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dirigés par Andrée CHARLES EHRESMANN

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## SOMMAIRE

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXIV-3 (2023)

# STONE DUALITY FOR TOPOLOGICAL CONVEXITY SPACES

*Toby Kenney*

**Résumé.** Un espace de convexité est un ensemble $X$ équipé d'une famille choisie de sous-ensembles (appelés les sous-ensembles convexes) fermée par intersections arbitraires et unions dirigées. On s'intéresse beaucoup aux espaces qui ont à la fois la structure d'espace de convexité et la structure d'espace topologique. Dans cet article, nous étudions la catégorie des espaces de convexité topologiques et étendons la dualité de Stone entre les coframes et les espaces topologiques á une adjonction entre la catégorie des espaces de convexité topologiques et la catégorie des treillis et des homomorphismes préservant le supremum. Cette adjonction peut etre factorisée à travers la catégorie des espaces de préconvexité (parfois appelés espaces de fermeture)
**Abstract.** A convexity space is a set $X$ with a chosen family of subsets (called convex subsets) that is closed under arbitrary intersections and directed unions. There is a lot of interest in spaces that have both a convexity space and a topological space structure. In this paper, we study the category of topological convexity spaces and extend the Stone duality between coframes and topological spaces to an adjunction between topological convexity spaces and sup-lattices. We factor this adjunction through the category of preconvexity spaces (sometimes called closure spaces).
**Keywords.** Stone duality; Topological Convexity Spaces; Sup-lattices; Preconvexity Spaces; Partial Sup-lattices
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## 1. Introduction

Stone duality is a contravariant equivalence of categories between categories of spaces and categories of lattices. The original Stone duality was between Stone spaces and Boolean algebras [15]. One of the most widely used extensions of Stone duality is between the categories of sober topological spaces and spatial coframes (or frames — since this is a 1-categorical duality, they are the same thing). This duality extends to an idempotent adjunction between topological spaces and coframes, given by the functors that send a topological space to its coframe of closed sets, and the functor that sends a coframe to its space of points.

In this paper, we develop an idempotent adjunction between topological convexity spaces and sup-lattices (the category whose objects are complete lattices, and morphisms are functions that preserve arbitrary suprema). Topological convexity spaces are sets equipped with both a chosen family of closed sets and a chosen family of convex sets. A canonical example is a metric space $X$ with the usual metric topology, and convex sets being sets closed under the betweenness relation given by $y$ is between $x$ and $z$ if $d(x, z) = d(x, y) + d(y, z)$. Many of the properties of metric spaces extend to topological convexity spaces. Homomorphisms of topological convexity spaces are continuous functions for which the inverse image of a convex set is convex.

Our approach to showing this adjunction goes via two equivalent intermediate categories. The first is the category of preconvexity spaces. A preconvexity space is a pair $(X, \mathcal{P})$ where $\mathcal{P}$ is a collection of subsets of $X$ that is closed under arbitrary intersections and empty unions. We will refer to sets $P \in \mathcal{P}$ as *preconvex* subsets of $X$. A homomorphism of preconvexity spaces $f : (X, \mathcal{P}) \longrightarrow (X', \mathcal{P}')$ is a function $f : X \longrightarrow X'$ such that for any $P \in \mathcal{P}'$, we have $f^{-1}(P) \in \mathcal{P}$. This category of preconvexity spaces was also studied by [4], and shown to be closed under arbitrary limits and colimits.

The second intermediate category that is equivalent to the category of $T_0$-preconvexity spaces, is a full subcategory of Distributive Partial Sup lattices. This category was studied in [11]. Objects of this category are complete lattices with a chosen family of suprema which distribute over arbitrary infima. Morphisms are functions that preserve all infima and the chosen suprema.

The motivation for partial sup lattices was an adjunction between partial sup lattices and preconvexity spaces, which is shown in [11].

Before we begin presenting the extension of Stone duality to topological convexity spaces, Section 2 provides a review of the main ingredients needed. While these reviews do not contain substantial new results, they are presented with a different focus from much of the literature, so we hope that the reviews offer a new perspective on these well-studied subjects. We first recap the basics of topological convexity spaces. We then review Stone duality for topological spaces. We then review the category of distributive partial sup-lattices. This category was defined in [11], with the motivation of modelling various types of preconvexity spaces. However, the definition presented in this review is changed from the original definition in that paper to make it cleaner in a categorical sense.

## 2. Preliminaries

### 2.1 Topological Convexity Spaces

**Definition 2.1.** *A* topological convexity space *is a triple* $(X, \mathcal{F}, \mathcal{C})$*, where* $X$ *is a set;* $\mathcal{F}$ *is a collection of subsets of* $X$ *that is closed under finite unions and arbitrary intersections, i.e. the collection of closed sets for some topology on* $X$*; and* $\mathcal{C}$ *is a collection of subsets of* $X$ *that is closed under directed unions and arbitrary intersections. Note that these include empty unions and intersections, so* $X$ *and* $\varnothing$ *are in both* $\mathcal{F}$ *and* $\mathcal{C}$*. Sets in* $\mathcal{F}$ *will be called* closed *subsets of* $X$ *and sets in* $\mathcal{C}$ *will be called* convex *subsets of* $X$*.*

The motivation here is that $(X, \mathcal{F})$ is a topological space, while $(X, \mathcal{C})$ is an abstract convexity space. Abstract convexity spaces are a generalisation of convex subsets of standard Euclidean spaces. Abstract convexity spaces were defined in [10], though in that paper, the definition did not require $\mathcal{C}$ to be closed under nonempty directed unions. Closure under directed unions was an additional property, called “domain finiteness”. Later authors incorporated closure under directed unions into the definition of an abstract convexity space, and used the term *preconvexity space* for a set with a chosen collection of subsets that is closed under arbitrary intersections and contains the empty set [4].

While the definition of an abstract convexity space captures many of the important properties of convex sets in geometry, it also allows a large number of interesting examples far beyond the original examples from classical geometry, including many examples from combinatorics and algebra. The resulting category of convexity spaces has many natural closure properties [4].

The definition above does not include any interaction between the topological and convexity structures on $X$. While it will be convenient to deal with such general spaces, it is also useful to include compatibility axioms between the convexity and topological structures. The following axioms from [16] are often used to ensure suitable compatibility between topology and convexity structure.

(i) All convex sets are connected.

(ii) All polytopes (convex closures of finite sets) are compact.

(iii) The hull operation is uniformly continuous relative to a metric which generates the topology.

We will modify the third condition to not require the topology to come from a metric space, giving the weaker condition that the convex closure operation preserves compact sets.

**Definition 2.2.** *We will call a topological convexity space* compatible *if it satisfies the two conditions*

*(i) All convex sets are connected.*

*(ii) The convex closure of a (topologically) closed compact set is (topologically) closed and compact.*

*We will call a topological convexity space* precompatible *if it satisfies the two conditions*

*(i) All convex sets are connected.*

*(ii') The convex closure of a finite set is (topologically) compact.*

At this point, we will introduce some notation for describing topological convexity spaces. For any subset $A \subseteq X$, we will write $[A]$ for the intersection of all convex sets containing $A$. To simplify notation, when $A$ is finite, we will write $[a_1, \ldots, a_n]$ instead of $[\{a_1, \ldots, a_n\}]$.

**Examples 2.3.**

1. If $(X, d)$ is a metric space, then setting $\mathcal{F}$ to be the closed sets for the metric topology, i.e.

$$\mathcal{F} = \left\{ A \subseteq X \middle| (\forall x \in X) \left( \bigwedge_{y \in A} d(x, y) = 0 \Rightarrow x \in A \right) \right\}$$

and

$$\mathcal{C} = \{A \subseteq X | (\forall x, y, z \in X) ((x, z \in A \wedge d(x, z) = d(x, y) + d(y, z)) \Rightarrow y \in A)\}$$

we have that $(X, \mathcal{F}, \mathcal{C})$ is a topological convexity space. To ensure that convex sets are connected, we will often assume that geodesics exist — that is, for any $r < d(x, y)$, there is some $z \in [x, y]$ such that $d(x, z) = r$ and $d(y, z) = d(x, y) - r$, to ensure that convex sets are connected. We will usually also require that open balls are convex, and that the set $\{z \in X | d(x, y) = d(x, z) + d(z, x)\}$ is convex (and therefore the interval $[x, y]$). For common examples where these conditions hold, the convex closure of a compact set is compact, so that the space is compatible. However, it is not easy to prove compatibility of these spaces under simple conditions, or to find examples of metric spaces where this structure is not compatible.

2. Let $L$ be a complete lattice. We define a topological convexity space structure by

$$\mathcal{F} = \left\{ \bigcap_{i \in I} F_i \middle| (\forall i \in I)(\exists x_1, \ldots, x_{n_i} \in X)(F_i = \downarrow\{x_1, \ldots, x_{n_i}\}) \right\} \cup \{\varnothing\}$$

and

$$\mathcal{C} = \{I \subseteq X | (\forall x_1, x_2 \in I) ((\forall y \leqslant x_1)(y \in I) \wedge (x_1 \vee x_2 \in I))\}$$

That is, $\mathcal{F}$ is the set of arbitrary intersections of finitely generated downsets, plus the emptyset (which are the closed sets for the weak topology [9]) and $\mathcal{C}$ is the set of (possibly empty) ideals of $L$. This topological convexity space is precompatible. To prove connectedness of convex sets, we want to show that an ideal cannot be covered by two disjoint weak-closed sets. Suppose $U$ and $V$ are disjoint weak-closed sets that cover $I$. Let $a \in I \cap U$ and $b \in I \cap V$. Then $a \vee b \in I$, and if $a \vee b \in U$, then $b \in U$ contradicting disjointness of $U$ and $V$. Similarly if $a \vee b \in V$ then $a \in V$. This contradicts disjointness of $U$ and $V$. The ideal generated by a finite set of elements in $L$ is clearly principal, and therefore closed and compact. $L$ is not in general compatible, since, for example, if $L$ is the powerset of $\mathbb{N}$, then singletons in $L$ are weak-closed, since for any set $X' \subseteq X$ containing two elements $a$ and $b$, the downset $\downarrow\{\{a\}^c, \{b\}^c\}$ is finitely generated, and contains all singletons, but does not contain $X'$.

3. Let $n \in \mathbb{Z}^+$ be a positive integer. Let $S_n$ be the group of permutations on $n$ elements. Let $\mathcal{F}$ consist of all subsets of $S_n$, and for any partial order $\preceq$ on $n$, let

$$P_{\preceq} = \{\sigma \in S_n | (\forall i, j \in \{1, \ldots, n\})(i \preceq j \Rightarrow \sigma(i) \leqslant \sigma(j))\}$$

where $\leqslant$ is the usual total order on $\mathbb{Z}^+$. That is $P_{\preceq}$ is the set of permutations $\sigma$ such that $\preceq$ is contained in $\sigma^{-1}(\leqslant)$. let

$$\mathcal{C} = \{P_{\preceq} | \preceq \text{ is a partial order on } \{1, \ldots, n\}\} \cup \{\varnothing\}$$

Since $S_n$ is finite, to prove that $(S_n, \mathcal{F}, \mathcal{C})$ is a convexity space, we just need to show that $\mathcal{C}$ is closed under intersection. This is straightforward. Since partial orders are closed under intersection, the poset of partial orders on $\{1, \ldots, n\}$, with a top element adjoined, is a lattice. Thus the intersection $P_{\preceq} \cap P_{\sqsubseteq} = P_{\preceq \vee \sqsubseteq}$, so $\mathcal{C}$ is closed under intersection. This is a metric topology, with the metric given by $d(\sigma, \tau)$ is the Cayley distance from $\sigma$ to $\tau$, under the Coxeter generators. That is, $d(\sigma, \tau)$ is the length of the shortest word equal to $\tau\sigma^{-1}$ in the generators $\{\tau_i | i = 1, \ldots, n-1\}$, where

$$\tau_i(j) = \begin{cases} i+1 & \text{if } j = i \\ i & \text{if } j = i+1 \\ j & \text{otherwise} \end{cases}$$

is the transposition of $i$ and $i + 1$.

4. If $G$ is a topological group, or more generally a universal algebra equipped with a suitable topology, then we can define a topological convexity space by making subgroups (or more generally subalgebras) and the empty set convex, and keeping the closed sets from the topology.

Having defined the objects in the category of topological convexity spaces, we need to define the morphisms.

**Definition 2.4.** *A* homomorphism $f : (X, \mathcal{F}, \mathcal{C}) \longrightarrow (X', \mathcal{F}', \mathcal{C}')$ *between topological convexity spaces is a function* $f : X \longrightarrow X'$ *such that for every* $F \in \mathcal{F}'$, $f^{-1}(F) \in \mathcal{F}$ *and for every* $C \in \mathcal{C}'$, $f^{-1}(C) \in \mathcal{C}$.

The condition that $f^{-1}(F) \in \mathcal{F}$ is the condition that $f$ is continuous as a function between topological spaces. The condition that $f^{-1}(C) \in \mathcal{C}$ is called *monotone* by [4], by analogy with the example of endofunctions of the real numbers. This was in the context of convexity spaces without topological structure. Dawson [4] uses the term $\mathcal{A}lign$ for the category of convexity spaces and monotone homomorphisms, and $\mathcal{C}onvex$ for the category of convexity spaces and functions whose forward image preserves convex sets. However, this terminology has not been widely used, and later authors have all considered the monotone homomorphisms as the natural homomorphisms of abstract convexity spaces. In the case of topological convexity spaces, the monotone condition is an even more natural choice because it aligns well with the continuity condition and leads to the Stone duality extension that we show in this paper.

**Examples 2.5.**

1. For the topological convexity space coming from a metric space, such that intervals are of the form $[x, y] = \{z \in X | d(x, z) + d(y, z) = d(x, y)\}$, a homomorphism is a function $f : X \longrightarrow Y$ such that whenever $d(x, z) = d(x, y) + d(y, z)$, we have $d(f(x), f(z)) = d(f(x), f(y)) + d(f(y), f(z))$. That is, $f$ embeds geodesics from $X$ into the geodesics in $Y$. To see that homomorphisms have this property, we have that $f^{-1}([f(x), f(z)])$ is convex, and contains $x$ and $z$, so if $d(x, z) =$

$d(x,y) + d(y,z)$, then $f^{-1}([f(x), f(z)])$ must contain $y$. This means $f(y) \in [f(x), f(z)] = \{v | d(f(x), f(z)) = d(f(x), v) + d(v, f(z))\}$. Conversely, if $f$ has the given property, then for any convex $A \subseteq Y$, if $x, z \in f^{-1}(A)$, then for any $y$ such that $d(x,z) = d(x,y) + d(y,z)$, we have $d(f(x), f(z)) = d(f(x), f(y)) + d(f(y), f(z))$, so by convexity, $f(y) \in A$, making $y \in f^{-1}(A)$, so $f^{-1}(A)$ is convex.

2. If $L$ and $M$ are complete lattices with the weak topology and convex sets are ideals, then topological convexity space homomorphisms from $L$ to $M$ are exactly sup-homomorphisms. To see this, let $f : L \longrightarrow M$ be a sup-homomorphism. Let $I \subseteq M$ be an ideal. Since $f$ is order-preserving, $f^{-1}(I)$ is clearly a downset, and for $a, b \in f^{-1}(I)$, $f(a \vee b) = f(a) \vee f(b) \in I$. Since inverse image preserves intersection, it is sufficient to show that the inverse image of a finitely-generated downset $F \subseteq M$ is weak-closed. Let $F = \downarrow\{m_1, \ldots, m_n\}$. For $i = 1, \ldots, n$, let $l_i = f_*(m_i)$, where $f_*$ is the order-theoretic right adjoint of $f$ (which exists because $f$ is a sup-homomorphism). We have $f(x) \leqslant m_i$, if and only if $x \leqslant l_i$. Thus, $f^{-1}(F) = \downarrow\{l_1, \ldots, l_n\}$. Conversely suppose $f : L \longrightarrow M$ is a topological convexity space homomorphism. Weak-closed ideals are easily seen to be principal ideals, since if $I$ is an ideal, and $I \subseteq \downarrow\{x_1, \ldots, x_n\}$, then if there are elements $y_i \in I$ with $y_i \nleqslant x_i$, then $y_1 \vee \cdots \vee y_n$ cannot be in $\downarrow\{x_1, \ldots, x_n\}$, which is a contradiction, so we must have $I \subseteq \downarrow x_i$ for some $i \in \{1, \ldots, n\}$. Thus the inverse image of a principal ideal is another principal ideal. In particular, $f^{-1}(\downarrow \bigvee\{f(a) | a \in A\}$ is a principal ideal containing $A$, so it contains $\bigvee A$, and thus $f(\bigvee A) \leqslant \bigvee\{f(a) | a \in A\}$ as required.

For the partial order convexity on $S_n$ from Example 2.3.3, describing the topological convexity space morphisms is more challenging. We start by looking at half-spaces (convex sets with convex complements). Half-spaces of $S_n$ are of the form $C_{ij} = P_{\preceq}$, where $\preceq$ is the partial order where the only non-trivial comparison is $i \preceq j$. That is, $C_{ij} = \{\sigma \in S_n | \sigma(i) \leqslant \sigma(j)\}$. We first consider automorphisms:

**Lemma 2.6.** *If $i$, $j$, $k$ and $l$ are distinct, then the only half-spaces that contain $C_{ij} \cap C_{kl}$ are $C_{ij}$ and $C_{kl}$.*

*Proof.* For any half-space $C_{st} \notin \{C_{ij}, C_{kl}\}$, we need to find some $\sigma \in C_{ij} \cap C_{kl}$ with $\sigma \notin C_{st}$. Suppose $s = j$ and $t \neq i$, then we can find a permutation

$\sigma$ such that $\sigma(i) < \sigma(j) < \sigma(t) < \sigma(k) < \sigma(l)$. This $\sigma$ is in $\mathcal{C}_{ij} \cap C_{kl}$, but not in $C_{st}$ as required. Similar permutations work for all combinations. □

**Lemma 2.7.** *An automorphism $f : (S_n, P(S_n), \mathcal{C}) \longrightarrow (S_n, P(S_n), \mathcal{C})$ is of the form $f(\sigma) = \theta\sigma\tau$ for some $\tau \in S_n$ and some $\theta \in \{e, \rho\}$ where $e$ is the identity permutation and $\rho$ is the permutation which reverses the order of all elements.*

*Proof.* It is easy to see that for $\tau \in S_n$, $f_\tau$ given by $f_\tau(\sigma) = \sigma\tau$ is an automorphism of $(S_n, P(S_n), \mathcal{C})$. Now we consider the stabiliser of the identity element. Since $\{C_{i(i+1)} | i = 1, \ldots, (n-1)\}$ is the only set of $n-1$ half-spaces whose intersection contains only the identity permutation, any automorphism which fixes the identity permutation must fix this set. Furthermore, since $C_{(i-1)i} \cap C_{i(i+1)} \subseteq C_{(i-1)(i+1)}$, it follows that

$$f^{-1}(C_{(i-1)i}) \cap f^{-1}(C_{i(i+1)}) \subseteq f^{-1}(C_{(i-1)(i+1)})$$

Since $f$ is an automorphism, $f^{-1}(C_{(i-1)(i+1)})$ cannot be either $f^{-1}(C_{(i-1)i})$ or $f^{-1}(C_{i(i+1)})$. By Lemma 2.6, it follows that $f^{-1}(C_{(i-1)i})$ and $f^{-1}(C_{i(i+1)})$ are adjacent half-spaces. Since the set of half-spaces

$$\{C_{i(i+1)} | i = 1, \ldots, (n-1)\}$$

is permuted by $f^{-1}$, the only possible permutations are the identity and the reversal $C_{i(i+1)} \mapsto C_{(n-i)(n+1-i)}$. This reversal sends a permutation $\sigma$ to $\rho\sigma\rho$.

We want to show that these are the only elements in the stabiliser of the identity. By applying $\rho\sigma\rho$ if necessary, we can change an element in the stabiliser of $e$ to one such that $f^{-1}$ fixes every $C_{i(i+1)}$. Now $C_{i(i+2)}$ is the unique half-space that contains $C_{i(i+1)} \cap C_{(i+1)(i+2)}$ that is not equal to either $C_{i(i+1)}$ or $C_{(i+1)(i+2)}$, so it is also fixed by $f^{-1}$. By induction, we can show that every $C_{ij}$ is fixed by $f^{-1}$, and thus $f$ is the identity. □

**Proposition 2.8.** *$f \colon S_n \longrightarrow S_m$ is a surjective topological convexity space homomorphism, if and only if there is an injective function $g : m \longrightarrow n$, such that $f$ is either given by*

1. *$f(\tau)(i) = |\{j \in \{1, \ldots, m\} | \tau(g(j)) \leqslant \tau(g(i))\}|$. That is, $f(\tau)$ is the automorphism part of the automorphism—order-preserving-inclusion factorisation of $\tau g$.*

$$\begin{tikzcd} n \arrow[r, "\tau"] & n \\ m \arrow[u, tail, "g"] \arrow[r, "f(\tau)"'] & m \arrow[u, tail, "i"'] \end{tikzcd}$$

*or*

2. $f(\tau)(i) = |\{j \in \{1,\ldots,m\}|\tau(g(j)) \geqslant \tau(g(i))\}|$. *That is, $f(\tau)$ is the automorphism part of the automorphism-order-preserving-inclusion factorisation of $\rho\tau g$, where $\rho$ is the order-reversing permutation on $n$.*

$$\begin{tikzcd} n \arrow[r, "\tau"] & n \arrow[r, "\rho"] & n \\ m \arrow[u, tail, "g"] \arrow[rr, "f(\tau)"'] & & m \arrow[u, tail, "i"'] \end{tikzcd}$$

*Proof.* Firstly, we show that for an injective function $g : m \longrightarrow n$, both the functions

$$\alpha_g(\sigma)(i) = |\{j \in \{1,\ldots,m\}|\sigma(g(j)) \leqslant \sigma(g(i))\}|$$

and

$$\delta_g(\sigma)(i) = |\{j \in \{1,\ldots,m\}|\sigma(g(j)) \geqslant \sigma(g(i))\}|$$

are surjective homomorphisms. We see that for any $i \neq j \in \{1,\ldots,m\}$,

$${\alpha_g}^{-1}(C_{ij}) = \{\sigma \in S_n|\alpha_g(\sigma)(i) < \alpha_g(\sigma)(j)\} = \{\sigma \in S_n|\sigma(g(i)) < \sigma(g(j))\} = C_{g(i)g(j)}$$

and

$${\delta_g}^{-1}(C_{ij}) = \{\sigma \in S_n|\delta_g(\sigma)(i) < \delta_g(\sigma)(j)\} = \{\sigma \in S_n|\sigma(g(i)) > \sigma(g(j))\} = C_{g(j)g(i)}$$

so $\alpha_g$ and $\delta_g$ are homomorphisms. For surjectivity, let $\phi \in S_m$. We need to show that $\phi = \alpha_g(\tau)$ for some $\tau \in S_n$. Given the injections $m \overset{g}{\rightarrowtail} n$ and $m \xrightarrow{\phi} m \xrightarrow{i} n$ for any injective order-preserving $m \overset{i}{\rightarrowtail} n$, $n \overset{g}{\leftarrowtail} m \overset{i\phi}{\rightarrowtail} n$ is a partial permutation of $n$, so it extends to a full permutation $\tau$ with $\alpha_g(\tau) = \phi$. Similarly, we have $\delta_g(\rho\tau) = \alpha_g(\tau) = \phi$, so $\alpha_g$ and $\delta_g$ are both surjective.

Conversely, let $f : S_n \longrightarrow S_m$ be a surjective homomorphism. Since $\{e\}$ is convex, where $e$ is the identity homomorphism, $f^{-1}(\{e\})$ is convex. Furthermore, $f^{-1}(\{e\}) = \bigcap_{i<j} f^{-1}(C_{ij})$. Since $f^{-1}$ preserves convex sets, for every $i \neq j \in \{1, \ldots, m\}$ $f^{-1}(C_{ij}) = C_{st}$ for some $s, t \in \{1, \ldots, n\}$. Furthermore, $f^{-1}(C_{ij} \cap C_{jk}) = C_{st} \cap C_{tu}$. Thus, we have $f^{-1}(\{e\}) = C_{i_1 i_2 \ldots i_m} = C_{i_1 i_2} \cap C_{i_2 i_3} \cdots \cap C_{i_{m-1} i_m}$. If $f^{-1}(C_{12}) = C_{i_1 i_2}$, then we can define $g(j) = i_j$, and we have that $f = \alpha_g$. If on the other hand $f^{-1}(C_{12}) = C_{i_{m-1} i_m}$, then we let $g(j) = i_{m+1-j}$ and we have $f = \delta_g$.

□

Describing general homomorphisms between these topological convexity spaces is more difficult, and outside the scope of this paper.

## 3. Preconvexity Spaces and the Adjunction with Topological Convexity Spaces

**Definition 3.1.** *A* preconvexity space *(sometimes called a* closure space*) is a pair $(X, \mathcal{P})$, where $X$ is a set and $\mathcal{P}$ is a collection of subsets of $X$ that is closed under arbitrary intersections and contains the empty set (since $X$ is an empty intersection, we also have $X \in \mathcal{P}$).*

This was [10]'s original definition of a convexity space. However, later authors decided that closure under directed unions should be a required property for a convexity space, and [4] introduced the term preconvexity space for these spaces that do not require closure under directed unions.

**Definition 3.2.** *A* homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ *of preconvexity spaces is a function $X \xrightarrow{f} X'$ such that for any preconvex set $P \in \mathcal{P}'$, the inverse image $f^{-1}(P) \in \mathcal{P}$.*

**Examples 3.3.**

If $(X, \mathcal{F}, \mathcal{C})$ is a topological convexity space, then $(X, \mathcal{F} \cap \mathcal{C})$ is a preconvexity space. The underlying function of any topological convexity space homomorphism $(X, \mathcal{F}, \mathcal{C}) \xrightarrow{f} (X', \mathcal{F}', \mathcal{C}')$ is a preconvexity homomorphism. Conversely, if $\mathcal{C}'$ consists of directed unions from $\mathcal{F}' \cap \mathcal{C}'$, and $\mathcal{F}'$

consists of intersections of finite unions from $\mathcal{F}' \cap \mathcal{C}'$, then any preconvexity homomorphism $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{g} (X', \mathcal{F}' \cap \mathcal{C}')$ is a topological convexity homomorphism.

Example 3.3 gives a functor $\mathcal{C}onvex\mathcal{T}op \xrightarrow{CC} \mathcal{P}reconvex$ that sends every topological convexity space to the preconvexity space of closed convex sets. The action on morphisms simply reinterprets the topological convexity homomorphism as a preconvexity homomorphism.

This closed-convex functor has a right adjoint, $IS$, which sends the preconvexity space $(X, \mathcal{P})$ to $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ where $\widetilde{\mathcal{P}}$ is the closure of $\mathcal{P}$ under directed unions, and $\overline{\mathcal{P}}$ is the closure of $\mathcal{P}$ under finite unions and arbitrary intersections. We will show that this defines a topological convexity space and is a right adjoint.

**Lemma 3.4.** *For any preconvexity space $(X, \mathcal{P})$, the set $\widetilde{\mathcal{P}}$ is the collection $\{\bigcup \mathcal{D} | \mathcal{D} \subseteq \mathcal{P}$ directed$\}$.*

*Proof.* Let $\mathcal{Q} = \{\bigcup \mathcal{D} | \mathcal{D} \subseteq \mathcal{P} \text{ directed}\}$. We need to show that $\mathcal{Q}$ is closed under directed unions. Let $\mathcal{D} \subseteq \mathcal{Q}$ be directed. For each $D \in \mathcal{D}$, there is a directed $\mathcal{D}_D \subseteq \mathcal{P}$ such that $D = \bigcup \mathcal{D}_D$. Let $\widetilde{\mathcal{D}}$ be the closure of $\bigcup \{\mathcal{D}_D | D \in D\}$ under finite joins in $\mathcal{P}$ (which exist because $\mathcal{P}$ is closed under arbitrary intersections). By definition, $\widetilde{\mathcal{D}}$ is directed. We will show that $\bigcup \mathcal{D} = \bigcup \widetilde{\mathcal{D}}$. Suppose $x \in \bigcup \mathcal{D}$. Then there is some $D \in \mathcal{D}$ with $x \in D$, and since $D = \bigcup \mathcal{D}_D$, there is some $P \in \mathcal{D}_D \subseteq \widetilde{\mathcal{D}}$ with $x \in P$, so $x \in \bigcup \widetilde{\mathcal{D}}$. Conversely, if $x \in \bigcup \widetilde{\mathcal{D}}$, then there is some $P_1, \ldots, P_n \in \bigcup \{\mathcal{D}_D | D \in \mathcal{D}\}$ such that $x \in P_1 \vee \cdots \vee P_n$. Now let each $P_i \in \mathcal{D}_{D_i}$ for some $D_i \in \mathcal{D}$. This means that $P_i \subseteq D_i$. Since $\mathcal{D}$ is directed, there is an element of $\mathcal{D}$ that contains $D_1, \ldots, D_n$, and which must therefore contain $P_1 \vee \cdots \vee P_n$. □

**Lemma 3.5.** *For any preconvexity space $(X, \mathcal{P})$, the set $\widetilde{\mathcal{P}}$ is closed under directed unions and arbitrary intersections.*

*Proof.* By definition, $\widetilde{\mathcal{P}}$ is closed under directed unions, so we just need to show that it is closed under intersections. Let $\{P_i | i \in I\}$ be a family of elements of $\widetilde{\mathcal{P}}$. By definition, for every $i \in I$, there is a directed $\mathcal{D}_i \subseteq \mathcal{P}$ with $P_i = \bigcup \mathcal{D}_i$. W.l.o.g. assume every $\mathcal{D}_i$ is down-closed in $\mathcal{P}$. We will show that

$$\bigcap_{i\in I} P_i = \bigcup_{\substack{f:I\longrightarrow\mathcal{P}\\(\forall i\in I)f(i)\in\mathcal{D}_i}} \bigcap_{i\in I} f(i) \tag{1}$$

That is, the intersection of the family $\{P_i|i \in I\}$ is the union over all choice functions $f$, of the intersection of $\{f(i)|i \in I\}$. Every $f(i) \in \mathcal{P}$, so this intersection $\bigcap_{i\in I} f(i)$ is also in $\mathcal{P}$, and the set of choice functions is directed, since every $\mathcal{D}_i$ is directed and down-closed, so for choice functions $f, g : I \longrightarrow \mathcal{P}$ the join $(f \vee g)(i) = f(i) \vee g(i)$ is also a choice function. Equation (1) therefore shows that $\bigcap_{i\in I} P_i \in \widetilde{\mathcal{P}}$.

To prove Equation (1), first let $x \in \bigcap_{i\in I} P_i$. Since $(\forall i)(x \in P_i)$, and $P_i = \bigcup \mathcal{D}_i$, there is some $D_{i,x} \in \mathcal{D}_i$ with $x \in D_{i,x}$. Thus, we can take the choice function $f_x(i) = D_{i,x}$, and deduce $x \in \bigcap_{i\in I} f_x(i)$. Conversely, let

$$x \in \bigcup_{\substack{f:I\longrightarrow\mathcal{P}\\(\forall i\in I)f(i)\in\mathcal{D}_i}} \bigcap_{i\in I} f(i)$$

There must be some choice function $f$ with $x \in \bigcap_{i\in I} f(i)$. Since $f(i) \in \mathcal{D}_i$, it follows that $f(i) \subseteq P_i$, so $x \in P_i$ for every $i \in I$. Thus $x \in \bigcap_{i\in I} P_i$. □

**Remark 3.6.** The proof of Lemma 3.5 does not actually require the axiom of choice, because there are canonical choices for all choice functions needed — for each $P_i$, we need to choose a directed family $\mathcal{D}_i$ with $P_i = \bigcup \mathcal{D}_i$. We can let $\mathcal{D}_i = \{P \in \mathcal{P}|P \subseteq P_i\}$, and since every $\mathcal{D}_i$ is a downset, we can set $D_{i,x} = \overline{\{x\}}$ for every $i \in I$, where $\overline{\{x\}}$ is the convex-closed closure of $\{x\}$.

**Lemma 3.7.** *Every $F \in \overline{\mathcal{P}}$ is of the form $\bigcap \mathcal{F}$, where*

$$\mathcal{F} \subseteq \{P_1 \cup \cdots \cup P_n|P_1, \ldots, P_n \in \mathcal{P}\}$$

*Proof.* Let $\widehat{\mathcal{P}} = \{P_1 \cup \cdots \cup P_n|P_1, \ldots, P_n \in \mathcal{P}\}$ be the set of finite unions from $\mathcal{P}$. We need to show that the set $\left\{\bigcap \mathcal{F}|\mathcal{F} \subseteq \widehat{\mathcal{P}}\right\}$ is closed under finite unions. (By definition, it is closed under arbitrary intersections.) Let $F_1 = \bigcap \mathcal{F}_1$ and $F_2 = \bigcap \mathcal{F}_2$ for $\mathcal{F}_1, \mathcal{F}_2 \subseteq \widehat{\mathcal{P}}$. Let

$$\mathcal{F}_{12} = \{P_1 \cup P_2|P_1 \in \mathcal{F}_1, P_2 \in \mathcal{F}_2\}$$

We will show that $F_1 \cup F_2 = \bigcap \mathcal{F}_{12}$. Clearly, for every $P_1 \in \mathcal{F}_1$, and $P_2 \in \mathcal{F}_2$, we have $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$, so $F_1 \cup F_2 \subseteq P_1 \cup P_2$. Conversely, suppose $x \notin F_1 \cup F_2$. Then there is some $P_1 \in \mathcal{F}_1$ and some $P_2 \in \mathcal{F}_2$ with $x \notin P_1$ and $x \notin P_2$. It follows that $x \notin P_1 \cup P_2 \in \mathcal{F}_{12}$, so $x \notin \bigcap \mathcal{F}_{12}$. □

**Lemma 3.8.**

1. *For a set $X$, the identity function on $X$ is a preconvexity homomorphism $(X, \mathcal{P}) \longrightarrow (X, \mathcal{P}')$ if and only if $\mathcal{P}' \subseteq \mathcal{P}$.*

2. *For a set $X$, the identity function on $X$ is a topological convexity homomorphism $(X, \mathcal{F}, \mathcal{C}) \longrightarrow (X, \mathcal{F}', \mathcal{C}')$ if and only if $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{C}' \subseteq \mathcal{C}$.*

*Proof.* This is immediate from the definition. □

**Proposition 3.9.** *The assignment $IS$ that sends the preconvexity space $(X, \mathcal{P})$ to the topological convexity space $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ and the preconvexity homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ to $f$ considered as a topological convexity homomorphism, is a functor, and is right adjoint to the functor $CC$ : $\mathcal{C}onvex\mathcal{T}op \longrightarrow \mathcal{P}reconvex$.*

*Proof.* Because the forgetful functor to **Set** sends $IS$ to the identity functor, the functoriality of $IS$ is automatic provided it is well-defined. That is, if any preconvexity homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ is a topological convexity homomorphism from $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ to $(X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$. For the adjunction, we need to demonstrate that for any topological convexity space $(X, \mathcal{F}, \mathcal{C})$ and any preconvexity space $(X', \mathcal{P}')$, a function $f : X \longrightarrow X'$ is a topological convexity space homomorphism $(X, \mathcal{F}, \mathcal{C}) \xrightarrow{f} (X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$ if and only if it is a preconvexity homomorphism $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{f} (X', \mathcal{P}')$. The "only if" part is obvious.

Suppose $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{f} (X', \mathcal{P}')$ is a preconvexity homomorphism. Let $F \in \overline{\mathcal{P}'}$. We want to show that $f^{-1}(F) \in \mathcal{F}$. Now $F \in \overline{\mathcal{P}'}$ means $F = \bigcap \mathcal{U}$ where $\mathcal{U} \subseteq \widehat{\mathcal{P}'}$. Now if $P_1 \cup \cdots \cup P_n \in \widehat{\mathcal{P}'}$, then $f^{-1}(P_1 \cup \cdots \cup P_n) = f^{-1}(P_1) \cup \cdots \cup f^{-1}(P_n)$ is a finite union of sets from $\mathcal{F} \cap \mathcal{C}$, so since $\mathcal{F}$ is closed under finite unions, $f^{-1}(P_1 \cup \cdots \cup P_n) \in \mathcal{F}$. Therefore $f^{-1}(F) =$

$\bigcap\{f^{-1}U|U \in \mathcal{U}\}$ and $\{f^{-1}U|U \in \mathcal{U}\} \subseteq \mathcal{F}$, so as $\mathcal{F}$ is closed under arbitrary intersections, $f^{-1}(F) \in \mathcal{F}$. Similarly, let $C = \bigcup \mathcal{D}$, where $\mathcal{D} \subseteq \mathcal{P}'$ is a directed downset. For every $D \in \mathcal{D}$, we have $f^{-1}(D) \in \mathcal{C}$, and for any $D_1, D_2 \in \mathcal{D}$, there is some $D_{12} \in \mathcal{D}$ with $D_1 \subseteq D_{12}$ and $D_2 \subseteq D_{12}$. It follows that $f^{-1}(D_1) \subseteq f^{-1}(D_{12})$ and $f^{-1}(D_2) \subseteq f^{-1}(D_{12})$. Therefore, $\{f^{-1}(D)|D \in \mathcal{D}\}$ is directed. Now

$$f^{-1}(C) = f^{-1}(\bigcup \mathcal{D}) = \bigcup\{f^{-1}(D)|D \in \mathcal{D}\}$$

Since $\{f^{-1}(D)|D \in \mathcal{D}\} \subseteq \mathcal{C}$, and $\mathcal{C}$ is closed under directed unions, it follows that $f^{-1}(C) \in \mathcal{C}$. Thus $f$ is a homomorphism of topological convexity spaces.

Well-definedness of the functor $IS$ also follows from the adjunction, because $\mathcal{P} \subseteq \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}$, so the identity function on $X$ is always a preconvexity homomorphism $(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}) \xrightarrow{i} (X, \mathcal{P})$. Thus the composite

$$(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}) \xrightarrow{i} (X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$$

is a preconvexity homomorphism, so by the adjunction, it is a topological convexity space homomorphism $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}}) \xrightarrow{f} (X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$

□

**Corollary 3.10.** *The adjunction $CC \dashv IS$ is idempotent.*

*Proof.* The counit and unit of the adjunction are both the identity function viewed as a homomorphism in the relevant category. The triangle identities for the adjunction therefore give an isomorphism of spaces, showing that the adjunction is idempotent. □

For an idempotent adjunction, a natural question is what are the fixed points?

