cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN

VOLUME LXIV-2, 2e Trimestre 2023

AMIENS

Cahiers de Topologie et Géométrie Différentielle Catégoriques

Directeur de la publication: Andrée C. EHRESMANN,
Faculté des Sciences, Mathématiques LAMFA
33 rue Saint-Leu, F-80039 Amiens.

Les "*Cahiers*" comportent un Volume par an, divisé en 4 fascicules trimestriels. Ils publient des articles originaux de Mathématiques, de préférence sur la Théorie des Catégories et ses applications, e.g. en Topologie, Géométrie Différentielle, Géométrie ou Topologie Algébrique, Algèbre homologique…
Les manuscrits soumis pour publication doivent être envoyés à l'un des Rédacteurs comme fichiers .pdf.

Depuis 2018, les "*Cahiers*" publient une **Edition Numérique en Libre Accès,** sans charge pour l'auteur : le fichier pdf du fascicule trimestriel est, dès parution, librement téléchargeable sur :

The "*Cahiers*" are a quarterly Journal with one Volume a year (divided in 4 issues). They publish original papers in Mathematics, the center of interest being the Theory of categories and its applications, e.g. in topology, differential geometry, algebraic geometry or topology, homological algebra…
Manuscripts submitted for publication should be sent to one of the Editors as pdf files.

From 2018 on, the "*Cahiers*" have also a **Full Open Access Edition (**without Author Publication Charge): the pdf file of each quarterly issue is immediately freely downloadable on:

https://ehres.pagesperso-orange.fr/Cahiers/Ctgdc.htm
and http://cahierstgdc.com/

cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LXIV – 2, 2ème trimestre 2023

SOMMAIRE

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXIV-2 (2023)

DOUBLE GROUPOIDS AND POSTNIKOV INVARIANTS

Antonio M. CEGARRA

Résumé. Dans cet article, nous prouvons un théorème de classification pour les groupoïdes doubles (satisfaisant à une condition de remplissage supplémentaire, tout à fait naturelle) au moyen de troisièmes classes de cohomologie de groupoïdes. Dans une seconde étape, indépendante, nous montrons que la classe de cohomologie associée à un groupoïde double coïncide avec l'unique k-invariant non trivial de sa réalisation géométrique.

Abstract. In this paper, we prove a classification theorem for double groupoids (satisfying an extra, quite natural, filling condition) by means of third cohomology classes of groupoids. In a second, independent, step, we prove that the cohomology class associated to a double groupoid coincides with the unique non-trivial k-invariant of its geometric realization.

Keywords. Double groupoid, Cohomology of groupoids, Postnikov invariant, weak equivalence, homotopy type.

Mathematics Subject Classification (2010). 18D05, 20L05, 55Q05, 55S45, 55U40.

Introduction and summary

Double groupoids (groupoid objects in the category of groupoids) go back to Ehresmann [14, 15, 16]. Roughly, they consist of objects, two kinds of morphisms between them, horizontal and vertical, and boxes whose boundaries are squares with morphisms as edges, usually depicted

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\alpha"] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd},$$

together with horizontal and vertical composition of morphisms and boxes giving compatible groupoid structures and obeying middle four interchange on boxes. The double groupoids we encounter in practice, and certainly in this work, are small and satisfy a natural *filling condition*: Any filling problem

$$\begin{tikzcd} d & \cdot \arrow[l, dotted] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\exists?"] & a \arrow[l, "g"] \arrow[u, dotted] \end{tikzcd},$$

finds a solution in the double groupoid. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in weak Hopf algebra theory or in differential geometry (see, for instance, Andruskiewitsch and Natale [1] and Mackenzie [23]), and it is satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.*, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see the surveys by Brown [3, 4, 5] and the references given there). Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see Brown and Higgins [6], Brown and Spencer [7], Brown, Hardie, Kamps and Porter [8] and Brown, Kamps and Porter [9]), for double groupoid objects in the category of groups (also termed cat^2-groups by Loday [22], see also Porter [25] and Bullejos, Cegarra and Duskin [10]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete (see for instance Moerdijk and Svensson [24] and Hardie, Kamps and Kieboom [20]).

Every (small) double groupoid $\mathcal{G}$ has a geometric realization, which is the topological space defined by first taking the double nerve $\mathrm{NN}\mathcal{G}$, which is a bisimplicial set, and then realizing geometrically the diagonal to obtain a space: $|\mathcal{G}| = |\triangle \, \mathrm{NN}\mathcal{G}|$. The usual definition of the homotopy invariants of a double groupoid $\mathcal{G}$ involves only its underlying topological space

$|\mathcal{G}|$ and does not take into account the algebraic structure. Our main goal in this paper is *to give a combinatorial definition of the* (unique) *Postnikov invariant of a double groupoid with the filling condition using only its algebraic structure*. Recall that a (2-dimensional) *Postnikov system* is a triple $(P, \mathcal{A}, \mathbf{k})$, where P is a groupoid, $\mathcal{A}$ is an abelian group valued functor on P, and $\mathbf{k} \in H^3(P, \mathcal{A})$ is a three-cohomology class of P with coefficients in $\mathcal{A}$. Our definitions and constructions here are suggested by previous work of the author and collaborators; particularly by the results in [11], where we address the homotopy types realized from double groupoids satisfying the filling condition. They are all the (not necessarily path-connected) homotopy 2-types, that is, the homotopy types of all CW-complexes whose homotopy groups at any base point vanish in degree 3 and higher.

After Section 1, where we briefly fix some notational conventions on double groupoids, in Sections 2 and 3, we review several needed definitions and results on the (algebraically defined) fundamental groupoid $\Pi\mathcal{G}$ and the homotopy groups $\pi_2(\mathcal{G}, a)$ of a double groupoid $\mathcal{G}$ satisfying the filling condition. Section 4 contains the new definition of the Postnikov invariant of such a double groupoid, which is the equivalence class of a Postnikov system $(\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ where $\mathbf{k}\mathcal{G} \in H^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ is a certain characteristic cohomology class of the fundamental groupoid of $\mathcal{G}$ with coefficients in the abelian group valued functor on $\Pi\mathcal{G}$ which assigns the homotopy group $\pi_2(\mathcal{G}, a)$ to each object a of $\mathcal{G}$. In Section 5, we mainly state and prove the expected classification result:

"The assignment $\mathcal{G} \mapsto (\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ induces a bijective correspondence between weak equivalence classes of double groupoids satisfying the filling condition and equivalence classes of Postnikov systems."

Finally, in Section 6 we prove

"The Postnikov invariant of a double groupoid $\mathcal{G}$ with the filling condition is equivalent to the Postnikov invariant of its geometric realization $|\mathcal{G}|$."

As a bonus, we find a new proof of the fact that the assignment $\mathcal{G} \mapsto |\mathcal{G}|$ induces a bijective correspondence between weak-equivalence classes of double groupoids satisfying the filling condition and homotopy 2-types.

1. Some conventions on double groupoids

The notion of double groupoid is well-known, we just specify in this preliminary section some basic terminology and notational conventions. We will work only with small double groupoids, so that in a double groupoid $\mathcal{G}$ we have a set of objects (usually denoted by $a, b, c, \ldots$), horizontal morphisms between them $(f, g, h, \ldots)$, vertical morphisms between them $(x, y, z, \ldots)$, both with composition written by juxtaposition, and boxes $(\alpha, \beta, \gamma, \ldots)$, usually depicted as

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd} \qquad (1)$$

where the horizontal morphisms f and g are, respectively, its vertical target and source and the vertical morphisms y and x are its respective horizontal target and source. The horizontal composition of boxes is denoted by the symbol $\circ_{\mathrm{h}}$:

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f'"'] & \cdot \arrow[l, "f'"'] \\ \cdot \arrow[u, "z"] & \cdot \arrow[l, "g'"] \arrow[u] \arrow[ul, phantom, "\alpha'"] & \cdot \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "f'f"'] \\ \cdot \arrow[u, "z"] & \cdot \arrow[l, "g'g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha'\circ_{\mathrm{h}}\alpha"] \end{tikzcd}$$

and, similarly, the vertical composition of boxes is denoted by the symbol $\circ_{\mathrm{v}}$:

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "y"] & \cdot \arrow[l] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \\ \cdot \arrow[u, "y'"] & \cdot \arrow[l, "h"] \arrow[u, "x'"'] \arrow[ul, phantom, "\alpha'"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "yy'"] & \cdot \arrow[l, "h"] \arrow[u, "xx'"'] \arrow[ul, phantom, "\alpha\circ_{\mathrm{v}}\alpha'"] \end{tikzcd}$$

Horizontal and vertical identities on objects and morphisms are respectively denoted by $\mathrm{I}^{\mathrm{h}}a$, $\mathrm{I}^{\mathrm{v}}a$, $\mathrm{I}^{\mathrm{h}}x$, $\mathrm{I}^{\mathrm{v}}f$, and $\mathrm{I}a := \mathrm{I}^{\mathrm{v}}\mathrm{I}^{\mathrm{h}}a = \mathrm{I}^{\mathrm{h}}\mathrm{I}^{\mathrm{v}}a$, depicted as

$$a = a \qquad \begin{tikzcd} a \arrow[d, equal] \\ a \end{tikzcd} \qquad \begin{tikzcd} \cdot & \cdot \arrow[l, equal] \\ \cdot \arrow[u, "x"] & \cdot \arrow[l, equal] \arrow[u, "x"'] \arrow[ul, phantom, "\mathrm{I}^{\mathrm{h}}x"] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[d, equal] & \cdot \arrow[l, "f"'] \arrow[d, equal] \\ \cdot & \cdot \arrow[l, "f"] \arrow[ul, phantom, "\mathrm{I}^{\mathrm{v}}f"] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[d, equal] & a \arrow[l, equal] \arrow[d, equal] \\ a & a \arrow[l, equal] \arrow[ul, phantom, "\mathrm{I}a"] \end{tikzcd}$$

and horizontal and vertical inverses of boxes are respectively denoted by $\alpha^{-\mathrm{h}}$, $\alpha^{-\mathrm{v}}$, and $\alpha^{-\mathrm{hv}} := (\alpha^{-\mathrm{h}})^{-\mathrm{v}} = (\alpha^{-\mathrm{v}})^{-\mathrm{h}}$; that is,

$$
\begin{tikzcd} \cdot & \cdot \arrow[l, "f^{-1}"'] \\ \cdot \arrow[u, "x"] \arrow[ur, phantom, "\alpha^{-\mathrm{h}}"] & \cdot \arrow[l, "g^{-1}"] \arrow[u, "y"'] \end{tikzcd}
\qquad
\begin{tikzcd} \cdot & \cdot \arrow[l, "g"'] \\ \cdot \arrow[u, "y^{-1}"] \arrow[ur, phantom, "\alpha^{-\mathrm{v}}"] & \cdot \arrow[l, "f"] \arrow[u, "x^{-1}"'] \end{tikzcd}
\qquad
\begin{tikzcd} \cdot & \cdot \arrow[l, "g^{-1}"'] \\ \cdot \arrow[u, "x^{-1}"] \arrow[ur, phantom, "\alpha^{-\mathrm{hv}}"] & \cdot \arrow[l, "f^{-1}"] \arrow[u, "y^{-1}"'] \end{tikzcd}
$$

We will use several times the coherence theorem by Dawson and Paré [13, Theorem 1.2], which assures us that *if a compatible arrangement of boxes in a double groupoid is composable in two different ways, the resulting pasted boxes are equal*. Throughout the paper, an equality between pasting diagrams of boxes in a double groupoid means that the resulting pasted boxes are the same.

The double groupoids we are interested in satisfy the so-called filling condition: *Any filling problem*

$$
\begin{tikzcd} \cdot & \cdot \arrow[l, dotted] \\ \cdot \arrow[u, "y"] \arrow[ur, phantom, "\exists ?"] & \cdot \arrow[l, "g"] \arrow[u, dotted] \end{tikzcd},
$$

has a solution; that is, for any horizontal morphism g and any vertical morphism y such that the source of y coincides with the target of g, there is a box whose vertical source is g and whose horizontal target is y. This condition is more symmetric than it appears thanks to the following lemma by Andruskiewitsch and Natale [1, Lemma 1.12].

Lemma 1.1. *A double groupoid satisfies the filling condition if and only if any filling problem such as the one below has a solution.*

$$
\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists ?"] & \cdot \arrow[l, dotted] \arrow[u, "x"'] \end{tikzcd},
\quad
\begin{tikzcd} \cdot & \cdot \arrow[l, dotted] \\ \cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists ?"] & \cdot \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd},
\quad
\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "y"] \arrow[ur, phantom, "\exists ?"] & \cdot \arrow[l, dotted] \arrow[u, dotted] \end{tikzcd},
$$

Throughout the paper we make the assumption that the double groupoids we work with are small and satisfy the filling condition.

2. The fundamental groupoid $\Pi\mathcal{G}$

Let $\mathcal{G}$ be a double groupoid. If a_0, a_1 are objects of $\mathcal{G}$, we define a *path* in $\mathcal{G}$ from a_0 to a_1 to be a diagram (f, b, x) of the form

$$\begin{tikzcd} a_1 & b \arrow[l, "f"'] \\ & a_0 \arrow[u, "x"'] \end{tikzcd}$$

that is, where b is an object, f a horizontal morphism from b to a_1, and x a vertical morphism from a_0 to b. Throughout the paper, we identify paths in $\mathcal{G}$ of the form

$$\begin{tikzcd} a_1 & a_0 \arrow[l, "f"'] \arrow[d, equal] \\ & a_0 \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[r, equal] & a_1 \\ & a_0 \arrow[u, "x"'] \end{tikzcd}$$

with the morphisms f and x respectively; that is, we write

$$f = (f, a_0, \mathrm{I}^{\mathrm{v}} a_0), \qquad x = (\mathrm{I}^{\mathrm{h}} a_1, a_1, x).$$

If (f, b, x) and (g, c, y) are two paths from a_0 to a_1, then we say that (f, b, x) is *homotopic* to (g, c, y), denoted by $(f, b, x) \simeq (g, c, y)$, if there is a box α in $\mathcal{G}$ of the form

$$\begin{tikzcd} b \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c \arrow[l, "f^{-1}g"'] \\ b \arrow[r, equal] & b \arrow[u, "yx^{-1}"'] \end{tikzcd} \tag{2}$$

that is, whose horizontal target and vertical source are identities, its horizontal source is yx^{-1}, and its its vertical target is $f^{-1}g$. We call such a box a *homotopy*, and we often write $\alpha : (f, b, x) \simeq (g, c, y)$ whenever we wish to display the homotopy.

Lemma 2.1. *Homotopy is an equivalence relation on the set of paths in $\mathcal{G}$ from a_0 to a_1.*

Proof. Reflexivity: For any path (f, b, x), clearly $\mathrm{I}b : (f, b, x) \simeq (f, b, x)$. *Symmetry*: If $\alpha : (f, b, x) \simeq (g, c, y)$ is a homotopy, then the pasted box of

$$\begin{tikzcd} c \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}(g^{-1}f)"] & b \arrow[l, "g^{-1}f"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha^{-\mathrm{v}}"] & b \\ c & b \arrow[l, "g^{-1}f"] & c \arrow[l, "f^{-1}g"] \arrow[u, "xy^{-1}"'] \end{tikzcd}$$

is a homotopy $(g,c,y)\simeq(f,b,x)$.
Transitivity: Assume that $\alpha:(f,b,x)\simeq(g,c,y)$ and $\beta:(g,c,y)\simeq(h,d,z)$. Then, we find a homotopy $\gamma:(f,b,x)\simeq(h,d,z)$ by pasting the diagram of boxes

$$\begin{tikzcd} b \arrow[d, equal] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}(f^{-1}g)"] & c \arrow[l, "f^{-1}g"'] \arrow[d, equal] \arrow[r, phantom, "\beta"] & d \arrow[l, "g^{-1}h"'] \\ b \arrow[d, equal] & c \arrow[l, "f^{-1}g"] \arrow[r, equal] \arrow[d, phantom, "\alpha"] & c \arrow[u, "zy^{-1}"'] \\ b \arrow[rr, equal] & & c \arrow[u, "yx^{-1}"'] \end{tikzcd}$$

$\square$

Let $[f,b,x]$ denote the homotopy class of a path (f,b,x) in $\mathcal{G}$.

We define the *fundamental groupoid* $\Pi\mathcal{G}$ of the double groupoid $\mathcal{G}$ to be a category having as objects all the objects of $\mathcal{G}$. An arrow in $\Pi\mathcal{G}$ from an object a_0 to an object a_1 is the homotopy class of a path in $\mathcal{G}$ from a_0 to a_1. Composition in $\Pi\mathcal{G}$ is as follows:

For each morphism in the fundamental groupoid $\rho\in\Pi\mathcal{G}(a_0,a_1)$, let us choose a representative path (f_ρ,b_ρ,x_ρ) of ρ,

$$\begin{tikzcd} a_1 & b_\rho \arrow[l, "f_\rho"'] \\ & a_0, \arrow[u, "x_\rho"'] \end{tikzcd} \tag{3}$$

that is, such that $\rho=[f_\rho,b_\rho,x_\rho]$. If $a_2\overset{\psi}{\leftarrow}a_1\overset{\rho}{\leftarrow}a_0$ are any two composable morphisms in $\Pi\mathcal{G}$, by the filling condition on $\mathcal{G}$, we can select a box θ in $\mathcal{G}$ whose horizontal target is x_ψ and whose vertical source is f_ρ. Thus, we have a diagram in $\mathcal{G}$ of the form

$$\begin{tikzcd} a_2 & b_\psi \arrow[l, "f_\psi"'] \arrow[r, phantom, "\theta"'] & b \arrow[l, "f"'] \\ & a_1 \arrow[u, "x_\psi"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\ & & a_0 \arrow[u, "x_\rho"'] \end{tikzcd} \tag{4}$$

and we define the composite $\psi\rho=[f_\psi f,b,xx_\rho]\in\Pi\mathcal{G}(a_0,a_2)$.

Lemma 2.2. *The composite $\psi\rho$ is well-defined, that is, it is independent of the choices of representative paths of ρ and ψ and of the choice of θ in* (4)*.*

Proof. Suppose that $\alpha_\rho : (f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$ and $\alpha_\psi : (f_\psi, b_\psi, x_\psi) \simeq (g_\psi, c_\psi, y_\psi)$ are homotopies and that we have selected boxes θ and θ' as in the diagrams below.

$$\begin{tikzcd}
a_2 & b_\psi \arrow[l, "f_\psi"'] & b \arrow[l, "f"'] \\
 & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\
 & & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_2 & c_\psi \arrow[l, "g_\psi"'] & c \arrow[l, "g"'] \\
 & a_1 \arrow[u, "y_\psi"] \arrow[ur, phantom, "\theta'"] & c_\rho \arrow[l, "g_\rho"] \arrow[u, "y"'] \\
 & & a_0 \arrow[u, "y_\rho"']
\end{tikzcd}$$

Then, we get a homotopy $\alpha : (f_\psi f, b, x x_\rho) \simeq (g_\psi g, c, y y_\rho)$ by pasting the diagram

$$\begin{tikzcd}
b & b_2 \arrow[l, "f^{-1}"'] \arrow[d, equal] \arrow[dr, phantom, "\alpha_\psi"] & c_\psi \arrow[l, "f_\psi^{-1} g_\psi"'] & c \arrow[l, "g"'] \\
\arrow[r, phantom, "\theta^{-\mathrm{h}}"] & b_\psi \arrow[r, equal] & b_\psi \arrow[u, "y_\psi x_\psi^{-1}"'] \arrow[r, phantom, "\theta'"] & {} \\
b_\rho \arrow[uu, "x"] \arrow[d, equal] & a_2 \arrow[l, "f_\rho^{-1}"] \arrow[u, "x_\psi"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}} x_\psi"] & a_2 \arrow[u, "x_\psi"'] & c_\rho \arrow[l, "g_\rho"] \arrow[uu, "y"'] \\
b_\rho \arrow[rrr, equal] \arrow[urrr, phantom, "\alpha_\rho"] & & & b_\rho \arrow[u, "y_\rho x_\rho^{-1}"'] \\
b \arrow[u, "x^{-1}"] \arrow[rrr, equal] \arrow[urrr, phantom, "\mathrm{I}^{\mathrm{h}} x^{-1}"] & & & b \arrow[u, "x^{-1}"']
\end{tikzcd}$$

□

For each object a of $\mathcal{G}$, let $id_a = [\mathrm{I}^{\mathrm{h}} a, a, \mathrm{I}^{\mathrm{v}} a] \in \Pi\mathcal{G}(a, a)$.

Theorem 2.3. *With these definitions, $\Pi\mathcal{G}$ is a groupoid.*

Proof. Identity: For every arrow $\rho = [f_\rho, b_\rho, x_\rho] \in \Pi\mathcal{G}(a_0, a_1)$, the diagrams in $\mathcal{G}$

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}} f_\rho"] & b_\rho \arrow[l, "f_\rho"'] \arrow[d, equal] \\
 & a_1 & b_\rho \arrow[l, "f_\rho"] \\
 & & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] \arrow[r, equal] & b_\rho \\
 & a_0 \arrow[u, "x_\rho"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}} x_\rho"] & a_0 \arrow[u, "x_\rho"'] \arrow[d, equal] \\
 & & a_0
\end{tikzcd}$$

show that $id_{a_1}\rho = \rho = \rho\, id_{a_0}$.

Associativity: if $a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1 \xleftarrow{\rho} a_0$ are any three composable morphisms in $\Pi\mathcal{G}$, we can choose boxes θ, θ' and θ'' as in the diagram

$$\begin{tikzcd}
a_3 & b_\phi \arrow[l, "f_\phi"'] \arrow[dr, phantom, "\theta''"] & b' \arrow[l, "f'"'] & b'' \arrow[l, "f''"'] \\
& a_2 \arrow[u, "x_\phi"] \arrow[ur, phantom, "\theta'"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x'"'] & b \arrow[l, "f"] \arrow[u, "x''"'] \\
& & a_1 \arrow[u, "x_2"] \arrow[ur, phantom, "\theta"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\
& & & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}$$

whence,

$$(\phi\psi)\rho = [f_\phi f', b', x'x_\psi]\,\rho = [f_\phi f' f'', b'', x''x'x_\rho] = \phi\,[f_\psi f, b, x_\rho] = \phi(\psi\rho).$$

Inverse: For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$, we can select a box γ in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a_0 & b \arrow[l, "f"'] \\
b_\rho \arrow[u, "x_\rho^{-1}"] \arrow[ur, phantom, "\gamma"] & a_1 \arrow[l, "f_\rho^{-1}"] \arrow[u, "x"']
\end{tikzcd}$$

and construct $\rho^{-1} = [f, b, x] \in \mathcal{G}(a_1, a_0)$. From the diagrams in $\mathcal{G}$

$$\begin{tikzcd}
a_0 & b \arrow[l, "f"'] & a_0 \arrow[l, "f^{-1}"'] \\
& a_1 \arrow[u, "x"] \arrow[ur, phantom, "\gamma^{-\mathrm{h}}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_\rho^{-1}"'] \\
& & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] & a_1 \arrow[l, "f_\rho^{-1}"'] \\
& a_0 \arrow[u, "x_\rho"] \arrow[ur, phantom, "\gamma^{-\mathrm{v}}"] & b \arrow[l, "f"] \arrow[u, "x^{-1}"'] \\
& & a_1 \arrow[u, "x"']
\end{tikzcd}$$

it follows that $\rho^{-1}\rho = id_{a_0}$ and $\rho\rho^{-1} = id_{a_1}$. $\square$

Lemma 2.4. (i) *For any two composable horizontal morphisms*

$$a_2 \xleftarrow{g} a_1 \xleftarrow{f} a_0$$

and for any two composable vertical morphisms

$$\begin{tikzcd} a_2 \\ a_1 \arrow[u, "y"'] \\ a_0 \arrow[u, "x"'] \end{tikzcd}$$

the equalities $[g][f] = [gf]$ *and* $[y][x] = [yx]$ *hold in* $\Pi\mathcal{G}$.

(ii) *For any path* (f, b, x) *in* $\mathcal{G}$, $[f, b, x] = [f][x]$.

(iii) *The filling problem in* $\mathcal{G}$

$$\begin{tikzcd} a_1 & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\exists ?"] & a_0 \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd}$$

has a solution if and only if $[y][g] = [f][x]$ *in* $\Pi\mathcal{G}$.

Proof. (i) follows from the existence of the first two diagrams below and (ii) by the third one.

$$\begin{tikzcd} a_2 & a_1 \arrow[l, "g"'] \arrow[d, equal] & a_0 \arrow[l, "f"'] \arrow[d, equal] \\ & a_1 \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}} f"] & a_0 \arrow[l, "f"] \arrow[d, equal] \\ & & a_0 \end{tikzcd}
\qquad
\begin{tikzcd} a_2 \arrow[r, equal] & a_2 \arrow[r, equal] & a_2 \\ & a_1 \arrow[u, "y"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}} y"] & a_1 \arrow[u, "y"'] \\ & & a_0 \arrow[u, "x"'] \end{tikzcd}
\qquad
\begin{tikzcd} a_1 & b \arrow[l, "f"'] \arrow[r, equal] \arrow[d, equal] & b \arrow[d, equal] \\ & b \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}b"] & b \\ & & a_0 \arrow[u, "x"'] \end{tikzcd}$$

For (iii), suppose first θ is any solution to the given filling problem. Then, the diagram

$$\begin{tikzcd} a_1 \arrow[r, equal] & a_1 & b \arrow[l, "f"'] \\ & c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[d, equal] \\ & & a_0 \end{tikzcd}$$

shows that $[y][g] = [f, b, x] \overset{(ii)}{=} [f][x]$. Conversely, assume that $[y][x] = [f][x] \overset{(ii)}{=} [f, b, x]$. By the filling condition on $\mathcal{G}$, we can select a box θ' of the form

$$\begin{tikzcd}
a_1 & b' \arrow[l, "f'"'] \\
c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a_0 \arrow[l, "g"] \arrow[u, "x'"']
\end{tikzcd}$$

whence, by the already proven part, $[y][g] = [f'][x'] = [f', b', x']$. It follows that $[f, b, x] = [f', b', x']$, and therefore there is a homotopy $\alpha : (f', b', x') \simeq (f, b, x)$ which gives us the solution θ that we are seeking for the filling problem by pasting the diagram

$$\begin{tikzcd}
a_1 & b' \arrow[l, "f'"'] & b \arrow[l, "f'^{-1}f"'] \\
d \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f'"] & b' \arrow[l, "f'"'] \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha"] & b' \arrow[u, "xx'^{-1}"'] \\
c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a_0 \arrow[l, "g"] \arrow[u, "x'"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x'"] & a_0 \arrow[u, "x'"']
\end{tikzcd}$$

□

3. The functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$

For each object a of $\mathcal{G}$, let $\pi_2(\mathcal{G}, a)$ denote the set of all boxes σ in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a \arrow[r, equal] & a \\
a \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & a \arrow[u, equal]
\end{tikzcd}$$

that is, whose horizontal source and target are both $\mathrm{I}^{\mathrm{v}}a$, the vertical identity of a, and whose vertical source and target are both $\mathrm{I}^{\mathrm{h}}a$, the horizontal identity of a. By the general Eckman-Hilton argument, the interchange law on $\mathcal{G}$ implies that both operations $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ on $\pi_2(\mathcal{G}, a)$ coincide and are commutative. Thus, $\pi_2(\mathcal{G}, a)$ is an abelian group with addition

$$\sigma + \tau := \sigma \circ_{\mathrm{h}} \tau = \sigma \circ_{\mathrm{v}} \tau,$$

zero $0 := \mathrm{I}a$, and opposites $-\sigma := \sigma^{-\mathrm{v}} = \sigma^{-\mathrm{h}}$.

The assignment $a \mapsto \pi_2(\mathcal{G}, a)$ is the function on objects of a functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$, which acts on morphism as follows. There is an abelian group valued functor on the groupoid of horizontal morphisms which assigns to each horizontal morphism $f : a_0 \to a_1$ the homomorphism

$$f_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$$

defined by $f_*\sigma = \mathrm{I}^{\mathrm{v}}f \circ_{\mathrm{h}} \sigma \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}}f^{-1}$,

$$\begin{tikzcd}
a_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] & \overset{f_*}{\mapsto} & a_1 \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & a_0 \arrow[l, "f"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}"] & a_1 \arrow[l, "f^{-1}"'] \arrow[d, equal] \\
a_0 \arrow[r, equal] & a_0 & & a_1 & a_0 \arrow[l, "f"] \arrow[r, equal] & a_0 & a_1 \arrow[l, "f^{-1}"]
\end{tikzcd}$$

and, similarly, there is an abelian group valued functor on the groupoid of vertical morphisms which assigns to each vertical morphism $x : a_0 \to a_1$ the homomorphism

$$x_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$$

defined by $x_*\sigma = \mathrm{I}^{\mathrm{h}}x \circ_{\mathrm{v}} \sigma \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}x^{-1}$,

$$\begin{tikzcd}
 & & & a_1 \arrow[r, equal] & a_1 \\
a_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] & \overset{x_*}{\mapsto} & a_0 \arrow[u, "x"] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x"] & a_0 \arrow[u, "x"'] \arrow[d, equal] \\
a_0 \arrow[r, equal] & a_0 & & a_0 \arrow[r, equal] & a_0 \\
 & & & a_1 \arrow[u, "x^{-1}"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x^{-1}"] & a_1 \arrow[u, "x^{-1}"']
\end{tikzcd}$$

Lemma 3.1. *If*

$$\begin{tikzcd}
a_1 \arrow[dr, phantom, "\theta"] & b \arrow[l, "f"'] \\
c \arrow[u, "y"] & a_0 \arrow[l, "g"] \arrow[u, "x"']
\end{tikzcd}$$

is any box in $\mathcal{G}$, then the diagram below commutes.

$$\begin{tikzcd}
\pi_2(\mathcal{G}, a_1) & \pi_2(\mathcal{G}, b) \arrow[l, "f_*"'] \\
\pi_2(\mathcal{G}, c) \arrow[u, "y_*"] & \pi_2(\mathcal{G}, a_0) \arrow[l, "g_*"] \arrow[u, "x_*"']
\end{tikzcd}$$

Proof. Let us consider, for any $\sigma \in \pi_2(\mathcal{G}, a_0)$, the following pasting diagram

$$\begin{tikzcd}
a_1 \arrow[dr, phantom, "\theta"] & b \arrow[l, "f"'] \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}x"] & b \arrow[dr, phantom, "\theta^{-\mathrm{h}}"] & a_1 \arrow[l, "f^{-1}"'] \\
c \arrow[u, "y"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}g"] & a_0 \arrow[l, "g"'] \arrow[u, "x"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[u, "x"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}g^{-1}"] & c \arrow[l, "g^{-1}"'] \arrow[u, "y"'] \arrow[d, equal] \\
c & a_0 \arrow[l, "g"'] \arrow[r, equal] \arrow[dl, phantom, "\theta^{-\mathrm{v}}"] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}x^{-1}"] & a_0 \arrow[dr, phantom, "\theta^{-\mathrm{hv}}"] & c \arrow[l, "g^{-1}"'] \\
a_1 \arrow[u, "y^{-1}"] & b \arrow[l, "f"] \arrow[u, "x^{-1}"] \arrow[r, equal] & b \arrow[u, "x^{-1}"'] & a_1 \arrow[l, "f^{-1}"] \arrow[u, "y^{-1}"']
\end{tikzcd}$$

The two natural ways to paste this diagram yield, on the one hand, $f_* x_* \sigma$ and, on other hand, $y_* g_* \sigma$. Hence $f_* x_* \sigma = y_* g_* \sigma$. □

For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$, we define the homomorphism

$$\rho_* := f_{\rho *} x_{\rho *} : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1),$$

where (f_ρ, b_ρ, x_ρ) is a representative path of ρ.

Lemma 3.2. *The homomorphism $\rho_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$ does not depend of the choice of representative path of ρ.*

Proof. If $(f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$, there is a box in $\mathcal{G}$ as below.

$$\begin{tikzcd}
b_\rho \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c \arrow[l, "f_\rho^{-1} g_\rho"'] \\
b_\rho \arrow[r, equal] & b_\rho \arrow[u, "y_\rho x_\rho^{-1}"']
\end{tikzcd}$$

Then, by Lemma 3.1, $f^{-1}_{\rho *} g_{\rho *} y_{\rho *} x^{-1}_{\rho *} = id_{\pi_2(\mathcal{G}, b_\rho)}$ or, equivalently, $g_{\rho *} y_{\rho *} = f_{\rho *} x_{\rho *}$. □

Theorem 3.3. *The assignments $a \mapsto \pi_2(\mathcal{G}, a)$, $\rho \mapsto \rho_*$, define a functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$.*

Proof. That $(id_a)_* = id$, for any object a of $\mathcal{G}$, is clear. Let $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ be two composable morphisms in $\Pi\mathcal{G}$. For any box θ as in (4), we have $\psi\rho = [f_\psi f, b, x x_\rho]$ Then, by Lemmas 3.2 and 3.1, $(\psi\rho)_* = f_{\psi *} f_* x_* x_{\rho *} = f_{\psi *} f_{\psi *} x_{\psi *} x_{\rho *} = \psi_* \phi_*$. □

3.1 The action of $\pi_2\mathcal{G}$ on boxes of $\mathcal{G}$

For any box in $\mathcal{G}$

$$\begin{tikzcd} d \arrow[dr, phantom, "\theta"] & b \arrow[l, "f"'] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd}$$

and any $\sigma \in \pi_2(\mathcal{G}, d)$, we define the box $\sigma + \theta$ (with the same edges as θ) by

$$\begin{tikzcd} d \arrow[dr, phantom, "\sigma+\theta"] & b \arrow[l, "f"'] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} := \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] & b \arrow[l, "f"'] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[r, phantom, "\theta"] & {} \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[uu, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[drr, phantom, "\theta"] & d & b \arrow[l] \\ c \arrow[u, "y"] & & a \arrow[ll, "g"] \arrow[u, "x"'] \end{tikzcd}$$

Clearly $0 + \theta = \theta$ and, for any $\tau, \sigma \in \pi_2(\mathcal{G}, d)$,

$$\tau + (\sigma + \theta) = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\tau"] & d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\tau+\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = (\tau + \sigma) + \theta. \tag{5}$$

Lemma 3.4. *For any $\sigma \in \pi_2(\mathcal{G}, d)$, any box θ as above, and any boxes*

$$\begin{tikzcd} c \arrow[dr, phantom, "\delta"] & a \arrow[l, "g"'] \\ \cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} b \arrow[dr, phantom, "\gamma"] & \cdot \arrow[l] \\ a \arrow[u, "x"] & \cdot \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[dr, phantom, "\alpha"] & d \arrow[l, "h"'] \\ \cdot \arrow[u] & c \arrow[l] \arrow[u, "y"'] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[dr, phantom, "\beta"] & d \arrow[l] \\ d \arrow[u, "z"] & b \arrow[l, "f"] \arrow[u] \end{tikzcd}$$

the following equalities hold,

$$(\sigma + \theta) \circ_{\mathrm{v}} \delta = \sigma + (\theta \circ_{\mathrm{v}} \delta), \tag{6}$$

$$(\sigma + \theta) \circ_{\mathrm{h}} \gamma = \sigma + (\theta \circ_{\mathrm{h}} \gamma), \tag{7}$$

$$\alpha \circ_{\mathrm{h}} (\sigma + \theta) = h_*\sigma + (\alpha \circ_{\mathrm{h}} \theta), \tag{8}$$

$$\beta \circ_{\mathrm{v}} (\sigma + \theta) = z_*\sigma + (\beta \circ_{\mathrm{v}} \theta). \tag{9}$$

Moreover,

$$(\sigma + \theta)^{-\mathrm{h}} = -f_*^{-1}\sigma + \theta^{-\mathrm{h}}, \tag{10}$$

$$(\sigma + \theta)^{-\mathrm{v}} = -y_*^{-1}\sigma + \theta^{-\mathrm{v}}. \tag{11}$$

Proof. (6) (the proof of (7) is dual):

$$(\sigma+\theta)\circ_{\mathrm{v}}\delta = \begin{tikzcd}
d \arrow[r, equal] & d & b \arrow[l, "f"'] \\
d \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & d \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I^v}f"] & b \arrow[l] \arrow[u, equal] \\
c \arrow[u, "y"] \arrow[urr, phantom, "\theta"] & & a \arrow[ll, "g"] \arrow[u, "x"'] \\
\cdot \arrow[u] \arrow[urr, phantom, "\delta"] & & \cdot \arrow[ll] \arrow[u]
\end{tikzcd} = \sigma + (\theta\circ_{\mathrm{v}}\delta).$$

(8) (the proof of (9) is dual):

$$\alpha+(\sigma+\theta) = \begin{tikzcd}
\cdot & d \arrow[l, "h"'] \arrow[r, equal] & d & b \arrow[l, "f"'] \\
 & d \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & d \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I^v}f"] & b \arrow[l] \arrow[u, equal] \\
\cdot \arrow[uu] \arrow[uur, phantom, "\alpha"] & c \arrow[l] \arrow[u, "y"] \arrow[urr, phantom, "\theta"] & & a \arrow[ll, "g"] \arrow[u, "x"']
\end{tikzcd} = \begin{tikzcd}
\cdot & d \arrow[l, "h"'] \arrow[r, equal] & d & \cdot \arrow[l, "h^{-1}"'] & d \arrow[l, "hf"'] \\
\cdot \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I^v}h"] & d \arrow[l] \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & d \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I^v}h^{-1}"] & \cdot \arrow[l] \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I^v}(hf)"] & d \arrow[l] \arrow[u, equal] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & c \arrow[l] \arrow[u, "y"] \arrow[urrr, phantom, "\theta"] & & & a \arrow[llll, "g"] \arrow[u, "x"']
\end{tikzcd}$$

$$= h_*(\sigma)+\alpha\circ_{\mathrm{h}}\theta.$$

(10) (the proof of (11) is dual):

$$\begin{aligned}
(\sigma+\theta)\circ_{\mathrm{h}}(-f_*\sigma+\theta^{-\mathrm{h}}) &\overset{(7)}{=} \sigma+\big(\theta\circ_{\mathrm{h}}(-f_*\sigma+\theta^{-\mathrm{h}})\big)\\
&\overset{(9)}{=} \sigma+\big(-f_*f_*^{-1}\sigma+\theta\circ_{\mathrm{h}}\theta^{-\mathrm{h}}\big)\\
&\overset{(5)}{=} (\sigma-\sigma)+\mathrm{I^h}y = 0+\mathrm{I^h}y=\mathrm{I^h}y.
\end{aligned}$$

$\square$

Lemma 3.5. *For any two boxes with the same edges*

$$\begin{tikzcd}
d & b \arrow[l, "f"'] \\
c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a \arrow[l, "g"] \arrow[u, "x"']
\end{tikzcd} \qquad \begin{tikzcd}
d & b \arrow[l, "f"'] \\
c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a \arrow[l, "g"] \arrow[u, "x"']
\end{tikzcd}$$

there is a unique $\sigma\in\pi_2(\mathcal{G},d)$ such that $\sigma+\theta=\theta'$.

Proof. Uniqueness. For any $\sigma \in \pi_2(\mathcal{G}, d)$ and θ as above,

$$(\sigma + \theta) \circ_{\mathrm{h}} \theta^{-\mathrm{h}} \overset{(8)}{=} \sigma + (\theta \circ_{\mathrm{h}} \theta^{-\mathrm{h}}) = \sigma + \mathrm{I}^{\mathrm{h}}y = \sigma \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}y.$$

Hence, $\sigma + \theta$ determines σ as $\sigma = \big((\sigma + \theta) \circ_{\mathrm{h}} \theta^{-\mathrm{h}}\big) \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}y^{-1}$.

