject Classification (2010). 18A25 , 18B40 , 18D05 , 58A05

Introduction

Differential Calculus is a central ingredient of modern mathematics. While the "working mathematician" takes this tool for granted, thinking about its conceptual foundations remains a potentially important topic. In the present work, we continue the line of research started with [BGN04, Be08, BeS14, Be17], and combine it with what Grothendieck once called the "simple idea of a good functor from rings to sets" (according to W. Lawvere, cf. `n-lab`)[1]. The "simple idea" mentioned by Grothendieck is currently used in algebraic

[1]Here the quote from the n-lab: "The 1973 Buffalo Colloquium talk by Alexander Grothendieck had as its main theme that the 1960 definition of scheme ... should be abandoned AS the FUNDAMENTAL one and replaced by the *simple idea of a good functor from rings to sets*. The needed restrictions could be more intuitively and more geometrically stated directly in terms of the topos of such functors, and of course the ingredients from the "baggage" could be extracted when needed as auxiliary explanations of already existing objects, rather than being carried always as core elements of the very definition."

geometry, and in Lie Theory, where one often considers a real "space" – for instance, a Lie group $\underline{G}$ – as set of "real points" $G_{\mathbb{R}}$ of a *complex* Lie group $G_{\mathbb{C}}$. This is a kind of non-linear analog of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector space (or of a real Lie algebra). Grothendieck's insight was that this idea of "complexification" should not be limited to *field* extensions, but enlarged to more general *ring* extensions, in order to incorporate operations belonging to *infinitesimal calculus*: a $\mathbb{K}$-Lie group G, or a general $\mathbb{K}$-smooth manifold M, should admit "scalar extensions" $M_{\mathbb{A}}$ akin to a hypothetic tensor product $M \otimes_{\mathbb{K}} \mathbb{A}$, for certain $\mathbb{K}$-algebras $\mathbb{A}$. The simplest example of such an extension is the one by *dual numbers*,

$$\mathbb{K}[\varepsilon] := \mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K} \quad (\varepsilon^2 = 0), \tag{0.1}$$

where the nilpotent element ε is the class $[X]$ modulo (X^2). Grothendieck, following ideas of Weil [We53], realized that the tangent bundle TM of a "space" M, which is "defined over $\mathbb{K}$", could be understood as something like $M \otimes_{\mathbb{K}} \mathbb{K}[\varepsilon]$. This idea has been used by Demazure and Gabriel in their theory of algebraic groups [DG], in differential calculus over general base field and rings [Be08], and in the approach to natural operations in differential geometry via the so-called *Weil functors* ([KMS93], cf. also [BeS14]). The most elaborate and systematic development of these ideas leads to what is called nowadays *synthetic differential geometry* (SDG, see [MR91]). The approach to be presented here pursues the same goals as SDG, but by different means: we keep closer to the idea of generalizing the algebraic tensor product. In a very direct sense, our problem is to generalize the algebraic scalar extension $V_{\mathbb{A}} := V \otimes_{\mathbb{K}} \mathbb{A}$ of a $\mathbb{K}$-module V, to more general spaces M, like, e.g., manifolds – where we face the problem that such an operation won't be possible for *all* $\mathbb{K}$-algebras $\mathbb{A}$, so we have to single out a good class (good category) of algebras for which such an extension is possible. Such a class, called *the category of (anchored) tangent algebras*, will be defined in this paper. It arises naturally, when questioning the very shape of differential calculus, instead of taking it for granted. Let us briefly explain the main ideas.

0.1 Topological differential calculus

In differential calculus we consider maps f whose domain U and codomain U' are *locally linear sets* – by this we mean $U \subset V$ and $U' \subset V'$ are non-empty subsets of linear (or affine, if one prefers) spaces V and V'. In this situation, we may define the *slope* or *difference quotient map*: when $t, s \in \mathbb{K}$ are such that $t - s$ is invertible, we look at the difference quotient

$$f^{[1]}(v_0, v_1; t, s) := f^{[1]}_{(t,s)}(v_0, v_1) := \frac{f(v_0 + tv_1) - f(v_0 + sv_1)}{t - s}. \tag{0.2}$$

To speak of *topological* calculus, we shall assume that V, V' are topological vector spaces or modules over topological fields or rings $\mathbb{K}$, and U, U' are open. For the moment, let's

consider the "classical case" $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$. Then the following holds (cf. [BGN04, Be08]): *The map f is of class C^1 if, and only if, the difference quotient map $f^{[1]}$ extends continuously to a map defined on the set*

$$U^{[1]} := \Big\{(v_0, v_1; t, s) \in V^2 \times \mathbb{K}^2 \Big| \begin{array}{l} v_0 + tv_1 \in U \\ v_0 + sv_1 \in U \end{array} \Big\}. \tag{0.3}$$

If this is the case, we denote still by $f^{[1]} : U^{[1]} \to U'$ the extended map. Then the differential df of f is given by $f^{[1]}(v_0, v_1; 0, 0) = df(v_0)v_1$. Now, these conditions make perfectly sense for any "good" topological ring $\mathbb{K}$ and for maps defined on open locally linear sets, and thus can serve as *definition* of differentiability over $\mathbb{K}$ – the "topological differential calculus" thus defined has excellent functorial properties allowing to give a "purely algebraic" presentation of certain features of usual calculus (see [BGN04, Be11]). To understand the structure of formulae like (0.2) and (0.3), the following *way of talking* turns out to be useful:

- call $\mathbf{v} = (v_0, v_1)$ "space variables", with v_0 the "foot point" and v_1 the "direction" (in which we differentiate),
- call (t, s) "time variables", and t "target time", and s "source time",
- call (t, s) "regular", or "finite", if $t - s$ is invertible in $\mathbb{K}$, and "singular" or "infinitesimal" else, with $t - s = 0$ being the "most singular value",
- call $v_0 + sv_1$ the "source", and $v_0 + tv_1$ the "target evaluation point",
- for fixed (t, s), call $\alpha\big((v_0, v_1)\big) := v_0 + sv_1$ the "source map", and define the "target map" $\beta\big((v_0, v_1)\big) := v_0 + tv_1$.

The slogan summarizing topological calculus is: *the slope extends continuously (jointly in space and time variables) from finite to singular times*. The notable difference with [BGN04, Be11] is that here we shall use a *pair* of time parameters (t, s), instead of a single parameter t as in loc. cit. Although the expression (0.2) is of course symmetric under switch of target and source time, it will be important to distinguish "target" and "source". The setting of [BGN04, Be11] is gotten by restricting to $s = 0$ (we call this "target calculus"); symmetrically, the theory could also be formulated when letting $t = 0$ ("source calculus"). But now we can take advantage to define a third calculus, the "symmetric calculus", which corresponds to the case $t = -s$: then $v_0 = \frac{v_0+sv_1+v_0+tv_1}{2}$, so the footpoint is the midpoint of target and source evaluation point – see Subsection 2.5.[2]

[2] A price has to be paid: one will have to require that 2 be invertible in $\mathbb{K}$. Analysts won't bother, some algebraists might...

0.2 The underlying algebraic structure: anchor

In the second section we shall carve out the algebraic structures underlying topological differential calculus. As in general groupoid theory, the pair (α, β) given by source and target will be called *anchor map*[3]. We use the same term when considering the pair of time variables (t, s) as a "frozen parameter" (temporarily considered to be fixed); then we write (t, s) as lower index – for instance,

$$U^{[1]}_{(t,s)} := \{(v_0, v_1) \mid (v_0, v_1; t, s) \in U^{[1]}\}. \tag{0.4}$$

For fixed (t, s), we call again *anchor* the (linear) map sending the space variables $\mathbf{v} = (v_0, v_1)$ to the pair of evaluation points:

$$\Upsilon_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U, \quad \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_0 + sv_1 \\ v_0 + tv_1 \end{pmatrix} = \begin{pmatrix} \alpha(\mathbf{v}) \\ \beta(\mathbf{v}) \end{pmatrix}. \tag{0.5}$$

Of course, a choice is made here: the "first" component of $U \times U$ shall be associated with "source", and the "second" with "target". One of our concerns in the sequel will be to formalize the levels on which such choices are operated. Anyhow, by direct computation, the anchor is seen to be invertible if, and only if, $t - s$ is invertible, and then its inverse is given by

$$\Upsilon^{-1}_{(t,s)} : U \times U \to U^{[1]}_{(t,s)}, \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \frac{1}{t-s} \begin{pmatrix} t & -s \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{tx_0 - sx_1}{t-s} \\ \frac{x_1 - x_0}{t-s} \end{pmatrix}. \tag{0.6}$$

The first component is an affine combination $v_0 = \frac{s}{s-t}x_1 + \frac{t}{t-s}x_0$, and the second a "difference quotient". From this, comparing with (0.2), we see that $f^{[1]}_{(t,s)}$ is precisely the second component of the map $f^{\{1\}}_{(t,s)} := \Upsilon^{-1}_{(t,s)} \circ (f \times f) \circ \Upsilon_{(t,s)}$, given by

$$f^{\{1\}}_{(t,s)} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{tf(v_0+sv_1) - sf(v_0+tv_1)}{t-s} \\ \frac{f(v_0+tv_1) - f(v_0+sv_1)}{t-s} \end{pmatrix}. \tag{0.7}$$

The big advantage is that $f^{\{1\}}_{(t,s)}$ *depends functorially on* f: the "chain rule" simply reads $(g \circ f)^{\{1\}}_{(t,s)} = g^{\{1\}}_{(t,s)} \circ f^{\{1\}}_{(t,s)}$. Now we can reformulate the property of being $C^1_{\mathbb{K}}$ (Lemma 1.2): *The map $f : U \to U'$ is of class $C^1_{\mathbb{K}}$ if, and only if, for all $(t, s) \in \mathbb{K}^2$ there exists a continuous map $f^{\{1\}}_{(t,s)} : U^{\{1\}}_{(t,s)} \to (U')^{\{1\}}_{(t,s)}$, jointly continuous also in the parameter $(t, s) \in \mathbb{K}^2$, such that*

$$\Upsilon_{(t,s)} \circ f^{\{1\}}_{(t,s)} = (f \times f) \circ \Upsilon_{(t,s)} : \qquad \begin{array}{ccc} U_{(t,s)} & \xrightarrow{f^{\{1\}}_{(t,s)}} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \xrightarrow{f \times f} & U \times U \end{array} \tag{0.8}$$

[3] This map is indeed the anchor map of a groupoid structure, see Subsection 4.2.

In a nutshell, this diagram contains the essential ingredients needed for our approach: our aim is to translate diagram (0.8) into a "categorical" formulation, so that it will make sense in an abstract setting, not requiring topology any more. In a first step, we generalize this diagram at higher order $n \in \mathbb{N}$ (Theorem 1.8): indeed, differentiability at order n is characterized by a diagram of the same kind, replacing $f^{\{1\}}_{(t,s)}$, etc., by higher order maps $f^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}$, etc., where $(\mathbf{t},\mathbf{s}) = (t_1,\dots,t_n;s_1,\dots,s_n) \in \mathbb{K}^{2n}$. Technically, we work with 2^n-fold direct products, which have to be indexed by elements A of the *n-hypercube* $\mathcal{P}(\mathsf{n})$ (power set of $\mathsf{n} = \{1,\dots,n\}$).

0.3 The simple idea of a good functor from rings to sets

In order to formalize the idea that the extended domains and maps $(U^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}, f^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})})$ are scalar extensions $(U \otimes_{\mathbb{K}} \mathbb{A}, f \otimes_{\mathbb{K}} \mathbb{A})$, we look at the case $U = \mathbb{K}$. From functoriality, it follows that the spaces $\mathbb{K}^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}$ are in fact $\mathbb{K}$-*algebras*, which can easily be identified,

1. in terms of polynomial rings: they are polynomial algebras $\mathbb{K}[X_1,\dots,X_n]$, quotiented by the relations $(X_i - t_i)(X_i - s_i) = 0$, for $i = 1,\dots,n$,
2. in terms of tensor products: they are n-fold tensor products of "first order algebras" $\mathbb{K}_{(t_1,s_1)} \otimes \dots \otimes \mathbb{K}_{(t_n,s_n)}$.