**Proposition 3.11.** *A topological convexity space $X = (X, \mathcal{F}, \mathcal{C})$ satisfies $IS \circ CC(X) = X$ if and only if $X$ satisfies the conditions:*

1. *Every convex set is a directed union of closed convex sets.*
2. *For every $V \in \mathcal{F}$ and any $x \in X \backslash V$, there are sets $C_1, \dots, C_n \in \mathcal{F} \cap \mathcal{C}$ such that $V \subseteq C_1 \cup \dots \cup C_n$ and $x \notin C_1 \cup \dots \cup C_n$.*

*Proof.* The counit of the adjunction is the identity function on the underlying sets. Thus $\overline{(\mathcal{F} \cap \mathcal{C})} \subseteq \mathcal{F}$ and $\widetilde{(\mathcal{F} \cap \mathcal{C})} \subseteq \mathcal{C}$. Let $A \in \mathcal{C}$ be convex in $X$. By Condition 1, $A$ is a directed union of sets in $\mathcal{F} \cap \mathcal{C}$. By definition, this is in $\widetilde{(\mathcal{F} \cap \mathcal{C})}$.

Now let $V \in \mathcal{F}$. For any $W = C_1 \cup \cdots \cup C_n$ with $C_i \in \mathcal{F} \cap \mathcal{C}$, $W \in \overline{(\mathcal{F} \cap \mathcal{C})}$ by definition. Thus, by Condition 2, for every $x \in X \backslash V$, there is some $W \in \overline{(\mathcal{F} \cap \mathcal{C})}$ with $V \subseteq W$ and $x \notin W$. Now, clearly $V$ is the intersection of all these $W$ for all $x \notin V$. Since $\overline{(\mathcal{F} \cap \mathcal{C})}$ is closed under arbitrary intersections, this implies $V \in \overline{(\mathcal{F} \cap \mathcal{C})}$.

Conversely, if $X$ is a fixed point of the adjunction, i.e. $IS \circ CC(X) = X$, then $\mathcal{C} = \widetilde{(F \cap C)}$, which is exactly Condition 1. Also $\mathcal{F} = \overline{(F \cap C)}$, meaning that for every $V \in \mathcal{F}$, we have $V = \bigcap \mathcal{U}$ where $\mathcal{U}$ is a family of finite unions of sets from $\mathcal{F} \cap \mathcal{C}$. Since $V = \bigcap \mathcal{U}$, for any $x \notin V$, there is some $U \in \mathcal{U}$ with $x \notin U$. By definition, $U = C_1 \cup \cdots \cup C_n$ for some $C_1, \ldots, C_n \in \mathcal{F} \cap \mathcal{C}$, which is Condition 2. $\square$

We will call a topological convexity space *teetotal* if the conditions of Proposition 3.11 hold. The teetotal conditions are closely related to the compatible conditions from Definition 2.2. However, there are compatible spaces which are not teetotal.

**Example 3.12.** $l^2$ is the vector-space of square-summable sequences of real numbers, with the $l^2$ norm. Since $l^2$ is a metric space, it is easy to check that it is a compatible topological convexity space.

Let $F$ be the unit sphere, which is a closed set, and let $x = 0$. In order for $l^2$ to be teetotal, we need to find a finite family of closed convex subsets $C_1, \ldots, C_n$ such that $F \subseteq C_1 \cup \cdots \cup C_n$ and $x \notin C_1 \cup \cdots \cup C_n$. For closed convex $C_i$ and $x \notin C_i$, since $C_i$ is closed, there is an open ball containing $x$ disjoint from $C_i$. Let $d = \sup\{r \in \mathbb{R} | B(x, r) \cap C_i = \varnothing\}$ be the distance from $x$ to $C_i$. Since $B(x, d)$ is the directed union of $\{B(x, r) | r < d\}$, it follows that $B(x, d) \cap C_i = \varnothing$.

We first show that if $C$ is a closed convex set that does not contain 0, then there is a unique $y \in C$ that minimises $\|y\|$. If there is no $y \in C$ that minimises $\|y\|$, then there must be a sequence $a_1, a_2, \ldots \in C$ such that $\|a_i\|$ is strictly decreasing and

$$\lim_{n \to \infty} \|a_n\| = \inf_{y \in C} \|y\|$$

Since $[a_1, \ldots, a_n]$ is compact for every $n$, there is a point $b_n \in [a_1, \ldots, a_n]$ that minimises $\|b\|$. In particular, this means that for any $i < n$ and any $0 < \epsilon < 1$, $\|b_n + \epsilon(b_i - b_n)\| \geqslant \|b_n\|$. Squaring both sides gives

$$2\epsilon\langle b_i, b_n\rangle - 2\epsilon\langle b_n, b_n\rangle + \epsilon^2\langle b_i - b_n, b_i - b_n\rangle > 0$$

Taking the limit as $\epsilon \to 0$ gives $\langle b_i, b_n\rangle > \langle b_n, b_n\rangle$. Thus

$$\begin{aligned}\|b_i - b_n\|^2 &= \|b_i\|^2 + \|b_n\|^2 - 2\langle b_i, b_n\rangle \\ &\leqslant \|b_i\|^2 - \|b_n\|^2\end{aligned}$$

Since $\|b_n\|^2$ is a decreasing sequence, bounded below by 0, it converges to some limit $r$. Thus $\|b_i - b_n\|^2 \leqslant \|b_i\|^2 - r$ for any $i < n$. Thus $b_n$ is a Cauchy sequence, so it converges to some limit $b_\infty$. Now since $C$ is closed, $b_\infty \in C$, and

$$\|b_\infty\| = \lim_{n\to\infty}\|b_n\| = \inf_{y\in C}\|y\|$$

Thus $b_\infty$ is a nearest point in $C$ to 0. If $y$ is another point with minimal norm, then $\frac{y+b_\infty}{2}$ must have smaller norm. Thus $b_\infty$ is the unique point with smallest norm.

Now for any $y \in C$, since $C$ is convex, we have that $\|b_\infty + \epsilon(y - b_\infty)\| > \|b_\infty\|$, and by the above argument, $\langle y, b_\infty\rangle \geqslant \langle b_\infty, b_\infty\rangle$. Thus $C \subseteq \{x \in l^2 | \langle x, b_\infty\rangle > \frac{1}{2}\|b_\infty\|^2\}$. That is, every closed convex set is contained in an open half-space that does not contain $x = 0$.

We can therefore find half-spaces $H_1, \ldots, H_n$ with $x \notin H_i$ and $C_i \subseteq H_i$. Thus, we may assume that $F \subseteq H_1 \cup \cdots \cup H_n$. Half-spaces that do not contain the origin are sets of the form $H_{w,a} = \{v \in l^2 | \langle v, w\rangle > a\}$ for some $w \in l^2$ and $a \in \mathbb{R}^+$. Given a finite family $H_1, \ldots, H_n = H_{w_1,a_1}, \ldots, H_{w_n,a_n}$, we can find a unit vector $w$ that is orthogonal to all of $w_1, \ldots, w_n$. This means that $w \notin H_i$ for all $i$, and $w \in F$, contradicting the assumption that $F \subseteq H_1 \cup \cdots \cup H_n$. Therefore, $l^2$ does not satisfy the teetotal axioms.

The teetotal interior $IS \circ CC(l^2)$ has the same convex sets, but closed sets are intersections of finite unions of closed half-spaces. We can check that this is the product topology on $l^2$ as a real vector space.

**Example 3.13.** Let $(X, d)$ be a metric space, where $X = \bigcup_{n\in\mathbb{N}}[n]^n$ is the set of finite lists with entries bounded by list length. The distance is given by $d(u, v) = l(u) + l(v) - l(u \cap v)$, where $l(u)$ is the length of the list $u$

and $u \cap v$ is the longest list which is an initial sublist of both $u$ and $v$. The induced topology is clearly discrete. The complement of the empty list is not contained in a finite union of convex subsets that does not contain the empty list. In particular, a convex subset of $X$ that does not contain $\varnothing$ must consist of lists that all start with the same first element. Since there are infinitely many possible first elements, a finite collection of convex sets that do not contain the empty list cannot cover $X\backslash\varnothing$.

The space $(X, d)$ is a metric space and every closed ball is compact. However, it is not a fixed point of the adjunction between $\mathcal{C}onvex\mathcal{T}op$ and $\mathcal{P}reconvex$.

For a (pre)compatible topological space to be teetotal, an additional property is needed.

**Proposition 3.14.** *If $(X, \mathcal{F}, \mathcal{C})$ is a precompatible topological convexity space with the following properties:*

- *There is a basis of open sets that are convex, whose closure is convex and compact.*
- *$(X, \mathcal{F})$ is Hausdorff.*
- *If $A$ is closed convex and $x \notin A$, then there is a closed convex set $H$ such that $H^c$ is convex, with $A \subseteq H$ and $x \notin H$. (This property, without the topological constraints, is often used in the literature, where it is called the Kakutani condition.)*

*then $(X, \mathcal{F}, \mathcal{C})$ is fixed by the adjunction.*

*Proof.* We need to show that for any closed $V \in \mathcal{F}$, and any $x \notin V$, there is a finite set of closed convex sets whose union covers $V$ but does not contain $x$. Let $U$ be an open subset of $V^c$, containing $x$ such that $U$ is convex and $\overline{U}$ is convex and compact. Let $A = \overline{U}\backslash U$. For any $a \in A$, by the Hausdorff property, we can find an open $U_a$ that contains $a$, whose closure does not contain $x$. Since convex open sets with convex closure form a basis of open sets, we can find a convex open $U'_a$ with convex closure that does not contain $x$. Since $A$ is compact, it is covered by a finite subset $U'_{a_1} \cup \cdots \cup U'_{a_n}$. Now each $\overline{U'_{a_i}}$ is contained in a closed convex $H_{a_i}$ which does not contain $x$, such that $H^c_{a_i}$ is also convex.

For any $y \in V$, since $[x, y]$ is connected (by compatibility), it cannot be the union $([x, y] \cap U) \cup \left([x, y] \cap \overline{U}^c\right)$, so $[x, y] \cap A \neq \emptyset$. Let $z \in [x, y] \cap A$. Since $H_{a_i}$ cover $A$, we have $z \in H_{a_i}$ for some $i$. Now if $y \in {H_{a_i}}^c$, then since ${H_{a_i}}^c$ is convex and contains $x$, it follows that $z \in {H_{a_i}}^c$ contradicting $z \in H_{a_i}$. Thus, we must have $y \in H_{a_i}$. Since $y \in V$ is arbitrary, we have that $V \subseteq H_{a_1} \cup \cdots \cup H_{a_n}$ as required.

We also need to show that every convex set is a directed union of closed convex sets. Let $C \in \mathcal{C}$ be a convex set. Let $\mathcal{D} = \{[F] | F \subseteq C, F \text{ finite}\}$ be the collection of finitely generated convex subsets of $C$. Since finite sets are closed under binary unions, $\mathcal{D}$ is directed. Since the convex closure of any finite set is closed, it follows that $C$ is a directed union of closed convex sets as required. $\square$

For a metric space, these conditions can be simplified to give more natural conditions.

**Lemma 3.15.** *If $X$ is a topological convexity space where intervals are closed, satisfying the Kakutani property that every pair of disjoint closed convex sets are separated by a closed half-space, then for any $x, s, t, p, q, r \in X$ with $s \in [x, p]$, $t \in [x, q]$ and $r \in [p, q]$, we have $[x, r] \cap [s, t] \neq \emptyset$.*

*Proof.* If $[x, r] \cap [s, t] = \emptyset$, then $[x, r]$ and $[s, t]$ are disjoint closed convex sets, so by the Kakutani propery, there is a closed half-space $H$ such that $[x, r] \subseteq H$ and $[s, t] \subseteq H^c$. Now if $p \in H$, then since $x \in H$ and $H$ is convex, we get $s \in H$, contradicting $[s, t] \subseteq H^c$. This is a contradiction, so we must have $p \in H^c$. A similar argument shows that $q \in H^c$. However, since $H^c$ is convex, it follows that $r \in H^c$, contradicting $[x, r] \subseteq H$. This contradiction disproves $[x, r] \cap [s, t] = \emptyset$, so $[x, r] \cap [s, t] \neq \emptyset$ $\square$

**Lemma 3.16.** *If $(X, d)$ is a metric space, such that every open ball is convex, every pair of disjoint closed convex sets are separated by a closed half-space (a closed convex set with convex complement), and every interval $[a, b]$ is isomorphic (as a topological convexity space) to the real interval $[0, 1]$ then for any convex compact $A \subseteq X$ and any $x \in X$, we have*

$$[x, A] = \bigcup\{[x, y] | y \in A\}$$

*Proof.* We need to show that $\bigcup\{[x, y] | y \in A\}$ is closed under the betweenness relation. Let $s, t \in \bigcup\{[x, y] | y \in A\}$, and let $z \in [s, t]$. Let $s \in [x, p]$

and $t \in [x, q]$ for $p, q \in A$. We will show that $z \in [x, r]$ for some $r \in [p, q]$. Since $[s, t] \cong [0, 1]$, we have that $[s, t] = [s, z] \cup [z, t]$. For $r \in [p, q]$, if $[x, r] \cap [s, z] \neq \varnothing$ and $[x, r] \cap [z, t] \neq \varnothing$, then clearly $z \in [x, r]$. Thus if $(\forall r \in [p, q])(z \notin [x, r])$, then

$$(\forall r \in [p, q])(([x, r] \cap [s, z] = \varnothing) \vee ([x, r] \cap [z, t] = \varnothing))$$

so

$$[p, q] = \{r \in [p, q] | [x, r] \cap [s, z] = \varnothing\} \cup \{r \in [p, q] | [x, r] \cap [z, t] = \varnothing\}$$

and this union is disjoint by Lemma 3.15. By connectedness of $[p, q]$, we just need to show that $\{r \in X | [x, r] \cap [s, z] = \varnothing\}$ and $\{r \in X | [x, r] \cap [z, t] = \varnothing\}$ are open to reach a contradiction, which would prove $z \in [x, r]$ for some $r \in [p, q]$. Let $U = \{r \in X | [x, r] \cap [s, z] = \varnothing\}$, and let $v \in U$. We want to show that there is some $\epsilon$ such that $B(v, \epsilon) \subseteq U$. Now $[s, z] \cap [x, v] = \varnothing$, which means $(\forall w \in [s, z])(d(x, w) + d(w, v) \neq d(x, v))$. Since $[s, z]$ is compact, the function $f(w) = d(x, w) + d(w, v) - d(x, v)$ is bounded away from zero on $[s, z]$. Let $\delta$ be a lower bound. Now if $v' \in B\left(v, \frac{\delta}{2}\right)$, then for any $w \in [s, z]$, we have

$$\begin{aligned} d(x, w) + d(w, v') &\geqslant d(x, w) + d(w, v) - d(v, v') \\ &> d(x, v) + \delta - \frac{\delta}{2} \\ &\geqslant d(x, v') - d(v', v) + \frac{\delta}{2} \\ &> d(x, v') \end{aligned}$$

Because the inequality is strict, we have $w \notin [x, v']$ for any $w \in [s, z]$, i.e. $v' \in U$. Thus $B\left(v, \frac{\delta}{2}\right) \subseteq U$, meaning $U$ is open as required. □

**Corollary 3.17.** *If $(X, d)$ is a metric space, such that every closed ball is compact, every open ball is convex, every pair of disjoint closed convex sets are separated by a closed half-space, and every interval $[a, b]$ is isomorphic to the real interval $[0, 1]$ then the induced topological convexity space is fixed by the adjunction.*

*Proof.* We will show that the conditions of Proposition 3.14 hold in this case. The Hausdorff condition is always true for metric spaces.

The open balls form a basis for the topology, and are convex. By connectedness, if $d(x,y) = r$, then since $[x,y]$ is connected, there is a sequence $y_1,\dots,y_n \to y$ in $[x,y]$, so $y_1,\dots,y_n \in B(x,r)$. Thus, we have $\overline{B(x,r)} = \overline{B}(x,r) = \{y \in X | d(x,y) \leqslant r\}$. Thus, the closure of an open ball is compact. Also, $\overline{B}(x,r) = \bigcap_{R>r} B(x,R)$ is an intersection of convex sets, so closed balls are convex. Thus open balls are a basis of the topology with the required property.

Next, we need to show that the convex closure of a finite set is compact. We will do this inductively. By Lemma 3.16, we have that $[x_1,\dots,x_n] = \bigcup\{[x_1,y] | y \in [x_2,\dots,x_n]\}$. By the induction hypothesis $[x_2,\dots,x_n]$ is compact. This means that $[x_2,\dots,x_n] \subseteq B(x_1,r)$ for some $r \in \mathbb{R}^+$. It follows that $[x_1,\dots,x_n] \subseteq B(x_1,r)$, since $B(x_1,r)$ is convex. Therefore, it is sufficient to prove that $[x_1,\dots,x_n]$ is closed.

Let $z \notin [x_1,\dots,x_n]$. We want to prove that there is some open ball about $z$ that is disjoint from $[x_1,\dots,x_n]$. For any $y \in [x_2,\dots,x_n]$, we know $z \notin [x_1,y]$, so $d(x_1,z) + d(z,y) - d(x_1,y) > 0$. For $y \in [x_2,\dots,x_n]$, let $f(y) = d(x_1,z) + d(z,y) - d(x_1,y)$. Then $f(y)$ is a continuous function $[x_2,\dots,x_n] \to \mathbb{R}^+$. Since $[x_2,\dots,x_n]$ is compact, $f$ attains its lower bound, so in particular, there is some $\epsilon > 0$ such that $f(y) > \epsilon$ for all $y \in [x_2,\dots,x_n]$. Now if $d(z,z') < \frac{\epsilon}{2}$, then for any $y \in [x_2,\dots,x_n]$,

$$d(x_1,z') + d(z',y) > d(x_1,z) - \frac{\epsilon}{2} + d(z,y) - \frac{\epsilon}{2} > d(x_1,y)$$

so $z' \notin [x_1,y]$ because the inequality is strict and open balls are convex. It follows that $z' \notin [x_1,\dots,x_n]$, so $[x_1,\dots,x_n]$ is closed, as required. □

In the other direction, it is natural to ask which preconvexity spaces are fixed by the monad $CC \circ IS$. The functor $CC \circ IS$ sends a preconvexity space, $(X,\mathcal{P})$ to the space $(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}})$. We will call a preconvexity space $(X,\mathcal{P})$ *geometric* if $\overline{\mathcal{P}} \cap \widetilde{\mathcal{P}} = \mathcal{P}$.

**Proposition 3.18.** *If $X$ is finite, then any preconvexity space $(X,\mathcal{P})$ is geometric.*

*Proof.* If $X$ is finite, then $\widetilde{\mathcal{P}} = \mathcal{P}$, so $\widetilde{\mathcal{P}} \cap \overline{\mathcal{P}} = \mathcal{P}$ as required. □

A natural question is whether this extends to topologically discrete spaces. In fact, there are preconvexity spaces where all sets are in both $\overline{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$, but not in $\mathcal{P}$.

**Example 3.19.** Let $X = \mathbb{N}$. Let $\mathcal{P}$ consist of all subsets of $\mathbb{N}$ whose complement is infinite or empty. Clearly every subset of $\mathbb{N}$ is a finite union from $\mathcal{P}$, and also a directed union from $\mathcal{P}$ (as $\mathcal{P}$ contains all finite sets). Thus $(X, \mathcal{P})$ is a non-geometric example where all sets are closed and all sets are convex.

**Proposition 3.20.** *Every $T_0$ preconvexity space (meaning for any $x \neq y$, there is a preconvex set containing exactly one of $x$ and $y$) embeds in a geometric preconvexity space.*

*Proof.* For a $T_0$ preconvexity space $(X, \mathcal{P})$, let $(Y, \mathcal{Q})$ be given by $Y = \mathcal{P}$ and $\mathcal{Q} = \big\{\{S \in \mathcal{P} | S \subseteq R\} | R \in \mathcal{P}\big\}$. Now the inclusion $X \xrightarrow{i} Y$ given by $i(x) = \bigcap\{P \in \mathcal{P} | x \in P\}$, is an embedding of preconvexity spaces, meaning that for $A \subseteq X$, we have $A \in \mathcal{P}$ if and only if $A = i^{-1}(B)$ for some $B \in \mathcal{Q}$. Clearly if $A \in \mathcal{P}$, then $\{S \in \mathcal{P} | S \subseteq A\} \in \mathcal{Q}$. Now it is easy to see that $a \in i^{-1}\big(\{S \in \mathcal{P} | S \subseteq A\}\big)$ if and only if $i(a) \subseteq A$, if and only if $a \in A$. Thus $A = i^{-1}\big(\{S \in \mathcal{P} | S \subseteq A\}\big)$. Conversely, let $\mathcal{R} \in \mathcal{Q}$. By definition, there is some $P \in \mathcal{P}$ such that $\mathcal{R} = \{S \in \mathcal{P} | S \subseteq P\}$. It is easy to see that $i^{-1}(\mathcal{R}) = P$.

We need to show that $(Y, \mathcal{Q})$ is geometric. $Y$ is a complete lattice, ordered by set-inclusion, and $\mathcal{Q}$ is the set of principal downsets of $Y$. This means that $\widetilde{\mathcal{Q}}$ is the set of ideals in $Y$, and $\overline{\mathcal{Q}}$ is the set of closed sets of the weak topology. From Examples 2.5.2, we know that the intersection of these is $\mathcal{Q}$. ☐

This leads to the natural question is what subspaces of a geometric preconvexity space are geometric.

**Proposition 3.21.** *If $(X, \mathcal{P})$ is a geometric preconvexity space and $A \in \mathcal{P}$, then the restriction $(A, \mathcal{P}|_A)$ is a geometric preconvexity space.*

*Proof.* Since $\mathcal{P}$ is closed under intersection, $\mathcal{P}|_A \subseteq \mathcal{P}$. Now let $C \subseteq A$ be both a directed union of sets from $\mathcal{P}|_A$ and an intersection of finite unions of sets from $\mathcal{P}|_A$. Since $\mathcal{P}|_A \subseteq \mathcal{P}$, $C$ is both a directed union of sets from $\mathcal{P}$ and an intersection of finite unions of sets from $\mathcal{P}$. Since $(X, \mathcal{P})$ is geometric, it follows that $C \in \mathcal{P}$, and since $C \subseteq A$, we have $C \in \mathcal{P}|_A$ as required. ☐

On the other hand, closed or convex subspaces of geometric preconvexity spaces are not necessarily geometric.

**Example 3.22.** Let $X = \mathbb{R}^2 \backslash \{(0, 0)\}$, and let

$$Y = \{(x, y) \in [0, 1]^2 | (|2x - 1| - 1)(|2y - 1| - 1) = 0\}$$

be the unit square with one corner at the origin. It is straightforward to check that $X$ and $Y$, with the preconvexities coming from closed convex subsets of $\mathbb{R}^2$, are geometric. However, $X \cap Y$ is a closed subspace of $X$, and a convex subspace of $Y$, but the subset $\{(x, y) \in X \cap Y | x > 0 \text{ or } y = 1\}$ is both closed and convex, but is not closed convex, so $X \cap Y$ is not geometric.

# 4. Stone Duality

## 4.1 Stone Duality for Topological Spaces

In this section, we review Stone duality for topological spaces. While a lot of what we review is well-known, some parts are written from an unusual perspective, and are not as well-known as they might be.

Given a topological space, the collection of closed sets form a coframe. (Many authors refer to the frame of open sets, but for our purposes the closed sets are more natural, and since we are not considering 2-categorical aspects, it does not matter since $\mathcal{C}\!\mathit{oframe} = \mathcal{F}\!\mathit{rame}^{\mathrm{co}}$.) Furthermore, the inverse image of a continuous function between topological spaces is by definition a coframe homomorphism between the coframes of closed spaces. This induces a functor $C : \mathcal{T}\!\mathit{op} \longrightarrow \mathcal{C}\!\mathit{oframe}^{\mathrm{op}}$. Not every coframe arises as closed sets of a topological space. Coframes that do arise in this way are called *spatial* and are said to "have enough points".

In some cases, there can be many topological spaces that have the same coframe of closed sets. If multiple points have the same closure, then there is no way to separate them by looking at the coframe of closed sets. Therefore, we restrict our attention to $T_0$ spaces, where the function from $X$ to $C(X)$ sending a point to its closure is injective. The functor $T_0\text{-}\mathcal{T}\!\mathit{op} \xrightarrow{C} \mathcal{C}\!\mathit{oframe}^{\mathrm{op}}$ is faithful.

We can recover a $T_0$ topological space from its lattice of closed sets and from the subset $S \subset C(X)$ consisting of the closures of singletons. For a coframe $L$, the elements which could arise as closures of singletons for a topological space corresponding to $L$ are non-zero elements that cannot be written as a join of two strictly smaller elements (called *join-irreducible* elements). These are called the "points" of $L$ since they correspond to coframe homomorphisms $f : L \longrightarrow 2$, where the 2-element coframe, 2, is the terminal object in $\mathcal{C}oframe^{\mathrm{op}}$. If we let $P_L \subseteq L$ be the set of points, a topological space $X$ corresponds to a coframe $L = C(X)$ with a chosen subset $S \subseteq P_L$ such that for every $x \in L$, we have $x = \bigvee(S \cap \downarrow x)$ (that is, $S$ is join-dense in $L$). Continuous functions $X \xrightarrow{g} Y$ correspond to coframe homomorphisms $C(Y) \xrightarrow{C(g)} C(X)$ whose left adjoint $C(X) \xrightarrow{C(g)^*} C(Y)$ (in the category of order-preserving maps) sends $S_X$ to $S_Y$. We can express this left adjoint condition topologically as: for every $s \in S_X$, $C(g)^{-1}(\downarrow s)$ is a principal downset in $C(Y)$, and the top element is in $S_Y$, where $S_X \subseteq P_{C(X)}$ and $S_Y \subseteq P_{C(Y)}$ are the chosen sets of points that correspond to elements of $X$ and $Y$ respectively.

More formally, let $\mathcal{S}patial\mathcal{C}oframe_*$ be the category of pointed spatial coframes. Objects are pairs $(L, S)$ where $L$ is a coframe and $S \subseteq P_L$ is a join dense set of points of $L$ (meaning $(\forall a \in L)(a = \bigvee(S \cap \downarrow a))$). These pairs are introduced in [6, 7], where they are called prime-based complete lattices. Morphisms $(M, T) \xrightarrow{g} (L, S)$ are coframe homomorphisms $M \xrightarrow{g} L$ whose left adjoint $L \xrightarrow{g^*} M$ as order-preserving homomorphisms restricts to a function $S \xrightarrow{g^*_S} T$.

**Proposition 4.1** ([6]). *The category of $T_0$ topological spaces and continuous functions is equivalent to the category $\mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}}$.*

*Proof.* The functor $C : T_0\text{-}\mathcal{T}op \longrightarrow \mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}}$ sends a topological space $X$ to the pair $\left(C(X), \left\{\overline{\{x\}}\middle| x \in X\right\}\right)$, where $C(X)$ is the coframe of closed subsets of $X$. It sends a continuous function $f : X \longrightarrow Y$ to $f^{-1} : C(Y) \longrightarrow C(X)$. We need to show that this is a homomorphism in $\mathcal{S}patial\mathcal{C}oframe_*$. It is clearly a coframe homomorphism, so we need to show that for any $x \in X$,

$$\bigwedge \left\{t \in \left\{\overline{\{y\}}\middle| y \in Y\right\}\middle|\overline{\{x\}} \leqslant f^{-1}(t)\right\} \in \{\overline{\{y\}}| y \in Y\}$$

We will show that $\bigwedge \left\{t \in \left\{\overline{\{y\}}\middle| y \in Y\right\}\middle|\overline{\{x\}} \leqslant f^{-1}(t)\right\} = \overline{\{f(x)\}}$. We need to show that $\overline{\{x\}} \subseteq f^{-1}\left(\overline{\{f(x)\}}\right)$, and if $\overline{\{x\}} \subseteq f^{-1}(A)$ for any closed

$A \subseteq Y$, then $\overline{\{f(x)\}} \subseteq A$. Clearly, $x \in f^{-1}\left(\overline{\{f(x)\}}\right)$, so $f^{-1}\left(\overline{\{f(x)\}}\right)$ is a closed set containing $x$, so $\overline{\{x\}} \leqslant f^{-1}\left(\overline{\{f(x)\}}\right)$. On the other hand, suppose $\overline{\{x\}} \leqslant f^{-1}(A)$. Then $x \in f^{-1}(A)$, so $f(x) \in A$, so $\overline{\{f(x)\}} \leqslant A$. Thus $f^{-1}$ is a morphism in $\mathcal{SpatialCoframe}_*^{\mathrm{op}}$.

In the opposite direction, the functor $P : \mathcal{SpatialCoframe}_*^{\mathrm{op}} \longrightarrow T_0\text{-}\mathcal{Top}$ sends the pair $(L, S)$ to the topological space with elements $S$ and closed sets $\{S \cap \downarrow a | a \in L\}$. For the morphism $(L, S) \xrightarrow{f} (M, T)$, we define $T \xrightarrow{f^*} S$ as the restriction of the left adjoint of $f$ to $T$. By definition of $\mathcal{SpatialCoframe}_*$, this is a well-defined function. For any $F \in L$, we have $f^*(t) \in \downarrow F$ if and only if $t \leqslant f(F)$ by definition, so $(f^*)^{-1}(\downarrow F \cap S) = T \cap \downarrow f(F)$ is a closed subset of $P(M, T)$. Thus $f^*$ is continuous.

Finally, we need to show that the two functors defined above form an equivalence. For a topological space $X$, we see that $PCX$ has the same elements as $X$ and closed sets of $PCX$ are of the form $\downarrow F \cap \left\{\overline{\{x\}}\middle| x \in X\right\}$ for $F \in C(X)$. It is clear that $\overline{\{x\}} \leqslant F$ if and only if $x \in F$, so closed sets of $PCX$ are exactly closed sets of $X$, so $PCX \cong X$.

For a coframe $L$ with a chosen subset $S \subseteq L$, we want to show that $CP(L, S) \cong (L, S)$. By definition, elements of $CP(L, S)$ are $\{\downarrow a \cap S | a \in L\}$. Since $(\forall a \in L)(a = \bigvee(\downarrow a \cap S))$, it follows that the coframe of $CP(L, S)$ is isomorphic to $L$. The chosen elements are $\left\{\overline{\{s\}}\middle| s \in S\right\}$, where $\overline{\{s\}}$ is the closure of $\{s\}$ in $P(L, S)$. Closed sets of $P(L, S)$ are of the form $\downarrow a \cap S$ for $a \in L$, so in particular $\overline{\{s\}} = \downarrow s \cap S$. This clearly induces an isomorphism $(L, S) \cong CP(L, S)$. □

An alternative approach due to [17] is take the embedding of the coframe of closed sets into the completely distributive lattice of arbitrary unions of closed sets. That is, for $(L, S)$ a pointed spatial coframe, we have the coframe inclusion $L \rightarrowtail DS$, where $DS$ is the completely distributive lattice of down-closed subsets of $S$ (where $S$ is viewed as a sub-poset of $L$). In topological terms, $S$ is the collection of points of the space, with the specialisation order. Downsets of $S$ correspond to arbitrary unions of closed sets, and the inclusion of $L$ into $DS$ is the obvious inclusion. In lattice theoretic terms, the inclusion $L \rightarrowtail DS$ sends $x \in L$ to $\{s \in S | s \leqslant x\}$. For a homomorphism $f : (M, T) \longrightarrow (L, S)$, the condition that $f^*$ restricts to a function $S \xrightarrow{f^*} T$ means that the inverse image function $Df^* : DT \longrightarrow DS$ is a complete lattice homomorphism. Furthermore, the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, tail] & L \arrow[d, tail] \\ DT \arrow[r, "Df^*"'] & DS \end{tikzcd}$$

commutes, since $Df^*$ sends $T \cap \downarrow x$ to

$$\{s \in S | f^*(s) \in T \cap \downarrow x\} = \{s \in S | f^*(s) \leqslant x\} = \{s \in S | s \leqslant f(x)\} = S \cap \downarrow f(x)$$

The condition that $S \subseteq L$ means that all totally compact elements of $DS$ (elements $x \in DS$ such that for any $A \subseteq DS$, if $\bigvee A \geqslant x$, then there is some $a \in A$ such that $a \geqslant x$) are in $L$, so every element of $DS$ is a join of elements in $L$. We will refer to such lattice inclusions as dense. Thus the category of $T_0$ topological spaces is equivalent to the category of dense inclusions of spatial coframes into totally compactly generated completely distributive lattices.

The collections $L^{\mathrm{op}}$ of open subsets of the topological space, and $DS$ of arbitrary unions of closed sets, generate the open sets of a larger topology, called the *Skula* topology [14]. Putting these three lattices together gives a structure called the Skula biframe. A *biframe* [2], consists of a frame $L_0$ with two chosen subframes $L_1$ and $L_2$, such that $L_1 \cup L_2$ generates $L_0$. The biframe $(L_0, L_1, L_2)$ is strictly zero-dimensional if every element of $L_1$ is complemented in $L_0$, and the complement is in $L_2$. Every zero-dimensional biframe is determined by the complement inclusion $(L_1)^{\mathrm{op}} \rightarrowtail L_2$, so the functor that sends a topological space to the Skula biframe is one half of an equivalence between the category of $T_0$ topological spaces and the category of strictly zero-dimensional biframes [13].

For all of these representations of $T_0$ topological spaces, the fibres of the forgetful functor

$$T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$$

correspond to additional structure on the coframe, and are partially ordered by inclusion of this additional structure. Every fibre has a top element, which gives an adjoint to the forgetful functor $C$, sending a spatial coframe to the top element of the fibre over it. (In fact, this adjoint extends to all coframes, because spatial coframes are reflective in all coframes). These top elements of the fibres are exactly the sober spaces.

Not all fibres have bottom elements. However, a large number of the fibres of the forgetful functor $T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$ do have bottom elements and are actually complete Boolean algebras. This is probably easiest to see from the representation as coframes with a chosen set of elements which are closures of points of the topological space. If $S_0$ is the smallest such set of closed sets that can arise as closures of points, and $S_1$ is the largest set, then any set between $S_0$ and $S_1$ is a valid set of points, making the poset of possible sets of points isomorphic to the Boolean algebra $P(S_1 \backslash S_0)$. The topological spaces that can occur as the bottom elements of fibres are spaces where the closure of every point cannot be expressed as a union of closed sets not containing that point. That is, for every $x \in X$, $\overline{\{x\}} \backslash \{x\}$ is closed. Spaces with this property are called $T_D$ spaces [1].

Clearly, all $T_1$ spaces are $T_D$ because in a $T_1$ space $\overline{\{x\}} \backslash \{x\} = \varnothing$ is closed. However, even if we restrict to $T_1$ spaces and atomic spatial coframes, the assignment of an atomic spatial coframe to the bottom element in the corresponding fibre is not functorial, since the adjoint to a coframe homomorphism between $T_D$ spaces does not necessarily preserve join-indecomposable elements, or even atoms. This is why the focus of attention in most of the literature has been on sober spaces, rather than $T_D$ spaces. In order to model the morphisms between $T_D$ spaces, we need to restrict to coframe homomorphisms whose adjoint preserves join-indecomposable elements. While most of the topological spaces of interest are $T_D$, many of the fibres of the forgetful functor $T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$ contain only a singleton $T_0$ topological space, which is therefore both sober and $T_D$. (Several equivalent characterisations of when this occurs are given in [8].) Thus many important topological spaces are sober.

### 4.2 Stone Duality for Preconvexity Spaces

There is in many ways, a very similar picture for the category of preconvexity spaces. Instead of the coframe of closed sets, the structure that defines the preconvexity spaces is the complete lattice of preconvex sets $\mathcal{P}$. Because the inverse image function for a preconvexity space homomorphism preserves preconvex sets, it induces an inf-homomorphism between the lattices of preconvex sets. Thus, we have a functor $\mathcal{P}reconvex \xrightarrow{P} \mathcal{I}nf^{\mathrm{op}}$, where $\mathcal{I}nf$ is the category of complete lattices with infimum-preserving homomorphisms between them, sending every preconvexity space to its lattice of preconvex sets, and every homomorphism to the inverse image function. This has many of the nice properties of the Stone duality functor $\mathcal{T}op \xrightarrow{F} \mathcal{C}oframe^{\mathrm{op}}$.

As in the topology case, there is an equivalent category of sup-lattices with a set of chosen elements. Let $\mathcal{T}CGP artial\mathcal{S}up$ be the category whose objects are pairs $(L, S)$, where $L$ is a complete lattice and $S \subseteq L \backslash \{\bot\}$ is sup-dense, i.e. $(\forall x \in L)(x = \bigvee(S \cap \downarrow x))$. Morphisms $(L, S) \xrightarrow{f} (M, T)$ in $\mathcal{T}CGP artial\mathcal{S}up$ are sup-homomorphisms $L \xrightarrow{f} M$ with the property that $(\forall x \in S)(f(x) \in T)$. These pairs are called *based complete lattices* in [6].

**Proposition 4.2** ([6]). *The categories $\mathcal{T}CGP artial\mathcal{S}up$ and $T_0$-$\mathcal{P}reconvex$ are equivalent.*

*Proof.* There is a functor $T_0$-$\mathcal{P}reconvex \xrightarrow{F} \mathcal{T}CGP artial\mathcal{S}up$ given by $F(X, \mathcal{P}) = \left(\mathcal{P}, \left\{\overline{\{x\}}\middle| x \in X\right\}\right)$ on objects and $F(f) \dashv f^{-1}$ on morphisms, where the adjoint is as a partial order homomorphism and exists because $f^{-1}$ is an inf-homomorphism. To show this is well-defined, since $F(f)$ is a left adjoint, it is a sup-homomorphism, and can be given explicitly by $F(f)(A) = \bigcap\{B \in \mathcal{P}' | A \subseteq f^{-1}(B)\}$. In particular, if $A = \overline{\{x\}}$, then

$$F(f)(A) = \bigcap\{B \in \mathcal{P}' | x \in f^{-1}(B)\} = \bigcap\{B \in \mathcal{P}' | f(x) \in B\} = \overline{\{f(x)\}}$$

To complete the proof that $F$ is well-defined, we need to show that $\left\{\overline{\{x\}}\middle| x \in X\right\}$ is sup-dense in $\mathcal{P}$. For any $P \in \mathcal{P}$, and any $x \in P$, we have $\overline{\{x\}} \subseteq P$. Thus $P = \bigcup\left\{\overline{\{x\}}\middle| x \in P\right\}$ as required. Thus $F$ is well-defined, and functoriality is obvious.