Existence. Taking

$$\sigma = \begin{tikzcd} d & b \arrow[l, "f"'] & d \arrow[l, "f^{-1}"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[ur, phantom, "\theta^{-\mathrm{h}}"] & c \arrow[l, "g^{-1}"] \arrow[u, "y"'] \\ d \arrow[u, "y^{-1}"] \arrow[rr, equal] \arrow[urr, phantom, "\mathrm{I}^{\mathrm{h}}y^{-1}"] & & d \arrow[u, "y^{-1}"'] \end{tikzcd}$$

we have $\sigma + \theta = \big((\theta' \circ_{\mathrm{h}} \theta^{-\mathrm{h}}) \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}y^{-1} \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}y\big) \circ_{\mathrm{h}} \theta = \theta' \circ_{\mathrm{h}} \theta^{-\mathrm{h}} \circ_{\mathrm{h}} \theta = \theta'$ $\square$

4. The Postnikov invariant $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$

Let P be a groupoid. The category Ab^P of functors $\mathcal{A} : P \to \mathsf{Ab}$ is abelian and it has enough injectives and projective objects [19]. We can, thus, form the right derived functors of the functor $\varprojlim : \mathsf{Ab}^P \to \mathsf{Ab}$, which is given by

$$\varprojlim(\mathcal{A}) = \Big\{(x_a) \in \prod_{a \in \mathrm{Ob}P} \mathcal{A}(a) \mid \rho_* x_a = x_b \text{ for every } \rho : a \to b \text{ in } P\Big\},$$

where we write $\rho_* x$ for $\mathcal{A}(\rho)(x)$. The *cohomology groups of the groupoid P with coefficients in a functor $\mathcal{A} : P \to \mathsf{Ab}$* [26], denoted by $H^n(P, \mathcal{A})$, are defined by

$$H^n(P, \mathcal{A}) = (R^n \varprojlim)(A), \qquad n = 0, 1, \cdots .$$

To exhibit an explicit cochain complex that computes these cohomology groups, let $\mathrm{N}P$ be the nerve of P. That is, the simplicial set whose m-simplices are the composable sequences $\beta = (\beta m \overset{\beta_m}{\leftarrow} \cdots \overset{\beta_1}{\leftarrow} \beta 0)$ of m arrows in P (objects of P if $m = 0$). The face $d_i\beta$, for $0 < i < m$, is obtained from β by replacing the morphisms β_{i+1} and β_i by their composition $\beta_{i+1}\beta_i$, while $d_0\beta$ and $d_m\beta$ are obtained by leaving out $\beta 0$ and βm, respectively. The degeneracies $s_i\beta$ are obtained by inserting in β the identity morphism $id_{\beta i}$. This simplicial set $\mathrm{N}P$ is a Kan complex whose fundamental groupoid is P (and whose homotopy groups vanish in degree 2 and higher). Thus, every functor $\mathcal{A} : P \to \mathsf{Ab}$ defines a local coefficient system on $\mathrm{N}P$ and the cohomology groups $H^n(\mathrm{N}P, \mathcal{A})$ are defined [17, 18, 21]. By Illusie [21, Chap.VI, (3.4.2)] and Gabriel and Zisman [17, Appendix II, Prop. 3.3], there are natural isomorphisms

$$H^n(P, \mathcal{A}) \cong H^n(\mathrm{N}P, \mathcal{A}) \cong H^n C^\bullet(P, \mathcal{A}), \qquad n = 0, 1, \cdots .$$

where

$$C^\bullet(P, \mathcal{A}) : 0 \to C^0(P, \mathcal{A}) \to \cdots \to C^{m-1}(P, \mathcal{A}) \xrightarrow{\partial} C^m(P, \mathcal{A}) \to \cdots ,$$

denotes the *complex of normalized cochains of P with coefficients in $\mathcal{A}$*. Here, a normalized m-cochain $c \in C^m(P, \mathcal{A})$ is a function

$$c : \mathrm{N}P_m \to \bigsqcup_{a \in \mathrm{Ob}P} \mathcal{A}(a)$$

such that $c(\beta) \in \mathcal{A}(\beta m)$ and $c(\beta) = 0$ whenever some β_i is an identity. Each $C^m(P, \mathcal{A})$ is an abelian group with pointwise addition, and the coboundary $\partial : C^{m-1}(P, \mathcal{A}) \to C^m(P, \mathcal{A})$ is given by

$$\partial c(\beta) = \sum_{i=0}^{m-1} c(d_i\beta) + (-1)^m \beta_{m*} c(d_m\beta).$$

As usually, we write $Z^n(P, \mathcal{A})$ for the groups of n-cocycles of the complex $C^\bullet(P, \mathcal{A})$.

In this paper, we will only use the cohomology groups $H^3(P, \mathcal{A})$. For future reference let us specify that a normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$ is a function assigning to each three composable morphisms in the groupoid $a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1 \xleftarrow{\rho} a_0$ an element $k(\phi, \psi, \rho) \in \mathcal{A}(a_3)$ such that, for any four composable morphisms $a_4 \xleftarrow{\delta} a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1 \xleftarrow{\rho} a_0$, the 3-cocycle condition

$$k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho) + k(\delta, \phi\psi, \rho) - k(\delta\phi, \psi, \rho) + \delta_* k(\phi, \psi, \rho) = 0.$$

holds, and $k(\phi, \psi, \rho) = 0$ if one of the morphisms ϕ, ψ or ρ is an identity.

A normalized 2-cochain $c \in C^2(P, \mathcal{A})$ is a function assigning to each pair of composable morphisms $a_2 \xleftarrow{\phi} a_1 \xleftarrow{\psi} a_0$ an element $c(\phi, \psi) \in \mathcal{A}(a_2)$, such that $c(\phi, \psi) = 0$ whenever $\phi = id_{a_1}$ or $\psi = id_{a_0}$. The coboundary of such a 2-cochain is the 3-cocycle ∂c given by

$$\partial c(\phi, \psi, \rho) = c(\phi, \psi) - c(\phi, \psi\rho) + c(\phi\psi, \rho) - \phi_* c(\psi, \rho).$$

Two normalized 3-cocycles $k, k' \in Z^3(P, \mathcal{A})$ are cohomologous if and only if there is a normalized 2-cochain $c \in C^2(P, \mathcal{A})$ such that $k' = k + \partial c$.

Definition 4.1. *A (2-dimensional) Postnikov system $(P, \mathcal{A}, \mathbf{k})$ consists of a groupoid P, an abelian group valued functor $\mathcal{A} : P \to \mathsf{Ab}$, and a cohomology class $\mathbf{k} \in H^3(P, \mathcal{A})$. Two such Postnikov systems $(P, \mathcal{A}, \mathbf{k})$ and $(P', \mathcal{A}', \mathbf{k}')$ are equivalent if there exists an equivalence $\mathfrak{f} : P \xrightarrow{\sim} P'$ and a natural isomorphism $\mathfrak{F} : \mathcal{A} \cong \mathfrak{f}^*\mathcal{A}'$ such that $\mathfrak{f}^*(\mathbf{k}') = \mathfrak{F}_*(\mathbf{k})$, where*

$$\mathfrak{f}^* : H^3(P', \mathcal{A}') \cong H^3(P, \mathfrak{f}^*\mathcal{A}'), \qquad \mathfrak{F}_* : H^3(P, \mathcal{A}) \cong H^3(P, \mathfrak{f}^*\mathcal{A}')$$

are the corresponding induced isomorphisms in cohomology.

Let $[P, \mathcal{A}, \mathbf{k}]$ denote the equivalence class of a Postnikov system $(P, \mathcal{A}, \mathbf{k})$.

Let $\mathcal{G}$ be a double groupoid. We associate to $\mathcal{G}$ a Postnikov system $(\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ as follows. For each morphism in the fundamental groupoid $\rho \in \Pi\mathcal{G}(a_0, a_1)$, let us choose a representative path (f_ρ, b_ρ, x_ρ) of ρ, as in (3). In particular, if $\rho = id_a$ for some object a of $\mathcal{G}$, we take $(\mathrm{I}^{\mathrm{h}}a, a, \mathrm{I}^{\mathrm{v}}a)$ as its representative path.

If $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any two composable morphisms in $\Pi\mathcal{G}$, by Lemma 2.4, we have $[f_\psi][x_\psi][f_\rho][x_\rho] = \psi\rho = [f_{\psi\rho}][x_{\psi\rho}]$, whence

$$\begin{aligned}[x_\psi][f_\rho] &= [f_\psi^{-1}][f_{\psi\rho}][x_{\psi\rho}][x_\rho^{-1}] = [f_\psi^{-1} f_{\psi\rho}][x_{\psi\rho} x_\rho^{-1}] \\ &= [f_\psi^{-1} f_{\psi\rho}, b_{\psi\rho}, x_{\psi\rho} x_\rho^{-1}],\end{aligned}$$

and therefore we can select a box $\theta_{\psi,\rho}$ in $\mathcal{G}$ as below.

$$\begin{tikzcd} b_\psi & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] \\ a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \end{tikzcd} \tag{12}$$

In particular, we choose

$$\theta_{id_{a_1},\rho} = \mathrm{I}^{\mathrm{v}} f_\rho, \qquad \theta_{\psi, id_{a_0}} = \mathrm{I}^{\mathrm{h}} x_\psi. \tag{13}$$

If $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any three composable morphisms in $\Pi\mathcal{G}$, pasting in the diagrams

$$
\begin{tikzcd}
b_{\phi\psi} & b_\phi \arrow[l, "f_{\phi\psi}^{-1}f_\phi"'] & b_{\phi\psi\rho} \arrow[l, "f_\phi^{-1}f_{\phi\psi\rho}"'] \\
b_\phi \arrow[u, "x_{\phi\psi}x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{h}}_{\phi,\psi}"] & a_2 \arrow[l, "f_\psi^{-1}"] \arrow[u] \arrow[ur, phantom, "\theta_{\phi,\psi\rho}"] & b_{\psi\rho} \arrow[l, "f_{\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x_{\psi\rho}^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
b_{\phi\psi} & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] \\
a_1 \arrow[u, "x_{\phi\psi}"] \arrow[ur, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[l] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \\
b_\psi \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\phi\psi} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"']
\end{tikzcd}
$$

yields two boxes with the same edges, and therefore, by Lemma 3.5 and the isomorphism $f_{\phi\psi*} : \pi_2(\mathcal{G}, b_{\phi\psi}) \cong \pi_2(\mathcal{G}, a_3)$, there is a unique element $k^{\mathcal{G}}(\phi, \psi, \rho) \in \pi_2(\mathcal{G}, a_3)$ such that

$$f_{\phi\psi*}^{-1}k^{\mathcal{G}}(\phi, \psi, \rho) + (\theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}) = \theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}. \tag{14}$$

Note that composing horizontally with $\theta_{\phi,\psi}$ on the left in (14), by (8), gives

$$f_{\phi*}^{-1}k^{\mathcal{G}}(\phi, \psi, \rho) + \theta_{\phi,\psi\rho} = \theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}). \tag{15}$$

More explicitly,

$$
k^{\mathcal{G}}(\phi, \psi, \rho) =
\begin{tikzcd}
a_3 \arrow[ddd, equal] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"'] \arrow[ddd, equal] \\
 & & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[ur, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] & \\
 & a_2 \arrow[uu, "x_\phi"] \arrow[uur, phantom, "\theta_{\phi,\psi}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & \\
a_3 \arrow[uuur, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[urr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \arrow[uuur, phantom, "\mathrm{I}^{\mathrm{v}}f_{\phi\psi\rho}^{-1}"] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"]
\end{tikzcd}
\tag{16}$$

In fact, composing vertically with $\theta^{-\mathrm{v}}_{\phi,\psi\rho}$ on the right in (15) yields

$$f_{\phi*}^{-1}k^{\mathcal{G}}(\phi, \psi, \rho) + \mathrm{I}^{\mathrm{v}}(f_\phi^{-1}f_{\phi\psi\rho}) = \big(\theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\phi,\psi\rho},$$

whence

$$\begin{aligned}
&\Gamma^{\mathrm{v}} f_\phi \circ_{\mathrm{h}} \Big(\big(\theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\phi,\psi\rho}\Big) \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f^{-1}_{\phi\psi\rho} \\
&\quad = \Gamma^{\mathrm{v}} f_\phi \circ_{\mathrm{h}} \big(f^{-1}_{\phi *} k(\phi,\psi,\rho) + \Gamma^{\mathrm{v}}(f^{-1}_\phi f_{\phi\psi\rho})\big) \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f^{-1}_{\phi\psi\rho} \\
&\quad \overset{(8)}{=} k^{\mathcal{G}}(\phi,\psi,\rho) + \Gamma^{\mathrm{v}}(f_\phi f^{-1}_\phi f_{\phi\psi\rho} f^{-1}_{\phi\psi\rho}) \\
&\quad = k^{\mathcal{G}}(\phi,\psi,\rho) + 0 = k^{\mathcal{G}}(\phi,\psi,\rho).
\end{aligned}$$

Lemma 4.2. *So defined, $k^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$, that is, $k^{\mathcal{G}}$ is normalized 3-cocycle of $\Pi\mathcal{G}$ with coefficients in $\pi_2\mathcal{G}$.*

Proof. That $k^{\mathcal{G}}$ is a normalized cochain, that is, $k^{\mathcal{G}}(\phi,\psi,\rho) = 0$ whenever one of the morphisms ϕ, ψ or ρ is an identity, follows from the selection in (13). For instance, if $\phi = id_{a_2}$, then $k^{\mathcal{G}}(id_{a_2},\psi,\rho) = 0$ since

$$\begin{aligned}
\theta^{-\mathrm{h}}_{id_{a_2},\psi} \circ_{\mathrm{h}} \theta_{id_{a_2},\psi\rho} &= \Gamma^{\mathrm{v}} f^{-1}_\psi \circ_{\mathrm{h}} \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\psi\rho} = \Gamma^{\mathrm{v}}(f^{-1}_\psi f_{\psi\rho}) = \theta_{\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho} \\
&= \theta_{id_{a_2}\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}.
\end{aligned}$$

To prove that $k^{\mathcal{G}}$ is a 3-cocycle, suppose $a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are morphisms in $\Pi\mathcal{G}$. By using first horizontal composition in the diagram below, we see, from (16), that the pasted boxes of the inner regions labeled with (A), (B) and (C) are

$$\begin{aligned}
(A) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta,\phi,\psi) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}, \\
(B) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta,\phi\psi,\rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}, \\
(C) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ \delta_* k^{\mathcal{G}}(\phi,\psi,\rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}.
\end{aligned}$$

$$
\begin{tikzcd}
 & b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] \arrow[dr, phantom, "\theta_{\delta\phi,\psi}"] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \arrow[rr, equal] \arrow[ddd, equal] \arrow[ddrr, phantom, "\mathrm{I}b_{\delta\phi\psi\rho}"] & & b_{\delta\phi\psi\rho} \arrow[ddd, equal] \\
(A) & \arrow[r, phantom, "\theta_{\delta,\phi}"] & a_2 \arrow[u, "x_{\delta\phi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi}"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_{\delta\phi\psi}x_\psi^{-1}"'] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}(f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho})"] & & & \\
 & a_3 \arrow[uu, "x_\delta"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u, "x_\psi x_{\phi\psi}^{-1}"'] & & & \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] & & b_{\delta\phi\psi} \arrow[ll, "f_\delta^{-1}f_{\delta\phi\psi}"] \arrow[u, "x_{\phi\psi}x_{\delta\phi\psi}^{-1}"'] \arrow[dr, phantom, "\theta_{\delta\phi\psi,\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"] \arrow[rr, equal] & & b_{\delta\phi\psi\rho} \\
(B) & \arrow[rr, phantom, "\theta_{\delta,\phi\psi}"] & & a_1 \arrow[u, "x_{\delta\phi\psi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \arrow[rr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x_{\phi\psi\rho}^{-1})"] & & \\
 & a_3 \arrow[uu, "x_\delta"] \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u, "x_{\phi\psi}^{-1}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_\rho x_{\phi\psi\rho}^{-1}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[uu, "x_{\delta\phi\psi\rho}x_{\phi\psi\rho}^{-1}"] \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta"] & & & & b_\delta \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x_{\delta\phi\psi\rho}^{-1}"] \arrow[ddddd, equal] \\
 & a_3 \arrow[u, "x_\delta"] \arrow[ddd, equal] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"'] \arrow[u, "x_\delta"'] \arrow[ddd, equal] & \\
(C) & \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & \arrow[r, phantom, "\theta_{\phi,\psi}"] & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}f_{\phi\psi\rho}^{-1}"] & \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi\psi\rho})"] & \\
 & & a_2 \arrow[uu, "x_\phi"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & & \\
 & a_3 \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & b_\phi \arrow[l, "f_\phi"'] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"] & \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] & & & & b_\delta \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"']
\end{tikzcd}
$$

Hence, using now vertical composition of inner boxes in it, we see that

$\mathrm{I}^{\mathrm{v}}f_\delta^{-1} \circ_{\mathrm{h}} \big(k^{\mathcal{G}}(\delta,\phi,\psi) + k^{\mathcal{G}}(\delta,\phi\psi,\rho) + \delta_* k^{\mathcal{G}}(\phi,\psi,\rho)\big) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}}f_{\delta\phi\psi\rho} =$

$$
\begin{tikzcd}
b_\delta \arrow[dr, phantom, "\theta_{\delta,\phi}"] & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] \arrow[dr, phantom, "\theta_{\delta\phi,\psi}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi,\psi}"] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] \arrow[dr, phantom, "\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \arrow[drr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x_{\phi\psi\rho}^{-1})"] & & b_{\delta\phi\psi\rho} \arrow[ll, equal] \\
a_3 \arrow[u, "x_\delta"] & b_\phi \arrow[l, "f_\phi"] \arrow[u] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"] \arrow[u] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"'] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[u, "x_{\delta\phi\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & & & & b_\delta \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x_{\delta\phi\psi\rho}^{-1}"'] \arrow[ddddd, equal] \\
a_3 \arrow[u, "x_\delta"] \arrow[ddd, equal] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta"] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & b_\phi \arrow[l, "f_\phi"'] \arrow[ddr, phantom, "\theta_{\phi,\psi}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f_{\phi\psi\rho}^{-1}"] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"'] \arrow[u, "x_\delta"'] \arrow[ddd, equal] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi\psi\rho})"] & \\
& & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] & & \\
& a_2 \arrow[uu, "x_\phi"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_\psi^{-1}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & & \\
a_3 & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"] & \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & & & & b_\delta \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"']
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] & & b_{\delta\phi\psi\rho} \arrow[ll, equal] \\
a_3 \arrow[u, "x_\delta"] \arrow[ur, phantom, "\theta_{\delta,\phi}"] \arrow[ddd, equal] & b_\phi \arrow[l] \arrow[u] \arrow[ur, phantom, "\theta_{\delta\phi,\psi}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi,\psi}"] & b_{\phi\psi} \arrow[l] \arrow[u] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l] \arrow[u] \arrow[urr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho})"] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"'] \arrow[ddd, equal] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[u, "x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho}"'] \arrow[ddd, equal] \\
\arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & & a_1 \arrow[u] \arrow[l, phantom, "\theta_{\phi,\psi}"] & b_\rho \arrow[l] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \arrow[ul, phantom, "\theta_{\phi\psi,\rho}"] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}_{\phi\psi\rho}"] & & \\
& a_2 \arrow[uu, "x_\phi"] & b_\psi \arrow[l] \arrow[u] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] & \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}f_{\phi\psi\rho}"] & \\
a_3 & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[urr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & & & & b_\delta \arrow[llll, equal] \arrow[u, "x_\delta^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"']
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \\
& & a_1 \arrow[u, "x_{\delta\phi}"] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}"] \arrow[l, phantom, "\theta_{\delta\phi,\psi}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \\
& a_2 \arrow[uu, "x_{\delta\phi}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\phi\rho}"] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] \\
a_3 \arrow[uuu, "x_\delta"] \arrow[uuur, phantom, "\theta_{\delta,\phi}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[urr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\phi}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[urrr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & & & b_{\delta\phi\psi\rho} \arrow[lll, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"']
\end{tikzcd}
\tag{17}
$$

Now, we realize that the diagram (17) above is also obtained by using vertical composition of inner boxes in the following diagram

$$
\begin{tikzcd}
 & b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \\
(D) & & & a_1 \arrow[u, "x_{\delta\phi\psi}"] \arrow[ul, phantom, "\theta_{\delta\phi,\psi}"] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \\
 & & a_2 \arrow[uu, "x_{\delta\phi}"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l, "f_\phi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] \\
 & b_\delta \arrow[uuu, equal] \arrow[uur, phantom, "\mathrm{I}^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi})"] & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] \arrow[u, "x_{\delta\phi}^{-1}"] & & b_{\delta\phi\psi\rho} \arrow[ll, "f_{\delta\phi}^{-1}f_{\delta\phi\psi\rho}"'] \arrow[u, "x_{\psi\rho}x_{\delta\phi\psi\rho}^{-1}"'] \arrow[ull, phantom, "\theta^{-\mathrm{v}}_{\delta\phi,\psi\rho}"] \\
(E) & & a_2 \arrow[u, "x_{\delta\phi}"] & & b_{\psi\rho} \arrow[ll, "f_{\phi\psi}"'] \arrow[u, "x_{\delta\phi\psi\rho}x_{\psi\rho}^{-1}"'] \arrow[ul, phantom, "\theta_{\delta\phi,\psi\rho}"] \\
 & a_3 \arrow[uu, "x_\delta"] \arrow[ur, phantom, "\theta_{\delta,\phi}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \arrow[ul, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] & & & b_{\delta\phi\psi\rho} \arrow[lll, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x_{\delta\phi\psi\rho}^{-1}"'] \arrow[ull, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"]
\end{tikzcd}
$$

where the pasted boxes of the inner regions labeled with (D) and (E) are easily recognized, by (16), to be

$$
\begin{aligned}
(D) &= \mathrm{I}^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}}\ k^{\mathcal{G}}(\delta\phi, \psi, \rho)\ \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}} f_{\delta\phi\psi\rho},\\
(E) &= \mathrm{I}^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}}\ k^{\mathcal{G}}(\delta, \phi, \psi\rho)\ \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}} f_{\delta\phi\psi\rho}.
\end{aligned}
$$

So, the resulting pasted box of the diagram (17) is also

$$\mathrm{I}^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}} (k^{\mathcal{G}}(\delta\phi, \psi, \rho) + k^{\mathcal{G}}(\delta, \phi, \psi\rho)) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}} f_{\delta\phi\psi\rho}.$$

This proves the 3-cocycle condition, that is,

$$k^{\mathcal{G}}(\delta\phi, \psi, \rho) + k^{\mathcal{G}}(\delta, \phi, \psi\rho) = k^{\mathcal{G}}(\delta, \phi, \psi) + k^{\mathcal{G}}(\delta, \phi\psi, \rho) + \delta_* k^{\mathcal{G}}(\phi, \psi, \rho).$$

□

Next, we observe the effect of different choices of (f_ρ, b_ρ, x_ρ) and $\theta_{\psi,\rho}$ in the construction of the 3-cocycle $k^{\mathcal{G}}$.

Lemma 4.3. (i) *If the choice of the boxes $\theta_{\psi,\rho}$ in* (12) *is changed, then $k^{\mathcal{G}}$ is changed to a cohomologous cocycle. By suitably changing the boxes $\theta_{\psi,\rho}$, $k^{\mathcal{G}}$ may by changed to any cohomologous cocycle.*

(ii) *If the choice of the representative paths* (f_ρ, b_ρ, x_ρ) *in* (3) *is changed, then a suitable new selection of the boxes* $\theta_{\psi,\rho}$ *leaves the cocycle* $k^{\mathcal{G}}$ *unaltered.*

Proof. (i) Let, for each two composable morphisms $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in $\Pi\mathcal{G}$,

$$\begin{tikzcd} b_\psi & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] \\ a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta'_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \end{tikzcd}$$

be any other selection of boxes in (3), and let $k'^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2)$ be the corresponding 3-cocycle.

By Lemma 3.5 and the isomorphism $f_{\psi *} : \pi_2(\mathcal{G}, b_\psi) \cong \pi_2(\mathcal{G}, a_2)$, we can write $\theta'_{\psi,\rho} = f_{\psi *}^{-1} c(\psi, \rho) + \theta_{\psi,\rho}$ for a unique element $c(\psi, \rho) \in \pi_2(\mathcal{G}, a_2)$, and a normalized 2-cochain $c \in C^2(\Pi\mathcal{G}, \pi_2\mathcal{G})$ becomes so defined. Then, for every composable morphisms $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, we have

$$\begin{aligned}
&f_{\phi*}^{-1} k'^{\mathcal{G}}(\phi, \psi, \rho) + f_{\phi*}^{-1} c(\phi, \psi\rho) + \theta_{\phi,\psi\rho} \\
&\overset{(15)}{=} (f_{\phi*}^{-1} c(\phi, \psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} \big((f_{\phi\psi*}^{-1} c(\phi\psi, \rho) + \theta_{\phi\psi,\rho}) \circ_{\mathrm{v}} (f_{\psi*}^{-1} c(\psi, \rho) + \theta_{\psi,\rho})^{-\mathrm{v}}\big) \\
&\overset{(11)}{=} (f_{\phi*}^{-1} c(\phi, \psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} \big((f_{\phi\psi*}^{-1} c(\phi\psi, \rho) + \theta_{\phi\psi,\rho}) \circ_{\mathrm{v}} (-x_{\psi*}^{-1} f_{\psi*}^{-1} c(\psi, \rho) + \theta_{\psi,\rho}^{-\mathrm{v}})\big) \\
&\overset{(9)}{=} (f_{\phi*}^{-1} c(\phi, \psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} (f_{\phi\psi*}^{-1} c(\phi\psi, \rho) - x_{\phi\psi*} x_{\psi*}^{-1} f_{\phi*}^{-1} c(\psi, \rho) + \theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta_{\psi,\rho}^{-\mathrm{v}}) \\
&\overset{(8)}{=} f_{\phi*}^{-1} c(\phi, \psi) + f_{\phi*}^{-1} c(\phi\psi, \rho) - f_{\phi*}^{-1} f_{\phi\psi*} x_{\phi\psi*} x_{\psi*}^{-1} f_{\psi*}^{-1} c(\psi, \rho) \\
&\qquad + \theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta_{\psi,\rho}^{-\mathrm{v}}) \\
&\overset{(15)}{=} f_{\phi*}^{-1} c(\phi, \psi) + f_{\phi*}^{-1} c(\phi\psi, \rho) - f_{\phi*}^{-1} f_{\phi\psi*} x_{\phi\psi*} x_{\psi*}^{-1} f_{\psi*}^{-1} c(\psi, \rho) \\
&\qquad + f_{\phi*}^{-1} k^{\mathcal{G}}(\phi, \psi, \rho) + \theta_{\phi,\psi\rho}
\end{aligned}$$

whence, by Lemma 3.5,

$$\begin{aligned}
&k'^{\mathcal{G}}(\phi, \psi, \rho) + c(\phi, \psi\rho) \\
&\qquad = c(\phi, \psi) + c(\phi\psi, \rho) - f_{\phi\psi*} x_{\phi\psi*} x_{\psi*}^{-1} f_{\psi*}^{-1} c(\psi, \rho) + k^{\mathcal{G}}(\phi, \psi, \rho).
\end{aligned}$$

As, by Theorem 3.3 and Lemma 2.4,

$$f_{\phi\psi*} x_{\phi\psi*} x_{\psi*}^{-1} = (\phi\psi)_* x_{\psi*}^{-1} = \phi_* \psi_* x_{\psi*}^{-1} = \phi_* f_{\psi*} x_{\psi*} x_{\psi*}^{-1} = \phi_* f_{\psi*},$$

we finally conclude that

$$\begin{aligned} k'^{\mathcal{G}}(\phi,\psi,\rho) &= c(\phi,\psi) - c(\phi,\psi\rho) + c(\phi\psi,\rho) - \phi_* c(\psi,\rho) + k^{\mathcal{G}}(\phi,\psi,\rho) \\ &= \partial c(\phi,\psi,\rho) + k^{\mathcal{G}}(\phi,\psi,\rho). \end{aligned}$$

Thus, $k'^{\mathcal{G}} = \partial c + k^{\mathcal{G}}$ and therefore $k^{\mathcal{G}}$ and $k'^{\mathcal{G}}$ are cohomologous.

Conversely, suppose $c \in C^2(\Pi\mathcal{G}, \pi_2\mathcal{G})$ is any normalized 2-cochain and $k = \partial c + k^{\mathcal{G}}$. Then $f_{\psi *}^{-1} c(\psi,\rho) + \theta_{\psi,\rho}$ is an allowable choice of $\theta'_{\psi,\rho}$, for each pair of composable morphisms (ψ,ρ) in $\Pi\mathcal{G}$, for which, by the already shown above, the corresponding 3-cocycle just becomes $k'^{\mathcal{G}} = \partial c + k^{\mathcal{G}} = k$.

(ii) Suppose we have choosen another representative path (g_ρ, c_ρ, y_ρ) of each morphism ρ in $\Pi\mathcal{G}$. Then, we can select homotopies $\alpha_\rho : (f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$ and construct, for each two morphisms $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, the box

$$\begin{tikzcd}
c_\psi \arrow[r, phantom, "\theta'_{\psi,\rho}"] & c_{\psi\rho} \arrow[l, "g_\psi^{-1} g_{\psi\rho}"'] \\
a_1 \arrow[u, "y_\psi"] & c_\rho \arrow[l, "g_\rho"] \arrow[u, "y_{\psi\rho} y_\rho^{-1}"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
c_\psi \arrow[dr, phantom, "\alpha_\psi^{-\mathrm{h}}"] & b_\psi \arrow[l, "g_\psi^{-1} f_\psi"'] \arrow[d, equal] & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] \arrow[d, equal] \arrow[dr, phantom, "\alpha_{\psi\rho}"] & c_{\psi\rho} \arrow[l, "f_{\psi\rho}^{-1} g_{\psi\rho}"'] \\
b_\psi \arrow[u, "y_\psi x_\psi^{-1}"] \arrow[r, equal] \arrow[ddr, phantom, "\mathrm{I}^{\mathrm{h}} x_\psi"] & b_\psi \arrow[r, phantom, "\theta_{\psi,\rho}"] & b_{\psi\rho} \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\psi\rho} x_\rho^{-1})"] & b_{\psi\rho} \arrow[u, "y_{\psi\rho} x_{\psi\rho}^{-1}"'] \\
 & a_1 \arrow[u, "x_\psi"] \arrow[d, equal] & b_\rho \arrow[u] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha_\rho^{-\mathrm{v}}"] & b_\rho \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \\
a_1 \arrow[uu, "x_\psi"] \arrow[r, equal] & a_1 & b_\rho \arrow[l, "f_\rho"] & c_\rho \arrow[l, "f_\rho^{-1} g_\rho"] \arrow[u, "x_\rho y_\rho^{-1}"']
\end{tikzcd}$$

which, by the already proven part (i), we can use to define the corresponding 3-cocycle $k'^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ from the new selected representative paths. Then, for any three composable morphisms $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$,

$$\mathrm{I}^{\mathrm{v}} g_\phi^{-1} \circ_{\mathrm{h}} k'^{\mathcal{G}}(\phi,\psi,\rho) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}} g_{\phi\psi\rho} = \big(\theta'_{\phi,\psi} \circ_{\mathrm{h}} (\theta'_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta'^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} \theta'^{-\mathrm{v}}_{\phi,\psi\rho} =$$

$$
\begin{tikzcd}
c_\phi & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] & c_{\phi\psi} \arrow[l, "f_{\phi\psi}^{-1}g_{\phi\psi}"'] & b_{\phi\psi} \arrow[l, "g_{\phi\psi}^{-1}f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"'] \\
b_\phi \arrow[u, "y_\phi x_\phi^{-1}"] \arrow[r, phantom, "\alpha_\phi^{-\mathrm{h}}"] \arrow[rdddd, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_\phi \arrow[l, equal] \arrow[u, equal] & b_{\phi\psi} \arrow[u, equal] \arrow[r, phantom, "\alpha_{\phi\psi}"] & b_{\phi\psi} \arrow[l, equal] \arrow[u] \arrow[r, phantom, "\alpha_{\phi\psi}^{-\mathrm{h}}"] & b_{\phi\psi} \arrow[l, equal] \arrow[u, equal] & b_{\phi\psi\rho} \arrow[u, equal] \arrow[r, phantom, "\alpha_{\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, equal] \arrow[u, "y_{\phi\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
 & & & & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[r, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[u] \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_\rho \arrow[l, equal] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \\
 & & & a_1 \arrow[uu, "x_{\phi\psi}"] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}"] \arrow[ul, phantom, "\mathrm{I}^{\mathrm{h}}"] & a_1 \arrow[l, equal] \arrow[u, equal] & b_\rho \arrow[l, "f_\rho"] \arrow[u, equal] \arrow[r, phantom, "\alpha_\rho^{-\mathrm{v}}"] & c_\rho \arrow[l] \arrow[u, "x_\rho y_\rho^{-1}"'] \\
 & & & & a_1 \arrow[u, equal] & b_\rho \arrow[u, equal] \arrow[r, phantom, "\alpha_\rho"] & b_\rho \arrow[l, equal] \arrow[u, "y_\rho x_\rho^{-1}"'] \\
 & a_2 \arrow[uuuu, "x_\phi"] \arrow[ruu, phantom, "\theta_{\phi,\psi}"] & b_\psi \arrow[uuuu] & b_\psi \arrow[l, equal] \arrow[uu, "x_\psi^{-1}"] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_\psi \arrow[l, equal] \arrow[u, "x_\psi^{-1}"] \arrow[r, phantom, "\theta_{\phi,\psi}^{-\mathrm{v}}"] & b_{\psi\rho} \arrow[u] \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_{\psi\rho} \arrow[l, equal] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] \\
a_2 \arrow[uuuuu, "x_\phi"] & a_2 \arrow[l, equal] \arrow[u, equal] & b_\psi \arrow[l, "f_\psi"] \arrow[u, equal] \arrow[r, phantom, "\alpha_\psi^{-\mathrm{v}}"] & c_\psi \arrow[l, "f_\psi^{-1}g_\psi"] \arrow[u] \arrow[r, phantom, "\alpha_\psi^{-\mathrm{vh}}"] & b_\psi \arrow[l, "g_\psi^{-1}f_\psi"] \arrow[u, equal] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, equal] \arrow[r, phantom, "\alpha_{\psi\rho}^{-\mathrm{v}}"] & c_{\psi\rho} \arrow[l] \arrow[u, "x_{\psi\rho}y_{\psi\rho}^{-1}"'] \\
 & a_2 \arrow[u, equal] & & & & b_{\psi\rho} \arrow[u, equal] \arrow[r, phantom, "\alpha_{\psi\rho}"] & b_{\psi\rho} \arrow[l, equal] \arrow[u, "y_{\psi\rho}x_{\psi\rho}^{-1}"'] \\
b_\phi \arrow[uu, "x_\phi^{-1}"] \arrow[ru, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_\phi \arrow[l, equal] \arrow[u, "x_\phi^{-1}"'] \arrow[rrrru, phantom, "\theta_{\phi,\psi\rho}^{-\mathrm{v}}"] & & & & b_{\phi\psi\rho} \arrow[u] \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_{\phi\psi\rho} \arrow[l, equal] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
c_\phi \arrow[u, "x_\phi y_\phi^{-1}"] \arrow[r, phantom, "\alpha_\phi^{-\mathrm{hv}}"] & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"] \arrow[u, equal] & & & & b_{\phi\psi\rho} \arrow[llll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, equal] \arrow[r, phantom, "\alpha_{\phi\psi\rho}^{-\mathrm{v}}"] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}y_{\phi\psi\rho}^{-1}"']
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
c_\phi \arrow[ddd, equal] \arrow[r, leftarrow, "g_\phi^{-1}f_\phi"] & b_\phi \arrow[r, leftarrow, "f_\phi^{-1}f_{\phi\psi}"] & b_{\phi\psi} \arrow[r, leftarrow, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[r, leftarrow, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"] & c_{\phi\psi\rho} \arrow[ddd, equal] \\
\arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}(g_\phi^{-1}f_\phi)"] & \arrow[r, phantom, "\theta_{\phi,\psi}"] & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[r, leftarrow, "f_\rho"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}(f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho})"] & \\
& a_2 \arrow[uu, "x_\phi"] \arrow[r, leftarrow, "f_\psi"'] & b_\psi \arrow[u, "x_\psi^{-1}"] \arrow[r, leftarrow, "f_\psi^{-1}f_{\psi\rho}"'] \arrow[d, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_{\psi\rho} \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & \\
c_\phi \arrow[r, leftarrow, "g_\phi^{-1}f_\phi"'] & b_\phi \arrow[u, "x_\phi^{-1}"] \arrow[rr, leftarrow, "f_\phi^{-1}f_{\phi\psi\rho}"'] & & b_{\phi\psi\rho} \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \arrow[r, leftarrow, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"'] & c_{\phi\psi\rho}
\end{tikzcd}
$$

$= \mathrm{I}^{\mathrm{v}}g_\phi^{-1} \circ_{\mathrm{h}} k'^{\mathcal{G}}(\phi,\psi,\rho) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}}g_{\phi\psi\rho}.$

Hence $k'^{\mathcal{G}}(\phi,\psi,\rho) = k^{\mathcal{G}}(\phi,\psi,\rho)$, and the 3-cocycle $k^{\mathcal{G}}$ is unchanged. $\square$

Lemmas 4.2 and 4.3 prove that each double groupoid $\mathcal{G}$ has a three-dimensional cohomology class $\mathbf{k}\mathcal{G} = [k^{\mathcal{G}}] \in H^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ associated with it. We refer to

$$[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$$

as the *Postnikov invariant* of $\mathcal{G}$.

A *double functor* $F : \mathcal{G} \to \mathcal{G}'$ between double groupoids takes objects, horizontal and vertical morphisms, and boxes in $\mathcal{G}$ to objects, horizontal and vertical morphisms, and squares in $\mathcal{G}'$, respectively, in such a way that all the structure categories are preserved. Clearly, such a double functor induces a functor $\Pi F : \Pi\mathcal{G} \to \Pi\mathcal{G}'$,

$$
\begin{bmatrix} a_1 \xleftarrow{f} b \\ \qquad\uparrow{\scriptstyle x} \\ \qquad a_0 \end{bmatrix}
\quad \mapsto \quad
\begin{bmatrix} Fa_1 \xleftarrow{Ff} b \\ \qquad\quad\uparrow{\scriptstyle Fx} \\ \qquad\quad Fa_0 \end{bmatrix},
$$

and a natural transformation $\pi_2 F : \pi_2\mathcal{G} \to (\Pi F)^*\pi_2\mathcal{G}'$, which consists of the homomorphisms $\pi_2(F, a) : \pi_2(\mathcal{G}, a) \to \pi_2(\mathcal{G}', Fa)$ given by

$$
\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array}
\quad \mapsto \quad
\begin{array}{ccc} Fa & = & Fa \\ \| & F\sigma & \| \\ Fa & = & Fa. \end{array}
$$

We say that the double functor F is a *weak equivalence*, and write

$$F : \mathcal{G} \xrightarrow{\sim} \mathcal{G}',$$

whenever ΠF is an equivalence of groupoids and $\pi_2 F$ is an isomorphism. If, for any double groupoid $\mathcal{G}$ we define

$$\pi_0\mathcal{G} = \pi_0(\Pi\mathcal{G}),$$

the set of iso-classes of objects of its fundamental groupoid, and, for each object a of $\mathcal{G}$,

$$\pi_1(\mathcal{G}, a) = \Pi\mathcal{G}(a, a),$$

the group of automorphisms of a in its fundamental groupoid, this notion of weak equivalence is similar to the usual topological notion. Indeed, one readily verifies that a double functor $F : \mathcal{G} \to \mathcal{G}'$ is a weak equivalence if and only if F induces an isomorphism of sets $\pi_0\mathcal{G} \cong \pi_0\mathcal{G}'$ and for every object a of $\mathcal{G}$ isomorphisms of groups $\pi_i(\mathcal{G}, a) \cong \pi_i(\mathcal{G}', Fa)$ for $i = 1, 2$ (cf. [11, 3.4]).