The second item shows that the collection of these algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ forms a *small monoidal category* with respect to the tensor product, where we define morphisms to be given by left or right multiplications coming from the monoid structure. This is the *category* $\mathbf{talg}_{\mathbb{K}}$ *of* $\mathbb{K}$*-tangent algebras*. Every such algebra admits an *anchor morphism* $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}$ to the *cube algebra* which is a direct product of copies of $\mathbb{K}$, indexed by the n-hypercube $\mathcal{P}(\mathsf{n})$. We compute an explicit formula describing $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Theorem 2.8). This anchor morphism is an isomorphism if, and only if, $(\mathbf{t},\mathbf{s})$ is *regular*, and we give an explicit formula for the inverse morphism (Theorem 2.9).

Now, the "simple idea of a good functor from rings to sets" is to view "$\mathbb{K}$-smooth spaces" as functors $\underline{M}$ from the category $\mathbf{talg}_{\mathbb{K}}$ to the category of sets, satisfying certain conditions specified in Subsection 3.5, and "$\mathbb{K}$-smooth maps" as certain *natural transformations* between functors $\underline{M}$ and $\underline{M'}$, behaving in all respects like a family of "algebraic scalar extensions" $f \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}$. Indeed, in the framework of topological differential calculus, for a smooth map $f : M \to M'$, the family $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ satisfies these conditions, and thus "topological calculus" imbeds into "categorical calculus".

In order to fully justify such a functorial approach to differential calculus, one usually requires in SDG that the model be *well-adapted*, that is, that we obtain a *full and faithful* imbedding of a "usual" category of differential calculus into the "functorial" one. We show that, for our setting, this is indeed the case (Theorem 3.11). The proof is much

easier than the one of analogs in SDG, because, in essence, the whole setting is designed for such a theorem to hold: it is merely the translation of Theorem 1.8 into a more abstract language.

0.4 Further topics

The aim of this work is to lay the basic framework for a purely categorical approach to calculus over general (commutative) base rings. In Section 4 we briefly indicate further questions and topics to be studied in this context: to study *natural transformations* in the sense of [KMS93], we have to introduce further morphisms in our categories, and in particular those arising via the natural *(higher order) groupoid structure* that exists on the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. Very likely, a good understanding requires to understand also the *full* iteration procedure, and not only the restricted one used here, so to include, for instance, also the *simplicial calculus* from [Be13]. Finally, we conjecture that, replacing the usual braiding of tensor products by the braiding defining the *graded tensor product*, the present approach will also turn out to be useful in a categorical approach to *super-calculus*.

Acknowledgment. Part of these results should have been presented at the CIMPA spring school "Lie groupoids and algebroids", which had to be cancelled due to the Covid-19 crisis. We thank the organisers for their work, and we hope that the school will take place soon after the end of this crisis. We also thank Alain Genestier for helpful discussions and the referee for his careful reading and useful comments on the manuscript.

Notation. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathsf{n} = \{1, 2, \ldots, n\}$. Categories are denoted in boldface characters: small letters for small categories, such as $\mathbf{talg}_{\mathbb{K}}$, and capital letters for large categories, such as $\mathbf{Sets}$ (category of sets). The letter $\mathbf{Fn}$ stands for "functor category", so $\mathbf{Fn}(\mathbf{c}, \mathbf{Sets}) = \mathbf{Sets}^{\mathbf{c}}$ is the category of (covariant) functors from a (small) category $\mathbf{c}$ to $\mathbf{Sets}$. Throughout, $\mathbb{K}$ is a commutative base ring with unit 1.

1. Topological differential calculus

In differential calculus, one usually is mostly interested in the *morphisms*, that is, in *maps of class C^n*. However, let us first say some words about the *objects*:

1.1 Locally linear sets, and the anchor

A *locally linear set* is a pair (U, V), where V is a $\mathbb{K}$-module, and $U \subset V$ a non-empty subset. We define the set $U^{[1]}$ by (0.3), and the *(full) anchor* by

$$\Upsilon : U^{[1]} \to (U \times \mathbb{K})^2, \quad (v_0, v_1; t, s) \mapsto (v_0 + sv_1, s; v_0 + tv_1, t). \tag{1.1}$$

When time parameters $(t,s) \in \mathbb{K}^2$ are fixed, we define $U_{(t,s)} := U^{[1]}_{(t,s)} := U^{\{1\}}_{(t,s)}$ by (0.4), and the *(restricted) anchor*

$$\Upsilon_{(t,s)} := \Upsilon^{\{1\}}_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U \tag{1.2}$$

is given by restricting the map Υ defined above, i.e., it is given by (0.5). Direct computation shows that $\Upsilon_{(t,s)}$ is invertible iff $s - t$ is invertible in $\mathbb{K}$, with inverse given by (0.6). Note that $(U^{[1]}_{(t,s)}, V^2)$ is again a locally linear set, and hence the construction can be iterated, with some new parameter (t_2, s_2), and so on. Explicit formulae, describing this, will be given later (restricted iteration, Def. 1.5).

1.2 The topological setting

In the remainder of this section we assume that $\mathbb{K}$ is a *good topological ring* (i.e., a topological ring whose unit group $\mathbb{K}^\times$ is open and dense, and inversion is a continuous map), that all $\mathbb{K}$-modules are topological modules, and that all locally linear sets (U, V), $(U', V'), \ldots$ are *open* inclusions.

Definition 1.1. *We say that $f : U \to V'$ is* of class $C^{\mathbb{K}}_1$ *if the* slope *given by (0.2) extends to a* continuous *map $f^{[1]} : U^{[1]} \to V'$. We then define, for all $(x, v) \in U \times V$,*

$$df(x)v := \partial_v f(x) := f^{[1]}(x, v; 0, 0).$$

Remark 1.1. Letting $s = 0$, the preceding definition clearly implies that f is of class $C^1_{\mathbb{K}}$ in the sense of [BGN04] or [Be08]. Conversely, the map denoted here by $f^{[1]}$ can be expressed by the one denoted $f^{[1]}$ in loc. cit., and hence the $C^1_{\mathbb{K}}$-notions used there are equivalent to the one given above. We call the calculus obtained by restricting to $s = 0$ *target calculus*. Recall from [BGN04] that, in the real or complex finite dimensional case this definition is equivalent to all usual ones, and in the infinite dimensional locally convex case it is equivalent to Keller's definition of differentiability.

Lemma 1.2. *For a map $f : U \to U'$, the following are equivalent:*

1. *f is $C^1_{\mathbb{K}}$,*
2. *for all $(t,s) \in \mathbb{K}^2$, there exists a (unique) map $f_{(t,s)} = f^{\{1\}}_{(t,s)} : U_{(t,s)} \to U'_{(t,s)}$, such that*
   (a) *the map $U^{[1]} \to (U')^{[1]}$, $(x, v; t, s) \mapsto f_{(t,s)}(x, v)$ is continuous,*
   (b) *for all $(t,s) \in \mathbb{K}^2$,*

$$\Upsilon_{(t,s)} \circ f^{\{1\}}_{(t,s)} = (f \times f) \circ \Upsilon_{(t,s)} : \quad \begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \xrightarrow{f \times f} & U' \times U' \end{array}$$

Proof. As we have already seen, when $t - s$ is invertible in $\mathbb{K}$, then $f_{(t,s)}$ is necessarily given by (0.7). Since its second component is the slope $f^{[1]}$, existence of $f_{(t,s)}$, jointly continuous in $(x, v; t, s)$, implies existence of a continuous extension of the slope, whence (2) $\Rightarrow$ (1). To prove the converse, assume (1) and write $f_{(t,s)}(x, v) = (w_0, w_1)$ with (w_0, w_1) given by (0.7). Assumption (1) means that $w_1 = w_1(x, v; t, s)$ admits a continuous extension. Let us show that $w_0 = w_0(x, v; t, s)$ also admits a continuous extension. To see this, let $x_0 := f(x + sv)$ and $x_1 := f(x + tv)$. Then $x_0 = w_0 + sw_1$, $x_1 = w_0 + tw_1$, whence

$$w_0 = x_1 - tw_1 = f(x + tv) - tf^{[1]}(x, v; t, s),$$

showing that $w_0(x, v)$ extends continuously for all (t, s) if so does $f^{[1]}(x, v; t, s)$. □

Example 1.1. If $f : V \to V'$ is *linear* and continuous, then direct computation using (0.7) shows that $f_{(t,s)}(v_0, v_1) = (f(v_0), f(v_1))$, so f is $C^1_{\mathbb{K}}$.

Remark 1.2. Letting $v_1 = 0$ in (0.7), we always get $f_{(t,s)}(v_0, 0) = (f(v_0), 0)$. In diagrammatic form, the map f itself imbeds into $f_{(t,s)}$: we define the imbedding

$$\iota_{(t,s)} : U \to U_{(t,s)}, \quad v_0 \mapsto (v_0, 0) \tag{1.3}$$

then the computation just mentioned shows that $f_{(t,s)} \circ \iota_{(t,s)} = \iota_{(t,s)} \circ f$:

$$\begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \iota \uparrow & & \uparrow \iota \\ U & \xrightarrow{f} & U \end{array} \tag{1.4}$$

Note that $\Upsilon \circ \iota$ is the diagonal imbedding $\Delta : U \to U \times U$, $x \mapsto (x, x)$.

In this setting, the usual rules of first order calculus hold (chain rule, product rule, linearity of first differential) – for a systematic exposition we refer to [BGN04, Be08, Be11]. Most important for our purposes is the Chain Rule, which we write in functorial form

$$\forall (t, s) \in \mathbb{K}^2 : \quad (g \circ f)_{(t,s)} = g_{(t,s)} \circ f_{(t,s)}. \tag{1.5}$$

This follows easily from Lemma 1.2: for invertible $t - s$, we have functoriality $(g \times g) \circ (f \times f) = (g \circ f) \times (g \circ f)$, and for general (t, s), it follows "by density".

1.3 Full versus restricted iteration

Higher order differentiability is defined by iterating first order differentiability. However, there are various ways of doing so, and it is important to distinguish them. In [BGN04], f is defined to be of class $C^2_{\mathbb{K}}$ if it is C^1 and if $f^{[1]}$ also is C^1, so that we can define $f^{[2]} := (f^{[1]})^{[1]}$, etc.:

Definition 1.3 (Full iteration)**.** *We say that f is* of class $C^n_{\mathbb{K}}$ *if: f is of class $C^1_{\mathbb{K}}$, and $f^{[1]}$ is of class $C^{n-1}_{\mathbb{K}}$. In this case we let $f^{[n]} := (f^{[1]})^{[n-1]}$.*

Remark 1.3. In the real or complex finite dimensional case this is equivalent to the usual definitions (see [BGN04, Be11]). However, since full iteration repeats the procedure for all variables together, the number of variables exploses, and it is hard to get control over the structure of the maps $f^{[n]}$ (see [Be15b]). To reduce the number of variables, in *restricted iteration* we consider in each step time variables to be frozen, and take difference quotients only with respect to space variables.

Notation. For each $k \in \mathbb{N}$, we denote by an upper index $\{k\}$ a copy of the procedure $\{1\}$ that has been defined above. An upper index $\{i,j\}$ $(i<j)$ indicates a double application of the procedure (first $\{i\}$, then $\{j\}$), etc. E.g., an upper index $\mathsf{n} := \{1,\dots,n\}$ indicates that we first apply $\{1\}$, then $\{2\}$, etc., and finally $\{n\}$.

To abbreviate, in the sequel, we let $(\mathbf{t},\mathbf{s}) = (t_1,\dots,t_n;s_1,\dots,s_n) \in \mathbb{K}^{2n}$.

Definition 1.4 (Restricted iterated domain)**.** *For $U \subset V$, define $U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \subset V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ by*

$$U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := U^{\{1,\dots,n\}}_{(\mathbf{t},\mathbf{s})} := (\dots(U^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\dots)^{\{n\}}_{(t_n,s_n)} = (U^{\{1\}}_{(t_1,s_1)})^{\{2,\dots,n\}}_{(t_2,\dots,t_n,s_2,\dots,s_n)}.$$

Note that $V_{(t_i,s_i)} \cong V^2$, so $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \cong V^{(2^n)}$.