In the other direction, we define $G : \mathcal{T}CGP artial\mathcal{S}up \longrightarrow \mathcal{P}reconvex$ by $G(L, S) = (S, \{S \cap \downarrow x | x \in L\})$ on objects and $G(f)(s) = f(s)$ on morphisms. To show well-definedness, for $(L, S) \xrightarrow{f} (M, T)$ a morphism of $\mathcal{T}CGP artial\mathcal{S}up$, we need to show that $G(f)$ is a preconvexity homomorphism. That is, for any $s \in S$, we have $f(s) \in T$, and for any $m \in M$,

$G(f)^{-1}(T \cap \downarrow m) = S \cap \downarrow x$ for some $x \in L$. The first condition is by definition of a homomorphism. Since $f$ is a sup-homomorphism, it has a right adjoint $f_*$ given by $f_*(m) = \bigwedge\{x \in L | f(x) \geqslant m\}$. Now

$$G(f)^{-1}(T \cap \downarrow m) = \{s \in S | f(s) \leqslant m\} = \{s \in S | s \leqslant f_*(m)\} = S \cap \downarrow f_*(m)$$

which gives the required homomorphism property. Finally, we want to show that $F$ and $G$ form an equivalence of categories. For a preconvexity space $(X, \mathcal{P})$, we have that

$$GF(X,\mathcal{P}) = G\left(\mathcal{P}, \left\{\overline{\{x\}} | x \in X\right\}\right) = \left(\left\{\overline{\{x\}} | x \in X\right\}, \left\{\left\{\overline{\{x\}} | x \in X\right\} \cap \downarrow P \middle| P \in \mathcal{P}\right\}\right)$$

It is obvious that the function sending $x$ to $\overline{\{x\}}$ is a natural isomorphism of preconvexity spaces. In the other direction, for $(L, S) \in \mathrm{ob}(\mathcal{T}CGPartial\mathcal{S}up)$, we have

$$FG(L,S) = F(S, \{S \cap \downarrow x | x \in L\}) = \left(\{S \cap \downarrow x | x \in L\}, \left\{\overline{\{s\}} \middle| s \in S\right\}\right)$$

For $s \in S$, $\overline{\{s\}} = S \cap \downarrow s$, so the function $L \xrightarrow{i} \{S \cap \downarrow x | x \in L\}$ given by $i(x) = S \cap \downarrow x$ is easily seen to be an isomorphism in $\mathcal{T}CGPartial\mathcal{S}up$. Thus we have shown the equivalence of categories. □

Under this equivalence (and the adjoint isomorphism $\mathcal{S}up \cong \mathcal{I}nf^{\mathrm{op}}$), the functor $T_0$-$\mathcal{P}reconvex \xrightarrow{P} \mathcal{I}nf^{\mathrm{op}}$ corresponds to the forgetful functor $\mathcal{T}CGPartial\mathcal{S}up \xrightarrow{U} \mathcal{S}up$ sending $(L, S)$ to $L$. As in the topological case, it is easy to see that the fibres of the functor $U$ are partial orders. Each fibre clearly has a top element setting $S = L$. This induces a right adjoint to $U$. Furthermore, we can show that this right adjoint extends to all preconvexity spaces.

**Proposition 4.3.** *The preconvex set lattice functor* $\mathcal{P}reconvex \xrightarrow{U} \mathcal{S}up$ *has a right adjoint* $\mathcal{S}up \xrightarrow{P} \mathcal{P}reconvex$*.*

*Proof.* The right adjoint $P$ is defined by $P(L) = (L, \{\varnothing\} \cup \{\downarrow x | x \in L\})$. That is, it sends the complete lattice $L$ to $L$ with the preconvexity where only principal downsets are preconvex. From the equivalence, between $T_0$-$\mathcal{P}reconvex$ and $\mathcal{T}CGPartial\mathcal{S}up$, this $P$ sends $L$ to the pair $(L, L)$, which is clearly the top element of the fibre of the forgetful functor, $U$, when restricted to $T_0$ preconvexity spaces. To show that $P$ is right adjoint to $U$, we need to show the hom-sets $\mathcal{S}up(UX, L)$ and $\mathcal{P}reconvex(X, PL)$ are naturally isomorphic. For $f \in \mathcal{S}up(UX, L)$, the corresponding element of $\mathcal{P}reconvex(X, PL)$ is $\hat{f}$ given by $\hat{f}(x) = f(\overline{\{x\}})$. It is easy to see that $\hat{f}$ is a preconvexity homormophism, since preconvex sets in $PL$ are principal downsets of $L$, and

$$(\hat{f})^{-1}(\downarrow(y)) = \{x \in X | f(x) \leqslant y\} = \{x \in X | \overline{\{x\}} \subseteq f_*(y)\} = f_*(y)$$

is preconvex. For $f \in \mathcal{P}reconvex(X, PL)$, the corresponding element of $\mathcal{S}up(UX, L)$ is $\tilde{f}$ given by $\tilde{f}(A) = \bigvee_{x \in A} f(x)$. Since $f$ is a preconvexity homomorphism, we have that $f^{-1}(\downarrow(y))$ is preconvex for any $y \in L$. For $\mathcal{A} \subseteq UX$, we want to show that $\tilde{f}(\bigvee \mathcal{A}) \leqslant \bigvee_{A \in \mathcal{A}} \tilde{f}(A)$. If $y$ is an upper bound for $\{\tilde{f}(A) | A \in \mathcal{A}\}$, then since $f$ is a preconvexity homomorphism, $f^{-1}(\downarrow y)$ is preconvex, and for any $A \in \mathcal{A}$, we have $\tilde{f}(A) \leqslant y$, so $A \subseteq f^{-1}(\downarrow y)$. Thus $f^{-1}(\downarrow y)$ is an upper bound for $\mathcal{A}$, in the lattice of preconvex subsets of $X$, so it contains $\bigvee \mathcal{A}$, as required. □

Bottom elements of the fibres are of the form $(L, S)$ where $S$ is the smallest subset of $L$ satisfying $(\forall x \in L)(x = \bigvee S \cap \downarrow x)$. For any $x \in L$, if we can find a downset $D \subseteq L$ with $\bigvee D = x$ and $x \notin D$, then clearly if $(L, S) \in \mathrm{ob}\,\mathcal{TCGP}artial\mathcal{S}up$, then $(L, (S \backslash \{x\}) \cup D) \in \mathrm{ob}\,\mathcal{TCGP}artial\mathcal{S}up$, so if there is a minimum set $S$, then we cannot have $x \in S$. Conversely, if the only downset whose supremum is $x$ is the principal downset $\downarrow x$, then for any $(L, S)$ in $\mathcal{TCGP}artial\mathcal{S}up$, we must have $x \in S$. Thus, if there is a smallest element of the fibre above $L$, it must be given by $(L, S)$, where

$$S = \left\{x \in L \middle| (\forall D \subseteq L)\left(\left(\bigvee D = x\right) \Rightarrow x \in D)\right)\right\}$$

This is similar to the total compactness condition on elements of a sup-lattice, but an element $x$ is called totally compact if it satisfies

$$(\forall D \subseteq L)\left(\left(\bigvee D \geqslant x\right) \Rightarrow (\exists y \in D)(x \leqslant y)\right)$$

which is a stronger condition. This condition is that $x$ is totally compact in the sub-lattice $\downarrow x$.

As in the topological case, when bottom elements of the fibre exist, they are usually the spaces of greatest interest. For example, spaces where every singleton set is preconvex are always the bottom elements of the corresponding fibre. However, the fibres of the forgetful functor are very rarely singletons, so the top elements of the fibres are not of as much interest as in the topological case.

It is also worth noting that we have the chain of adjunctions

$$\mathcal{C}onvex\mathcal{T}op \underset{IS}{\overset{CC}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{P}reconvex \underset{P}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{S}up$$

which gives an adjunction between the category of topological convexity spaces and the category of sup-lattices. This adjunction sends a topological convexity space $(X, \mathcal{F}, \mathcal{C})$ to the lattice of sets $\mathcal{F} \cap \mathcal{C}$ ordered by set inclusion, and a topological convexity space homomorphism to the left adjoint of its inverse image. The right adjoint sends a sup-lattice $L$ to the topological convexity space $(L, \mathcal{S}, \mathcal{I})$, where $\mathcal{S}$ is the set of weak-closed subsets of $L$, namely intersections of finitely-generated downsets in $L$, and $\mathcal{I}$ is the set of ideals in $L$.

**Theorem 4.4.** *There is an adjunction between the category of topological convexity spaces and the category of sup-lattices. The left adjoint sends a topological convexity space $(X, \mathcal{F}, \mathcal{C})$ to the lattice $\mathcal{F} \cap \mathcal{C}$ of closed convex sets, ordered by inclusion, and a topological convexity space homomorphism $X \xrightarrow{f} Y$ to the adjoint of its inverse image function. The right adjoint sends a sup-lattice $L$ to the topological convexity space $(L, \mathcal{S}, \mathcal{I})$ from Example 2.3(2), and a sup-homomorphism $L \xrightarrow{f} K$ to $f$ viewed as a topological convexity space homomorphism.*

*Proof.* It is straightforward to check that these functors are the composites of the adjunctions

$$\mathcal{C}onvex\mathcal{T}op \underset{IS}{\overset{CC}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{P}reconvex \underset{P}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{S}up$$

shown in Proposition 3.9 and Proposition 4.3. □

**Remark 4.5.** In the abstract, we described the relation between topological convexity spaces and inf-lattices as an extension to the Stone duality between topological spaces and coframes. Any topological space is a topological convexity space with the discrete convexity, where all sets are convex. Similarly, the category of coframes is a subcategory of the category of inf-lattices. The following diagram commutes:

$$\begin{tikzcd} \mathcal{T}op \arrow[r, tail, "D"] \arrow[d, "C"'] & \mathcal{C}onvex\mathcal{T}op \arrow[d, "UCC"] \\ \mathcal{C}oframe^{\mathrm{op}} \arrow[r, tail] & \mathcal{I}nf^{\mathrm{op}} \end{tikzcd}$$

However, the adjoint $ISP$ to $UCC$ does not restrict to an adjoint to the closed set coframe functor, $C$, because $ISP(L)$ is not in general a topological space, even if $L$ is a coframe. For $ISP(L)$ to be a topological space, all subsets of $L$ would need to be ideals, which is impossible for non-trivial lattices. Thus only the forgetful functor is truly an extension, and the duality is not an extension.

### 4.3 Distributive Partial-Sup Lattices

The equivalence $T_0$-$\mathcal{P}reconvex \cong \mathcal{T}CGPartialSup$ is based on previous work [11]. We present this work in a more abstract framework here. The idea is that for a preconvexity space $(X, \mathcal{P})$, the sets in $\mathcal{P}$ are partially ordered by inclusion. This partial order has an infimum operation given by intersection, but union of sets only gives a partial supremum operation because a union of preconvex sets is not necessarily preconvex. (Because of the existence of arbitrary intersections, there is a supremum operation given by union followed by the induced closure operation, but this supremum is not related to the structure of the preconvexity space. Unions of preconvex sets better reflect the structure of the preconvexity space. We therefore add a partial operation to the structure to describe these unions where possible.) For a preconvexity space, the operations are union and intersection, so we have a distributivity law between the partial join operation and the infimum operation. This can be neatly expressed by saying that the partial join structure is actually a partial join structure in the category $\mathcal{I}nf$. We define a partial join structure as a partial algebra for the downset monad. The downset monad exists in the category of partial orders, and also in the category of inf-lattices.

We begin by recalling the following definitions:

**Definition 4.6** ([12])**.** *A* KZ-doctrine *on a 2-category $\mathcal{C}$ is a monad $(T, \eta, \mu)$ on $\mathcal{C}$ with a modification $T\eta \overset{\lambda}{\Longrightarrow} \eta_T$ such that $\lambda\eta$, $\mu\lambda$ and $\mu T\mu\lambda_T$ are all identity 2-cells.*

**Definition 4.7** ([3])**.** *A 2-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is* sinister *if for every morphism $f$ in $\mathcal{C}$, $Ff$ has a right adjoint in $\mathcal{D}$.*

In particular, if $F$ is sinister, then it gives rise to a functor from the category of partial maps in $\mathcal{C}$, to $\mathcal{D}$, sending the partial map $X \xleftarrow{a} A \xrightarrow{f} Y$ to the map $FX \xrightarrow{(Fa)_*} FA \xrightarrow{Ff} FY$.

**Definition 4.8.** *A* lax partial algebra *for a sinister KZ-doctrine in an order-enriched category is a partial map $TX \xrightarrow{\theta} X$ such that*

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "1_X"'] & TX \arrow[d, "\theta"] \\ & X \end{tikzcd}$$

*commutes and there is a 2-cell*

$$\begin{tikzcd} TTX \arrow[r, "T\theta"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Leftarrow"] & TX \arrow[d, "\theta"] \\ TX \arrow[r, "\theta"'] & X \end{tikzcd}$$

*A* homomorphism *of lax partial algebras from $(X, \theta)$ to $(Y, \tau)$ is a morphism $X \xrightarrow{f} Y$, together with a 2-cell*

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Rightarrow"] & TY \arrow[d, "\tau"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

**Remark 4.9.** It is possible to define lax partial algebras for KZ monads in general 2-categories. However, this requires more careful consideration of coherence conditions, so to focus on the particular case of distributive partial sup-lattices, we have restricted attention to $\mathcal{O}rd$-enriched categories, where $\mathcal{O}rd$ is the category of partially-ordered sets and order-preserving functions between them.

**Definition 4.10.** *A* partial sup-lattice *is a lax partial algebra for the sinister KZ doctrine* $(D, \downarrow, \bigcup)$ *in* $\mathcal{O}rd$, *where* $D$ *is the downset functor,* $\downarrow_X$ *is the function sending an element* $x \in X$ *to the principal downset it generates, and* $\bigcup_X : DDX \longrightarrow DX$ *sends a collection of downsets to its union.*

*A* distributive partial sup-lattice *is a lax partial algebra for the sinister KZ doctrine* $(D, \downarrow, \bigcup)$ *in* $\mathcal{I}nf$, *where* $D$ *is the downset functor,* $\downarrow_X$ *is the function sending an element* $x \in X$ *to the principal downset it generates, and* $\bigcup_X : DDX \longrightarrow DX$ *sends a collection of downsets to its union.*

The definition given in [11] is

**Definition 4.11** ([11])**.** *A* partial sup lattice *is a pair* $(L, J)$ *where* $L$ *is a complete lattice,* $J$ *is a collection of downsets of* $L$ *with the following properties:*

- $J$ *contains all principal downsets.*
- $J$ *is closed under arbitrary intersections.*
- *If* $A \in J$ *has supremum* $x$, *then any downset* $B$ *with* $A \subseteq B \subseteq \downarrow x$ *has* $B \in J$.
- *If* $\mathcal{A} \subseteq J$ *is down-closed,* $Y \in J$ *has* $\bigvee Y = x$ *and for any* $a \in Y$, *there is some* $A \in \mathcal{A}$ *with* $\bigvee A \geqslant a$, *then there is some* $B \subseteq \bigcup \mathcal{A}$ *with* $B \in J$ *and* $\bigvee B \geqslant x$.

*A partial sup-lattice,* $(L, J)$, *is* distributive *if for any* $\mathcal{D} \subseteq J$, *we have* $\bigwedge \{\bigvee D | D \in \mathcal{D}\} = \bigvee \bigcap \mathcal{D}$.

*An inf-homomorphism* $L \xrightarrow{f} M$ *is a* partial sup-lattice homomorphism $(L, J) \xrightarrow{f} (M, K)$ *if for any* $A \in J$, *we have* $\downarrow\{f(a) | a \in A\} \in K$, *and* $\bigvee \downarrow\{f(a) | a \in A\} = f\left(\bigvee A\right)$.

**Proposition 4.12.** *Definitions 4.10 and 4.11 give equivalent definitions of distributive partial sup lattices.*

*Proof.* We need to show that if $DL \xrightarrow{\theta} L$ is a lax partial algebra for the downset monad in $\mathcal{I}nf$, then there is some $J \subseteq DL$ satisfying the conditions of Definition 4.11. We will show that setting $J$ as the domain of the partial algebra morphism $DL \xrightarrow{\theta} L$ works. We will let $j$ denote the inclusion

$J \rightarrowtail DL$, and write $j^{-1}$ for the inverse image map $DDL \xrightarrow{j^{-1}} DJ$ that is right adjoint to $Dj$.

From the unit condition

$$\begin{tikzcd} L \arrow[r, "\downarrow"] \arrow[drr, "1_L"'] & DL & J \arrow[l, tail, "j"'] \arrow[d, "\theta"] \\ & & L \end{tikzcd}$$

we have that all principal downsets must be contained in $J$. This allows us to show that $\theta$ is the join whenever it is defined. For $A \in J$, if $x = \bigvee A$, then $A \leqslant {\downarrow} x$ in $J$, and for any $a \in A$, we have ${\downarrow} a \leqslant A$ in $J$. Since $\theta$ is order-preserving, this gives $a = \theta({\downarrow} a) \leqslant \theta(A) \leqslant \theta({\downarrow} x) = x$, so $\theta(A)$ is an upper bound of $A$, and is below $x = \bigvee A$. Thus $\theta(A) = \bigvee A$. Since the inclusion $J \overset{j}{\rightarrowtail} DL$ is an inf-homomorphism, we get that $J$ is closed under arbitrary intersections. Suppose $A \in J$ has supremum $x$, and $B \in DL$ satisfies $A \subseteq B \subseteq {\downarrow} x$. We want to show that $B \in J$.

The lax partial algebra condition gives

$$\begin{tikzcd} DDL \arrow[r, "j^{-1}"] \arrow[dd, "\bigcup"'] & DJ \arrow[r, "D\theta"] & DL \\ & \geqslant & J \arrow[u, tail, "j"'] \arrow[d, "\theta"] \\ DL & J \arrow[l, tail, "j"'] \arrow[r, "\theta"] & L \end{tikzcd}$$

In particular, since $A \in J \cap {\downarrow} B$, we have $D\theta(J \cap {\downarrow} B) = {\downarrow} x$, and since $\theta({\downarrow} x) = x$ is defined, we have that the upper composite partial morphism is defined on $J \cap {\downarrow} B$. For the lower composite, we have $\bigcup(J \cap {\downarrow} B) = B$, so for the lower composite to be defined, we must have $B \in J$.

Finally if $\mathcal{A} \in DJ$, $Y \in J$ has $\bigvee Y = x$ and for any $a \in Y$, there is some $A_a \in \mathcal{A}$ with $\bigvee A_a \geqslant a$, then clearly $A_a \cap {\downarrow} a \in J$, and since $J \xrightarrow{\theta} L$ is an inf-homomorphism, $\bigvee(A_a \cap {\downarrow} a) = \theta(A_a \cap {\downarrow} a) = a$. Thus, setting $\mathcal{B} = {\downarrow}\{A_a \cap {\downarrow} a | a \in Y\}$ gives $D\theta(\mathcal{B}) = Y$, so the upper composite is defined for $\mathcal{B}$, and is equal to $x$. Thus, the lower composite gives $B = \bigcup \mathcal{B} \in J$ with $\theta(B) = x$, which proves the last condition.

Conversely, suppose that $(L, J)$ is a distributive partial sup lattice as in Definition 4.11. We want to show that $DL \overset{j}{\leftarrowtail} J \xrightarrow{\bigvee} L$ is a lax partial algebra for the downset KZ monad. That is, we want to show that

$$\begin{tikzcd} X \arrow[r, "\downarrow"] \arrow[dr, "1_X"'] & DX \arrow[d, "\theta"] \\ & X \end{tikzcd}$$

commutes, and

$$\begin{tikzcd} DDX \arrow[r, "D\theta"] \arrow[d, "\bigcup"'] \arrow[dr, phantom, "\geqslant"] & DX \arrow[d, "\theta"] \\ DX \arrow[r, "\theta"'] & X \end{tikzcd}$$

We expand the partial morphisms to get the following diagrams

$$\begin{tikzcd} L \arrow[r] \arrow[d, equal] & J \arrow[d, hook, "j"'] \arrow[ddr, "\bigvee"] & \\ L \arrow[r, "\downarrow"] \arrow[drr, "1_L"'] & DL & \\ & & L \end{tikzcd}
\qquad
\begin{tikzcd} DDL \arrow[r, "j^{-1}"] \arrow[dd, "\mu_L"'] \arrow[ddrr, phantom, "\geqslant"] & DJ \arrow[r, "D\bigvee"] & DL \\ & & J \arrow[u, hook, "j"'] \arrow[d, "\bigvee"] \\ DL & J \arrow[l, hook', "j"] \arrow[r, "\bigvee"'] & L \end{tikzcd}$$

The first diagram commutes because $J$ contains all principal downsets. For the second diagram, if the upper-right composite of the diagram is defined for $\mathcal{A}$, we have $j^{-1}(\mathcal{A}) = \mathcal{A} \cap J$, and $Y = {\downarrow}\{\bigvee A | A \in \mathcal{A} \cap J\} \in J$. By definition, for every $a \in Y$, there is some $A_a \in \mathcal{A} \cap J$ such that $\bigvee A_a \geqslant a$. Now, by the fourth condition in Definition 4.11, there is some $B \subseteq \bigcup \mathcal{A}$, with $B \in J$ and $\bigvee B \geqslant \bigvee Y$. For $x \in \bigcup \mathcal{A}$, we have ${\downarrow}(x) \in \mathcal{A} \cap J$, so $x \in Y$, and therefore $\bigvee Y = \bigvee (\bigcup \mathcal{A})$, so $\bigvee B = \bigvee Y = \bigvee (\bigcup \mathcal{A})$. Now by the third condition of Definition 4.11, it follows that $\bigcup \mathcal{A} \in J$, so the lower-left composite is defined for $\mathcal{A}$, giving the required inequality of partial maps. □

**Proposition 4.13.** *The definition of distributive partial sup-lattice homomorphisms given in Definition 4.11 is equivalent to a lax partial algebra homomorphism between lax partial algebras.*

*Proof.* Because $\theta$ is the restriction of the supremum operation, the lax partial algebra homomorphism condition is exactly that $J$ factors through the pullback

$$\begin{tikzcd} K^* \arrow[r] \arrow[d, tail] & K \arrow[d, tail] \\ DL \arrow[r, "Df"'] & DM \end{tikzcd}$$

and for any $A \in J$, $f\left(\bigvee A\right) \leqslant \bigvee {\downarrow}\{f(a)|a \in A\}$.

The pullback is given by $K^* = \{A \in DL|Df(A) \in K\}$. Thus the inclusion is equivalent to the condition for any $A \in J$, ${\downarrow}\{f(a)|a \in A\} \in K$.

Since $f$ is order-preserving, for $a \in A$, we have that $f(a) \leqslant f\left(\bigvee A\right)$, so $f\left(\bigvee A\right)$ is an upper bound for ${\downarrow}\{f(a)|a \in A\}$, and thus $\bigvee {\downarrow}\{f(a)|a \in A\} \leqslant f\left(\bigvee A\right)$. Thus the second condition that $\bigvee {\downarrow}\{f(a)|a \in A\} \geqslant f\left(\bigvee A\right)$ is equivalent to $\bigvee {\downarrow}\{f(a)|a \in A\} = f\left(\bigvee A\right)$ as required.

□

**Definition 4.14.** *An element $a$ of a partial sup-lattice $(L, J)$ is* totally compact *if for any downset $D \in J$, $\bigvee D \geqslant a \Rightarrow a \in D$. (Note that $\varnothing \in J$, so $\perp$ is not totally compact.) A partial sup-lattice $(L, J)$ is* totally compactly generated *if for any $x \in L$, there is some $C \subseteq L$ such that every $c \in C$ is totally compact, and such that ${\downarrow} C \in J$ and $\bigvee C = x$.*

**Proposition 4.15.** *The full subcategory of totally compactly generated distributive partial sup-lattices and partial sup-lattice homomorphisms is equivalent to the category $\mathcal{TCGP}artial\mathcal{S}up^{\mathrm{op}}$ defined at the start of Section 4.2.*

*Proof.* Given a totally compactly generated distributive partial sup-lattice $(L, J)$, let $K \subseteq L$ be the set of totally compact elements of $(L, J)$. Then $(L, K)$ is an element of $\mathcal{TCGP}artial\mathcal{S}up$. Conversely, for the object $(L, S) \in \mathrm{ob}\,\mathcal{TCGP}artial\mathcal{S}up$, let $J = \{D \in DL|S \cap {\downarrow}\left(\bigvee D\right) \subseteq D\}$ be the set of downsets of $L$ that contain all totally compact elements below their supremum. It is clear that performing these two constructions gives an isomorphic structure. To show an equivalence of categories, we need to show that $L \xrightarrow{f} M$ is a distributive partial sup-lattice homomorphism if and only if it is a morphism in $\mathcal{TCGP}artial\mathcal{S}up^{\mathrm{op}}$. Since distributive partial sup-lattice homomorphisms preserve infima, they have left adjoints. If $f$ is a partial sup-lattice homomorphism, and $f^*$ is its left adjoint, then $f^*$ is a sup-homomorphism, and for any totally compact $a \in M$, if $B \in J$ has $\bigvee B \geqslant f^*(a)$, then the adjunction gives $f\left(\bigvee B\right) \geqslant a$. Since $f$ is a partial sup-homomorphism, we have $f\left(\bigvee B\right) = \bigvee\{f(b)|b \in B\} \geqslant a$. As $a$ is totally compact, we must have $a \leqslant f(b)$ for some $b \in B$. By the adjunction, this gives $f^*(a) \leqslant b$. Thus we have shown that if $B \in J$ has $\bigvee B \geqslant f^*(a)$, then $f^*(a) \in B$. That is, $f^*(a)$ is totally compact.

Conversely, if $g$ is a sup-homomorphism between totally compactly generated distributive partial sup-lattices, that preserves totally compact elements, then its right adjoint is a partial sup-homomorphism, since if $B \in J$ has $\bigvee B = x$, then if $a \leqslant g_*(x)$ is totally compact, then $g(a) \leqslant x$ is also totally compact, so $g(a) \in B$. It follows that $a \in \downarrow\{g_*(b)|b \in B\}$, so $\bigvee \downarrow\{g_*(b)|b \in B\} = g_*(x)$ as required. $\square$

## 5. Final Remarks and Future Work

We have extended the left adjoint functor from Stone duality, sending a topological space to its coframe of closed sets to a functor sending a topological convexity spaces to its sup-lattice of closed convex sets. As in the topological Stone duality, this functor has a right adjoint. This right adjoint is not an extension of the topological case.

In many ways, the theory is nicer in this situation than in the topological case. For example, there are no non-spatial sup-lattices: every sup-lattice arises as the closed convex sets of a topological convexity space. However, in some ways this nicer theory makes the results less useful, because in topology, the non-spatial locales fill some problematic gaps in the category of topological spaces. With every sup-lattice arising as the closed convex sets of a topological convexity space, there are no new spaces to be added, so we are not filling the gaps.

Another significant difference between this and Stone duality for topological spaces is that for the topological Stone duality, many interesting topological spaces are in the singleton fibres of the functor, meaning that the closed set functor is full and faithful for these spaces, so we can study the categorical structure of large classes of interesting topological spaces using the category of coframes. For topological convexity spaces, there are no singleton fibres, and the top elements of fibres (on which the functor is full and faithful) are not very interesting topological convexity spaces. The most interesting topological convexity spaces are the bottom elements of their fibres, and when we restrict the functor to these spaces, it is faithful, but not full, meaning that from a categorical perspective, $\mathcal{C}onvex\mathcal{T}op$ and $\mathcal{S}up$ are not so closely related.

The adjunction between topological convexity spaces and sup-lattices factors through the category of preconvexity spaces, or the equivalent category of totally compactly-generated distributive partial sup-lattices. The adjunction between topological convexity spaces and preconvexity spaces is potentially more interesting, with most interesting topological convexity spaces being fixed-points of the induced comonad on $\mathcal{C}onvex\mathcal{T}op$. We have characterised which topological convexity spaces are fixed by this comonad in Proposition 3.11, and given some important examples in Proposition 3.14. In the opposite direction, for the question of which preconvexity spaces are fixed by the induced monad on $\mathcal{P}reconvex$, we have only been able to show this for a few special cases.

### 5.1 Future Work

The study of topological convexity spaces is an extremely promising area of research, including some classical geometric examples and also some very interesting combinatorial examples. The adjunctions from this paper are likely to prove extremely valuable in the study of topological convexity spaces. In this section, we discuss a number of important problems about topological convexity spaces that may be addressed using these adjunctions.

#### 5.1.1 Restricting this to a Duality

In topology, it is often convenient to restrict Stone duality to an isomorphism of categories between sober topological spaces and spatial locales. Sober topological spaces can be described in a number of topologically natural ways. Similarly, spatial locales can be easily described. It is easy to describe the topological convexity spaces from this adjunction, as they come directly from lattices. However, for the intermediate adjunction between topological convexity spaces and preconvexity spaces, the conditions for fixed points are less clear. The characterising conditions in Proposition 3.11 are not particularly natural, while the natural and commonly used conditions in Proposition 3.14 exclude a number of interesting combinatorial examples. A result between these two that includes the interesting combinatorial examples but also consists of natural, easy-to-understand conditions would be extremely valuable. In the other direction, describing the geometric preconvexity spaces is more challenging, and could lead to a lot of fruitful research.

### 5.1.2 Euclidean Spaces

The motivating examples for topological convexity spaces are real vector spaces, particularly finite-dimensional ones. The author has nearly completed a characterisation of these spaces within the category of topological convexity spaces, which will be presented in another paper.

### 5.1.3 Convexity Manifolds

In differential geometry, a manifold is a space which has a local differential structure. That is, the space is covered by a family of open sets, each of which has a local differential structure. There are examples of spaces with a cover by open subsets with a local convexity structure. The motivating example here is real projective space. We cannot assign a global convexity structure to projective space, but if we remove a line from the projective plane, then the remaining space is isomorphic to the Euclidean plane, and so has a canonical convexity space. Furthermore, these convexity spaces have a certain compatibility condition — given a subset $C$ of the intersection which is convex in both convexity spaces, the convex subsets of $C$ are the same in both spaces. This gives us the outline for a definition of convexity manifolds. Further work is needed to identify the Euclidean projective spaces within the category of convexity manifolds, and to determine what geometric structure is retained at this level of generality.

### 5.1.4 Metrics and Measures

There are connections between metrics and measures. For example, on the real line, any metric that induces the usual convexity space structure corresponds to a monotone function $\mathbb{R} \xrightarrow{d} \mathbb{R}$ with 0 as a fixed point. Such a function naturally induces a measure on the Lebesgue sets of $\mathbb{R}$. Conversely, for every measure on the Lebesgue sets of $\mathbb{R}$, we obtain a monotone endofunction of $\mathbb{R}$ by integrating. Thus, for the real numbers, there is a bijective correspondence between metrics that induce the usual topological convexity structure and measures on $\mathbb{R}$. This property is specific to $\mathbb{R}$, and does not generalise to other spaces like $\mathbb{R}^2$.

There is a more general connection between topological convexity spaces, sigma algebras, measures and metrics.

**Example 5.1.** Let $(X, \mathcal{B})$ be a $\Sigma$-algebra. There is a topological convexity space $(\mathcal{B}, \mathcal{F}, \mathcal{I})$ where $\mathcal{I}$ is the set of intervals in the lattice $\mathcal{B}$ and $\mathcal{F}$ is the set of collections of measurable sets closed under limits of characteristic functions. That is, for $B_1, B_2, \ldots \in \mathcal{B}$, say $B$ is the *limit* of $B_1, B_2, \ldots$ if for any $x \in B$, there is some $k \in \mathbb{Z}^+$ such that $x \notin B_i \Rightarrow i < k$, and for any $x \notin B$, there is some $k \in \mathbb{Z}^+$ such that $x \in B_i \Rightarrow i < k$. $\mathcal{F}$ is the collection of subsets $F \subseteq \mathcal{B}$ such that for any $B_1, B_2, \ldots \in F$, if $B$ is the limit of $B_1, B_2, \ldots$, then $B \in F$.

**Proposition 5.2.** *If $\mu$ is a finite measure on $(X, \mathcal{B})$ such that no non-empty set has measure zero, then $d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$ given by $d(A, B) = \mu(A \triangle B)$, is a metric and induces the topological convexity space structure $(\mathcal{B}, \mathcal{F}, \mathcal{I})$ or a finer structure. Furthermore, all metrics inducing this topological convexity space structure on $\mathcal{B}$ are of this form.*

*Proof.* We have $d(A, A) = \mu(\varnothing) = 0$ and $d(A, B) = d(B, A)$, so we need to prove the triangle inequality. That is, for $A, B, C \in \mathcal{B}$, we have $d(A, C) \leqslant d(A, B) + d(B, C)$. This is clear because $A \triangle C \subseteq A \triangle B \cup B \triangle C$. Thus $d$ is a metric. To prove that it induces this topological convexity structure, we note that $d(A, C) = d(A, B) + d(B, C)$ if and only if $A \triangle C = A \triangle B \amalg B \triangle C$. This only happens if $A \cap C \subseteq B \subseteq A \cup C$, which means that convex sets must be intervals. Finally, we need to show the topology from the metric is finer than $\mathcal{F}$. That is, if $B$ is the limit of $B_1, B_2, \ldots$, then $d(B_i, B) \to 0$. By definition, $\bigcap_{i=1}^{\infty} B_i \triangle B = \varnothing$. Thus, we need to show that for a sequence $A_i = B_i \triangle B$ of measurable sets with empty intersection, $\mu(A_i) \to 0$. Let $C_i = \bigcup_{j \geqslant i} A_j$. Since $A_i \to \varnothing$, we get $\bigcap_{i=1}^{\infty} C_i = \varnothing$. Since the $C_i$ are nested, we have $\lim_{i \to \infty} \mu(C_i) = \mu\left(\bigcap_{i=1}^{\infty} C_i\right) = 0$.

To show that every metric is of that form, let $d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$ be a metric on $\mathcal{B}$ whose induced topology and convexity are finer than $\mathcal{F}$ and $\mathcal{I}$ respectively. We want to show that there is a finite measure $\mu$ on $(X, \mathcal{B})$ such that $d(A, B) = \mu(A \triangle B)$. By the convexity, whenever $A = B \amalg C$ is a disjoint union, we have $d(A, B) + d(B, \varnothing) = d(\varnothing, A) = d(\varnothing, C) + d(C, A)$

and $d(B,\varnothing)+d(\varnothing,C)=d(B,C)=d(B,A)+d(A,C)$. It follows that

$$2d(A,B)+d(B,\varnothing)+d(A,C)=d(A,C)+d(B,\varnothing)+2d(\varnothing,C)$$

so $d(A,B)=d(C,\varnothing)$. For general $B$, we have $A\cap B$ is between $A$ and $B$, so

$$d(A,B)=d(A,A\cap B)+d(A\cap B,B)=d(\varnothing,A\backslash B)+d(\varnothing,B\backslash A)=d(\varnothing,A\triangle B)$$

Thus, if we define $\mu(B)=d(\varnothing,B)$, then $d$ is defined by $d(A,B)=\mu(A\triangle B)$. We need to show that $\mu$ is a measure on $(X,\mathcal{B})$. That is, that if $A$ and $B$ are disjoint, we have $\mu(A\cup B)=\mu(A)+\mu(B)$ and if $B_1\subseteq B_2\subseteq\cdots$, then $\mu(\bigcup_{i=1}^{\infty}B_i)=\lim_{i=1}^{\infty}\mu(B_i)$. We have already shown that the first of these comes from the convexity. The second comes from the topology. Consider the sequence $A_i=\left(\bigcup_{j=1}^{\infty}B_j\right)\backslash B_i$. By the convexity, we have $\mu(A_i)=\mu\left(\bigcup_{j=1}^{\infty}B_j\right)-\mu(B_i)$, so it is sufficient to show that $\mu(A_i)\to 0$, when $(A_i)_{i=1}^{\infty}$ is a decreasing sequence with empty intersection. If $(A_i)_{i=1}^{\infty}$ is a decreasing sequence with empty intersection, then for any $x\in X$, we have $(\exists k\in\mathbb{Z}^+)(x\notin A_k)$. Thus $\varnothing$ is a limit of $(A_i)_{i=1}^{\infty}$. Thus we have $\mu(A_i)\to\mu(\varnothing)=0$ as required.

□

### 5.1.5 Sheaves

A lot of information about topological spaces can be obtained by studying their categories of sheaves. A natural question is whether a similar category of sheaves can be constructed for a topological convexity space. Part of the difficulty here is that the usual construction of the sheaf category is described in terms of open sets. However, for topological convexity spaces, closed sets are more fundamental, so it is necessary to redefine sheaves in terms of closed sets. This is conceptually strange. One interpretation of sheaves is as sets with truth values given by open sets. In this interpretation, closed sets correspond to the truth values of negated statements, such as inequality. [5] argues that inequality is a more fundamental concept for studying lattices of equivalence relations as a form of logical statement, so a definition of sheaves in terms of closed sets could be linked to this work.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXIV-3 (2023)

# PICARD GROUPOIDS AND $\Gamma$-CATEGORIES

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**Résumé.** Dans cet article, nous construisons une catégorie de modèles fermée monoïdale symétrique de groupoïdes de Picard commutatifs cohérents. Nous construisons une autre structure de catégorie de modèles sur la catégorie des (petites) catégories permutatives dont les objets fibrants sont des groupoïdes de Picard (permutatifs). Le résultat principal de cet article est que le foncteur nerf de Segal est un foncteur de Quillen droit d'une équivalence de Quillen entre les deux catégories de modèles susmentionnées. Sur la base de notre résultat principal, nous donnons une nouvelle preuve du résultat classique selon lequel les groupoïdes de Picard modélisent des monotypes d'homotopie stables.

**Abstract.** In this paper we construct a symmetric monoidal closed model category of coherently commutative Picard groupoids. We construct another model category structure on the category of (small) permutative categories whose fibrant objects are (permutative) Picard groupoids. The main result of this paper is that the Segal's nerve functor is a right Quillen functor of a Quillen equivalence between the two aforementioned model categories. Based on our main result, we give a new proof of the classical result that Picard groupoids model stable homotopy one-types.