We define two double groupoid $\mathcal{G}$ and $\mathcal{G}'$ to be *weak equivalent* if there exists a zig-zag chain of weak equivalences

$$\mathcal{G} = \mathcal{G}_0 \xrightarrow{\sim} \mathcal{G}_1 \xleftarrow{\sim} \mathcal{G}_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{G}_k = \mathcal{G}'.$$

connecting $\mathcal{G}$ and $\mathcal{G}'$ (see Corollary 5.4).

Let $[\mathcal{G}]$ denote the weak equivalence class of a double groupoid $\mathcal{G}$.

Theorem 4.4. *The Postnikov invariant* $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$ *of a double groupoid* $\mathcal{G}$ *only depends on its weak equivalence class* $[\mathcal{G}]$.

Proof. Let $F : \mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ be a weak equivalence between double groupoids. Suppose that the construction of $k^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ has been made by means of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$, as in (3), and boxes $\theta_{\psi,\rho}$ for each pair of composable morphisms (ψ, ρ), as in (12). Then, for the construction of $k^{\mathcal{G}'} \in Z^3(\Pi\mathcal{G}', \pi_2\mathcal{G}')$, we can choose $(Ff_\rho, Fb_\rho, Fx_\rho)$ as representative paths of the morphisms $\Pi F\rho$ in $\Pi\mathcal{G}'$ as well as the boxes $\theta_{\Pi F\psi, \Pi F\rho} = F\theta_{\psi,\rho}$. If we do this, it follows from (16) that, for any triplet (ϕ, ψ, ρ) of composable morphisms in $\Pi\mathcal{G}$,

$$k^{\mathcal{G}'}(\Pi F\phi, \Pi F\psi, \Pi F\rho) = Fk^{\mathcal{G}}(\phi, \psi, \rho).$$

This means that $(\Pi F)^*(k^{\mathcal{G}'}) = (\pi_2 F)_*(k^{\mathcal{G}})$, whence $(\Pi F)^*(\mathbf{k}\mathcal{G}) = (\pi_2 F)_*(\mathbf{k}\mathcal{G})$. Thus, $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}] = [\Pi\mathcal{G}', \pi_2\mathcal{G}', \mathbf{k}\mathcal{G}']$. $\square$

5. The Classification Theorem

Theorem 5.1. *The mapping* $[\mathcal{G}] \mapsto [\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathrm{k}\mathcal{G}]$ *establishes a bijective correspondence between weak equivalence classes of double groupoids and equivalence classes of Postnikov systems.*

Proof. This follows from the following construction of a double groupoid $\mathcal{G}^k$ associated to each normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$, of any groupoid P with coefficients in a functor $\mathcal{A} : P \to \mathsf{Ab}$, and Proposition 5.3 bellow. □

Let P be a groupoid, $\mathcal{A} : P \to \mathsf{Ab}$ a functor and $k \in Z^3(P, \mathcal{A})$ a normalized 3-cocycle of P with coefficients in $\mathcal{A}$. We construct a double groupoid, denoted by $\mathcal{G}^k$, as follows.

• The objects of $\mathcal{G}^k$ are the arrows of P.

• For any two arrows of P, there is a unique horizontal (resp. vertical) morphism in $\mathcal{G}^k$ between them whenever they have the same target (resp. source), whereas if they have different target (resp. source) then there are no horizontal (resp. vertical) morphisms between them. Compositions and identities are defined in the obvious manner. Thus, a path in $\mathcal{G}^k$

$$\begin{tikzcd} \xi_1 & \eta \arrow[l] \\ & \xi_0 \arrow[u] \end{tikzcd}$$

consists of three morphisms of P such that ξ_0 and η have the same source and η and ξ_1 have the same target. Notice that such a path writes uniquely as

$$\begin{tikzcd} \xi_1 & \eta \arrow[l] \\ & \xi_0 \arrow[u] \end{tikzcd} = \begin{tikzcd} \phi\psi & \phi\psi\rho \arrow[l] \\ & \psi\rho \arrow[u] \end{tikzcd}$$

with $\rho = \xi_1^{-1}\eta$, $\phi = \eta\xi_0^{-1}$ and $\psi = \xi_0\eta^{-1}\xi_1$ are three composable arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in the groupoid P.

• A box $(\phi, \psi, \rho; u)$ in $\mathcal{G}^k$, with boundary as below

$$\begin{tikzcd} \phi\psi & \phi\psi\rho \arrow[l] \\ \psi \arrow[u] & \psi\rho, \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle(\phi,\psi,\rho;u)}"] \end{tikzcd} \tag{18}$$

consists of three composable arrows in P, $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, together with an element $u \in \mathcal{A}(a_3)$.

• For any four composable arrows in P, $a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, $u \in \mathcal{A}(a_3)$ and $v \in \mathcal{A}(a_4)$, the vertical composition of the boxes

$$\begin{tikzcd} \delta\phi\psi & \delta\phi\psi\rho \arrow[l] \\ \phi\psi \arrow[u] \arrow[r, phantom, "{\scriptstyle (\delta,\phi\psi,\rho;v)}"] & \phi\psi\rho, \arrow[l] \arrow[u] \\ \psi \arrow[u] \arrow[r, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"] & \psi\rho \arrow[l] \arrow[u] \end{tikzcd} \tag{19}$$

is given by

$$\begin{aligned}(\delta,\phi\psi,\rho;v)\circ_{\mathrm{v}}&(\phi,\psi,\rho;u)\\ &= \big(\delta\phi,\psi,\rho; v+\delta_* u + k(\delta,\phi,\psi) - k(\delta,\phi,\psi\rho)\big).\end{aligned} \tag{20}$$

• For any four composable arrows in P, $a_4 \overset{\phi}{\leftarrow} a_3 \overset{\psi}{\leftarrow} a_2 \overset{\rho}{\leftarrow} a_1 \overset{\lambda}{\leftarrow} a_0$, and $u, v \in \mathcal{A}(a_4)$, the horizontal composition of the boxes

$$\begin{tikzcd} \phi\psi & \phi\psi\rho \arrow[l] & \phi\psi\rho\lambda \arrow[l] \\ \psi \arrow[u] \arrow[r, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"] & \psi\rho \arrow[l] \arrow[u] \arrow[r, phantom, "{\scriptstyle (\phi,\psi\rho,\lambda;v)}"] & \psi\rho\lambda, \arrow[l] \arrow[u] \end{tikzcd} \tag{21}$$

is given by

$$(\phi,\psi,\rho;u)\circ_{\mathrm{h}}(\phi,\psi\rho,\lambda;v) = \big(\phi,\psi,\rho\lambda;u+v\big). \tag{22}$$

• The vertical and horizontal identity boxes are respectively defined by

$$\mathrm{I}^{\mathrm{v}}(\phi\leftarrow\phi\psi) = \begin{tikzcd} \phi & \phi\psi \arrow[l] \\ \phi \arrow[u, equal] \arrow[r, phantom, "{\scriptstyle (id_{a_2},\phi,\psi;0)}"] & \phi\psi \arrow[l] \arrow[u, equal] \end{tikzcd} \qquad \mathrm{I}^{\mathrm{h}}\binom{\phi\psi}{\overset{\uparrow}{\psi}} = \begin{tikzcd} \phi\psi \arrow[r, equal] & \phi\psi \\ \psi \arrow[u] \arrow[r, equal] \arrow[r, phantom, shift left=3, "{\scriptstyle (\phi,\psi,id_{a_0};0)}"] & \psi \arrow[u] \end{tikzcd} \tag{23}$$

for any two composable arrows $a_2 \overset{\phi}{\leftarrow} a_1 \overset{\psi}{\leftarrow} a_0$ in P.

Lemma 5.2. *With these definitions, $\mathcal{G}^k$ is a double groupoid (satisfying the filling condition).*

Proof. We first observe that the vertical composition of boxes in $\mathcal{G}^k$ is associative thanks to the 3-cocycle condition of k. In fact, let

$$\begin{array}{ccc}
\gamma\delta\phi\psi & \longleftarrow & \gamma\delta\phi\psi\rho \\
\uparrow & {\scriptstyle (\gamma,\gamma\delta\phi\psi,\rho;w)} & \uparrow \\
\delta\phi\psi & \longleftarrow & \delta\phi\psi\rho \\
\uparrow & {\scriptstyle (\delta,\phi\psi,\rho;v)} & \uparrow \\
\phi\psi & \longleftarrow & \phi\psi\rho, \\
\uparrow & {\scriptstyle (\phi,\psi,\rho;u)} & \uparrow \\
\psi & \longleftarrow & \psi\rho
\end{array}$$

be three vertically composable boxes, defined by five arrows

$$a_5 \overset{\gamma}{\leftarrow} a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$$

of P and elements $u \in \mathcal{A}(a_3)$, $v \in \mathcal{A}(a_4)$ and $w \in \mathcal{A}(a_5)$. Then,

$$\begin{aligned}
&\big((\gamma,\gamma\delta\phi\psi,\rho;w)\circ_{\mathrm{v}}(\delta,\phi\psi,\rho;v)\big)\circ_{\mathrm{v}}(\phi,\psi,\rho;u)\\
&\qquad=\big(\gamma\delta,\phi\psi,\rho;w+\gamma_*v+k(\gamma,\delta;\phi\psi)-k(\gamma,\delta,\phi\psi\rho)\big)\circ_{\mathrm{v}}(\phi,\psi,\rho;u)\\
&\qquad=\big(\gamma\delta\phi,\psi,\rho;w+\gamma_*v+k(\gamma,\delta,\phi\psi)-k(\gamma,\delta;\phi\psi\rho)+\gamma_*\delta_*u\\
&\qquad\qquad\qquad\qquad\qquad\qquad+k(\gamma\delta,\phi,\psi)-k(\gamma\delta,\phi,\psi\rho)\big),
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
&(\gamma,\gamma\delta\phi\psi,\rho;w)\circ_{\mathrm{v}}\big((\delta,\phi\psi,\rho;v)\circ_{\mathrm{v}}(\phi,\psi,\rho;u)\big)\\
&\qquad=(\gamma,\gamma\delta\phi\psi,\rho;w)\circ_{\mathrm{v}}\big(\delta\phi,\psi,\rho;v+\delta_*(u)+k(\delta,\phi,\psi)-k(\delta,\phi,\psi\rho)\big)\\
&\qquad=\big(\gamma\delta\phi,\psi,\rho;w+\gamma_*v+\gamma_*\delta_*u+\gamma_*k(\delta,\phi,\psi)-\gamma_*k(\delta,\phi,\psi\rho)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+k(\gamma,\delta\phi,\psi)-k(\gamma,\delta\phi,\psi\rho)\big).
\end{aligned}$$

Hence the result follows by comparison, using that the cocycle condition of k applied to the lists of arrows $a_5 \overset{\gamma}{\leftarrow} a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi\rho}{\leftarrow} a_0$ and $a_5 \overset{\gamma}{\leftarrow} a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1$ gives the equalities

$$\begin{aligned}
\gamma_*k(\delta,\phi,\psi\rho)+k(\gamma,\delta\phi,\psi\rho)&=k(\gamma\delta,\phi,\psi\rho)+k(\gamma,\delta,\phi\psi\rho)-k(\gamma,\delta,\phi),\\
\gamma_*k(\delta,\phi,\psi)+k(\gamma,\delta\phi,\psi)&=k(\gamma\delta,\phi,\psi)+k(\gamma,\delta,\phi\psi)-k(\gamma,\delta,\phi).
\end{aligned}$$

The associativity of the horizontal composition of boxes is easier. Let

$$\begin{tikzcd}
\phi\psi & \phi\psi\rho \arrow[l] & \phi\psi\rho\lambda \arrow[l] & \phi\psi\rho\lambda\mu \arrow[l] \\
\psi \arrow[u] \arrow[r, phantom, "{(\phi,\psi,\rho;u)}"] & \psi\rho \arrow[l] \arrow[u] \arrow[r, phantom, "{(\phi,\psi\rho,\lambda;v)}"] & \psi\rho\lambda \arrow[l] \arrow[u] \arrow[r, phantom, "{(\phi,\psi\rho\lambda,\mu;w)}"] & \psi\rho\lambda\mu \arrow[l] \arrow[u]
\end{tikzcd}$$

be boxes, defined by arrows $a_5 \overset{\phi}{\leftarrow} a_4 \overset{\psi}{\leftarrow} a_3 \overset{\rho}{\leftarrow} a_2 \overset{\lambda}{\leftarrow} a_1 \overset{\mu}{\leftarrow} a_0$ of P and elements $u, v, w \in \mathcal{A}(a_5)$. Then,

$$\begin{aligned}
\big((\phi,\psi,\rho;u) \circ_{\mathrm{h}} (\phi,\psi\rho,\lambda;v)\big) \circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w)& \\
&= (\phi,\psi,\rho\lambda;u+v) \circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w)\\
&= (\phi,\psi,\rho\lambda\mu;u+v+w) = (\phi,\psi,\rho;u) \circ_{\mathrm{h}} (\phi,\psi\rho,\lambda\mu;v+w)\\
&= (\phi,\psi,\rho;u) \circ_{\mathrm{h}} \big((\phi,\psi\rho,\lambda;v)\big) \circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w)\big).
\end{aligned}$$

For any box $(\phi,\psi,\rho;u)$ as in (18), its respective vertical and horizontal inverses

$$\begin{tikzcd}
\psi & \psi\rho \arrow[l] \\
\phi\psi \arrow[u] \arrow[r, phantom, "{(\phi,\psi,\rho;u)^{-\mathrm{v}}}"] & \phi\psi\rho \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi\psi\rho & \phi\psi \arrow[l] \\
\psi\rho \arrow[u] \arrow[r, phantom, "{(\phi,\psi,\rho;u)^{-\mathrm{h}}}"] & \psi, \arrow[l] \arrow[u]
\end{tikzcd}$$

are given by

$$\begin{cases}(\phi,\psi,\rho;u)^{-\mathrm{v}} = (\phi^{-1},\phi\psi,\rho;k(\phi^{-1},\phi,\psi\rho) - k(\phi^{-1},\phi,\psi) - \phi_*^{-1}u),\\ (\phi,\psi,\rho;u)^{-\mathrm{h}} = (\phi,\psi,\rho^{-1};-u).\end{cases} \tag{24}$$

The only non-straightforward verification here is that

$$(\phi,\psi,\rho;u) \circ_{\mathrm{v}} (\phi,\psi,\rho;u)^{-\mathrm{v}} = \mathrm{I}^{\mathrm{v}}(\phi\psi \leftarrow \phi\psi\rho).$$

which is as follows

$$\begin{aligned}
(\phi,\psi,\rho;u) \circ_{\mathrm{v}} (\phi^{-1},\phi\psi,\rho;k(\phi^{-1},\phi,\psi\rho) - k(\phi^{-1},\phi,\psi) - \phi_*^{-1}u)&\\
&= \big(id_d,\psi\psi,\rho;u + \phi_* k(\phi^{-1},\phi,\psi\rho) - \phi_* k(\phi^{-1},\phi,\psi) - u\\
&\qquad\qquad + k(\phi^{-1},\phi,\phi\psi) - k(\phi^{-1},\phi,\phi\psi\rho)\big)\\
&= \big(id_{a_3},\psi\psi,\rho;\phi_* k(\phi^{-1},\phi,\psi\rho) - \phi_* k(\phi^{-1},\phi,\psi) + k(\phi^{-1},\phi,\phi\psi)\\
&\qquad\qquad - k(\phi^{-1},\phi,\phi\psi\rho)\big)\\
&\overset{(25)}{=} \big(id_{a_3},\phi\psi,\rho;k(\phi,\phi^{-1},\phi) - k(\phi,\phi^{-1},\phi)\big) = (id_{a_3},\phi\psi,\rho;0)\\
&= \mathrm{I}^{\mathrm{v}}(\phi\psi \leftarrow \phi\psi\rho),
\end{aligned}$$

where we have used the equality

$$\phi_* k(\phi^{-1}, \phi, \psi) - k(\phi, \phi^{-1}, \phi\psi) = k(\phi, \phi^{-1}, \phi) \tag{25}$$

which follows from the 3-cocycle and normalization conditions of k for the sequence of arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\phi^{-1}}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1$ in P.

All other requirements are easily verified, except perhaps the interchange law which is proved as follows. Suppose given boxes

$$\begin{array}{ccccc}
\delta\phi\psi & \longleftarrow & \delta\phi\psi\rho & \longleftarrow & \delta\phi\psi\rho\lambda \\
\uparrow & {\scriptstyle (\delta,\phi\psi,\rho;v)} & \uparrow & {\scriptstyle (\delta,\phi\psi\rho,\lambda;v')} & \uparrow \\
\phi\psi & \longleftarrow & \phi\psi\rho, & \longleftarrow & \phi\psi\rho\lambda \\
\uparrow & {\scriptstyle (\phi,\psi,\rho;u)} & \uparrow & {\scriptstyle (\phi,\psi\rho,\lambda;u')} & \uparrow \\
\psi & \longleftarrow & \psi\rho & \longleftarrow & \psi\rho\lambda
\end{array}$$

defined by arrows of P, $a_5 \overset{\delta}{\leftarrow} a_4 \overset{\phi}{\leftarrow} a_3 \overset{\psi}{\leftarrow} a_2 \overset{\rho}{\leftarrow} a_1 \overset{\lambda}{\leftarrow} a_0$, and elements $v, v' \in \mathcal{A}(a_5)$ and $u, u' \in \mathcal{A}(a_4)$. Then,

$$\begin{aligned}
&\big((\delta, \phi\psi, \rho; v) \circ_{\mathrm{v}} (\phi, \psi, \rho; u)\big) \circ_{\mathrm{h}} \big((\delta, \phi\psi\rho, \lambda; v') \circ_{\mathrm{v}} (\phi, \psi\rho, \lambda; u')\big) \\
&\quad = \big(\delta\phi, \psi, \rho; v + \delta_* u + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho)\big) \circ_{\mathrm{h}} \big(\delta\phi, \psi\rho, \lambda; v' \\
&\qquad\qquad + \delta_* u' + k(\delta, \phi, \psi\rho) - k(\delta, \phi, \psi\rho\lambda)\big) \\
&\quad = \big(\delta\phi, \psi, \rho\lambda; v + \delta_* u + v' + \delta_* u' + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho\lambda)\big) \\
&\quad = (\delta, \phi\psi, \rho\lambda; v + v') \circ_{\mathrm{v}} (\phi, \psi, \rho\lambda; u + u') \\
&\quad = \big((\delta, \phi\psi, \rho; v) \circ_{\mathrm{h}} (\delta, \phi\psi\rho, \lambda; v')\big) \circ_{\mathrm{v}} \big((\phi, \psi, \rho; u) \circ_{\mathrm{h}} (\phi, \psi\rho, \lambda; u')\big).
\end{aligned}$$

$\square$

Proposition 5.3. (i) *Let* $(P, \mathcal{A}, \mathbf{k})$ *be a Postnikov system. For any representative* 3*-cocycle* $k \in Z^3(P, \mathcal{A})$ *of* $\mathbf{k}$*, the Postnikov invariant of the double groupoid* $\mathcal{G}^k$ *is equivalent to* $(P, \mathcal{A}, \mathbf{k})$*, that is,*

$$[\Pi\mathcal{G}^k, \pi_2\mathcal{G}^k, \mathbf{k}\mathcal{G}^k] = [P, \mathcal{A}, \mathbf{k}].$$

(ii) *Suppose* $(P, \mathcal{A}, \mathbf{k})$ *and* $(P', \mathcal{A}', \mathbf{k}')$ *are equivalent Postnikov systems. Then, for any representative* 3*-cocycles* $k \in Z^3(P, \mathcal{A})$ *and* $k' \in Z^3(P', \mathcal{A}')$ *of* $\mathbf{k}$ *and* $\mathbf{k}'$ *respectively, there is a weak equivalence* $\mathcal{G}^k \overset{\sim}{\to} \mathcal{G}^{k'}$.

(iii) *Let $\mathcal{G}$ be a double groupoid. For any 3-cocycle $k \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ representative of the cohomology class $\mathbf{k}\mathcal{G}$, there is a weak equivalence $\mathcal{G}^k \stackrel{\sim}{\to} \mathcal{G}$.*

Proof. Firstly notice that the homotopy relation between paths in $\mathcal{G}^k$ is trivial. In fact, suppose

$$\begin{array}{ccc} \xi_1 \longleftarrow \eta & & \xi_1 \longleftarrow \mu \\ \uparrow & \simeq & \uparrow \\ \xi_0 & & \xi_0 \end{array}$$

are two homotopic paths in $\mathcal{G}^k$. This means that there is a box in $\mathcal{G}^k$ of the form

$$\begin{tikzcd} \eta & \mu \arrow[l] \\ \eta \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"] & \eta \arrow[u] \end{tikzcd}$$

for some composable arrows $a_3 \stackrel{\phi}{\leftarrow} a_2 \stackrel{\psi}{\leftarrow} a_1 \stackrel{\rho}{\leftarrow} a_0$ in P and some $u \in \mathcal{A}(a_3)$. But then, we have the equalities $\psi = \eta = \psi\rho = \phi\psi$ and $\phi\psi\rho = \mu$ which imply $\eta = \mu$.

(i) There is a functor $\mathfrak{f}^k : P \to \Pi\mathcal{G}^k$ which carries each object a of P to the identity morphism id_a, regarded as an object of $\mathcal{G}^k$, and carries a morphism $\rho : a_0 \to a_1$ of P to the path

$$\mathfrak{f}^k\rho = \begin{array}{c} id_{a_1} \longleftarrow \rho \\ \qquad\qquad \uparrow \\ \qquad\qquad id_{a_0}. \end{array}$$

If $\psi : a_1 \to a_2$ is another morphism in P, the equality $\mathfrak{f}^k(\psi\rho) = \mathfrak{f}^k\psi\ \mathfrak{f}^k\rho$ follows from the diagram in $\mathcal{G}$

$$\begin{tikzcd} id_{a_2} & \psi \arrow[l] & \psi\rho \arrow[l] \\ & id_{a_1} \arrow[u] \arrow[ur, phantom, "{\scriptstyle (\psi,id_{a_1},\rho;0)}"] & \rho \arrow[l] \arrow[u] \\ & & id_{a_0} \arrow[u] \end{tikzcd}$$

and, for any object a of P,

$$\mathfrak{f}^k id_a = \begin{array}{c} id_a = id_a \\ \qquad \| \\ \qquad id_a \end{array} = id_{\mathfrak{f}^k a}.$$

So, $\mathfrak{f}^k$ is actually a functor which is clearly fully faithful. Indeed, it is an equivalence of groupoids since any object $\rho : a_0 \to a_1$ is isomorphic in $\Pi\mathcal{G}^k$ to the object $\mathfrak{f}^k a_0 = id_{a_0}$ because of the path

$$\begin{array}{ccc} id_{a_0} & = & id_{a_0} \\ & & \uparrow \\ & & \rho. \end{array}$$

Now, for any object a of P, the abelian group $\pi_2(\mathcal{G}^k, \mathfrak{f}a)$ just consists of all the boxes in $\mathcal{G}^k$ of the form

$$\begin{array}{ccc} id_a & = & id_a \\ \| & {\scriptstyle (id_a, id_a, id_a; u)} & \| \\ id_a & = & id_a \end{array}$$

with $u \in \mathcal{A}(a)$. The mapping $\mathfrak{F}^k : \mathcal{A}(a) \to \pi_2(\mathcal{G}^k, \mathfrak{f}^k a)$,

$$u \mapsto (id_a, id_a, id_a; u),$$

is clearly an isomorphism of groups, for any object a of P, and thus we see that we are in presence of a natural isomorphism $\mathfrak{F}^k : \mathcal{A} \cong \mathfrak{f}^* \pi_2 \mathcal{G}^k$.

To complete the proof, it is enough to prove that $\mathfrak{f}^{k*}(\mathbf{k}\mathcal{G}^k) = \mathfrak{F}^k_*(\mathbf{k})$. Indeed, we are going to prove that $\mathfrak{f}^{k*}(k^{\mathcal{G}^k}) = \mathfrak{F}^k_*(k)$ once we select, for each pair of composable arrows $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in P, the box

$$\theta_{\mathfrak{f}^k\psi, \mathfrak{f}^k\rho} = \begin{array}{ccc} \psi & \longleftarrow & \psi\rho \\ \uparrow & {\scriptstyle (\psi, id_{a_1}, \rho; 0)} & \uparrow \\ id_{a_1} & \longleftarrow & \rho \end{array}$$

in the construction of the 3-cocycle $k^{\mathcal{G}^k}$. In fact, for any given composable arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in P, by (16), the element

$$\mathfrak{f}^{k*}(k^{\mathcal{G}^k})(\phi, \psi, \rho) = k^{\mathcal{G}^k}(\mathfrak{f}^k\phi, \mathfrak{f}^k\psi, \mathfrak{f}^k\rho) \in \pi_2(\mathcal{G}^k, \mathfrak{f}^k a_3)$$

is the pasted box of the diagram

$$\begin{tikzcd}
id_{a_3} \arrow[ddd, equal] & \phi \arrow[l] & \phi\psi \arrow[l] \arrow[dr, phantom, "{\scriptstyle (\phi\psi,id_{a_2},\rho;0)}"] & \phi\psi\rho \arrow[l] & id_{a_3} \arrow[l] \arrow[ddd, equal] \\
 & & id_{a_1} \arrow[u] \arrow[l, phantom, "{\scriptstyle (\phi,id_{a_2},\psi;0)}"] & \rho \arrow[l] \arrow[u] \arrow[r, phantom, "{\scriptstyle \mathrm{I}^{\mathrm{v}}(\phi\psi\rho\leftarrow id_{a_3})}"] & \\
 & id_{a_2} \arrow[uu] \arrow[ul, phantom, "{\scriptstyle \mathrm{I}^{\mathrm{v}}(id_{a_3}\leftarrow\phi)}"] & \psi \arrow[l] \arrow[u] \arrow[ur, phantom, "{\scriptstyle (\psi,id_{a_1},\rho;0)^{-\mathrm{v}}}"] & \psi\rho \arrow[l] \arrow[u] \arrow[dll, phantom, "{\scriptstyle (\phi,id_{a_2},\psi\rho;0)^{-\mathrm{v}}}"] & \\
id_{a_3} & \phi \arrow[l] \arrow[u] & & \phi\psi\rho \arrow[ll] \arrow[u] & id_{a_2} \arrow[l]
\end{tikzcd}$$

Since, by (23) and (24),

$$\begin{cases}
\mathrm{I}^{\mathrm{v}}(id_{a_3} \leftarrow \phi) = (id_{a_3}, id_{a_3}, \phi; 0), \\
\mathrm{I}^{\mathrm{v}}(\phi\psi\rho \leftarrow id_{a_3}) = (id_{a_3}, \phi\psi\rho, (\phi\psi\rho)^{-1}; 0), \\
(\psi, id_{a_1}, \rho; 0)^{-\mathrm{v}} = (\psi^{-1}, \psi, \rho; -k(\psi^{-1}, \psi, \rho)), \\
(\phi, id_{a_2}, \psi\rho; 0)^{-\mathrm{v}} = (\phi^{-1}, \phi, \psi\rho; -k(\phi^{-1}, \phi, \psi\rho)),
\end{cases}$$

a direct computation, using (22) and (22), gives

$$\begin{aligned}
&k^{\mathcal{G}^k}(\mathfrak{f}\phi, \mathfrak{f}\psi, \mathfrak{f}\rho) \\
&= \Big(id_{a_3}, id_{a_3}, id_{a_3}; -\phi_*\psi_* k(\psi^{-1}, \psi, \rho) + k(\phi\psi, \psi^{-1}, \psi\rho) - k(\phi\psi, \psi^{-1}, \psi) \\
&\qquad\qquad - \phi_* k(\phi^{-1}, \phi, \psi\rho) + k(\phi, \phi^{-1}, \phi\psi\rho) - k(\phi, \phi^{-1}, \phi)\Big) \\
&\overset{(25)}{=} \big(id_{a_3}, id_{a_3}, id_{a_3}; -\phi_*\psi_* k(\psi^{-1}, \psi, \rho) + k(\phi\psi, \psi^{-1}, \psi\rho) - k(\phi\psi, \psi^{-1}, \psi)\big).
\end{aligned}$$

Moreover, since the cocycle condition of k on the sequence

$$a_3 \overset{\phi\psi}{\leftarrow} a_1 \overset{\psi^{-1}}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$$

yields

$$\phi_*\psi_* k(\psi^{-1}, \psi, \rho) - k(\phi, \psi, \rho) - k(\phi\psi, \psi^{-1}, \psi\rho) + k(\phi\psi, \psi^{-1}, \psi) = 0$$

we conclude that

$$\begin{aligned}
k^{\mathcal{G}^k}(\mathfrak{f}^k\phi, \mathfrak{f}^k\psi, \mathfrak{f}^k\rho) &= \big(id_{a_3}, id_{a_3}, id_{a_3}; k(\phi, \psi, \rho)\big) = \mathfrak{F}^k k(\phi, \psi, \rho) \\
&= \mathfrak{F}^k_*(k)(\phi, \psi, \rho).
\end{aligned}$$

(ii) By hypothesis, there is an equivalence $\mathfrak{f} : P \xrightarrow{\sim} P'$, a natural isomorphism $\mathfrak{F} : \mathcal{A} \cong \mathfrak{f}^*\mathcal{A}'$, and a normalized 2-cochain $c \in C^2(P, \mathfrak{f}^*\mathcal{A}')$ such that $\mathfrak{f}^*(k') = \mathfrak{F}_*(k) + \partial c$. A weak equivalence $F : \mathcal{G}^k \xrightarrow{\sim} \mathcal{G}^{k'}$ is then defined by the following assignments on objets, horizontal and vertical morphisms, and boxes

$$\rho \mapsto \mathfrak{f}\rho, \quad \big(\psi \leftarrow \psi\rho\big) \mapsto \big(\mathfrak{f}\psi \leftarrow \mathfrak{f}\psi\,\mathfrak{f}\rho\big), \quad \begin{pmatrix} \phi\psi \\ \uparrow \\ \psi \end{pmatrix} \mapsto \begin{matrix} \mathfrak{f}\phi\,\mathfrak{f}\psi \\ \uparrow \\ \mathfrak{f}\psi \end{matrix}$$

$$\begin{tikzcd} \phi\psi & \phi\psi\rho \arrow[l] \\ \psi \arrow[u] & \psi\rho, \arrow[l] \arrow[u] \arrow[ul, phantom, "(\phi,\psi,\rho;u)"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathfrak{f}\phi\,\mathfrak{f}\psi & \mathfrak{f}\phi\,\mathfrak{f}\psi\,\mathfrak{f}\rho \arrow[l] \\ \mathfrak{f}\psi \arrow[u] & \mathfrak{f}\psi\,\mathfrak{f}\rho. \arrow[l] \arrow[u] \arrow[ul, phantom, "\big(\mathfrak{f}\phi,\mathfrak{f}\psi,\mathfrak{f}\rho;\mathfrak{F}u+c(\phi,\psi)-c(\phi,\psi\rho)\big)"] \end{tikzcd}$$

So defined, one verifies easily that $F : \mathcal{G}^k \to \mathcal{G}^{k'}$ is actually a double functor. That F is a weak equivalence follows from the commutativity of the diagrams

$$\begin{tikzcd} P \arrow[r, "\mathfrak{f}^k"] \arrow[d, "\mathfrak{f}"'] & \Pi\mathcal{G}^k \arrow[d, "\Pi F"] \\ P' \arrow[r, "\mathfrak{f}^{k'}"] & \Pi\mathcal{G}^{k'} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(a) \arrow[r, "\mathfrak{F}^k"] \arrow[d, "\mathfrak{F}"'] & \pi_2(\mathcal{G}^k, \mathfrak{f}^k a) \arrow[d, "\pi_2 F"] \\ \mathcal{A}'(\mathfrak{f}a) \arrow[r, "\mathfrak{F}^{k'}"] & \pi_2(\mathcal{G}^{k'}, \mathfrak{f}^{k'} a) \end{tikzcd}$$

where $\mathfrak{f}$, $\mathfrak{f}^k$ and $\mathfrak{f}^{k'}$ are equivalences of groupoids and, for any object a of P, $\mathfrak{F}$, $\mathfrak{F}^k$ and $\mathfrak{F}^{k'}$ are isomorphisms of groups.

(iii) By Lemma 4.3 (i), we can assume that $k = k^{\mathcal{G}}$ for a certain selection of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$ and the boxes $\theta_{\psi,\rho}$, as in (3) and (12). Then, a double functor $F : \mathcal{G}^k \xrightarrow{\sim} \mathcal{G}$ is defined by the following assignments on objets, horizontal and vertical morphisms, and boxes

$$F(\rho) = b_\rho, \quad F(\psi \leftarrow \psi\rho) = \big(b_\psi \xleftarrow{f_\psi^{-1} f_{\psi\rho}} b_{\psi\rho}\big)\ , \quad F\begin{pmatrix} \psi\rho \\ \uparrow \\ \rho \end{pmatrix} = \begin{matrix} b_{\psi\rho} \\ \uparrow{\scriptstyle x_{\psi\rho} x_\rho^{-1}} \\ b_\rho \end{matrix}\ ,$$

$$F\left(\begin{array}{ccc}\phi\psi & \longleftarrow & \phi\psi\rho \\ \uparrow & {\scriptstyle (\phi,\psi,\rho;u)} & \uparrow \\ \psi & \longleftarrow & \psi\rho\end{array}\right) = \begin{array}{ccc} b_{\phi\psi} & \xleftarrow{\;f^{-1}_{\phi\psi}f_{\phi\psi\rho}\;} & b_{\phi\psi\rho} \\ {\scriptstyle x_{\phi\psi}x_\psi^{-1}}\uparrow & {\scriptstyle f^{-1}_{\phi\psi*}(\sigma)+\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho}} & \uparrow{\scriptstyle x_{\phi\psi\rho}x^{-1}_{\psi\rho}} \\ b_\psi & \xleftarrow[\;f^{-1}_{\psi}f_{\psi\rho}\;]{} & b_{\psi\rho}\end{array}.$$

Many of the details to confirm that F, so defined, is a double functor are routine and easily verifiable, so are left to the reader. For instance, if ϕ,ψ,ρ are any three composable morphisms in $\mathcal{G}$,

$$F(\phi\leftarrow\phi\psi)\,F(\phi\psi\leftarrow\phi\psi\rho) = f^{-1}_{\phi}f^{-1}_{\phi\psi}f_{\phi\psi}f^{-1}_{\phi\psi\rho} = f^{-1}_{\phi}f^{-1}_{\phi\psi\rho} = F(\phi\leftarrow\phi\psi\rho)$$

and thus we see that F preserves horizontal composition of morphisms. The proof that F preserves composition of boxes is as follows. Suppose two vertically composable boxes in $\mathcal{G}^k$, as in (19). Then,

$$\begin{aligned}
&F(\delta,\phi\psi,\rho;\tau)\circ_{\mathrm{v}}F(\phi,\psi,\rho;\tau)\\
&=(f^{-1}_{\delta\phi\psi*}\tau+\theta^{-\mathrm{h}}_{\delta,\phi\psi}\circ_{\mathrm{h}}\theta_{\delta,\phi\psi\rho})\circ_{\mathrm{v}}(f^{-1}_{\phi\psi*}\sigma+\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\\
&\overset{(14)}{=}\big(f^{-1}_{\delta\phi\psi*}(\tau-k(\delta,\phi\psi,\rho))+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}\big)\circ_{\mathrm{v}}\big(f^{-1}_{\phi\psi*}(\sigma-k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad+\theta_{\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho}\big)\\
&\overset{(9)}{=}f^{-1}_{\delta\phi\psi*}(\tau-k(\delta,\phi\psi,\rho))+x_{\delta\phi\psi*}x^{-1}_{\phi\psi*}f^{-1}_{\phi\psi*}(\sigma-k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho}\\
&\overset{2.4}{=}f^{-1}_{\delta\phi\psi*}(\tau-k(\delta,\phi\psi,\rho))+f^{-1}_{\delta\phi\psi*}f_{\delta*}x_{\delta*}(\sigma-k(\phi,\psi,\rho))+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho}\\
&\overset{2.4}{=}f^{-1}_{\delta\phi\psi*}(\tau-k(\delta,\phi\psi,\rho))+f^{-1}_{\delta\phi\psi*}\delta_*(\sigma-k(\phi,\psi,\rho))+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho}\\
&=f^{-1}_{\delta\phi\psi*}(\tau+\delta_*\sigma)+f^{-1}_{\delta\phi\psi*}(-k(\delta,\phi\psi,\rho)-\delta_*k(\phi,\psi,\rho))+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho},
\end{aligned}$$

$$\begin{aligned}
&F\big((\delta,\phi\psi,\rho;\tau)\circ_{\mathrm{v}}(\phi,\psi,\rho;\tau)\big)\\
&\overset{(20)}{=}F(\delta\phi,\psi,\rho;\tau+\delta_*\sigma+k^{\mathcal{G}}(\delta,\phi,\psi)-k^{\mathcal{G}}(\delta,\phi,\psi\rho))\\
&=f^{-1}_{\delta\phi\psi*}(\tau+\delta_*\sigma+k^{\mathcal{G}}(\delta,\phi,\psi)-k^{\mathcal{G}}(\delta,\phi,\psi\rho))+\theta^{-\mathrm{h}}_{\delta\phi,\psi}\circ_{\mathrm{h}}\theta_{\delta\phi,\psi\rho}\\
&\overset{(14)}{=}f^{-1}_{\delta\phi\psi*}(\tau+\delta_*\sigma)+f^{-1}_{\delta\phi\psi*}\big(k^{\mathcal{G}}(\delta,\phi,\psi)-k^{\mathcal{G}}(\delta,\phi,\psi\rho))-k^{\mathcal{G}}(\delta\phi,\psi,\rho)\big)\\
&\qquad\qquad\qquad\qquad+\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\psi,\rho},
\end{aligned}$$

and the result follows from Lemma 3.5, thanks to the 3-cocycle condition of k. To prove that Γ preserves horizontal composition of boxes is easier. Suppose two horizontally composable boxes in $\mathcal{G}^k$, as in (21). Then,

$$\begin{aligned}
&F(\phi,\psi,\rho;\sigma)\circ_{\mathrm{h}} F(\phi,\psi\rho,\lambda;\tau)\\
&\qquad = (f^{-1}_{\phi\psi *}\sigma + \theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{h}}(f^{-1}_{\phi\psi\rho *}\tau + \theta^{-\mathrm{h}}_{\phi,\psi\rho}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho\lambda})\\
&\qquad \overset{(8)}{=} f^{-1}_{\phi\psi *}\sigma + f^{-1}_{\phi\psi *}\tau + \theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho}\circ_{\mathrm{h}}\theta^{-\mathrm{h}}_{\phi,\psi\rho}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho\lambda}\\
&\qquad = f^{-1}_{\phi\psi *}(\sigma+\tau) + \theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho\lambda}\\
&\qquad = F(\phi,\psi,\rho\lambda;\sigma+\tau) = F\big((\phi,\psi,\rho;\sigma)\circ_{\mathrm{h}}(\phi,\psi\rho,\lambda;\tau)\big).
\end{aligned}$$