Definition 1.5 (Restricted iteration)**.** *A map $f : U \to U'$ is called* of class $C_{\mathbb{K},n}$ *if: it is of class $C^1_{\mathbb{K}}$, and, for all $(t_1,s_1) \in \mathbb{K}^2$, the map $f^{\{1\}}_{(t_1,s_1)}$ is of class $C_{\mathbb{K},n-1}$. In this case we define inductively*

$$f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := (f^{\{1\}}_{(t_1,s_1)})^{\{2,\dots,n\}}_{(t_2,\dots,t_n,s_2,\dots,s_n)} = (\dots(f^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\dots)^{\{n\}}_{(t_n,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

We also require that $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ be jointly continuous both in space and in time variables.

Theorem 1.6. *When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and V is a locally convex topological vector space, then the conditions $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ are both equivalent to the usual (Keller's) definition of C^n-maps.*

Proof. As already mentioned, $C^n_{\mathbb{K}}$ clearly implies $C_{\mathbb{K},n}$. Equivalence of $C^n_{\mathbb{K}}$ with Keller's definition has been proved in [BGN04]. On the other hand, $C_{\mathbb{K},n}$ obviously implies Keller's C^n-definition, which arises simply by taking $(\mathbf{t},\mathbf{s}) = (0,\dots,0)$ in the $C_{\mathbb{K},n}$-condition. Thus all three conditions are equivalent. □

Remark 1.4. For general $\mathbb{K}$, properties $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ cease te be equivalent: in *positive characteristic*, condition $C^n_{\mathbb{K}}$ appears to be strictly stronger than $C_{\mathbb{K},n}$ (cf. the proof of the general Taylor formula in [BGN04, Be11], which really uses *full* iteration; concerning this item, cf. also [Be13]). It would be interesting to have a criterion allowing to decide when $C^n_{\mathbb{K}}$ and $C_{n,\mathbb{K}}$ are equivalent.

Definition 1.7. *For all* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*, the* n-th order anchor *of* $U \subset V$ *is defined as follows: for all locally linear sets* $(U,V),(U',V')$*, we consider the map*

$$(U \times U')_{(t,s)} \to U_{(t,s)} \times U'_{(t,s)}, \quad ((v_0,v_0'),(v_1,v_1')) \mapsto ((v_0,v_1),(v_0',v_1'))$$

as identification. Under such identifications, the map $\Upsilon := \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} :=$

$$(\Upsilon^{\{1\}}_{(t_1,s_1)})^{\{2,\dots,n\}}_{(t_2,\dots,t_n,s_2,\dots,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U^{\{1\}}_{(t_1,s_1)})^{\{2,\dots,n\}}_{(t_2,\dots,t_n,s_2,\dots,s_n)} \times (U^{\{1\}}_{(t_1,s_1)})^{\{2,\dots,n\}}_{(t_2,\dots,t_n,s_2,\dots,s_n)}$$

inductively gives rise to a map $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to U^{2^n}$ *which we call the* n-fold anchor.

Remark 1.5. In order to fully formalize this definition, we need an explicit labelling of the 2^n copies of U in U^{2^n}. For the moment, this is not needed, and will be taken up later (Def. 2.7). Let us, however, give the result for $n = 2$: space variables have labels $0,1,2,12$ corresponding to the subsets of $\{1,2\}$, so we write $\mathbf{v} = (v_0,v_1,v_2,v_{12}) \in U^{\{1,2\}}_{(t_1,t_2,s_1,s_2)}$. Then iteration shows that the linear map Υ is given by the (block) matrix (Kronecker product of two first-order anchors)

$$\begin{pmatrix} 1 & s_1 \\ 1 & t_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & s_2 \\ 1 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & s_1 & s_2 & s_1s_2 \\ 1 & t_1 & s_2 & t_1s_2 \\ 1 & s_1 & t_2 & s_1t_2 \\ 1 & t_1 & t_2 & t_1t_2 \end{pmatrix}, \tag{1.6}$$

so we have four "evaluation points" given by the four lines of the (block) matrix:

$$\begin{array}{rcl} \Upsilon_\emptyset(\mathbf{v}) &=& v_\emptyset + s_1v_1 + s_2v_2 + s_1s_2v_{12}, \\ \Upsilon_1(\mathbf{v}) &=& v_\emptyset + t_1v_1 + s_2v_2 + t_1s_2v_{12}, \\ \Upsilon_2(\mathbf{v}) &=& v_\emptyset + s_1v_1 + t_2v_2 + s_1t_2v_{12}, \\ \Upsilon_{12}(\mathbf{v}) &=& v_\emptyset + t_1v_1 + t_2v_2 + t_1t_2v_{12}. \end{array} \tag{1.7}$$

The inverse matrix of (1.6) is the Kronecker product of the inverses of the respective first order anchors (when these are invertible): it is given by

$$\frac{1}{t_1-s_1}\begin{pmatrix} t_1 & -s_1 \\ -1 & 1 \end{pmatrix} \otimes \frac{1}{t_2-s_2}\begin{pmatrix} t_2 & -s_2 \\ -1 & 1 \end{pmatrix} = \frac{1}{(\mathbf{t}-\mathbf{s})_2}\begin{pmatrix} t_1t_2 & -s_1t_2 & -t_1s_2 & s_1s_2 \\ -t_2 & t_2 & s_2 & -s_2 \\ -t_1 & s_1 & t_1 & -s_1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1.8}$$

where $(\mathbf{t}-\mathbf{s})_2 := (t_1-s_1)(t_2-s_2)$. For the general case, see Theorem 2.9.

Theorem 1.8. *For a map* $f : U \to U'$*, the following are equivalent:*

1. f *is* $C_{\mathbb{K},n}$,

2. *for all* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$, *there exists a (unique) map* $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, *such that*

   (a) $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v})$ *is jointly continuous in space and time variables* $(\mathbf{v};\mathbf{t},\mathbf{s})$,

   (b) *for all* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$, $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{2^n} \circ \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$:

$$\begin{array}{ccc} U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} & \xrightarrow{f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} & (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ \Upsilon^{\mathsf{n}}_{\mathbf{t},\mathbf{s}} \downarrow & & \downarrow \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ U^{2^n} & \xrightarrow{f^{2^n}} & (U')^{2^n}. \end{array}$$

The map $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *depends functorially on* f: $(f \circ g)^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ g^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Chain Rule).

Proof. By induction: for $n = 1$, this is Lemma 1.2. Assume the claim holds on level $n-1$ and apply it to f replaced by $f^{\{1\}}_{(t_1,s_1)}$. From the inductive definitions, it follows readily that the properties are again equivalent on level n. The (higher order) Chain Rule now also follows by induction. □

Example 1.2. Using Formula (1.8), let us give explicit formulae for $n = 2$:

$$f^2_{(t_1,t_2,s_1,s_2)}(\mathbf{v}) = \Upsilon^{-1}\big(f(\Upsilon_\emptyset(\mathbf{v})), f(\Upsilon_1(\mathbf{v})), f(\Upsilon_2(\mathbf{v})), f(\Upsilon_{12}(\mathbf{v}))\big)$$

$$= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathbf{2}}} \begin{pmatrix} t_1t_2f(\Upsilon_\emptyset\mathbf{v}) - s_1t_2f(\Upsilon_1\mathbf{v}) - t_1s_2f(\Upsilon_2\mathbf{v}) + s_1s_2f(\Upsilon_{12}\mathbf{v}) \\ -t_2f(\Upsilon_\emptyset\mathbf{v}) + t_2f(\Upsilon_1\mathbf{v}) + s_2f(\Upsilon_2\mathbf{v}) - s_2f(\Upsilon_{12}\mathbf{v}) \\ -t_1f(\Upsilon_\emptyset\mathbf{v}) + s_1f(\Upsilon_1\mathbf{v}) + t_1f(\Upsilon_2\mathbf{v}) - s_1f(\Upsilon_{12}\mathbf{v}) \\ f(\Upsilon_\emptyset\mathbf{v}) - f(\Upsilon_1\mathbf{v}) - f(\Upsilon_2\mathbf{v}) + f(\Upsilon_{12}\mathbf{v}) \end{pmatrix} \tag{1.9}$$

Since $(\mathbf{t}-\mathbf{s})_{\mathbf{2}} = (t_1 - s_1)(t_2 - s_2)$, the first term is in fact an affine combination of values of f at the four evaluation points, whereas the other three terms are "zero-sum combinations" of these values, and hence correspond to "true" difference quotients. In order to state results at arbitrary order, we need some notation:

1.4 Hypercube notation, and formula for higher order slopes

Definition 1.9. *We call* n-hypercube *the power set* $\mathcal{P}(\mathsf{n}) = \mathcal{P}(\{1,\dots,n\})$. *It serves as index set for space variables, which we write in the form* $\mathbf{v} = (v_A)_{A\in\mathcal{P}(\mathsf{n})}$. *Recall that* $\mathcal{P}(\mathsf{n})$ *is a semigroup for union* $\cup$ *and intersection* $\cap$, *and a group with respect to the* symmetric difference

$$A\Delta B = (A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (B \cap A^c),$$

where $A^c = \mathsf{n} \setminus A$ *is the complement of* A *in* n. *Recall also that* $A^c\Delta B^c = A\Delta B$, *and that* $A\Delta B^c = (A\Delta B)^c = A^c\Delta B$, *whence* $|A\Delta B^c| = n - |A\Delta B|$.

Definition 1.10. *For all* $\mathbf{t}, \mathbf{s} \in \mathbb{K}^n$ *and* $A \in \mathcal{P}(\mathsf{n})$*, we let* $\mathbf{t}_\emptyset = 1 = \mathbf{s}_\emptyset$*, and*

$$\mathbf{t}_A = \prod_{k\in A} t_k, \qquad \mathbf{s}_A = \prod_{k\in A} s_k, \qquad (\mathbf{t}-\mathbf{s})_A = \prod_{k\in A}(t_k - s_k).$$

Call $(\mathbf{t}, \mathbf{s})$ regular*, or* finite*, if,* $\forall i = 1, \dots, n : (t_i - s_i) \in \mathbb{K}^\times$*, and* singular *if* $\forall i = 1, \dots, n : (t_i - s_i) \notin \mathbb{K}^\times$*, and* mixed *else.*

Theorem 1.11. *Let* $f : U \to U'$ *be of class* $C_{\mathbb{K},n}$*. Then, for all regular* $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$*, and all* $B \in \mathcal{P}(\mathsf{n})$*, the component* $(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B$ *is given by*

$$(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{A\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} \mathbf{s}_{B^c\cap A}\mathbf{t}_{B^c\cap A^c}\, f\Big(\sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} v_C\Big).$$

The proof will be given in Subsection 2.4. For $B = \emptyset$, the component is an affine combination of values of f at the 2^n evaluation points, and for all other components it is again a "zero sum combination".

1.5 Categories of locally linear sets and $C_{\mathbb{K},n}$-maps

To summarize, we have defined a *category of locally linear sets and their morphisms*:

Definition 1.12. *We denote by* $\mathbf{Llin}_{\mathbb{K},n}$ *the category whose objects are pairs* (U, V)*, where* V *is a topological* $\mathbb{K}$*-module and* $U \subset V$ *a non-empty open subset, and morphisms are* $C_{\mathbb{K},n}$*-maps* $f : U \to U'$*. (For* $n = 0$*, morphisms are continuous maps, and for* $n = \infty$*, these are maps that are* $C_{\mathbb{K},n}$ *for all* $n \in \mathbb{N}$*.)*

Definition 1.13. *For* $m \geq n$ *and* $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$*, the* $(n; \mathbf{t}, \mathbf{s})$-tangent functor *is the functor from* $\mathbf{Llin}_{\mathbb{K},m}$ *to* $\mathbf{Llin}_{\mathbb{K},m-n}$ *given by* $(U, V) \mapsto (U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}, V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})})$ *and* $f \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$*.*

Remark 1.6 (Manifolds). By the usual glueing procedures, one may now define $C_{\mathbb{K},n}$*-manifolds over* $\mathbb{K}$, modelled on locally linear sets – since these methods are independent of the particular form of differential calculus, we do not wish to go here into details (see [Be16] for a formulation of such principles, adapted to most general contexts). The $(n; \mathbf{t}, \mathbf{s})$-tangent functor then carries over to manifolds : for every $\mathbb{K}$-smooth manifold M we have a "generalized higher order tangent bundle" $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, depending functorially on M, and coming with an anchor map $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to M^{2^n}$.