**Keywords.** Your keywords come here.
**Mathematics Subject Classification (2010).** Your MSC numbers come here.

## 1. Introduction

Picard groupoids are interesting objects both in topology and algebra. A major reason for interest in topology is because they model stable homotopy 1-types which is a classical result appearing in various parts of the literature [JO12][Pat12][GK11][MOP$^+$20]. The category of Picard groupoids is the archetype example of a 2-Abelian category, see [Dup08]. A theory of 2-chain complexes of Picard groupoids was developed in [dRMMV05]. A simplicial cohomology with coefficients in Picard groupoids was introduced in the paper [CMM04].

This cohomology was used in [SV] to construct a TQFT called the *Dijkgraaf-Witten* theory. This (Picard) groupoidification of cohomology played a vital role in explaining a mysterious *integration theory* introduced in [FQ93], [Shab] which is pivotal in constructing the aforementioned TQFT functor.

A tensor product of Picard groupoids was defined in [Sch08]. However, a shortcoming of the category of Picard groupoids remains: unlike the category of abelian groups, it is not a symmetric monoidal closed category. In this paper we address this problem by proposing another model for Picard groupoids based on $\Gamma$- categories. A $\Gamma$- category is a functor from the (skeletal) category of finite based sets $\Gamma^{op}$ into the category of all (small) categories $\mathbf{Cat}$. We denote the category of all $\Gamma$- categories and natural transformations between them by $\Gamma\mathbf{Cat}$. Along the lines of the construction of the stable Q-model category in [Sch99], we construct a symmetric monoidal closed model category $\Gamma\mathcal{P}ic$. The underlying category of $\Gamma\mathcal{P}ic$ is $\Gamma\mathbf{Cat}$ and we refer to it as the model category structure of *coherently commutative Picard groupoids*. A $\Gamma$- category $X$ is called a *coherently commutative Picard groupoid* if it satisfies the Segal condition, see [Seg74] and moreover it has homotopy inverses. These $\Gamma$- categories are fibrant objects in our model category $\Gamma\mathcal{P}ic$. The main objective of this paper is to compare a (model) category of all (small) Picard groupoids with the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$. We construct another model category structure on $\mathbf{Perm}$ whose fibrant objects are (permutative) Picard groupoids. This model category is denoted by $(\mathbf{Perm}, \mathbf{Pic})$ and called the model category of Picard groupoids. The main result of this paper, theorem 5.2, states that the following adjoint pair is a Quillen equivalence:

$$\mathcal{L} : \Gamma\mathcal{P}ic \rightleftarrows (\mathbf{Perm}, \mathbf{Pic}) : \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is the classical Segal's nerve functor which was originally defined in [Seg74] and a different description of it has recently appeared in[Shac].

A second aspect of our paper is about establishing a Quillen equivalence between a second pair of model category structures on the same two underlying categories, namely $\mathbf{Perm}$ and $\Gamma\mathbf{Cat}$. We first construct another cartesian closed (combinatorial) model category structure on $\mathbf{Cat}$, denoted by $(\mathbf{Cat}, \mathbf{Gpd})$, whose fibrant objects are groupoids. We then transfer this model category structure on the category of all permutative categories $\mathbf{Perm}$. Fibrant objects in this model category are permutative groupoids and it is denoted by $(\mathbf{Perm}, \mathbf{Gpd})$. We localize the model category of coherently commutative monoidal categories to get another symmetric monoidal closed model category $\Gamma\mathbf{Gpd}^{\otimes}$. The fibrant objects of this model category can be described as coherently commutative monoidal groupoids. These two model categories are instrumental in the construction of the model categories featuring in our main result. A second prominent result of this paper is that the following adjunction is a Quillen equivalence:

$$\mathcal{L} : \Gamma\mathbf{Gpd}^{\otimes} \rightleftarrows (\mathbf{Perm}, \mathbf{Gpd}) : \mathcal{K} \tag{2}$$

The Quillen equivalences (1) and (2) are proved in section 5. We first prove (2) and then use that to prove our main result (1).

In the last section of this paper we use our main result to prove a version of the *stable homotopy hypothesis* for Picard groupoids: the homotopy category of $(\mathbf{Perm}, \mathbf{Pic})$ is equivalent to a homotopy category of stable homotopy one-types. We recall that a stable homotopy one-type is a connective spectrum whose stable homotopy groups are trivial in all degrees greater than one.

We would like to mention that the Quillen equivalence (2) proved in theorem 5.1 can also be proved by an adaptation of some results of [dBM17] to (a suitable model category of) groupoids. This approach goes through the theory of dendroidal groupoids, more concretely, the approach appeals to the abstract relations between dendroidal groupoids and algebras over operads in groupoids and dendroidal groupoids and $\Gamma$-groupoids. In this paper we establish this Quillen equivalence without using any theory of dendroidal groupoids. Our approach is direct and is based on the results of [Shac].

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## 2. The Setup

In this section we will collect the machinery needed for the development of this paper. We begin with a review of permutative categories. We will also give a quick review of $\Gamma$- categories and collect some useful results about them. We will also construct a cartesian closed (simplicial) model category structure on the category of (small) categories $\mathbf{Cat}$ which will be used throughout this paper.

### 2.1 Review of Permutative categories

In this subsection we will briefly review the theory of permutative categories and monoidal and oplax functors between them. The definitions reviewed here and the notation specified here will be used throughout this paper.

**Definition 2.1.** A symmetric monoidal category $C$ is called a *permutative* category or a *strict* symmetric monoidal category if its monoidal structure is strictly associative and unital.

**Definition 2.2.** An *oplax symmetric monoidal* functor $F$ is a triple $(F, \lambda_F, \epsilon_F)$, where $F : C \to D$ is a functor between symmetric monoidal categories $C$ and $D$,

$$\lambda_F : F \circ (- \underset{C}{\otimes} -) \Rightarrow (- \underset{D}{\otimes} -) \circ (F \times F)$$

is a natural transformation and $\epsilon_F : F(1_C) \to 1_D$ is a morphism in $D$, such that the following three conditions $OL.1$, $OL.2$ and $OL.3$ in [Shac, Defn. 2.4] are satisfied.

**Notation 2.3.** We will say that a functor $F : C \to D$ between two symmetric monoidal categories is *unital* or *normalized* if it preserves the unit of the symmetric monoidal structure *i.e.* $F(1_C) = 1_D$. In particular, we will say that an oplax symmetric monoidal functor is a unital (or normalized) oplax symmetric monoidal functor if the morphism $\epsilon_F$ is the identity.

**Definition 2.4.** An oplax symmetric monoidal functor $F = (F, \lambda_F, \epsilon_F)$ is called a *strong* symmetric monoidal functor (or just a symmetric monoidal functor) if $\lambda_F$ is a natural isomorphism and $\epsilon_F$ is also an isomorphism.

**Definition 2.5.** An oplax symmetric monoidal functor $F = (F, \lambda_F, \epsilon_F)$ is called a *strict* symmetric monoidal functor if it is unital and $\lambda_F$ is the identity natural transformation.

**Definition 2.6.** The *category of elements* of a $\mathbf{Cat}$ valued functor $F : C \to \mathbf{Cat}$, denoted by $\int^{c \in C} F(c)$ or *elF*, is a category whose objects are pairs $(c, d)$, where $c \in C$ and $d \in F(c)$. A map from $(c, d)$ to $(a, b)$ in $\int^{c \in C} F(c)$ is a pair $(f, \alpha)$, where $f : c \to a$ is an arrow in $C$ and $\alpha : F(f)(d) \to b$ is an arrow in $F(a)$.

**Notation 2.7.** Throughout this paper we will denote by $J : \mathbf{Cat} \to \mathbf{Gpd}$, a right adjoint of the inclusion functor $i : \mathbf{Gpd} \to \mathbf{Cat}$. For a category $C$, the groupoid $J(C)$ is obtained by discarding all non-invertible arrows of $C$.

## 2.2 Review of Γ- categories

In this subsection we will briefly review the theory of Γ- categories. We begin by introducing some notations which will be used throughout the paper.

**Notation 2.8.** We will denote by $\underline{n}$ the finite set $\{1, 2, \ldots, n\}$ and by $n^+$ the based set $\{0, 1, 2, \ldots, n\}$ whose basepoint is the element $0$.

**Notation 2.9.** We will denote by $\mathcal{N}$ the skeletal category of finite unbased sets whose objects are $\underline{n}$ for all $n \geq 0$ and maps are functions of unbased sets. The category $\mathcal{N}$ is a (strict) symmetric monoidal category whose symmetric monoidal structure will be denoted by $+$. For two objects $\underline{k}, \underline{l} \in \mathcal{N}$ their *tensor product* is defined as follows:

$$\underline{k} + \underline{l} := \underline{k + l}.$$

**Notation 2.10.** We will denote by $\Gamma^{op}$ the skeletal category of finite based sets whose objects are $n^+$ for all $n \geq 0$ and maps are functions of based sets.

**Notation 2.11.** Given a morphism $f : n^+ \to m^+$ in $\Gamma^{op}$, we denote by $\mathrm{Supp}(f)$ the largest subset of $\underline{n}$ whose image under $f$ does not contain the basepoint of $m^+$. The set $\mathrm{Supp}(f)$ inherits an order from $\underline{n}$ and therefore could be regarded as an object of $\mathcal{N}$. We denote by $\mathrm{Supp}(f)^+$ the based set $\mathrm{Supp}(f) \sqcup \{0\}$ regarded as an object of $\Gamma^{op}$ with order inherited from $\underline{n}$.

**Definition 2.12.** A map $f : n^+ \to m^+$ in $\Gamma^{op}$ is called *inert* if its restriction to the set $\mathrm{Supp}(f)^+$ is a bijection.

**Definition 2.13.** A morphism $f$ in $\Gamma^{op}$ is called *active* if $f^{-1}(\{0\}) = \{0\}$ *i.e.* the pre-image of $\{0\}$ is the singleton set $\{0\}$.

**Notation 2.14.** A map $f : \underline{n} \to \underline{m}$ in the category $\mathcal{N}$ uniquely determines an active map in $\Gamma^{op}$ which we will denote by $f^+ : n^+ \to m^+$. This map agrees with $f$ on non-zero elements of $n^+$.

*Remark* 1. Each morphism in $\Gamma^{op}$ can be factored into a composite of an inert map followed by an active map in $\Gamma^{op}$. The factorization is unique up to a unique isomorphism.

**Definition 2.15.** Each $n^+ \in \Gamma^{op}$ determines $n$ *projection* maps $\delta^n_i : n^+ \to 1^+$ for $1 \leq i \leq n$ which are defined by $\delta^n_i(i) = 1$ and $\delta^n_i(j) = 0$ for $j \neq i$ and $j \in n^+$.

**Definition 2.16.** Each $n^+ \in \Gamma^{op}$ determines a *multiplication* map $m_n : n^+ \to 1^+$ which is the unique active map from $n^+$ to $1^+$.

## 2.3 The model category structure of groupoids on $\mathrm{Cat}$

In this subsection we will construct another model category structure on the category of all small categories $\mathbf{Cat}$ wherein an object is fibrant if and only if it is a groupoid and which we will refer to as the *model category structure of groupoids*. We remark that the model structure constructed here is different from the two well known model structures on $\mathbf{Cat}$, namely the natural model structure wherein all categories are fibrant and the Thomason model category structure which is Quillen equivalent to the Kan model category of simplicial sets $(\mathbf{sSets}, \mathbf{Kan})$. We will show that the weak equivalences in this model structure are those functors which induce a weak homotopy equivalence on their nerve. The model category structure is obtained by a *left Bousfield localization* of the *natural* model category structure on $\mathbf{Cat}$ with respect to the singleton set $\{i : 0 \to I$, where $I$ is the category $0 \to 1$ and $i(0) = 0$. We review the definition and an existence result of left Bousfield localizations of model categories in appendix A.

**Proposition 2.17.** A category $C$ is local with respect to the singleton set $\{i : 0 \to I\}$ if and only if it is a groupoid.

*Proof.* Let $\mathcal{J}$ denote the groupoid $0 \cong 1$. This groupoid is equipped with an inclusion functor $\iota : I \hookrightarrow \mathcal{J}$. A category $C$ is a groupoid if and only if $J([\iota, C]) : J([\mathcal{J}, C]) \to J([I, C])$ is an equivalence of categories.

Since each object of the natural model category $\mathbf{Cat}$ is both cofibrant and fibrant, for any pair of categories $C$ and $D$, the homotopy function complex is given as follows:

$$Map^h_{\mathbf{Cat}}(C, D) = N(J([C, D])).$$

This implies that a category $C$ is $\{i\}$-local if and only if the following functor is an equivalence of groupoids:

$$J([i, C]) : J([I, C]) \to J(C).$$

Now we consider the following commutative diagram:

$$\begin{tikzcd} J([I,C]) \arrow[r, "J([i,C])"] & J(C) \\ & J([\mathcal{J},C]) \arrow[ul, "J([\iota,C])"] \arrow[u, "J([j,C])"'] \end{tikzcd}$$

where $j$ is the inclusion functor $0 \hookrightarrow \mathcal{J}$. In light of the observation that the functor $J([j, C])$ is an equivalence of groupoids, the result now follows from the above commutative diagram of groupoids. $\square$

**Theorem 2.18.** There is a combinatorial model category structure on the category of (small) categories $\mathbf{Cat}$ in which a functor $F : A \to B$ is

1. a cofibration if it is monic on objects.

2. a weak equivalence if the following functor

$$[i, F] : [B, Z] \to [A, Z]$$

is an equivalence of categories for each groupoid $Z$.

3. a fibration if it has the right lifting property with respect to functors which satisfy both $(1)$ and $(2)$.

*Proof.* We want to carry out a left Bousfield localization of the natural model category of (small) categories with respect to the singleton set $\{i : 0 \to I$. The existence of this localization follows from theorem A.2. $(1)$ follows from the aforementioned theorem. $(2)$ follows from proposition 2.17 and [Shac, Lemma E.4]. $(3)$ follows from the fact that fibrations in any model category are completely determined by cofibrations and weak equivalences. □

**Notation 2.19.** We will refer to the above model category structure as the model category structure of *groupoids* on $\mathbf{Cat}$ and denote the model category by $(\mathbf{Cat}, \mathbf{Gpd})$. We will refer to a fibration in this model category as a *path fibration* of categories and refer to a weak equivalence as a *groupoidal equivalence* of categories.

*Remark* 2. Every category is cofibrant in the model category of groupoids. A category is fibrant if and only if it is a groupoid.

*Remark* 3. A groupoidal equivalence between groupoids is an equivalence of categories.

**Proposition 2.20.** The nerve of a path fibration of categories between two groupoids is a Kan fibration of simplicial sets.

*Proof.* Let $p : C \to D$ be a path fibration of categories such that both $C$ and $D$ are groupoids. Since $C$ and $D$ are fibrant in $(\mathbf{Cat}, \mathbf{Gpd})$, which is a left Bousfield localization of the natural model category structure on $\mathbf{Cat}$, therefore $p$ is an isofibration from [Sha20, Lem. 4.17].

The nerve functor takes an isofibrations to a pseudo-fibration *i.e.* a fibration in the Joyal model category on simplicial sets so $N(p) : N(C) \to N(D)$ is a pseudo-fibration. However both $N(C)$ and $N(D)$ are Kan complexes. Now it follows that $N(p)$ is a Kan fibration , by the same aforementioned result [Sha20, Lem. 4.17] because $(\mathbf{sSets}, \mathbf{Kan})$ is a left Bousfield localization of the Joyal model category of quasi-categories $(\mathbf{sSets}, \mathbf{Q})$. □

Next we are interested in providing a characterization of weak equivalences and fibrations in the model category of groupoids. We first recall the notion of a *homotopy reflection*:

**Definition 2.21.** A Quillen adjunction $(F, G)$ is called a *homotopy reflection* if the right derived functor of $G$ is fully-faithful.

**Lemma 2.22.** The adjunction $\tau_1 : \mathbf{sSets} \rightleftarrows \mathbf{Cat} : N$ is a Quillen adjunction between the model category of groupoids and the Kan model category of simplicial sets. Further the adjunction is also a homotopy reflection.

*Proof.* The first statement follows from the observation that the adjunction in context is a composite of the following two Quillen adjunctions:

$$\tau_1 : (\mathbf{sSets}, \mathbf{Kan}) \rightleftarrows \mathbf{Cat} : N$$

and

$$id : \mathbf{Cat} \rightleftarrows (\mathbf{Cat}, \mathbf{Gpd}) : id$$

where $\mathbf{Cat}$ denotes the natural model category of (small) categories.

The second statement follows from the observation that both of the aforementioned Quillen adjunctions are homotopy reflections and the fact that a composite of homotopy reflections is again a homotopy reflection.

□

The following corollary is an easy consequence of the above lemma:

**Corollary 2.23.** A functor $F : G \to H$ between groupoids is a groupoidal equivalence if and only if it's nerve, $N(F)$, is a homotopy equivalence of Kan complexes.

The inclusion functor $\mathbf{Gpd} \to \mathbf{Cat}$, where $\mathbf{Gpd}$ is the full subcategory whose objects are groupoids, has a left adjoint which we denote by $\Pi_1 : \mathbf{Cat} \to \mathbf{Gpd}$. The groupoid $\Pi_1(C)$ is obtained from the category $C$ by formally inverting all arrows in $C$ *i.e.* $\Pi_1(C) = C[Ar(C)^{-1}]$.

*Remark* 4. In the paper [JT08] a model category structure was constructed on the full subcategory of $\mathbf{Cat}$ whose objects are groupoids $\mathbf{Gpd}$. We will refer to this model category as the *natural model category of groupoids*. The functor $\Pi_1$ is a left Quillen functor of a Quillen adjunction

$$\Pi_1 : \mathbf{Cat} \rightleftharpoons \mathbf{Gpd} : i$$

where $\mathbf{Cat}$ is endowed with the model category structure of groupoids and $\mathbf{Gpd}$ is the natural model category of groupoids. This Quillen adjunction is a Quillen equivalence.

The following proposition will be used repeatedly in this paper:

**Proposition 2.24.** The free groupoid functor $\Pi_1 : \mathbf{Cat} \to \mathbf{Gpd}$ preserves products.

**Proposition 2.25.** A functor $F : C \to D$ is a groupoidal equivalence if and only if the induced functor $\Pi_1(F) : \Pi_1(C) \to \Pi_1(D)$ is an equivalence of categories.

*Proof.* The unit of the adjunction $\Pi_1 \dashv i$ gives the following commutative diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "F"'] & \Pi_1(C) \arrow[d, "\Pi_1(F)"] \\ D \arrow[r] & \Pi_1(D) \end{tikzcd}$$

where both vertical functors are inclusions. We will first prove that these two inclusion maps are both weak equivalences. Since $\Pi_1$ is a left adjoint to the inclusion functor $i$ therefore the inclusion functor $\iota_C : C \to \Pi_1(C)$ induces the following bijection for each groupoid $G$:

$$\mathbf{Cat}(\Pi_1(C), G) \cong \mathbf{Cat}(C, G).$$

Consider the following chain of bijections:

$$\begin{aligned} \mathbf{Cat}(I, [\Pi_1(C), G]) &\cong \mathbf{Cat}(I \times \Pi_1(C), G) \cong \mathbf{Cat}(\Pi_1(C), [I, G]) \\ &\cong \mathbf{Cat}(C, [I, G]) \cong \mathbf{Cat}(I \times C, G) \cong \mathbf{Cat}(I, [C, G]). \end{aligned}$$

The above two bijections together imply that we have the following equivalence of functor categories:

$$[\iota_C, G] : [\Pi_1(C), G] \to [C, G].$$

Now Theorem 2.18 $(2)$ implies that the two inclusion maps are weak equivalences in the model category structure of groupoids. Now the theorem follows from the two out of three property of weak equivalences in a model category. $\square$

Finally we would like to show that the groupoidal model category structure on $\mathbf{Cat}$ is cartesian closed.

**Proposition 2.26.** The groupoidal model category structure on $\mathbf{Cat}$ is cartesian closed.

*Proof.* The proposition follows from an application of theorem A.3 to the cartesian closed natural model category $\mathbf{Cat}$ with respect to the singleton set of maps $\{i_0\}$. $\square$

**Proposition 2.27.** The model category of groupoids is a simplicial model category.

*Proof.* The proposition follows by an application of [Bar07, lem. 3.6] to the Quillen adjunction $(\tau_1, N)$ from lemma 2.22. The simplicial Hom is defined by the composite bifunctor:

$$\mathbf{Cat}^{op} \times \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Cat} \overset{N}{\to} \mathbf{sSets}.$$

The cotensor is defined by the following bifunctor:

$$\mathbf{sSets}^{op} \times \mathbf{Cat} \to \mathbf{Cat}^{op} \overset{\tau_1 \times id}{\times} \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Cat}$$

The tensor product bifunctor is defined by the following composite:

$$\mathbf{Cat} \times \mathbf{sSets} \overset{id \times \tau_1}{\to} \mathbf{Cat} \times \mathbf{Cat} \overset{- \times -}{\to} \mathbf{Cat}.$$

□

## 3. Two model category structures on $\mathbf{Perm}$

We denote by $\mathbf{Perm}$ the category whose objects are *permutative* categories, namely symmetric monoidal categories which are strictly unital and strictly associative. The morphisms of this category are *strict* symmetric monoidal functors, namely those symmetric monoidal functors which preserve the symmetric monoidal structure strictly. A model category structure on $\mathbf{Perm}$ was described in [Shac, Thm. 3.1]. This model category structure was obtained by transferring the natural model category structure on $\mathbf{Cat}$ to $\mathbf{Perm}$ and therefore it is aptly called the *natural* model category structure of permutative categories. In this section we will describe two new model category structures on $\mathbf{Perm}$ which can be described as the model category of permutative groupoids and the model category of (permutative) Picard groupoids.

### 3.1 The model category structure of Permutative groupoids

In this subsection we will construct the desired model category structure of permutative groupoids on $\mathbf{Perm}$ namely a model category structure whose fibrant objects are groupoids equipped with a permutative or strict symmetric monoidal structure. Before doing so we recall the following adjunction and also a permutative groupoid structure inherited by the fundamental groupoid of a permutative category:

$$\mathcal{F} : \mathbf{Cat} \rightleftharpoons \mathbf{Perm} : i \tag{3}$$

where $i$ is the forgetful functor and $\mathcal{F}$ is its left adjoint namely the *free permutative category* functor. The following lemma recalls the aforementioned permutative structure:

**Lemma 3.1.** The fundamental groupoid of a permutative category is a permutative groupoid.

*Proof.* Let $C$ be a permutative category and let $-\otimes- : C \times C \to C$ be bifunctor giving the permutative structure. From proposition 2.24, we have the isomorphism $\Pi_1(C \times C) \cong \Pi_1(C) \times \Pi_1(C)$. Since $\Pi_1(C)$ is a groupoid, the universal property of $\Pi_1(C \times C)$ and the above isomorphism imply that we have a dotted arrow in the following diagram which makes the diagram commutative:

$$\begin{tikzcd} C \times C \arrow[r, "-\otimes-"] \arrow[d] & C \arrow[d] \\ \Pi_1(C) \times \Pi_1(C) \arrow[r, dashed] & \Pi_1(C) \end{tikzcd}$$

This bifunctor, represented by the dotted arrow in the above diagram, provides a permutative structure on the groupoid $\Pi_1(C)$. The symmetry natural transformation of $C$ is a functor

$$\gamma_C : C \times C \times J \to C$$

Once again by proposition 2.24 the free groupoid generated by $C \times C \times J$ is $\Pi_1(C) \times \Pi_1(C) \times J$. Again, the universal property of $\Pi_1(C \times C \times J)$ and the above isomorphism imply that we have a dotted arrow in the following diagram:

$$\begin{tikzcd} C \times C \times J \arrow[r, "\gamma_C"] \arrow[d] & C \arrow[d] \\ \Pi_1(C) \times \Pi_1(C) \times J \arrow[r, dashed, "\gamma_{\Pi_1(C)}"'] & \Pi_1(C) \end{tikzcd}$$

which is the symmetry natural isomorphism of $\Pi_1(C)$. □

*Remark* 5. The functor $\Pi_1$ restricts to a functor on $\mathbf{Perm}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, "\Pi_1"] & \mathbf{Gpd} \\ \mathbf{Perm} \arrow[u, "i"] \arrow[r, "\Pi_1"'] & \mathbf{PGpd} \arrow[u, "i"'] \end{tikzcd}$$

where $\mathbf{PGpd}$ denotes the category of permutative groupoids *i.e.*, the full subcategory of $\mathbf{Perm}$ having as objects those permutative categories whose underlying categories are groupoids.

We recall from [Shac] that $\Gamma^1$ is the representable $\Gamma$- category which is also the unit of the symmetric monoidal structure on the functor category $\Gamma\mathbf{Cat}$ which is tensored over $\mathbf{Cat}$. The inclusion map $i_0 : 0 \hookrightarrow I$ gives us the following map of $\Gamma$- categories by *tensoring* with $\Gamma^1$:

$$\Gamma^1 \otimes i_0 : \Gamma^1 \to \Gamma^1 \otimes I \tag{4}$$

We further recall from [Shac] the Quillen equivalence $\mathcal{L} : \Gamma\mathbf{Cat} \rightleftarrows \mathbf{Perm} : \mathcal{K}$. The image of the above map under the left Quillen functor $\mathcal{L}$ gives us the following strict symmetric monoidal functor which is the generator of the model structure to be constructed later in this subsection:

$$\mathcal{L}(\Gamma^1 \otimes i_0) : \mathcal{L}(\Gamma^1) \to \mathcal{L}(\Gamma^1 \otimes I) \tag{5}$$

*Remark* 6. The above strict symmetric monoidal functor $\mathcal{L}(\Gamma^1 \otimes i_0)$ has cofibrant domain and codomain.

Now we state the main theorem of this subsection:

**Theorem 3.2.** There is a model category structure on the category of all small permutative categories and strict symmetric monoidal functors $\mathbf{Perm}$ in which

1. A cofibration is a strict symmetric monoidal functor which is a cofibration in the natural model category structure on $\mathbf{Perm}$
2. A weak-equivalence is an $\{\mathcal{L}(\Gamma^1 \otimes i_0)\}$-local equivalence.
3. A fibration is a strict symmetric monoidal functor having the right lifting property with respect to all maps which are both cofibrations and weak equivalences.

Further, this model category structure is combinatorial and left-proper.

*Proof.* The desired model category structure is a left-Bousfield localization of the left-proper, combinatorial natural model category structure on $\mathbf{Perm}$ with respect to the singleton $\{\mathcal{L}(\Gamma^1 \otimes i_0)\}$. The existence follows from A.2. $\square$

The following proposition characterizes fibrant objects of the above model category:

**Proposition 3.3.** A permutative category is fibrant in the above model category if and only if it's underlying category is a groupoid.

*Proof.* Let $C$ be a permutative category. In light of 2.17, it is sufficient to show that $C$ is fibrant in the above model category if and only if $J([i, C]) : J([I, C]) \to J(C)$ is an equivalence of groupoids. It follows from the results of [Shac, appendix D] that the two homotopy function complexes in context can be defined as follows:

$$Map^h(\mathcal{L}(\Gamma^1 \otimes I), C) = N(J([\mathcal{L}(\Gamma^1 \otimes I), C]^{str}_{\otimes}))$$

and

$$Map^h(\mathcal{L}(\Gamma^1), C) = N(J([\mathcal{L}(\Gamma^1), C]^{str}_{\otimes})).$$

Further, the simplicial map $Map^h(\mathcal{L}(\Gamma^1 \otimes i_0), C)$ is an equivalence of Kan complexes if and only if $J([\mathcal{L}(\Gamma^1 \otimes i_0), C]^{str}_{\otimes})$ is an equivalence of groupoids. Now the result can be deduced by the following commutative diagram in the category of groupoids:

$$\begin{tikzcd}
J([\mathcal{L}(\Gamma^1 \otimes I), C]^{str}_{\otimes}) \arrow[r, "\cong"] \arrow[dd, "J({[\mathcal{L}(\Gamma^1 \otimes i_0), C]^{str}_{\otimes}})"'] & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, \mathcal{K}(C))) \arrow[d, "\cong"] & \\
 & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(I, \mathcal{K}(C)))) \arrow[r, "\cong"] \arrow[d, "J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, \mathcal{K}(C))))"'] & J([I, C]) \arrow[d, "J({[i, C]})"] \\
J([\mathcal{L}(\Gamma^1), C]^{str}_{\otimes}) \arrow[r, "\cong"'] & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathcal{K}(C))) \arrow[r, "\cong"'] & J(C)
\end{tikzcd}$$

where $- \otimes -$, $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-, -)$ and $\mathbf{hom}_{\Gamma\mathbf{Cat}}(-, -)$ are the tensor product, categorical Hom and cotensor of $\Gamma\mathbf{Cat}$ over $\mathbf{Cat}$. See [Shac, Sec. 4] for details. $\square$

**Notation 3.4.** We will refer to the above model category as the model category of *permutative groupoids* and will be denoted by $(\mathbf{Perm}, \mathbf{Gpd})$.

The following proposition presents a characterization of weak-equivalences in $(\mathbf{Perm}, \mathbf{Gpd})$:

**Proposition 3.5.** A strict symmetric monoidal functor $F : C \to D$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if its image $U(F)$, under the forgetful functor $U : \mathbf{Perm} \to \mathbf{Cat}$, is a groupoidal equivalence of (ordinary) categories.

*Proof.* The proof has two parts. In the first part we show that $(\Pi_1, i)$ is a fibrant replacement functor on $(\mathbf{Perm}, \mathbf{Gpd})$. To show this it suffices to show that for each permutative category $C$, the map $i : C \to \Pi_1(C)$ is an acyclic cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$. It is easy to see that $i$ is a cofibration in $\mathbf{Perm}$ in light of the observation that $Ob(i)$ is the identity function and [Shac, Lem. 3.8]. In order to show that $i$ is also a weak-equivalence we observe the following diagram:

$$\begin{tikzcd} C \arrow[r, "i_c"] \arrow[d, "i"'] & C^f \arrow[d, "i^f"] \\ \Pi_1(C) \arrow[ur, dashed] \arrow[r, equal] & \Pi_1(C) \end{tikzcd}$$

where $i = i^f \circ i_c$ is a factorization of $i$ into an acyclic cofibration $i_c$ and a fibration $i^f$ in $(\mathbf{Perm}, \mathbf{Gpd})$. Since $i^f$ is a fibration in $(\mathbf{Perm}, \mathbf{Gpd})$ therefore $C^f$ is a permutative groupoid. Further, a fibration between permutative groupoids is a fibration in the natural model category $\mathbf{Perm}$ which is an isofibration. Thus $i_f$ is a fibration in $(\mathbf{Cat}, \mathbf{Gpd})$. Since $i$ is an acyclic cofibration in $(\mathbf{Cat}, \mathbf{Gpd})$ therefore we have a (dotted) lifting arrow which makes the whole diagram commutative. Since the two horizontal arrows in the above diagram are weak-equivalences in $(\mathbf{Perm}, \mathbf{Gpd})$, by the two out of six property of weak-equivalences in model categories we conclude that $i$ is also a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$.

In the second part of the proof we establish the desired result. Since $(\Pi_1, i)$ is a fibrant replacement functor therefore $F$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if $\Pi_1(F)$ is one. However, $\Pi_1(F)$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if it is a weak-equivalence in the natural model category $\mathbf{Perm}$. Thus $\Pi_1(F)$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if $U(\Pi_1(F))$ is an equivalence of categories. Now the result follows from proposition 2.25.

□

The natural model category structure on $\mathbf{Perm}$ is a $\mathbf{Cat}$-model category structure [Shac, Thm. 3.1]. We recall that the *cotensor* of this enrichment is given by the bifunctor

$$[-,-] : \mathbf{Cat}^{op} \times \mathbf{Perm} \overset{id \times U}{\to} \mathbf{Cat} \times \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Perm} \tag{6}$$

where $[-,-]$ is the internal Hom of $\mathbf{Cat}$ but it takes values in $\mathbf{Perm}$ if the codomain category is permutative. The categorical Hom is the category of strict symmetric monoidal functors is given by the bifunctor

$$[-,-]^{str}_{\otimes} : \mathbf{Perm}^{op} \times \mathbf{Perm} \to \mathbf{Cat} \tag{7}$$

The *tensor product* of this enrichment is does not have a simple description but we will denote it as follows:

$$- \boxtimes - : \mathbf{Cat} \times \mathbf{Perm} \to \mathbf{Perm} \tag{8}$$

**Proposition 3.6.** The model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category.

*Proof.* Let $i : U \to V$ be a functor which is monic on objects and $j : W \to X$ be a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$. We will show that the following map in $\mathbf{Perm}$ is a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$ which is acyclic whenever $i$ or $j$ is acyclic:

$$i \square j : (V \boxtimes W) \underset{U \boxtimes W}{\sqcup} (U \boxtimes X) \to V \boxtimes X.$$

Since the cofibrations in the natural model category structure on $\mathbf{Perm}$ are the same as those in $(\mathbf{Perm}, \mathbf{Gpd})$ and the natural model category is a $\mathbf{Cat}$-model category, therefore $i \square j$ is a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$.

Let us further assume that $i$ is an acyclic cofibration in $(\mathbf{Cat}, \mathbf{Gpd})$. We will show that now $i \square j$ is an acyclic cofibration. We recall a well-known fact that a map is an acyclic cofibration in a model category if and only if it has the left-lifting property with respect to all fibrations between fibrant objects. Let $p : A \to B$ be a fibration between permutative groupoids. By adjointness, the map $i \square j$ has the left lifting property with respect to $p$ if and only if there exists a (dotted) lifting arrow in the following diagram:

$$\begin{tikzcd} U \arrow[rr] \arrow[d, "i"'] & & {[X, A]^{str}_{\otimes}} \arrow[d, "(j^*, p_*)"] \\ V \arrow[r] \arrow[urr, dashed, "L"] & {[X, B]^{str}_{\otimes}} & \underset{[W,B]^{str}_{\otimes}}{\times} {[W, A]^{str}_{\otimes}} \end{tikzcd}$$

Since the natural model category $\mathbf{Perm}$ is a $\mathbf{Cat}$-model category with categorical Hom given by $[-,-]^{str}_{\otimes}$, therefore the assumptions on $j$ and $p$ together imply that the map $(j^*, p_*)$ is a fibration in the natural model structure on $\mathbf{Cat}$ namely an isofibration. However, it is an isofibration between groupoids, therefore it is a fibration in $(\mathbf{Cat}, \mathbf{Gpd})$. Hence there exists a(dotted) lifting arrow $L$ which makes the whole diagram commutative. Thus, we have shown that $i \square j$ is an acyclic cofibration when $i$ is one.

A similar argument applied to $j$ shows that if $j$ is an acyclic cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$, then so is $i \square j$.

$\square$

The $(\mathbf{Cat}, \mathbf{Gpd})$-model category structure described in the proposition above induces a simplicial model category structure on $(\mathbf{Perm}, \mathbf{Gpd})$:

**Proposition 3.7.** The model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a simplicial model category.

*Proof.* The proof follows from [Bar07, lem. 3.6] and the $\mathbf{Cat}$-model category structure on the natural model category $\mathbf{Perm}$. However, we will describe the three bifunctors involved in this enrichment: The simplicial Hom bifunctor is defined to be the composite:

$$\mathbf{Perm}^{op} \times \mathbf{Perm} \overset{[-,-]_{\otimes}^{str}}{\to} \mathbf{Cat} \overset{N}{\to} \mathbf{sSets}. \tag{9}$$

This *cotensor* is defined as follows:

$$\mathbf{sSets}^{op} \times \mathbf{Perm} \overset{\tau_1^{op} \times id}{\to} \mathbf{Cat}^{op} \times \mathbf{Perm} \overset{[-,-]}{\to} \mathbf{Perm} \tag{10}$$

The *tensor product* bifunctor is defined by the following composite:

$$\mathbf{sSets} \times \mathbf{Perm} \overset{\tau_1 \times id}{\to} \mathbf{Cat} \times \mathbf{Perm} \overset{-\boxtimes-}{\to} \mathbf{Perm}. \tag{11}$$

□

## 3.2 The model category of Picard groupoids

In this subsection we will construct yet another model category structure on $\mathbf{Perm}$ in which the fibrant objects are Picard groupoids. We obtain the desired model category by carrying out a left Bousfield localization of the model category constructed in the previous subsection, namely $(\mathbf{Perm}, \mathbf{Gpd})$. The model category we construct inherits an enrichment over $(\mathbf{Cat}, \mathbf{Gpd})$ and the Kan model category of simplicial sets from its parent model category.

**Definition 3.8.** A *Picard groupoid* $G$ is a permutative groupoid such that one of the following two functors is an equivalences of categories:

$$G \times G \overset{(-\underset{G}{\otimes}-, p_1)}{\to} G \times G \quad \textit{and} \quad G \times G \overset{(-\underset{G}{\otimes}-, p_2)}{\to} G \times G, \tag{12}$$

where $p_1$ and $p_2$ are the two obvious projection maps.

*Remark* 7. If one of the two functors in the above definition is an equivalence of categories, then the permutative structure on the groupoid $G$ in the above definition implies the other functor is also an equivalence.

*Remark* 8. A permutative groupoid is a Picard groupoid if and only if for each object $g \in Ob(G)$ there exists another object $g^{-1} \in Ob(G)$ and the following two isomorphisms in $G$:

$$g \underset{G}{\otimes} g^{-1} \cong 1_G, \qquad g^{-1} \underset{G}{\otimes} g \cong 1_G$$

We recall the construction of the permutative category $\mathcal{L}(1)$ from [Shac]. The permutative category $\mathcal{L}(1)$ is a groupoid whose object set consists of all finite sequences $(s_1, s_2, \dots, s_r)$, where either $s_i = 1$ or $s_i = 0$ for all $1 \leq i \leq r$. For an object $S = (s_1, s_2, \dots, s_r)$ in $\mathcal{L}(1)$ we denote by $\underline{S}$ the sum $\overset{r}{\underset{i=1}{+}} s_i$. A map $S = (s_1, s_2, \dots, s_r) \to T = (t_1, t_2, \dots, t_k)$ in $\mathcal{L}(1)$ is a bijection $f : \underline{S} \to \underline{T}$. The symmetric monoidal structure on $\mathcal{L}(1)$ is given by concatenation. It follows from [Shac, Lem. 3.8] that $\mathcal{L}(1)$ is cofibrant in the natural model category $\mathbf{Perm}$.

**Proposition 3.9.** For any permutative groupoid $G$, the evaluation map

$$ev_{(1)} : [\mathcal{L}(1), G]^{str}_{\otimes} \to G$$

is an equivalence of categories.