That F preserves identity boxes is also easily checked. For instance,

$$\begin{aligned}
F\mathrm{I}^{\mathrm{v}}(\phi\leftarrow\phi\psi) &= F(id,\phi,\psi;0) = \theta^{-\mathrm{h}}_{id,\phi}\circ_{\mathrm{h}}\theta_{id,\phi\psi} = \mathrm{I}^{\mathrm{v}}f^{-1}_{\phi}\,\mathrm{I}^{\mathrm{v}}f_{\phi\psi}\\
&= \mathrm{I}^{\mathrm{v}}(f^{-1}_{\phi}f_{\phi\psi}) = \mathrm{I}^{\mathrm{v}}F(\phi\leftarrow\phi\psi).
\end{aligned}$$

This double functor F is a weak equivalence. In fact, the induced functor on fundamental groupoids $\Pi F:\Pi\mathcal{G}^k\to\Pi\mathcal{G}$ is an equivalence since its composition with the equivalence $\mathfrak{f}^k:\Pi\mathcal{G}\simeq\Pi\mathcal{G}^k$ is the identity functor on $\Pi\mathcal{G}$: for any morphism $\rho\in\Pi\mathcal{G}(a,b)$,

$$\Pi F(\mathfrak{f}^k\rho) = \Pi F\begin{pmatrix} id_b \longleftarrow \rho \\ \hfill\uparrow \\ \hfill id_a\end{pmatrix} = \begin{bmatrix} b \xleftarrow{f_\rho} b_\rho \\ \hfill\uparrow^{x_\rho} \\ \hfill a \end{bmatrix} = [f_\rho, b_\rho, x_\rho] = \rho.$$

Furthermore, for any object a of $\mathcal{G}$, the induced map

$$\pi_2F:\pi_2(\mathcal{G}^k,id_a)\to\pi_2(\mathcal{G},a)$$

is the obvious isomorphism

$$\begin{array}{ccc} id_a & = & id_a \\ \| & {\scriptstyle (id_a,id_a,id_a;\sigma)} & \| \\ id_a & = & id_a \end{array} \;\mapsto\; \begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array}.$$

□

Corollary 5.4. *Two double groupoids $\mathcal{G}$ and $\mathcal{G}'$ are weak equivalent if and only if there is a double groupoid $\mathcal{G}''$ with weak equivalences $\mathcal{G}\xleftarrow{\sim}\mathcal{G}''\xrightarrow{\sim}\mathcal{G}'$.*

Proof. Suppose $\mathcal{G}$ and $\mathcal{G}'$ are weak equivalent. By Theorem 4.4, they have the same Postnikov invariant, that is, the Postnikov systems $(\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ and $(\Pi\mathcal{G}', \pi_2\mathcal{G}', \mathbf{k}\mathcal{G}')$ are equivalent. Then, by Proposition 5.3 (ii) and (iii), for any representative 3-cocycles of $\mathbf{k}\mathcal{G}$ and $\mathbf{k}\mathcal{G}'$, say k and k' respectively, there is a sequence of weak equivalences

$$\mathcal{G} \overset{\sim}{\leftarrow} \mathcal{G}^k \overset{\sim}{\to} \mathcal{G}^{k'} \overset{\sim}{\to} \mathcal{G}'.$$

□

6. Geometric realization

Theorem 6.1. *The Postnikov invariant of a double groupoid $\mathcal{G}$ agrees with the Postnikov invariant of its geometric realization $|\mathcal{G}|$.*

Proof. This follows from Proposition 6.2 below. □

For a groupoid P, let us recall from the beginning of Section 4 that $\mathrm{N}P$ denotes its nerve, that is, the simplicial set with m-simplices the composable sequences $\beta = (\beta m \overset{\beta_m}{\leftarrow} \cdots \overset{\beta_1}{\leftarrow} \beta 0)$ of m arrows in P. If $(P, \mathcal{A}, \mathbf{k})$ is any Postnikov system and we select any normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$ representative of the cohomology class $\mathbf{k} \in H^3(P, \mathcal{A})$, then the equivalence class $[P, \mathcal{A}, \mathbf{k}]$ is justly realized as the unique Postnikov invariant of (the geometric realization of) the simplicial set homotopy colimit of the functor

$$K(\mathcal{A}, 2) : P \to \mathbf{Sset}, \qquad a \mapsto K(\mathcal{A}(a), 2),$$

twisted by the 3-cocycle k (see, for instance, Goerss and Jardine [18, Chapter VI, Lemma 5.8]). This simplicial set, which we denote by

$$\operatorname*{hocolim}_{P} K(\mathcal{A}, 2; k), \tag{26}$$

has the same simplices as the ordinary homotopy colimit $\operatorname*{hocolim}_{P} K(\mathcal{A}, 2)$, that is, its set of m-simplices is

$$\bigsqcup_{\beta \in \mathrm{N}P_m} K(\mathcal{A}(\beta m), 2)_m.$$

Its face and degeneracy maps are also the same as those of non-twisted homotopy colimit, except the last face maps which are here canonically affected by the cocycle k. This twisted homotopy colimit (26) becomes a Kan complex that is coskeletal in dimensions higher than three and whose 3-truncation can be described explicitly as below

$$\bigsqcup_{\beta\in \mathrm{N}P_3} \mathcal{A}(\beta 3)^3 \underset{d_3}{\overset{d_0}{\Rrightarrow}} \bigsqcup_{\beta\in \mathrm{N}P_2} \mathcal{A}(\beta 2) \underset{d_2}{\overset{d_0}{\Rrightarrow}} \mathrm{N}P_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathrm{N}P_0$$

where, for any $\beta \in \mathrm{N}P_2$ and $\sigma \in \mathcal{A}(\beta 2)$

$$d_i(\beta,\sigma) = d_i\beta, \quad 0 \le i \le 2,$$

for any $\beta \in \mathrm{N}P_3$ and $(\sigma_0,\sigma_1,\sigma_2) \in \mathcal{A}(\beta 3)^3$,

$$d_i(\beta,\sigma_0,\sigma_1\sigma_2) = \begin{cases} (d_i\beta,\sigma_i) & \text{if} \quad 0 \le i \le 2, \\ \big(d_3\beta, \beta_{3*}^{-1}(k(\beta)+\sigma_2-\sigma_1+\sigma_0)\big) & \text{if} \quad i = 3, \end{cases}$$

for any $\beta \in \mathrm{N}P_1$,

$$s_i(\beta) = (s_i\beta, 0), \quad i = 0,1,$$

and, for any $\beta \in \mathrm{N}P_2$ and $\sigma \in \mathcal{A}(\beta 2)$,

$$s_i(\beta,\sigma) = \begin{cases} (s_0\beta,\sigma,\sigma,0) & \text{if} \quad i = 0, \\ (s_1\beta,0,\sigma,\sigma) & \text{if} \quad i = 1, \\ (s_2\beta,0,0,\sigma) & \text{if} \quad i = 2. \end{cases}$$

Now, for a double groupoid $\mathcal{G}$, let $\mathrm{NN}\mathcal{G}$ denote its double nerve, that is, the bisimplicial set where a (p,q)-simplex is a subdivision of a box of $\mathcal{G}$ as a matrix of $p \times q$ horizontally and vertically composable boxes of the form

$$\begin{tikzcd}
a_{pq} & a_{p-1q} \arrow[l, "f_{pq}"'] & \cdots & a_{1q} & a_{0q} \arrow[l, "f_{1q}"'] \\
a_{pq-1} \arrow[u, "x_{pq}"] \arrow[r, phantom, "\theta_{p,q}"] & a_{p-1q-1} \arrow[l, "f_{pq-1}"] \arrow[u, "x_{p-1q}"'] & \cdots & a_{1q} \arrow[u, "x_{1q}"] \arrow[r, phantom, "\theta_{1,q}"] & a_{0q-1} \arrow[l, "f_{1q-1}"] \arrow[u, "x_{0q}"'] \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] \\
a_{p1} \arrow[u] & a_{p-11} \arrow[l, "f_{p1}"'] \arrow[u] & \cdots & a_{11} \arrow[u] & a_{01} \arrow[l, "f_{11}"'] \arrow[u] \\
a_{p0} \arrow[u, "x_{p1}"] \arrow[r, phantom, "\theta_{p,1}"] & a_{p-10} \arrow[l, "f_{p0}"] \arrow[u, "p-11"'] & \cdots & a_{10} \arrow[u, "x_{11}"] \arrow[r, phantom, "\theta_{1,1}"] & a_{00} \arrow[l, "f_{10}"] \arrow[u, "x_{01}"']
\end{tikzcd}$$

The bisimplicial face maps are the natural ones, induced by horizontal and vertical composition of boxes in $\mathcal{G}$, and the degeneracy ones by appropriate identity boxes. We picture $\mathrm{NN}\mathcal{G}$ so that the set of (p, q)-simplices is the set in the p-th row and q-th column. Thus, its p-th column, $\mathrm{NN}\mathcal{G}_{p\bullet}$, is the nerve of the "vertical" groupoid whose objects are strings $\cdot \overset{f_p}{\leftarrow} \cdots \overset{f_1}{\leftarrow} \cdot$ of p composable horizontal arrows in $\mathcal{G}$ and whose arrows are length p sequences of horizontally composable boxes

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g_p"'] & \cdots & \cdot & \cdot \arrow[l] & \cdot \arrow[l, "g_1"'] \\
\cdot \arrow[u] \arrow[r, phantom, "\theta_p"] & \cdot \arrow[l, "f_p"] \arrow[u] & \cdots & \cdot \arrow[u] \arrow[r, phantom, "\theta_2"] & \cdot \arrow[l] \arrow[u] \arrow[r, phantom, "\theta_1"] & \cdot \arrow[l, "f_1"] \arrow[u]
\end{tikzcd}$$

Similarly, the q-th column, $\mathrm{NN}\mathcal{G}_{\bullet q}$, is the nerve of the "horizontal" groupoid whose objects are length q sequences of composable vertical morphisms in $\mathcal{G}$ and whose arrows are sequences of q vertically composable boxes. In particular, $\mathrm{NN}\mathcal{G}_{0\bullet}$ and $\mathrm{NN}\mathcal{G}_{\bullet 0}$ are, respectively, the nerves of the groupoids of vertical and horizontal morphisms of $\mathcal{G}$.

The geometric realization $|\mathcal{G}|$ of the double groupoid $\mathcal{G}$ is, by definition, the geometric realization of the simplicial set *diagonal* of its double nerve, that is, $|\mathcal{G}| = |\triangle\, \mathrm{NN}\mathcal{G}|$. By Cegarra-Remedios [12, Therem 1.1] or Zisman [27], $|\mathcal{G}|$ can be also realized, up to homotopy equivalence, as the geometric realization of the Artin-Mazur *total* simplicial set [2, Section III] (aka *codiagonal* or $\overline{W}$) of the double nerve, $\nabla\,\mathrm{NN}\mathcal{G}$. A direct analysis of this simplicial set tell us that it is a Kan complex in which any simplex of dimension higher than two is determined by any three of its faces. In particular, it is coskeletal in dimensions higher than 3, so that it is completely determined by its 3-truncation, which is explicitly described as follows. Its vertices are the objects a of $\mathcal{G}$. The 1-simplices ξ_1 are the paths of $\mathcal{G}$

$$\xi_1 : \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \\ & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

whose faces are $d_0\xi_1 = a_{11}$ and $d_1\xi_1 = a_{00}$. The 2-simplices ξ_2 are the diagrams in $\mathcal{G}$

$$\xi_2 : \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

with faces

$$d_0\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] \\ & a_{11}, \arrow[u, "x_{12}"'] \end{tikzcd} \quad d_1\xi_2 = \begin{tikzcd} a_{22} & a_{02} \arrow[l, "f_{22}f_{12}"'] \\ & a_{00}, \arrow[u, "x_{02}x_{01}"'] \end{tikzcd} \quad d_2\xi_2 = \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \\ & a_{00}, \arrow[u, "x_{01}"'] \end{tikzcd}$$

and its 3-simplices ξ_3 are the diagrams in $\mathcal{G}$

$$\xi_3 : \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{13} \arrow[l, "f_{23}"'] & a_{03} \arrow[l, "f_{13}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}"] & a_{12} \arrow[l, "f_{22}"] \arrow[u, "x_{13}"'] \arrow[ur, phantom, "\theta_{13}"] & a_{02} \arrow[l, "f_{12}"] \arrow[u, "x_{03}"'] \\ & & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

with faces

$$d_0\xi_3 = \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{13} \arrow[l, "f_{23}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}"] & a_{12} \arrow[l, "f_{22}"] \arrow[u, "x_{13}"'] \\ & & a_{11}, \arrow[u, "x_{12}"'] \end{tikzcd} \qquad d_1\xi_3 = \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{03} \arrow[l, "f_{23}f_{13}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}\circ_{\mathrm{h}}\theta_{13}"] & a_{02} \arrow[l, "f_{22}f_{12}"] \arrow[u, "x_{03}"'] \\ & & a_{00}, \arrow[u, "x_{02}x_{01}"'] \end{tikzcd}$$

$$d_2\xi_3 = \begin{tikzcd} a_{33} & a_{13} \arrow[l, "f_{33}f_{23}"'] & a_{03} \arrow[l, "f_{13}"'] \\ & a_{11} \arrow[u, "x_{13}x_{12}"] \arrow[ur, phantom, "\theta_{13}\circ_{\mathrm{v}}\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{03}x_{02}"'] \\ & & a_{00}, \arrow[u, "x_{01}"'] \end{tikzcd} \qquad d_3\xi_3 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & a_{00}. \arrow[u, "x_{01}"'] \end{tikzcd}$$

Degeneracies are defined by

$$s_0 a = \begin{tikzcd} a \arrow[r, equal] & a \arrow[d, equal] \\ & a \end{tikzcd} \quad s_0\xi_1 = \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \arrow[r, equal] & a_{01} \\ & a_{11} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{01}"] & a_{01} \arrow[u, "x_{01}"'] \arrow[d, equal] \\ & & a_{00} \end{tikzcd} \quad s_1\xi_1 = \begin{tikzcd} a_{11} \arrow[r, equal] & a_{11} \arrow[d, equal] & a_{01} \arrow[l, "f_{11}"'] \arrow[d, equal] \\ & a_{11} \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{11}"] & a_{01} \arrow[l] \\ & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

$$s_0\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \arrow[r, equal] & a_{02} \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{02}"] & a_{01} \arrow[u, "x_{02}"'] \\ & & a_{00} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{01}"] & a_{00} \arrow[u, "x_{01}"'] \arrow[d, equal] \\ & & & a_{00} \end{tikzcd} \qquad s_1\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] \arrow[r, equal] & a_{12} & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{12}"] & a_{11} \arrow[u] \arrow[d, equal] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l] \arrow[u, "x_{02}"'] \arrow[d, equal] \\ & & a_{11} \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{11}"] & a_{01} \arrow[l, "f_{11}"] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

$$s_2\xi_2 = \begin{tikzcd} a_{22} \arrow[r, equal] & a_{22} \arrow[d, equal] & a_{12} \arrow[l, "f_{22}"'] \arrow[d, equal] & a_{02} \arrow[l, "f_{12}"'] \arrow[d, equal] \\ & a_{22} \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{22}"] & a_{12} \arrow[l] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{12}"] & a_{02} \arrow[l] \\ & & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

Proposition 6.2. *Let $\mathcal{G}$ be a double groupoid. For any normalized 3-cocycle $k \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ representing the cohomology class $\mathbf{k}\mathcal{G}$, there is a weak equivalence of simplicial sets*

$$\Gamma : \underset{\Pi\mathcal{G}}{\operatorname{hocolim}}\, K(\pi_2\mathcal{G}, 2; k) \xrightarrow{\sim} \nabla\, \mathrm{NN}\mathcal{G}.$$

Proof. By Lemma 4.3 (i), we can assume that $k = k^{\mathcal{G}}$ for a certain selection of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$ and the boxes $\theta_{\psi,\rho}$, as in (3) and (12). The claimed simplicial map Γ, which is completely defined by its 3-truncation

$$\begin{tikzcd}[column sep=large]
\bigsqcup\limits_{\beta\in \mathrm{N}\Pi\mathcal{G}_3} \pi_2(\mathcal{G},\beta 3)^3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d_3"'] \arrow[d, "\Gamma_3"'] & \bigsqcup\limits_{\beta\in \mathrm{N}\Pi\mathcal{G}_2} \pi_2(\mathcal{G},\beta 2) \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=30] \arrow[l, bend right=20, "s_2"] \arrow[d, "\Gamma_2"] & \mathrm{N}\Pi\mathcal{G}_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[l, bend right=35, "s_0"'] \arrow[l, bend right=20, "s_1"] \arrow[d, "\Gamma_1"] & \mathrm{N}\Pi\mathcal{G}_0 \arrow[l, bend right=30, "s_0"'] \arrow[d, equal, "\Gamma_0"] \\
\nabla\mathrm{NN}\mathcal{G}_3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d_3"'] & \nabla\mathrm{NN}\mathcal{G}_2 \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=30] \arrow[l, bend right=20, "s_2"] & \nabla\mathrm{NN}\mathcal{G}_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[l, bend right=35, "s_0"'] \arrow[l, bend right=20, "s_1"] & \nabla\mathrm{NN}\mathcal{G}_0, \arrow[l, bend right=30, "s_0"']
\end{tikzcd}$$

is given as follows: Γ_0 is the identity map on the objects of the double groupoid $\mathcal{G}$. For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$,

$$\Gamma_1(\rho) = \begin{tikzcd} a_1 & b_\rho \arrow[l, "f_\rho"'] \\ & a_0, \arrow[u, "x_\rho"'] \end{tikzcd}$$

If $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any two morphisms in $\Pi\mathcal{G}$ and $\sigma \in \pi_2(\mathcal{G}, a_2)$,

$$\Gamma_2(\psi,\rho;\sigma) = \begin{tikzcd} a_2 & b_\psi \arrow[l, "f_\psi"'] & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] \\ & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "f_{\psi *}^{-1}(\sigma)+\theta_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho}x_\rho^{-1}"'] \\ & & a_0, \arrow[u, "x_\rho"'] \end{tikzcd}$$

and, for any $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in $\Pi\mathcal{G}$ and $(\sigma_0, \sigma_1, \sigma_2) \in \pi_2(\mathcal{G}, a_3)^3$,

$$
\Gamma_3(\phi, \psi, \rho; \sigma_0, \sigma_1, \sigma_2) =
\begin{tikzcd}
a_3 & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1} f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1} f_{\phi\psi\rho}"'] \\
& a_2 \arrow[u, "x_\phi"] \arrow[ur, phantom, "f_{\phi*}^{-1}(\sigma_0)+\theta_{\phi,\psi}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u] \arrow[ur, phantom, "\substack{f_{\phi\psi*}^{-1}(\sigma_1-\sigma_0)\\ + \\ \theta_{\phi,\psi}^{-\mathrm{h}} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}}"] & b_{\psi\rho} \arrow[l] \arrow[u, "x_{\phi\psi\rho} x_{\psi\rho}^{-1}"'] \\
& & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\substack{f_{\psi*}^{-1}\phi_*^{-1}(\sigma_0-\sigma_1+\sigma_2)\\ + \\ f_{\psi*}^{-1}\phi_*^{-1}k(\phi,\psi,\rho) \\ + \\ \theta_{\psi,\rho}}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \\
& & & a_0. \arrow[u, "x_\rho"']
\end{tikzcd}
$$

So defined, all the simplicial identities to verify that Γ is actually a simplicial map are easily checked, except perhaps that $d_i\Gamma_3 = \Gamma_2 d_i$ for $i = 1, 2$. For $i = 1$, the required equality $d_1\Gamma_3(\phi, \psi, \rho, \sigma_0, \sigma_1, \sigma_2) = \Gamma_2(\phi, \psi\rho, \sigma_1)$ follows from the equalities

$$
\begin{aligned}
&\big(f_{\phi*}^{-1}(\sigma_0)+\theta_{\phi,\psi}\big) \circ_{\mathrm{h}} \big(f_{\phi\psi*}^{-1}(\sigma_1 - \sigma_0) + \theta_{\phi,\psi}^{-\mathrm{h}} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}\big) \\
&\qquad \overset{(7)(8)}{=} f_{\phi*}^{-1}(\sigma_0) + f_{\phi*}^{-1}(\sigma_1 - \sigma_0) + \theta_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi}^{-\mathrm{h}} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho} \\
&\qquad = f_{\phi*}^{-1}(\sigma_1) + \theta_{\phi,\psi\rho}.
\end{aligned}
$$

The case $i = 2$ is somewhat more complicated. In this case, the required equality

$$d_2\Gamma_3(\phi, \psi, \rho, \sigma_0, \sigma_1, \sigma_2) = \Gamma_2(\phi\psi, \rho, \sigma_2)$$

follows from the equalities

$$
\begin{aligned}
\big(f^{-1}_{\phi\psi*}(\sigma_1{-}\sigma_0)+\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho}\big)&\circ_{\mathrm{v}}\big(f^{-1}_{\psi*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho)+\theta_{\psi,\rho}\big)\\
&\overset{(9)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}x^{-1}_{\psi*}f^{-1}_{\psi_*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad + (\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\overset{2.4(ii)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}\psi^{-1}_{*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad + (\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\overset{3.3}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}(\phi\psi)^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad + (\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\overset{2.4(ii)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}x^{-1}_{\phi\psi*}f^{-1}_{\phi\psi*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad + (\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&= f^{-1}_{\phi\psi*}\sigma_2+f^{-1}_{\phi\psi*}k(\phi,\psi,\rho)+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\overset{(14)}{=} f^{-1}_{\phi\psi*}\sigma_2+\theta_{\phi\psi,\rho}.
\end{aligned}
$$

That Γ induces an isomorphism on the fundamental groupoids follows from the observation that homotopies $(f,b,x)\simeq(g,c,y)$ in $\mathcal{G}$ between two paths from an object a_0 to an object a_1, as in (2), are in bijection with homotopies $(f,b,x)\simeq(g,c,y)$ in the simplicial set $\nabla\,\mathrm{NN}\mathcal{G}$, by the mapping

$$
\begin{tikzcd}
b \arrow[d, equal] \arrow[r, phantom, "\alpha"] & c \arrow[l, "f^{-1}g"'] \\
b \arrow[r, equal] & b \arrow[u, "yx^{-1}"']
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[d, equal] \arrow[r, phantom, "\Gamma^{\mathrm{v}} f\circ_{\mathrm{h}}\alpha"] & c \arrow[l, "g"'] \\
 & a_1 & b \arrow[l, "f"] \arrow[u, "yx^{-1}"'] \\
 & & a_0 \arrow[u, "x"']
\end{tikzcd}
$$

Furthermore, for any object a of $\mathcal{G}$, the induced homomorphism by Γ on the second homotopy groups with base a,

$$\pi_2\big(\operatorname*{hocolim}_{\Pi\mathcal{G}} K(\pi_2\mathcal{G},2;k),a\big)\to\pi_2(\nabla\mathrm{NN}\mathcal{G},a),$$

is explicitly given by

$$\begin{tikzcd} & a \arrow[r, equal] & a \arrow[r, equal] \arrow[dr, phantom, "\sigma"] & a \\ (id_a, id_a; \sigma) \arrow[r, mapsto, "\Gamma_2"] & {} & a \arrow[u, equal] \arrow[r, equal] & a \arrow[u, equal] \arrow[d, equal] \\ & & & a \end{tikzcd}$$

and clearly is an isomorphism.

Since the homotopy groups of $\operatorname*{hocolim}_{\Pi\mathcal{G}} K(\pi_2\mathcal{G}, 2; k)$ and of $\nabla\mathrm{NN}\mathcal{G}$ vanish in degree 3 and higher, Γ is actually a weak homotopy equivalence. $\square$

As a consequence of Theorems 5.1 and 6.1, we get a new proof of the following fact (cf. [12, Theorem 13] for a more general result).

Corollary 6.3. *The mapping $\mathcal{G} \mapsto |\mathcal{G}|$ induces a bijective correspondence between weak equivalence classes of double groupoids and weak homotopy classes of 2-types.*

References

[1] N. Andruskiewitsch and S. Natale, Tensor categories attached to double groupoids. *Adv. Math.* 200 (2006), 539-583.

[2] M. Artin and B. Mazur, On the van Kampen theorem. *Topology* 5 (1966), 179-189.

[3] R. Brown, From groups to groupoids, a brief survey. *Bull. London Math. Soc.* 19 (1987) 113-134.

[4] R. Brown, Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems, *Fields Inst. Commun.* 43, 101-130, Amer. Math. Soc. (2004).

[5] R. Brown, 'Double modules', double categories and groupoids, and a new homotopical double groupoid, *Preprint* 2009 (arXiv:0903.2627v2).

[6] R. Brown and P.J. Higgins, On the algebra of cubes, *J. Pure Appl. Algebra* 21 (1981), 233-260.

[7] R. Brown and C.B. Spencer, Double groupoids and crossed modules, *Cahiers Topologie Géom. Différentielle* 17 (1976), 343-362.

[8] R. Brown, K.A. Hardie, K.H. Kamps and T. Porter, A homotopy double groupoid of a Hausdorff space. *Theory Appl. Categ.* 10 (2002), 71-93.

[9] R. Brown, K.H. Kamps and T. Porter, A homotopy double groupoid of a Hausdorff space. II. A van Kampen theorem. *Theory Appl. Categ.* 14 (2005), 200-220.

[10] M. Bullejos, A.M. Cegarra and J. Duskin, On cat^n-groups and homotopy types. *J. Pure Appl. Algebra* 86 (1993), 135-154.

[11] A.M. Cegarra, B.A. Heredia and J. Remedios, Double Groupoids and Homotopy 2-types. *Appl. Categ. Structures* 20 (2012), 323-378 .

[12] A.M. Cegarra and J. Remedios. The relationship between the diagonal and the bar constructions on a bisimplicial set. *Topology Appl.* 153 (2005), 21-51.

[13] R. Dawson and R. Paré. General associativity and general composition for double categories. *Cahiers Topologie Géom. Différentielle Catég.* 34 (1993), 57-79.

[14] C. Ehresmann, Catégories doubles et catégories structurées, *C. R. Acad. Sci. Paris* 256 (1963), 1198-1201.

[15] C. Ehresmann, Catégories Structurées, *Ann. Sci. Ecole Normale Sup.* 80 (1963), 349-425.

[16] C. Ehresmann, *Catégories et structures*, Dunod, Paris (1965).

[17] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band* 35 Springer-Verlag New York, Inc., New York, 1967.

[18] P.G. Goerss and J.F. Jardine, Simplicial homotopy theory. *Modern Birkhäuser Classics*, Birkhäuser Verlag, Basel, 2009.

[19] A. Grothendieck, Sur quelques points d'algère homologique. *Tohoku Math. J.* 9 (1957), 119-221.

[20] K.A. Hardie, K.H. Kamps and R.W. Kieboom, A homotopy 2-groupoid of a Hausdorff space, *Appl. Categ. Structures* 8 (2000), 209-234.

[21] L. Illusie, Complex Cotangent et Déformations II. *Lecture Notes in Math.* 283, Springer, Berlin, 1972.

[22] J.-L. Loday, Spaces with finitely many nontrivial homotopy groups, *J. Pure Appl. Algebra* 24 (1982), 179-202.

[23] K. Mackenzie, Double Lie algebroids and Second-order Geometry, II, *Adv. Math.* 154 (2000), 46-75.

[24] I. Moerdijk and J.-A. Svensson, Algebraic classification of equivariant homotopy 2-types. I. *J. Pure Appl. Algebra* 89 (1993), 187-216.

[25] T. Porter, n-types of simplicial groups and crossed n-cubes. *Topology* 32 (1993), 5-24.

[26] J.-E. Roos, Sur les foncteurs dérivés de $\varprojlim$. Applications. *C. R. Acad. Sci. Paris* 252 (1961), 3702-3704.

[27] M. Zisman, Comparaison de deux diagonales pour un ensemble bisimplicial, *J. Homotopy Relat. Struct.* 10 (2015), 1013-1021.

Antonio M. Cegarra
Department of Algebra, Faculty of Sciences
University of Granada
18071 Granada, Spain
acegarra@ugr.es

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXIV-2 (2023)

COMPARING THE NON-UNITAL AND UNITAL SETTINGS FOR DIRECTED HOMOTOPY

Philippe GAUCHER

Résumé. Cette note explore le lien entre la structure de modèles de type Quillen des flots et la structure de modèles de Ilias sur les petites catégories enrichies sur les espaces topologiques. Les deux ont des équivalences faibles qui induisent des équivalences sur les (semi)catégories fondamentales. La structure de modèles de Ilias ne peut pas être transférée sur les flots le long de l'adjoint à gauche qui ajoute le morphismes identité. La structure de modèles minimale sur les flots ayant comme cofibrations le transfert le long de ce foncteur des cofibrations de la structure de modèles de Ilias a comme catégorie homotopique l'ensemble totalement ordonné à 3 éléments. La structure de modèles de type Quillen des flots peut être transférée le long de l'adjoint à droite oubliant les morphismes identité. On obtient une catégorie de modèle minimale telle que les équivalences faibles induisent une équivalence sur les catégories fondamentales. Le foncteur identité de la catégorie des petites catégories enrichies sur les espaces topologiques n'est ni un adjoint de Quillen à gauche, ni un adjoint de Quillen à droite entre la structure de modèles de type Quillen et la structure de modèles de Ilias.

Abstract. This note explores the link between the q-model structure of flows and the Ilias model structure of topologically enriched small categories. Both have weak equivalences which induce equivalences of fundamental (semi)categories. The Ilias model structure cannot be left-lifted along the left adjoint adding identity maps. The minimal model structure on flows having as cofibrations the left-lifting of the cofibrations of the Ilias model structure has a homotopy category equal to the 3-element totally ordered set. The q-model structure of flows can be right-lifted to a q-model structure of topologically enriched small categories which is minimal and such that the weak equivalences induce equivalences of fundamental categories. The identity functor of topologically enriched small categories is neither a left Quillen adjoint nor a right Quillen adjoint between the q-model structure and the Ilias model structure.

Keywords. Directed homotopy, flow, Dwyer-Kan equivalence, combinatorial model category, minimal model category, locally presentable category, topologically enriched category.

Mathematics Subject Classification (2020). 18C35,18D20,55U35,68Q85.

1. Introduction

1.1 Presentation

The time flow of a concurrent process can be modelled by a topologically enriched small semicategory [8] or by a topologically enriched small category [4, 27]. The objects represent the *states* of the concurrent process and the nonidentity morphisms represent the *execution paths*, the topology modelling concurrency [6]. The primary reason for excluding identity morphisms in [8, Definition 4.11] is to obtain *functorial* constructions for the branching and merging homology theories (see [8, Section 20]). It enables us to prove the invariance by refinement of observation in [10, Corollary 11.3], and therefore to fix Goubault-Jensen's construction of [19]. The main technical tool is the minimal model category introduced in [8], called the *q-model structure (of flows)* after [17, Theorem 7.6]. The examples coming from computer science are non-unital as well because they are modelled by *precubical* sets (e.g. [12, 13, 20, 31]) and because precubical sets have non-unital geometric realizations [12, Definition 7.2]. The transverse degeneracy maps of precubical sets, introduced for the *functorial* formalization of the parallel product with synchronization of process algebra [13, Theorem 3.1.15 and Definition 4.2.2], belong to the non-unital world as well. The transverse degeneracy maps lead to a vast generalization of Raussen's notion of natural d-path in [18]. The non-unital setting is also necessary to construct the underlying homotopy type functor which is geometrically the homotopy type of the space obtained after forgetting the temporal information [11, Section 6] [15, Proposition 8.16].

On the other hand, the mathematical literature provides several constructions of model structures on enriched small categories such that the weak equivalences are the so-called Dwyer-Kan equivalences of [5]: for simplicially enriched small categories [3], for topologically enriched small categories [25] and for small categories enriched in a given monoidal model category [2]. The generating cofibrations of the q-model structure of flows of [8] are almost those obtained by transfer along the left adjoint formally adding identity maps from the generating cofibrations of the Ilias model structure constructed in [25]. The only difference is the presence of the flow cofibration $R : \{0,1\} \to \{0\}$ which has no counterpart in the Ilias model structure (see Proposition 3.3). This leads to the question of comparing the model structures on flows and on topologically enriched small categories. The following sequence of theorems answers the question.

Theorem. *(Theorem 3.5) The Ilias model structure on topologically enriched small categories [25] cannot be transferred to the category of flows along the left adjoint formally adding identity maps.*

Theorem. *(Corollary 4.10) The minimal model structure on flows with respect to the transfer of the cofibrations of the Ilias model structure along the left adjoint formally adding identity maps has three homotopy types.*

Theorem. *(Theorem 5.2) The q-model structure of flows can be transferred along the right adjoint forgetting the identity maps to the category of topologically enriched small categories. We obtain a combinatorial model structure which is minimal. Its weak equivalences induce equivalences of fundamental categories. The left Quillen adjoint formally adding identity maps from flows to enriched small categories is not a left Quillen equivalence.*

The model category of Theorem 5.2 on topologically enriched small categories seems to be new. With the same argument, the h-model structure and the m-model structure of flows constructed in [17, Theorem 7.4] can be transferred along the right adjoint forgetting the identity maps to the category of topologically enriched small categories as well. We obtain a h-model structure and a m-model structure on topologically enriched small categories which are both accessible as model categories.

The following table of minimal model categories summarizes the results of this note. The symbol ● means that the weak equivalences induce equivalences of fundamental (semi)categories. The symbol ○ means that they do not.

	R is a cofibration	R is not a cofibration
Flow	● q-model structure of [8]	○ Corollary 4.10
Cat	● Theorem 5.2	● Ilias model structure of [25]

The conclusion that must be drawn from this note is that the flow cofibration $R : \{0,1\} \to \{0\}$ is much more important in a globular approach of directed homotopy than what was expected in [8].

1.2 Prerequisites and notations

We refer to [1] for locally presentable categories, to [7, 22, 29] for combinatorial and accessible model categories. We refer to [23, 24] for more general model categories. We work with a locally presentable convenient category of topological spaces $\mathbf{Top}$ for doing algebraic topology. The internal hom is denoted by $\mathbf{TOP}(-,-)$. The category of *Δ-generated spaces* or of *Δ-Hausdorff Δ-generated spaces* (cf. [16, Section 2 and Appendix B]) are two such examples. The category $\mathbf{Top}$ is equipped with its q-model structure (we use the terminology of [26]). What follows is some notations and conventions: $\varnothing$ is the initial object, $\mathbf{1}$ is the final object, Id_X is the identity of X. A model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ means that the class of cofibrations is $\mathcal{C}$, that the class of fibrations is $\mathcal{F}$ and that the class of weak equivalences is $\mathcal{W}$. A combinatorial model structure on $\mathcal{K}$ is *minimal* if the class of weak equivalences is the smallest Grothendieck localizer with respect to its set of generating cofibrations [21, 30]. Note that in [30], the adjective *left-determined* is used instead. When all objects of a model category are fibrant, any Grothendieck localizer which is strictly smaller than the class of weak equivalences never induces a model structure. By [21, Theorem 1.4], every tractable combinatorial model category with fibrant objects is minimal. The notation $f \boxslash g$ means that g satisfies the right lifting property (RLP) with respect to f; ${}^{\boxslash}\mathcal{C} = \{g, \forall f \in \mathcal{C}, g \boxslash f\}$; $\mathcal{C}^{\boxslash} = \mathbf{inj}(\mathcal{C}) = \{g, \forall f \in \mathcal{C}, f \boxslash g\}$; $\mathbf{cof}(\mathcal{C}) = {}^{\boxslash}(\mathcal{C}^{\boxslash})$; $\mathbf{cell}(\mathcal{C})$ is the class of transfinite compositions of pushouts of elements of $\mathcal{C}$. A *cellular* object X

of a combinatorial model category is an object such that the canonical map $\varnothing \to X$ belongs to $\mathbf{cell}(I)$ where I is the set of generating cofibrations.

In this paper, the transfer of a model structure of $\mathbf{Flow}$ along the right adjoint $\mathbf{Cat} \subset \mathbf{Flow}$ of Proposition 2.7 is called the *right-lifting* of the model structure of $\mathbf{Flow}$. Similarly, the transfer of a model structure of $\mathbf{Cat}$ (of a weak factorization system resp.) along the left adjoint $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ of Proposition 2.7 is called the *left-lifting* of the model structure of $\mathbf{Cat}$ (of the weak factorization system resp.). See the introductions of [7, 22] and the beginning of [7, Section 2] for further explanations.

1.3 Acknowledgments

I thank Simon Henry for a useful discussion about semicategories. I thank the anonymous referee for the suggestions to improve and clarify the presentation.

2. The adjunction $\mathrm{I}^+ : \mathbf{Flow} \leftrightarrows \mathbf{Cat} :\supset$

Definition 2.1. *[8, Definition 4.11] A* flow *is a small semicategory enriched over the closed monoidal category* $(\mathbf{Top}, \times)$. *The corresponding category is denoted by* $\mathbf{Flow}$.

A *flow* X consists of a topological space $\mathbb{P}X$ of *execution paths*, a discrete space X^0 of *states*, two continuous maps s and t from $\mathbb{P}X$ to X^0 called the source and target map respectively, and a continuous and associative map $* : \{(x, y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$ such that $s(x * y) = s(x)$ and $t(x * y) = t(y)$. Let $\mathbb{P}_{\alpha,\beta}X = \{x \in \mathbb{P}X \mid s(x) = \alpha \text{ and } t(x) = \beta\}$. Note that the composition is denoted by $x * y$, not by $y \circ x$.

Every set can be viewed as a flow with an empty space of execution paths. The obvious functor $\mathbf{Set} \subset \mathbf{Flow}$ from the category of sets to that of flows is limit-preserving and colimit-preserving. The following examples of flows are important for the sequel:

Example 2.2. For a topological space Z, let $\mathrm{Glob}(Z)$ be the flow defined by

$$\mathrm{Glob}(Z)^0 = \{0, 1\},\ \mathbb{P}\mathrm{Glob}(Z) = \mathbb{P}_{0,1}\mathrm{Glob}(Z) = Z,\ s = 0,\ t = 1.$$

This flow has no composition law. The directed segment is the flow $\vec{I} = \mathrm{Glob}(\{0\})$.

Example 2.3. Denote by B (like branching) the flow $1 \leftarrow 0 \rightarrow 1$ with three states and two execution paths. This flow has no composition law.

Notation 2.4. Let $n \geqslant 1$. Denote by $\mathbf{D}^n = \{b \in \mathbb{R}^n, |b| \leqslant 1\}$ the n-dimensional disk, and by $\mathbf{S}^{n-1} = \{b \in \mathbb{R}^n, |b| = 1\}$ the $(n-1)$-dimensional sphere. By convention, let $\mathbf{D}^0 = \{0\}$ and $\mathbf{S}^{-1} = \varnothing$.

Notation 2.5. Let

$$\begin{aligned}
&I^{gl} = \{c_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n) \mid n \geqslant 0\},\\
&J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1] \mid n \geqslant 0\},\\
&C : \varnothing \to \{0\}, R : \{0,1\} \to \{0\}.
\end{aligned}$$

Notation 2.6. The category of small categories enriched over $\mathbf{Top}$ is denoted by $\mathbf{Cat}$. The set of objects of an enriched small category X is denoted by $\mathrm{Obj}(X)$ and the space of morphisms from A to B by $X(A, B)$.