2. The rings of calculus: tangent algebras

Our next aim is to understand the $(n; \mathbf{t}, \mathbf{s})$-tangent functor as a *functor of scalar extension*, from $\mathbb{K}$ to a ring denoted by $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, and which we shall define next.

2.1 The scaloid, and the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

The scaloid is the index set that will be used in the following construction of *tangent algebras*:

Definition 2.1. *We call* scaloid *the* free monoid over $\mathbb{K}^2$*, that is, the disjoint union over* $n \in \mathbb{N}_0$ *of all* $\mathbb{K}^{2n}$*:*

$$\mathbf{scal} := \mathbf{scal}_{\mathbb{K}} := \coprod_{n\in\mathbb{N}_0} \mathbb{K}^{2n}$$

(in the following, we write $(\mathbf{t},\mathbf{s})$ *with* $\mathbf{t},\mathbf{s} \in \mathbb{K}^n$ *for elements of* $\mathbb{K}^{2n}$*), together with its monoid structure given by juxtaposition, and denoted by*

$$(\mathbf{t},\mathbf{s}) \oplus (\mathbf{t}',\mathbf{s}') = (t_1,\ldots,t_n,t'_1,\ldots,t'_m;s_1,\ldots,s_n,s'_1,\ldots,s'_m) = (\mathbf{t}\oplus\mathbf{t}',\mathbf{s}\oplus\mathbf{s}')\ .$$

We denote by $\mathbb{K}[X_1,\ldots,X_n]$ the algebra of polynomials in n variables with coefficients in $\mathbb{K}$. It can be defined inductively by using the isomorphisms, where $\otimes_{\mathbb{K}}$ denotes the tensor product of two associative $\mathbb{K}$-algebas,

$$\mathbb{K}[X_1,X_2] \cong (\mathbb{K}[X_1])[X_2] \cong \mathbb{K}[X_1]\otimes_{\mathbb{K}}\mathbb{K}[X_2], \tag{2.1}$$

so, by induction, we have an iterated tensor product of algebras

$$\mathbb{K}[X_1,\ldots,X_n] \cong \mathbb{K}[X_1]\otimes_{\mathbb{K}}\ldots\otimes_{\mathbb{K}}\mathbb{K}[X_n]. \tag{2.2}$$

Definition 2.2. *For* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*, we define the* $(\mathbf{t},\mathbf{s})$-tangent algebra

$$\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := \mathbb{K}[X_1,\ldots,X_n]/((X_i-t_i)(X_i-s_i), i=1,\ldots,n)$$

(quotient by the ideal $I_{(\mathbf{t},\mathbf{s})}$ *generated by all* $(X_i-t_i)(X_i-s_i), i=1,\ldots,n$*).*

Lemma 2.3. *The algebra* $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *is a free* $\mathbb{K}$*-module of dimension* 2^n*, having a canonical basis indexed by elements* A *of the* n*-cube* $\mathcal{P}(\mathsf{n})$*,*

$$e_A := [X^A], \qquad X^A = \prod_{k\in A} X_k.$$

It is also isomorphic to an n*-fold tensor product of first order tangent algebras* $\mathbb{K}^{\{i\}}_{(t_i,s_i)} = \mathbb{K}[X_i]/((X_i-s_i)(X_i-t_i))$*:*

$$\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \mathbb{K}^{\{1\}}_{(t_1,s_1)}\otimes\ldots\otimes\mathbb{K}^{\{n\}}_{(t_n,s_n)}.$$

Proof. For $n=1$, the claim is obviously true: a polynomial algebra $\mathbb{K}[X]$ quotiented by the ideal generated by a polynomial of degree 2 is of dimension 2, with $\mathbb{K}$-basis the classes $[1]$ and $[X]$. For $n>1$, the claim follows by induction using (2.1). □

Theorem 2.4. *Assume $\mathbb{K}$ is a good topological ring. Then the structure maps $+$ and $\cdot$ of the ring $\mathbb{K}$ are of class $C_{\mathbb{K},\infty}$, and applying n-fold restricted iteration with parameters $(\mathbf{t},\mathbf{s})$ yields a good topological ring which is canonically isomorphic to $\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ (whence in particular is a free $\mathbb{K}$-module of dimension 2^n)*

Proof. The structure maps are continuous and (bi)-linear, hence smooth (both in the full and restricted sense, cf. [BGN04]). By functoriality, and applying, concerning Cartesian products, the convention from Def. 1.7, rings are transformed by the iterated functors into rings. We have to show that the ring structure on the underlying set of $\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ is precisely the one defined above. For $n = 1$ and regular $(t_1, s_1) = (t, s)$, this follows from the explicit formulae for difference calculus : slightly more general, given a bilinear continuous map $\beta : V \times W \to Y$, thought of as a "product", so let us write $v \bullet w := \beta(v, w)$, we compute

$$\beta^{\{1\}}_{(t,s)} : V_{(t,s)} \times W_{(t,s)} \to Y_{(t,s)}, \quad (\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}) \mapsto \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

which by an explicit computation using Formula (0.7) is given by

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} v_0 \bullet w_0 - st\, v_1 \bullet w_1 \\ v_0 \bullet w_1 + v_1 \bullet w_0 + (s+t) v_1 \bullet w_1 \end{pmatrix}. \tag{2.3}$$

Now, decomposing the product of $\mathbb{K}^{\{1\}}_{(t,s)}$ according to the canonical basis $e_0 = [1], e_1 = [X]$, we get exactly the same formula, whence the claim for $n = 1$ and regular (t, s). By density, the claim follows for all (t, s), and by straightforward induction, using Lemma 2.3, it now follows for all elements $(\mathbf{t}, \mathbf{s})$ of the scaloid. Finally, by general argments ([Be08, Be11]), the ring $\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ is again "good". □

By exactly the same arguments we see also that the structure maps $V \times V \to V$ and $\mathbb{K} \times V \to V$ of a topological $\mathbb{K}$-module are smooth, and give by restricted iteration rise to the corresponding structure maps of the scalar-extended module $V^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})} = V \otimes_{\mathbb{K}} \mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$; also, if $f : V \to V'$ is *linear*, then $f^{\mathbf{n}}_{\mathbf{t},\mathbf{s}}$ coincides with the algebraic scalar extension $f \otimes \mathrm{id}_{\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}}$.

2.2 Source and target

Evaluation of a class $[P] \in \mathbb{K}[X]/((X - s)(X - t))$ at elements $x \in \mathbb{K}$ is in general not well-defined, but it is so for $x = s$ and $x = t$. Thus we get two algebra morphisms $\alpha, \beta : \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}$, called *source* and *target*

$$\alpha([P]) = P(s), \qquad \beta([P]) = P(t). \tag{2.4}$$

(Note that α is coupled with s and β with t, so the order of (s, t) matters.) With respect to the basis $e_0 = [1]$, $e_1 = [X]$, we have $\alpha(v_0 + v_1 e_1) = v_0 + s v_1$, $\beta(v_0 + v_1 e_2) = v_0 + t v_1$,

which is in keeping with the definitions in Subsection 0.2. In Appendix B we describe the structure of $\mathbb{K}^{\{1\}}_{(t,s)}$ in an intrinsic way, via α and β; this may be useful for a further structure theory, but is not directly needed in the sequel.

2.3 The anchor

Putting source and target together, the *first order anchor* is the algebra morphism defined by

$$\Upsilon^{\{1\}}_{(t,s)} : \mathbb{K}_{(t,s)} \to \mathbb{K} \times \mathbb{K}, \quad [P] \mapsto (\alpha(P), \beta(P)) = (P(s), P(t)).$$

Lemma 2.5. *The first order anchor is an isomorphism if, and only if,* (t, s) *is regular, i.e., iff* $t - s \in \mathbb{K}^\times$.

Proof. The $\mathbb{K}$-linear map $\Upsilon_{(t,s)}$ is bijective iff its determinant $t - s \in \mathbb{K}^\times$, see Subsection 0.2. □

Higher order anchors can be defined in two (equivalent) ways: either by evaluating (classes of) polynomials in several variables on a hypercube of evaluation points, or by tensoring first order anchors. Here we choose the latter approach. For this, we need some definitions and conventions:

Definition 2.6 (Hypercubic spaces and algebras)**.** *Let* $N \subset \mathbb{N}$ *be a finite subset of cardinal* n*. The* hypercubic space, based on N*, is by definition the free* $\mathbb{K}$*-module* $\mathbb{K}^{\mathcal{P}(N)}$ *of dimension* 2^n *of functions from* $\mathcal{P}(N)$ *to* $\mathbb{K}$*, with its canonical basis*

$$E_A = E^N_A : \mathcal{P}(N) \to \mathbb{K}, \qquad E_A(A) = 1,\ \forall B \neq A : E_A(B) = 0.$$

A hypercubic space carries several important algebra structures. When equipping $\mathbb{K}^{\mathcal{P}(N)}$ *with its* pointwise algebra structure*, i.e., considering it as the algebra of functions from* $\mathcal{P}(N)$ *to* $\mathbb{K}$*, so that the product of the canonical basis elements is*

$$E^N_A \cdot E^N_B = \delta_{A,B} E^N_A,$$

we say that $\mathbb{K}^{\mathcal{P}(N)}$ *is the* N-hypercube algebra*. When* $N = \mathsf{n} = \{1, \dots, n\}$*, we often omit the upper index* n*, and just speak of* the n-hypercube algebra*.*

Remark 2.1. See Appendix A for some basic facts about linear algebra on hypercubic spaces (independent of the algebra structure). For induction procedures, the following remark is useful: If N_1 and N_2 are disjoint subsets of $\mathbb{N}$, then

$$\mathcal{P}(N_1) \times \mathcal{P}(N_2) \to \mathcal{P}(N_1 \sqcup N_2), \quad (A, B) \mapsto A \cup B$$

is a bijection, whence we get an isomorphism (of modules, and of cube-algebras)

$$\mathbb{K}^{\mathcal{P}(N_1)} \otimes \mathbb{K}^{\mathcal{P}(N_2)} \cong \mathbb{K}^{\mathcal{P}(N_1 \sqcup N_2)}.$$

In particular, by induction, there is a canonical isomorphism

$$\mathbb{K}^{\mathcal{P}(\{1,\dots,n\})} \cong \mathbb{K}^{\mathcal{P}(\{1\})} \otimes \dots \otimes \mathbb{K}^{\mathcal{P}(\{n\})}.$$

Note that the neutral element of $\mathbb{K}^{\mathcal{P}(N)}$ is the function that is 1 everywhere, that is

$$1 = \sum_{A\in\mathcal{P}(N)} E^N_A.$$

Definition 2.7. *The* n-fold anchor *is the tensor product of* n *copies of the first order anchor: it is the algebra morphism*

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := \otimes_{i=1}^n \Upsilon^{\{i\}}_{(t_i,s_i)} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})},$$

where for each $k \in \mathbb{N}$, $\Upsilon^{\{k\}}_{(t_k,s_k)} : \mathbb{K}^{\{k\}}_{(t_k,s_k)} \to \mathbb{K}^{\mathcal{P}(\{k\})}$ *is a copy of the first order anchor. Thus, by definition,*

$$\Upsilon^{\{k\}}_{(t_k,s_k)}(e_\emptyset) = E^k_\emptyset + E^k_k, \qquad \Upsilon^{\{k\}}_{(t_k,s_k)}(e_k) = s_k E^k_\emptyset + t_k E^k_k.$$

For the categorical approach, it is not strictly necessary to have an explicit formula for the higher order anchor; however, such a formula allows to derive the explicit formula for the higher order slopes, and thus makes the whole procedure algorithmic and computable. Recall Formula (1.6) for the matrix of the second order anchor, which is the Kronecker product of two first-order anchors. Note that, when $s_1 = 1 = s_2$, then this matrix is a *symmetric matrix*, whereas for $t_1 = 1 = t_2$, this is not the case. Using notation introduced above, we generalize:

Theorem 2.8. *Fix* $n \in \mathbb{N}$, *and* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$. *With respect to the bases* $(e_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its domain and* $(E_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its range, the* n*-fold anchor is given by*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}\ e^*_A \otimes E_B.$$

In other terms, it is characterized by the following equivalent conditions:

1. $\Upsilon(e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} E_B$,
2. $\Upsilon(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \big(\sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} v_A\big) E_B$,
3. *the matrix of* Υ *with respect to these bases has coefficients*

$$\Upsilon_{(B,A)} := E^*_B(\Upsilon(e_A)) = \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}, \qquad (A,B) \in \mathcal{P}(\mathsf{n})^2.$$

In particular, in the symmetric case $\mathbf{s} = -\mathbf{t}$*, we have* $\Upsilon_{(B,A)} = (-1)^{|A\cap B|}\mathbf{s}_A$*, so*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(-\mathbf{s},\mathbf{s})} = \sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{s}_A \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|} e_A^* \otimes E_B.$$

Proof. This is the special case of Theorem A.1 for $\mathbf{a} = 1 = \mathbf{c}$, $\mathbf{b} = \mathbf{s}$, $\mathbf{d} = \mathbf{t}$. □

Next, to compute the inverse of the anchor, in the regular case, recall Formula (1.8) concerning the case $n = 2$. This generalizes as follows:

Theorem 2.9. *Fix* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*. Recall the notation* $(\mathbf{t}-\mathbf{s})_{\mathsf{n}} = \prod_{k=1}^n (t_k - s_k)$*. The anchor map* $\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *is invertible if, and only if,* $(\mathbf{t},\mathbf{s})$ *is regular, i.e.,* $t_k - s_k$ *is invertible for all* $k = 1,\dots,n$*, and then its inverse map is given by the formula*

$$\Upsilon^{-1} = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} (-1)^{|A\Delta B|} \mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\ E_A^*\otimes e_B.$$

Equivalently,

1. $\Upsilon^{-1}(E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|}\mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\ e_B$,
2. $\Upsilon^{-1}(\sum_{A\in\mathcal{P}(\mathsf{n})} y_A E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} y_A \mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\ e_B$.

In particular, in case $\mathbf{s} = -\mathbf{t}$*, we get (using* $(A\Delta B)\sqcup(A^c\cap B^c) = (A\cap B)^c$*)*

$$\Upsilon^{-1}(E_A) = \frac{1}{(-2)^n \mathbf{s}_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|}\mathbf{s}_{B^c}\, e_B.$$

Proof. This is a special case of Theorem A.2. □

2.4 The n-th order restriced slope map

Now we prove the already anounced formula from Theorem 1.11 for $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ when $(\mathbf{t},\mathbf{s})$ is regular. We decompose $\mathbf{v} \in V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V\otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ in the form $\mathbf{v} = \sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A$, and $\Upsilon(\mathbf{v}) = \sum_{A\in\mathcal{P}(\mathsf{n})} \Upsilon_A(\mathbf{v}) E_A$, with the 2^n *evaluation points* given by

$$\Upsilon_A(\mathbf{v}) = \sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} v_C.$$

Then

$$\begin{aligned} f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A\Big) &= \Upsilon^{-1}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} f\big(\Upsilon_A(\mathbf{v})\big)\Big) \\ &= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} e_B\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|}\mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\big(\Upsilon_A(\mathbf{v})\big)\Big) \\ &= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} e_B\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|}\mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\Big(\sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{C\cap A}\mathbf{s}_{C\cap A^c} v_C\Big)\Big). \end{aligned}$$

2.5 Target calculus, source calculus, and symmetric calculus

There are three special cases of calculus, as defined here, that deserve attention:

1. *target calculus*, obtained when $\mathbf{s} = \mathbf{0}$, i.e., $\forall i,\ s_i = 0$;
2. *source calculus*, obtained when $\mathbf{t} = \mathbf{0}$,
3. *symmetric calculus*, obtained when $\mathbf{s} = -\mathbf{t}$, i.e., $\forall i,\ s_i + t_i = 0$.

In these cases, the range of scaloid parameter reduces to $\mathbb{K}^n$ instead of $\mathbb{K}^{2n}$, and the relations satisfied by the canonical basis $(e_A)_{A\in\mathcal{P}(\mathsf{n})}$ are relatively simple:

1. *target calculus*, $e_i^2 = t_i e_i$, whence $e_A^2 = \mathbf{t}_A e_A$ and $e_A e_B = \mathbf{t}_{A\cap B} e_{A\cup B}$,
2. *source calculus*, same, with $\mathbf{s}$ instead of $\mathbf{t}$,
3. *symmetric calculus*, $e_i^2 = 4t_i^2$, so $e_A^2 = 4^{|A|}\mathbf{t}_A^2,\ e_A e_B = 4^{|A\cap B|}\mathbf{t}_{A\cap B}^2 e_{A\Delta B}$.

The "most singular value" is in all cases $\mathbf{t} = \mathbf{0} = \mathbf{s}$, whereas the "unit value" is

1. *target calculus*, "unit" $\mathbf{t} = \mathbf{1} = (1,\dots,1)$, $\mathbf{s} = \mathbf{0}$,
2. *source calculus*, "unit" $\mathbf{t} = \mathbf{0}$, $\mathbf{s} = \mathbf{1}$,
3. *symmetric calculus*, "unit" $\mathbf{t} = \mathbf{1}, \mathbf{s} = -\mathbf{1} = (-1,\dots,-1)$ (another convention would be to divide this by 2, if 2 is invertible in $\mathbb{K}$).

Thus, taking for $(\mathbf{t},\mathbf{s})$ the unit value, the algebra $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ with its canonical basis,

1. in *target calculus*, is the *semigroup algebra of the monoid* $(\mathcal{P}(\mathsf{n}),\cup)$,
2. idem in *source calculus*,
3. in *symmetric calculus*, after normalizing by division by 2, is the *group algebra of the group* $(\mathcal{P}(\mathsf{n}),\Delta)$ *with group law given by the symmetric difference* Δ.

In all three cases, the anchor, being a morphism to the multiplicative algebra of functions on $\mathcal{P}(\mathsf{n})$, plays the rôle of a *Fourier transform*. Namely, for $A \in \mathcal{P}(\mathsf{n})$, the linear form $E_A^* : \mathbb{K}^{\mathcal{P}(\mathsf{n})} \to \mathbb{K}$ is the A-projection, which is a *character*, i.e., an algebra morphism into the base ring. Thus the 2^n components of Υ,

$$\Upsilon_A := E_A^* \circ \Upsilon : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}, \quad x \mapsto \sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} x_C$$

also are characters (for $n = 1$, these are just the source and target projections; for $n \geq 1$, they can be considered as higher order versions of source and target maps). For instance,

when $\mathbf{t} = -\mathbf{s}$ is constant $\frac{1}{2}$, then from the explicit formula above we get all 2^n characters of the group $(\mathcal{P}(\mathsf{n}), \Delta)$ (for $A \in \mathcal{P}(\mathsf{n})$),

$$\Upsilon_A = \chi_A : \mathcal{P}(\mathsf{n}) \to \{\pm 1\}, \quad B \mapsto \chi_A(B) = (-1)^{|A \Delta B|} \tag{2.5}$$

Thus the matrix of Υ is the character table of the abelian group $(\mathcal{P}(\mathsf{n}), \Delta)$, which is also the matrix of the Fourier transform when identifying this group with its dual group.

3. The categorical approach

In the preceding section we have described how to define, starting with a $\mathbb{K}$-smooth function f, a family of functions $(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})})_{(\mathbf{t},\mathbf{s};n)\in \mathbf{scal}_{\mathbb{K}}}$, behaving well with tangent algebras, anchors, and their corresponding scalar extensions. In the present section, we describe an abstract, categorical setting capturing the main features of these constructions. The procedure is very much like the classical one, starting from polynomial functions, to define abstract polynomial rings. In general, one cannot recover all abstract polynomials by polynomial functions; for this we need assumptions on $\mathbb{K}$ (e.g., of topological nature).

3.1 The small monoidal categories in question

Let $\mathbf{c}$ be a monoid, with "product" denoted by $\oplus$ and neutral element 0. It gives rise to a small category that shall also be denoted by $\mathbf{c}$: its objects are elements $t \in \mathbf{c}$, and morphisms are given by compositions of left- and right multiplications in the monoid, i.e., of the form

$$t \to t_1 \oplus t \oplus t_2, \quad t, t_1, t_2 \in \mathbf{c}.$$

The monoids we are interested in will all be *left and right cancellative*, that is, $t \oplus s = t' \oplus s \Rightarrow t = t'$ and $t \oplus s = t \oplus s' \Rightarrow s = s'$; thus the small category $\mathbf{c}$ is *skeletal* in the sense of [CWM], p. 93: two objects are isomorphic iff they are equal. Now, here are the cases we are interested in:

1. The monoid $\mathbb{N}_0$ with its usual addition, and neutral element 0.
2. Recall from Definition 2.1 that objects of the scaloid $\mathbf{scal}_{\mathbb{K}}$ are elements $(\mathbf{t}, \mathbf{s})$ of the free monoid over $\mathbb{K}^2$. The neutral element is the empty word. Morphisms are now defined as above.
3. The *small category of* $\mathbb{K}$*-tangent algebras* $\mathbf{talg}_{\mathbb{K}}$ has objects the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ defined in Def. 2.2, together with their label $(\mathbf{t}, \mathbf{s})$. The monoidal structure is given by the tensor product of associative $\mathbb{K}$-algebras, which now serves to define also the morphisms in this category. The neutral element is $\mathbb{K}$, labelled by the empty word.

Lemma 3.1. *The small monoidal categories* $\mathbf{talg}_{\mathbb{K}}$ *and* $\mathbf{scal}_{\mathbb{K}}$ *are isomorphic (in the sense defined in [CWM], p. 92): under this bijection,* $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *corresponds to* $(\mathbf{t},\mathbf{s})$.

Proof. By the definitions given above, the map $\mathbf{talg}_{\mathbb{K}} \to \mathbf{scal}_{\mathbb{K}}$ is well-defined, its inverse map is $(\mathbf{t},\mathbf{s}) \mapsto \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. As we have seen in Lemma 2.3, this bijection then is an isomorphism of monoids. □

Lemma 3.2. *The "length" or "degree" map* $\ell : \mathbf{scal}_{\mathbb{K}} \to \mathbb{N}_0$*, associating to each word its length, is a monoid morphism, and defines a functor of monoidal categories.*

Proof. Obviously, ℓ is a morphism, and by routine computation such a morphism induces a morphism (functor) of the corresponding monoidal categories. □

3.2 Functor categories

Next we consider *functor categories*. We mostly follow notations and conventions from [CWM, MM92]. Thus, we denote by $\mathbf{Sets}$ the (large) category of sets and set-maps, and (following notation from [MM92], p. 25) by $\mathbf{Sets}^{\mathbf{2}}$ the (large) category of *anchored sets*, that is, objects (M,γ,M') are maps $\gamma : M \to M'$, where morphisms are *anchor-compatible pairs of maps* $\Phi : M \to N$, $\Phi' : M' \to N'$, i.e. $\gamma_N \circ \Phi = \Phi' \circ \gamma_M$.

Functors from a category C to a category B, together with their natural transformations, form a *functor category* $\mathbf{Fn}(C,B) = B^C$ (see e.g. [CWM], II.4, or [MM92]). Specifically, for $\mathbf{c}$ one of the small monoidal categories mentioned above, we are interested in functor categories $\mathbf{Fn}(\mathbf{c},\mathbf{Sets}) = \mathbf{Sets}^{\mathbf{c}}$ or $\mathbf{Fn}(\mathbf{c},\mathbf{Sets}^2)$. If $\underline{M} : \mathbf{c} \to \mathbf{Sets}$ is a functor, then for every object $a \in \mathbf{c}$ we write $M_a := \underline{M}(a)$ (the set obtained by applying $\underline{M}$ to a), and for every morphism $\phi : a \to b$ of $\mathbf{c}$, we write $M_\phi : M_a \to M_b$ for the induced set-map. Likewise, for each natural transformation $\underline{f} : \underline{M} \to \underline{N}$, we write $f_a : M_a \to N_a$ for the corresponding set-map from $\underline{M}(a)$ to $\underline{N}(a)$. The compatibility condition then is

$$\forall \phi : a \to b,\ \forall \underline{f} : \quad N_\phi \circ f_a = f_b \circ M_\phi.$$

Composition of natural transformations is defined "pointwise", i.e., for two laws $\underline{f} : \underline{M} \to \underline{N}$, $g : \underline{N} \to \underline{P}$ and all objects a of $\mathbf{c}$, we have $(\underline{g} \circ \underline{f})_a := g_a \circ f_a : M_a \to P_a$.