*Proof.* The free permutative category $\mathcal{F}(\underline{1})$, see (3), can be described as follow: The objects are finite sets $\underline{n}$ for all $n \geq 0$. A morphism is a bijection between finite sets. The permutative category $\mathcal{F}(1)$ is cofibrant in the natural model category $\mathbf{Perm}$. This category has the property that the evaluation functor on the object $\underline{1}$:

$$ev_{\underline{1}} : [\mathcal{F}(\underline{1}), C]^{str}_{\otimes} \to C$$

is an isomorphism for any permutative category $C$. This category is equipped with an inclusion functor

$$i : \mathcal{F}(\underline{1}) \to \mathcal{L}(1),$$

such that $i(\underline{1}) = (1)$, which is an equivalence of categories. Now the 2 out of 3 and the following commutative diagram prove the proposition:

$$\begin{tikzcd}
{[\mathcal{L}(1), G]^{str}_{\otimes}} \arrow[r, "ev_{(1)}"] \arrow[d, "{[i,G]^{str}_{\otimes}}"'] & G \\
{[\mathcal{F}(\underline{1}), G]^{str}_{\otimes}} \arrow[ur, "\cong"']
\end{tikzcd}$$

□

The maps of finite sets $m_2 : 2^+ \to 1^+$, $\delta^2_1 : 2^+ \to 1^+$ and $\delta^2_2 : 2^+ \to 1^+$ together induce the following two maps in $\mathbf{Perm}$

$$\mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))}{\to} \mathcal{L}(2) \quad \textit{and} \quad \mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2))}{\to} \mathcal{L}(2) \qquad (13)$$

*Remark* 9. For each $n \in \mathbb{N}$, the permutative groupoid $\mathcal{L}(n)$ [Shac, Defn. 5.4] is canonically isomorphic to the permutative groupoid $\mathcal{L}(\Gamma^n)$, where $\mathcal{L} : \Gamma\mathbf{Cat} \to \mathbf{Perm}$ is the left adjoint of the Segal's nerve functor, see [Shac, Sec. 5]. This implies that the map $(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))$ is isomorphic to $\mathcal{L}(\Gamma^{(m_2,\delta^2_1)})$ and $(\mathcal{L}(m_2), \mathcal{L}(\delta^2_2))$ is isomorphic to $\mathcal{L}(\Gamma^{(m_2,\delta^2_2)})$, where the maps $\Gamma^{(m_2,\delta^2_1)}$ and $\Gamma^{(m_2,\delta^2_2)}$ are defined in (17) and (18) respectively.

*Remark* 10. The above remark and the fact that $\mathcal{L}$ is a left Quillen functor together imply that the permutative groupoid $\mathcal{L}(n)$ is cofibrant in the natural model category $\mathbf{Perm}$. The symmetric monoidal structure on $\mathcal{L}(n)$ is concatenation.

By [Shac, Lemma 5.29] the strict symmetric monoidal functor $(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2))$ is an acyclic cofibration in the natural model category structure on $\mathbf{Perm}$. This implies that for any permutative category $C$, we have the following equivalence of categories:

$$[(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2)), C]^{str}_{\otimes} : [\mathcal{L}(2), C]^{str}_{\otimes} \to [\mathcal{L}(1), C]^{str}_{\otimes} \times [\mathcal{L}(1), C]^{str}_{\otimes}. \tag{14}$$

**Lemma 3.10.** A permutative groupoid $G$ is a Picard groupoid if and only if it is a $\{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))\}$-local object.

*Proof.* The permutative groupoid $G$ is $\{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))\}$-local if and only if we have the following weak homotopy equivalence of simplicial sets:

$$Map^h((\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G) : Map^h(\mathcal{L}(2), G) \to Map^h(\mathcal{L}(1) \vee \mathcal{L}(1), G)$$

We recall that the function complex for a pair of permutative categories $C$ and $D$ in $(\mathbf{Perm}, \mathbf{Gpd})$, where $C$ is cofibrant and $D$ is a permutative groupoid is defined as follows:

$$\mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(C, D) := N([C, D]^{str}_{\otimes})$$

which implies that $Map^h((\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G)$ is a homotopy equivalence if and only if the functor:

$$[(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G]^{str}_{\otimes} : [\mathcal{L}(2), G]^{str}_{\otimes} \to [\mathcal{L}(1) \vee \mathcal{L}(1), G]^{str}_{\otimes} \cong [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes}.$$

is an equivalence of categories. Thus we get the following (composite) weak equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$:

$$[\mathcal{L}(2), G]^{str}_{\otimes} \xrightarrow{p} [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes} \xrightarrow{(ev_{(1)}, ev_{(1)})} G \times G \tag{15}$$

where $p = [(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G]^{str}_{\otimes}$. There is another composite map in $\mathbf{Perm}$ which is the following:

$$[\mathcal{L}(2), G]^{str}_{\otimes} \xrightarrow{q} [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes} \xrightarrow{(ev_{(1)}, ev_{(1)})} G \times G \xrightarrow{r} G \times G \tag{16}$$

where $q = [(\mathcal{L}(\delta_1^2), \mathcal{L}(\delta_2^2)), G]_{\otimes}^{str}$ and the map $r = (- \underset{G}{\otimes} -, p_2)$. We will now construct a natural isomorphism (in $\mathbf{Cat}$) $H : (ev_{(1)}, ev_{(1)}) \circ p \Rightarrow r \circ (ev_{(1)}, ev_{(1)}) \circ q$ between the above two functors. For each $F \in [\mathcal{L}(2), G]_{\otimes}^{str}$ let us denote $F((\underline{2}))$ by $g_{12}$. The isomorphism $p_{12} : (\underline{2}) \cong (\{1\}, \{2\})$ in $\mathcal{L}(2)$ gives an isomorphism $F(p_{12}) : g_{12} \cong g_1 \otimes g_2$, where $g_1 = F((\{1\}))$ and $g_2 = F((\{2\}))$. We observe that $r \circ (ev_{(1)}, ev_{(1)}) \circ q(F) = (g_1 \otimes g_2, g_1)$ and $(ev_{(1)}, ev_{(1)}) \circ p(F) = (g_{12}, g_1)$. We define $H(F) := F(p_{12})$. Let $\sigma : F \Rightarrow G$ be a (monoidal) natural transformation and denoting $G((\underline{2}))$ by $g'_{12}$, $G((1))$ by $g'_1$ and $G((2))$ by $g'_2$ we get an isomorphism $G(p_{12}) : g'_{12} \cong g'_1 \otimes g'_2$. The following diagram commutes:

$$\begin{tikzcd}
g_{12} \arrow[r, "H(F)"] \arrow[d, "\sigma((\underline{2}))"'] & g_1 \otimes g_2 \arrow[d, "\sigma((\{1\},\{2\}))"] \\
g'_{12} \arrow[r, "H(G)"'] & g'_1 \otimes g'_2
\end{tikzcd}$$

because $\sigma$ is a natural isomorphism. Hence we have constructed the desired natural isomorphism $H$. The construction of $H$ implies that the strict symmetric monoidal functor $r \circ (ev_{(1)}, ev_{(1)}) \circ q$ is a groupoidal equivalence if and only if $(ev_{(1)}, ev_{(1)}) \circ p$ is one. We know that the functors $q$ and $(ev_{(1)}, ev_{(1)})$ are both equivalence of categories. Let us assume that $G$ is a aforementioned local object then $(ev_{(1)}, ev_{(1)}) \circ p$ is a groupoidal equivalence and, by the above argument, so is the composite functor $r \circ (ev_{(1)}, ev_{(1)}) \circ q$. By two out of three property of weak equivalences this implies that $r$ is a weak equivalence which implies that $G$ is a Picard groupoid. Conversely, let us assume that $G$ is a Picard groupoid in which case $r$ is a groupoidal equivalence which means that both $r \circ (ev_{(1)}, ev_{(1)}) \circ q$ and $(ev_{(1)}, ev_{(1)}) \circ p$ are groupoidal equivalences. Again by the two out of three property, $p$ is a groupoidal equivalence which implies that $G$ is local.

□

**Theorem 3.11.** There is a combinatorial model category structure on the category of (small) permutative categories $\mathbf{Perm}$ in which a functor $F : A \to B$ is

1. a cofibration if it is a cofibration in the natural model category structure on $\mathbf{Perm}$.

2. a weak equivalence if the following functor

   $$\mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(F, P) : \mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(B, P) \to \mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(A, P)$$

   is a homotopy equivalence of simplicial sets for each Picard groupoid $P$.

3. a fibration if it has the right lifting property with respect to the set of maps which are both cofibrations and weak equivalences.

A permutative category is a fibrant objects of this model category if and only if it ia a Picard groupoid.

*Proof.* We will prove this theorem by localizing the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ with respect to the map

$$\mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(m_2), \mathcal{L}(\delta_1^2))}{\rightarrow} \mathcal{L}(2).$$

The existence of this left Bousfield localization follows from theorem A.2. A left Bousfield localization preserves cofibrations therefore the cofibrations in the new model category are the same as those in $(\mathbf{Perm}, \mathbf{Gpd})$. Lemma 3.10 above tells us that a permutative groupoid is a $\{(\mathcal{L}(m_2), \mathcal{L}(\delta_1^2))\}$-local object if and only if it is a Picard groupoid.

□

**Notation 3.12.** We will refer to the above model category as the model category of *Picard groupoids*. We denote this model category by $(\mathbf{Perm}, \mathbf{Pic})$.

Adaptations of arguments used in the proof of propositions 3.6 and 3.7, to the model category $(\mathbf{Perm}, \mathbf{Pic})$ prove the following two analogous propositions:

**Proposition 3.13.** The bifunctors (8), (6) and (7) equip the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ with a $(\mathbf{Cat}, \mathbf{Gpd})$-model category structure.

and

**Proposition 3.14.** The bifunctors (11), (10) and (9) equip the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ with a simplicial model category structure.

## 4. The model category structures

A $\Gamma$- category is a functor from $\Gamma^{op}$ to $\mathbf{Cat}$. The category of functors from $\Gamma^{op}$ to $\mathbf{Cat}$ and natural transformations between them $[\Gamma^{op}, \mathbf{Cat}]$ will be denoted by $\Gamma\mathbf{Cat}$. The main objective of this section is to construct two new symmetric monoidal closed model category structures on $\Gamma\mathbf{Cat}$. Some notations used in this section have been defined in [Shac, Sec. 4]. We recall the following definition:

**Definition 4.1.** A $Q$-cofibration is a cofibration in the strict (or projective) model category structure on $\Gamma\mathbf{Cat}$.

## 4.1 Coherently commutative monoidal groupoids

In the paper [Shac, Sec. 4] a symmetric monoidal closed model category structure was constructed on $\Gamma\mathbf{Cat}$ whose fibrant objects are coherently commutative monoidal categories, see definition [Shac, Defn. 4.15]. These objects should be understood as categories equipped with a multiplication which is associative, unital and commutative only up to higher coherence data. In this subsection we want to construct another symmetric monoidal closed model category structure on $\Gamma\mathbf{Cat}$ whose fibrant objects are groupoids equipped with a multiplication which is associative, unital and commutative only up to higher coherence data. In other words, the underlying category of a fibrant object in the desired model category is a fibrant object in the groupoidal model category $(\mathbf{Cat}, \mathbf{Gpd})$. We will construct the desired model category as a left Bousfield localization of the model category of coherently commutative monoidal categories with respect to the map $\Gamma^1 \otimes i_0 : \Gamma^1 \to \Gamma^1 \otimes I$, see (4).

**Definition 4.2.** We will refer to a $\{\Gamma^1 \otimes i_0\}$-local equivalence as an equivalence of coherently commutative monoidal groupoids.

**Definition 4.3.** We will refer to a fibrant $\{\Gamma^1 \otimes i_0\}$-local object as a coherently commutative monoidal groupoid.

**Proposition 4.4.** A $\Gamma$- category $X$ is a coherently commutative monoidal groupoid if and only if the following two conditions are satisfied:

1. For each $k^+ \in Ob(\Gamma^{op})$, $X(k^+)$ is a groupoid.
2. For each $k^+, l^+ \in Ob(\Gamma^{op})$
$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \to X(k^+) \times X(l^+)$$
is a groupoidal equivalence.

*Proof.* The model category of coherently commutative monoidal categories is a $\mathbf{Cat}$-model category and its categorical Hom $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-,-)$ is defined in [Shac, Sec. 4]. Now it follows from [Shac, Appendix D] that for any cofibrant $C$ and fibrant $X$ in the model category of coherently commutative monoidal categories

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(C, X) = N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(C, X)\right)\right).$$

We begin by observing that conditions $(1)$ and $(2)$ are satisfied by $X$ if and only if $X$ is a coherently commutative monoidal category and $X(1^+)$ is a groupoid. Now it is sufficient to show that $X$ is a coherently commutative monoidal groupoid if and only if $X(1^+)$ is a groupoid and $X$ is a coherently commutative monoidal category. The $\Gamma$- category $X$ is a coherently commutative monoidal groupoid if and only if it is a coherently commutative monoidal category and the following simplicial map is a weak homotopy equivalence:

$$N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes i_0, X)\right)\right) : N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, X)\right)\right) \to N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right).$$

This simplicial map of Kan complexes is a homotopy equivalence if and only if the following functor between groupoids is a groupoidal equivalence:

$$\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes i_0, X)\right)\right) : \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, X)\right)\right) \to \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right).$$

By adjointness, the above functor is a groupoidal equivalence if and only if the following functor is a groupoidal equivalence:

$$\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, X))\right)\right) : \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(I, X))\right)\right) \to \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right) \cong J\left(X(1^+)\right).$$

Unwinding definition, the functor $\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, X))\right)\right)$ is isomorphic to the following functor:

$$J[i_0; X(1^+)] : J[I; X(1^+)] \to J(X(1^+)).$$

This implies that $X$ is a coherently commutative monoidal groupoid if and only if it is a coherently commutative monoidal category and $X(1^+)$ is a groupoid *i.e.* a $i_0$-local object in $\mathbf{Cat}$.

□

A left-Bousfield localization with respect to the map $\{\Gamma^1 \otimes i_0\}$ gives us the following model category.

**Theorem 4.5.** There is a left proper, combinatorial model category structure on the category of $\Gamma$- categories, $\Gamma\mathbf{Cat}$, in which

1. The class of cofibrations is the same as the class of Q-cofibrations of $\Gamma$- categories.

2. The weak equivalences are equivalences of coherently commutative monoidal groupoids.

An object is fibrant in this model category if and only if it is a coherently commutative monoidal groupoid.

*Proof.* The model category structure follows from an application of A.2 to the model category of coherently commutative monoidal categories with respect to the singleton set $\{\Gamma^1 \otimes i_0\}$. The characterization of fibrant objects also follows from the same theorem.

□

**Notation 4.6.** We will refer to the above model category as the model category of coherently commutative monoidal categories and denote it by $\Gamma\mathbf{Gpd}^{\otimes}$

The following proposition will be useful in proving the main result of this subsection:

**Proposition 4.7.** The model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category.

The rest of this subsection is devoted to showing that the model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a symmetric monoidal closed model category under the Day convolution. In order to do so we will need the following result:

**Lemma 4.8.** For each $Q$-cofibrant $\Gamma$- category $W$, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid if $A$ is one.

*Proof.* Since $A$ is also a coherently commutative monoidal category *i.e.* a fibrant object in the model category of coherently commutative monoidal categories, the symmetric monoidal closed structure on the aforementioned model category, [Shac, Thm. 4.27], implies that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal category. Now, in light of proposition 4.4, it is sufficient to show that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)(k^+)$ is a groupoid, for all $k^+ \in \Gamma^{op}$. Since $W$ is cofibrant, therefore we have the following equality:

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)(k^+) = \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^k, A).$$

We recall that $\Gamma^k$ is a $Q$-cofibrant $\Gamma$- category. Since $W$ is $Q$-cofibrant by assumption therefore $W * \Gamma^k$ is also $Q$-cofibrant by [Shac, Thm. 4.27]. The result now follows from the above observation that the domain $\Gamma$- category $W * \Gamma^k$ is $Q$-cofibrant and the model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category which together imply that the category $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^k, A)$ is a groupoid.

□

The following theorem is the main result of this section:

**Theorem 4.9.** The model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ is a symmetric monoidal closed model category under the Day convolution product.

*Proof.* The generating cofibrations in the model category of coherently commutative monoidal categories are maps between $Q$-cofibrant objects. For a $Q$-cofibrant object $W$ and a coherently commutative monoidal groupoid $A$, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid by lemma 4.8. The model category of coherently commutative monoidal categories is symmetric monoidal closed under the Day convolution product by [Shac, Thm. 4.27]. Now the result follows from Theorem A.3.

□

## 4.2 Coherently commutative Picard groupoids

In this subsection we will introduce a notion of a *coherently commutative* Picard groupoid. We will go on to construct another model category structure on $\Gamma\mathbf{Cat}$ whose fibrant objects are the aforementioned objects. A prominent result of this section is that this new model category is symmetric monoidal closed under the Day convolution product thereby giving us a tensor product of Picard groupoids.

The mode of construction of this new model category will be localization. The following two pairs of maps, see definitions 2.16 and 2.15, of based sets:

$$m_2 : 2^+ \to 1^+ \quad \textit{and} \quad \delta^2_1 : 2^+ \to 1^+$$

and

$$m_2 : 2^+ \to 1^+ \quad \textit{and} \quad \delta^2_2 : 2^+ \to 1^+$$

induce two maps of $\Gamma$- categories

$$\Gamma^{(m_2, \delta^2_1)} : \Gamma^1 \vee \Gamma^1 \to \Gamma^2 \tag{17}$$

and

$$\Gamma^{(m_2, \delta^2_2)} : \Gamma^1 \vee \Gamma^1 \to \Gamma^2 \tag{18}$$

*Remark* 11. We recall that for each $k \geq 0$, the representatble $\Gamma$- category $\Gamma^k$ is $Q$-cofibrant. Further the coproduct of two $Q$-cofibrant $\Gamma$- categories is again $Q$-cofibrant. This implies that the above two maps are between $Q$-cofibrant $\Gamma$- categories.

**Notation 4.10.** We denote the set $\{\Gamma^{(m_2, \delta^2_1)}\}$ by $\mathcal{P}_\infty$.

**Definition 4.11.** A *coherently commutative Picard groupoid* is a coherently commutative monoidal groupoid which is also a $\mathcal{P}_\infty$-local object.

Unravelling the above definition gives us the following characterization of a coherently commutative Picard groupoid:

**Proposition 4.12.** A $\Gamma$- category $X$ is a coherently commutative Picard groupoid if and only if it satisfies the following three conditions:

1. For each $k^+ \in Ob(\Gamma^{op})$, $X(k^+)$ is a groupoid.
2. For each $k^+, l^+ \in Ob(\Gamma^{op})$
$$(X(\delta_k^{k+l}), X(\delta_l^{k+l})) : X((k+l)^+) \to X(k^+) \times X(l^+)$$
is a groupoidal equivalence.
3. One of the following two maps, and hence both maps, are groupoidal equivalences:
$$(X(m_2), X(\delta_1^2)) : X(2^+) \to X(1^+) \times X(1^+) \text{ and } (X(m_2), X(\delta_2^2)) : X(2^+) \to X(1^+) \times X(1^+)$$

**Definition 4.13.** A stable equivalence of $\Gamma$- categories is a $\mathcal{P}_\infty$-local equivalence.

An application of theorem A.2 to the model category $\Gamma\mathbf{Gpd}^\otimes$ with respect to the set $\mathcal{P}_\infty$ gives us the following model category:

**Theorem 4.14.** There is a left proper, combinatorial model category structure on the category of $\Gamma$- categories, $\Gamma\mathbf{Cat}$, in which

1. The class of cofibrations is the same as the class of Q-cofibrations of $\Gamma$- categories.
2. The weak equivalences are stable equivalences of $\Gamma$- categories.

An object is fibrant in this model category if and only if it is a coherently commutative Picard groupoid.

**Notation 4.15.** We denote the above model category by $\Gamma\mathcal{P}ic$.

The following lemma will be useful in the proof of the main result of this section:

**Lemma 4.16.** For each $Q$-cofibrant $\Gamma$- category $W$, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative Picard groupoid if $A$ is one.

*Proof.* If $A$ is a coherently commutative Picard groupoid then it is also a fibrant object in the model category of coherently commutative monoidal groupoids, the symmetric monoidal closed structure on the aforementioned model category, 4.9, implies that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid because $W$ is $Q$-cofibrant by assumption. Thus we have verified $(1)$ and $(2)$ in proposition 4.12. In order to verify $(3)$ in the same proposition we need to show that the following functor is a groupoidal equivalence:

$$\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^{(m_2,\delta_1^2)}, A) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^2, A) \to \\ \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^1, A) \times \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^1, A)$$

By adjointness, the morphism of $\Gamma$- categories $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^{(m_2,\delta_1^2)}, A)$ is a groupoidal equivalence if and only if its adjunct map

$$\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A)) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)) \to \\ \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)) \times \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A))$$

is one. Since $W$ is $Q$-cofibrant, it is sufficient to show that the morphism

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A) : \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A) \to \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A) \times \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)$$

is a strict equivalence of $\Gamma$- groupoids. Since the $\Gamma$- categories $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)$ and $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)$ are both coherently commutative monoidal groupoids therefore the morphism $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A)$ will be a strict equivalence of $\Gamma$- groupoids if and only if $(\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A))(1^+)$ is a groupoidal equivalence. The following commutative diagram :

$$\begin{tikzcd}
\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)(1^+) \arrow[r, "U"] \arrow[d, "\cong"'] & \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)(1^+) \times \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)(1^+) \arrow[d, "\cong"] \\
A(2^+) \arrow[r, "A((m_2,\delta_1^2))"'] & A(1^+) \times A(1^+)
\end{tikzcd}$$

where $U = (\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A))(1^+)$, implies that this map is a groupoidal equivalence because $A$ is a coherently commutative Picard groupoid by assumption. □

**Theorem 4.17.** The model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ is a symmetric monoidal closed model category under the Day convolution product.

*Proof.* The generating cofibrations of $\Gamma\mathbf{Gpd}^{\otimes}$ are maps between $Q$-cofibrant objects. For a $Q$-cofibrant object $W$ and a coherently commutative Picard groupoid $A$, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative Picard groupoid by lemma 4.16. Now an application of theorem A.3 to the model category $\Gamma\mathbf{Gpd}^{\otimes}$ with the set of morphisms $\mathcal{S} = \mathcal{P}_{\infty}$, see (4.10), proves the theorem. □

## 5. The Quillen equivalences

In this section we prove that the following two adjoint pairs are Quillen equivalences :

$$\mathcal{L} : \Gamma\mathcal{P}ic \rightleftarrows (\mathbf{Perm}, \mathbf{Pic}) : \mathcal{K}$$

and

$$\mathcal{L} : \Gamma\mathbf{Gpd}^{\otimes} \rightleftarrows (\mathbf{Perm}, \mathbf{Gpd}) : \mathcal{K}$$

where $\mathcal{K}$ is the classical Segal's nerve functor, see [Seg74], [Man10],[EM06] [Shac]. We begin with a proof of the later result:

**Theorem 5.1.** The Quillen pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative permutative groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$.

*Proof.* We recall that the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ is a left Bousfield localization of the model category of coherently commutative monoidal categories [Shac, Thm. 4.20] with respect to a single map $\Gamma^1 \otimes i_0$, see (4). By remark 11, this is a map between $Q$-cofibrant $\Gamma$- categories. We further recall that the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a left Bousfield localization of the natural model category $\mathbf{Perm}$ with respect to the image of $\Gamma^1 \otimes i_0$ under the left adjoint $\mathcal{L}$. Moreover, the adjoint pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative monoidal categories and the natural model category $\mathbf{Perm}$ [Shac, Cor. 6.19]. Now the result follows from [Hir02, Thm. 3.3.20.]. □

Now we prove the main result of this paper:

**Theorem 5.2.** The Quillen pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Pic})$.

*Proof.* We recall that the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ is a left Bousfield localization of the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ with respect to a single map $\Gamma^{(m_2, \delta_1^2)}$, see (17).

We observe that this is a map between $Q$-cofibrant $\Gamma$- categories. We further recall that the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ is a left Bousfield localization of the model category $(\mathbf{Perm}, \mathbf{Gpd})$ with respect to a single map $(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))$ which is isomorphic to the image of $\Gamma^{(m_2, \delta^2_1)}$ under the left adjoint $\mathcal{L}$, see remark (9). Further, the adjoint pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category $(\mathbf{Perm}, \mathbf{Gpd})$ by theorem 5.1. Now the result follows from [Hir02, Thm. 3.3.20.]. □

In light of the natural weak-equivalence [Shac, Cor. 6.19] between the Segal's nerve functor $\mathcal{K}$ and the thickened Segal's nerve functor $\overline{\mathcal{K}}$, constructed in [Shac, Sec. 6], the following two theorems follow from the above two Quillen equivalences:

**Theorem 5.3.** The Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the model category of coherently commutative permutative groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$.

**Theorem 5.4.** The Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the model category of coherently commutative Picard groupoids and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Pic})$.

## 6. Stable homotopy hypothesis for Picard groupoids

In this section we give a new proof of the classical result that Picard groupoids model *stable homotopy one-types*. This result has been referred to in the literature as the *stable homotopy hypothesis* for Picard groupoids. The main objective of this section is to show that the homotopy category of our model category $\Gamma\mathcal{P}ic$ is equivalent to a (suitably defined) homotopy category of stable homotopy one-types. We use the language of *relative categories* in this section, see [BK12]. We define two relative categories for the objects in context: a relative category $(\mathbf{Pic}, \mathit{Str})$ of Picard groupoids whose homotopy category is equivalent to that of $(\mathbf{Perm}, \mathbf{Pic})$ and another relative category of stable homotopy one-types $(\Gamma\mathcal{S}^f_\bullet[1], \mathit{Str})$. We prove a stronger result, namely we establish a homotopy equivalence of the two aforementioned relative categories which implies that their homotopy categories are equivalent. Our proof of the homotopy equivalence is based on the main result of this paper, namely theorem 5.4. A short time before the first version of this paper was released, a different proof of another version of the aforementioned homotopy equivalence was given in the paper [MOP$^+$20]. This proof is based the stable homotopy hypothesis proved in the same paper. In this section we will be dealing with the model category of pointed spaces $(\mathbf{sSets_\bullet}, \mathbf{Kan})$ and we recall that a map in this model category is a weak equivalence if and only if its underlying (unpointed) simplicial map is a weak homotopy equivalence.

**Definition 6.1.** A stable homotopy one type is a functor $X : \Gamma^{op} \to \mathbf{sSets_\bullet}$ such that the following conditions are satisfied:

1. For each $n^+ \in \Gamma^{op}$, the (pointed) simplicial set $X(n^+)$ is a Kan complex.
2. All homotopy groups of pointed simplicial set $X(1^+)$ vanish in degree greater than one *i.e.*, $\pi_n(X(1^+)) = *$ for $n \geq 2$.
3. For each pair of objects $k^+, l^+ \in \Gamma^{op}$, the following simplicial map is a weak homotopy equivalence:
$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \to X(k^+) \times X(l^+)$$
4. One of the following two maps, and hence both maps:
$(X(m_2), X(\delta^2_1)) : X(2^+) \to X(1^+) \times X(1^+)$ *and* $(X(m_2), X(\delta^2_2)) : X(2^+) \to X(1^+) \times X(1^+)$
are homotopy equivalences of pointed simplicial sets.

*Remark* 12. Each stable homotopy one type is a fibrant object in the stable $Q$-model category constructed in [Sch99].

*Remark* 13. Each stable homotopy one-type determines a *connective spectrum* with at most two non-trivial homotopy groups in degree zero or one, see [BF78].

*Remark* 14. The adjoint pair of functors $(\tau, N)$ induce an adjunction
$$[\Gamma^{op}, \tau] : \Gamma\mathbf{Cat} \rightleftharpoons \Gamma\mathcal{S} : [\Gamma^{op}, N]$$
This adjunction is a Quillen pair with respect to the strict (or projective) model category structure on the two functor categories, see [Lur09, Remark A.2.8.6]. Since the counit of $(\tau, N)$ is the identity, therefore the counit of the induced adjunction is also identity.

We recall from [Shaa] the adjoint pair $((-)^{nor}, U)$ which determines a Quillen equivalence between the $JQ$-model category of $\Gamma$-spaces [Sha20, Notation 4.11] and the $JQ$-model category of normalized $\Gamma$-spaces [Sha20, Notation C.19]. We recall from [Sha20] that a normalized $\Gamma$-space is a functor from $X : \Gamma^{op} \to \mathbf{sSets_\bullet}$ such that $X(0^+) = *$. It is easy to see that each coherently commutative monoidal Picard groupoid $X$ determines a $\Gamma$-space upon composition with the nerve functor, we denote this $\Gamma$-space by $N(X)$. Applying the left adjoint gives us a normalized $\Gamma$-space $(N(X))^{nor}$. This leads us to the following proposition:

**Proposition 6.2.** For each coherently commutative Picard groupoid $X$, the normalized $\Gamma$-space $(N(X))^{nor}$ is a stable homotopy one-type.

*Proof.* The nerve functor preserves products, maps groupoids to Kan complexes and also maps groupoidal equivalences between two groupoids to homotopy equivalences of simplicial sets therefore $N(X)$ is a coherently commutative monoidal quasi-category in which $N(X)(k^+)$ is a Kan complex for each $k^+ \in \Gamma^{op}$. It follows that the following simplicial maps:

$$N(X(m_2), X(\delta_1^2)) : N(X(2^+)) \to N(X(1^+)) \times N(X(1^+))$$

and

$$N(X(m_2), X(\delta_2^2)) : N(X(2^+)) \to N(X(1^+)) \times N(X(1^+))$$

are both homotopy equivalences of Kan complexes. It follows from [Shaa, Prop. 6.6] that the unit simplicial map $\eta_{N(X)} : N(X) \to U((N(X))^{nor})$ is a strict $JQ$-equivalence of $\Gamma$-spaces. This implies that the normalized $\Gamma$-space $(N(X))^{nor}$ is a stable homotopy one-type.

□

We recall that a *relative category* $C = (C, W)$ consists of a pair of categories $(C, W)$ which have the same set of objects and the set arrows of $W$ is a subset of arrows of $C$ and the maps of $W$ are called *weak-equivalences* of $C$. A morphism of relative categories $F : (C, W) \to (D, X)$ is a functor $F : C \to D$ that preserves weak-equivalences *i.e.* $F(W) \subseteq X$. A morphism of relative categories is called a *functor of relative categories*.

**Definition 6.3.** A *strict homotopy* between two functors of relative categories $F : (C, W) \to (D, X)$ and $G : (C, W) \to (D, X)$ is a natural transformation $H : F \Rightarrow G$ such that for each object $c \in C$, the map $H(c)$ lies in $X$, *i.e.*, it is a weak-equivalence in $D$.

More generally, we will say that there exists a *homotopy* between $F$ and $G$ if they can be joined by a finite zig-zag of strict homotopies.

Based on the notion of homotopy, we define another notion of homotopy equivalence:

**Definition 6.4.** A functor of relative categories $F : (C, W) \to (D, X)$ is called a *strict homotopy equivalence* if there exists another functor of relative categories $F^{-1} : (D, X) \to (C, W)$ and two strict homotopies $\eta : id \Rightarrow F^{-1} \circ F$ and $\epsilon : F \circ F^{-1} \Rightarrow id$.

$F$ will be called a *homotopy equivalence* if $\eta$ and $\epsilon$ are just homotopies, namely, zig-zags of strict homotopies.

*Remark* 15. A homotopy equivalence induces an equivalence on the homotopy categories of its domain and codomain relative categories.

Next we will construct three relative categories:

**Definition 6.5.** We denote by $(\mathbf{Pic}, \mathit{Str})$ the relative category in which $\mathbf{Pic}$ is the category whose objects are permutative Picard groupoids and arrows are strict symmetric monoidal functors. The morphisms of *Str* are those strict symmetric monoidal functors whose underlying functors are equivalences of categories.

*Remark* 16. The homotopy category of the relative category $(\mathbf{Pic}, \mathit{Str})$ is equivalent to the homotopy category of the model category $(\mathbf{Perm}, \mathbf{Pic})$.

**Definition 6.6.** We denote by $(\Gamma\mathcal{P}ic^f, \mathit{Str})$ the relative category in which $\Gamma\mathcal{P}ic^f$ is the full subcategory of $\Gamma\mathbf{Cat}$ whose objects are coherently commutative Picard groupoids. The morphisms of *Str* are strict equivalences of $\Gamma$- categories.

*Remark* 17. The homotopy category of the relative category $(\Gamma\mathcal{P}ic^f, \mathit{Str})$ is equivalent to the homotopy category of the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$.

**Definition 6.7.** We denote by $(\Gamma\mathcal{S}^f_\bullet[1], \mathit{Str})$ the relative category in which $\Gamma\mathcal{S}^f_\bullet[1]$ is the full subcategory of $\Gamma\mathcal{S}_\bullet$ (the category of normalized $\Gamma$-spaces, see [Shaa]) whose objects are stable homotopy one types, see definition (6.1). The morphisms of *Str* are strict $JQ$-equivalences of normalized $\Gamma$-spaces, see [Shaa].

*Remark* 18. The homotopy category of the relative category $(\Gamma\mathcal{S}^f_\bullet[1], \mathit{Str})$ is equivalent to the full subcategory of the homotopy category of the stable $Q$-model category, constructed in [Sch99], whose objects are normalized $\Gamma$-spaces having at most two non-zero stable homotopy groups only in degree zero or one.

We recall the classical result that the homotopy theory of one-types *i.e.*, Kan complexes (fibrant simplicial sets) whose homotopy groups are trivial in degrees 2 and above is equivalent to the homotopy theory of groupoids. This result can be expressed by the following (strict) homotopy equivalence:

$$\tau_1 : (\mathbf{sSets}^1, \mathit{WH}) \rightleftharpoons (\mathbf{Gpd}, \mathit{Eq.}) : N \tag{19}$$

where $\mathbf{sSets}^1$ denotes the full subcategory of $\mathbf{sSets}$ whose objects are one-types and the maps in *WH* are homotopy equivalences of simplicial sets. The functors in *Eq* are equivalences of categories.

**Notation 6.8.** We denote by $N^{nor}(-)$ the composite functor

$$\Gamma\mathbf{Cat} \xrightarrow{N} \Gamma\mathcal{S} \xrightarrow{(-)^{nor}} \Gamma\mathcal{S}_\bullet$$

where $N$ denotes the functor $[\Gamma^{op}, N] : \Gamma\mathbf{Cat} \to \Gamma\mathcal{S}$.

Proposition 6.2 above implies that the functor $N^{nor}(-)$ restricts to:

$$N^{nor}(-) : \Gamma\mathcal{P}ic^f \to \Gamma\mathcal{S}^f_\bullet[1]. \tag{20}$$

**Lemma 6.9.** The functor $N^{nor}(-)$ is a homotopy equivalence of relative categories.

*Proof.* We begin by observing that the following composite functor:

$$\Gamma\mathcal{S}_\bullet \xrightarrow{U} \Gamma\mathcal{S} \xrightarrow{\tau_1} \Gamma\mathbf{Cat},$$

restricts to a functor

$$\tau^{un}(-) : \Gamma\mathcal{S}^f_\bullet[1] \to \Gamma\mathcal{P}ic^f.$$

This follows by an argument similar to the one in the proof of Proposition 6.2 based on the fact that $U$ and $\tau_1$ preserve strict $JQ$-equivalences as well as products. We claim that this functor $\tau^{un}(-)$ is a homotopy inverse of $N^{nor}(-)$. We observe that the functor $N^{nor}(-)$ is a functor of relative categories because $N = [\Gamma^{op}, N]$ is a right Quillen functor and therefore preserves weak-equivalences (strict equivalences) between fibrant objects. The functor $(-)^{nor}$ preserves strict equivalences by [Shaa, Prop. 6.2]. Similarly, the functor $\tau^{un}(-)$ preserves strict equivalences because both $U$ and $\tau_1 = [\Gamma^{op}, \tau_1]$ do so.

Next , we will construct a (strict) homotopy $\beta^c : id \Rightarrow \tau^{un}(-) \circ N^{nor}(-)$ with the identity (relative) functor on $(\Gamma\mathcal{P}ic^f, Str)$. For each $X \in Ob(\Gamma\mathcal{P}ic^f)$, the unit of the Quillen equivalence $((-)^{nor}, U)$ provides a strict equivalence of $\Gamma$-spaces $\eta_X : N(X) \to U(N(X)^{nor})$. Applying the left Quillen functor $\tau_1$, we get a weak equivalence in $(\Gamma\mathcal{P}ic^f, Str)$, namely, $\tau_1(\eta_X) : X = \tau_1(N(X)) \to \tau_1(U(N(X)^{nor}))$. We define $\beta^c_X = \tau_1(\eta_X)$. One can easily check that this defines a natural transformation $\beta^c$. Now we define a (strict) homotopy $\beta^u : id \Rightarrow N^{nor}(-) \circ \tau^{un}(-)$. Let $Y$ be a stable homotopy one type. The unit map of the Quillen adjunction $(\tau_1, N)$ gives a map $\eta_Y : U(Y) \to N(\tau_1(U(Y)))$. Since $Y$ is a stable homotopy one type therefore this map is a weak homotopy equivalence by (19). Now applying the functor $(-)^{nor}$, we get a weak homotopy equivalence

$$(\eta_Y)^{nor} : Y = (U(Y))^{nor} \to N(\tau(U(Y)))^{nor}.$$

Now we define $\beta^u_Y = (\epsilon_Y)^{nor}$. One can easily check that this defines a natural transformation. Thus we have established a (strict) homotopy equivalence. $\square$

It follows from theorem [Shac, 6.17] that the left adjoint functor $\overline{\mathcal{L}}$ restricts to a functor of relative categories

$$\overline{\mathcal{L}} : (\Gamma\mathcal{P}ic^f, Str) \to (\mathbf{Pic}, Str).$$

Further, it follows from [Shac, lem. 6.14] that the right Quillen functor $\overline{\mathcal{K}}$ restricts to a functor of relative categories:

$$\overline{\mathcal{K}} : (\mathbf{Pic}, Str) \to (\Gamma\mathcal{P}ic^f, Str).$$

This leads us to the final lemma of this section:

**Lemma 6.10.** The pair of functors of relative categories $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ determines a (strict) homotopy equivalence between the relative categories $(\Gamma\mathcal{P}ic^f, Str)$ and $(\mathbf{Pic}, Str)$.