Proposition 2.7. *(well-known) The inclusion* $\mathbf{Cat} \subset \mathbf{Flow}$ *has a left adjoint* [1] *denoted by* $\mathrm{I}^+ : \mathbf{Flow} \longrightarrow \mathbf{Cat}$. *It consists of adding identity maps as isolated points in the spaces of morphisms. This functor is faithful.*

What follows is an adaptation of [6, Definition 4.37].

Definition 2.8. *Let X be an object of* $\mathbf{Flow}$. *The* fundamental semicategory *of X is the small semicategory $\vec{\pi}_1(X)$ having X^0 for the set of objects and the set of morphisms between two objects is the set of path-connected components of the space of execution paths between these two objects. If $X \in \mathbf{Cat} \subset \mathbf{Flow}$, then $\vec{\pi}_1(X)$ is a small category which is called the* fundamental category *of X.*

For all $X \in \mathbf{Flow}$, $\mathrm{I}^+(\vec{\pi}_1(X))$ is also a small category which is called the fundamental category of X. For $X \in \mathbf{Cat}$, the canonical map $\mathrm{I}^+(\vec{\pi}_1(X)) \to \vec{\pi}_1(X)$ is not an equivalence of categories.

[1] It has also a right adjoint, the enriched small category of idempotents of a flow, which is not used in this note.

3. Left-lifting the Ilias model structure

Theorem 3.1. *[25, Theorem 2.4] There exists one and only one combinatorial model structure $(\mathbf{Cat})_{DK} = (\mathcal{C}_{DK}, \mathcal{F}_{DK}, \mathcal{W}_{DK})$ on $\mathbf{Cat}$ with the following properties:*

- *A set of generating cofibrations is the set of maps $\mathrm{I}^+(I^{gl} \cup \{C\})$.*
- *The weak equivalences are the DK-equivalences: there are the maps of enriched functors $F : \mathcal{C} \to \mathcal{D}$ such that $\overrightarrow{\pi}_1(F) : \overrightarrow{\pi}_1(\mathcal{C}) \to \overrightarrow{\pi}_1(\mathcal{D})$ is an equivalence of categories and such that for all pairs of objects (α, β) of $\mathcal{C}$, there is a weak homotopy equivalence $\mathcal{C}(\alpha, \beta) \to \mathcal{D}(F(\alpha), F(\beta))$.*
- *A set of generating trivial cofibrations is given by the set of maps $\mathrm{I}^+(J^{gl}) \cup \{\mathrm{I}^+(\{0\}) \to (\{0 \cong 1\})^{cof}\}$ where $\{0 \cong 1\}$ is the small category with two isomorphic objects 0 and 1.*

It is called the Ilias model structure. *All objects are fibrant.*

Theorem 3.1 is the topological analogue of the Bergner model structure on simplicially enriched small categories [3]. The weak equivalences are the *Dwyer-Kan equivalences* of [5]. The combinatorial model category is minimal since all objects are fibrant. The weak equivalences of $\mathcal{W}_{DK}$ induce equivalences of fundamental categories by definition.

Proposition 3.2. *Let $f : X \to Y$ be a map of flows. Let $i : A \to B \in I^{gl} \cup \{C\}$. Consider a commutative square of* $\mathbf{Cat}$

$$\begin{tikzcd}
\mathrm{I}^+(A) \arrow[r, "\phi"] \arrow[d, "\mathrm{I}^+(i)"'] & \mathrm{I}^+(X) \arrow[d, "\mathrm{I}^+(f)"] \\
\mathrm{I}^+(B) \arrow[r, "\overline{\phi}"] & \mathrm{I}^+(Y)
\end{tikzcd}$$

Then either $\mathrm{I}^+(B) \sqcup_{\mathrm{I}^+(A)} \mathrm{I}^+(X) \cong \mathrm{I}^+(X)$, or the canonical map $\mathrm{I}^+(B) \sqcup_{\mathrm{I}^+(A)} \mathrm{I}^+(X) \to \mathrm{I}^+(Y)$ is of the form $\mathrm{I}^+(g)$ for some unique map of flows $g : B \sqcup_A X \to Y$.

Proof. That there is at most one such a map g is a consequence of the fact that I^+ is faithful. A commutative diagram of enriched small categories of the form

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \mathrm{I}^+(X) \arrow[d, "\mathrm{I}^+(f)"] \\
\mathrm{I}^+(\{0\}) \arrow[r] & \mathrm{I}^+(Y)
\end{tikzcd}$$

is the image by the functor $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ of the commutative diagram of flows

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\{0\} \arrow[r] & Y
\end{tikzcd}$$

Thus, in this case, g exists by the universal property of the pushout. Consider now a commutative diagram (C) of enriched small categories of the form

$$\begin{tikzcd}
\mathrm{I}^+(\mathrm{Glob}(\mathbf{S}^{n-1})) \arrow[r, "\phi"] \arrow[d] & \mathrm{I}^+(X) \arrow[d, "\mathrm{I}^+(f)"] \\
\mathrm{I}^+(\mathrm{Glob}(\mathbf{D}^n)) \arrow[r, "\overline{\phi}"] & \mathrm{I}^+(Y)
\end{tikzcd}$$

with $n \geqslant 0$. If $\phi(0) \neq \phi(1)$ are two different objects of $\mathrm{I}^+(X)$, then the latter commutative diagram of enriched small categories is the image by the functor $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ of the commutative diagram (D) of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Glob}(\mathbf{D}^n) \arrow[r] & Y
\end{tikzcd}$$

We conclude the existence of g as above. It remains the case $\phi(0) = \phi(1)$. In this case, we have the commutative diagram of topological spaces

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "\phi"] \arrow[d] & \{\mathrm{Id}_{\phi(0)}\} \sqcup \mathbb{P}_{\phi(0),\phi(1)}X \arrow[d, "f"] \\ \mathbf{D}^n \arrow[r, "\overline{\phi}"] & \{\mathrm{Id}_{\phi(0)}\} \sqcup \mathbb{P}_{\phi(0),\phi(1)}Y \end{tikzcd}$$

If $n \geqslant 1$ and since $\mathbf{D}^n$ is connected, then either $\overline{\phi}(\mathbf{D}^n) \subset \{\mathrm{Id}_{\phi(0)}\}$ and $\phi(\mathbf{S}^{n-1}) \subset \{\mathrm{Id}_{\phi(0)}\}$ or $\overline{\phi}(\mathbf{D}^n) \subset \mathbb{P}_{\phi(0),\phi(1)}Y$ and $\phi(\mathbf{S}^{n-1}) \subset \mathbb{P}_{\phi(0),\phi(1)}X$. If $n = 0$, then $\mathbf{S}^{n-1} = \varnothing$ and either $\overline{\phi}(\mathbf{D}^n) \subset \{\mathrm{Id}_{\phi(0)}\}$ or $\overline{\phi}(\mathbf{D}^n) \subset \mathbb{P}_{\phi(0),\phi(1)}Y$.

In the first alternative in both cases, there is the pushout diagram of enriched small categories

$$\begin{tikzcd} \mathrm{I}^+(\mathrm{Glob}(\mathbf{S}^{n-1})) \arrow[r, "\phi"] \arrow[d] & \mathrm{I}^+(X) \arrow[d, "\mathrm{I}^+(f)"] \\ \mathrm{I}^+(\mathrm{Glob}(\mathbf{D}^n)) \arrow[r] & \mathrm{I}^+(X). \end{tikzcd}$$

In the second alternative in both cases, the commutative diagram (C) is the image by the functor $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ of the commutative diagram (D) and we conclude the existence of g as above. $\square$

By [7, Theorem 2.6], the left-lifting of the small weak factorization system $(\mathcal{C}_{DK}, \mathcal{F}_{DK} \cap \mathcal{W}_{DK})$ along the left adjoint $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ exists and is accessible. In fact, we have the proposition:

Proposition 3.3. *The left-lifting of the small weak factorization system* $(\mathcal{C}_{DK}, \mathcal{F}_{DK} \cap \mathcal{W}_{DK})$ *along the left adjoint* $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ *is small, being generated by* $I^{gl} \cup \{C\}$.

Proof. It suffices to prove that ${\mathrm{I}^+}^{-1}(\mathcal{C}_{DK}) = \mathbf{cof}(I^{gl} \cup \{C\})$. We have $\mathrm{I}^+(I^{gl} \cup \{C\}) \subset \mathcal{C}_{DK}$ by Theorem 3.1. Since $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$ is a left adjoint, we obtain the inclusion $\mathbf{cell}(I^{gl} \cup \{C\}) \subset {\mathrm{I}^+}^{-1}(\mathcal{C}_{DK})$. And using the fact that every map of $\mathbf{cof}(I^{gl} \cup \{C\})$ is a retract of a map of $\mathbf{cell}(I^{gl} \cup \{C\})$, we obtain the inclusion $\mathbf{cof}(I^{gl} \cup \{C\}) \subset {\mathrm{I}^+}^{-1}(\mathcal{C}_{DK})$ since the class of maps $\mathcal{C}_{DK}$ is closed under retract. Conversely, let $f : X \to Y$ be a map of flows such that $\mathrm{I}^+(f) : \mathrm{I}^+(X) \to \mathrm{I}^+(Y)$ is a cofibration of $\mathbf{Cat}$. By using the small object argument of [24, Theorem 2.1.14], we factor $\mathrm{I}^+(f)$ as a composite $\mathrm{I}^+(X) \to Z \to \mathrm{I}^+(T)$ such that the map $\mathrm{I}^+(X) \to Z$ belongs to $\mathbf{cell}(\mathrm{I}^+(I^{gl} \cup \{C\}))$ and such that the map $Z \to \mathrm{I}^+(T)$ belongs to $\mathbf{inj}(\mathrm{I}^+(I^{gl} \cup \{C\}))$. Since I^+ is a left adjoint, by an immediate transfinite induction, there exists a transfinite tower $(X_\alpha)_{\alpha<\lambda}$ of $\mathbf{Flow}$ with $X = X_0$ and $Z = \mathrm{I}^+(X_\lambda)$ such that each map $X_\alpha \to X_{\alpha+1}$ for $\alpha < \lambda$ is a pushout of a map of $I^{gl} \cup \{C\}$. By induction on $\alpha \geqslant 0$, let us prove that the map of enriched small categories $\mathrm{I}^+(X_\alpha) \to \mathrm{I}^+(T)$ is the image by the functor I^+ of a map of flows $g_\alpha : X_\alpha \to T$. There is nothing to prove for $\alpha = 0$. The passage from α to $\alpha + 1$ is ensured by Proposition 3.2. Finally, the statement holds for a limit ordinal α since I^+ is colimit-preserving. We deduce that the map of enriched small categories $Z \to \mathrm{I}^+(T)$ is of the form $\mathrm{I}^+(g)$ for some map of flows $g : X_\lambda \to T$: take $g = g_\lambda$. The lift ℓ in the commutative diagram of enriched small categories

$$\begin{tikzcd}
\mathrm{I}^+(X) \arrow[r] \arrow[d, "\mathrm{I}^+(f)"'] & \mathrm{I}^+(X_\lambda) \arrow[d, "\mathrm{I}^+(g)"] \\
\mathrm{I}^+(T) \arrow[r, equal] \arrow[ur, "\ell"] & \mathrm{I}^+(T)
\end{tikzcd}$$

exists since $\mathrm{I}^+(f)$ is a cofibration of $\mathbf{Cat}$ by hypothesis. For all $\alpha \in T^0$, the commutativity of the diagram of spaces

$$\begin{tikzcd}
\{\mathrm{Id}_\alpha\} \sqcup \mathbb{P}_{\alpha,\alpha}T \arrow[r, "\ell"] \arrow[rr, bend left, equal] & \{\mathrm{Id}_{\ell(\alpha)}\} \sqcup \mathbb{P}_{\ell(\alpha),\ell(\alpha)}X_\lambda \arrow[r, "\mathrm{I}^+(g)"] & \{\mathrm{Id}_\alpha\} \sqcup \mathbb{P}_{\alpha,\alpha}T
\end{tikzcd}$$

implies that $\ell(\mathbb{P}_{\alpha,\alpha}T) \subset \mathbb{P}_{\ell(\alpha),\ell(\alpha)}X_\lambda$, and therefore that $\ell = \mathrm{I}^+(\overline{\ell})$ for some map of flows $\overline{\ell} : T \to X_\lambda$. Since the functor I^+ is faithful, we obtain the commutative diagram of flows

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & X_\lambda \arrow[d, "g"] \\
T \arrow[r, equal] \arrow[ur, "\overline{\ell}"] & T
\end{tikzcd}$$

It means that the map of flows $f : X \to T$ is a retract of the map of flows $X \to X_\lambda$. We deduce $f \in \mathbf{cof}(I^{gl} \cup \{C\})$, the map $X \to X_\lambda$ belonging to $\mathbf{cell}(I^{gl} \cup \{C\})$ by construction. We deduce the inclusion $\mathrm{I}^{+^{-1}}(\mathcal{C}_{DK}) \subset \mathbf{cof}(I^{gl} \cup \{C\})$. $\square$

Lemma 3.4. *Let $f : X \to Y$ be a map of $\mathbf{Flow}$ such that Y is a set. Then X is a set as well.*

Proof. It is a consequence of the lack of identity maps for the objects of $\mathbf{Flow}$. $\square$

Theorem 3.5. *The model category $(\mathbf{Cat})_{DK}$ cannot be left-lifted along the left adjoint $\mathrm{I}^+ : \mathbf{Flow} \to \mathbf{Cat}$.*

Proof. By Proposition 3.3 and Lemma 3.4, the map $R : \{0,1\} \to \{0\}$ satisfies the RLP with respect to $\mathrm{I}^{+^{-1}}(\mathcal{C}_{DK})$ because it satisfies the RLP with respect to $C : \varnothing \to \{0\}$. But $\mathrm{I}^+(R) \notin \mathcal{W}_{DK}$. It means that the left acyclicity condition $\mathrm{I}^{+^{-1}}(\mathcal{C}_{DK})^{\boxslash} \subset \mathrm{I}^{+^{-1}}(\mathcal{W}_{DK})$ fails and that the left-induced model structure does not exist by [7, Proposition 2.3]. $\square$

Theorem 3.5 can be proved without using Proposition 3.3. Indeed, thanks to Lemma 3.4, the only maps of flows f belonging to $\mathrm{I}^{+^{-1}}(\mathcal{C}_{DK})$ such that there exists a morphism in the category of maps of flows from f to R are the set maps of $\mathbf{cell}(C) = \mathbf{cof}(C)$, i.e. the one-to-one set maps. Proposition 3.3 is proved because it is used in Corollary 4.10.

4. Left-lifting the cofibrations of the Ilias model structure

We need to recall:

Theorem 4.1. *[17, Theorem 7.6] There exists one and only one combinatorial model structure $(\mathbf{Flow})_q$ on $\mathbf{Flow}$ with the following properties:*

- *A set of generating cofibrations is the set of maps $I^{gl} \cup \{C, R\}$.*
- *The weak equivalences are the maps of flows $f : X \to Y$ inducing a bijection $f^0 : X^0 \cong Y^0$ and a weak homotopy equivalence $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$.*

- *A set of generating trivial cofibrations is given by the set of maps J^{gl}.*

It is called the q-model structure. *The cofibrations (fibrations resp.) are called q-cofibrations (q-fibrations resp.). All flows are q-fibrant.*

The weak equivalences of $(\mathbf{Flow})_q$ induce isomorphisms of fundamental semicategories. The q-model structure of flows is minimal by [21, Theorem 1.4] since it is combinatorial and all its objects are fibrant [2].

Definition 4.2. *The class of maps of flows $\overline{\mathcal{W}}_{DK}$ consists of the maps of flows $f : X \to Y$ such that either $X = Y = \varnothing$, or X and Y are both nonempty sets, or X and Y both contain at least one execution path.*

As an immediate consequence of the definition above, we obtain:

Proposition 4.3. *All maps of flows*

$$\begin{aligned} &I^{gl}_{\geqslant 1} = \{c_n \mid n \geqslant 1\}, C^+ : \{0\} \subset \{0,1\},\\ &c_0^+ = \mathrm{Id}_{\overrightarrow{I}} \sqcup c_0 : \overrightarrow{I} \sqcup \mathrm{Glob}(\mathbf{S}^{-1}) \subset \overrightarrow{I} \sqcup \mathrm{Glob}(\mathbf{D}^0)\end{aligned}$$

belong to $\overline{\mathcal{W}}_{DK}$.

We recall the four following propositions for the convenience of the reader.

Proposition 4.4. *[8, Proposition 13.2] Let $i : U \to V$ be a map of $\mathbf{Top}$. A morphism of flows $f : X \to Y$ satisfies the RLP with respect to $\mathrm{Glob}(i) : \mathrm{Glob}(U) \to \mathrm{Glob}(V)$ if and only if for all $(\alpha, \beta) \in X^0 \times X^0$, the map $\mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to i.*

Proposition 4.5. *([24, Theorem 2.1.19]) Let I and J be two sets of maps of a locally presentable category $\mathcal{K}$. Let $\mathcal{W}$ be a class of maps satisfying the two-out-of-three property and which is closed under retract. If $\mathbf{cell}(J) \subset \mathcal{W} \cap \mathbf{cof}(I)$, $\mathbf{inj}(I) \subset \mathcal{W} \cap \mathbf{inj}(J)$ and $\mathcal{W} \cap \mathbf{cof}(I) \subset \mathbf{cof}(J)$, then $(\mathbf{cof}(I), \mathbf{inj}(J), \mathcal{W})$ is a model structure on $\mathcal{K}$.*

[2] [14, Theorem 4.3] gives another argument which does not require to use a locally presentable setting.

Proposition 4.6. *([24, Lemma 5.2.6]) Let $\mathcal{M}$ be a model category. Consider a pushout diagram of $\mathcal{M}$ of the form*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\simeq"'] & Y \arrow[d] \\ Z \arrow[r] & T \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

such that X, Y, Z are cofibrant, such that the top horizontal map is a cofibration and such that the left vertical map is a weak equivalence. Then the right vertical map $Y \to T$ is a weak equivalence.

Proposition 4.7. *[14, Proposition 3.7] The globe functor* $\mathrm{Glob} : \mathbf{Top} \to \mathbf{Flow}$ *preserves connected colimits (i.e. colimits such that the underlying small category is connected).*

Notation 4.8. Let $\underline{\mathbf{3}}$ be the small category associated with the poset $\{0 \leqslant 1 \leqslant 2\}$.

Theorem 4.9. *There exists one and only one model structure on* $\mathbf{Flow}$ *such that*

- *A set of generating cofibrations is $I^{gl} \cup \{C\}$.*
- *A set of generating trivial cofibrations is $\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1}$.*
- *The class of weak equivalences is $\overline{\mathcal{W}}_{DK}$.*
- *The homotopy category of this model structure is the category $\underline{\mathbf{3}}$: every flow is weakly equivalent either to the initial or terminal flow, or to a singleton.*
- *The cofibrant flows are the q-cofibrant flows.*
- *The fibrant flows are the flows X such that $\mathbb{P}X = \varnothing$ (i.e. the sets) and the flows X such that for all $(\alpha, \beta) \in X^0 \times X^0$, the space $\mathbb{P}_{\alpha,\beta}X$ is contractible. In particular, not all flows are fibrant.*

Moreover, this combinatorial model structure is minimal.

Proof. The uniqueness comes from the fact that a model structure is characterized by its cofibrations and its trivial cofibrations. Note that $\underline{\mathbf{3}}$ is the full subcategory of $\mathbf{Flow}$ generated by the initial and terminal flows and by the singleton. Consider the functor $\underline{w} : \mathbf{Flow} \to \underline{\mathbf{3}}$ characterized as the unique functor which takes a flow X to 0 if $X^0 = \varnothing$, to 1 if $X^0 \neq \varnothing$ and $\mathbb{P}X = \varnothing$, and to 2 otherwise. Then $\overline{\mathcal{W}}_{DK}$ is the inverse image by $\underline{w}$ of the identity maps of $\underline{\mathbf{3}}$. We deduce that the class $\overline{\mathcal{W}}_{DK}$ has the two-out-of-three property and that it is closed under retract.

All maps of $\mathbf{cell}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1})$ are q-cofibrations which are one-to-one on states, which implies $\mathbf{cell}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1}) \subset \mathbf{cof}(I^{gl} \cup \{C\})$. Every map of $\mathbf{cell}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1})$ is either between nonempty sets or between flows containing execution paths, hence the inclusion $\mathbf{cell}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1}) \subset \overline{\mathcal{W}}_{DK}$.

We obtain the inclusion $\mathbf{cell}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1}) \subset \overline{\mathcal{W}}_{DK} \cap \mathbf{cof}(I^{gl} \cup \{C\})$. An element $f : X \to Y$ of $\mathbf{inj}(I^{gl} \cup \{C\})$ is surjective on states. Therefore $X^0 = \varnothing$ if and only if $Y^0 = \varnothing$ and $f \in \mathbf{inj}(C^+)$. By Proposition 4.4, every map $\mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ for all $(\alpha, \beta) \in X^0 \times X^0$ is a trivial q-fibration of spaces. Consequently, X contains execution paths if and only if Y contains execution paths. We deduce that $f \in \overline{\mathcal{W}}_{DK}$. By Proposition 4.4 again, we deduce that $f \in \mathbf{inj}(J^{gl} \cup I^{gl})$. We obtain the inclusions $\mathbf{inj}(I^{gl} \cup \{C\}) \subset \overline{\mathcal{W}}_{DK} \cap \mathbf{inj}(\{C^+\} \cup J^{gl} \cup I^{gl}) \subset \overline{\mathcal{W}}_{DK} \cap \mathbf{inj}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1})$.

Finally, a map $f \in \overline{\mathcal{W}}_{DK} \cap \mathbf{cof}(I^{gl} \cup \{C\})$ is a q-cofibration which is one-to-one on states such that either the source and the target are empty, or the source and the target are nonempty set (in this case, f belongs to $\mathbf{cof}(\{C^+\})$), or such that both the source and the target contain execution paths. In the latter case, it belongs to $\mathbf{cof}(\{c_0^+\} \cup I^{gl}_{\geqslant 1})$. We deduce that $\overline{\mathcal{W}}_{DK} \cap \mathbf{cof}(I^{gl} \cup \{C\}) \subset \mathbf{cof}(\{C^+, c_0^+\} \cup J^{gl} \cup I^{gl}_{\geqslant 1})$.

The proof of the existence of the model structure is complete thanks to Proposition 4.5.

Since all flows are q-fibrant, a flow X is fibrant if and only if the canonical map $X \to \mathbf{1}$ satisfies the RLP with respect to $\{C^+, c_0^+\} \cup I^{gl}_{\geqslant 1}$. Since $\mathbf{inj}(C^+) \cap \mathbf{Set}$ is equal to the surjective set maps union the set maps starting from the empty set by [9, Lemme 4.4(3)], the canonical map $X \to \mathbf{1}$ always satisfies the RLP with respect to C^+. Thus a flow X is fibrant if and only if the canonical map $X \to \mathbf{1}$ satisfies the RLP with respect to $\{c_0^+\} \cup I^{gl}_{\geqslant 1}$.

We deduce that all sets viewed as flows are fibrant. Consider now a flow X such that $\mathbb{P}X \neq \varnothing$. Then the map $X \to \mathbf{1}$ satisfies the RLP with respect to c_0^+ if and only it satisfies the RLP with respect to c_0. The characterization of fibrant objects is complete thanks to Proposition 4.4.

Since not all flows are fibrant for this model structure, an additional argument is required to prove that it is indeed minimal.

Consider a model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on $\mathbf{Flow}$ such that $\mathcal{C} = \mathbf{cof}(I^{gl} \cup \{C\})$. The cofibrant flows are the q-cofibrant flows and the cofibrations are the q-cofibrations which are one-to-one on states. All trivial q-fibrations are trivial fibrations since they satisfy the RLP with respect to $I^{gl} \cup \{C\} \subset I^{gl} \cup \{C, R\}$.

Observe at first that $R : \{0,1\} \to \{0\}$ is a trivial fibration. We have $R.C^+ = \mathrm{Id}_{\{0\}}$. By the two-out-of-three property, we deduce that $C^+ : \{0\} \subset \{0,1\}$ is a weak equivalence. It means that two nonempty sets viewed as flows are always weakly equivalent.

We are going to prove by induction on $n \geqslant 1$ that the map

$$c_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n)$$

is a trivial cofibration. From the pushout diagram (see Example 2.3)

$$\begin{tikzcd}
\{1\} \sqcup \{1\} \arrow[r] \arrow[d] & B \arrow[d] \\
\{1\} \arrow[r] & \mathrm{Glob}(\mathbf{S}^0) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and Proposition 4.6, we deduce that the map $B \to \mathrm{Glob}(\mathbf{S}^0)$ is a weak equivalence. From the fact that the composite map $B \to \mathrm{Glob}(\mathbf{S}^0) \to \overrightarrow{I}$ is a trivial fibration and the two-out-of-three property, we deduce that the unique map of flows $\mathrm{Glob}(\mathbf{S}^0) \to \overrightarrow{I}$ is a weak equivalence. Consider the commutative diagram of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^0) \arrow[r, equal] \arrow[d, "c_1"'] & \mathrm{Glob}(\mathbf{S}^0) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^1) \arrow[r] & \overrightarrow{I} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The bottom horizontal map $\mathrm{Glob}(\mathbf{D}^1) \to \overrightarrow{I}$ is a weak equivalence, being a trivial q-fibration. By the two-out-of-three property, we deduce that $c_1 : \mathrm{Glob}(\mathbf{S}^0) \to \mathrm{Glob}(\mathbf{D}^1)$ is a weak equivalence as well, and therefore a trivial cofibration since it is a q-cofibration which is one-to-one on states. The induction hypothesis is therefore proved for $n = 1$. Suppose that the induction hypothesis is proved for $n \geqslant 1$. Using Proposition 4.7 and the pushout diagram of spaces

$$\begin{tikzcd}
\mathbf{S}^{n-1} \arrow[r] \arrow[d] & \mathbf{D}^n \arrow[d] \\
\mathbf{D}^n \arrow[r] & \mathbf{S}^n
\end{tikzcd}$$

we obtain the commutative diagram of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "c_n"] \arrow[d, "c_n"'] & \mathrm{Glob}(\mathbf{D}^n) \arrow[rr] \arrow[d] & & \overrightarrow{I} \arrow[d, equal] \\
\mathrm{Glob}(\mathbf{D}^n) \arrow[r] & \mathrm{Glob}(\mathbf{S}^n) \arrow[r, "c_{n+1}"] & \mathrm{Glob}(\mathbf{D}^{n+1}) \arrow[r] & \overrightarrow{I}
\end{tikzcd}$$

Using the induction hypothesis, we deduce that the map $\mathrm{Glob}(\mathbf{D}^n) \to \mathrm{Glob}(\mathbf{S}^n)$ is a trivial cofibration, being a pushout of the trivial cofibration c_n. All maps $\mathrm{Glob}(\mathbf{D}^N) \to \overrightarrow{I}$ for $N \geqslant 0$ are trivial q-fibrations, and hence trivial fibrations. Using the two-out-of-three property, we obtain the induction hypothesis for $n + 1$. We have proved that all maps of $\mathbf{cell}(I^{gl}_{\geqslant 1})$ are trivial cofibrations.

Now we can conclude the proof as follows. Let X be a flow containing at least one execution path and let X^{cof} be a q-cofibrant replacement of X. Consider the flow $\mathrm{Mon}(X^{cof})$ defined by the pushout diagram of flows

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X^{cof} \arrow[d] \\
\{0\} \arrow[r] & \mathrm{Mon}(X^{cof}).
\end{tikzcd}$$

By Proposition 4.6, the canonical map $X^{cof} \to \mathrm{Mon}(X^{cof})$ is a weak equivalence. Consequently, we can suppose without loss of generality that

$X^0 = \{0\}$ and that X is a cellular object of the q-model structure of flows. Write the canonical map $\varnothing \to X$ as a composite $\varnothing \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X$ such that the map $X^0 \to X^1$ belongs to $\mathbf{cell}(\{c_0\})$ and such that $X^1 \to X$ belongs to $\mathbf{cell}(I^{gl}_{\geqslant 1})$. In particular, the map $X^1 \to X$ is a trivial cofibration by the first part of the proof. Factor the canonical map $X^1 \to \mathbf{1}$ as a composite $X^1 \to X^\infty \to \mathbf{1}$ such that the left-hand map belongs to $\mathbf{cell}(I^{gl}_{\geqslant 1})$ and such that the right-hand map belongs to $\mathbf{inj}(I^{gl}_{\geqslant 1})$. It means that X is weakly equivalent to X^∞. Since the map $X^\infty \to \mathbf{1}$ is bijective on states, it is injective with respect to $C : \varnothing \to \{0\}$. Since, moreover, X^∞ contains an execution path, it is also injective with respect to $c_0 : \mathrm{Glob}(\mathbf{S}^{-1}) \subset \mathrm{Glob}(\mathbf{D}^0)$. Thus, the map $X^\infty \to \mathbf{1}$ is a weak equivalence, being a trivial fibration. We deduce that every flow in $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is weakly equivalent to $\varnothing$, $\{0\}$ or $\mathbf{1}$. Since the full subcategory of $\mathbf{Flow}$ generated by the three objects $\varnothing$, $\{0\}$ and $\mathbf{1}$ is $\underline{\mathbf{3}}$, the homotopy category of $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is then a categorical localization of $\underline{\mathbf{3}}$. We deduce the inclusion $\overline{\mathcal{W}}_{DK} \subset \mathcal{W}$. The set of generating cofibrations $I^{gl} \cup \{C\}$ is tractable. Therefore, by [21, Theorem 1.4], there exists a minimal model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ with respect to the set of generating cofibrations $I^{gl} \cup \{C\}$. In this case, there is also the inclusion $\mathcal{W} \subset \overline{\mathcal{W}}_{DK}$ and the proof is complete since a model structure is characterized by its classes of cofibrations and weak equivalences. □

Corollary 4.10. *The minimal model structure on flows with respect to the left-lifting of the cofibrations of the Ilias model structure has three homotopy types.*

Proof. It is a consequence of Proposition 3.3 and Theorem 4.9. □

5. Right-lifting the q-model structure of flows

We want to prove that the q-model structure of flows can be transferred along the right adjoint $\mathbf{Cat} \subset \mathbf{Flow}$. At first, we recall:

Theorem 5.1. *(Kan-Quillen, see [28, proof of Theorem 1 of Section II.4] and [23, Theorem 11.3.2] or for an abstract presentation [22, Theorem 2.2.1]) Let $\mathcal{M}$ and $\mathcal{N}$ be two locally presentable categories. Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a combinatorial model structure of $\mathcal{M}$ such that all objects are fibrant. Consider a categorical adjunction $L : \mathcal{M} \dashv \mathcal{N} : U$. Suppose that there exists a* *factorization of the diagonal of $\mathcal{N}$ as a composite $X \xrightarrow{\tau} \mathrm{Path}(X) \xrightarrow{\pi} X \times X$ such that $U(\tau)$ is a weak equivalence of $\mathcal{M}$ and such that $U(\pi)$ is a fibration of $\mathcal{M}$ for all objects X of $\mathcal{N}$. Then there exists a unique combinatorial model structure on $\mathcal{N}$ such that the class of fibrations is $U^{-1}(\mathcal{F})$ and such that the class of weak equivalences is $U^{-1}(\mathcal{W})$. If the set of generating (trivial resp.) cofibrations of $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is I (J resp.), then the set of generating (trivial resp.) cofibrations of the model structure of $\mathcal{N}$ is $L(I)$ ($L(J)$ resp.).*

In the terminology of this note, Theorem 5.2 means that the q-model structure of flows has a right-lifting to the category of small topologically enriched categories which is minimal.

Theorem 5.2. *There exists a unique model structure $(\mathbf{Cat})_q = (\mathcal{C}_q, \mathcal{F}_q, \mathcal{W}_q)$ on $\mathbf{Cat}$ such that:*

- *The set of generating cofibrations is $\{\mathrm{I}^+(\mathrm{Glob}(\mathbf{S}^{n-1})) \subset \mathrm{I}^+(\mathrm{Glob}(\mathbf{D}^n)) \mid n \geqslant 0\} \cup \{\mathrm{I}^+(C), \mathrm{I}^+(R)\}$.*
- *The set of generating trivial cofibrations is $\{\mathrm{I}^+(\mathrm{Glob}(\mathbf{D}^n \times \{0\})) \subset \mathrm{I}^+(\mathrm{Glob}(\mathbf{D}^n \times [0,1])) \mid n \geqslant 0\}$.*
- *A map of small enriched categories $f : X \to Y$ is a weak equivalence if and only if $\mathrm{Obj}(f) : \mathrm{Obj}(X) \to \mathrm{Obj}(Y)$ is a bijection and for all $(\alpha, \beta) \in \mathrm{Obj}(X) \times \mathrm{Obj}(X)$, the continuous map $X(\alpha, \beta) \to X(f(\alpha), f(\beta))$ is a weak homotopy equivalence.*
- *A map of small enriched categories $f : X \to Y$ is a fibration if and only if for all $(\alpha, \beta) \in \mathrm{Obj}(X) \times \mathrm{Obj}(X)$, the continuous map $X(\alpha, \beta) \to X(f(\alpha), f(\beta))$ is a q-fibration of spaces.*

Moreover, this model structure is minimal and all objects are fibrant. The left Quillen adjoint $\mathrm{I}^+ : (\mathbf{Flow})_q \to (\mathbf{Cat})_q$ is not a left Quillen equivalence.

Proof. Consider the right adjoint $\mathbf{Cat} \subset \mathbf{Flow}$. Let X be a small enriched category. Let $\mathrm{Path}(X)$ be the small enriched category having the same objects as X and such that the space of morphisms $\mathrm{Path}(X)(\alpha, \beta)$ is equal to the topological space $\mathbf{TOP}([0,1], X(\alpha, \beta))$ with the continuous composition law defined for any triple (α, β, γ) of objects of X as the composite:

$$\begin{aligned}\mathbf{TOP}([0,1], X(\alpha,\beta)) \times \mathbf{TOP}([0,1], X(\beta,\gamma)) \\ \cong \mathbf{TOP}([0,1], X(\alpha,\beta) \times X(\beta,\gamma)) \\ \longrightarrow \mathbf{TOP}([0,1], X(\alpha,\gamma)).\end{aligned}$$

The composition law is clearly associative. The identity of $\mathrm{Path}(X)(\alpha,\alpha)$ (the space of morphisms in $\mathrm{Path}(X)$ from α to itself) is the constant map $\mathrm{Id}_\alpha : [0,1] \to X(\alpha,\alpha)$. For all small enriched categories X, for all $(\alpha,\beta) \in \mathrm{Obj}(X) \times \mathrm{Obj}(X)$, the map $X(\alpha,\beta) \cong \mathbf{TOP}(\{0\}, X(\alpha,\beta)) \to \mathbf{TOP}([0,1], X(\alpha,\beta)) = \mathrm{Path}(X)(\alpha,\beta)$ is a trivial q-fibration of spaces and the map

$$\begin{aligned}\mathrm{Path}(X)(\alpha,\beta) = \mathbf{TOP}([0,1], X(\alpha,\beta)) \to \mathbf{TOP}(\{0,1\}, X(\alpha,\beta)) \\ \cong X(\alpha,\beta) \times X(\alpha,\beta)\end{aligned}$$

is a q-fibration of spaces. Using Theorem 5.1, the q-model structure of $\mathbf{Flow}$ right induces a combinatorial model structure on $\mathbf{Cat}$. The model structure is minimal because it is combinatorial and all its objects are fibrant.

Let X be an enriched small category. In $\mathbf{Flow}$, the map $X^{cof} \to X$ is a trivial q-fibration of flows. It means that for all $\alpha \in \mathrm{Obj}(X)$, $\mathbb{P}_{\alpha,\alpha}X^{cof} \to \mathbb{P}_{\alpha,\alpha}X$ is a trivial q-fibration of spaces. Therefore the map $\mathrm{I}^+(X^{cof})(\alpha,\alpha) = \{\mathrm{Id}_\alpha\} \sqcup \mathbb{P}_{\alpha,\alpha}X^{cof} \to X(\alpha,\alpha) = \mathbb{P}_{\alpha,\alpha}X$ cannot be a weak homotopy equivalence. It implies that the map $\mathrm{I}^+(X^{cof}) \to X$ cannot be a weak equivalence of $(\mathbf{Cat})_q$. We deduce that the left Quillen adjoint $(\mathbf{Flow})_q \to (\mathbf{Cat})_q$ is not homotopically surjective, and therefore that it is not a left Quillen equivalence. □

We have $\mathrm{I}^+(\{0\}) \to (\{0 \cong 1\})^{cof} \in (\mathcal{C}_{DK} \cap \mathcal{W}_{DK}) \backslash (\mathcal{C}_q \cap \mathcal{W}_q)$. Thus, $\mathrm{Id} : (\mathbf{Cat})_{DK} \to (\mathbf{Cat})_q$ cannot be a left Quillen adjoint. We have $R : \{0,1\} \to \{0\} \in \mathcal{C}_q \backslash \mathcal{C}_{DK}$. It implies that $\mathrm{Id} : (\mathbf{Cat})_q \to (\mathbf{Cat})_{DK}$ cannot be a left Quillen adjoint either.

References

[1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*. Cambridge University Press, Cambridge, 1994.

[2] C. Berger and I. Moerdijk. On the homotopy theory of enriched categories. *Q. J. Math.*, 64(3):805–846, 2013.

[3] J. E. Bergner. A model category structure on the category of simplicial categories. *Trans. Am. Math. Soc.*, 359(5):2043–2058, dec 2006.

[4] J. Dubut, E. Goubault, and J. Goubault-Larrecq. The directed homotopy hypothesis. In *Computer science logic 2016*, volume 62 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 9, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

[5] W.G. Dwyer and D.M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.

[6] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, and M. Raussen. *Directed algebraic topology and concurrency. With a foreword by Maurice Herlihy and a preface by Samuel Mimram*. SpringerBriefs Appl. Sci. Technol. Springer, 2016.

[7] R. Garner, M. Kędziorek, and E. Riehl. Lifting accessible model structures. *J. Topol.*, 13(1):59–76, 2020.

[8] P. Gaucher. A model category for the homotopy theory of concurrency. *Homology Homotopy Appl.*, 5(1):p.549–599, 2003.

[9] P. Gaucher. Flow does not model flows up to weak dihomotopy. *Appl. Categ. Structures*, 13(5-6):371–388, 2005.

[10] P. Gaucher. T-homotopy and refinement of observation (III) : Invariance of the branching and merging homologies. *New York J. Math.*, 12:319–348, 2006.

[11] P. Gaucher. T-homotopy and refinement of observation (IV): Invariance of the underlying homotopy type. *New York J. Math.*, 12:63–95, 2006.

[12] P. Gaucher. Towards a homotopy theory of process algebra. *Homology Homotopy Appl.*, 10(1):353–388, 2008.

[13] P. Gaucher. Combinatorics of labelling in higher-dimensional automata. *Theor. Comput. Sci.*, 411(11-13):1452–1483, March 2010.

[14] P. Gaucher. Flows revisited: the model category structure and its left determinedness. *Cah. Topol. Géom. Différ. Catég.*, LXI-2:208–226, 2020.

[15] P. Gaucher. Homotopy theory of Moore flows (II). *Extr. Math.*, 36(2):157–239, 2021.

[16] P. Gaucher. Left properness of flows. *Theory Appl. Categ.*, 37(19):562–612, 2021.

[17] P. Gaucher. Six model categories for directed homotopy. *Categ. Gen. Algebr. Struct. Appl.*, 15(1):145–181, 2021.

[18] P. Gaucher. Directed degeneracy maps for precubical sets, 2022. arXiv.