Definition 3.3. *For each object* a *of* $\mathbf{c}$*,* evaluation at level a*, defined by*

$$\mathrm{ev}_a : \underline{M} \mapsto M_a,\ \underline{f} \mapsto f_a,$$

is a functor from $\mathbf{Fn}(\mathbf{c},\mathbf{Sets})$ *to* $\mathbf{Sets}$*. In particular, when* $\mathbf{c}$ *is monoidal with neutral element* 0*, we call simply* evaluation *the evaluation* ev_0 *at* 0*.*

In the following, our concern will be to define ("extension") functors that go in the direction opposite to $\mathrm{ev}_0 : \mathbf{Fn}(\mathbf{c},\mathbf{Sets}) \to \mathbf{Sets}$.

3.3 Cubic extensions of sets.

For each set M and $n \in \mathbb{N}$, we have a hypercube of sets $M^{\mathcal{P}(\mathsf{n})} \cong M^{2^n}$. This gives rise to a "cubic extension functor":

Lemma 3.4. *Let us define*

$$\iota : \mathbf{Sets} \to \mathbf{Sets}^{\mathbb{N}_0}, \quad \begin{array}{rcl} M \mapsto \underline{\underline{M}} & := & (0 \mapsto M,\, n \mapsto M^{\mathcal{P}(\mathsf{n})}) \\ f \mapsto \underline{\underline{f}} & := & (0 \mapsto f,\, n \mapsto f^{\mathcal{P}(\mathsf{n})}) \end{array}.$$

Then $\underline{\underline{M}} : \mathbb{N}_0 \to \mathbf{Sets}$ is a functor, and (for $f : M \to N$), $\underline{\underline{f}} : \underline{\underline{M}} \to \underline{\underline{N}}$ is a natural transformation, and ι is a functor from $\mathbf{Sets}$ to $\mathbf{Fn}(\mathbb{N}_0, \mathbf{Sets})$ such that $\mathrm{ev}_0 \circ \iota = I_{\mathbf{Sets}}$ is the identity functor on $\mathbf{Sets}$.

Proof. The main point is to see that $\underline{\underline{M}}$ is a functor. Indeed, this follows from the identifications $(M^A)^B = M^{A\times B}$ together with $\mathcal{P}(\mathsf{n}+\mathsf{m}) = \mathcal{P}(\mathsf{n}) \times \mathcal{P}(\mathsf{m})$:

$$M^{\mathcal{P}(\mathsf{n}+\mathsf{m})} = M^{\mathcal{P}(\mathsf{n})\times\mathcal{P}(\mathsf{m})} = (M^{\mathcal{P}(\mathsf{n})})^{\mathcal{P}(\mathsf{m})}.$$

(In particular, for $n = 0$, this means that $M = M_0 \to M^{\mathcal{P}(\mathsf{m})}$ is the diagonal imbedding: an element $x \in M$ corresponds to the constant function $x : \mathcal{P}(\mathsf{m}) \to M$ having value x.) Next, the properties of a natural transformation for $\underline{\underline{f}}$ are easily checked, as are those saying that ι is a functor. Finally, by definition, for the neutral element, $\underline{\underline{M}}_0 = M$, whence $\mathrm{ev}_0 \circ \iota(M) = M$. □

Definition 3.5. *Let us call* cubic set *the realisation $\underline{\underline{M}}$ of a set M as a functor described by the lemma, and denote by $\mathbf{CubeSet}$ the image of ι, the* cubic realisation of the category $\mathbf{Sets}$.

3.4 Scalar extensions of modules

On the category $\mathbf{Mod}_{\mathbb{K}}$ of $\mathbb{K}$-modules with $\mathbb{K}$-linear maps, we also have the "usual" algebraic scalar extension functor:

Lemma 3.6. *Let us define*

$$\tau : \mathbf{Mod}_{\mathbb{K}} \to \mathbf{Sets}^{\mathbf{scal}_{\mathbb{K}}}, \quad \begin{array}{rcl} V \mapsto \underline{V} & := & (n, \mathbf{t}, \mathbf{s}) \mapsto V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ f \mapsto \underline{\underline{f}} & := & (n, \mathbf{t}, \mathbf{s}) \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} \end{array}.$$

This defines a functor from the category $\mathbf{Mod}_{\mathbb{K}}$ to $\mathbf{Fn}(\mathbf{scal}_{\mathbb{K}}, \mathbf{Sets})$ such that $\mathrm{ev}_0 \circ \tau$ is the identity functor on $\mathbf{Mod}_{\mathbb{K}}$.

Proof. All of this is clear from properties of algebraic scalar extensions, along with the isomorphism of categories $\mathbf{scal}_{\mathbb{K}} \cong \mathbf{talg}_{\mathbb{K}}$. (As in the preceding proof, the main point is that $\underline{V}$ is a functor. In the present case, this holds more generally for general ring morphisms, and not only those coming from the monoidal structure of $\mathbf{scal}_{\mathbb{K}} \cong \mathbf{talg}_{\mathbb{K}}$.) □

Remark 3.1. Clearly, as morphisms in $\mathbf{Mod}_{\mathbb{K}}$ one could also use affine maps instead of linear ones. More generally, following N. Roby [Ro63], one could replace linear maps f by polynomial morphisms, corresponding to "polynomial laws" as defined in loc. cit.

3.5 $\mathbb{K}$-space laws

Now we define a functor category $\mathbf{Space}_{\mathbb{K}}$ of *smooth $\mathbb{K}$-space laws*. One could do so for each fixed $n \in \mathbb{N}$, defining *$\mathbb{K}$-space laws of class C^n*, but it is quicker and clearer to do this for all $n \in \mathbb{N}_0$ together.

Definition 3.7. *Objects of $\mathbf{Space}_{\mathbb{K}}$ are pairs $(\underline{M}, \underline{\Upsilon})$, where $\underline{M} : \mathbf{scal}_{\mathbb{K}} \to \mathbf{Sets}$ is a functor and $\underline{\Upsilon} : \underline{M} \to \underline{\underline{M_0}}$ is a natural transformation, and morphisms of $\mathbf{Space}_{\mathbb{K}}$ are natural transformations $\underline{f} : \underline{M} \to \underline{M}'$ commuting with anchors in the sense that*

$$\underline{\Upsilon}' \circ \underline{f} = \underline{\underline{f_0}} \circ \underline{\Upsilon} : \ \underline{M} \to \underline{\underline{M'_0}}.$$

We require that $\mathbf{Mod}_{\mathbb{K}}$ is a subcategory of $\mathbf{Space}_{\mathbb{K}}$, in the sense that on $\mathbf{Mod}_{\mathbb{K}}$ the extensions coincide with algebraic scalar extensions coming from the corresponding ring extensions: when V is a $\mathbb{K}$-module, then $\underline{\Upsilon} : \underline{V} \to \underline{\underline{V_0}}$ is, for each $(n, \mathbf{t}, \mathbf{s}) \in \mathbf{scal}_{\mathbb{K}}$, given by the anchor of scalar extensions $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to V^{\mathcal{P}(\mathsf{n})}$.

Equivalently, a $\mathbb{K}$-space law $(\underline{M}, \Upsilon_M)$ could also be defined as a functor from $\mathbf{scal}_{\mathbb{K}}$ to $\mathbf{Sets}^{\mathbf{2}}$, the category of "anchored sets", satisfying certain properties. The present formulation features the anchor as a kind of "underlying morphism" of functor categories $\mathbf{Space}_{\mathbb{K}} \to \mathbf{CubeSet} \cong \mathbf{Sets}$. At this point, the situation is quite similar to the one given by abstract polynomials $P \in \mathbb{K}[X]$, to which we can associate, by evaluation on $\mathbb{K}$, an underlying set-map $\tilde{P} : \mathbb{K} \to \mathbb{K}$. In order to define a functor in the other direction, we need assumptions.

3.6 The topological case

Let's return to the topological case, and assume that $\mathbb{K}$ is a good topological ring. Recall from Definition 1.12 the category $\mathbf{Llin}_{\mathbb{K},n}$ of locally linear sets with $C_{\mathbb{K},n}$-maps as morphisms ($n \in \mathbb{N}$, or $n = \infty$).

Definition 3.8 (Prolongation functor)**.** *We define a* prolongation functor

$$\iota : \mathbf{Llin}_{\mathbb{K},\infty} \to \mathbf{Fn}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets})$$

by associating to an object (U, V) (i.e., U open in a topological $\mathbb{K}$-module V) the functor $\underline{U}$ defined by $(\mathbf{t}, \mathbf{s}) \mapsto U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Def. 1.5), and to a $C_{\mathbb{K},n}$-map $f : U \to U'$ the natural transformation defined by restricted iteration (Def. 1.5)

$$\underline{f} : \qquad f_{\mathbb{K}} = f, \quad f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

Lemma 3.9. *The correspondence ι defined above defines a $\mathbb{K}$-space, it is indeed a functor, and*

$$\mathrm{ev}_0 \circ \iota = \mathrm{id}_{\mathbf{Llin}_{\mathbb{K},\infty}}.$$

Proof. First of all, $\underline{U}$ defines a $\mathbb{K}$-space: there is an anchor having the required properties; it is a functor: it is compatible with left and right tensoring, and similarly, $C_{\mathbb{K}}$-maps indeed induce natural transformations. Finally, the evaluation functor clearly gives us back the original objects and morphisms, $U_{\mathbb{K}} = U$, $f_{\mathbb{K}} = f$. □

For the moment, the composition $\iota \circ \mathrm{ev}_{\mathbb{K}}$ is not even defined, since the evaluation $\mathrm{ev}_0(\underline{f})$ has no reason to be a *smooth* function. Thus our concern will be to define a subcategory where this is the case. Since the local linear structure plays a decisive role here, we restrict our attention to this situation, allowing us to state the result even as an *isomorphism of categories*.

Definition 3.10. *We define the functor category* $\mathbf{CSpace}_{\mathbb{K}}$ *of* continuous $\mathbb{K}$-space laws *to be the subcategory of* $\mathbf{Space}_{\mathbb{K}}$ *defined as follows:*

1. *categories* $\mathbf{Sets}$ *and* $\mathbf{Sets}^{\mathbf{2}}$ *are replaced by* $\mathbf{Toplin}$ *and* $\mathbf{Toplin}^{\mathbf{2}}$ *(open sets in topological $\mathbb{K}$-modules, and the corresponding continuous anchors and continuous morphisms, meaning that all $\Upsilon_{\mathbb{A}}$ and $f_{\mathbb{A}}$ are continuous maps),*

2. *morphisms $\underline{f}$ are moreover* jointly continuous in the scaloid*, i.e.: for all locally linear sets (U, V) and morphisms $\underline{f}$, the following map is continuous (where $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \cong V^{2^n}$ via the e-basis, and likewise for W^{2^n}):*

$$\mathbb{K}^{2n} \times V^{2^n} \supset \{(\mathbf{t}, \mathbf{s}; \mathbf{v}) \mid \mathbf{v} \in U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\} \to W^{2^n}, \quad (\mathbf{t}, \mathbf{s}; \mathbf{v}) \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}).$$

Inclusions of (non-empty) open sets in topological $\mathbb{K}$-modules, $U \subset V$, then induce morphisms $\underline{U} \to \underline{V}$, which again will be called "inclusions". The whole set-up of our theory is designed such that the following result becomes essentially a tautology:

Theorem 3.11. *We have two well-defined and mutually inverse functors* ev *and* ι*, defining an isomorphism of categories*

$$\mathbf{Llin}_{\mathbb{K},\infty} \cong \mathbf{CSpace}_{\mathbb{K}}.$$

In particular, ι *defines a full and faithful imbedding of* $\mathbf{Llin}_{\mathbb{K}}$ *into a functor category.*

Proof. Note that in the present case we can speak of equality of objects on both sides in question, and hence the notion of "isomorphism" of these categories makes sense (cf. [CWM], p. 92-93).