The proof follows from the two observations above, namely $\overline{\mathcal{K}}$ and $\overline{\mathcal{L}}$ are functors of relative categories and theorem 5.4. Now the previous two lemms give us the main result of this section:

**Theorem 6.11.** The composite functor of relative categories $N^{nor}(-) \circ \overline{\mathcal{K}}$ is a homotopy equivalence between the relative categories $(\mathbf{Pic}, Str)$ and $(\Gamma\mathcal{S}^f_\bullet[1], Str)$.

## A. Localization in model categories

In this appendix we review the definition and a fundamental existence theorem of a *left Bousfield localization* of a model category. The original result of this section is theorem A.3 which formulates a condition on a symmetric monoidal closed model category so that a left Bousfield localization preserves the symmetric monoidal closed structure. A thorough exposition on homotopy function complexes in model categories can be found in [Hir02], [DK80].

**Definition A.1.** Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure $L_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

1. The class of cofibrations of $L_{\mathcal{S}}\mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.

2. A map $f : A \to B$ is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ if it is an $\mathcal{S}$-local equivalence, namely, for every fibrant $\mathcal{S}$-local object $X$, the induced map on homotopy function complexes

$$f^* : Map^h_{\mathcal{M}}(B, X) \to Map^h_{\mathcal{M}}(A, X)$$

is a weak homotopy equivalence of simplicial sets. Recall that an object $X$ is called fibrant $\mathcal{S}$-local if $X$ is fibrant in $\mathcal{M}$ and for every element $g : K \to L$ of the set $\mathcal{S}$, the induced map on homotopy function complexes

$$g^* : Map^h_{\mathcal{M}}(L, X) \to Map^h_{\mathcal{M}}(K, X)$$

is a weak homotopy equivalence of simplicial sets.

We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [Smi] but a proof was later provided by Barwick in [Bar07].

**Theorem A.2.** [Bar07, Theorem 2.11] If $\mathcal{M}$ is a left proper, combinatorial model category and $\mathcal{S}$ is a small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ along any set representing $\mathcal{S}$ exists and satisfies the following conditions.

1. The model category $L_{\mathcal{S}}\mathcal{M}$ is left proper and combinatorial.
2. As a category, $L_{\mathcal{S}}\mathcal{M}$ is simply $\mathcal{M}$.
3. The cofibrations of $L_{\mathcal{S}}\mathcal{M}$ are exactly those of $\mathcal{M}$.
4. The fibrant objects of $L_{\mathcal{S}}\mathcal{M}$ are the fibrant $\mathcal{S}$-local objects $Z$ of $\mathcal{M}$.
5. The weak equivalences of $L_{\mathcal{S}}\mathcal{M}$ are the $\mathcal{S}$-local equivalences.

The next theorem provides a condition for a left Bousfield localization to preserves the symmetric monoidal model category structure:

**Theorem A.3.** Let $\mathcal{M}_{\mathcal{O}}$ be a combinatorial model category such that the generating cofibrations in $\mathcal{M}_{\mathcal{O}}$ are maps between cofibrant objects. Let the underlying category of $\mathcal{M}_{\mathcal{O}}$, denoted by $\mathcal{M}$, have a symmetric monoidal closed structure which endows on $\mathcal{M}_{\mathcal{O}}$ a symmetric monoidal closed model category structure. Let us denote by $\mathcal{M}_{\mathcal{S}}$ the model category, which is a left Bousfield localization of $\mathcal{M}_{\mathcal{O}}$, with respect to a set of maps $\mathcal{S}$ in $\mathcal{M}$. If the internal mapping object $\mathcal{M}ap_{\mathcal{M}}(X, Y)$ is an $\mathcal{S}$-local object whenever $X$ is cofibrant in $\mathcal{M}_{\mathcal{O}}$ and $Y$ is an $\mathcal{S}$-local object, then the model category $\mathcal{M}_{\mathcal{S}}$ is also symmetric monoidal closed.

*Proof.* Let $i : U \to V$ be a cofibration in $\mathcal{M}_{\mathcal{S}}$ and $j : Y \to Z$ be another cofibration in $\mathcal{M}_{\mathcal{S}}$. We will prove the theorem by showing that the following *pushout product* morphism

$$i \square j : U * Z \coprod_{U * Y} V * Y \to V * Z$$

is a cofibration in $\mathcal{M}_{\mathcal{S}}$ which is also an $\mathcal{S}$-local equivalence whenever either $i$ or $j$ is one. We first deal with the case of $i$ being a generating cofibration in $\mathcal{M}_{\mathcal{O}}$. The assumption of a symmetric monoidal closed model category structure on $\mathcal{M}_{\mathcal{O}}$ implies that $i\Box j$ is a cofibration in $\mathcal{M}_{\mathcal{O}}$ and we recall that the cofibrations in $\mathcal{M}_{\mathcal{S}}$ are exactly cofibration in $\mathcal{M}_{\mathcal{O}}$. Thus $i\Box j$ is a cofibration in $\mathcal{M}_{\mathcal{S}}$. Let us assume that $j$ is an acyclic cofibration *i.e.* $j$ is a cofibration in $\mathcal{M}_{\mathcal{S}}$ and also an $\mathcal{S}$-local equivalence. We recall that the fibrant objects of $\mathcal{M}_{\mathcal{S}}$ are exactly $\mathcal{S}$-local objects and fibrations in $\mathcal{M}_{\mathcal{S}}$ between $\mathcal{S}$-local objects are fibrations in $\mathcal{M}_{\mathcal{O}}$. According to [Shac, Proposition 4.22] the cofibration $i\Box j$ is an $\mathcal{S}$-local equivalence if and only if it has the left lifting property with respect to all fibrations in $\mathcal{M}_{\mathcal{S}}$ between $\mathcal{S}$-local objects. Let $p : W \to X$ be a fibration in $\mathcal{M}_{\mathcal{S}}$ between two $\mathcal{S}$-local objects. A (dotted) lifting arrow would exists in the following diagram

$$\begin{tikzcd}
U * Z \coprod_{U*Y} V * Y \arrow[r] \arrow[d] & W \arrow[d, "p"] \\
V * Z \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

if and only if a (dotted) lifting arrow exists in the following adjoint commutative diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[d, "j"'] & & \mathcal{M}ap_{\mathcal{M}}(V, W) \arrow[d, "(i^*, p^*)"] \\
Y \arrow[r] \arrow[urr, dotted] & \mathcal{M}ap_{\mathcal{M}}(U, X) & \underset{\mathcal{M}ap_{\mathcal{M}}(U,Y)}{\times} \ \mathcal{M}ap_{\mathcal{M}}(V, Y)
\end{tikzcd}$$

The map $(i^*, p^*)$ is a fibration in $\mathcal{M}_{\mathcal{O}}$ by [Hov99, lem. 4.2.2] and the assumption that $\mathcal{M}_{\mathcal{O}}$ is a symmetric monoidal closed model category with internal Hom denoted by $\mathcal{M}ap_{\mathcal{M}}(-,-)$. Further the assumption of cofibrancy on both $V$ and $U$ and the assumption on internal mapping objects together imply that $(j^*, p^*)$ is a fibration in $\mathcal{M}_{\mathcal{O}}$ between $\mathcal{S}$-local objects and therefore a fibration in the model category $\mathcal{M}_{\mathcal{S}}$. Since $j$ is an acyclic cofibration in $\mathcal{M}_{\mathcal{S}}$ by assumption, therefore the (dotted) lifting arrow exists in the above diagram. Thus, we have shown that if $i$ is a generating cofibration in $\mathcal{M}_{\mathcal{O}}$ and $j$ is a cofibration in $\mathcal{M}_{\mathcal{O}}$ which is also an $\mathcal{S}$-local equivalence then $i\Box j$ is an acyclic cofibration in the model category $\mathcal{M}_{\mathcal{S}}$. Now we deal with the general case of $i$ being an arbitrary cofibration in $\mathcal{M}_{\mathcal{O}}$. Consider the following set:

$$\mathcal{I} = \{i : U \to V \,|\, i\Box j \ \textit{ is an acyclic cofibration in } \ \mathcal{M}_{\mathcal{S}}\}$$

We have proved above that the set $\mathcal{I}$ contains all generating cofibration in $\mathcal{M}_{\mathcal{S}}$. We observe that the set $\mathcal{I}$ is closed under pushouts, transfinite compositions and retracts. Thus, $I$ contains all cofibration in $\mathcal{M}_{\mathcal{O}}$. Thus, we have proved that $i\Box j$ is a cofibration which is acyclic if $j$ is acyclic. The same argument as above when applied to the second argument of the Box product (*i.e.*, in the variable $j$) shows that $i\Box j$ is an acyclic cofibration whenever $i$ is an acyclic cofibration in $\mathcal{M}_{\mathcal{S}}$.

$\square$

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# FUNCTORIALITY OF PRINCIPAL BUNDLES AND CONNECTIONS

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**Résumé.** L'une des plus importantes contributions de la théorie de jauge en mathématiques est de souligner l'importance des foncteurs d'association. En mettant l'accent sur la théorie des catégories nous caractérisons ces derniers en utilisant deux de leurs propriétés naturelles. Cette caractérisation est ensuite utilisée pour établir une équivalence entre la catégorie des fibrés principaux et une certaine catégorie de foncteurs. Du point de vue de la géométrie differentielle nous décrivons la particularisation des connexions non–linéaires ou d'Ehresmann au cas principal ou linéaire. La propriété d'universalité des courbures principales, par ailleurs bien connue et largement utilisée, est alors employée afin de caractériser les fibrés vectoriels dans l'image d'un foncteur d'association donné.

**Abstract.** Perhaps the most important contribution of gauge theory to general mathematics is to point out the importance of association functors. Emphasizing category theory we characterize association functors by two of their natural properties and use this characterization to establish an equivalence between the category of principal bundles and a suitably defined category of functors. From the point of view of differential geometry we detail the specialization of non–linear or Ehresmann to principal and linear connections and discuss the widely known and very useful universality of principal curvature in order to characterize the vector bundles in the image of a given association functor.

**Keywords.** Principal Bundles, Connections, Association Functor.

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# 1 Introduction

Principal bundles and their association functors play a fundamental role in differential geometry and mathematical physics. Spin structures in pseudo–Riemannian geometry are defined right away as special principal bundles and the basic tenet of harmonic analysis is that the canonical association functor of a pointed homogeneous space is an equivalence of categories to the category of homogeneous fiber bundles. Last but not least the choice of principal bundle corresponds to the choice of vacuum sector in quantum field gauge theories. Nevertheless principal bundles tend to obfuscate calculations due to some inevitable arbitrariness, as one can see for example in Cartan geometries and in the botched proof of Blunder 5.24 in the otherwise excellent reference [LM]. Explicit calculations are more easily done using only the existence of association functors and the universality of curvature, arguably one of the most useful theorems in all of differential geometry.

En nuce this article brings these reservations against the use of principal bundles to a point: We show that a principal bundle $GM$ over a manifold $M$ is completely determined by its association functor $\mathrm{Ass}_{GM}$. Conversely every functor $\mathfrak{F}$ from a suitable category of model fibers to the category of fiber bundles over $M$ satisfying two more or less self–evident axioms agrees with the association functor for some principal bundle over $M$. Under natural transformations the class of all such functors $\mathfrak{F}$ becomes the category $\mathbf{GTS}_M$ of gauge theory sectors over $M$, which turns out to be equivalent to the category $\mathbf{PB}_M$ of principal bundles over the manifold $M$.

Category theory is usually not considered to be of particular importance to differential geometry, the text books [KMS] and [L] as well as the article [SM] are notable exceptions to this rule. Besides the characterization of associated vector bundles as geometric vector bundles in the sense of [SW] the common differential geometer may find little of interest in this article. Our main motivation for studying categorical properties of principal bundles nevertheless is the need to formulate the proper analogue of the concepts of principal bundles and connections in non–commutative geometry along the lines of [D1], [D2] and [D3]. Every definition of quantum bundles with quantum connections like the one presented in [Sa] will necessarily reflect functorial properties of principal bundles in classical differential geometry.

In order to provide a more detailed outline of this article we consider a Lie group $G$ and the category $\mathbf{MF}_G$ of manifolds $\mathscr{F}$ endowed with a smooth left action $\star : G \times \mathscr{F} \longrightarrow \mathscr{F}$ under smooth $G$–equivariant maps. Every principal $G$–bundle $GM$ over a manifold $M$ defines a functor from the category $\mathbf{MF}_G$ of model fibers to the category $\mathbf{FB}_M$ of fiber bundles over $M$

$$\mathrm{Ass}_{GM} : \ \mathbf{MF}_G \longrightarrow \mathbf{FB}_M, \qquad \mathscr{F} \longmapsto GM \times_G \mathscr{F} ,$$

which we may promote to a functor $\mathrm{Ass}^{\omega}_{GM} : \mathbf{MF}_G \longrightarrow \mathbf{FB}^{\nabla}_M$ to the category of fiber bundles with connections in the presence of a principal connection $\omega$ on $GM$. This association functor maps Cartesian products in $\mathbf{MF}_G$ to Cartesian products in $\mathbf{FB}^{\nabla}_M$ and maps a manifold $\mathscr{F}$ endowed with the trivial $G$–action to the trivial fiber bundle $M \times \mathscr{F}$. Our first main theorem stipulates that these two properties already characterize association functors as the reader can appreciate in Theorem 5.1

Let us now consider the category $\mathbf{PB}^{\nabla}_M$ of principal bundles with connections over $M$: Objects are triples $(\,G,\, GM,\, \omega\,)$ formed by a Lie group $G$ and a principal $G$–bundle $GM$ over $M$ endowed with a principal connection $\omega$, while morphisms are tuples $(\,\varphi_{\mathrm{grp}},\, \varphi\,)$ consisting of a parallel map $\varphi : GM \longrightarrow \hat{G}M$ of the underlying principal bundles, which is equivariant over the homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$ of Lie groups. The canonical factorization of the model homomorphism $\varphi_{\mathrm{grp}}$ entails a factorization of $\varphi$

$$\varphi : \ GM \xrightarrow{\ \mathrm{pr}\ } GM/_{\ker^\circ \varphi_{\mathrm{grp}}} \xrightarrow{\ \overline{\mathrm{pr}}\ } GM/_{\ker\ \varphi_{\mathrm{grp}}} \xrightarrow{\ \overline{\varphi}\ } \hat{G}M$$

into a parallel projection, a covering and a parallel injective immersion. In this sense every morphism in the category $\mathbf{PB}^{\nabla}_M$ of principal bundles with connections over $M$ is a product of just three basic types: The removal of a connected isospin subgroup, a covering $\overline{\mathrm{pr}}$ of principal bundles, a generalized spin structure, and a holonomy reduction $\overline{\varphi}$.

In order to translate this description of generalized spin structures and holonomy reductions as basic type morphism between principal bundles into a truly functorial description we consider the category $\mathbf{GTS}^{\nabla}_M$ of gauge theory sectors with connections over $M$. Its objects are tuples $(\,G,\, \mathfrak{F}\,)$ of a Lie group $G$ together with a functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}^{\nabla}_M$ satisfying the assumptions of Theorem 5.1. A morphism $(\,\varphi_{\mathrm{grp}},\, \Phi\,)$ between two such gauge theory sectors is a natural transformation $\Phi : \mathfrak{F} \circ \varphi^*_{\mathrm{grp}} \longrightarrow \hat{\mathfrak{F}}$ between the functors twisted by the pull back $\varphi_{\mathrm{grp}}^* : \mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{MF}_G$ of the action along the homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$ of Lie groups:

**Corollary 5.2 (Association Functor as Equivalence of Categories)**
*For every smooth manifold $M$ the association functor* $\mathrm{Ass}$ *provides us with an equivalence of categories from the category $\mathbf{PB}_M^\nabla$ of principal bundles to the category $\mathbf{GTS}_M^\nabla$ of gauge theory sectors with connections:*

$$\mathrm{Ass} : \ \mathbf{PB}_M^\nabla \ \stackrel{\simeq}{\longrightarrow} \ \mathbf{GTS}_M^\nabla, \qquad (\, G,\, GM,\, \omega \,) \ \longmapsto \ (\, G,\, \mathrm{Ass}_{GM}^\omega \,) \ .$$

*In particular two principal $G$–bundles endowed with principal connections on $M$ are isomorphic via a parallel, $G$–equivariant homomorphism of fiber bundles, if and only if their association functors are naturally isomorphic.*

A direct consequence of Corollary 5.2 is that association functors are not in general full functors, this is they are not surjective on morphisms, simply because the action pull back functor $\varphi_{\mathrm{grp}}^* : \mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{MF}_G$ is not a full functor unless the image of $G$ in $\hat{G}$ is dense. In other words there will be more parallel smooth homomorphisms of associated fiber bundles than there are $G$–equivariant smooth maps between their model fibers unless the principal connection $\omega$ has dense holonomy group.

According to Corollary 5.2 a spin structure on an oriented pseudo–Riemannian manifold $(\, M,\, g \,)$ can be defined as a functor extending the association functor $\mathbf{MF}_{\mathbf{SO}(T)} \longrightarrow \mathbf{FB}_M^\nabla$ determined by the oriented orthonormal frame bundle of $M$ to a functor $\mathbf{MF}_{\mathbf{Spin}(T)} \longrightarrow \mathbf{FB}_M^\nabla$ still satisfying the assumptions of Theorem 5.1, the corresponding spinor bundle $\$M$ is simply the image of the irreducible Clifford module under the extended functor. A fundamental problem in differential geometry related to spin structures is to characterize the vector and fiber bundles in the image of a given association functor. A partial answer to this problem is given in Proposition 4.6, which opens the way to an axiomatic characterization of spinor bundles and highlights the universality of principal curvatures.

This paper breaks down into five sections. Section 2 is a leisurely introduction to non–linear or Ehresmann connections on fiber bundles; we relate their curvature to the commutator of iterated covariant derivatives and discuss how non–linear connections specialize to principal and linear connections. In Section 3 we generalize objects of group type in categories with Cartesian products to principal objects. Association functors are studied in Section 4, the universality of curvature is formulated in Proposition 4.4. Having proved Theorem 5.1 in Section 5 we define the category of gauge theory sectors and establish the equivalence of categories formulated in Corollary 5.2.

The research project described in this article was inspired by the first part of the article [N] and can be seen as a direct analogue of this work in the framework of differential instead of algebraic geometry, moreover we address the additional complications brought about by the presence of connections.

# 2 Fiber Bundles and Non–Linear Connections

Perhaps the single most important concept in differential geometry is the notion of connections or the closely related notion of covariant derivatives on a vector or more general on a fiber bundle over a fixed manifold $M$. In this section we will modify the standard category $\mathbf{FB}_M$ of fiber bundles over $M$ to a category more useful for our study, the category $\mathbf{FB}_M^\nabla$ of fiber bundles with non–linear connections over $M$. Moreover we will discuss principal and linear connections in the framework of this category.

In general a fiber bundle over a manifold $M$ with model fiber manifold $\mathscr{F}$ is a manifold $\mathscr{F}M$ endowed with a smooth projection map $\pi : \mathscr{F}M \longrightarrow M$, which is locally trivializable. The preimage of a point $p \in M$ under $\pi$ is called the fiber of the bundle over $p$, it is a submanifold $\mathscr{F}_pM := \pi^{-1}(\,p\,) \subset \mathscr{F}M$ of the total space $\mathscr{F}M$ diffeomorphic to the model fiber $\mathscr{F}$. Fiber bundles over $M$ are the objects in the category $\mathbf{FB}_M$, morphisms in this category are smooth maps $\varphi : \mathscr{F}M \longrightarrow \hat{\mathscr{F}}M$ between the total spaces which commute with the respective projections $\hat{\pi} \circ \varphi = \pi$ and thus map the fibers of $\mathscr{F}M$ to the fibers of $\hat{\mathscr{F}}M$. Terminal objects in the category $\mathbf{FB}_M$ correspond to diffeomorphisms $\pi : \hat{M} \longrightarrow M$ thought of as fiber bundles over $M$ with single point fiber.

The Cartesian product of two fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ in $\mathbf{FB}_M$ is called the fibered product in differential geometry $\mathscr{F}M \times_M \hat{\mathscr{F}}M$ and it is defined as the equalizer of $\pi \circ \mathrm{pr}_L$ and $\hat{\pi} \circ \mathrm{pr}_R$ in the manifold product $\mathscr{F}M \times \hat{\mathscr{F}}M$.

In order to study connections in the context of category theory we prefer the following definition:

**Definition 2.1 (Non–linear Connections on Fiber Bundles)**
*A non–linear connection on a fiber bundle $\mathscr{F}M$ over a manifold $M$ is a field $\mathbb{P}^\nabla \in \Gamma(\mathscr{F}M, \operatorname{End} T\mathscr{F}M)$ of projections $(\mathbb{P}^\nabla)^2 = \mathbb{P}^\nabla$ on the tangent bundle $T\mathscr{F}M$ such that its image distribution equals the vertical foliation:*

$$\operatorname{im}\Big(\,\mathbb{P}^\nabla_f : \; T_f\mathscr{F}M \longrightarrow T_f\mathscr{F}M\,\Big) \;\stackrel{!}{=}\; \operatorname{Vert}_f\mathscr{F}M\,.$$

Every non–linear connection $\mathbb{P}^\nabla$ on a fiber bundle $\mathscr{F}M$ allows us to define the first order differential operator

$$D^\nabla : \; \Gamma(M, TM) \times \Gamma_{\mathrm{loc}}(M, \mathscr{F}M) \longrightarrow \Gamma_{\mathrm{loc}}(M, \operatorname{Vert}\mathscr{F}M) \tag{1}$$

such that

$$(D^\nabla_X f)_p \;:=\; \Big(\,T_pM \stackrel{f_{*,p}}{\longrightarrow} T_{f(p)}\mathscr{F}M \stackrel{\mathbb{P}^\nabla_{f(p)}}{\longrightarrow} \operatorname{Vert}_{f(p)}\mathscr{F}M\,\Big)\,X_p\,,$$

which is the non–linear analogue of the classical definition of covariant derivatives on vector bundles. Somewhat annoyingly this covariant derivative $D^\nabla_X f$ contains the redundant information $f = \pi_{\mathscr{F}M} \circ D^\nabla_X f$, where $\pi_{\mathscr{F}M}$ denotes the vertical tangent bundle projection $\operatorname{Vert}\mathscr{F}M \longrightarrow \mathscr{F}M$, the simplicity of linear and principal connections stems from the fact that we can get rid of this redundancy altogether, the reduced covariant derivative $\nabla_X f$ captures only the partial derivatives of the section $f$.

The Nijenhuis or curvature tensor of a non–linear connection $\mathbb{P}^\nabla$ on a fiber bundle $\mathscr{F}M$ over a manifold $M$ is the horizontal 2–form $R^\nabla$ on the total space $\mathscr{F}M$ of the fiber bundle with values in the vertical tangent bundle defined for two arbitrary vector fields $X, Y$ on $\mathscr{F}M$ by:

$$R^\nabla(X, Y) \;=\; -\,\mathbb{P}^\nabla[\,(\mathrm{id} - \mathbb{P}^\nabla)X,\, (\mathrm{id} - \mathbb{P}^\nabla)Y\,]\,. \tag{2}$$

In particular the curvature $R^\nabla$ measures exactly the failure of the horizontal distribution $\ker \mathbb{P}^\nabla \subseteq T\mathscr{F}M$ associated to $\mathbb{P}^\nabla$ to be integrable. An interpretation of the curvature tensor along classical lines as a commutator of covariant derivatives is shown in [SaW]

**Definition 2.2 (Parallel Homomorphisms between Fiber Bundles)**
*A parallel homomorphism between fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ over the*

*same manifold $M$ endowed with connections $\mathbb{P}^\nabla$ and $\mathbb{P}^{\hat\nabla}$ respectively is a homomorphism $\varphi : \mathscr{F}M \longrightarrow \hat{\mathscr{F}}M$ of fiber bundles such that the following diagram commutes:*

$$\begin{tikzcd} T\,\mathscr{F}M \arrow[r, "\varphi_*"] \arrow[d, "\mathbb{P}^\nabla"'] & T\,\hat{\mathscr{F}}M \arrow[d, "\mathbb{P}^{\hat\nabla}"] \\ T\,\mathscr{F}M \arrow[r, "\varphi_*"] & T\,\hat{\mathscr{F}}M \end{tikzcd}$$

The constraint $\hat\pi \circ \varphi = \pi$ characterizing homomorphisms of fiber bundles in the category $\mathbf{FB}_M$ readily implies $\varphi_*(\,\mathrm{Vert}\,\mathscr{F}M\,) \subset \mathrm{Vert}\,\hat{\mathscr{F}}M$, hence the homomorphism $\varphi$ of fiber bundles is parallel, if and only if $\varphi_*$ maps the horizontal distribution of $\mathscr{F}M$ to the horizontal distribution of $\hat{\mathscr{F}}M$:

$$\varphi \text{ parallel} \qquad\Longleftrightarrow\qquad \varphi_*(\,\ker\,\mathbb{P}^\nabla\,) \;\subset\; \ker\,\mathbb{P}^{\hat\nabla}\,.$$

Modifying the category $\mathbf{FB}_M$ we define the category $\mathbf{FB}_M^\nabla$ of fiber bundles with connection over $M$, in this category morphisms are parallel homomorphisms of fiber bundles.

In the resulting category terminal objects are still diffeomorphisms considered as fiber bundles with single point fibers endowed with the zero connection $\mathbb{P}^\nabla = 0$. Besides terminal objects the category $\mathbf{FB}_M^\nabla$ has Cartesian products: The fibered product $\mathscr{F}M \times_M \hat{\mathscr{F}}M$ of two fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ over $M$ with connections $\mathbb{P}^\nabla$ and $\mathbb{P}^{\hat\nabla}$ carries the product connection $(\mathbb{P}^\nabla \oplus \mathbb{P}^{\hat\nabla}) : T(\mathscr{F}M \times_M \hat{\mathscr{F}}M) \longrightarrow \mathrm{Vert}\,\mathscr{F}M \oplus \mathrm{Vert}\,\hat{\mathscr{F}}M$ defined by

$$\frac{d}{dt}\bigg|_0 (\,f_t,\,\hat f_t\,) \;\longmapsto\; \mathbb{P}^\nabla\Big(\,\frac{d}{dt}\bigg|_0 f_t\Big) \,\oplus\, \mathbb{P}^{\hat\nabla}\Big(\,\frac{d}{dt}\bigg|_0 \hat f_t\Big)\,,$$

where $t \longmapsto f_t$ and $t \longmapsto \hat f_t$ are smooth curves in $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ subject to the fibered product constraint $\pi(f_t) = \hat\pi(\hat f_t)$ for all $t$. In light of all these definitions the Cartesian product with the base manifold $M$ becomes a functor from the category $\mathbf{MF}$ of smooth manifolds to the category $\mathbf{FB}_M^\nabla$

$$M\times : \;\; \mathbf{MF} \;\longrightarrow\; \mathbf{FB}_M^\nabla, \qquad \mathscr{F} \;\longmapsto\; M \times \mathscr{F}\,, \tag{3}$$

because every trivial fiber bundle $M \times \mathscr{F}$ over $M$ comes along with the trivial connection $\mathbb{P}^{\mathrm{triv}}$, namely the projection to the tangent bundle of $\mathscr{F}$:

$$T(\,M \times \mathscr{F}\,) \;\stackrel{\cong}{\longrightarrow}\; TM \times T\mathscr{F} \;\stackrel{\pi\times\mathrm{id}}{\longrightarrow}\; M \times T\mathscr{F} \;\cong\; \mathrm{Vert}(\,M \times \mathscr{F}\,)\,.$$

Evidently the horizontal distribution $TM \times \mathscr{F} \subset T(M \times \mathscr{F})$ is an integrable foliation with leaves $M \times \{f\}$ for every trivial connection $\mathbb{P}^{\text{triv}}$, in consequence $R^{\text{triv}} = 0$ vanishes necessarily. The product functor $M\times$ defined in equation (3) will feature prominently in Sections 3 and 5.

Having discussed general non–linear connections on fiber bundles in some detail we now want to specialize to principal and linear connections in the second part of this section. Recall first of all that a principal bundle modeled on a Lie group $G$ is a smooth fiber bundle $GM$ with model fiber $G$ endowed with a smooth right $\rho$, fiber preserving action of $G$ on its total space $GM$. Also it is possible to define the affine product $\backslash : GM \times_M GM \longrightarrow G$.

The automorphism group bundle of a principal bundle $GM$ over a manifold $M$ is the Lie group bundle $\mathrm{Aut}\; GM$ over $M$ defined by

$$\mathrm{Aut}\, GM \;:=\; \{\; (p, \psi) \;|\; \psi : G_pM \longrightarrow G_pM \text{ is } G\text{–equivariant } \} \quad (4)$$

with the bundle projection $\pi_{\mathrm{Aut}\, GM} : \mathrm{Aut}\, GM \longrightarrow M,\ (p,\, \psi) \longmapsto p$. In mathematical physics the Fréchet–Lie group $\Gamma(\, M,\ \mathrm{Aut}\, GM\,)$ of all global sections of the automorphism bundle is called the gauge group of $GM$.

The fiber of the Lie group bundle $\mathrm{Aut}\; GM$ over a point $p \in M$ is a Lie group $\mathrm{Aut}_p GM$ isomorphic, although not canonically so, to the original group $G$, in particular its Lie algebra $\mathfrak{aut}_p GM \cong \mathfrak{g}$ is isomorphic to the Lie algebra of $G$. All these Lie algebras assemble into a smooth Lie algebra bundle $\mathfrak{aut}\, GM$, whose global sections $\Gamma(\, M,\ \mathfrak{aut}\, GM\,)$ form the Fréchet–Lie algebra of the gauge group $\Gamma(\, M,\ \mathrm{Aut}\, GM\,)$ of the principal bundle $GM$.

**Definition 2.3 (Principal Connections)**
*A principal connection on a principal $G$–bundle $GM$ over a manifold $M$ is a non–linear connection $\mathbb{P}^{\nabla}$ on the fiber bundle $GM$, which is invariant under the right action of $G$ on $GM$ in the sense that the right translations $R_\gamma : GM \longrightarrow GM,\ g \longmapsto g\gamma$, are parallel automorphisms for all $\gamma \in G$.*

In difference to general fiber bundles the vertical tangent bundle of a principal bundle $GM$ is trivializable by

$$\mathrm{vtriv} : \mathrm{Vert}\, GM \longrightarrow GM \times \mathfrak{g},\ \left.\frac{d}{dt}\right|_0 g_t \longmapsto (g_0,\, \left.\frac{d}{dt}\right|_0 g_0^{-1} g_t).$$

This allows to establish the following well–known result

**Lemma 2.4 (Principal Connection Axiom)**
*On every principal $G$–bundle $GM$ the association $\mathbb{P}^\nabla \longleftrightarrow \omega$ characterized by $\omega := \mathrm{vtriv} \circ \mathbb{P}^\nabla$ induces a bijection between principal connections in the sense of Definition 2.3 and $\mathfrak{g}$–valued $1$–forms $\omega$ on $GM$ satisfying the axiom*

$$\omega_{g_0\gamma_0}\Big( \frac{d}{dt}\Big|_0 g_t\,\gamma_t \Big) \;=\; \mathrm{Ad}_{\gamma_0^{-1}}\,\omega_{g_0}\Big( \frac{d}{dt}\Big|_0 g_t \Big) \;+\; \frac{d}{dt}\Big|_0 \gamma_0^{-1}\gamma_t$$

*for all choices of smooth curves $t \longmapsto g_t$ in $GM$ and curves $t \longmapsto \gamma_t$ in $G$.*

Cartan's Second Structure Equation [B] is a convenient description of the image of the composition of the curvature tensor $R^\nabla$ with the vertical trivialization $\mathrm{vtriv}$ in terms of the exterior derivative of the connection form

$$\Omega \;:=\; \mathrm{vtriv} \circ R^\nabla \;\overset{!}{=}\; d\omega \;+\; \frac{1}{2}[\,\omega \wedge \omega\,]\,, \tag{5}$$

where $\frac{1}{2}[\,\omega \wedge \omega\,](X, Y) \;:=\; [\,\omega(X),\,\omega(Y)\,]$.

The strategy persued for linear connections on vector bundles $VM$ follows the model of principal connections closely. The tangent bundle of a vector space is canonically trivializable $TV \;\cong\; V \times V$ by taking actual derivatives and this becomes via $[\,\mathrm{Vert}\,VM\,]_p \;=\; T(\,V_pM\,)$ the vertical trivialization

$$\mathrm{vtriv} : \mathrm{Vert}\,VM \overset{\cong}{\longrightarrow} VM \oplus VM,\;\; \frac{d}{dt}\Big|_0 v_t \longmapsto v_0 \oplus \lim_{t\to 0} \frac{1}{t}(v_t - v_0).$$

This map can be used to project out $\nabla_X v \;:=\; \mathrm{vtriv}(D^\nabla_X v)$ the redundant information from the covariant derivative $D^\nabla_X v$ of a section $v \;\in\; \Gamma(\,M,\,VM\,)$:

**Definition 2.5 (Linear Connections on Vector Bundles)**
*A linear connection on a vector bundle $VM$ on $M$ is a non–linear connection $\mathbb{P}^\nabla$ on $VM$ such that the reduced covariant derivative is $\mathbb{R}$–bilinear:*

$$\nabla :\;\; \Gamma(\,M, TM\,) \;\times\; \Gamma(\,M, VM\,) \;\longrightarrow\; \Gamma(\,M, VM\,)\,.$$

In [SaW] it is showed a proof of the following lemma.

**Lemma 2.6 (Characterization of Linear Connections)**
*A non–linear connection $\mathbb{P}^\nabla$ on a vector bundle gives rise to an $\mathbb{R}$–bilinear covariant derivative $\nabla : \Gamma(\,M, TM\,) \times \Gamma(\,M,\,VM\,) \longrightarrow \Gamma(\,M,\,VM\,)$, if and only if the multiplication by every $\lambda \,\in\, \mathbb{R}$ is a parallel endomorphism:*

$$\Lambda_\lambda :\;\; VM \;\longrightarrow\; VM, \qquad v \;\longmapsto\; \lambda\,v\,.$$

# 3 Principal Objects in Categories

In every category $\mathscr{C}$ with terminal objects and Cartesian products the notion of a group so fundamental to all of mathematics can be generalized to the notion of a group like object in $\mathscr{C}$. In this section we take this beautiful idea to characterize homogeneous spaces with trivial stabilizers, generally known as principal homogeneous or affine group spaces, in terms of their structure morphisms. Moreover we apply this characterization of affine group spaces to the category $\mathbf{FB}_M^\nabla$ of fiber bundles with connections over a manifold $M$ in order to characterize principal bundles with principal connections.

A group like object in a category $\mathscr{C}$ with terminal objects and Cartesian products is an object $G \in \text{OBJ}\,\mathscr{C}$ together a choice of structure morphisms

$$m: \ G \times G \longrightarrow G \qquad\qquad \iota: \ G \longrightarrow G \qquad\qquad \epsilon: \ * \longrightarrow G$$

in $\mathscr{C}$ called the multiplication, the inverse and the neutral element respectively with an arbitrary fixed terminal object $*$ such that the three diagrams

$$\begin{tikzcd} G\times G\times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{id}\times e"] \arrow[d, "e\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{id}\times\iota"] \arrow[d, "\iota\times\mathrm{id}"'] \arrow[dr, "e"] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd}$$

all commute, where $e = \epsilon \circ \mathrm{term}$ equals the composition of $\epsilon$ with the terminal morphism $\mathrm{term}: G \longrightarrow *$. In the category $\mathbf{Set}$ of sets for example the terminal objects are sets with exactly one element, hence $\epsilon: * \longrightarrow G$ essentially corresponds to an element of $G$. In turn the commutative diagrams above convert respectively into the associativity, the existence of a neutral element and the existence of inverses axiom in the definition of a group. In other words group like objects in $\mathbf{Set}$ are just plain groups.

In categories more complicated than $\mathbf{Set}$ the classification of group like objects can be simplified by the use of functors: Every covariant functor $\mathfrak{F}: \mathscr{C} \longrightarrow \hat{\mathscr{C}}$, which maps terminal objects to terminal objects and preserves Cartesian products, maps group like objects in the category $\mathscr{C}$ to group like objects in $\hat{\mathscr{C}}$. The standard forgetful functor $\mathbf{MF} \longrightarrow \mathbf{Set}$ from manifolds to sets for examples maps a group like object in $\mathbf{MF}$ to a group, albeit a Lie group whose multiplication and inverse are smooths maps.

In the same vein group like objects $G$ in the category $\mathbf{Grp}$ of groups carry two different group structures, one for being an object in $\mathbf{Grp}$ and the other due to the forgetful functor $\mathbf{Grp} \longrightarrow \mathbf{Set}$. It is a rather insightful exercise to verify that these two group structures actually agree so that $G$ is necessarily abelian, because its multiplication $m : G \times G \longrightarrow G$ is a morphism in $\mathbf{Grp}$. In consequence the fundamental group $\pi_1(G, e)$ of a topological group $G$ is always abelian, because the functor $\pi_1$ maps terminal objects to terminal objects and preserves Cartesian products.

With these examples of the usefulness of functors in combination with a categorical definition of groups in mind we want to describe the concept of an affine group or principal homogeneous space in terms of category theory. Given a group like object $G$ in a category $\mathscr{C}$ we define a (right) principal $G$–object to be an object $X \in \mathrm{OBJ}\,\mathscr{C}$ endowed with two structure morphisms

$$\rho : X \times G \longrightarrow X \qquad \backslash : X \times X \longrightarrow G \tag{6}$$

in $\mathscr{C}$ called action and left division respectively such that the action diagrams

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}\times e"] \arrow[dr, "\mathrm{id}"'] & X \times G \arrow[d, "\rho"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \times G \times G \arrow[r, "\mathrm{pr}_L \times m"] \arrow[d, "\rho\times\mathrm{pr}_R"'] & X \times G \arrow[d, "\rho"] \\ X \times G \arrow[r, "\rho"] & X \end{tikzcd} \tag{7}$$

and the following diagrams encoding simple transitivity all commute:

$$\begin{tikzcd} X \times X \arrow[rr, "\mathrm{pr}_L \times \backslash"] \arrow[dr, "\mathrm{pr}_R"'] & & X \times G \arrow[dl, "\rho"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} X \times G \arrow[rr, "\mathrm{pr}_L \times \rho"] \arrow[dr, "\mathrm{pr}_R"'] & & X \times X \arrow[dl, "\backslash"] \\ & G & \end{tikzcd} \tag{8}$$

In these diagrams $\mathrm{pr}_L$ and $\mathrm{pr}_R$ denote the projections to the leftmost and rightmost factor respectively, moreover $m : G \times G \longrightarrow G$ denotes multiplication in $G$ and $e : X \longrightarrow G$ the composition of $\epsilon$ with the terminal morphism $\mathrm{term} : X \longrightarrow *$. Left principal objects can be defined in complete analogy simply by switching left and right factors.