[19] E. Goubault and T. P. Jensen. Homology of higher-dimensional automata. In *CONCUR'92 (Stony Brook, NY, 1992)*, volume 630 of *Lecture Notes in Comput. Sci.*, pages 254–268. Springer, Berlin, 1992.

[20] E. Goubault and S. Mimram. Directed homotopy in non-positively curved spaces. *Log. Methods Comput. Sci.*, 16(3):55, 2020. Id/No 4.

[21] S. Henry. Minimal model structures, 2020. arXiv.

[22] K. Hess, M. Kędziorek, E. Riehl, and B. Shipley. A necessary and sufficient condition for induced model structures. *J. Topol.*, 10(2):324–369, 2017.

[23] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[24] M. Hovey. *Model categories*. American Mathematical Society, Providence, RI, 1999.

[25] A. Ilias. Model structure on the category of small topological categories. *J. Homotopy Relat. Struct.*, 10(1):63–70, June 2013.

[26] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[27] T. Porter. Enriched categories and models for spaces of evolving states. *Theor. Comput. Sci.*, 405(1-2):88–100, 2008.

[28] D. Quillen. *Homotopical algebra*. Springer-Verlag, Berlin, 1967.

[29] J. Rosický. On combinatorial model categories. *Appl. Categ. Structures*, 17(3):303–316, 2009.

[30] J. Rosický and W. Tholen. Left-determined model categories and universal homotopy theories. *Trans. Am. Math. Soc.*, 355(9):3611–3623, May 2003.

[31] K. Ziemiański. On execution spaces of PV-programs. *Theor. Comput. Sci.*, 619:87–98, 2016.

Philippe Gaucher
Université Paris Cité
CNRS
IRIF
F-75013 Paris
France
http://www.irif.fr/~gaucher

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES
VOLUME LXIV-2 (2023)

Cartesian Differential Comonads and New Models of Cartesian Differential Categories

Sacha IKONICOFF and Jean-Simon Pacaud LEMAY

Résumé. Les catégories différentielles cartésiennes (CDC) sont équipées d'un combinateur différentiel qui formalise l'opération de dérivation du calcul différentiel à plusieurs variables, et fournissent aussi la sémantique du lambda-calcul différentiel. Une source importante d'exemple de CDCs provient des catégories coKleisli des comonades structurelles des categories différentielles, ce dernier concept fournissant la sémantique catégorique de la logique linéaire différentielle. Dans cet article, nous généralisons cette construction en introduisant la notion de comonade différentielle cartésienne, qui sont précisemment les comonades dont la catégorie de coKleisli est une CDC, ce qui offre une plus large gamme d'exemples. Nous construisons ainsi de nouveaux exemples de CDC provenant de comonades différentielles cartésiennes faisant intervenir les séries formelles, les algèbres à puissances divisées, et les algèbres de Zinbiel.

Abstract. Cartesian differential categories (CDC) come equipped with a differential combinator that formalizes the derivative from multi-variable differential calculus, and also provide the categorical semantics of the differential λ-calculus. An important source of examples of CDCs are the coKleisli categories of the comonads of differential categories, where the latter concept provides the categorical semantics of differential linear logic. In this paper, we generalize this construction by introducing Cartesian differential comonads, which are precisely the comonads whose coKleisli categories are CDCs, and thus allows for a wider variety of examples of

CDCs. As such, we construct new examples of CDCs from Cartesian differential comonads based on power series, divided power algebras, and Zinbiel algebras.

Keywords. Cartesian Differential Categories, Cartesian Differential Comonads, Power Series, Divided Powers, Zinbiel Algebras.

Mathematics Subject Classification (2010). 13F25, 18B99, 18C20, 18D99.

1. Introduction

Cartesian differential categories (CDC), introduced by Blute, Cockett, and Seely in [4], formalize the theory of multivariable differential calculus by axiomatizing the (total) derivative, and also provide the categorical semantics of the differential λ-calculus, as introduced by Ehrhard and Regnier in [18]. Briefly, a CDC (Def 2.3) is a category with finite products such that each homset is a commutative monoid, which allows for zero maps and sums of maps (Def 2.1), and equipped with a differential combinator D, which for every map $f : A \to B$ produces its derivatives $\mathsf{D}[f] : A \times A \to B$. The differential combinator satisfies seven axioms, known as **[CD.1]** to **[CD.7]**, which formalize the basic identities of the (total) derivative from multi-variable differential calculus such as the chain rule, linearity in the vector argument, symmetry of the partial derivatives, etc. Two main examples of CDCs are the category of Euclidean spaces and real smooth functions between them (Ex 2.7), and the Lawvere Theory of polynomials over a commutative (semi)ring (Ex 2.6). An important class of examples of CDCs, especially for this paper, are the coKleilsi categories of the comonads of differential categories [4, Propostion 3.2.1].

Differential categories, introduced by Blute, Cockett, and Seely in [3], provide the algebraic foundations of differentiation and the categorical semantics of differential linear logic [17]. Briefly, a differential category (Ex 3.12) is a symmetric monoidal category with a comonad $!$, with comonad structure maps $\delta_A : !(A) \to !!(A)$ and $\varepsilon_A : !(A) \to A$, such that for each object A, $!(A)$ is a cocommutative comonoid with comultiplication $\Delta_A : !(A) \to !(A) \otimes !(A)$ and counit $e_A : !(A) \to I$, and equipped with a deriving transformation, which is a natural transformation $\mathsf{d}_A : !(A) \otimes A \to !(A)$. The deriving transformation satisfies five axioms, this time called **[d.1]** to **[d.5]**, which formalize basic identities of differentiation such as the chain rule and the product rule. In the opposite category of a differential category, called a codifferential category, the deriving transformation is a derivation in the classical algebra sense. Examples of differential categories include the opposite category of the category of vector spaces over a field where ! is induced by the free symmetric algebra [3, 6], as well as the opposite category of the category of real vector spaces where ! is instead induced by free $\mathcal{C}^\infty$-rings [15].

In a differential category, a smooth map from A to B is a map of type $!(A) \to B$. In other words, the (infinitely) differentiable maps are precisely the coKleisli maps. The interpretation of coKleisli maps as smooth can be made precise when the differential category has finite (bi)products where one uses the deriving transformation to define a differential combinator on the coKleisli category. Briefly, for a coKleisli map $f : !(A) \to B$ (which is a map of type $A \to B$ in the coKleisli category), its derivative $\mathsf{D}[f] : !(A \times A) \to B$ (which is a map of type $A \times A \to B$ in the coKleisli category) is defined as $[\![f]\!] \circ \mathsf{d}_A \circ (1_{!(A)} \otimes \varepsilon_A) \circ (!(\pi_0) \otimes !(\pi_1)) \circ \Delta_{A \times A}$, where composition $\circ$ is the one of the base category and where π_i are the product projection maps. One then uses the five deriving transformations axioms **[d.1]** to **[d.5]** to prove that D satisfies the seven differential combinator axioms **[CD.1]** to **[CD.7]**. Thus, for a differential category with finite (bi)products, its coKleisli category is a CDC. For the examples where ! is the free symmetric algebra or given by free $\mathcal{C}^\infty$-rings, the resulting coKleisli category can respectively be interpreted as the category of polynomials or real smooth functions over possibly infinite variables (but that will still only depend on a finite number of them), of which the Lawvere theory of polynomials or category of real smooth functions is a sub-CDC.

Let us take another look at the construction of the differential combinator for the coKleisli category. Define the natural transformation $\partial_A : !(A \times A) \to !(A)$ as $\partial_A = \mathsf{d}_A \circ (1_{!(A)} \otimes \varepsilon_A) \circ (!(\pi_0) \otimes !(\pi_1)) \circ \Delta_{A \times A}$. Then the differential combinator is simply defined by precomposing a coKleisli map $f : !(A) \to B$ with ∂, so $\mathsf{D}[f] := f \circ \partial_A$. It is important to stress that this is the composition in the base category and not the composition in the coKleisli category. Thus, the properties of the differential combinator D in the coKleisli category are fully captured by the properties of the natural transformation ∂ in the base category, which in turn are a result of the axioms of the deriving transformation d. However, observe that the type of $\partial_A : !(A \times A) \to !(A)$ does not involve any monoidal structure. In fact, if one starts with a comonad whose coKleisli category is a CDC, it is always possible to construct ∂, and to show that $\mathsf{D}[-] = - \circ \partial$, but it is not always possible to extract a monoidal structure on the base category. Thus, if one's goal is simply to build CDCs from coKleisli categories, then a monoidal structure $\otimes$ or a deriving transformation d, or even a comonoid structure Δ and e, are not always necessary. Therefore, the objective of this paper is to precisely characterize the comonads whose coKleisli categories are CDCs. To this end, in this paper we introduce the novel notion of a Cartesian differential comonad.

Cartesian differential comonads are precisely the comonads whose coKleisli categories are CDCs. Briefly, a Cartesian differential comonad is a comonad ! on a category with finite biproducts equipped with a differential combinator transformation, which is a natural transformation $\partial_A : !(A \times A) \to !(A)$ which satisfies six axioms called **[dc.1]** to **[dc.6]** (Def 3.1). The axioms of a differential combinator transformation are analogues of the axioms of a differential combinator. Thus, the coKleisli category of a Cartesian differential comonad is a CDC where the differential combinator is defined by precomposition with the differential combinator transformation (Thm 3.4). This is proven by reasonably straightforward calculations, but one must be careful when translating back and forth between the base category and the coKleisli category. Conversely, a comonad on a category with finite biproduct whose coKleisli category is a CDC is in fact a Cartesian differential comonad, where the differential combinator transformation is the derivative of the identity map $1_{!(A)} : !(A) \to !(A)$ seen as a coKleisli map $A \to !(A)$ (Prop 3.5). Using this, since we already know that the coKleisli category of a differential category is a CDC, it immediately follows that the comonad of a differential category is a Cartesian differential comonad, where the differential combinator transformation is precisely the one defined above. Therefore, Cartesian differential comonads and differential combinator transformations are indeed generalizations of differential categories and deriving transformations. However, Cartesian differential comonads are a strict generalization since, as mentioned, they can be defined without the need of a monoidal structure. A very simple separating example is the identity comonad on any category with finite biproducts, where the differential combinator transformation is simply the second projection map (Ex 3.15). While this example is trivial, it recaptures the fact that any category with finite biproducts is a CDC and this example clearly works without any extra monoidal structure, and thus is not a differential category example. Therefore, Cartesian differential comonads allow for a wider variety of examples of CDCs. As such, in this paper we present three new interesting examples of Cartesian differential comonads, which are not differential categories, and their induced CDCs. These three examples are respectively based on formal power series, divided power algebras, and Zinbiel algebras. It is worth mentioning that these new examples arise more naturally as coCartesian differential monads (Ex 3.13), the dual notion of Cartesian differential comonads, and thus it is the opposite of the Kleisli category which is a CDC.

The first example (Sec 5) is based on reduced power series. Recall that a formal power series is said to be reduced if it has no constant/degree 0 term. While the composition of arbitrary multivariable formal power series is not always well defined, due to their constant terms, the composition of reduced multivariable power series is always well-defined [7, Sec 4.1], and so we may construct categories of reduced power series. Also, it is well known that power series are always and easily differentiable, similarly to polynomials, and that the derivative of a reduced multivariable power series is again reduced. Motivated by capturing power series differentiation, we show that the free reduced power series algebra monad [20, Sec 1.4.3] is a coCartesian differential monad whose monad structure is based on reduced power series composition and whose differential combinator transformation is induced by standard power series differentiation (Prop 5.1). Furthermore, the Lawvere theory of reduced power series (Ex 5.2) is a sub-CDC of the opposite category of the resulting Kleisli category.

The second new example (Sec 6) is based on divided power algebras. Divided power algebras, defined by Cartan [8], are commutative non-unital associative algebras equipped with additional operations $(-)^{[n]}$ for all strictly positive integer n, satisfying some relations (Def 6.1). In characteristic 0, divided power algebras correspond precisely to commutative non-unital associative algebras. In positive characteristics, however, the two notions diverge. There exist free divided power algebras and we show that the free divided power algebra monad [28, Sec 10, Théorème 1 and 2] is a coCartesian differential monad (Prop 6.2). Free divided power algebras correspond to the algebra of reduced divided power polynomials. Thus the differential combinator transformation of this example captures differentiating divided power polynomials [25]. In particular, the Lawvere theory of reduced divided power polynomials (Ex 6.3) is a sub-CDC of the opposite category of the Kleisli category of the free divided power algebra monad.

The third new example (Sec 7), and perhaps the most exotic example in this paper, is based on Zinbiel algebras. The notion of Zinbiel algebra was introduced by Loday [27] and also further studied by Dokas [16]. A Zinbiel algebra is a vector space A endowed with a non-associative and non-commutative bilinear operation $<$. Using the Zinbiel product, every Zinbiel algebra can be turned into a commutative non-unital associative algebra. The underlying vector space of free Zinbiel algebras is the same as the underlying vector space of the non-unital tensor algebra. Therefore, free Zinbiel algebras are spanned by (non-empty) associative words and equipped with a product $<$ (which is sometimes referred to as the semi-shuffle product). The resulting commutative associative algebra is then precisely the non-unital shuffle algebra over V. We show that the free Zinbiel algebra monad [27, Prop 1.8] is a coCartesian differential monad whose differential combinator transformation (Prop 7.2) corresponds to differentiating non-commutative polynomials with respect to the Zinbiel product. The resulting CDC can be understood as the category of reduced non-commutative polynomials where the composition is defined using the Zinbiel product, which we simply call Zinbiel polynomials. As such, the Lawvere theory of Zinbiel polynomials is a new exotic example of a CDC. It is worth mentioning that the shuffle algebra has been previously studied as an example of another generalization of differential categories in [1], but not from the point of view of Zinbiel algebras.

An important class of maps in a CDC are the D-linear maps (Def 2.4), also often simply called linear maps [4]. A map $f : A \to B$ is D-linear if its derivative $\mathsf{D}[f] : A \times A \to B$ is equal to f evaluated in its second argument, that is, $\mathsf{D}[f] = f \circ \pi_1$ (where π_1 is the projection map of the *second* argument). A D-linear map should be thought of as being of degree 1, and thus does not have any higher-order derivative. Thus, in many examples, D-linearity often coincides with the classical notion of linearity. For example, in the CDC of real smooth functions, a smooth function is D-linear if and only if it is $\mathbb{R}$-linear. For a Cartesian differential comonad, every map of the base category provides a D-linear map in the coKleisli category. However, it is not necessarily the case that the base category is isomorphic to the subcategory of D-linear maps of the coKleisli category. Indeed, a simple example of such a case is the trivial Cartesian differential comonad which maps every object to the zero object and thus every coKleisli map is a zero map. Clearly, if the base category is non-trivial it will not be equivalent to the subcategory of D-linear maps. Instead, it is possible to provide necessary and sufficient conditions for the base category to be isomorphic to the subcategory of D-linear maps of the coKleisli category. It turns out that this is precisely the case when the Cartesian differential comonad comes equipped with a D-linear unit, which is a natural transformation $\eta_A : A \to !(A)$ satisfying two axioms **[du.1]** and **[du.2]** (Def 3.7). If it exists, a D-linear unit is unique and it is equivalent to an isomorphism between the base category and the subcategory of D-linear maps of the coKleisli category (Prop 3.10). In the context of differential categories, specifically in categorical models of differential linear logic, the D-linear unit is precisely the codereliction [3, 6, 17]. The Cartesian differential comonads based on power series, or divided power algebras, or Zinbiel algebras all come equipped with D-linear units.

In [5], Blute, Cockett, and Seely give a characterization of the CDCs which are the coKleisli categories of differential categories. Generalizing their approach, it is also possible to precisely characterize the CDCs which are the coKleisli categories of Cartesian differential comonads (Sec 4). To this end, we must work with abstract coKleisli categories (Def 4.1), which gives a description of coKleisli categories without starting from a comonad. Abstract coKleisli categories are the dual notion of Führmann's thunk-force-categories [22], which instead do the same for Kleisli categories. Every abstract coKleisli category is canonically the coKleisli category of a comonad on a certain subcategory (Lem 4.3), and conversely, the coKleisli category of any comonad is an abstract coKleisli category (Lem 4.11). In this paper, we introduce Cartesian differential abstract coKleisli categories (Def 4.8) which are abstract coKleisli categories that are also CDCs such that the differential combinator and the abstract coKleisli structure are compatible. Every Cartesian differential abstract coKleisli category is canonically the coKleisli category of a Cartesian differential comonad over a certain subcategory of D-linear maps (Prop 4.9), and conversely, the coKleisli category of a Cartesian differential comonad is a Cartesian differential abstract category (Prop 4.15).

In conclusion, Cartesian differential comonads give a minimum general construction to build coKleisli categories which are CDCs. The theory of Cartesian differential comonads also highlights the interaction between the coKleisli structure and the differential combinator. While Cartesian differential comonads recapture some of the notions of differential categories, they are more general. Therefore, Cartesian differential comonads open the door to a variety of new, interesting, and exotic examples of CDCs. New examples will be particularly important and of interest, especially since applications of CDCs keep being developed, especially in the fields of machine learning and automatic differentiation.

Remark: In order to stay within the journal's page limits, the majority of the heavy technical proofs have been removed (as approved by the editors). All proofs in full details and extra commutative diagrams for definitions can be found in an extended version of this paper here [24].

Conventions: In an arbitrary category, we use the classical notation for composition as opposed to diagrammatic order which was used in other papers on differential categories (such as in [4, 26] for example). The composite map $g \circ f : A \to C$ is the map that first does $f : A \to B$ then $g : B \to C$. We denote identity maps as $1_A : A \to A$.

2. Cartesian Differential Categories

In this background section we review CDCs [4].

The underlying structure of a CDC is that of a Cartesian left additive category (CLAC), which in particular allows one to have zero maps and sums of maps, while also allowing for maps which do not preserve said sums or zeros. Maps which do preserve the additive structure are called *additive* maps. Then a CLAC is a left additive category with finite products such that the product structure is compatible with the commutative monoid structure, that is, the projection maps are additive. Note that since we are working with commutative monoids, we do not assume that our CLACs necessarily come equipped with additive inverses, or in other words negatives. For a category with (chosen) finite products we denote the (chosen) terminal object as $\top$, the binary product of objects A and B by $A \times B$ with projection maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ and pairing operation $\langle -, - \rangle$, so that for maps $f : C \to A$ and $g : C \to B$, $\langle f, g \rangle : C \to A \times B$ is the unique map such that $\pi_0 \circ \langle f, g \rangle = f$ and $\pi_1 \circ \langle f, g \rangle = g$. As such, the product of maps $h : A \to B$ and $k : C \to D$ is the map $h \times k : A \times C \to B \times D$ defined as $h \times k = \langle h \circ \pi_0, k \circ \pi_1 \rangle$.

Definition 2.1. *A **left additive category** [4, Def 1.1.1] is a category $\mathbb{X}$ such that each hom-set $\mathbb{X}(A, B)$ is a commutative monoid, with binary addition $+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B)$, $(f, g) \mapsto f + g$ and zero $0 \in \mathbb{X}(A, B)$, and such that pre-composition preserves the additive structure, that is, for any maps $f : A \to B$, $g : A \to B$, and $x : A' \to A$, we have that $(f + g) \circ x = f \circ x + g \circ x$ and $0 \circ x = 0$. A map $f : A \to B$ is said to be **additive** [4, Def 1.1.1] if post-composition by f preserves the additive structure, that is, for any maps $x : A' \to A$ and $y : A' \to A$, we have that $f \circ (x + y) = f \circ x + f \circ y$ and $f \circ 0 = 0$. A **Cartesian left additive category** (CLAC) [26, Def 2.3] is a left additive category $\mathbb{X}$ which has finite products and such that all the projection maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ are additive.*

We note that the definition of a CLAC presented here is not precisely that given in [4, Def 1.2.1], but was shown to be equivalent in [26, Lem 2.4]. Also note that in a CLAC, the unique map to the terminal object $\top$ is the zero map $0 : A \to \top$. Here are now some important maps for CDCs that can be defined in any CLAC:

Definition 2.2. *In a CLAC $\mathbb{X}$, define the **injection maps** $\iota_0 : A \to A \times B$ and $\iota_1 : B \to A \times B$ as $\iota_0 := \langle 1_A, 0 \rangle$ and $\iota_1 := \langle 0, 1_B \rangle$; the **sum map** $\nabla_A : A \times A \to A$ as $\nabla_A := \pi_0 + \pi_1$; the **lifting map** $\ell_A : A \times A \to (A \times A) \times (A \times A)$ as $\ell := \iota_0 \times \iota_1$; and lastly the **interchange map** $c_A : (A \times A) \times (A \times A) \to (A \times A) \times (A \times A)$ as $c_A := \langle \pi_0 \times \pi_0, \pi_1 \times \pi_1 \rangle$.*

It is important to note that while c is natural in the expected sense, the injection maps ι_j, the sum map ∇, and the lifting map ℓ are not natural transformations. Instead, they are natural only with respect to additive maps. In particular, since the injection maps are not natural map for arbitrary maps, it follows that these injection maps do not make the product a coproduct, and therefore not a biproduct. However, the biproduct identities still hold in a CLAC in the sense that $\pi_i \circ \iota_j = 0$ if $i \neq j$ and $\pi_i \circ \iota_i = 1$, and also $\iota_0 \circ \pi_0 + \iota_1 \circ \pi_1 = 1_{A \times B}$. With all this said, it turns out that a category with finite biproducts is precisely a CLAC where every map is additive [23, Ex 2.3.(ii)]. In that case, note the injection maps and the sum map as defined above are precisely the injection maps and codiagonal of the coproduct.

CDCs are CLACs which come equipped with a differential combinator, which in turn is axiomatized by the basic properties of the directional derivative from multivariable differential calculus. There are various equivalent ways of expressing the axioms of a CDC. Here we have chosen the one found in [26, Def 2.6] (using the notation for CLACs introduced above). It is important to notice that in the following definition, unlike in the original paper [4] and other early works on CDCs, we use the convention used in the more recent works where the linear argument of $\mathsf{D}[f]$ is its second argument rather than its first argument.

Definition 2.3. *A* ***Cartesian differential category*** *(CDC) [4, Def 2.1.1] is a CLAC* $\mathbb{X}$ *equipped with a* ***differential combinator*** D*, which is a family of operators* $\mathsf{D} : \mathbb{X}(A, B) \to \mathbb{X}(A \times A, B)$*, which sends a map* $f : A \to B$ *to a map* $\mathsf{D}[f] : A \times A \to B$*, and such that the following seven axioms hold:*
[CD.1] $\mathsf{D}[f + g] = \mathsf{D}[f] + \mathsf{D}[g]$ *and* $\mathsf{D}[0] = 0$
[CD.2] $\mathsf{D}[f] \circ (1_A \times \nabla_A) = \mathsf{D}[f] \circ (1_A \times \pi_0) + \mathsf{D}[f] \circ (1_A \times \pi_1)$ *and* $\mathsf{D}[f] \circ \iota_0 = 0$
[CD.3] $\mathsf{D}[1_A] = \pi_1$, $\mathsf{D}[\pi_0] = \pi_0 \circ \pi_1$, *and* $\mathsf{D}[\pi_1] = \pi_1 \circ \pi_1$

[CD.4] $\mathsf{D}[\langle f, g \rangle] = \langle \mathsf{D}[f], \mathsf{D}[g] \rangle$ **[CD.6]** $\mathsf{D}\left[\mathsf{D}[f]\right] \circ \ell_A = \mathsf{D}[f]$
[CD.5] $\mathsf{D}[g \circ f] = \mathsf{D}[g] \circ \langle f \circ \pi_0, \mathsf{D}[f] \rangle$ **[CD.7]** $\mathsf{D}\left[\mathsf{D}[f]\right] \circ c_A = \mathsf{D}\left[\mathsf{D}[f]\right]$

For a map $f : A \to B$*,* $\mathsf{D}[f] : A \times A \to B$ *is called the derivative of* f*.*

A discussion on the intuition for the differential combinator axioms can be found in [4, Remark 2.1.3]. It is also worth mentioning that there is a sound and complete term logic for CDCs [4, Sec 4]. An important class of maps in a CDC is the class of linear maps. In this paper, however, we borrow the terminology from [23] and will instead call them D-linear maps. This terminology will help distinguish between the classical notion of linearity from commutative algebra and the CDC notion of linearity.

Definition 2.4. *In a CDC $\mathbb{X}$ with differential combinator D, a map f is said to be* D***-linear*** *[4, Def 2.2.1] if $\mathsf{D}[f] = f \circ \pi_1$. Define the subcategory of linear maps $\mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}]$ to be the category whose objects are the same as $\mathbb{X}$ and whose maps are D-linear in $\mathbb{X}$, and let $\mathsf{U} : \mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}] \to \mathbb{X}$ be the obvious forgetful functor.*

By [4, Lem 2.2.2], every D-linear is additive, and therefore it follows that $\mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}]$ has finite biproducts, and is thus also a CLAC (where every map is additive) such that the forgetful functor $\mathsf{U} : \mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}] \to \mathbb{X}$ preserves the Cartesian left additive structure strictly. It is important to note that although additive and linear maps often coincide in many examples of CDC, in an arbitrary CDC, not every additive map is necessarily linear. However it is always possible to linearize a map. For any map $f : A \to B$, define $\mathsf{L}[f] : A \to B$, called the linearization of f [14, Def 3.1], as $\mathsf{L}[f] = \mathsf{D}[f] \circ \iota_1$. Then $\mathsf{L}[f]$ is D-linear, and $f : A \to B$ is D-linear if and only if $f = \mathsf{L}[f]$. For other properties of linear maps, see [4, Cor 2.2.3].

We conclude this section with some examples of well-known CDCs and their D-linear maps. The first three examples are based on the standard notions of differentiating linear functions, polynomials, and smooth functions respectively.

Example 2.5. Any category $\mathbb{X}$ with finite biproduct is a CDC where the differential combinator is defined by precomposing with the second projection map: $\mathsf{D}[f] = f \circ \pi_1$. In this case, every map is D-linear by definition and so $\mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}] = \mathbb{X}$. As a particular example, let $\mathbb{F}$ be a field and let $\mathbb{F}\text{-}\mathsf{VEC}$ be the category of $\mathbb{F}$-vector spaces and $\mathbb{F}$-linear maps between them. Then $\mathbb{F}\text{-}\mathsf{VEC}$ is a CDC where for an $\mathbb{F}$-linear map $f : V \to W$, its derivative $\mathsf{D}[f] : V \times V \to W$ is defined as $\mathsf{D}[f](v, w) = f(w)$.

Example 2.6. Let $\mathbb{F}$ be a field. Define the category $\mathbb{F}\text{-}\mathsf{POLY}$ whose object are $n \in \mathbb{N}$, where a map $P : n \to m$ is a m-tuple of polynomials in n variables, that is, $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$ with $p_i(\vec{x}) \in \mathbb{F}[x_1, \dots, x_n]$. $\mathbb{F}\text{-}\mathsf{POLY}$ is a CDC where the differential combinator is given by the standard differentiation of polynomials, that is, for a map $P : n \to m$, with $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$, its derivative $\mathsf{D}[P] : n \times n \to m$ is defined as the tuple of the sum of the partial derivatives of the polynomials $p_i(\vec{x})$, $\mathsf{D}[P](\vec{x}, \vec{y}) := \left(\sum_{i=1}^{n} \frac{\partial p_j(\vec{x})}{\partial x_i} y_i \right)_{j=1}^{m}$. A map $P : n \to m$ is D-linear if it of the form: $P = \langle \sum_{i=0}^{n} r_{i,m} x_i \rangle_{j=1}^{m}$. In other words, $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle$ is D-linear if and only if each $p_i(\vec{x})$ induces an $\mathbb{F}$-linear map $\mathbb{F}^n \to \mathbb{F}$. As such, $\mathsf{D}\text{-lin}[\mathbb{F}\text{-}\mathsf{POLY}]$ is equivalent to the category $\mathbb{F}\text{-}\mathsf{LIN}$ whose objects are the finite powers $\mathbb{F}^n$ for each $n \in \mathbb{N}$ (including the singleton $\mathbb{F}^0 = \{0\}$) and whose maps are $\mathbb{F}$-linear maps $\mathbb{F}^n \to \mathbb{F}^m$. We note that this example can be generalized to the category of polynomials over an arbitrary commutative (semi)ring.

Example 2.7. Let $\mathbb{R}$ be the set of real numbers. Define SMOOTH as the category whose objects are the Euclidean real vector spaces $\mathbb{R}^n$ and whose maps are the real smooth functions $F : \mathbb{R}^n \to \mathbb{R}^m$ between them. SMOOTH is a CDC, arguably the canonical example, where the differential combinator is defined as the directional derivative of a smooth function. So for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative is the smooth function $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ defined as: $\mathsf{D}[F](\vec{x}, \vec{y}) := \left\langle \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\vec{x}) y_i \right\rangle_{j=1}^{m}$. Note that $\mathbb{R}\text{-}\mathsf{POLY}$ is a sub-CDC of SMOOTH. A smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is D-linear if and only if it is $\mathbb{R}$-linear in the classical sense. Therefore, $\mathsf{D}\text{-lin}[\mathsf{SMOOTH}] = \mathbb{R}\text{-}\mathsf{LIN}$.

Example 2.8. An important source of examples of CDCs, especially for this paper, are those which arise as the coKleisli category of a differential category [3, 5]. We will review this example in Ex 3.12.

There are many other interesting (and sometimes very exotic) examples of CDCs in the literature. See [14, 23] for lists of more examples of CDCs.

3. Cartesian Differential Comonads

In this section, we introduce the main novel concept of study in this paper: Cartesian differential comonads, which are precisely the comonads whose coKleisli category is a CDC. This is a generalization of [4, Prop 3.2.1], which states that the coKleisli category of the comonad of a differential category is a CDC. The generalization comes from the fact that a Cartesian differential comonad can be defined without the need for a monoidal product or cocommutative comonoid structure on the comonad's coalgebras. As such, this allows for a wider variety of examples of CDCs. Briefly, a Cartesian differential comonad is a comonad on a category with finite biproducts, which comes equipped with a differential combinator transformation, which generalizes the notion of a deriving transformation in a differential category [3, 6]. The induced differential combinator is defined by precomposing a coKleisli map with the differential combinator transformation (with respect to composition in the base category). Conversely, a comonad whose coKleisli category is a CDC is a Cartesian differential comonad, where the differential combinator transformation is defined using the coKleisli category's differential combinator. We point out that this statement, regarding comonads whose coKleisli categories are CDCs, is a novel observation and shows us that even if one cannot extract a monoidal product on the base category from the coKleisli category, it is possible to obtain a natural transformation which captures differentiation. Lastly, we will also study the case where the D-linear maps of the coKleisli category correspond to the maps of the base category. The situation arises precisely in the presence of what we call a D-linear unit, which generalizes the notion of a codereliction from differential linear logic [3, 6, 17, 19].

If only to introduce notation, recall that a comonad on a category $\mathbb{X}$ is a triple $(!, \delta, \varepsilon)$ consisting of a functor $! : \mathbb{X} \to \mathbb{X}$, and two natural transformations $\delta_A : !(A) \to !!(A)$, called the comonad comultiplication, and $\varepsilon_A : !(A) \to A$, called the comonad counit, and such that $\delta_{!(A)} \circ \delta_A = !(\delta_A) \circ \delta_A$ and $\varepsilon_{!(A)} \circ \delta_A = 1_{!(A)} = !(\varepsilon_A) \circ \delta_A$.

Definition 3.1. *For a comonad $(!, \delta, \varepsilon)$ on a category $\mathbb{X}$ with finite biproducts, a **differential combinator transformation** on $(!, \delta, \varepsilon)$ is a natural transformation $\partial_A : !(A \times A) \to !(A)$ such that the following equalities hold (where ι_j, ∇, ℓ, and c are defined as in Def 2.2):*

[dc.1] *Zero Rule:* $\partial_A \circ !(\iota_1) = 0$;

[dc.2] *Additive Rule:* $\partial_A \circ !(1_A \times \nabla_A) = \partial_A \circ (!(1_A \times \pi_0) + !(1_A \times \pi_1))$;

[dc.3] *Linear Rule:* $\varepsilon_A \circ \partial_A = \pi_1 \circ \varepsilon_{A \times A}$;

[dc.4] *Chain Rule:* $\delta_A \circ \partial_A = \partial_{!(A)} \circ ! (\langle !(\pi_0), \partial_A \rangle) \circ \delta_{A \times A}$;

[dc.5] *Lift Rule:* $\partial_A \circ \partial_{A \times A} \circ !(\ell_A) = \partial_A$;

[dc.6] *Symmetry Rule:* $\partial_A \circ \partial_{A \times A} \circ !(c_A) = \partial_A \circ \partial_{A \times A}$.

*A **Cartesian differential comonad** on a category $\mathbb{X}$ with finite biproducts is a quadruple $(!, \delta, \varepsilon, \partial)$ consisting of a comonad $(!, \delta, \varepsilon)$ and a differential combinator transformation ∂ on $(!, \delta, \varepsilon)$.*

For commutative diagram versions of the axioms **[dc.1]** to **[dc.6]** see the extended version [24]. As the name suggests, the differential combinator transformations axioms correspond to some of the axioms a differential combinator. The zero rule **[dc.1]** and the additive rule **[dc.2]** correspond to **[CD.2]**, the linear rule **[dc.3]** corresponds to **[CD.3]**, the chain rule **[dc.4]** corresponds to **[CD.5]**, the lift rule corresponds to **[CD.6]**, and lastly the symmetry rule **[dc.6]** corresponds to **[CD.7]**.

Our goal is now to show that the coKleisli category of a Cartesian differential comonad is a CDC. As we will be working with coKleisli categories, we will use the notation found in [5] and use interpretation brackets $[\![-]\!]$ to help distinguish between composition in the base category and coKleisli composition. So for a comonad $(!, \delta, \varepsilon)$ on a category $\mathbb{X}$, let $\mathbb{X}_!$ denote its coKleisli category, which is the category whose objects are the same as $\mathbb{X}$ and where a map $A \to B$ in the coKleisli category is map of type $!(A) \to B$ in the base category, that is, $\mathbb{X}_!(A, B) = \mathbb{X}(!(A), B)$. Composition of coKleisli maps $[\![f]\!] : !(A) \to B$ and $[\![g]\!] : !(B) \to C$ is defined as $[\![g \circ f]\!] = [\![g]\!] \circ ! ([\![f]\!]) \circ \delta_A$. The identity maps in the coKleisli category is given by the comonad counit: $[\![1_A]\!] := \varepsilon_A$. Let $\mathsf{F}_! : \mathbb{X} \to \mathbb{X}_!$ be the standard inclusion functor which is defined on objects as $\mathsf{F}_!(A) = A$ and on maps $f : A \to B$ as follows: $[\![\mathsf{F}_!(f)]\!] = f \circ \varepsilon_A$. A key map in this story is the coKleisli map whose interpretation is the identity map in the base category. So for every object A, define the map $\varphi_A : A \to !(A)$ in the coKleisli category as $[\![\varphi_A]\!] = 1_{!(A)}$. It is a well-known result that if the base category has finite products, then so does the coKleisli category.

Lemma 3.2. *[30, Dual of Proposition 2.2] Let $(!, \delta, \varepsilon)$ be a comonad on a category $\mathbb{X}$ with finite products. Then the coKleisli category $\mathbb{X}_!$ has finite products where the product $\times$ on objects and terminal object are defined as as in $\mathbb{X}$ and the projection maps $[\![\pi_0]\!] : !(A \times B) \to A$ and $[\![\pi_1]\!] : !(A \times B) \to B$ are defined respectively as $[\![\pi_i]\!] = \pi_i \circ \varepsilon_{A \times B}$. Furthermore, $\mathsf{F}_! : \mathbb{X} \to \mathbb{X}_!$ preserves the finite product strictly, that is, $\mathsf{F}_!(A \times B) = A \times B$ and $\mathsf{F}_!(\top) = \top$, and also that $[\![\mathsf{F}_!(\pi_i)]\!] = [\![\pi_i]\!]$, $[\![\mathsf{F}_! (\langle f, g \rangle)]\!] = [\![\langle \mathsf{F}_!(f), \mathsf{F}_!(g) \rangle]\!]$, and $[\![\mathsf{F}_! (f \times g)]\!] = [\![\mathsf{F}_!(f) \times \mathsf{F}_!(g)]\!]$.*

If the base category is also Cartesian left additive, then so is the coKleisli category in a canonical way, that is, where the additive structure is simply that of the base category.

Lemma 3.3. *[4, Prop 1.3.3] Let $(!, \delta, \varepsilon)$ be a comonad on a CLAC $\mathbb{X}$ with finite products. Then the coKleisli category $\mathbb{X}_!$ is a CLAC where the finite product structure is given in Lem 3.2, the sum of coKleisli maps $[\![f]\!] : !(A) \to B$ and $[\![g]\!] : !(A) \to B$ is defined as in $\mathbb{X}$, $[\![f + g]\!] = [\![f]\!] + [\![g]\!]$, and the zero $[\![0]\!] : !(A) \to B$ is the same as in $\mathbb{X}$, $[\![0]\!] = 0$. Furthemore, $\mathsf{F}_! : \mathbb{X} \to \mathbb{X}_!$ preserves the additive structure strictly, that is, $[\![\mathsf{F}_!(0)]\!] = 0$ and $[\![\mathsf{F}_!(f + g)]\!] = [\![\mathsf{F}_!(f) + \mathsf{F}_!(g)]\!]$.*

Now since every category $\mathbb{X}$ with finite biproducts is a CLAC, it follows that for every comonad $(!, \delta, \varepsilon)$ on $\mathbb{X}$, the coKleisli category $\mathbb{X}_!$ is a CLAC. It is important to point out that even if all maps in $\mathbb{X}$ are additive maps, the same is not true for $\mathbb{X}_!$. This is due to the fact that $!(f + g)$ and $!(0)$ do not necessarily equal $!(f) + !(g)$ and 0 respectively.

We now provide the first main result of this paper: that the coKleisli category of a Cartesian differential comonad is a CDC.

Theorem 3.4. *Let $(!, \delta, \varepsilon, \partial)$ be a Cartesian differential comonad on a category $\mathbb{X}$ with finite biproducts. Then the coKleisli category $\mathbb{X}_!$ is a CDC where the Cartesian left additive structure is defined as in Lem 3.3 and the differential combinator D is defined as follows: for a map $[\![f]\!] : !(A) \to B$, its derivative $[\![\mathsf{D}[f]]\!] : !(A \times A) \to B$ is defined as $[\![\mathsf{D}[f]]\!] = [\![f]\!] \circ \partial_A$. Furthermore:*

(i) *For every object A in $\mathbb{X}$, $[\![\mathsf{D}[\varphi_A]]\!] = \partial_A$.*

(ii) *A coKleisli map $[\![f]\!] : !(A) \to B$ is D-linear in $\mathbb{X}_!$ if and only if the following equality holds: $[\![f]\!] \circ \partial_A \circ !(\iota_1) = [\![f]\!]$;*

(iii) *For every map $f : A \to B$ in $\mathbb{X}$, $[\![\mathsf{F}_!(f)]\!]$ is D-linear in $\mathbb{X}_!$.*

(iv) *There is a functor $\mathsf{F}_{\mathsf{D}\text{-lin}} : \mathbb{X} \to \mathsf{D}\text{-lin}[\mathbb{X}_!]$ which is defined on objects as $\mathsf{F}_{\mathsf{D}\text{-lin}}(A) = A$ and on maps $f : A \to B$ as $[\![\mathsf{F}_{\mathsf{D}\text{-lin}}(f)]\!] = f \circ \varepsilon_A = [\![\mathsf{F}_!(f)]\!]$, and such that $\mathsf{F}_! = \mathsf{U} \circ \mathsf{F}_{\mathsf{D}\text{-lin}}$.*

Proof. See extended version [24]. □

The converse of Thm 3.4 is also true and states that a comonad whose coKleisli category is a CDC is indeed a Cartesian differential comonad.