Starting with a $C_{\mathbb{K},n}$-function f, it follows from Lemma 3.9 that f can be identified with evaluation at level 0 of the natural transformation $\underline{f}$ defined by f.

To prove the converse, let $\underline{f} : \underline{U} \to \underline{U'}$ be a continuous morphism of laws. We have to show that $\underline{f}$ is induced by a map of class $C_{\mathbb{K},n}$; more precisely, we show that the underlying map $f = f_0 : U_{\mathbb{K}} = U \to U' = (U')_{\mathbb{K}}$ is of class $C_{\mathbb{K},n}$, and that it induces $\underline{f}$. As required in Definition 3.7, the anchor of $V^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ is given by $\mathrm{id}_V \otimes \Upsilon_{\mathbb{K}}$, and via inclusions, the anchor of $\underline{U}$ is given by restricting the anchor of $\underline{V}$. Since $\underline{f}$ is a morphism, it commutes with the anchor in the sense that

$$\Upsilon \circ f_{\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}} = f_{\mathbb{K}^{\mathcal{P}(\mathbf{n})}} \circ \Upsilon.$$

By the continuity property (2) from Definition 3.10, these maps are continuous and jointly continuous also in $(\mathbf{t}, \mathbf{s})$, whence satisfy the condition from Theorem 1.8, showing that the base map $f = f_{\mathbb{K}}$ is of class $C_{\mathbb{K},\infty}$, with the components of $\underline{f}$ given by the construction from topological differential calculus; thus $f_{\mathbb{K}}$ induces the natural transformation $\underline{f}$. □

Remark 3.2. As usual for "tautological" results, the main work lies in the preceding definitions and auxiliary results. To make this yet more plain, let's write G for the monoid $\mathbf{talg}_{\mathbb{K}} \cong \mathbf{scal}_{\mathbb{K}}$ (Lemma 3.1) and C for some subcategory of $\mathbf{Sets}^{\mathbf{2}}$. Assuming C to be small, we may consider the set C^G of all functions from G to C. Clearly, evaluation at the neutral element $o \in G$ defines a map $\mathrm{ev}_o : C^G \to C$. The natural candidate for a map in the other direction is sending C to the "constants" $C \to C^G$, $f \mapsto (g \mapsto f)$. The problem is that the meaning of "constants" has to be carefully defined in a categorical context.

Remark 3.3 (Infinitesimal vs. local and global). A remark on comparison with the case of *Weil laws* as defined in [Be14] is in order here. Taking for $\mathbf{c}_{\mathbb{K}}$ the category of *Weil algebras*, instead of our tangent algebras, we get a formally quite similar theory. However, the anchor becomes "invisible" (for a Weil algebra, it degenerates to a single character), and one may say that Weil algebras are by nature *infinitesimal objects* (because of the nilpotency condition). Thus the link with the local and global theory is not encoded by algebra (as in our approach), and in order to get a *well-adapted* model one has to use more analytic tools (so it is not clear how far these can be generalized beyond the case of real or complex base field) – see [Du79, MR91]. Nevertheless, it might be interesting to look for a category of algebras comprising both Weil algebras and our tangent algebras – in order

to prepare the ground, in Appendix B, we describe some algebraic structures that might be useful for such an approach.

4. Further directions

With Theorem 3.11, we have shown that the functor category $\mathbf{Space}_{\mathbb{K}}$ can be considered as a "well adapted model" for general differential calculus. In subsequent work, we will develop the theory further: on the one hand, comparing with SDG, we will investigate categorical questions, on the other hand, by enriching the structure of our category of algebras, the theory naturally offers links with *higher algebra* and with *super-calculus*. We give some short comments on these items.

4.1 Natural transformations, morphisms

In the preceding formulation, we have limited morphisms in the monoidal categories $\mathbf{scal}_{\mathbb{K}}$, resp. $\mathbf{talg}_{\mathbb{K}}$, to the strict minimum necessary to state the general form of the theory. However, in differential geometry, other algebra morphisms play a rôle by inducing *natural transformations*, as explained by the theory of Weil-functors (see [KMS93]). These algebra morphisms appear already on the level of difference calculus: for instance, the automorhism κ (inversion, see Theorem B.1) corresponds to the *exchange automorphism* on the level of $\mathbb{K}^{\mathcal{P}(1)} \cong \mathbb{K}^2$, inducing a global automorphism on the level of the functor categories. Likewise, our monoidal categories are moreover *symmetric braided monoidal*, via the usual braiding $\mathbb{A}\otimes\mathbb{B} \cong \mathbb{B}\otimes\mathbb{A}$ of associative algebras: again, this gives rise to globally defined morphisms (Schwarz's Theorem, and the "canonical flip" of higher tangent bundles) which together with the inversions, generate at n-th order level an automorphism group which is a *hyperoctahedral group* (automorphism group of a hypercube).

4.2 Groupoids, and higher algebra

In topological calculus, the extended domains $U^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ carry a natural structure of *n-fold groupoid* (by iteration from Item (5) of Theorem B.1; see [Be15a, Be15b, Be17], for the case of target calculus). This is related to the preceding item: indeed, one can show that the groupoid structure on $\mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})}$ is internal to the category of algebras, i.e., all structure maps of the groupoid are algebra morphisms. However, in order to "categorify" this, one needs to enlarge our small category of algebras so that it becomes stable under more general operations than just tensor products, such as *fiber products*. This will be taken up in subsequent work.

4.3 Graded calculus

We insist on the importance of the monoidal structure of the categories $\mathbf{talg}_{\mathbb{K}}$ and $\mathbf{scal}_{\mathbb{K}}$, with the aim to adapt the present approach for giving a functorial approach to *super*-calculus. In principle, it seems that the basic structure outlined in Remark 3.2 can be transposed to the monoidal category of *graded algebras and graded tensor products* generated by $\Upsilon_{t,s}$. It remains to understand the precise relation of such a graded categorical calculus with supercalculus, as it is currently presented. To do this, on should concentrate on symmetric calculus ($\mathbf{t} = -\mathbf{s}$), since in this case the groupoid inversion κ (which becomes the grading automorphism of superalgebras) is given by the simple formula $\kappa(v_0 + ev_1) = v_0 - ev_1$ (cf. Theorem B.1).

4.4 Full iteration, and simplicial calculus

As mentioned in Remark 1.3, *full iteration* leads to higher order "tangent maps" $f^{\{1,\dots,n\}}$ having a very complicated structure. In principle, this structure can also be interpreted in terms of higher groupoids (see [Be15b]). In this setting, the analog of the tangent algebra category $\mathbf{talg}_{\mathbb{K}}$ will be some small higher order category, whose structure remains to be understood yet. Restricting again variables to certain subspaces, one can obtain a sufficiently simple calculus, called *simplicial* in [Be13], and corresponding to the classical concept of *divided differences*. It is certainly possible to put this simplicial calculus into a categorical form, essentially as done in this work for restricted iteration. The advantage should be a better compatibility of calculus with algebra in *positive* characteristic, but the drawback is that the close link with the tensor product, featured in the present approach, gets lost: iteration is no longer given by subsequent tensor products.

A. Hypercubic linear algebra

In this appendix, "linear spaces" are modules over a commutative ring $\mathbb{K}$. Recall Definition 2.6 of a *hypercubic space based on* $N \in \mathcal{P}(\mathbb{N})$. Changing slightly our viewpoint, every free $\mathbb{K}$-module V with basis indexed by $\mathcal{P}(N)$ is isomorphic to $\mathbb{K}^{\mathcal{P}(N)}$ and hence will also be called *hypercubic space*.

When $f : V \to W$ is linear, for bases $(b_j)_{j\in J}$ in V and $(c_i)_{i\in I}$ in W, we denote by $f_{i,j} := c_i^*(f(b_j))$ its *matrix coefficients* (where $(c_i^*)_{i\in I}$ is the dual basis of c). We write also $(\phi \otimes v)(x) = \phi(x) \cdot v$. Then

$$f = \sum_{(i,j)\in I\times J} f_{i,j}\, b_j^* \otimes c_i, \qquad f(b_k) = \sum_i f_{i,k} c_k.$$

When writing a matrix in the usual way as rectangular number array, we use the natural

total order on the index set – that is, the *lexicographic order*; for instance,

$$\mathcal{P}(\{1,2\}) = (\emptyset, \{1\}, \{2\}, \{1,2\}).$$

In the following, for an n-tuple $\mathbf{a} = (a_i)_{i\in N} \in \mathbb{K}^n$, we use the notation $\mathbf{a}_N := \prod_{i\in N} a_i$, in the same way as we do for $\mathbf{t}, \mathbf{s} \in \mathbb{K}^n$ in the main text. When N is considered to be fixed, and $A \subset N$, we denote by $A^c = N \setminus A$ its complement.

The following result allows to put hands on induction procedures using iterated tensor products, cf. Remark 2.1.

Theorem A.1. *Let $N = \{k_1, \dots, k_n\}$ and $f_i : \mathbb{K}^{\mathcal{P}(\{k_i\})} \to \mathbb{K}^{\mathcal{P}(\{k_i\})}$ linear, with matrix*

$$f_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} : \qquad E^i_\emptyset \mapsto a_i E^i_\emptyset + c_i E^i_i, \quad E^i_i \mapsto b_i E^i_\emptyset + d_i E^i_i.$$

Then the matrix of the linear map $f := \otimes_{i=1}^n f_i : \mathbb{K}^{\mathcal{P}(N)} \to \mathbb{K}^{\mathcal{P}(N)}$ is given by the matrix coefficients, for $(A, B) \in \mathcal{P}(N)^2$,

$$f_{A,B} = E^*_A\big(f(E_B)\big) = \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}.$$

In other terms, $f(E^N_B) = \sum_{A\in\mathcal{P}(N)} \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}\ E^N_A$, or

$$f = \sum_{(A,B)\in\mathcal{P}(N)^2} \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}\ (E^N_B)^* \otimes E^N_A.$$

Proof. When the cardinality n of N is equal to one, then the claim is true, directly by definition of the matrix coefficients. For $n = 2$, the matrix of $f_1 \otimes f_2$ is

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & b_1a_2 & a_1b_2 & b_1b_2 \\ c_1a_2 & d_1a_2 & c_1b_2 & d_1b_2 \\ a_1c_2 & b_1c_2 & a_1d_2 & b_1d_2 \\ c_1c_2 & d_1c_2 & c_1d_2 & d_1d_2 \end{pmatrix}$$

("Kronecker product"). For instance, when $B = \emptyset$, so $B^c = \{1, 2\}$,

$$f(E^{\{12\}}_\emptyset) = a_{12}E_\emptyset + c_1a_2E_1 + a_1c_2E_2 + c_{12}E_{12},$$

in keeping with the claim. In the general case, we expand the expression

$$f = \otimes_i f_i = \otimes_i \Big(a_i(E^i_\emptyset)^* \otimes E^i_\emptyset + b_i(E^i_\emptyset)^* \otimes E^i_i + c_i(E^i_i)^* \otimes E^i_\emptyset + d_i(E^i_i)^* \otimes E^i_i \Big)$$

by distributivity: we get a sum of 4^n terms, which correspond exactly to the 4^n terms in the last formula of the claim. (E.g., for $n = 2$, there are 16 terms, corresponding to expanding the product $(a_1+b_1+c_1+d_1)(a_2+b_2+c_2+d_2)$ by distributivity, giving the 16 matrix coefficients shown above. The first column contains the 4 terms from expanding $(a_1 + c_1)(a_2 + c_2)$, etc.) □

To memorise the formula: for 2×2-matrices and indices, the correspondence is

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \quad : \quad \begin{pmatrix} A^c \cap B^c & A^c \cap B \\ A \cap B^c & A \cap B \end{pmatrix}.$$