Intuitively, a principal object is essentially the group object itself, where we have forgotten the neutral element, in fact every group like object $G$ in a category $\mathscr{C}$ becomes a principal object over itself under the two structure morphisms $\rho := m$ and $\backslash := m \circ (\iota \times \mathrm{id})$. In the category $\mathbf{Set}$ of sets for example a principal object over a group $G$ is a set $X$ endowed with a right action $\rho : X \times G \longrightarrow X$, $(x, g) \longmapsto xg$, due to the commutative diagrams in (7) and an additional application $\backslash : X \times X \longrightarrow G$, $(x, y) \longmapsto x^{-1}y$, such that the following two axioms are met for all $x,\, y \in X$ and $g \in G$

$$x\,(\,x^{-1}\,y\,) \;=\; y \qquad\qquad x^{-1}\,(\,x\,g\,) \;=\; g\,,$$

which reflect the commutative diagrams in (8). In consequence the right action of $G$ on $X$ is transitive with trivial stabilizers, once we have declared an arbitrary point $x \in X$ to be the neutral element a principal object $X \neq \emptyset$ becomes indiscernible from the group $G$. In linear algebra for example it would be appropriate to define an affine space to be a principal object $\mathscr{V} \neq \emptyset$ under the additive group underlying a vector space $V$ over a field $\mathbb{K}$.

**Lemma 3.1 (Group Like and Principal Objects in $\mathbf{FB}^{\nabla}_{M}$)**
*For every Lie group $G$ the trivial fiber bundle $M \times G$ over a manifold $M$ endowed with the trivial connection and the obvious structure morphisms is a group like object in the category $\mathbf{FB}^{\nabla}_{M}$ of fiber bundles with non–linear connections over $M$. Principal $M \times G$–objects are exactly the principal $G$–bundles $GM$ over $M$ endowed with a principal connection $\omega$.*

**Proof:** The product functor $M\times : \mathbf{MF} \longrightarrow \mathbf{FB}^{\nabla}_{M}$, $\mathscr{F} \longmapsto M \times \mathscr{F}$, maps of course terminal objects in $\mathbf{MF}$ to terminal objects in $\mathbf{FB}^{\nabla}_{M}$ and preserves Cartesian products, hence it maps the Lie group $G$, a group like object in the category $\mathbf{MF}$, to the group like object $M \times G$ in the category $\mathbf{FB}^{\nabla}_{M}$. Consider now a principal $M \times G$–object in $\mathbf{FB}^{\nabla}_{M}$, this is a fiber bundle $GM$ over $M$ endowed with a non–linear connection $\nabla$ and structure homomorphisms:

$$\rho :\; GM \times_M (\,M \times G\,) \;\longrightarrow\; GM \qquad \backslash :\; GM \times_M GM \;\longrightarrow\; M \times G\,.$$

The obvious diffeomorphism $GM \times_M (M \times G) \cong GM \times G$ of fiber bundles provides $GM$ with a fiber preserving right action $\rho : GM \times G \longrightarrow GM$ such that each fiber $G_pM$ becomes a principal $G$–object in the category $\mathbf{Set}$, this is to say that the action $\rho$ is simply transitive on fibers. For every $\gamma \in G$

the element morphism $\gamma : \{*\} \longrightarrow G$ in the category $\mathbf{MF}$ induces moreover a parallel homomorphism in the category $\mathbf{FB}^{\nabla}_{M}$ of fiber bundles

$$GM \overset{\mathrm{id}\times\mathrm{term}}{\longrightarrow} GM \times_{M} (M \times \{*\}) \overset{\mathrm{id}\times\gamma}{\longrightarrow} GM \times_{M} (M \times G) \overset{\rho}{\longrightarrow} GM\,,$$

which is just the right multiplication $R_\gamma : GM \longrightarrow GM,\ g \longmapsto g\gamma$. In turn the non–linear connection $\mathbb{P}^{\nabla}$ present on the object $GM$ in $\mathbf{FB}^{\nabla}_{M}$ arises from a principal connection $\omega$ in the sense of Definition 2.3. ■

A general group like object in the category $\mathbf{FB}_M$ of fiber bundles over a manifold $M$ is just a bundle of Lie groups over $M$, a fiber bundle $GM$ endowed with the structure of a Lie group on each fiber such that the multiplication $m : GM \times_M GM \longrightarrow GM$, the inverse $\iota : GM \longrightarrow GM$ and the neutral element section $\epsilon : M \longrightarrow GM$ are smooth. Somewhat stronger is the concept of a Lie group bundle: A bundle $GM$ of Lie groups, which can be trivialized locally by group isomorphisms. Evidently this stronger condition is necessary and sufficient for the existence of a non–linear connection $\mathbb{P}^{\nabla}$, under which $GM$ becomes a group like object in the category $\mathbf{FB}^{\nabla}_{M}$.

# 4 Association Functors and Principal Bundles

Principal bundles are in a sense universal fiber bundles, every given principal bundle induces myriad fiber bundles with a large variety of model fibers over the same base manifold. The construction of all these fiber bundles is functorial in nature and best thought of as a functor, the association functor $\mathrm{Ass}^{\omega}_{GM}$, from a suitably defined category $\mathbf{MF}_G$ of model fibers to the category $\mathbf{FB}^{\nabla}_{M}$ of fiber bundles with connections over a manifold $M$. In this section we study the more important properties of association functors, the universality of principal connections and their curvature and characterize all vector bundles in the image of a fixed association functor.

Besides the categories $\mathbf{FB}_M$ and $\mathbf{FB}^{\nabla}_{M}$ of fiber bundles we are interested in the category $\mathbf{MF}_G$ of manifolds $\mathscr{F}$ acted upon by a fixed Lie group $G$ under smooth $G$–equivariant maps $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ as morphisms. Terminal objects are one point manifolds $\{*\}$ and Cartesian products in the category $\mathbf{MF}_G$ see $G$ acting diagonally on the Cartesian product $\mathscr{F} \times \hat{\mathscr{F}}$ of the manifolds underlying two objects $\mathscr{F}$ and $\hat{\mathscr{F}}$. Interestingly the category $\mathbf{MF}_G$ comes along with a canonical endofunctor, the tangent bundle endofunctor

$$T: \ \mathbf{MF}_G \longrightarrow \mathbf{MF}_G, \qquad \mathscr{F} \longrightarrow T\mathscr{F} ,$$

which sends an object $\mathscr{F} \in \mathrm{OBJ}\,\mathbf{MF}_G$ to the tangent bundle of its underlying manifold considered as a manifold $T\mathscr{F}$ in its own right, on which the Lie group $G$ acts by the differential of its characteristic action $\star$ on $\mathscr{F}$:

$$\star_{T\mathscr{F}}: \ G \times T\mathscr{F} \longrightarrow T\mathscr{F}, \qquad \Big(\gamma, \frac{d}{dt}\Big|_0 f_t\Big) \longmapsto \frac{d}{dt}\Big|_0 \gamma \star f_t \, .$$

In order to define the tangent bundle functor on morphisms we observe that the differential $\varphi_* : T\mathscr{F} \longrightarrow T\hat{\mathscr{F}}$ of a $G$–equivariant map $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ is again $G$–equivariant and this observation suggests $T\varphi := \varphi_*$. It should be noted that the Lie group $G$ provides a distinguished object in the category $\mathbf{MF}_G$, namely its Lie algebra $\mathfrak{g} := T_eG$ considered just as a manifold endowed with the adjoint representation $\mathrm{Ad} : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$. The infinitesimal action links this distinguished object to the tangent bundle endofunctor:

**Definition 4.1 (Infinitesimal Action)**
*Consider a smooth left action $\star : G \times \mathscr{F} \longrightarrow \mathscr{F}$, $(\gamma, f) \longmapsto \gamma \star f$, of a Lie group $G$ on a smooth manifold $\mathscr{F}$. The infinitesimal action of the Lie algebra $\mathfrak{g}$ of the group $G$ associated to this smooth action $\star$ is defined by*

$$\star_{\mathrm{inf}} : \ \mathfrak{g} \times \mathscr{F} \longrightarrow T\mathscr{F}, \qquad \Big(\frac{d}{dt}\Big|_0 \gamma_t, f\Big) \longmapsto \frac{d}{dt}\Big|_0 \gamma_t \star f \, ,$$

*where $t \longmapsto \gamma_t$ with $\gamma_0 = e$ represents the tangent vector $\frac{d}{dt}\big|_0 \gamma_t \in \mathfrak{g}$.*

En nuce the infinitesimal action is a natural transformation from the endofunctor $\mathfrak{g}\times$ to the tangent bundle endofunctor. In fact $\star_{\mathrm{inf}} : \mathfrak{g} \times \mathscr{F} \longrightarrow T\mathscr{F}$ is $G$–equivariant and thus a morphism in $\mathbf{MF}_G$ for all objects $\mathscr{F}$ due to

$$\gamma \star_{T\mathscr{F}} (X \star_{\mathrm{inf}} f) \; = \; \frac{d}{dt}\Big|_0 (\gamma\gamma_t\gamma^{-1}) \star (\gamma \star f) \; = \; (\mathrm{Ad}_\gamma X) \star_{\mathrm{inf}} (\gamma \star f)$$

for all $f \in \mathscr{F}$ and all tangent vectors $X = \frac{d}{dt}\big|_0 \gamma_t$ at $\gamma_0 = e$, moreover $\star_{\mathrm{inf}}$ intertwines with the differential $\varphi_*$ of every $G$–equivariant smooth map $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ in the identity $\varphi_*(X \star_{\mathrm{inf}} f) = X \star_{\mathrm{inf}} \varphi(f)$.

**Definition 4.2 (Association Functor)**
*Consider a Lie group $G$ and a principal $G$–bundle $GM$ over a manifold $M$. Every smooth action $\star : G \times \mathscr{F} \longrightarrow \mathscr{F}$ of the group $G$ on a manifold $\mathscr{F}$ extends to a free and smooth right action of the group $G$ on the Cartesian product $GM \times \mathscr{F}$ via $(g, f) \star \gamma := (g\gamma, \gamma^{-1} \star f)$. The quotient of $GM \times \mathscr{F}$ by this free action is a fiber bundle over $M$ with model fiber $\mathscr{F}$*

$$\mathrm{Ass}_{GM}(\mathscr{F}) \;=\; GM \times_G \mathscr{F} \;:=\; (GM \times \mathscr{F})/_G$$

*called the fiber bundle associated to $GM$ and $\mathscr{F} \in$ OBJ $\mathbf{MF}$. Every $G$–equivariant map $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ induces a homomorphism of fiber bundles*

$$\mathrm{Ass}_{GM}(\varphi) : \; GM \times_G \mathscr{F} \longrightarrow GM \times_G \hat{\mathscr{F}}, \qquad [g, f] \longmapsto [g, \varphi(f)],$$

*which is well–defined in terms of representatives $(g, f)$ of the equivalence class $[g, f]$. In other words $\mathrm{Ass}_{GM} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M$, $\mathscr{F} \longmapsto GM \times_G \mathscr{F}$, is a functor from $\mathbf{MF}_G$ to the category $\mathbf{FB}_M$ of fiber bundles over $M$.*

Recall now that the each of the categories $\mathbf{MF}_G$ and $\mathbf{FB}_M$ has a canonical endofunctor associated with it, namely the tangent bundle endofunctor $T$ for the category $\mathbf{MF}_G$ of manifolds with $G$–action and the vertical tangent bundle functor $\mathrm{Vert}$ for the category $\mathbf{FB}_M$. Considered as a fiber bundle over $M$ the vertical tangent bundle has fiber $[\,\mathrm{Vert}\,\mathscr{F}M\,]_p = T[\,\mathscr{F}_pM\,]$ over every $p \in M$ and so we may suspect that the following diagram commutes

$$\begin{tikzcd} \mathbf{MF}_G \arrow[r, "\mathrm{Ass}_{GM}"] \arrow[d, "T"'] & \mathbf{FB}_M \arrow[d, "\mathrm{Vert}"] \\ \mathbf{MF}_G \arrow[r, "\mathrm{Ass}_{GM}"] & \mathbf{FB}_M\,. \end{tikzcd} \tag{9}$$

up to a natural isomorphism $\mathrm{Vert}(GM \times_G \mathscr{F}) \xrightarrow{\cong} GM \times_G T\mathscr{F}$ given by:

$$\frac{d}{dt}\bigg|_0 \Big[\, g_t,\; f_t \,\Big] \;\longmapsto\; \Big[\, g_0,\; \frac{d}{dt}\bigg|_0 (\, g_0^{-1}\, g_t\,) \,\star\, f_t \,\Big]\,. \tag{10}$$

Of course this isomorphism is motivated by $[\,g_t, f_t\,] = [\,g_0, (g_0^{-1}g_t) \star f_t\,]$, whenever the representative curve $t \longmapsto [\,g_t,\, f_t\,]$ for a vertical tangent vector to $GM \times_G \mathscr{F}$ has been chosen such that $g_t$ stays in the fiber of $g_0$ for all $t$.

**Remark 4.3 (Action of Automorphism Group Bundle)**
*The automorphism group bundle of a principal bundle $GM$ acts naturally $\star : \operatorname{Aut} GM \times_M \mathscr{F} M \longrightarrow \mathscr{F} M$ on every fiber bundle $\mathscr{F} M := GM \times_G \mathscr{F}$ associated to $GM$ and an object $\mathscr{F} \in$ OBJ $\mathbf{MF}_G$ by means of*

$$(p, \psi) \star [g, f] := [\psi(g), f]$$

*for all $(p, \psi) \in \operatorname{Aut}_p GM$ and $[g, f] \in \mathscr{F}_{\pi(g)} M$ in the fibers of* $\operatorname{Aut} GM$ *and $\mathscr{F} M$ over the same point $p = \pi(g)$ of the base manifold $M$.*

In concrete examples the automorphism group bundle $\operatorname{Aut} GM$ is usually more readily identified than the principal bundle $GM$ itself due to its omnipresent action on associated fiber bundles. Consider the orthonormal frame bundle of a pseudo–Riemannian manifold $(M, g)$ for example

$$\mathbf{O}(M, g) := \{ (p, F) \mid p \in M \text{ and } F : T \longrightarrow T_p M \text{ isometry } \},$$

where $T$ is a pseudo–euclidean model vector space of the correct signature and $\mathbf{O}(T)$ acts from the right by precomposition $(p, F)\gamma = (p, F \circ \gamma)$. The automorphism group bundle of the orthonormal frame bundle $\mathbf{O}(M, g)$ equals the Lie group bundle of all infinitesimal isometries of tangent spaces

$$\mathbf{O}(TM, g) := \{ (p, \psi) \mid \psi : T_p M \longrightarrow T_p M \text{ isometry } \}$$

acting by postcomposition $(p, \psi) \star (p, F) = (p, \psi \circ F)$; it just as well acts on the tangent bundle $TM$ and all kinds of the tensor bundles etc.

For a general principal bundle $GM$ we can use the same idea to identify the automorphism group bundle $\operatorname{Aut} GM$ as a Lie group bundle over $M$ with the image of a group object in the category $\mathbf{MF}_G$. Letting $G$ act on itself by conjugation $\star : G \times G \longrightarrow G$, $(\gamma, g) \longmapsto \gamma g \gamma^{-1}$, we obtain in fact a group object $G^{\mathrm{ad}} \in$ OBJ $\mathbf{MF}_G$, whose image under the association functor is a Lie group bundle $\operatorname{Ass}_{GM}(G^{\mathrm{ad}})$ over $M$ acting $G$–equivariantly on $GM$ by

$$\operatorname{Ass}_{GM}(G^{\mathrm{ad}}) \times_M GM \longrightarrow GM, \quad ([g, \gamma], \hat{g}) \longmapsto g\gamma(g^{-1}\hat{g}) \quad (11)$$

for all $\gamma \in G$ and all $g, \hat{g} \in GM$ in the same fiber. In particular $\operatorname{Aut}_p GM$ is isomorphic, but not naturally so, to the Lie group $G$ in every $p \in M$.

Under this identification $\mathrm{Ass}_{GM}(G^{\mathrm{ad}}) = \mathrm{Aut}\, GM$ of Lie group bundles the natural action of $\mathrm{Aut}\, GM$ on associated fiber bundles $GM \times_G \mathscr{F}$ pointed out in Remark 4.3 becomes the functorial extension of the original action $\star$ considered as a $G$–equivariant smooth map $\star : G^{\mathrm{ad}} \times \mathscr{F} \longrightarrow \mathscr{F}$. In the same vein the functor $\mathrm{Ass}_{GM}$ converts the infinitesimal action of Definition 4.1 considered as a $G$–equivariant map $\star_{\mathrm{inf}} : \mathfrak{g} \times \mathscr{F} \longrightarrow T\mathscr{F}$ into

$$\mathrm{Ass}_{GM}(\star_{\mathrm{inf}}) : \quad (GM \times_G \mathfrak{g}) \times_M (GM \times_G \mathscr{F}) \longrightarrow (GM \times_G T\mathscr{F}),$$

which in turn becomes the infinitesimal action associated to Remark 4.3:

$$\star_{\mathrm{inf}} : \quad \mathfrak{aut}\, GM \times_M (GM \times_G \mathscr{F}) \longrightarrow \mathrm{Vert}(GM \times_G \mathscr{F}).$$

Before we proceed to prove the universality of principal connections and their curvature we want to digress a little to discuss the gauge principle, a fundamental principle in the study of principal bundles allowing us to translate calculations on $GM$ to statements about $M$. In its most basic formulation the gauge principle is the assertion that we have a canonical bijection

$$[\Omega^\bullet_{\mathrm{hor}}(GM, V)]^G \xrightarrow{\cong} \Omega^\bullet(M, GM \times_G V), \qquad \eta \longmapsto \mathrm{GP}[\eta] \tag{12}$$

between the horizontal differential forms $\eta \in \Omega^\bullet_{\mathrm{hor}}(GM, V)$ on $GM$ with values in some representation $V$ of $G$ satisfying $R^*_\gamma \eta = \gamma \star \eta$ for all $\gamma \in G$ and general differential forms on the base manifold $M$ with values in the associated vector bundle $GM \times_G V$. Explicitly this gauge principle reads

$$\mathrm{GP}[\eta]_p(X_1, \ldots, X_r) := [g, \eta_g(\tilde{X}_1, \ldots, \tilde{X}_r)]$$

for arbitrary lifts $\tilde{X}_1, \ldots, \tilde{X}_r \in T_g GM$ of the argument tangent vectors $X_1, \ldots, X_r \in T_p M$ to an arbitrary point $g \in G_p M$ in the fiber over $p \in M$. Due to horizontality the resulting differential form $\mathrm{GP}[\eta]$ does not depend on the choice of lifts and the assumption $R^*_\gamma \eta = \gamma \star \eta$ ensures that $\mathrm{GP}[\eta]$ does not depend on the choice of $g \in G_p M$ either. The gauge principle converts the curvature 2–form $\Omega \in \Omega^2_{\mathrm{hor}}(GM, \mathfrak{g})$ of Cartan's Second Structure Equation (5) into a 2–form on $M$ with values in $\mathfrak{aut}\, GM$:

$$R^\omega := \mathrm{GP}[\Omega] \in \Omega^2(M, \mathfrak{aut}\, GM) \tag{13}$$

**Proposition 4.4 (Universality of Principal Curvature)**
*Every choice of a principal connection $\omega$ on a principal $G$–bundle $GM$ allows us to promote the association functor $\mathrm{Ass}_{GM} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M$ to a functor to the category of fiber bundles over $M$ with non–linear connections:*

$$\mathrm{Ass}^{\omega}_{GM} : \ \mathbf{MF}_G \ \longrightarrow \ \mathbf{FB}^{\nabla}_M, \qquad \mathscr{F} \ \longmapsto \ GM \times_G \mathscr{F} \ .$$

*In other words $\omega$ induces a natural connection $\nabla$ on $GM \times_G \mathscr{F}$ for every $G$–manifold $\mathscr{F} \in \mathrm{OBJ}\,\mathbf{MF}_G$. The curvature $R^{\nabla}$ of this induced connection is determined by the infinitesimal action of the Lie algebra bundle $\mathfrak{aut}\,GM$*

$$\star_{\mathrm{inf}} : \ \ \mathfrak{aut}\,GM \ \times_M \ (\,GM \times_G \mathscr{F}\,) \ \longrightarrow \ \mathrm{Vert}(\,GM \times_G \mathscr{F}\,)$$

*and the 2–form $R^{\omega} \in \Omega^2(\,M,\,\mathfrak{aut}\,GM\,)$. More precisely for all local sections $f \in \Gamma_{\mathrm{loc}}(\,M, GM \times_G \mathscr{F}\,)$ and all $X,\,Y \in \Gamma(\,M, TM\,)$ we find:*

$$R^{\nabla}_{X,Y} f \ \ = \ \ R^{\omega}(\,X,\,Y\,) \star_{\mathrm{inf}} \ f \ .$$

**Proof:** By definition $GM \times_G \mathscr{F}$ is the quotient of the Cartesian product $GM \times \mathscr{F}$ by a free right action of the Lie group $G$. In turn the canonical projection $\mathrm{pr} : GM \times \mathscr{F} \longrightarrow GM \times_G \mathscr{F}$ defines a tower of fiber bundles

$$\begin{tikzcd} GM \times \mathscr{F} \arrow[d, "\mathrm{pr}"] \\ GM \times_G \mathscr{F} \arrow[d, "\pi"] \\ M \end{tikzcd} \tag{14}$$

over $M$, which becomes $U \times (G \times \mathscr{F}) \xrightarrow{\mathrm{pr}} U \times \mathscr{F} \xrightarrow{\pi} U$ in a local equivariant trivialization of $GM$. The central idea of the proof is to choose the connection $\mathbb{P}^{\nabla}$ on $GM \times_G \mathscr{F}$ such that $\mathrm{pr}$ is parallel with respect to the product $\mathbb{P}^{\omega} \times \mathbb{P}^{\mathrm{triv}}$ of the principal connection $\omega$ on $GM$ and the trivial connection $\mathbb{P}^{\mathrm{triv}}$ on $M \times \mathscr{F}$.

For this purpose we consider a curve $t \longmapsto (\,g_t, f_t\,)$ in $GM \times \mathscr{F}$ and choose a curve $t \longmapsto \gamma_t$ in $G$ with $\gamma_0 = e$ representing the tangent vector $\frac{d}{dt}\big|_0 \gamma_t = \omega_{g_0}(\,\frac{d}{dt}\big|_0 g_t\,) \in \mathfrak{g}$. The Principal Connection Axiom 2.4 ensures

$$\omega_{g_0 e}\Big(\ \frac{d}{dt}\Big|_0 g_t\,\gamma_t^{-1}\ \Big) \ \ = \ \ \mathrm{Ad}_e^{-1}\,\omega_{g_0}\Big(\ \frac{d}{dt}\Big|_0 g_t\ \Big) \ + \ \frac{d}{dt}\Big|_0 e^{-1}\,\gamma_t^{-1} \ \ = \ \ 0$$

and so $t \longmapsto g_t\gamma_t^{-1}$ represents a horizontal tangent vector. In turn

$$\begin{aligned}(\mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}})\Big(\,\frac{d}{dt}\Big|_0 (\,g_t,\,f_t\,)\Big) \\ &= \;(\mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}})\Big(\,\frac{d}{dt}\Big|_0 (\,g_t\gamma_t^{-1}\gamma_0,\,f_0\,)\;+\;\frac{d}{dt}\Big|_0 (\,g_0\,\gamma_0^{-1}\,\gamma_t,\,f_t\,)\Big) \\ &= \;\frac{d}{dt}\Big|_0 (\,g_0\,\gamma_t,\,f_t\,)\,,\end{aligned}$$

because the first summand is horizontal and the second vertical in $GM \times \mathscr{F}$. Projecting this identity to equivalence classes in $GM \times_G \mathscr{F}$ we find

$$\begin{aligned}\mathbb{P}^\nabla\Big(\,\frac{d}{dt}\Big|_0 [\,g_t,\,f_t\,]\Big) \;&:=\; \frac{d}{dt}\Big|_0 [\,g_0,\,\gamma_t\,\star\,f_t\,] \\ &=\; \Big[\,g_0,\;\frac{d}{dt}\Big|_0 f_t\;+\;\omega_{g_0}\Big(\,\frac{d}{dt}\Big|_0 g_t\Big)\,\star_{\mathrm{inf}}\,f_0\,\Big]\end{aligned}$$

due to the Definition 4.1 of the infinitesimal action and the choice of the curve $t \longmapsto \gamma_t$. In light of the isomorphism (10) the right hand side denotes a vertical tangent vector to $GM \times_G \mathscr{F}$ and so the latter formula defines a non–linear connection $\mathbb{P}^\nabla$ on the fiber bundle $GM \times_G \mathscr{F}$.

With respect to this non–linear connection $\mathbb{P}^\nabla$ the canonical projection $\mathrm{pr} \,:\, GM \times \mathscr{F} \longrightarrow GM \times_G \mathscr{F}$ is parallel, because it maps horizontal tangent vectors $\frac{d}{dt}\big|_0 [\,g_t, f_0\,]$ with $\omega_{g_0}(\frac{d}{dt}\big|_0 g_t) \;=\; 0$ to horizontal vectors. The construction of $\mathbb{P}^\nabla$ is natural in the category $\mathbf{MF}_G$ as well: The functorial extension $\mathrm{Ass}_{GM}(\,\varphi\,) :\, GM \times_G \mathscr{F} \longrightarrow GM \times_G \hat{\mathscr{F}},\; [\,g, f\,] \longmapsto [\,g, \varphi(f)\,]$, of every $G$–equivariant smooth map $\varphi :\, \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ is parallel

$$\begin{aligned}\mathbb{P}^{\hat\nabla}\Big(\,\frac{d}{dt}\Big|_0 [\,g_t, \varphi(f_t)\,]\Big) \;&=\; \Big[\,g_0,\;\frac{d}{dt}\Big|_0 \varphi(f_t)\;+\;\omega_{g_0}\Big(\,\frac{d}{dt}\Big|_0 g_t\Big)\,\star_{\mathrm{inf}}\,\varphi(f_0)\Big] \\ &=\; \mathrm{Ass}_{GM}(\,\varphi_*\,)\,\mathbb{P}^\nabla\Big(\,\frac{d}{dt}\Big|_0 [\,g_t,\,f_t\,]\Big)\end{aligned}$$

due to the infinitesimal equivariance $X \star_{\mathrm{inf}} \varphi(f) \;=\; \varphi_*(X \star_{\mathrm{inf}} f)$. In order to calculate the curvature of the connection $\mathbb{P}^\nabla$ we use the fact that in a tower of fiber bundles like (14) with a parallel submersion $\mathrm{pr}$ the curvature of the image connection $\mathbb{P}^\nabla$ is just the image of the preimage connection $\mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}}$

under the differential $\mathrm{pr}_*$. Using arbitrary lifts $\tilde{X},\, \tilde{Y} \,\in\, T_gGM$ of tangent vectors $X,\, Y \,\in\, T_pM$ to a point $g \,\in\, G_pM$ we calculate in this way

$$\begin{aligned} R^{\nabla}_{[\,g,\,f\,]}(\,\tilde{X},\,\tilde{Y}\,) \;&=\; \mathrm{pr}_{*,\,(\,g,\,f\,)}R^{\omega\times\mathrm{triv}}_{(\,g,\,f\,)}(\,\tilde{X},\,\tilde{Y}\,) \\ &=\; \frac{d}{dt}\bigg|_0 \Big[\, g\,\exp\Big(\,t\,\Omega_g(\,\tilde{X},\,\tilde{Y}\,)\Big),\; f\,\Big] \\ &=\; [\,g,\,\Omega_g(\tilde{X},\tilde{Y})\,]\,\star_{\mathrm{inf}}\,[\,g,f\,] \;=\; R^{\omega}_p(X,Y)\,\star_{\mathrm{inf}}\,[\,g,f\,]\,, \end{aligned}$$

where $R^{\omega} \;:=\; \mathrm{GP}[\,\Omega\,] \;\in\; \Omega^2(\,M,\mathfrak{aut}\,GM\,)$ is the 2–form with values in $\mathfrak{aut}\,GM$ the gauge principle (12) associates to $\Omega \;:=\; d\omega \,+\, \frac{1}{2}[\,\omega\wedge\omega\,]$. Formulated in terms of local sections $f \;\in\; \Gamma_{\mathrm{loc}}(\,M,\,GM\,\times_G\,\mathscr{F}\,)$ the latter identity becomes $R^{\nabla}_{X,\,Y}f \;=\; R^{\omega}(X,Y)\,\star_{\mathrm{inf}}\,f$. ■

One of the most important properties of association functors is that they intertwine the actions of smooth functors on the categories $\mathbf{Rep}_G$ and $\mathbf{VB}_M$. A smooth functor is an endofunctor $\mathbb{S}\,:\;\mathbf{Vect}^{\times}_{\mathbb{R}}\longrightarrow\mathbf{Vect}^{\times}_{\mathbb{R}}$ of the category of finite dimensional vector spaces under linear isomorphisms such that

$$\mathrm{MOR}_{\mathbf{Vect}^{\times}_{\mathbb{R}}}(\,V,\,V\,)\;\longrightarrow\;\mathrm{MOR}_{\mathbf{Vect}^{\times}_{\mathbb{R}}}(\,\mathbb{S}\,V,\,\mathbb{S}\,V\,),\qquad \varphi\;\longmapsto\;\mathbb{S}(\,\varphi\,),$$

is a smooth map between the smooth manifolds $\mathrm{MOR}_{\mathbf{Vect}^{\times}_{\mathbb{R}}}(V,V)\;=\;\mathbf{GL}\,V$ and $\mathrm{MOR}_{\mathbf{Vect}^{\times}_{\mathbb{R}}}(\mathbb{S}\,V,\mathbb{S}\,V)$ for every finite dimensional vector space $V$ over $\mathbb{R}$. Smooth functors extend naturally to endofunctors of the category $\mathbf{Rep}_G$ of representations $V$ of a Lie group $G$ by letting $G$ act on $\mathbb{S}\,V$ via:

$$\star\;:\;\;G\,\times\,\mathbb{S}V\;\longrightarrow\;\mathbb{S}V,\qquad(\,\gamma,\,s\,)\;\longmapsto\;\mathbb{S}(\,\gamma\star:\;V\;\xrightarrow{\;\cong\;}\;V\,)\,s\,.$$

This extension to representations makes the classification of smooth functors an exercise in the representation theory of general linear groups: Every smooth functor is naturally isomorphic $\mathbb{S}\;\cong\;\mathbb{S}_1\oplus\ldots\oplus\mathbb{S}_r$ to a finite direct sum of Schur functors $\mathbb{S}_1,\,\ldots,\,\mathbb{S}_r$ twisted by density lines [FH].

In the same vein every smooth functor $\mathbb{S}$ extends naturally to an endofunctor of the category $\mathbf{VB}^{\nabla}_M$ of vector bundles with connections over a manifold $M$. The smoothness of $\mathbb{S}$ allows us to define a differentiable structure on the disjoint union of vector spaces obtained by applying $\mathbb{S}$ fiberwise

$$\mathbb{S}\,VM\;\;:=\;\;\bigcup_{p\,\in\,M}\mathbb{S}\,V_pM$$

to obtain a new vector bundle $\mathbb{S}\,VM$ over $M$; every connection $\nabla$ on the original vector bundle $VM$ extends naturally to a connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}\,VM$ by the requirement that parallel transport with respect to this connection along an arbitrary curve $t \longmapsto p_t$ in the manifold $M$ is simply the image

$$\Big(\,\mathbf{PT}_t^{\nabla^{\mathbb{S}}}:\ \mathbb{S}\,V_{p_0}M \xrightarrow{\cong} \mathbb{S}\,V_{p_t}M\,\Big) \;=\; \mathbb{S}\,\Big(\,\mathbf{PT}_t^{\nabla}:\ V_{p_0}M \xrightarrow{\cong} V_{p_t}M\,\Big)$$

of parallel transport with respect to $\nabla$ under the functor $\mathbb{S}$. Because parallel transport in associated vector bundles is essentially the image of parallel transport in the principal bundle $GM$ itself, every association functor $\mathrm{Ass}^{\omega}_{GM}$ intertwines the two extensions of a smooth functor $\mathbb{S}$ to the categories $\mathbf{Rep}_G$ of representations and $\mathbf{VB}^{\nabla}_M$ of vector bundles with connections:

$$\begin{tikzcd} \mathbf{Rep}_G \arrow[r, "\mathrm{Ass}^{\omega}_{GM}"] \arrow[d, "\mathbb{S}"'] & \mathbf{VB}^{\nabla}_M \arrow[d, "\mathbb{S}"] \\ \mathbf{Rep}_G \arrow[r, "\mathrm{Ass}^{\omega}_{GM}"] & \mathbf{VB}^{\nabla}_M\ . \end{tikzcd} \tag{15}$$

Classically the vector bundles of the form $\mathbb{S}\,TM$ on a manifold $M$ with a smooth functor $\mathbb{S}$ are called pseudotensor bundles, their sections pseudotensors, and they comprise exactly the natural vector bundles of order one. Some modern authors however seem to confuse the classical concept of tensors with the property of having a value defined at every point.

**Lemma 4.5 (Properties of Association Functors)**
*Consider a principal $G$–bundle $GM$ over a manifold $M$ endowed with a principal connection $\omega$ and the corresponding association functor from the category $\mathbf{MF}_G$ of manifolds endowed with smooth $G$–actions to the category $\mathbf{FB}^{\nabla}_M$ of fiber bundles over $M$ endowed with non–linear connections:*

1. *The association functor $\mathrm{Ass}^{\omega}_{GM}$ preserves Cartesian products:*

$$GM\ \times_G\ (\,\mathscr{F}\,\times\,\hat{\mathscr{F}}\,)\ =\ (\,GM\ \times_G \mathscr{F}\,)\times_M(\,GM\ \times_G \hat{\mathscr{F}}\,)\,.$$

2. *On the full subcategory $\mathbf{MF}\ \subset\ \mathbf{MF}_G$ of manifolds with trivial $G$–action the association functor $\mathrm{Ass}^{\omega}_{GM}$ agrees with the product functor:*

$$\mathrm{Ass}^{\omega}_{GM}|_{\mathbf{MF}}:\ \mathbf{MF}\ \longrightarrow\ \mathbf{FB}^{\nabla}_M,\qquad \mathscr{F}\ \longmapsto\ M\times\mathscr{F}\,.$$

3. *Restricted to the subcategory* $\mathbf{Rep}_G \subset \mathbf{MF}_G$ *of finite dimensional smooth representations of the Lie group* $G$ *under* $G$*–equivariant linear maps the association functor* $\mathrm{Ass}^{\omega}_{GM}$ *takes values in the subcategory* $\mathbf{VB}^{\nabla}_M$ *of vector bundles over* $M$ *endowed with linear connections:*

$$\mathrm{Ass}^{\omega}_{GM}|_{\mathbf{Rep}_G} : \ \mathbf{Rep}_G \longrightarrow \mathbf{VB}^{\nabla}_M, \qquad V \longmapsto GM \times_G V \ .$$

**Proof:** Of course all three statements of this lemma are easily proved directly by unwrapping all the definitions made above; the second statement for example is an elaborate description of the trivial fiber bundle isomorphism

$$GM \times_G \mathscr{F} \stackrel{=}{\longrightarrow} GM/_G \times \mathscr{F} \stackrel{\cong}{\longrightarrow} M \times \mathscr{F} \ ,$$

whenever $G$ acts trivially on $\mathscr{F}$ and thus effectively only on the first factor of $GM \times \mathscr{F}$ in the construction of the quotient $GM \times_G \mathscr{F}$. This fiber bundle isomorphism is evidently natural, it is compatible with all the fiber bundle homomorphisms induced by smooth maps $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ between manifolds $\mathscr{F}$ and $\hat{\mathscr{F}}$ with trivial $G$–action.

Nevertheless we think the lemma is quite interesting, because the third is actually a consequence of the first two statements. Combining the existence of additive inverses and the unity axiom $\forall v : \ 1 \cdot v = v$ into the axiom $\forall v : \ v + (-1) \cdot v = 0$ we see that only three structure maps are needed to formulate all axioms for a vector space object $V$ in a category $\mathscr{C}$ in terms of commutative diagrams provided we have specified a field object $\mathbb{K}$:

$$\cdot : \ \mathbb{K} \times V \longrightarrow V \qquad + : \ V \times V \longrightarrow V \qquad 0 : \ \{*\} \longrightarrow V \ .$$

In the category $\mathbf{MF}_G$ for example we may take the manifold $\mathbb{R}$ with the trivial $G$–action as the field object $\mathbb{K} = \mathbb{R}^{\mathrm{triv}}$, the corresponding vector space objects are smooth representations of the Lie group $G$ over $\mathbb{R}$.

On the other hand the first and second statement of the lemma assert that the association functor $\mathrm{Ass}^{\omega}_{GM}$ preserves Cartesian products and agrees with the product functor $M \times$ on the full subcategory $\mathbf{MF} \subset \mathbf{MF}_G$. In consequence $\mathrm{Ass}^{\omega}_{GM}$ sends terminal objects in $\mathbf{MF}_G$ to terminal objects in $\mathbf{FB}^{\nabla}_M$ and a representation $V$ to a fiber bundle $VM := GM \times_G V$ with three parallel structure maps, the zero section $0 : M \longrightarrow VM$ and:

$$\cdot : \ \mathbb{R} \times VM \longrightarrow VM \qquad + : \ VM \times_M VM \longrightarrow VM \ .$$

According to Lemma 2.6 the parallelity of the scalar multiplication map alone suffices to force the non–linear connection $\mathbb{P}^\nabla$ on $VM \in$ OBJ $\mathbf{FB}_M^\nabla$ to be a linear connection in the sense of Definition 2.5. ■

Historically the concept of principal bundles and principal connections arose from Cartan's beautiful idea of moving frames, which asserts that every vector bundle $VM$ with connection $\nabla$ lies in the image of the association functor $\mathrm{Ass}^\omega_{GM}$ for some principal bundle with connection. A suitable choice for the principal bundle $GM$ is the frame bundle with model vector space $V$

$$\mathbf{GL}(\,M,\,VM\,) \;:=\; \{\;(\,p,\,F\,)\;|\;p\in M \text{ and } F:\,V\xrightarrow{\cong}V_pM\;\}\,,$$

which is a principal $\mathbf{GL}\,V$–bundle over $M$ with right multiplication given by precomposition $(p,F)\,\gamma \,=\, (p, F\circ\gamma)$. The tautological diffeomorphism

$$\mathbf{GL}(\,M,\,VM\,)\;\times_{\mathbf{GL}\,V}V\;\xrightarrow{\cong}\;VM, \qquad [\,(\,p,\,F\,),\,v\,]\;\longmapsto\;Fv\,,$$

is a parallel isomorphism for the principal connection on $\mathbf{GL}(M,VM)$

$$\omega\Big(\;\frac{d}{dt}\Big|_0(\,p_t,\,F_t\,)\Big)\;:=\;\frac{d}{dt}\Big|_0\;F_0^{-1}\circ(\,\mathbf{PT}_t^\nabla\,)^{-1}\circ F_t\;\;\in\;\;\mathrm{End}\,V$$

constructed from the parallel transport $\mathbf{PT}_t^\nabla:\,V_{p_0}M\longrightarrow V_{p_t}M$ with respect to $\nabla$ along the curve $t\longmapsto p_t$; the principal connection axiom of Lemma 2.4 is particularly easy to verify using this definition for $\omega$.