Proposition 3.5. *Let $\mathbb{X}$ be a category with finite biproducts and let $(!, \delta, \varepsilon)$ be a comonad on $\mathbb{X}$. Suppose that the coKleisli category $\mathbb{X}_!$ is a CDC with differential combinator D such that the underlying Cartesian left additive structure of $\mathbb{X}_!$ is the one from Lem 3.3 and for every map $f : A \to B$ in $\mathbb{X}$, $\llbracket \mathsf{F}_!(f) \rrbracket$ is a D-linear map in $\mathbb{X}_!$. Define the natural transformation $\partial_A : !(A \times A) \to !(A)$ as $\partial_A = \llbracket \mathsf{D}[\varphi_A] \rrbracket$. Then $(!, \delta, \varepsilon, \partial)$ is a Cartesian differential comonad and furthermore for every coKleisli map $\llbracket f \rrbracket : !(A) \to B$, $\llbracket \mathsf{D}[f] \rrbracket = \llbracket f \rrbracket \circ \partial_A$.*

Proof. See extended version [24]. □

As a result, we obtain the following bijective correspondence:

Corollary 3.6. *Let $\mathbb{X}$ be a category with finite biproducts and let $(!, \delta, \varepsilon)$ be a comonad on $\mathbb{X}$. Then there is a bijective correspondence between differential combinator transformations ∂ on $(!, \delta, \varepsilon)$ and differential combinators D on the coKleisli category $\mathbb{X}_!$ with respect to the Cartesian left additive structure from Lem 3.3 and such that for every map f in $\mathbb{X}$, $\llbracket \mathsf{F}_!(f) \rrbracket$ is a D-linear map in $\mathbb{X}_!$ via the constructions of Thm 3.4 and Prop 3.5. via*

Proof. See extended version [24]. □

We now turn our attention back to the D-linear maps in the coKleisli category of a Cartesian differential comonad. Specifically, we wish to provide necessary and sufficient conditions for when the subcategory of D-linear maps is isomorphic to the base category. Explicitly, we wish to study when $\mathsf{F}_{\mathsf{D}\text{-lin}} : \mathbb{X} \to \mathsf{D}\text{-lin}[\mathbb{X}_!]$ as defined in Thm 3.4.(iv) is an isomorphism. The answer, as it turns out, is requiring that the comonad counit has a section.

Definition 3.7. *Let $(!, \delta, \varepsilon, \partial)$ be a Cartesian differential comonad on a category $\mathbb{X}$ with finite biproducts. A* D***-linear unit*** *on $(!, \delta, \varepsilon, \partial)$ is a natural transformation $\eta_A : A \to !(A)$ such that the following equalities hold:*

[du.1] *Linear Rule: $\varepsilon_A \circ \eta_A = 1_A$;*

[du.2] *Linearization Rule: $\varepsilon_A \circ \eta_A = \partial_A \circ !(\iota_1)$.*

For commutative diagram versions of the axioms **[du.1]** to **[du.6]** see the extended version [24]. Note that the definition of a D-linear unit essentially says that $\partial_A \circ !(\iota_1)$ is a split idempotent via η_A and ε_A. Our first observation is that D-linear units are unique.

Lemma 3.8. *For a Cartesian differential comonad, if a* D*-linear unit exists, then it is unique.*

Proof. See extended version [24]. □

For a Cartesian differential comonad with a D-linear unit, the D-linear maps in the coKleisli category correspond precisely to the maps in the base category. We also have the following useful identity:

Lemma 3.9. *Let* $(!, \delta, \varepsilon, \partial)$ *be a Cartesian differential comonad on a category* $\mathbb{X}$ *with finite biproducts. Then* $[\![\mathsf{L}[\varphi_A]]\!] = \partial_A \circ !(\iota_1)$.

Proof. See extended version [24]. □

Proposition 3.10. *Let* $(!, \delta, \varepsilon, \partial)$ *be a Cartesian differential comonad on a category* $\mathbb{X}$ *with finite biproducts. Then* $\mathsf{F}_{\mathsf{D}\text{-lin}} : \mathbb{X} \to \mathsf{D}\text{-lin}[\mathbb{X}_!]$ *is an isomorphism (where* $\mathsf{F}_{\mathsf{D}\text{-lin}}$ *is defined as in Thm 3.4.(iv)) if and only if* $(!, \delta, \varepsilon, \partial)$ *has a* D*-linear unit* $\eta_A : A \to !(A)$.

Proof. See extended version [24]. □

As a result, in the presence of a D-linear unit, we obtain the following characterizations of D-linear maps.

Corollary 3.11. *Let* $(!, \delta, \varepsilon, \partial)$ *be a Cartesian differential comonad on a category* $\mathbb{X}$ *with finite biproducts. If* $(!, \delta, \varepsilon, \partial)$ *has a* D*-linear unit* η*, then the following are equivalent for a coKleisli map* $[\![f]\!] : !(A) \to B$*,*

(i) $[\![f]\!]$ *is* D*-linear in* $\mathbb{X}_!$

(ii) *There exists a (necessarily unique) map* $g : A \to B$ *in* $\mathbb{X}$ *such that* $[\![f]\!] = g \circ \varepsilon_A = [\![\mathsf{F}_!(g)]\!]$.

(iii) $[\![f]\!] \circ \eta_A \circ \varepsilon_A = [\![f]\!]$

We conclude this section with some examples.

Example 3.12. The main example of a Cartesian differential comonad is the comonad of a differential category. Briefly, a differential category [3, Def 2.4] is an additive symmetric monoidal category $\mathbb{X}$ equipped with a comonad $(!, \delta, \varepsilon)$, two natural transformations $\Delta_A : !(A) \to !(A) \otimes !(A)$ and $e_A : !(A) \to I$ such that $!(A)$ is a cocommutative comonoid, and a natural transformation called a deriving transformation $\mathsf{d}_A : !(A) \otimes A \to !(A)$ satisfying certain coherences which capture the basic properties of differentiation [6, Def 7]. By [4, Prop 3.2.1], for a differential category $\mathbb{X}$ with finite products, its coKleisli category $\mathbb{X}_!$ is a CDC where the differential combinator is defined using the deriving transformation. For a coKleisli map $[\![f]\!] : !A \to B$, its derivative $[\![\mathsf{D}[f]]\!] : !(A \times A) \to B$ is defined as: $[\![\mathsf{D}[f]]\!] = [\![f]\!] \circ \mathsf{d}_A \circ (1_{!(A)} \otimes \varepsilon_A) \circ (!(\pi_0) \otimes !(\pi_1)) \circ \Delta_{A \times A}$. Applying Prop 3.5, we obtain a differential combinator transformation: $\mathsf{d}_A \circ (1_{!(A)} \otimes \varepsilon_A) \circ (!(\pi_0) \otimes !(\pi_1)) \circ \Delta_{A \times A}$. Furthermore, if there exists a natural transformation $u_A : I \to !(A)$ such that $e_A \circ u_A = 1_I$ and $u_A \circ e_A = !(0)$, then we obtain a D-linear unit defined as $\eta_A = \mathsf{d}_A \circ (u_A \otimes 1_A); \lambda_A^{-1}$, where $\lambda_A : I \otimes A \cong A$. Readers familiar with differential linear logic will note that any differential *storage* category [3, Def 4.10] has such a map u and that in this case the D-linear unit is precisely the codereliction [6, Sec 5]. However, we stress that it is possible to have a D-linear unit for differential categories that are not differential storage categories. We invite the reader to see [6, Sec 9] and [23, Ex 4.7] for lists of examples of differential categories.

Example 3.13. Our three main novel examples of Cartesian differential comonads that we introduce in Sec 5, 6, and 7 below, arise instead more naturally as the dual notion, which we simply call **coCartesian differential monads**. Following the convention in the differential category literature for the dual notion of differential categories, we have elected to keep the same terminology and notation for the dual notion of a differential combinator transformation. Briefly, a coCartesian differential monad on a category $\mathbb{X}$ with finite biproducts is a quadruple $(\mathsf{S}, \mu, \eta, \partial)$ consisting of a monad (S, μ, η) (where $\mu_A : \mathsf{SS}(A) \to \mathsf{S}(A)$ and $\eta_A : A \to \mathsf{S}(A)$) and a natural transformation $\partial_A : \mathsf{S}(A) \to \mathsf{S}(A \times A)$, again called a differential combinator transformation, such that the dual diagrams of Def 3.1 commute. By the dual statement of Prop 3.5, the opposite category of the

Kleisli category of a coCartesian differential monad is a CDC. The dual notion of a D-linear unit is called a D-linear counit, which would be a natural transformation $\varepsilon_A : \mathsf{S}(A) \to A$ such that the dual diagrams of Def 3.7 commute. By the dual statement of Prop 3.10, the existence of a D-linear counit implies that the opposite of the base category is isomorphic to the subcategory of the D-linear of the opposite of the Kleisli category.

The following are two "trivial" examples of CDCs any category with finite biproducts. While both are "trivial" in their own way, they both provide simple separating examples. Indeed, the first is an example of a Cartesian differential comonad without a D-linear unit, while the second is a Cartesian differential comonad which is not induced by a differential category.

Example 3.14. Let $\mathbb{X}$ be a category with finite biproducts, and let $\top$ be the chosen zero object. Then the constant comonad C which sends every object to the zero object $\mathsf{C}(A) = \top$ and every map to zero maps $\mathsf{C}(f) = 0$ is a Cartesian differential comonad whose differential combinator transformation is simply 0. This Cartesian differential comonad has a D-linear unit if and only if every object of $\mathbb{X}$ is a zero object.

Example 3.15. Let $\mathbb{X}$ be a category with finite biproducts. Then the identity comonad $1_{\mathbb{X}}$ is a Cartesian differential comonad whose differential combinator transformation is the second projection $\pi_1 : A \times A \to A$ and has a D-linear unit given by the identity map $1_A : A \to A$. The resulting coKleisli category is simply the entire base category $\mathbb{X}$ and whose differential combinator the same as in Ex 2.5. As such, this example recaptures Ex 2.5 that every category with finite biproducts is a CDC where every map is D-linear.

4. Cartesian Differential Abstract coKleisli Categories

The goal of this section is to give a precise characterization of the CDCs which are the coKleisli categories of Cartesian differential comonads. This is a generalization of the work done by Blute, Cockett, and Seely in [5], where they characterize which CDCs are the coKleisli categories of the comonads of differential categories. This was achieved using the concept of abstract coKleisli categories [5, Sec 2.4], which is the dual notion of thunk-force-categories as introduced by Führmann in [22]. Abstract coKleisli categories provide a direct description of the structure of coKleisli categories in such a way that the coKleisli category of a comonad is an abstract coKleisli category and, conversely, every abstract coKleisli category is canonically the coKleisli category of a comonad on a certain subcategory. As such, here we introduced Cartesian differential abstract coKleisli categories which, as the name suggests, are abstract coKleisli categories that are also CDCs such that the differential combinator and abstract coKleisli structure are compatible. We show that the coKleisli category of a Cartesian differential comonad is a Cartesian differential abstract coKleisli categories and that, conversely, every Cartesian differential abstract coKleisli category is canonically the coKleisli category of a Cartesian differential comonad on a certain subcategory. We will also study the D-linear maps of Cartesian differential abstract coKleisli categories.

We will start from the abstract coKleisli side of the story.

Definition 4.1. *An* ***abstract coKleisli structure*** *on a category* $\mathbb{X}$ *is a triple* $(!, \varphi, \epsilon)$ *consisting of an endofunctor* $! : \mathbb{X} \to \mathbb{X}$*, a natural transformation* $\varphi_A : A \to !(A)$*, and a family of maps* $\epsilon_A : !(A) \to A$ *(which are not necessarily natural), such that* $\epsilon_{!(A)} : !!(A) \to !(A)$ *is a natural transformation, and that* $\epsilon_A \circ \varphi_A = 1_A = \epsilon_{!A} \circ !(\varphi_A)$ *and* $\epsilon_A \circ \epsilon_{!A} = \epsilon_A \circ !(\epsilon_A)$ *hold. An* ***abstract coKleisli category*** *[5, Def 2.4.1] is a category* $\mathbb{X}$ *equipped with an abstract coKleisli structure* $(!, \varphi, \epsilon)$.

Below in Lem 4.11, we will review how every coKleisli category is an abstract coKleisli category. In order to obtain the converse, we first need from an abstract coKleisli category to construct a category with comonad. In an abstract coKleisli category, there are an important class of maps called the ϵ-natural maps (which are the dual of thunkable maps in thunk-force categories [22, Def 7]). These ϵ-natural maps form a subcategory which comes equipped with a comonad, and the coKleisli category of this comonad is the starting abstract coKleisli category.

Definition 4.2. *In an abstract coKleisli category* $\mathbb{X}$ *with abstract coKleisli structure* $(!, \varphi, \epsilon)$*, a map* $f : A \to B$ *is said to* ***ϵ-natural*** *if* $\varepsilon_B \circ !(f) = f \circ \varepsilon_A$*. Define the subcategory of ϵ-natural maps* $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ *to be the category whose* *objects are the same as* $\mathbb{X}$ *and whose maps are* ϵ*-natural in* $\mathbb{X}$*, and let* $\mathsf{U}_\epsilon : \epsilon\text{-}\mathsf{nat}[\mathbb{X}] \to \mathbb{X}$ *be the obvious forgetful functor.*

As we will discuss in Lem 4.12, in the context of a coKleisli category of a comonad, these ϵ-natural maps should be thought of as the maps in the base category. We now review in detail how every abstract coKleisli category is isomorphic to the coKleisli category of a canonical comonad on the subcategory of ϵ-natural maps.

Lemma 4.3. *[22, Dual of Thm 4] Let* $\mathbb{X}$ *be an abstract coKleisli category with abstract coKleisli structure* $(!, \varphi, \epsilon)$*. Define the natural transformation* $\beta_A : !(A) \to !!(A)$ *as* $\beta_A = !(\varphi_A)$*. Then* $(!, \beta, \epsilon)$ *is a comonad on* $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ *such that the functor* $\mathsf{G}_\epsilon : \mathbb{X} \to \epsilon\text{-}\mathsf{nat}[\mathbb{X}]_!$ *defined on objects as* $\mathsf{G}_\epsilon(A) = A$ *and on a map* $f : A \to B$ *as* $[\![\mathsf{G}_\epsilon(f)]\!] = \epsilon_B \circ !(f)$*, is an isomorphism with inverse* $\mathsf{G}_\epsilon^{-1} : \epsilon\text{-}\mathsf{nat}[\mathbb{X}]_! \to \mathbb{X}$ *defined on objects as* $\mathsf{G}_\epsilon(A) = A$ *and on a coKleisli map* $[\![f]\!] : !(A) \to B$ *as* $\mathsf{G}_\epsilon^{-1}([\![f]\!]) = [\![f]\!] \circ \varphi_A$*.*

We now wish to equip abstract coKleisli categories with Cartesian differential structure. To do so, we must first discuss Cartesian left additive structure for abstract coKleisli categories. We start with the finite product structure:

Definition 4.4. *A* ***Cartesian abstract coKleisli category*** *[5, Def 2.4.1] is an abstract coKleisli category* $\mathbb{X}$ *with abstract coKleisli structure* $(!, \varphi, \epsilon)$ *such that* $\mathbb{X}$ *has finite products and all the projection maps* $\pi_0 : A \times B \to A$ *and* $\pi_1 : A \times B \to B$ *are* ϵ*-natural.*

For a Cartesian abstract coKleisli category $\mathbb{X}$, it follows that ϵ-natural maps are closed under the finite product structure.

Lemma 4.5. *[5, Sec 2.4] Let* $\mathbb{X}$ *be a Cartesian abstract coKleisli category with abstract coKleisli structure* $(!, \varphi, \epsilon)$*. Then* $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ *has finite products (which is defined as in* $\mathbb{X}$*).*

Next we discuss Cartesian left additive structure for abstract coKleisli categories, where we require that ϵ-natural maps are closed under the additive structure.

Definition 4.6. *A* ***Cartesian left additive abstract coKleisli category*** *is a Cartesian abstract coKleisli category $\mathbb{X}$ with abstract coKleisli structure $(!, \varphi, \epsilon)$ such that $\mathbb{X}$ is also a CLAC, zero maps $0 : A \to B$ are ϵ-natural, and if $f : A \to B$ and $g : A \to B$ are ϵ-natural, then their sum $f + g : A \to B$ is ϵ-natural.*

For a Cartesian left additive abstract coKleisli category, the subcategory of ϵ-natural maps also form a CLAC. It is important to stress however that ϵ-natural maps are not assumed to be additive, and therefore the subcategory of ϵ-natural maps does not necessarily have biproducts.

Lemma 4.7. *Let $\mathbb{X}$ be a Cartesian left additive abstract coKleisli category with abstract coKleisli structure $(!, \varphi, \epsilon)$. Then $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ is a CLAC (where the necessary structure is defined as in $\mathbb{X}$). Furthermore, $\epsilon_A \circ !(0) = 0$ and if $f : A \to B$ and $g : A \to B$ are ϵ-natural, then $\varepsilon_B \circ !(f + g) = \epsilon_B \circ !(f) + \epsilon_B \circ !(g)$.*

Proof. See extended version [24]. □

We are now in a position to define Cartesian differential abstract coKleisli categories.

Definition 4.8. *A* ***Cartesian differential abstract coKleisli category*** *is a CDC $\mathbb{X}$, with differential combinator D, such that $\mathbb{X}$ is also a Cartesian left additive abstract coKleisli category with abstract coKleisli structure $(!, \varphi, \epsilon)$ and every ϵ-natural map is D-linear.*

We will now show that for a Cartesian differential abstract coKleisli category, the canonical comonad on the subcategory of ϵ-natural maps is a Cartesian differential comonad and that the coKleisli category is isomorphic to the starting Cartesian differential abstract coKleisli category.

Proposition 4.9. *Let $\mathbb{X}$ be a Cartesian differential abstract coKleisli category with differential combinator D and abstract coKleisli structure $(!, \varphi, \epsilon)$. Then $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ is a category with finite biproducts and $(!, \beta, \epsilon, \partial)$ (where $(!, \beta, \epsilon)$ is defined as in Lem 4.3) is a Cartesian differential comonad on $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]$ where the differential combinator transformation $\partial_A : !(A) \to !(A \times A)$ is defined as $\partial_A = \epsilon_{!(A)} \circ ! (\mathsf{D}[\varphi_A])$. Furthermore, $\mathsf{G}_\epsilon : \mathbb{X} \to \epsilon\text{-}\mathsf{nat}[\mathbb{X}]_!$ is a Cartesian differential isomorphism, so*

$[\![\mathsf{G}_\epsilon(\mathsf{D}[f])]\!] = [\![\mathsf{D}[\mathsf{G}_\epsilon(f)]]\!]$ *and* $\mathsf{G}_\epsilon^{-1}([\![\mathsf{D}[f]]\!]) = \mathsf{D}[\mathsf{G}_\epsilon^{-1}([\![f]\!])]$*, where the differential combinator on the coKleisli category* $\epsilon\text{-}\mathsf{nat}[\mathbb{X}]_!$ *is defined as in Thm 3.4.*

Proof. See extended version [24]. □

It is important to note that while ϵ-natural maps are assumed to be D-linear, the converse is not necessarily true. It turns out that all D-linear maps are ϵ-natural precisely when the Cartesian differential comonad has a D-linear unit.

Lemma 4.10. *Let* $\mathbb{X}$ *be a Cartesian differential abstract coKleisli category with differential combinator* D *and abstract coKleisli structure* $(!, \varphi, \epsilon)$*. Define the natural transformation* $\eta_A : A \to !(A)$ *as* $\eta_A := \mathsf{L}[\varphi_A]$*. Then the following are equivalent:*

(i) $\epsilon\text{-}\mathsf{nat}[\mathbb{X}] = \mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}]$*, that is, every* D*-linear map is* ϵ*-natural;*

(ii) *For every object* A*,* η_A *is* ϵ*-natural;*

(iii) η *is a* D*-linear unit for* $(!, \beta, \epsilon, \partial)$*.*

Proof. See extended version [24]. □

We turn our attention to the converse of Prop 4.9. We will now explain how every coKleisli category of a Cartesian differential comonad is a Cartesian differential abstract coKleisli category. To do so, let us first quickly review how every coKleisli category is an abstract coKleisli category.

Lemma 4.11. *[5, Prop 2.6.3] Let* $(!, \delta, \varepsilon)$ *be a comonad on a category* $\mathbb{X}$*. Then define the endofunctor* $!_! : \mathbb{X}_! \to \mathbb{X}_!$ *on objects as* $!_!(A) = !(A)$ *and on a coKleisli map* $[\![f]\!] : !(A) \to B$ *as* $[\![!_!(f)]\!] = !([\![f]\!]) \circ \delta_A \circ \varepsilon_{!(A)}$*. Also define the family of coKleisli maps* $[\![\epsilon_A]\!] : !!(A) \to A$ *as* $[\![\epsilon_A]\!] = \varepsilon_A \circ \varepsilon_{!(A)}$*. Then the coKleisli category* $\mathbb{X}_!$ *is an abstract coKleisli category with abstract coKleisli structure* $(!_!, \varphi, \epsilon)$*, where* φ *is defined as* $[\![\varphi_A]\!] = 1_{!(A)}$*. Furthermore,*

(i) *A coKleisli map* $[\![f]\!] : !(A) \to B$ *is* ϵ*-natural if and only if* $[\![f]\!] \circ \varepsilon_{!(A)} = [\![f]\!] \circ !(\varepsilon_A)$*.*

(ii) *For every map $f : A \to B$ in $\mathbb{X}$, $[\![\mathsf{F}_!(f)]\!] : !(A) \to B$ is ϵ-natural;*

(iii) *There is a functor $\mathsf{F}_\epsilon : \mathbb{X} \to \epsilon\text{-}\mathsf{nat}[\mathbb{X}_!]$ which is defined on objects as $\mathsf{F}_\epsilon(A) = A$ and on maps $f : A \to B$ as $[\![\mathsf{F}_\epsilon(f)]\!] = f \circ \varepsilon_A = [\![\mathsf{F}_!(f)]\!]$, and such that $\mathsf{F}_! = \mathsf{U} \circ \mathsf{F}_\epsilon$.*

A natural question to ask is when the subcategory of ϵ-natural maps of a coKleisli category is isomorphic to the base category. The answer is when the comonad is exact (for monads, this is called the equalizer requirement [22, Def 8]).

Lemma 4.12. *[22, Dual of Thm 9] Let $(!, \delta, \varepsilon)$ be a comonad on a category $\mathbb{X}$. Then $\mathsf{F}_\epsilon : \mathbb{X} \to \epsilon\text{-}\mathsf{nat}[\mathbb{X}_!]$ is an isomorphism if and only if the comonad $(!, \delta, \varepsilon)$ is exact [5, Sec 2.6], that is, the following is a coequalizer diagram:*

$$\begin{tikzcd} !!(A) \arrow[r, shift left, "\varepsilon_{!(A)}"] \arrow[r, shift right, "!(\varepsilon_A)"'] & !(A) \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

In the case of an exact comonad, the base category can be recovered from the coKleisli category using the subcategory of ϵ-natural maps. For abstract coKleisli categories, note that the comonad from Lem 4.3 is always exact.

For a comonad on the category with finite products, the coKleisli category is a Cartesian abstract coKleisli category.

Lemma 4.13. *[5, Sec 2.6] Let $(!, \delta, \varepsilon)$ be a comonad on a category $\mathbb{X}$ with finite products. Then the coKleisli category $\mathbb{X}_!$ is a Cartesian abstract coKleisli category with abstract coKleisli structure as defined in Lem 4.11.*

For a comonad on a CLAC, the coKleisli category is a Cartesian left additive abstract coKleisli category.

Lemma 4.14. *Let $(!, \delta, \varepsilon)$ be a comonad on a CLAC $\mathbb{X}$. Then the coKleisli category $\mathbb{X}_!$ is a Cartesian left additive abstract coKleisli category with abstract coKleisli structure as defined in Lem 4.11 and Cartesian left additive structure as defined in Lem 3.3.*

Proof. See extended version [24]. □

We will now show that for a Cartesian differential comonad, its coKleisli category is a Cartesian differential abstract coKleisli category.

Proposition 4.15. *Let $(!, \delta, \varepsilon)$ be a Cartesian differential comonad on a category $\mathbb{X}$ with finite biproducts. Then $\mathbb{X}_!$ is a Cartesian differential abstract coKleisli category with Cartesian differential structure defined in Thm 3.4 and abstract coKleisli structure $(!_!, \varphi, \epsilon)$ as defined in Lem 4.11.*

Proof. See extended version [24]. □

We conclude this section by showing that for a Cartesian differential comonad with a D-linear unit, the underlying comonad is exact and that a coKleisli map is D-linear if and only if it ϵ-natural.

Lemma 4.16. *Let $(!, \delta, \varepsilon, \partial)$ be a Cartesian differential comonad on a category $\mathbb{X}$ with finite biproducts. Then $(!, \delta, \varepsilon, \partial)$ has a* D*-linear unit $\eta_A : A \to !(A)$ if and only if $(!, \delta, \varepsilon)$ is exact and for each object A, the* D*-linear map $[\![\mathsf{L}[\varphi_A]]\!] : !(A) \to !(A)$ is ϵ-natural.*

Proof. See extended version [24]. □

Corollary 4.17. *Let $(!, \delta, \varepsilon, \partial)$ be a Cartesian differential comonad with a* D*-linear unit η on a category $\mathbb{X}$ with finite biproducts. Then for a coKleisli map $[\![f]\!] : !(A) \to B$, $[\![f]\!]$ is* D*-linear in $\mathbb{X}_!$ if and only if $[\![f]\!]$ is ϵ-natural in $\mathbb{X}_!$. As such, $\mathbb{X} \cong \epsilon\text{-}\mathsf{nat}[\mathbb{X}_!] \cong \mathsf{D}\text{-}\mathsf{lin}[\mathbb{X}_!]$*

5. Example: Reduced Power Series

In this section we construct a Cartesian differential comonad (in the opposite category) based on *reduced* formal power series, which therefore induces a CDC of *reduced* formal power series. To the extent of the authors' knowledge, this is a new observation. This is an interesting and important non-trivial example of a Cartesian differential comonad which does not arise from a differential category. Unsurprisingly, the differential combinator will reflect the standard differentiation of arbitrary multivariable power series. However, the problem with arbitrary power series lies with composition. Indeed, famously, power series with degree 0 coefficients, also called constant terms, cannot be composed, since in general this results in an infinite non-converging sum in the base field. Thus, multivariable formal power series do not form a category, since their composition may be undefined. *Reduced* formal power series are power series with no constant term. These can be composed [7, Sec 4.1] and thus, we obtain a Lawvere theory of reduced power series. The total derivative of a reduced power series is again reduced, and therefore, we obtain a CDC of reduced power series. Futhermore, this CDC of reduced power series is in fact a subcategory of the opposite category of the Kleisli category of the coCartesian differential monad P, the free reduced power series algebra monad, which can be seen as the free complete algebra functor induced by the operad of commutative algebras [20, Sec 1.4.4]. Lastly, it is worth mentioning that, while in this section we will work with vector spaces over a field, we note that all the constructions easily generalize to the category of modules over a commutative (semi)ring.

Let $\mathbb{F}$ be a field. For an $\mathbb{F}$-vector space V, define $\mathsf{P}(V)$ as $\mathsf{P}(V) = \prod_{n=1}^{\infty} (V^{\otimes n})_{S(n)}$ where $(V^{\otimes n})_{S(n)}$ denotes the vector space of symmetrized n-tensors, that is, classes of tensors of length n under the action of the symmetric group which permutes the factors in $V^{\otimes n}$. An arbitrary element $\mathfrak{t} \in \mathsf{P}(V)$ is then an infinite ordered list $\mathfrak{t} = (\mathfrak{t}(n))_{n=1}^{\infty}$ where $\mathfrak{t}(n) \in (V^{\otimes n})_{S(n)}$. Therefore, an arbitrary element of $\mathsf{P}(V)$ can be written in the form $\mathfrak{t} = (\mathfrak{t}(n))_{n=1}^{\infty} = \left(\sum_{i=1}^{m} v_{(n,i,1)} \dots v_{(n,i,n)}\right)_{n=1}^{\infty}$ where $v_{(n,k,1)} \dots v_{(n,k,n)}$ denotes the class of $v_{(n,k,1)} \otimes \dots \otimes v_{(n,k,n)} \in V^{\otimes n}$ under the action of the symmetric group. If X is basis of V, then $\mathsf{P}(V) \cong \mathbb{F}[\![X]\!]_+$ [20, Sec 1.4.4], where $\mathbb{F}[\![X]\!]_+$ is the non-unital associative ring of reduced power series over X, that is, power series over X with no constant/degree 0 term. Therefore, $\mathsf{P}(V)$ is a non-unital associative $\mathbb{F}$-algebra. The algebra structure is induced by concatenation of classes of tensors $* : v_1 \dots v_n \otimes w_1 \dots w_k \mapsto v_1 \dots v_n w_1 \dots w_k$, which provides a commutative, associative multiplication: $* : (V^{\otimes n})_{S(n)} \otimes (V^{\otimes k})_{S(k)} \to (V^{\otimes n+k})_{S(n+k)}$. It is worth pointing out that $\mathsf{P}(V)$ does not have a unit element. More specifically, $\mathsf{P}(V)$ will not come equipped with a natural map of type $\mathbb{F} \to \mathsf{P}(V)$. So $\mathsf{P}(V)$ will not induce an algebra modality, and therefore will not induce a differential category structure on $\mathbb{F}\text{-}\mathsf{VEC}^{op}$.

This induces a monad P on $\mathbb{F}\text{-}\mathsf{VEC}$ [20, Sec 1.4.3]. Define the functor $\mathsf{P} : \mathbb{F}\text{-}\mathsf{VEC} \to \mathbb{F}\text{-}\mathsf{VEC}$ as mapping an $\mathbb{F}$-vector space V to $\mathsf{P}(V)$, as defined above, and mapping an $\mathbb{F}$-linear map $f : V \to W$ to the $\mathbb{F}$-linear map $\mathsf{P}(f) : \mathsf{P}(V) \to \mathsf{P}(W)$ defined on elements $\mathfrak{t}$ as above by $\mathsf{P}(f)(\mathfrak{t}) = \left(\sum_{i=1}^{m} f(v_{(n,i,1)}) \dots f(v_{(n,i,n)})\right)_{n=1}^{\infty}$. Define the monad unit

$\eta_V : V \to \mathsf{P}(V)$ by $\eta_V(v) = (v, 0, 0, \ldots)$. From a power series point of view, if X is a basis of V, η_V maps a basis element $x \in X$ to its associated monomial of degree 1. For the monad multiplication, let us first consider an element $\mathfrak{s} \in \mathsf{PP}(V)$, which is a list of symmetrized tensor products of lists of symmetrized tensor products, $\mathfrak{s} = (\mathfrak{s}(n))_{n=1}^{\infty}$, $\mathfrak{s}(n) \in ((\mathsf{P}(V))^{\otimes n})_{S(n)}$ and thus, $\mathfrak{s}(n)$ is of the form $\mathfrak{s}(n) = \sum_{i=1}^{m} \mathfrak{s}(n)_{(i,1)} \ldots \mathfrak{s}(n)_{(i,n)}$ for some $\mathfrak{s}(n)_{(i,j)} \in \mathsf{P}(V)$. Now for every partition of n not involving 0, that is, for every $n_1 + \ldots + n_k = n$ with $n_j \geq 1$, define $\mathfrak{s}(n_1, \ldots, n_k) \in (V^{\otimes n})_{S(n)}$ as $\mathfrak{s}(n_1, \ldots, n_k) = \sum_{i=1}^{m} \mathfrak{s}(k)_{(i,1)}(n_1) * \ldots * \mathfrak{s}(k)_{(i,k)}(n_k)$, where $*$ is the concatenation multiplication defined above. Lastly, define $\mu_V : \mathsf{PP}(V) \to \mathsf{P}(V)$ as $\mu_V(\mathfrak{s}) = \left(\sum_{k=1}^{n} \sum_{n_1+\ldots+n_k=n} \mathfrak{s}(n_1, \ldots, n_k)\right)_{n=1}^{\infty}$ This monad multiplication corresponds to the composition of multivariable reduced power series, as defined explicitly in [7, Sec 4.1].

We now introduce the differential combinator transformation for P, that will correspond to differentiating power series. Define the map $\partial_V : \mathsf{P}(V) \to \mathsf{P}(V \times V)$ by setting:

$$\partial_V(\mathfrak{t}) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left((v_{(n,i,1)}, 0) \ldots (\widehat{v_{(n,i,j)}, 0}) \ldots (v_{(n,i,n)}, 0) \right) (0, v_{n,i,j}) \right)_{n=1}^{\infty},$$

where $\mathfrak{t}$ is an arbitrary element of $\mathsf{P}(V)$ as above and $(\widehat{v_{(n,i,j)}, 0})$ indicates the omission of the factor $(v_{(n,i,j)}, 0)$ in the product. If X is a basis of V, the differential combinator transformation can described as a map $\partial_V : \mathbb{F}[\![X]\!]_+ \to \mathbb{F}[\![X \sqcup X]\!]_+$ which maps a reduced power series $\mathfrak{t}(\vec{x})$ to its sum of its partial derivatives: $\partial_V(\mathfrak{t}(\vec{x})) = \sum_{x_i \in \vec{x}} \frac{\partial \mathfrak{t}(\vec{x})}{\partial x_i} x_i^*$, where x_i^* denotes the element x_i in the second copy of X in the disjoint union $X \sqcup X$. Note that even if $\mathfrak{t}(\vec{x})$ depends on an infinite list of variables, $\partial_V(\mathfrak{t}(\vec{x}))$ is well-defined as a formal power series. It is worth insisting on the fact that ∂ cannot be induced by a deriving transformation in the sense of Ex 3.12. Indeed, as a map, ∂ does not factor through a map $\mathsf{P}(V) \to \mathsf{P}(V) \otimes V$. Note that a power series could have infinite partial derivatives and, since infinite sums and $\otimes$ are generally incompatible, the derivative of a power series could not be described as an element of $\mathsf{P}(V) \otimes V$. Moreover, we already noted the lack of unit: a differential operator of type $\mathsf{P}(V) \to \mathsf{P}(V) \otimes V$ would not be able to properly derive degree 1 monomials without a unit argument to put in the $\mathsf{P}(V)$ component. We also have a D-linear counit $\varepsilon_V : \mathsf{P}(V) \to V$ defined as simply the projection onto V: $\varepsilon_V(\mathfrak{t}) = \mathfrak{t}(1)$. From a power series point of view, ε projects out the degree 1 coefficients of a reduced power series. So $(\mathsf{P}, \mu, \eta, \partial)$ is a coCartesian differential monad with D-linear counit ε, or in other words:

Proposition 5.1. *$(\mathsf{P}, \mu, \eta, \partial)$ is a Cartesian differential comonad on $\mathbb{F}\text{-}\mathsf{VEC}^{op}$ with D-linear unit ε. Therefore $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}$ is a CDC and $\mathsf{D}\text{-lin}\left[\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}\right] \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}$.*

Proof. See extended version [24]. □

The CDC $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}$ can be interpreted as the category whose objects are $\mathbb{F}$-vector spaces and whose maps are reduced power series between them. As a result, focusing on the finite-dimensional vector spaces, specifically $\mathbb{F}^n$, one obtains a CDC of reduced power series over finite variables. We describe this category in detail.

Example 5.2. Let $\mathbb{F}$ be a field. Define the category $\mathbb{F}\text{-}\mathsf{POW}_{red}$ whose object are $n \in \mathbb{N}$, where a map $\mathfrak{P} : n \to m$ is a m-tuple of reduced power series (i.e. power series with no degree 0 coefficients) in n variables, that is, $\mathfrak{P} = \langle \mathfrak{p}_1(\vec{x}), \dots, \mathfrak{p}_m(\vec{x}) \rangle$ with $\mathfrak{p}_i(\vec{x}) \in \mathbb{F}[\![x_1, \dots, x_n]\!]_+$. The identity maps $1_n : n \to n$ are the tuples $1_n = \langle x_1, \dots, x_n \rangle$ and where composition is given by multivariable power series substitution [7, Sec 4.1]. $\mathbb{F}\text{-}\mathsf{POW}_{red}$ is a CLAC where the finite product structure is given by $n \times m = n + m$ with projection maps $\pi_0 : n \times m \to n$ and $\pi_1 : n \times m \to m$ defined as the tuples $\pi_0 = \langle x_1, \dots, x_n \rangle$ and $\pi_1 = \langle x_{n+1}, \dots, x_{n+m} \rangle$, and where the additive structure is defined coordinate-wise via the standard sum of power series. $\mathbb{F}\text{-}\mathsf{POW}_{red}$ is also a CDC where the differential combinator is given by the standard differentiation of power series, that is, for a map $\mathfrak{P} : n \to m$, with $\mathfrak{P} = \langle \mathfrak{p}_1(\vec{x}), \dots, \mathfrak{p}_m(\vec{x}) \rangle$, its derivative $\mathsf{D}[\mathfrak{P}] : n \times n \to m$ is defined as the tuple of the sum of the partial derivatives of the power series $\mathfrak{p}_i(\vec{x})$, so $\mathsf{D}[\mathfrak{P}](\vec{x}, \vec{y}) := \left\langle \sum_{i=1}^{n} \frac{\partial \mathfrak{p}_j(\vec{x})}{\partial x_i} y_i \right\rangle_{j=1}^{m}$. It is important to note that even if $\mathfrak{p}_j(\vec{x})$ has terms of degree 1, every partial derivative $\frac{\partial \mathfrak{p}_j(\vec{x})}{\partial x_i} y_i$ will still be reduced (even if $\frac{\partial \mathfrak{p}_j(\vec{x})}{\partial x_i}$ has a degree 0 term), and thus the differential combinator D is indeed well-defined. A map $\mathfrak{P} : n \to m$ is D-linear if it of the form $\mathfrak{P} = \langle \sum_{i=0}^{n} r_{i,j} x_i \rangle_{j=1}^{m}$. Thus $\mathsf{D}\text{-lin}[\mathbb{F}\text{-}\mathsf{POW}_{red}]$ is equivalent to $\mathbb{F}\text{-}\mathsf{LIN}$ (as defined in Ex 2.6). We note that this example generalize to the category of reduced formal power over an arbitrary commutative (semi)ring.

Observe that $\mathbb{F}\text{-}\mathsf{POW}_{red}(n,1) = \mathbb{F}[\![x_1,\ldots,x_n]\!]_+ \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}(\mathbb{F}^n,\mathbb{F})$, which then implies that $\mathbb{F}\text{-}\mathsf{POW}_{red}(n,m) \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}(\mathbb{F}^n,\mathbb{F}^m)$. Thus we have that $\mathbb{F}\text{-}\mathsf{POW}_{red}$ is isomorphic to the full subcategory of $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}$ whose objects are the finite dimensional $\mathbb{F}$-vector spaces. In the finite dimensional case, the differential combinator transformation corresponds precisely to the differential combinator on $\mathbb{F}\text{-}\mathsf{POW}_{red}$: $\partial_{\mathbb{F}^n}(\mathfrak{p}(\vec{x})) = \mathsf{D}[\mathfrak{p}](\vec{x},\vec{y})$. Therefore, $\mathbb{F}\text{-}\mathsf{POW}_{red}$ is a sub-CDC of $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}$, where the latter allows for power series over infinite variables.