Next, we give a formula for the *inverse* of f, when its determinant is invertible. From well-known properties of the Kronecker product it follows that

$$\det(f) = \det(\otimes_{i=1}^n f_i) = (\prod_{i=1}^n \det(f_i))^{2^{n-1}},$$

whence the first statement of the following theorem:

Theorem A.2. *Let N and $f = \otimes_{i=1}^n f_i$ be as in the preceding theorem. Then f is invertible if, and only if, all f_i are invertible, and then its inverse is given by the matrix coeffients, for $(A, B) \in \mathcal{P}(N)^2$ (recall $A\Delta B$ is the symmetric difference)*

$$(f^{-1})_{A,B} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} f_{B^c,A^c} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} \mathbf{a}_{A\cap B} \cdot \mathbf{b}_{A\cap B^c} \cdot \mathbf{c}_{A^c\cap B} \cdot \mathbf{d}_{A^c\cap B^c}.$$

Proof. Assume each f_i is invertible. For $n = 1$, $N = \{k\}$, the inverse is

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}^{-1} = \frac{1}{(a_k d_k - b_k c_k)} \begin{pmatrix} d_k & -b_k \\ -c_k & a_k \end{pmatrix}. \tag{A.1}$$

For $n = 2$, the matrix of the inverse is the Kronecker product of the inverses

$$\frac{1}{\det(f_1)\det(f_2)} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \otimes \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} =$$

$$\frac{1}{\det(f_1)\det(f_2)} \begin{pmatrix} d_1d_2 & -b_1d_2 & -d_1b_2 & b_1b_2 \\ -c_1d_2 & a_1d_2 & c_1b_2 & -a_1b_2 \\ -d_1c_2 & b_1c_2 & d_1a_2 & -b_1a_2 \\ c_1c_2 & -a_1c_2 & -c_1a_2 & a_1a_2 \end{pmatrix}$$

which is in keeping with the formula announced in the claim. To put this computation into a conceptual framework, note that the inverse in (A.1) is obtained by first taking the adjugate matrix, and then dividing by the determinant. The adjugate $X^\sharp$ of a 2×2-matrix X, in turn, is given by

$$X^\sharp = JX^\top J^{-1},$$

where $X^\top$ is the transposed matrix, $(X^\top)_{(A,B)} = X_{(B,A)}$, and

$$I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.2}$$

i.e., J sends $E_\emptyset \mapsto E_1$, $E_1 \mapsto -E_\emptyset$ (so $X^\sharp$ is the adjoint of X with respect to the canonical symplectic form on $\mathbb{K}^2$; call it "symplectic adjoint"). For each 2×2-matrix M let

$$M_{\mathsf{n}} = \otimes_{i=1}^n M : \mathbb{K}^{\mathcal{P}(\mathsf{n})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}.$$

Then, for the matrices I, J, K defined by (A.2), the effect on E_A is

$$I_{\mathsf{n}}(E_A) = (-1)^{|A|}E_A, \quad K_{\mathsf{n}}(E_A) = E_{A^c}, \quad J_{\mathsf{n}}(E_A) = (-1)^{|A^c|}E_{A^c}, \tag{A.3}$$

The inverse of J_{n} is $J_{\mathsf{n}}^{-1}(E_A) = K_{\mathsf{n}}I_{\mathsf{n}}(E_A) = (-1)^{|A|}E_{A^c} = (-1)^n J_{\mathsf{n}}(E_A)$. Using this, we compute

$$\begin{aligned} f^\sharp(E_A) &= J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1}(E_A) = (-1)^{|A|} J_{\mathsf{n}} \circ f^\top(E_{A^c}) \\ &= (-1)^{|A|} J_n \sum_B f^\top_{A^c,B} E_B \\ &= (-1)^{|A|} \sum_B f_{B,A^c}(-1)^{|B^c|}E_{B^c} = (-1)^{|A|} \sum_B f_{B^c,A^c}(-1)^{|B|}E_B \\ &= \sum_B (-1)^{|A|}(-1)^{|B|} \mathbf{a}_{A\cap B} \cdot \mathbf{b}_{A\cap B^c} \cdot \mathbf{c}_{A^c\cap B} \cdot \mathbf{d}_{A^c\cap B^c} E_B \end{aligned}$$

which together with $|A|+|B| \equiv |A\Delta B| \mod (2)$, so $(-1)^{|A|}(-1)^{|B|} = (-1)^{|A\Delta B|}$, gives us the adjugate and the claim. □

Remark A.1. In the same way, it follows that, even if f is not invertible, we have

$$f \circ J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1} = \prod_{i=1}^n \det(f_i) \cdot \mathrm{id}.$$

B. On the structure of tangent algebras

One may be interested in defining a class of algebras, generalizing the by now classical *Weil algebras* (see [KMS93, MR91]), and the *bundle algebras* from [Be14], incorporating also algebras arising from difference calculus. The following structure theorem might help to select structural features that could be used for defining such a category. We use notation defined in Subsection 2.2.

Theorem B.1 (Structure of the first order tangent algebra $\mathbb{K}^{\{1\}}_{(t,s)}$)**.**

1. *The ideals* $\ker(\alpha)$ *and* $\ker(\beta)$ *satisfy* $\ker(\alpha) \cdot \ker(\beta) = 0$.
2. *The product of* $w, v \in \mathbb{K}^{\{1\}}_{(t,s)}$ *is given by the "fundamental relation"*

$$w \cdot v = \alpha(w)v - \alpha(w)\beta(v) + \beta(v)w.$$

3. *The map*

$$\kappa : \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}^{\{1\}}_{(t,s)}, \quad v \mapsto (\alpha+\beta)(v)\cdot 1 - v$$

is an algebra automorphism of order 2 *such that* $\alpha \circ \kappa = \beta$. *Moreover,*

$$\forall v \in \mathbb{K}^{\{1\}}_{(t,s)} : \quad v \cdot \kappa(v) = \alpha(v)\beta(v)1.$$

4. *An element* v *is invertible in* $\mathbb{K}^{\{1\}}_{(t,s)}$ *if, and only if,* $\alpha(v)\beta(v) \in \mathbb{K}^\times$, *and then the inverse is*

$$v^{-1} = \frac{1}{\alpha(v)\beta(v)}\kappa(v) = (\frac{1}{\alpha(v)} + \frac{1}{\beta(v)})1 - \frac{v}{\alpha(v)\beta(v)}.$$

5. *The set* $\mathbb{K}^{\{1\}}_{(t,s)}$, *equipped with the following product* $*$ *(for* (u,w) *such* $\alpha(u) = \beta(w)$*), inversion* κ, *and units* $\lambda 1$ *(*$\lambda \in \mathbb{K}$*), is a groupoid:*

$$u * w = u - \alpha(u)1 + w.$$

Proof. (1) $\ker(\alpha) = \mathbb{K}(e-s)$ and $\ker(\beta) = \mathbb{K}(e-t)$, and, by the defining relation of the algebra, $(e-s)(e-t) = [(X-t)(X-s)] = 0$.

(2) Since $\alpha(v - \alpha(v)1) = 0$ and $\beta(w - \beta(w)1) = 0$, the preceding item implies

$$0 = (v-\alpha(v))(w-\beta(w)) = vw - \alpha(v)w - \beta(w)v + \alpha(v)\beta(w).$$

(3) Note that $\kappa(1) = 1+1-1 = 1$ and $\kappa(e) = s+t-e$, whence $\kappa(\kappa(e)) = s+t-(s+t-e) = e$, so $\kappa^2 = \mathrm{id}$. Next, $\alpha(\kappa(v)) = (\alpha+\beta)(v) - \alpha(v) = \beta(v)$. To prove that κ is an automorphism, since $\kappa(1) = 1$, it suffices to show that $\kappa(e^2) = \kappa(e)^2$. Indeed, $\kappa(e)^2 = (t+s)^2 - 2(t+s)e + e^2 = (t+s)^2 - ts - (t+s)e$ and $\kappa(e^2) = \kappa(-ts+(t+s)e) = -ts + (t+s)\kappa(e) = -ts + (t+s)^2 - (t+s)e$. Finally,

$$v\cdot\kappa(v) = \alpha(v)\kappa(v) - \alpha(v)\beta(\kappa v) + \beta(\kappa v)v = \alpha(v)\beta(v)1.$$

(4) If v is invertible, then applying the morphisms α and β, it follows that both $\alpha(v)$ and $\beta(v)$ are invertible. Conversely, the last formula from (3) shows that under this condition v has an inverse given by v^{-1} as in the claim.

(5) The defining properties of a groupoid are easily checked by direct computation, cf. [Be15a, Be17]. □

It is then true, moreover, that the groupoid law $*$ is an algebra morphism from the fiber product algebra $\mathbb{K}_{(t,s)} \times_{\alpha,\beta} \mathbb{K}_{(t,s)}$ to $\mathbb{K}_{(t,s)}$, and thus is "internal" to a certain category of algebras.

References

[Be08] Bertram, W., *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings*, Memoirs of the AMS **192**, no. 900 (2008). https://arxiv.org/abs/math/0502168

[Be11] Bertram, W., *Calcul différentiel topologique élémentaire*, Calvage et Mounet, Paris 2011

[Be13] Bertram, W., "Simplicial differential calculus, divided differences, and construction of Weil functors", Forum Math. **25** (1) (2013), 19–47. http://arxiv.org/abs/1009.2354

[Be14] Bertram, W., "Weil Spaces and Weil-Lie Groups", http://arxiv.org/abs/1402.2619

[Be15a] Bertram, W., "Conceptual Differential Calculus. I : First order local linear algebra" http://arxiv.org/abs/1503.04623

[Be15b] Bertram, W., "Conceptual Differential Calculus. II : Cubic higher order calculus." http://arxiv.org/abs/1510.03234

[Be16] Bertram, W., "A precise and general notion of manifold." http://arxiv.org/abs/1605.07745

[Be17] Bertram, W., "Lie Calculus." Proceedings of 50. Seminar Sophus Lie, Banach Center Publications 113 (2017), 59-85 https://arxiv.org/abs/1702.08282

[BGN04] Bertram, W., H. Gloeckner and K.-H. Neeb, "Differential Calculus over general base fields and rings", Expo. Math. 22 (2004), 213 –282. http://arxiv.org/abs/math/0303300

[BeS14] Bertram, W, and A. Souvay, "A general construction of Weil functors", Cahiers Top. et Géom. Diff. Catégoriques **LV**, Fasc. 4, 267 – 313 (2014), arxiv: math.GR/1201.6201

[CWM] Mac Lane, S., *Categories for the Working Mathematician*, Springer 1998 Second Edition

[DG] Demazure, M., and P. Gabriel, *Groupes Algébriques. I*, Masson, Paris 1970

[Du79] Dubuc, E.J., "Sur les modèles de la géométrie différentielle synthétique", Cahiers Top. et Géom. diff. **20** (1979), 231 – 279.

[Ko06] Kock, A., *Synthetic Differential Geometry*, Cambridge University Press, 2nd edition 2006.

[Ko10] Kock, A., *Synthetic Geometry of Manifolds*, Cambridge Tracts in Mathematics **180**, Cambridge 2010

[KMS93] Kolar, I, P. Michor and J. Slovak, *Natural Operations in Differential Geometry*, Springer, Berlin 1993.

[MM92] Mac Lane, S., and I. Moerdijk, *Sheaves in Geometry and Logic*, Springer, New York 1992

[MR91] Moerdijk, I, and G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer 1991

[Ro63] Roby, N., "Lois polynomes et lois formelles en théorie des modules", Ann. Sci. E.N.S., **80** (1963), 213 – 348.

[We53] Weil, A., "Théorie des points proches sur les variétés différentiables", Coll. Géo. Diff., CNRS 1953 (see also: A. Weil, Œuvres complets, Springer).

Institut Élie Cartan de Lorraine
Université de Lorraine at Nancy, CNRS, INRIA
Boulevard des Aiguillettes, B.P. 239
F-54506 Vandœuvre-lès-Nancy, France
mail : wolfgang.bertram, jeremy.haut@univ-lorraine.fr