In consequence of this moving frames argument it does not make too much sense to ask, whether or not a vector bundle with connection is in the image of *some* association functor. The appropriate answer to this question for an association functor fixed in advance is definitely more interesting and was given in the master thesis of one of the authors. A closely related concept is the concept of geometric vector bundles defined in [SW]:

**Proposition 4.6 (Images of Association Functors)**
*Let $G$ be a simply connected Lie group and let $GM$ be a principal $G$–bundle over a simply connected manifold $M$ endowed with a principal connection $\omega$. A vector bundle $VM$ with a linear connection $\mathbb{P}^\nabla$ over $M$ is isomorphic in the vector bundle category $\mathbf{VB}_M^\nabla$ to a vector bundle in the image of the association functor $\mathrm{Ass}^\omega_{GM}$, if and only if there exists a parallel bilinear map*

$$\star_{\mathrm{inf}}\;:\;\;\mathfrak{aut}\,GM\;\times_M\;VM\;\longrightarrow\;VM,\qquad(\,X,\,v\,)\;\longmapsto\;X\star v\,,$$

*which is a representation of the Lie algebra $\mathfrak{aut}_p GM$ at every $p \in M$*

$$(\star_{\mathrm{inf}})_p : \ \mathfrak{aut}_p GM \times V_p M \longrightarrow V_p M$$

*with the additional property that the curvature of the given connection $\nabla$ agrees with the pointwise action of the curvature $R^\omega \in \Omega^2(M, \mathfrak{aut}\, GM)$:*

$$R^\nabla_{X,Y} v = R^\omega_{X,Y} \star_{\mathrm{inf}} v \, .$$

**Proof:** Consider to begin with the vector bundle $VM := GM \times_G V$ associated to a representation $V$ of the Lie group $G$. According to our discussion of the infinitesimal action following Definition 4.1 the composition

$$\star_{\mathrm{inf}} : \ \mathfrak{g} \times V \xrightarrow{\star_{\mathrm{inf}}} TV \xrightarrow{\cong} V \times V \xrightarrow{\mathrm{pr}_R} V$$

is $G$–equivariant and thus gives rise to a parallel $\mathbb{R}$–bilinear map, which is a representation $(\star_{\mathrm{inf}})_p$ of the Lie algebra $\mathfrak{aut}_p GM$ on $V_p M$ in every point:

$$\star_{\mathrm{inf}} : \ \mathfrak{aut}\, GM \times_M VM \longrightarrow VM \, .$$

Conversely assume that $\star_{\mathrm{inf}} : \mathfrak{aut}\, GM \times_M VM \longrightarrow VM$ is a parallel representation of the Lie algebra bundle $\mathfrak{aut}\, GM$ on a vector bundle $VM$ with a linear connection $\mathbb{P}^\nabla$. According to equation (11) the fiber Lie group $\mathrm{Aut}_p GM$ is isomorphic to $G$ in every point $p \in M$ and so simply connected, in consequence the infinitesimal action $(\star_{\mathrm{inf}})_p$ of its Lie algebra $\mathfrak{aut}_p GM$ integrates to a representation of the Lie group $\mathrm{Aut}_p GM$ on the vector space $V_p M$. Though slightly technical it is straightforward to prove that the integrated representation depends smoothly on the point $p \in M$

$$\star : \ \mathrm{Aut}\, GM \times_M VM \longrightarrow VM \, , \tag{16}$$

the details of this argument are left to the reader. In addition to the vector bundle $VM$ with its connection $\mathbb{P}^\nabla$ we consider the vector bundle $GM \times_G V$ associated to some representation $V$ of $G$ endowed with the linear connection $\mathbb{P}^\omega$ induced by the principal connection $\omega$ in Proposition 4.4. The two connections determine a linear connection $\mathbb{P}^{(\omega,\nabla)}$ on the vector bundle $\mathrm{Hom}(GM \times_G V, VM)$ characterized by the fact that its parallel transport

$$\mathbf{PT}^{(\omega,\nabla)}_t : \ \mathrm{Hom}(G_{p_0} M \times_G V, V_{p_0} M) \longrightarrow \mathrm{Hom}(G_{p_t} M \times_G V, V_{p_t} M)$$

along an arbitrary curve $t \longmapsto p_t$ makes the following diagram commute

$$\begin{tikzcd} G_{p_0}M \times_G V \arrow[rr, "F"] \arrow[d, "\mathbf{PT}^\omega_t"'] & & V_{p_0}M \arrow[d, "\mathbf{PT}^\nabla_t"] \\ G_{p_t}M \times_G V \arrow[rr, "\mathbf{PT}^{(\omega,\nabla)}_t F"] & & V_{p_t}M \end{tikzcd} \tag{17}$$

for all linear maps $F : G_{p_0}M \times_G V \longrightarrow V_{p_0}M$, where $\mathbf{PT}^\omega_t$ and $\mathbf{PT}^\nabla_t$ are the parallel transports along the same curve with respect to $\mathbb{P}^\omega$ and $\mathbb{P}^\nabla$.

The principal idea of the proof is now to construct a parallel and actually flat vector subbundle of the vector bundle $\mathrm{Hom}(\, GM \times_G V,\, VM\,)$ over $M$. For this purpose we consider the family of vector subspaces of the fibers

$$\Big[\,\mathrm{Hom}_{\mathrm{Aut}\,GM}(\, GM \times_G V,\, VM\,)\,\Big]_p$$
$$:= \;\{\; F : G_pM \times_G V \longrightarrow V_pM \;\mid\; \text{linear and } \mathrm{Aut}_p GM \text{ equivariant } \;\}$$

of the vector bundle $\mathrm{Hom}(\, GM \times_G V,\, VM\,)$ in each point $p \in M$. In order to show that this family of subspaces is the family of fibers of a vector subbundle of $\mathrm{Hom}(\, GM \times_G V,\, VM\,)$ we observe that the parallel transport

$$\mathbf{PT}^\omega_t : \; G_{p_0}M \times_G V \stackrel{\cong}{\longrightarrow} G_{p_t}M \times_G V \qquad \mathbf{PT}^\nabla_t : \; V_{p_0}M \stackrel{\cong}{\longrightarrow} V_{p_t}M$$

in both vector bundles $GM \times_G V$ and $VM$ along a curve $t \longmapsto p_t$ is equivariant over the parallel transport with respect to the Lie group connection $\mathbb{P}^\omega$ on the automorphism bundle $\mathrm{Aut}\, GM$ induced by $\omega$. More precisely we find

$$\mathbf{PT}^\nabla_t\big(\,(\,p_0,\,\psi\,) \star v\,\big) \;=\; \mathbf{PT}^\omega_t(\,p_0,\,\psi\,) \star \mathbf{PT}^\nabla_t v$$

for the vector bundle $VM$, because $\star_{\mathrm{inf}} : \mathfrak{aut}\, GM \times_M VM \longrightarrow VM$ is parallel by assumption. In consequence the parallel transport $\mathbf{PT}^{(\omega,\nabla)}$ with respect to the linear connection $\mathbb{P}^{(\omega,\nabla)}$ specified in diagram (17) induces for all $t \in \mathbb{R}$ vector space isomorphisms $F \longmapsto \mathbf{PT}^\nabla_t \circ F \circ (\,\mathbf{PT}^\omega_t\,)^{-1}$ between:

$$\Big[\,\mathrm{Hom}_{\mathrm{Aut}\,GM}(\, GM \times_G V, VM\,)\,\Big]_{p_0} \stackrel{\cong}{\longrightarrow} \Big[\,\mathrm{Hom}_{\mathrm{Aut}\,GM}(\, GM \times_G V, VM\,)\,\Big]_{p_t}.$$

By assumption the underlying manifold $M$ is (simply) connected, and hence all vector subspaces $[\,\mathrm{Hom}_{\mathrm{Aut}\,GM}(\, GM \times_G V,\, VM\,)\,]_p$ have the same dimension. With parallel transport depending smoothly on the curve we conclude that $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\, GM \times_G V,\, VM\,)$ is a genuine vector subbundle of

$\mathrm{Hom}(\,GM \times_G V,\, VM\,)$, moreover it is a parallel subbundle as it is invariant under parallel transport along arbitrary curves.

On the other hand the curvature of the linear connection $\mathbb{P}^{(\omega,\nabla)}$ on the vector bundle $\mathrm{Hom}(\,GM \times_G V,\, VM\,)$ is determined by the universality of principal curvature discussed in Proposition 4.4, namely it holds true that

$$R^{(\omega,\nabla)}_{X,\,Y} F \;=\; R^{\nabla}_{X,\,Y} \circ F \;-\; F \circ (\,R^{\omega}_{X,\,Y} \star_{\mathrm{inf}}\,)$$

for all tangent vectors $X,\, Y \;\in\; T_pM$ and $F \;\in\; \mathrm{Hom}_p(\,GM \times_G V,\, VM\,)$. Due to equivariance the curvature of the connection $\mathbb{P}^{(\omega,\nabla)}$ restricted to the parallel vector subbbundle $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ vanishes identically, put differently $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ is a flat vector bundle over $M$ under the restriction of the connection $\mathbb{P}^{(\omega,\nabla)}$.

In the argument presented so far the actual choice of the representation $V$ did not play any role. In order to make a diligent choice we fix a frame $g \;\in\; G_pM$ over a point $p \;\in\; M$ and consider the Lie group isomorphism

$$\Phi:\;\; G \;\stackrel{\cong}{\longrightarrow}\; \mathrm{Aut}_pGM, \qquad \gamma \;\longmapsto\; \big(p,\, \hat{g} \;\longmapsto\; g\gamma(g^{-1}\hat{g})\big)\;,$$

which is essentially the Lie group bundle isomorphism (11) restricted to the fiber of $p$. This Lie group isomorphism allows us to pull back the integrated representation (16) of $\mathrm{Aut}_pGM$ on the vector space $V \;:=\; V_pM$ to a smooth representation $\star:\; G \times V \longrightarrow V$ enjoying the critical property that

$$\overline{\Phi}:\;\; G_pM \times_G V \;\stackrel{\cong}{\longrightarrow}\; V \;\stackrel{=}{\longrightarrow}\; V_pM, \qquad [\,\hat{g},\, v\,] \;\longmapsto\; \Phi(\,g^{-1}\hat{g}\,) \star v$$

is an equivariant vector space isomorphism under $\mathrm{Aut}_pGM$ in the sense:

$$\begin{aligned}\overline{\Phi}\big(\,\Phi(\,\gamma\,)\,[\,\hat{g},\, v\,]\,\big) \;&=\; \overline{\Phi}\big(\,[\,g\gamma(g^{-1}\hat{g}),\, v\,]\,\big)\\ &=\; \Phi\big(g^{-1}g\gamma(g^{-1}\hat{g})\big) \,\star\, v \;\;=\;\; \Phi(\,\gamma\,)\,\overline{\Phi}\big(\,[\,\hat{g},\, v\,]\,\big)\;.\end{aligned}$$

In consequence the fiber of the vector bundle $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ over the chosen point $p \;\in\; M$ contains the vector space isomorphism $\overline{\Phi}$, which translates under parallel transport along arbitrary curves with respect to the flat connection $\mathbb{P}^{(\omega,\nabla)}$ into a parallel, globally defined section $\overline{\Phi}$ on the simply connected manifold $M$. Evaluation of this parallel section in the points of $M$ converts it into a parallel isomorphism of vector bundles:

$$\overline{\Phi}:\;\; GM \times_G V \;\stackrel{\cong}{\longrightarrow}\; VM, \qquad [\,\hat{g},\, v\,] \;\longmapsto\; \overline{\Phi}_{\pi(\hat{g})}[\,\hat{g},\, v\,]\,. \qquad \blacksquare$$

# 5 The Category of Gauge Theory Sectors

Every association functor is in a sense a reproducing functor, there exists in its source category an object, whose image in its target category is isomorphic to the principal bundle defining the association functor in the first place. Based on this simple observation we characterize the association functors among all functors from $\mathbf{MF}_G$ to $\mathbf{FB}_M^\nabla$ in this section, moreover we establish an equivalence of categories between the category of principal bundles and a suitably defined category of functors called gauge theory sectors.

Consider the smooth action of a given Lie group $G$ on its underlying manifold by left multiplication $\star : G \times G \longrightarrow G$, $(\gamma, g) \longmapsto \gamma g$, which defines an object $G^{\mathrm{left}} \in \mathrm{OBJ}\,\mathbf{MF}_G$ in the category of $G$–manifolds. The image of $G^{\mathrm{left}}$ under the functor $\mathrm{Ass}^\omega_{GM}$ is isomorphic as a fiber bundle to $GM$

$$\mathrm{Ass}^\omega_{GM}(\,G^{\mathrm{left}}\,) \;\stackrel{\cong}{\longrightarrow}\; GM, \qquad [\,g,\,\gamma\,] \;\longmapsto\; g\,\gamma\,, \tag{18}$$

and the inverse isomorphism $g \longmapsto [g, e]$ is easily verified to be parallel with

$$\mathbb{P}^\nabla\Big(\;\frac{d}{dt}\Big|_0\,[\,g_t,\,e\,]\,\Big) \;=\; \Big[\;g_0,\;\frac{d}{dt}\Big|_0\,e\;+\;\omega\Big(\;\frac{d}{dt}\Big|_0\,g_t\,\Big)\,\star_{\mathrm{inf}}\,e\,\Big] \;=\; 0$$

whenever $\frac{d}{dt}\big|_0\,g_t$ is horizontal in the sense $\omega(\,\frac{d}{dt}\big|_0\,g_t) \;=\; 0$. This reproducing property of $\mathrm{Ass}^\omega_{GM}$ lies at the heart of the proof of the following theorem:

**Theorem 5.1 (Characterization of Association Functors)**
*Consider a covariant functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ from the category of $G$–manifolds to the category of fiber bundles with connection over $M$. If the functor $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor*

$$M\times :\;\; \mathbf{MF} \;\longrightarrow\; \mathbf{FB}_M^\nabla, \qquad \mathscr{F} \;\longmapsto\; M\,\times\,\mathscr{F}\,,$$

*on the full subcategory $\mathbf{MF} \;\subset\; \mathbf{MF}_G$ of manifolds with trivial $G$–action, then $\mathfrak{F}$ is naturally isomorphic to the association functor corresponding to some principal $G$–bundle $GM$ endowed with a principal connection $\omega$.*

**Proof:** Consider a functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ from the category of $G$–manifolds to the category of fiber bundles over $M$ endowed with non–linear connections, which preserves Cartesian products and agrees with the product functor $M\times\ :\ \mathbf{MF}\longrightarrow\mathbf{FB}_M^\nabla$ on the full subcategory of trivial $G$–manifolds. At least three different objects in the domain category $\mathbf{MF}_G$ of the functor $\mathfrak{F}$ have underlying manifold equal to the Lie group $G$:

$$G^{\mathrm{left}}\qquad\qquad G^{\mathrm{ad}}\qquad\qquad G^{\mathrm{triv}}\ .$$

The difference between these three objects in $\mathbf{MF}_G$ resides in their actions, which is by left multiplication $\gamma\star g\ :=\ \gamma g$ and conjugation $\gamma\star g\ :=\ \gamma g\gamma^{-1}$ respectively for $G^{\mathrm{left}}$ and $G^{\mathrm{ad}}$, whereas $G$ acts trivially on $G^{\mathrm{triv}}$. Every terminal object in the category $\mathbf{MF}_G$ is a zero–dimensional manifold point $\{*\}$ with necessarily trivial $G$–action, hence $\mathfrak{F}$ maps it to the terminal object $M\times\{*\}$ in the category $\mathbf{FB}_M^\nabla$. In other words the functor $\mathfrak{F}$ maps terminal objects to terminal objects and preserves Cartesian products and in consequence turns group like and principal objects in the category $\mathbf{MF}_G$ into group like and principal objects in the category $\mathbf{FB}_M^\nabla$.

With $G$ acting by automorphisms on both $G^{\mathrm{ad}}$ and $G^{\mathrm{triv}}$ both objects are group like objects in the category $\mathbf{MF}_G$ under the multiplication and inverse inherited from $G$. The significance of the group like object $\mathfrak{F}(\,G^{\mathrm{ad}}\,)$ in the category $\mathbf{FB}_M^\nabla$ may be somewhat obscure at this point, the group like object $\mathfrak{F}(\,G^{\mathrm{triv}}\,)\ =\ M\times G$ however is just the trivial $G$–bundle over $M$ endowed with the trivial connection. Moreover the original Lie group multiplication defines $G$–equivariant structure maps in analogy to definition (6)

$$\rho:\ \ G^{\mathrm{left}}\ \times\ G^{\mathrm{triv}}\ \longrightarrow\ G^{\mathrm{left}}\qquad\qquad\qquad \backslash:\ \ G^{\mathrm{left}}\ \times\ G^{\mathrm{left}}\ \longrightarrow\ G^{\mathrm{triv}}$$

by means of $\rho(g,\hat{g})\ :=\ g\hat{g}$ and $\backslash(g,\hat{g})\ :=\ g^{-1}\hat{g}$, which naturally enough turn $G^{\mathrm{left}}$ into a $G^{\mathrm{triv}}$–principal object in the category $\mathbf{MF}_G$. According to Lemma 3.1 the image of $G^{\mathrm{left}}$ is a principal $G$–bundle $GM\ :=\ \mathfrak{F}(\,G^{\mathrm{left}}\,)$ over the manifold $M$ endowed with a principal connection $\omega$. In passing we observe that the group like object $G^{\mathrm{ad}}$ acts $G$–equivariantly on $G^{\mathrm{left}}$ via

$$\star:\ \ G^{\mathrm{ad}}\ \times\ G^{\mathrm{left}}\ \longrightarrow\ G^{\mathrm{left}},\qquad (\,\gamma,\,g\,)\ \longmapsto\ \gamma\,g\ ,$$

and this action identifies the group like object $\mathfrak{F}(\,G^{\mathrm{ad}}\,)$ in the category $\mathbf{FB}_M^\nabla$ with the gauge group bundle $\mathrm{Aut}\,GM$ of $GM$ by means of the action:

$$\mathfrak{F}(\star):\ \ \mathfrak{F}(\,G^{\mathrm{ad}}\,)\ \times_M\ GM\ \longrightarrow\ GM\ .$$

It remains to show that the original functor $\mathfrak{F}$ is naturally isomorphic to the association functor $\mathrm{Ass}^{\omega}_{GM}$. For this purpose we consider a general object $\mathscr{F} \in$ OBJ $\mathbf{MF}_G$; replacing its $G$–action by the trivial $G$–action on the same underlying manifold we project it to an object $\mathscr{F}^{\mathrm{triv}} \in$ OBJ $\mathbf{MF}$ in the subcategory of manifolds with trivial $G$–action. The $G$–equivariant map

$$\Psi : \; G^{\mathrm{left}} \times \mathscr{F} \xrightarrow{\;\cong\;} G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}}, \qquad (\, g,\, f\,) \longmapsto (\, g,\, g^{-1} \star f\,)$$

is actually an isomorphism in $\mathbf{MF}_G$ with inverse $(\, g,\, f\,) \longmapsto (\, g,\, g \star f\,)$, which fits for an arbitrary element $\gamma \in G$ into the commutative diagram

$$\begin{tikzcd}
G^{\mathrm{left}} \times \mathscr{F} \arrow[rr, "\Psi"] \arrow[dd, "\rho_\gamma \times \mathrm{id}"'] \arrow[dr, "\mathrm{pr}_R"'] & & G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}} \arrow[dd, "\rho_\gamma \times (\gamma^{-1}\star)"] \arrow[dl, "\star"] \\
& \mathscr{F} & \\
G^{\mathrm{left}} \times \mathscr{F} \arrow[rr, "\Psi"] \arrow[ur, "\mathrm{pr}_R"] & & G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}} \arrow[ul, "\star"']
\end{tikzcd} \tag{19}$$

in the category $\mathbf{MF}_G$, where $\rho_\gamma : G^{\mathrm{left}} \longrightarrow G^{\mathrm{left}}$, $g \longmapsto g\gamma$, denotes the right multiplication by $\gamma$ and $\star$ the original $G$–action characterizing the object $\mathscr{F}$ thought of as a $G$–equivariant (sic!) map $\star : G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}} \longrightarrow \mathscr{F}$. Writing the right multiplication $\rho_\gamma$ in the category $\mathbf{MF}_G$ as a composition

$$G^{\mathrm{left}} \xrightarrow{\mathrm{id}\times\mathrm{term}} G^{\mathrm{left}} \times \{*\} \xrightarrow{\mathrm{id}\times\gamma} G^{\mathrm{left}} \times G^{\mathrm{triv}} \xrightarrow{\;\rho\;} G^{\mathrm{left}}$$

factorizing over the element morphism $\gamma : \{*\} \longrightarrow G^{\mathrm{triv}}$ in the subcategory $\mathbf{MF} \subset \mathbf{MF}_G$ we conclude that $\mathfrak{F}(\, \rho_\gamma\,) : GM \longrightarrow GM$ agrees with the right multiplication $R_\gamma : GM \longrightarrow GM$, $g \longmapsto g\gamma$, in the principal bundle $GM$ induced by $\mathfrak{F}(\, \rho\,) : GM \times G \longrightarrow GM$, because $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor $M\times$ on the trivial $G$–manifolds $\{*\}$ and $G^{\mathrm{triv}}$. In consequence the commutative diagram (19) translates under the functor $\mathfrak{F}$ into the following commutative diagram

$$\begin{tikzcd}
GM \times_M \mathscr{F}M \arrow[rr, "\Psi"] \arrow[dd, "R_\gamma \times \mathrm{id}"'] \arrow[dr, "\mathrm{pr}_R"'] & & GM \times \mathscr{F} \arrow[dd, "R_\gamma \times (\gamma^{-1}\star)"] \arrow[dl, "\mathfrak{F}(\star)"] \\
& \mathscr{F}M & \\
GM \times_M \mathscr{F}M \arrow[rr, "\Psi"] \arrow[ur, "\mathrm{pr}_R"] & & GM \times \mathscr{F} \arrow[ul, "\mathfrak{F}(\star)"']
\end{tikzcd} \tag{20}$$

in the category $\mathbf{FB}_M^\nabla$ with $\mathscr{F}M := \mathfrak{F}(\mathscr{F})$, because $\mathfrak{F}$ preserves Cartesian products, hence preserves projections and agrees on manifolds with trivial $G$–action like $\mathscr{F}^{\mathrm{triv}}$ with the product functor $M\times$. The parallel homomorphism $\mathfrak{F}(\star) : GM \times \mathscr{F} \longrightarrow \mathscr{F}M$ thus descends to the quotient

$$\overline{\mathfrak{F}(\star)} : \quad GM \times_G \mathscr{F} \longrightarrow \mathscr{F}M$$

of $GM \times \mathscr{F}$ by the right $G$–action defining the associated fiber bundle $GM \times_G \mathscr{F}$, which lets $\gamma \in G$ act by $R_\gamma \times (\gamma^{-1}\star)$. It goes without saying that the projection $\mathrm{pr}_R : GM \times_M \mathscr{F}M \longrightarrow \mathscr{F}M$ factors through the quotient of $GM \times_M \mathscr{F}M$ by the right $G$–action on the principal bundle $GM$, the commutative diagram (20) ensures moreover that the quotient diagram

$$\begin{tikzcd}
(GM/_G) \times_M \mathscr{F}M \arrow[rr, "\overline{\Psi}"] \arrow[dr, "\mathrm{pr}_R"'] & & GM \times_G \mathscr{F} \arrow[dl, "\overline{\mathfrak{F}(\star)}"] \\
& \mathscr{F}M &
\end{tikzcd}$$

still commutes. With $\mathrm{pr}_R : M \times_M \mathscr{F}M \xrightarrow{\cong} \mathscr{F}M$ and $\overline{\Psi}$ being parallel diffeomorphisms of fiber bundles with connections over $M$ we conclude that

$$\overline{\mathfrak{F}(\star)} : \quad GM \times_G \mathscr{F} \xrightarrow{\cong} \mathscr{F}M$$

is actually an isomorphism in the category $\mathbf{FB}_M^\nabla$, moreover the construction of this parallel fiber bundle isomorphism $\overline{\mathfrak{F}(\star)} : \mathrm{Ass}^\omega_{GM}\mathscr{F} \longrightarrow \mathfrak{F}(\mathscr{F})$ for a given object $\mathscr{F} \in \mathrm{OBJ}\ \mathbf{MF}_G$ is natural under morphisms in $\mathbf{MF}_G$ and comprises a natural isomorphism $\overline{\mathfrak{F}(\,\cdot\,)} : \mathrm{Ass}^\omega_{GM} \longrightarrow \mathfrak{F}$ of functors. ■

In order to press the point of Theorem 5.1 home let us define two rather special categories associated to a smooth manifold $M$. Objects in the category $\mathbf{PB}_M^\nabla$ of principal bundles with connections over $M$ are triples $(G, GM, \omega)$ formed by a Lie group $G$ and a principal $G$–bundle $GM$ over $M$ endowed with a principal connection $\omega$. Every morphism between two such objects

$$(\varphi_{\mathrm{grp}}, \varphi) : \quad (G, GM, \omega) \longrightarrow (\hat{G}, \hat{G}M, \hat{\omega})$$

consists of a parallel homomorphism $\varphi : GM \longrightarrow \hat{G}M$ of fiber bundles which is $G$–equivariant over the Lie group homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$.

Objects in the category $\mathbf{GTS}_M^\nabla$ of gauge theory sectors on $M$ with connections are on the other hand tuples $(\,G,\,\mathfrak{F}\,)$ formed by a Lie group $G$ and a covariant functor $\mathfrak{F}:\ \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ which preserves Cartesian products and agrees with the product functor on the full subcategory $\mathbf{MF}\ \subset\ \mathbf{MF}_G$ of manifolds with trivial $G$–action. In $\mathbf{GTS}_M^\nabla$ morphisms are again tuples

$$(\,\varphi_{\mathrm{grp}},\,\Phi\,):\ \ (\,G,\,\mathfrak{F}\,)\ \longrightarrow\ (\,\hat{G},\,\hat{\mathfrak{F}}\,)$$

consisting of a group homomorphism $\varphi_{\mathrm{grp}}:\ G \longrightarrow \hat{G}$ between the two Lie groups and a natural transformation $\Phi\ :\ \mathfrak{F}\,\circ\,\varphi_{\mathrm{grp}}^* \longrightarrow \hat{\mathfrak{F}}$ between the two functors $\mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{FB}_M^\nabla$ involved, where the action pull back functor

$$\varphi_{\mathrm{grp}}^*:\ \ \mathbf{MF}_{\hat{G}}\ \longrightarrow\ \mathbf{MF}_G, \qquad (\,\hat{\mathscr{F}},\,\star_{\hat{G}}\,)\ \longrightarrow\ (\,\hat{\mathscr{F}},\,\star_G\,)$$

induced by $\varphi_{\mathrm{grp}}$ lets $G$ act via $g\,\star_G\,f\ :=\ \varphi_{\mathrm{grp}}(g)\,\star_{\hat{G}}\,f$ on a $\hat{G}$–manifold $\hat{\mathscr{F}}$. We want to interpret the construction of the association functor as a functor

$$\mathrm{Ass}:\ \ \mathbf{PB}_M^\nabla\ \longrightarrow\ \mathbf{GTS}_M^\nabla$$

with $(\,G,\,GM,\,\omega\,) \longmapsto (\,G,\,\mathrm{Ass}_{GM}^\omega\,)$ on objects, hence we still have to specify $\mathrm{Ass}$ on morphisms: Every morphism in the source category $\mathbf{PB}_M^\nabla$ is a parallel fiber bundle homomorphism $\varphi\ :\ GM\ \longrightarrow\ \hat{G}M$ equivariant over $\varphi_{\mathrm{grp}}\ :\ G\ \longrightarrow\ \hat{G}$, in the the target category $\mathbf{GTS}_M^\nabla$ such a morphism becomes the natural transformation $\Phi_\varphi$ defined for $\hat{\mathscr{F}}\ \in\ \mathrm{O{\scriptsize BJ}}\,\mathbf{MF}_{\hat{G}}$ by:

$$\Phi_\varphi(\,\hat{\mathscr{F}}\,):\ \ GM\,\times_G\hat{\mathscr{F}}\ \longrightarrow\ \hat{G}M\,\times_{\hat{G}}\hat{\mathscr{F}}, \qquad [\,g,\,\hat{f}\,]\ \longmapsto\ [\,\varphi(g),\,\hat{f}\,]\,.$$

**Corollary 5.2 (Association Functor as Equivalence of Categories)**
*For every smooth manifold $M$ the association functor* $\mathrm{Ass}$ *provides an equivalence of categories from the category $\mathbf{PB}_M^\nabla$ of principal bundles to the category $\mathbf{GTS}_M^\nabla$ of gauge theory sectors over $M$ with connections:*

$$\mathrm{Ass}:\ \ \mathbf{PB}_M^\nabla\ \stackrel{\simeq}{\longrightarrow}\ \mathbf{GTS}_M^\nabla, \qquad (\,G,\,GM,\,\omega\,)\ \longmapsto\ (\,G,\,\mathrm{Ass}_{GM}^\omega\,)\,.$$

*In particular two principal $G$–bundles endowed with principal connections on $M$ are isomorphic via a parallel, $G$–equivariant homomorphism of fiber bundles, if and only if their association functors are naturally isomorphic.*

**Proof:** According to Theorem 5.1 every gauge theory sector with connection $(\,G,\,\mathfrak{F}\,)$ is isomorphic in the category $\mathbf{GTS}^\nabla_M$ to an association functor $\mathrm{Ass}^\omega_{GM}$ for a suitable principal $G$–bundle $GM$ with a principal connection $\omega$. In order to prove Corollary 5.2 we thus need to show that the association functor Ass induces for two arbitrary objects in $\mathbf{PB}^\nabla_M$ a bijection of sets:

$$\begin{aligned}\mathrm{Ass}:\quad & \mathrm{MOR}_{\mathbf{PB}^\nabla_M}\Big(\,(\,G,\,GM,\,\omega\,),\;(\,\hat{G},\,\hat{G}M,\,\hat{\omega}\,)\,\Big)\\ &\qquad\stackrel{\cong}{\longrightarrow}\;\mathrm{MOR}_{\mathbf{GTS}^\nabla_M}\Big(\,(\,G,\,\mathrm{Ass}^\omega_{GM}\,),\;(\,\hat{G},\,\mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}\,)\,\Big)\end{aligned}$$

Consider for this purpose a morphism $(\,\varphi_{\mathrm{grp}},\,\Phi\,)$ in the category $\mathbf{GTS}^\nabla_M$ from the image object $(\,G,\,\mathrm{Ass}^\omega_{GM}\,)$ to the image object $(\,\hat{G},\,\mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}\,)$. The natural transformation $\Phi$ applies to every object in $\mathbf{MF}_{\hat{G}}$, specifically for the object $\hat{G}^{\mathrm{left}}$ describing the action of $\hat{G}$ on itself by left multiplication the natural transformation $\Phi$ provides a parallel homomorphism of fiber bundles

$$\Phi(\,\hat{G}^{\mathrm{left}}\,):\;\; GM\times_G\hat{G}\;\longrightarrow\;\hat{G}M\times_{\hat{G}}\hat{G}\;,$$

which we may use to define $\varphi:\;GM\longrightarrow\hat{G}M$ as the composition:

$$\begin{array}{cccccccc}\varphi: & GM & \longrightarrow & GM\times_G\hat{G} & \stackrel{\Phi(\,\hat{G}^{\mathrm{left}}\,)}{\longrightarrow} & \hat{G}M\times_{\hat{G}}\hat{G} & \stackrel{\cong}{\longrightarrow} & \hat{G}M\\ & g & \longmapsto & [\,g,\,\hat{e}\,] & & [\,\hat{g},\,\hat{\gamma}\,] & \longmapsto & \hat{g}\,\hat{\gamma}\,.\end{array} \tag{21}$$

The argument we used in equation (18) to show that the right hand side isomorphism $\hat{G}M_{\hat{G}}\hat{G}\longrightarrow\hat{G}M$ is parallel implies that $GM\longrightarrow GM\times_G\hat{G}$ is parallel as well, in consequence $\varphi:\;GM\longrightarrow\hat{G}M$ is a parallel homomorphism of fiber bundles.

In order to show that $\varphi$ is equivariant over the group homomorphism $\varphi_{\mathrm{grp}}:\;G\longrightarrow\hat{G}$ we use the characteristic property of natural transformations like $\Phi$ for the right multiplication morphism $\rho_{\hat{\gamma}}:\;\hat{G}^{\mathrm{left}}\longrightarrow\hat{G}^{\mathrm{left}}$, $\hat{g}\longmapsto\hat{g}\hat{\gamma}$:

$$\begin{tikzcd} GM\times_G\hat{G} \arrow[r, "\Phi(\,\hat{G}^{\mathrm{left}}\,)"] \arrow[d, "(\mathrm{Ass}^\omega_{GM}\circ\varphi^*_{\mathrm{grp}})(\,\rho_{\hat{\gamma}}\,)"'] & \hat{G}M\times_{\hat{G}}\hat{G} \arrow[d, "\mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}(\,\rho_{\hat{\gamma}}\,)"] \\ GM\times_G\hat{G} \arrow[r, "\Phi(\,\hat{G}^{\mathrm{left}}\,)"] & \hat{G}M\times_{\hat{G}}\hat{G} \end{tikzcd}$$

Of course the association functors $\mathrm{Ass}^{\omega}_{GM} \circ \varphi^*_{\mathrm{grp}}$ and $\mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}$ are explicitly specified on morphisms in Definition 4.2 and both vertical arrows turn out to be the right multiplication $[\,g,\,\hat{\mathrm{r}}\,] \longmapsto [\,g,\,\hat{\mathrm{r}}\hat{\gamma}\,]$ by $\hat{\gamma} \in \hat{G}$. In turn we find

$$\varphi(g\gamma) \;=\; \Phi(\hat{G}^{\mathrm{left}})\,[\,g\gamma,\hat{e}\,] \;=\; \Phi(\hat{G}^{\mathrm{left}})\,[\,g,\varphi_{\mathrm{grp}}(\gamma)\,] \;=\; \varphi(g)\,\varphi_{\mathrm{grp}}(\gamma)$$

for all $g \in GM$ and $\gamma \in G$ and conclude that $\varphi$ is equivariant over $\varphi_{\mathrm{grp}}$. Eventually we consider for an arbitrary object $\hat{\mathscr{F}} \in \textsc{Obj}\,\mathbf{MF}_{\hat{G}}$ the orbit map $\mathrm{orb}_{\hat{f}} : \hat{G}^{\mathrm{left}} \longrightarrow \hat{\mathscr{F}},\ \hat{\gamma} \longmapsto \hat{\gamma} \star \hat{f}$, associated to an element $\hat{f} \in \hat{\mathscr{F}}$ as a morphism in the category $\mathbf{MF}_{\hat{G}}$ with associated commutative diagram:

$$\begin{array}{ccc} GM \times_G \hat{G} & \xrightarrow{\;\Phi(\,\hat{G}^{\mathrm{left}}\,)\;} & \hat{G}M \times_{\hat{G}} \hat{G} \\ {\scriptstyle (\mathrm{Ass}^{\omega}_{GM} \circ \varphi^*_{\mathrm{grp}})(\,\mathrm{orb}_{\hat{f}}\,)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}(\,\mathrm{orb}_{\hat{f}}\,)} \\ GM \times_G \hat{\mathscr{F}} & \xrightarrow{\;\Phi(\,\hat{\mathscr{F}}\,)\;} & \hat{G}M \times_{\hat{G}} \hat{\mathscr{F}}\,. \end{array}$$

Definition 4.2 provides again an explicit description of the two vertical arrows and the top arrow reads $[\,g,\,\hat{\gamma}\,] \longmapsto [\,\varphi(g),\,\hat{\gamma}\,]$, the commutativity of the diagram thus implies that $\Phi(\,\hat{\mathscr{F}}\,)$ is given by $[\,g,\,\hat{f}\,] \longmapsto [\,\varphi(g),\,\hat{f}\,]$. In other words the two natural transforms $\Phi$ and $\Phi_\varphi$ agree on arbitrary objects and so the functor $\mathrm{Ass}$ is full, this is surjective on morphisms. In order to show that $\mathrm{Ass}$ is injective on morphisms or faithful the reader may simply verify that the equivariant map $GM \longrightarrow \hat{G}M$ defined in equation (21) equals $\varphi$ in case we start with the natural transformation $\Phi = \Phi_\varphi$. ■

Mutatis mutandis the arguments presented in this section work without taking connections into account: A functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M$ is naturally isomorphic to the association functor $\mathrm{Ass}_{GM}$ for some principal bundle $GM$, if and only if $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor $M\times : \mathbf{MF} \longrightarrow \mathbf{FB}_M$ on the full subcategory of trivial $G$–manifolds. Suitably defined categories of principal bundles and gauge theory sectors then turn the association functor into an equivalence of categories:

$$\mathrm{Ass} : \;\mathbf{PB}_M \;\stackrel{\simeq}{\longrightarrow}\; \mathbf{GTS}_M, \qquad (\,G,\,GM\,) \;\longmapsto\; (\,G,\,\mathrm{Ass}_{GM}\,)\,.$$

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