6. Example: Divided Power Algebras

In this section, we show that the free divided power algebra monad is a coCartesian differential monad, and therefore, we obtain a CDC of divided power polynomials [29, Sec 12]. Divided power algebras were introduced by Cartan [8] to study the homology of Eilenberg-MacLane spaces with coefficients in a prime field of positive characteristic. Such structures appear notably on the homotopy of simplicial algebras [8, 21], and in the study of D-modules and crystalline cohomology [2]. The free divided power algebra monad Γ was first introduced by Roby in [28] and generalized in the context of operads by Fresse in [21]. Much as for reduced power series, the composition of divided power polynomials is only well-defined when they are reduced, that is, have no constant term. More generally, the study of divided power algebras has been widely developed in the non-unital setting [21]. Since the monad we study encodes a structure of non-unital algebras, this provides another example of a Cartesian differential comonad which is not induced by a differential category. We begin by reviewing the definition of a divided power algebra.

Definition 6.1. *Let $\mathbb{F}$ be a field.* ***A divided power algebra*** *[8, Sec 2] over $\mathbb{F}$ is a commutative associative (non-unital) $\mathbb{F}$-algebra $(A,*)$, where A is the underlying $\mathbb{F}$-vector space and $*$ is the $\mathbb{F}$-bilinear multiplication, which comes equipped with a divided power structure, that is, a family of functions $(-)^{[n]} : A \to A$, $a \mapsto a^{[n]}$, indexed by strictly positive integers n, such that the following identities hold:*

[dp.1] $(\lambda a)^{[n]} = \lambda^n a^{[n]}$ *for all* $a \in A$ *and* $\lambda \in \mathbb{F}$.

[dp.2] $a^{[m]} * a^{[n]} = \binom{m+n}{m} a^{[m+n]}$ *for all* $a \in A$.

[dp.3] $(a+b)^{[n]} = a^{[n]} + \left(\sum_{l=1}^{n-1} a^{[l]} * b^{[n-l]}\right) + b^{[n]}$ *for all* $a \in A$, $b \in A$.

[dp.4] $a^{[1]} = a$ *for all* $a \in A$.

[dp.5] $(a * b)^{[n]} = n! a^{[n]} * b^{[n]} = a^{*n} * b^{[n]} = a^{[n]} * b^{*n}$ *for all* $a \in A$, $b \in A$.

[dp.6] $(a^{[n]})^{[m]} = \frac{(mn)!}{m!(n!)^m} a^{[mn]}$ *for all* $a \in A$.

The function $(-)^{[n]}$ *is called the* n*-th divided power operation.*

When the base field $\mathbb{F}$ is of characteristic 0, the only divided power structure on a commutative associative algebra $(A, *)$ is given by $a^{[n]} = \frac{a^{*n}}{n!}$, which justifies the name "divided powers". Therefore, in the characteristic 0 case, a divided power algebra is simply a commutative associative (non-unital) algebra. However, in general, for non-zero characteristics, the two notions diverge. Examples of divided power algebras include the homology of Eilenberg-MacLane spaces [8, Sec 5 and 8], the homotopy of simplicial commutative algebras [8, Théorème 1], and all Zinbiel algebras (which we review in the next section) [16, Thm 3.4]. Furthermore, there exists a notion of free divided power algebras, which we review now.

Let $\mathbb{F}$ be a field. For an $\mathbb{F}$-vector space V, define $\mathsf{\Gamma}_n(V) = (V^{\otimes n})^{S(n)} \subseteq V^{\otimes n}$ as the subspace of tensors of length n of V which are fixed under the action of the symmetric group $S(n)$, that is, invariant under all n-permutations $\sigma \in S(n)$. Categorically speaking, $\mathsf{\Gamma}_n(V)$ is the joint equalizer of the n-permutations. Define $\mathsf{\Gamma}(V)$ as $\mathsf{\Gamma}(V) = \bigoplus_{n=1}^{\infty} \mathsf{\Gamma}_n(V)$. The vector space $\mathsf{\Gamma}(V)$ is endowed with a divided power algebra structure, and is the free divided power algebra over V [8, Sec 2]. Explicitly, the divided power operations and the product are defined on generators $v, w \in V$ by: $v^{[n]} = v^{\otimes n}$ and $v * w = v \otimes w + w \otimes v$. An arbitrary element of $\mathsf{\Gamma}(V)$ can then be expressed as a finite sum of divided power monomials [9, Sec 4], which are elements of the form: $v_1^{[r_1]} * \ldots * v_n^{[r_n]}$ for $v_1, \ldots, v_n \in V$, where $*$ is the multiplication of $\mathsf{\Gamma}(V)$, and $(-)^{[r_j]}$ are the divided power operations. Note that this decomposition in monomials is not unique in general. Later on, we will define the differential combinator on monomials. In order to check that this combinator is well defined, one can use the explicit form of such a monomial $v_1^{[r_1]} * \ldots * v_n^{[r_n]} = \sum_{\sigma \in S(n)/S(r_1,\ldots,r_n)} \sigma(v_1^{\otimes r_1} \otimes \ldots \otimes v_n^{\otimes r_n})$, where $S(r_1, \ldots, r_n) = S(r_1) \times \ldots \times S(r_p)$ is the Young subgroup of the symmetric group $S(r_1 + \ldots + r_p)$.

Free divided power algebras induce a monad Γ on $\mathbb{F}$-VEC [21, Prop 1.2.3]. Note that it is sufficient to define the monad structure maps on divided power monomials and then extend by linearity. Define the endofunctor $\Gamma : \mathbb{F}\text{-VEC} \to \mathbb{F}\text{-VEC}$ which sends a $\mathbb{F}$-vector space V to its free divided power algebra $\Gamma(V)$, and which sends an F-linear map $f : V \to W$ to the F-linear map $\Gamma(f) : \Gamma(V) \to \Gamma(W)$ defined on divided powers monomials as $\Gamma(f)(v_1^{[r_1]} * \ldots * v_p^{[r_n]}) = (f(v_1))^{[r_1]} * \ldots * (f(v_n))^{[r_n]}$, which we then extend by linearity. The monad unit $\eta_V : V \to \Gamma(V)$ is the injection map of V into $\Gamma(V)$: $\eta_V(v) = v^{[1]}$. Note that, with this notation, the zero element of $\Gamma(V)$ will here be denoted by $0^{[1]}$. The monad multiplication $\mu_V : \Gamma(\Gamma(V)) \to \Gamma(V)$ is defined as follows on divided power monomials of divided power monomials, using **[dp.5]** and **[dp.6]**:

$$\mu_V \left((v_{1,1}^{[q_{1,1}]} * \ldots * v_{1,k_1}^{[q_{1,k_1}]})^{[r_1]} * \ldots * (v_{p,1}^{[q_{p,1}]} * \ldots * v_{p,k_p}^{[q_{p,k_p}]})^{[r_p]} \right)$$
$$= \left(\prod_{i=1}^{p} \frac{1}{r_i!} \prod_{j=1}^{k_i} \frac{(r_i q_{i,j})!}{q_{i,j}!^{r_i}} \right) v_{1,1}^{[r_1 q_{1,1}]} * \ldots * v_{1,k_1}^{[r_1 q_{1,k_1}]} * \ldots * v_{p,k_p}^{[r_p q_{p,k_p}]}$$

which we then extend by linearity. Note that the functor Γ, and the monad structure we described, can be constructed from the operad of commutative (non-unital) algebras [21, Prop 1.2.3]. Furthermore, note that the algebras of the monad Γ are precisely the dividied power algebras [28, Sec 10, Thm 1 and 2].

Observe that Γ will not be an algebra modality since $\Gamma(V)$ is non-unital. Therefore, Γ will provide an example of Cartesian differential comonad that is not induced from a differential category structure. We now define the differential combinator transformation for Γ. Define $\partial_V : \Gamma(V) \to \Gamma(V \times V)$ as follows on divided power monomials:

$$\partial_V(v_1^{[r_1]} * \ldots * v_n^{[r_n]})$$
$$= \sum_{i=1}^{n} (v_1, 0)^{[r_1]} * \ldots * (v_i, 0)^{[r_i - 1]} * \ldots * (v_n, 0)^{[r_n]} * (0, v_i)^{[1]}$$

which we then extend by linearity. If $r_i = 1$, we use the following convention:

$$(v_1, 0)^{[r_1]} * \ldots * (v_i, 0)^{[r_i - 1]} * \ldots * (v_n, 0)^{[r_n]} * (0, v_i)^{[1]}$$
$$= (v_1, 0)^{[r_1]} * \ldots * (v_{i-1}, 0)^{[r_{i-1}]} * (v_{i+1}, 0)^{[r_{i+1}]} * \ldots * (v_n, 0)^{[r_n]} * (0, v_i)^{[1]}$$

We will see below that ∂ corresponds to taking the sum of the partial derivatives of divided power polynomials. Note that a consequence of the lack of a unit in $\Gamma(V)$ is that ∂_V does not factor through a map $\Gamma(V) \to \Gamma(V) \otimes V$ since such a map would be undefined on the divided power monomials of degree 1, $v^{[1]}$. We also have a D-linear counit $\varepsilon_V : \Gamma(V) \to V$ defined as follows on divided power monomials: $\varepsilon_V(v^[1]) = v$, and $\varepsilon_V(v_1^{[r_1]} * \ldots * v_n^{[r_n]}) = 0$ otherwise, which we extend by linearity. Thus ε_V picks out the divided power monomials of degree 1, $v^{[1]}$ for all $v \in V$, while mapping the rest to zero.

Proposition 6.2. *$(\Gamma, \mu, \eta, \partial)$ is a Cartesian differential comonad on $\mathbb{F}$-VEC^{op} with D-linear unit ε. Therefore $\mathbb{F}$-$\mathsf{VEC}^{op}_{\Gamma}$ is a CDC and $\mathsf{D}\text{-lin}\left[\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\Gamma}\right] \cong$ $\mathbb{F}$-VEC^{op}.*

Proof. See extended version [24]. □

The Kleisli category $\mathbb{F}$-VEC_{Γ} is closely related to the notion of (reduced) divided power polynomials. For a set X, we denote by $\mathbb{F}\lceil X \rceil$ the ring of reduced divided power polynomials over the set X, which is by definition the free divided power algebra over the $\mathbb{F}$-vector space with basis X [29, Sec 12]. In other words, a reduced divided polynomial with variables in X is an $\mathbb{F}$-linear composition of commutative monomials of the type $x_1^{[k_1]} \ldots x_n^{[k_n]}$ where $x_1, \ldots, x_n$ is a tuple of n different elements of X and $k_1, \ldots, k_n$ are strictly positive integers. By reduced, we mean that these polynomials do not have degree 0 terms. Multiplication is given by concatenation, multilinearity and the relation **[dp.2]** of Def 6.1. Composition of divided polynomials can be deduced from the relations **[dp.1]**, **[dp.3]**, **[dp.5]** and **[dp.6]** of 6.1. For example, if $p(x) = x^{[n]}$, and $q(y, z) = y^{[m]} z$, then: $p(q(y, z)) = \frac{(mn)!}{(m!)^n} y^{[mn]} z^{[n]}$. We can define a notion of formal partial derivation for divided polynomials. For $x \in X$, define the linear map $\frac{d}{dx} : \mathbb{F}\lceil X \rceil \to \mathbb{F}\lceil X \rceil \oplus \mathbb{F}$ on monomials (which we then extend by linearity). For all monomial $m = x_1^{[k_1]} \dots x_n^{[k_n]}$, (i) $\frac{d}{dx}(m) = 0$ if $x \neq x_i$ for all $i \in \{1, \dots, n\}$; (ii) $\frac{d}{dx}(m) = x_1^{[k_1]} \dots x_{j-1}^{[k_{j-1}]} x_j^{[k_j - 1]} x_{j+1}^{[k_{j+1}]} \dots x_n^{[k_n]}$ if $x = x_j$ and $k_j > 1$; (iii) $\frac{d}{dx}(m) = x_1^{[k_1]} \dots x_{j-1}^{[k_{j-1}]} x_{j+1}^{[k_{j+1}]} \dots x_n^{[k_n]}$ if $x = x_j$, $k_j = 1$, and $n > 1$; and finally (iv) $\frac{d}{dx}(x) = 1_{\mathbb{F}}$ where $1_{\mathbb{F}} \in \mathbb{F}$ is a generator of the second term of the direct sum $\mathbb{F}\lceil X \rceil \oplus \mathbb{F}$ given by the unit of $\mathbb{F}$. We note that, in the case where X is a singleton, these definitions correspond to the notion of derivation for formal divided power series, also called Hurwitz series, as defined by Keigher and Pritchard in [25]. We can restrict to the finite dimensional case and obtain a sub-CDC of $\mathbb{F}\text{-}\mathsf{VEC}_\Gamma^{op}$ which is isomorphic to the Lawvere theory of reduced divided power polynomials.

Example 6.3. Let $\mathbb{F}$ be a field. Define the category $\mathbb{F}\text{-}\mathsf{DPOLY}$ whose object are $n \in \mathbb{N}$, where a map $P : n \to m$ is a m-tuple of reduced divided polynomials in n variables, that is, $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$ with $p_i(\vec{x}) \in \mathbb{F}\lceil x_1, \dots, x_n \rceil$.The identity maps $1_n : n \to n$ are the tuples of the form $1_n = \langle x_1^{[1]}, \dots, x_n^{[1]} \rangle$ and composition is given by divided power polynomial substitution as explained above. $\mathbb{F}\text{-}\mathsf{DPOLY}$ is a CLAC where the finite product structure is given by $n \times m = n + m$ with projection maps $\pi_0 : n \times m \to n$ and $\pi_1 : n \times m \to m$ defined as the tuples $\pi_0 = \langle x_1^{[1]}, \dots, x_n^{[1]} \rangle$ and $\pi_1 = \langle x_{n+1}^{[1]}, \dots, x_{n+m}^{[1]} \rangle$, and where the additive structure is defined coordinate-wise via the standard sum of divided power polynomials. $\mathbb{F}\text{-}\mathsf{DPOLY}$ is also a CDC where for a map $P : n \to m$, with $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$, its derivative $\mathsf{D}[P] : n \times n \to m$ is defined as the tuple of the sum of the partial derivatives of the divided power polynomials $p_i(\vec{x})$: $\mathsf{D}[P](\vec{x}, \vec{y}) := \left(\sum_{i=1}^n \frac{dp_j(\vec{x})}{dx_i} y_i^{[1]} \right)_{j=1}^m$. It is important to note that even if $p_j(\vec{x})$ has terms of degree 1, every partial derivative $\frac{dp_j(\vec{x})}{dx_i} y_i^{[1]}$ will still be reduced (even if $\frac{dp_j(\vec{x})}{dx_i}$ may have a degree 0 term), and thus, the differential combinator D is indeed well-defined. A map $P : n \to m$ is D-linear if it of the form $P = \left\langle \sum_{i=0}^n \lambda_{i,j} x_i^{[1]} \right\rangle_{j=1}^m$. Thus, $\mathsf{D}\text{-}\mathsf{lin}[\mathbb{F}\text{-}\mathsf{DPOLY}]$ is equivalent to $\mathbb{F}\text{-}\mathsf{LIN}$ (as defined in Ex 2.6).

We have that $\mathbb{F}\text{-}\mathsf{DPOLY}(n, 1) = \mathbb{F}\lceil x_1, \dots, x_n \rceil \cong \mathbb{F}\text{-}\mathsf{VEC}_\Gamma^{op}(\mathbb{F}^n, \mathbb{F})$, which then implies that $\mathbb{F}\text{-}\mathsf{DPOLY}(n, m) \cong \mathbb{F}\text{-}\mathsf{VEC}_\Gamma^{op}(\mathbb{F}^n, \mathbb{F}^m)$. Therefore, $\mathbb{F}\text{-}\mathsf{DPOLY}$ is indeed isomorphic to the full subcategory of $\mathbb{F}\text{-}\mathsf{VEC}_\Gamma^{op}$ whose objects are the finite dimensional $\mathbb{F}$-vector spaces. In the finite dimensional case, the differential combinator transformation corresponds precisely to the differential combinator on $\mathbb{F}$-DPOLY: $\partial_{\mathbb{F}^n}(p(\vec{x})) = \mathsf{D}[p](\vec{x}, \vec{y})$. Thus, $\mathbb{F}$-DPOLY is a sub-CDC of $\mathbb{F}\text{-VEC}_{\mathsf{P}}^{op}$, where the latter allows for divided power polynomials over infinite variables (but will still only depend on a finite number of them).

7. Example: Zinbiel Algebras

In this section, we show that the free Zinbiel algebra monad is a coCartesian differential monad, and therefore we construct a CDC based on non-commutative polynomials equipped with the half-shuffle product. Zinbiel algebras were introduced by Loday in [27], as Koszul dual to the classical notion of Leibniz algebra. Zinbiel algebras were further studied by Dokas [16], who shows that they are closely related to divided power algebras. The free Zinbiel algebra is given by the non-unital shuffle algebra. Therefore, this example corresponds to differentiating non-commutative polynomials with a type of polynomial composition defined using the Zinbiel product. Due to the strangeness of the composition, the differential combinator transformation is very different from previous examples. Nevertheless, this is yet another interesting Cartesian differential comonad which does not arise as a differential category. Furthermore, it is worth mentioning that, while the (unital) shuffle algebra has been previously studied in a generalization of differential categories in [1], the Zinbiel algebra perspective was not considered. In future work, it would be interesting to study the link between these two notions.

Definition 7.1. *Let $\mathbb{F}$ be a field. A* ***Zinbiel algebra*** *[27, Def 1.2] over $\mathbb{F}$, also called* ***dual Leibniz algebra****, is an $\mathbb{F}$-vector space A equipped with a bilinear operation $<$ such that $(a < b) < c = (a < (b < c)) + (a < (c < b))$ for all $a, b, c \in A$.*

It is important to insist on the fact that the bilinear product $<$ is not assumed to be associative, commutative, or have a distinguished unit element. That said, it is interesting to point out that any Zinbiel algebra is equipped with an associative and commutative bilinear product $*$ defined as $a * b = a < b + b < a$. Thus, a Zinbiel algebra is also a non-unital commutative, associative algebra [27, Prop 1.5]. The underlying vector space of free Zinbiel algebras is the same as for free non-unital tensor algebras. Readers familiar with the latter will note that the tensor algebra is non-commutative when the multiplication is given by concatenation. However, there is another possible multiplication which is commutative, called the shuffle product. The tensor algebra equipped with the shuffle product is called the shuffle algebra. Furthermore, it turns out that the shuffle product is the commutative associative multiplication $*$ one obtains from the free Zinbiel algebra. Thus, the free Zinbiel algebra and the shuffle algebra are the same object. For the purposes of this paper, we only need to work with the Zinbiel product $<$.

Let $\mathbb{F}$ be a field. For an $\mathbb{F}$-vector space V, define $\mathsf{Zin}(V)$ as $\mathsf{Zin}(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$. It is known that $\mathsf{Zin}(V)$ is the free Zinbiel algebra over V [27, Prop 1.8] with Zinbiel product $<$ defined on pure tensors by $(v_1 \otimes \ldots \otimes v_n) < (w_1 \otimes \ldots \otimes w_m) = \sum_{\sigma \in S(n+m)/S(n)\times S(m)} v_1 \otimes \sigma \cdot (v_2 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m)$, which we then extend by linearity. Thus, $\mathsf{Zin}(V)$ is spanned by words of elements of V. Free Zinbiel algebras induce a monad Zin on $\mathbb{F}$-VEC [27, Prop 1.8]. Similar to previous examples, note that it is sufficient to define the monad structure maps on pure tensors and then extend by linearity. Define the endofunctor $\mathsf{Zin} : \mathbb{F}\text{-}\mathsf{VEC} \to \mathbb{F}\text{-}\mathsf{VEC}$ which sends an $\mathbb{F}$-vector space V to its free Zinbiel algebra $\mathsf{Zin}(V)$, and which sends an $\mathbb{F}$-linear map $f : V \to W$ to the $\mathbb{F}$-linear map $\mathsf{Zin}(f) : \mathsf{Zin}(V) \to \mathsf{Zin}(W)$ defined on pure tensors as $\mathsf{Zin}(f)\,(v_0 \otimes \ldots \otimes v_n) = f(v_0) \otimes \ldots \otimes f(v_n)$, which we then extend by linearity. The monad unit $\eta_V : V \to \mathsf{Zin}(V)$ is the injection of V into $\mathsf{Zin}(V)$, $\eta_V(v) = v$, and the monad multiplication $\mu_V : \mathsf{Zin}\mathsf{Zin}(V) \to \mathsf{Zin}(V)$ is defined on pure tensors by taking their Zinbiel product starting from the right, so defined on a pure tensor $\mathfrak{v}_1 \otimes \ldots \otimes \mathfrak{v}_n \in \mathsf{Zin}\mathsf{Zin}(V)$, where $\mathfrak{v}_i \in \mathsf{Zin}(V)$, by $\mu_V\,(\mathfrak{v}_1 \otimes \ldots \otimes \mathfrak{v}_n) = \mathfrak{v}_1 < (\ldots(\mathfrak{v}_{n-1} < \mathfrak{v}_n)\ldots)$, which we then extend by linearity. Unsurprisingly, the algebras of the monad Zin are precisely the Zinbiel algebras. Similar to the other examples, due to a lack of unit, Zin will not be an algebra modality and therefore this will result in another example of a Cartesian differential comonad which does not come from a differential category.

We can now define the differential combinator transformation for Zin. Define $\partial_V : \mathsf{Zin}(V) \to \mathsf{Zin}(V \times V)$ on pure tensors as follows:

$$\partial_V(v_1 \otimes v_2 \ldots \otimes v_n) = (0, v_1) \otimes (v_2, 0) \otimes \ldots \otimes (v_n, 0)$$

which we then extend by linearity. Note that this differential combinator transformation is quite different from the other examples in appearance. Below, we will explain how this differential combinator transformation corresponds to a sum of partial derivative for non-commutative polynomials with the multiplication given by the Zinbiel product. We also have a D-linear counit $\varepsilon_V : \mathsf{Zin}(V) \to V$ which projects out the V component of $\mathsf{Zin}(V)$, that is, it is defined on pure tensors as $\varepsilon(v) = v$ and $\varepsilon_V(v_1 \otimes \ldots \otimes v_n) = 0$ otherwise, and which we extend by linearity.

Proposition 7.2. $(\mathsf{Zin}, \mu, \eta, \partial)$ *is a Cartesian differential comonad on* $\mathbb{F}\text{-}\mathsf{VEC}^{op}$ *with* D*-linear unit* ε. *Therefore,* $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{Zin}}$ *is a CDC and* $\mathsf{D\text{-}lin}\left[\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{Zin}}\right] \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}$.

Proof. See extended version [24]. □

The Kleisli category $\mathbb{F}\text{-}\mathsf{VEC}_{\mathsf{Zin}}$ is closely related to non-commutative polynomials. For a set X, let $\mathbb{F}\langle X \rangle$ be the set of non-commutative polynomials and $\mathbb{F}\langle X \rangle_+$ be the set of reduced non-commutative polynomials, that is, those without any constant terms. As a vector space, $\mathbb{F}\langle X \rangle_+$ over a set X is isomorphic to the underlying vector space of the free Zinbiel algebra over the free vector space generated by X. Thus, to distinguish between polynomials and non-commutative polynomials, we will use the tensor product $\otimes$. For example, $xy = yx$ is the commutative polynomial, while $x \otimes y$ and $y \otimes x$ are two different non-commutative polynomials. Composition in the Kleisli category corresponds to using the Zinbiel product $<$ to define a new kind of substitution of non-commutative polynomials. We use the term Zinbiel polynomials to refer to reduced non-commutative polynomials with the Zinbiel product and the Zinbiel substitution. We are now in a position to define partial derivatives on non-commutative polynomials. For $x \in X$, define $\frac{d}{dx} : \mathbb{F}\langle X \rangle \to \mathbb{F}\langle X \rangle$ as follows on Zinbiel monomials (which we then extend by linearity): $\frac{d(x_1 \otimes x_2 \otimes \ldots \otimes x_n)}{dx} = x_2 \otimes \ldots \otimes x_n$ if $x_1 = x$ and $\frac{d(x_1 \otimes x_2 \otimes \ldots \otimes x_n)}{dx} = 0$ otherwise. We use the convention that $\frac{d(x)}{dx} = 1$. We can also restrict to the finite-dimensional case and obtain a sub-CDC $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{Zin}}$ which is isomorphic to the Lawvere theory of Zinbiel polynomials, and where the differential combinator is defined using their partial derivatives.

Example 7.3. Let $\mathbb{F}$ be a field. Define the category $\mathbb{F}$-ZIN whose object are natural numbers $n \in \mathbb{N}$, where a map $P : n \to m$ is an m-tuple of reduced non-commutative polynomials in n variables, so $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$ with $p_i(\vec{x}) \in \mathbb{F}\langle x_1, \dots, x_n \rangle_+$. The identity maps $1_n : n \to n$ are the tuples $1_n = \langle x_1, \dots, x_n \rangle$ and where composition is given by Zinbiel substitution, as defined above. $\mathbb{F}$-ZIN is a CLAC where the finite product structure is given by $n \times m = n + m$ with projection maps $\pi_0 : n \times m \to n$ and $\pi_1 : n \times m \to m$ defined as the tuples of the form $\pi_0 = \langle x_1, \dots, x_n \rangle$ and $\pi_1 = \langle x_{n+1}, \dots, x_{n+m} \rangle$, and where the additive structure is defined coordinate wise via the standard sum of non-commutative polynomials. $\mathbb{F}$-ZIN is also a CDC where the differential combinator is given by the differentiation of Zinbiel polynomial given above, that is, for a map $P : n \to m$, with $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$, its derivative $\mathsf{D}[P] : n \times n \to m$ is defined as the tuple of the sum of the partial derivatives of the Zinbiel polynomials $p_i(\vec{x})$, $\mathsf{D}[P](\vec{x}, \vec{y}) := \left\langle \sum_{i=1}^{n} y_i \otimes \frac{dp_j(\vec{x})}{dx_i} \right\rangle_{j=1}^{m}$. It is important to note that even if $p_i(\vec{x})$ has terms of degree 1, every partial derivative $y_i \otimes \frac{dp_j(\vec{x})}{dx_i}$ will still be reduced. Indeed, the polynomial of the form $y_i \otimes 1 \in \mathbb{F}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ are identified with the reduced polynomial $y_i \in \mathbb{F}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle_+$, and so, for example, $y_i \otimes \frac{d(x)}{x} = y_i$. Thus, the differential combinator D is indeed well-defined. A map $P : n \to m$ is D-linear if it of the form $P = \langle \sum_{i=0}^{n} r_{i,j} x_i \rangle_{j=1}^{m}$. Thus $\mathsf{D}\text{-lin}[\mathbb{F}\text{-}\mathsf{ZIN}]$ is equivalent to $\mathbb{F}$-LIN (as defined in Ex 2.6). We note that this example generalize to the category of Zinbiel polynomials over an arbitrary commutative (semi)ring.

Observe that $\mathbb{F}\text{-}\mathsf{ZIN}(n, 1) = F\langle x_1, \dots, x_n \rangle_+ \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{Zin}}(\mathbb{F}^n, \mathbb{F})$, which then implies that $\mathbb{F}\text{-}\mathsf{Zin}(n, m) \cong \mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{Zin}}(\mathbb{F}^n, \mathbb{F}^m)$. Therefore, we have that $\mathbb{F}$-Zin is isomorphic to the full subcategory of $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\Gamma}$ whose objects are the finite dimensional $\mathbb{F}$-vector spaces. In the finite dimensional case, the differential combinator transformation corresponds precisely to the differential combinator on $\mathbb{F}$-ZIN: $\partial_{\mathbb{F}^n}(p(\vec{x})) = \mathsf{D}[p](\vec{x}, \vec{y})$. Thus, $\mathbb{F}$-ZIN is a sub-CDC of $\mathbb{F}\text{-}\mathsf{VEC}^{op}_{\mathsf{P}}$, where the latter allows for Zinbiel polynomials over infinite variables (but will still only depend on a finite number of them).

It is worth noting the link between divided power algebras and Zinbiel algebra. Any Zinbiel algebra is endowed with a divided power algebra structure [16, Thm 3.4], and this results in an inclusion of the free divided power algebra into the free Zinbiel algebra, $\mathsf{\Gamma}(V) \to \mathsf{Zin}(V)$ [16, Sec 5]. As such, this inclusion can be extended to a monic monad morphism $\mathsf{\Gamma} \Rightarrow \mathsf{Zin}$. However, it is not compatible with the differential combinators. For instance, let V be the vector space spanned by x and y, and let $\partial^{\mathsf{\Gamma}}$ and ∂^{Zin} denote the differential combinator transformation for the respective monad. Let $p(x,y) = x^{[1]} * y^{[1]} \in \mathsf{\Gamma}(V)$. On one hand, the injection $\mathsf{\Gamma}(V) \to \mathsf{Zin}(V)$ identifies $p(x,y)$ to $p(x,y) = x \otimes y + y \otimes x$ and so we have that $\partial^{\mathsf{Zin}}_V(p)(x,y,x^*,y^*) = x^* \otimes y + y^* \otimes x$. On the other hand, we have that $\partial^{\mathsf{\Gamma}}_V(p)(x,y,x^*,y^*) = (x^*)^{[1]} * y^{[1]} + (y^*)^{[1]} * x^{[1]}$, which the injection $\mathsf{\Gamma}(V \times V) \to \mathsf{Zin}(V \times V)$ identifies to $\partial^{\mathsf{Zin}}_V(p)(x,y,x^*,y^*) = x^* \otimes y + y \otimes x^* + y^* \otimes x + x \otimes y^*$.

8. Future Work

Beyond finding and constructing new interesting examples of Cartesian differential comonads, and therefore also new examples of CDCs, there are many other interesting possibilities for future work with Cartesian differential comonads. We conclude this paper by listing three potential ideas.

I. In [23], it was shown that every CDC embeds into the coKleisli category of a differential (storage) category [23, Thm 8.7]. In principle, this already implies that every CDC embeds into the coKleisli category of a Cartesian differential comonad. However, Cartesian differential comonads can be defined without the need for a symmetric monoidal structure. Thus, it is reasonable to expect that there is a finer (and possibly simpler) embedding of a CDC into the coKleisli category of a Cartesian differential comonad.

II. In this paper, we studied the (co)Kleisli categories of (co)Cartesian differential (co)monads. A natural follow-up question to ask is: what can we say about the (co)Eilenberg-Moore categories of (co)Cartesian differential (co)monads? As discussed in [13], for differential categories the answer is tangent categories [10]. Indeed, the Eilenberg-Moore category of any codifferential category is always a tangent category [13, Thm 22], while the coEilenberg-Moore category of a differential (storage) category with sufficient limits is a (representable) tangent category [13, Thm 27]. As such, it is reasonable to expect the same to be true for (co)Cartesian differential (co)monads, that is, that the (co)Eilenberg-Moore category of (co)Cartesian differential (co)monad is a tangent category by generalizing the constructions found in [13]

III. An important part of the theory of calculus is integration, specifically its relationship to differentiation given by antiderivatives and the Fundamental Theorems of Calculus. Integration and antiderivatives have found their way into the theory of differential categories [12, 17] and CDCs [11]. In future work, it would therefore be of interest to define integration and antiderivatives for (co)Cartesian differential (co)monads. We conjecture that integration in this setting would be captured by an *integral combinator transformation*, which should be a natural transformation of the opposite type of the differential combinator transformation, that is, of type $\int_A : !(A) \to !(A \times A)$. The axioms of an integral combinator transformation should be analogue to the axioms of an integral combinator [11, Sec 5] in the coKleisli category. Some of the examples presented in this paper seem to come equipped with an integral combinator transformation. For example, there is a well-established notion of integration for power series which should induce integral combinator transformations in an obvious way. In the case of divided power polynomial, there is a notion of integration in the one-variable case (see [25] for the integration of formal divided power series in one variable). However, it is unclear to us how integration for multivariable divided power polynomials would be defined, which is necessary if we wish to construct an integral combinator transformation. In the case of Zinbiel algebras, we conjecture that $\int_V : \mathsf{Zin}(V \times V) \to \mathsf{Zin}(V)$ defined as: $\int (a_{1,0}, a_{1,1}) \otimes \ldots \otimes (a_{n,0}, a_{n,1}) = \sum_{f:\{1,\ldots,n\}\to\{0,1\}} a_{1,f(1)} \otimes \ldots \otimes a_{n,f(n)}$ is a candidate for an integral combinator transformation (in the dual sense). In a differential category, one way to build an integration operator is via the notion of antiderivatives [12, Def 6.1], which is the assumption that a canonical natural transformation $\mathsf{K}_A : !(A) \to !(A)$ be a natural isomorphism. Another goal for future work would be to generalize antiderivatives (in the differential category sense) for Cartesian differential comonads.

In conclusion, there are many potential interesting paths to take for future work with Cartesian differential comonads.

Acknowledgements

The authors would like to thank Kristine Bauer and Robin Cockett for useful discussions. The authors would also like to thank Emily Riehl and the anonymous reviewers for editorial comments and suggestions. For this research, the first author was financially supported by a PIMS–CNRS Postdoctoral Fellowship, and the second author was financially supported by a NSERC Postdoctoral Fellowship - Award #:456414649.

References

[1] Bagnol, M. and Blute, R. F. and Cockett, J. R. B. and Lemay, J.-S. P. The shuffle quasimonad and modules with differentiation and integration. ENTC 325, 29–45.(2016)

[2] Berthelot, P. Cohomologie cristalline des schémas de caractéristique $p > 0$. Lecture Notes in Mathematics 407, 1–604. (1974)

[3] Blute, R. F. and Cockett, J. R. B. and Seely, R. A. G. Differential Categories. MSCS 16 (06), 1049–1083. (2006)

[4] Blute, R. F. and Cockett, J. R. B. and Seely, R. A. G. CDCs. TAC 22 (23), 622–672. (2009)

[5] Blute, R. F. and Cockett, J. R. B. and Seely, R. A. G. Cartesian Differential Storage Categories. TAC 30 (18), 620–686. (2015)

[6] Blute, R. F., Cockett, J. R. B., Lemay, J.-S. P. and Seely, R. A. G. Differential Categories Revisited. ACS 28, 171–235. (2020)

[7] Brewer, T. S. Algebraic properties of formal power series composition. PhD Thesis, University of Kentucky. (2014)

[8] Cartan, H. Puissances divisées. Sém. H. Cartan 7 (1), 1–11. (1954)

[9] Cartan, H. Relations entre les opérations précédentes et les opérations de Bockstein; algèbre universelle d'un module libre gradué. Sém. H. Cartan 7 (1), 1–9. (1954)

[10] Cockett, J. R. B. and Cruttwell, G. S. H. Differential Structure, Tangent Structure, and SDG. ACS 22 (2), 331–417. (2014)

[11] Cockett, J. R. B. and Lemay, J.-S. P. Cartesian Integral Categories and Contextual Integral Categories. Proceedings of MFPS34. ENTCS 341, 45 –72. (2018)

[12] Cockett, J. R. B. and Lemay, J.-S. Integral categories and Calculus categories. MSCS 29 (2), 243–308. (2019)

[13] Cockett, J. R. B. and Lemay, J.-S. P. and Lucyshyn-Wright, R. B. B. Tangent Categories from the Coalgebras of Differential Categories. Proceedings of CSL34. LIPI 152, 17:1–17:17. (2020)

[14] Cockett, J. R. B. and Lemay, J.-S. P. Linearizing Combinators. TAC 38 (13), 374–431. (2022)

[15] Cruttwell, G. S. H. and Lemay, J-S. P. and Lucyshyn-Wright, R. B. B. Integral and differential structure on the free C^∞-ring modality. Cahiers de top. et géo. diff. cat. 62 (2), 116–176. (2021)

[16] Dokas, I. Zinbiel algebras and commutative algebras with divided powers. GMJ 52 (2), 303–313. (2010)

[17] Ehrhard, T. An introduction to Differential Linear Logic: proof-nets, models and antiderivatives. MSCS 28 (7), 1–66. (2017)

[18] Ehrhard, T. and Regnier, L. The differential lambda-calculus. TCS 309 (1), 1–41. (2003)

[19] Fiore, M. P. Differential structure in models of multiplicative biadditive intuitionistic linear logic. Proceedings of TLCA, 163–177. (2007)

[20] Fresse, B. Lie theory of formal groups over an operad. Journal of Alg. 202 (2), 455–511. (1998)

[21] Fresse, B. On the homotopy of simplicial algebras over an operad. Transactions of the American Math. Soc. 352 (9), 4113–4141. (2000)

[22] Führmann, C. Direct models of the computational lambda-calculus. ENTCS 20, 245–292. (1999)

[23] Garner, R. and Lemay, J.-S. P. CDCs as skew enriched categories. ACS 29, 1099–1150. (2021)

[24] Iknonicoff, S. and Lemay, J.-S. P. Cartesian Differential Comonads and New Models of CDCs (Extended Version). arxiv 2108.04304. (2023)

[25] Keigher, W. F. and Pritchard, F. L. Hurwitz series as formal functions. JPAA 146 (3), 291–304. (2000)

[26] Lemay, J.-S. P. A Tangent Category Alternative to the Faa Di Bruno Construction. TAC 33 (35), 1072–1110. (2018)

[27] Loday, J.-L. Cup-product for Leibniz cohomology and dual Leibniz algebras. Mathematica Scandinavica 77 (2), 189–196. (1995)

[28] Roby, N. Les algèbres à puissances divisées. Bulletin des Sciences Mathématiques. 2e Série 89, 75–91. (1965)

[29] Roby, N. Construction de certaines algèbres à puissances divisées. Bulletin de la Société Mathématique de France 96, 97–113. (1968)

[30] Szigeti, J. On limits and colimits in the Kleisli category. Cahiers de top. et géo. diff. cat. 24 (4), 381–391. (1983)

Sacha Ikonicoff
Department of Mathematics and Statistics
University of Ottawa
Ottawa ON, K1N 6N5
Canada
sikonico@uottawa.ca

Jean-Simon Pacaud Lemay
Research Institute of Mathematical Sciences
Kyoto University
Kyoto, 606-8267
Japan
jsplemay@kurims.kyoto-u.ac.jp

Backsets and Open Access

All the papers published in the "*Cahiers*" since their creation are freely downloadable on the site of NUMDAM for

Volumes I to VII and Volumes VIII to LII

and, from Volume L up to now on the 2 sites of the "*Cahiers*"

https://ehres.pagesperso-orange.fr/Cahiers/Ctgdc.htm

http://cahierstgdc.com/

Are also freely downloadable the *Supplements* published in 1980-83

Charles Ehresmann: Œuvres Complètes et Commentées

These Supplements (edited by Andrée Ehresmann) consist of 7 books collecting all the articles published by the mathematician Charles Ehresmann (1905-1979), who created the Cahiers in 1958. The articles are followed by long comments (in English) to update and complement them.

Part I: 1-2. *Topologie et Géométrie Différentielle*
Part II: 1. *Structures locales*
2. *Catégories ordonnées; Applications en Topologie*
Part III: 1. *Catégories structurées et Quotients*
2. *Catégories internes ett Fibrations*
Part IV: 1. *Esquisses et Complétions.*
2. *Esquisses et structures monoïdales fermées*

Mme Ehresmann, Faculté des Sciences, LAMFA.
33 rue Saint-Leu, F-80039 Amiens. France.
ehres@u-picardie.fr

Commission paritaire n° 58964
ISSN 1245-530X (IMPRIME)
1SSN 2681-2363 (EN LIGNE)

.