FUNCTORIALITY OF PRINCIPAL BUNDLES AND CONNECTIONS

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Résumé. L'une des plus importantes contributions de la théorie de jauge en mathématiques est de souligner l'importance des foncteurs d'association. En mettant l'accent sur la théorie des catégories nous caractérisons ces derniers en utilisant deux de leurs propriétés naturelles. Cette caractérisation est ensuite utilisée pour établir une équivalence entre la catégorie des fibrés principaux et une certaine catégorie de foncteurs. Du point de vue de la géométrie differentielle nous décrivons la particularisation des connexions non–linéaires ou d'Ehresmann au cas principal ou linéaire. La propriété d'universalité des courbures principales, par ailleurs bien connue et largement utilisée, est alors employée afin de caractériser les fibrés vectoriels dans l'image d'un foncteur d'association donné.

Abstract. Perhaps the most important contribution of gauge theory to general mathematics is to point out the importance of association functors. Emphasizing category theory we characterize association functors by two of their natural properties and use this characterization to establish an equivalence between the category of principal bundles and a suitably defined category of functors. From the point of view of differential geometry we detail the specialization of non–linear or Ehresmann to principal and linear connections and discuss the widely known and very useful universality of principal curvature in order to characterize the vector bundles in the image of a given association functor.

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1 Introduction

Principal bundles and their association functors play a fundamental role in differential geometry and mathematical physics. Spin structures in pseudo–Riemannian geometry are defined right away as special principal bundles and the basic tenet of harmonic analysis is that the canonical association functor of a pointed homogeneous space is an equivalence of categories to the category of homogeneous fiber bundles. Last but not least the choice of principal bundle corresponds to the choice of vacuum sector in quantum field gauge theories. Nevertheless principal bundles tend to obfuscate calculations due to some inevitable arbitrariness, as one can see for example in Cartan geometries and in the botched proof of Blunder 5.24 in the otherwise excellent reference [LM]. Explicit calculations are more easily done using only the existence of association functors and the universality of curvature, arguably one of the most useful theorems in all of differential geometry.

En nuce this article brings these reservations against the use of principal bundles to a point: We show that a principal bundle GM over a manifold M is completely determined by its association functor Ass_{GM}. Conversely every functor $\mathfrak{F}$ from a suitable category of model fibers to the category of fiber bundles over M satisfying two more or less self–evident axioms agrees with the association functor for some principal bundle over M. Under natural transformations the class of all such functors $\mathfrak{F}$ becomes the category $\mathbf{GTS}_M$ of gauge theory sectors over M, which turns out to be equivalent to the category $\mathbf{PB}_M$ of principal bundles over the manifold M.

Category theory is usually not considered to be of particular importance to differential geometry, the text books [KMS] and [L] as well as the article [SM] are notable exceptions to this rule. Besides the characterization of associated vector bundles as geometric vector bundles in the sense of [SW] the common differential geometer may find little of interest in this article. Our main motivation for studying categorical properties of principal bundles nevertheless is the need to formulate the proper analogue of the concepts of principal bundles and connections in non–commutative geometry along the lines of [D1], [D2] and [D3]. Every definition of quantum bundles with quantum connections like the one presented in [Sa] will necessarily reflect functorial properties of principal bundles in classical differential geometry.

In order to provide a more detailed outline of this article we consider a Lie

group G and the category $\mathbf{MF}_G$ of manifolds $\mathscr{F}$ endowed with a smooth left action $\star : G \times \mathscr{F} \longrightarrow \mathscr{F}$ under smooth G–equivariant maps. Every principal G–bundle GM over a manifold M defines a functor from the category $\mathbf{MF}_G$ of model fibers to the category $\mathbf{FB}_M$ of fiber bundles over M

$$\mathrm{Ass}_{GM} : \quad \mathbf{MF}_G \; \longrightarrow \; \mathbf{FB}_M, \qquad \mathscr{F} \; \longmapsto \; GM \times_G \mathscr{F} \, ,$$

which we may promote to a functor $\mathrm{Ass}_{GM}^{\omega} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^{\nabla}$ to the category of fiber bundles with connections in the presence of a principal connection ω on GM. This association functor maps Cartesian products in $\mathbf{MF}_G$ to Cartesian products in $\mathbf{FB}_M^{\nabla}$ and maps a manifold $\mathscr{F}$ endowed with the trivial G–action to the trivial fiber bundle $M \times \mathscr{F}$. Our first main theorem stipulates that these two properties already characterize association functors as the reader can appreciate in Theorem 5.1

Let us now consider the category $\mathbf{PB}_M^{\nabla}$ of principal bundles with connections over M: Objects are triples $(\, G,\, GM,\, \omega\,)$ formed by a Lie group G and a principal G–bundle GM over M endowed with a principal connection ω, while morphisms are tuples $(\, \varphi_{\mathrm{grp}},\, \varphi\,)$ consisting of a parallel map $\varphi : GM \longrightarrow \hat{G}M$ of the underlying principal bundles, which is equivariant over the homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$ of Lie groups. The canonical factorization of the model homomorphism φ_{grp} entails a factorization of φ

$$\varphi : \quad GM \; \xrightarrow{\;\mathrm{pr}\;} \; GM\big/_{\ker^{\circ} \varphi_{\mathrm{grp}}} \; \xrightarrow{\;\overline{\mathrm{pr}}\;} \; GM\big/_{\ker \; \varphi_{\mathrm{grp}}} \; \xrightarrow{\;\overline{\varphi}\;} \; \hat{G}M$$

into a parallel projection, a covering and a parallel injective immersion. In this sense every morphism in the category $\mathbf{PB}_M^{\nabla}$ of principal bundles with connections over M is a product of just three basic types: The removal of a connected isospin subgroup, a covering $\overline{\mathrm{pr}}$ of principal bundles, a generalized spin structure, and a holonomy reduction $\overline{\varphi}$.

In order to translate this description of generalized spin structures and holonomy reductions as basic type morphism between principal bundles into a truly functorial description we consider the category $\mathbf{GTS}_M^{\nabla}$ of gauge theory sectors with connections over M. Its objects are tuples $(\, G,\, \mathfrak{F}\,)$ of a Lie group G together with a functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^{\nabla}$ satisfying the assumptions of Theorem 5.1. A morphism $(\, \varphi_{\mathrm{grp}},\, \Phi\,)$ between two such gauge theory sectors is a natural transformation $\Phi : \mathfrak{F} \circ \varphi_{\mathrm{grp}}^{*} \longrightarrow \hat{\mathfrak{F}}$ between the

functors twisted by the pull back $\varphi_{\mathrm{grp}}^* : \mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{MF}_G$ of the action along the homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$ of Lie groups:

Corollary 5.2 (Association Functor as Equivalence of Categories)
For every smooth manifold M the association functor Ass provides us with an equivalence of categories from the category $\mathbf{PB}_M^\nabla$ of principal bundles to the category $\mathbf{GTS}_M^\nabla$ of gauge theory sectors with connections:

$$\mathrm{Ass} : \quad \mathbf{PB}_M^\nabla \; \xrightarrow{\simeq} \; \mathbf{GTS}_M^\nabla, \qquad (\, G, \, GM, \, \omega \,) \longmapsto (\, G, \, \mathrm{Ass}_{GM}^\omega \,) \, .$$

In particular two principal G–bundles endowed with principal connections on M are isomorphic via a parallel, G–equivariant homomorphism of fiber bundles, if and only if their association functors are naturally isomorphic.

A direct consequence of Corollary 5.2 is that association functors are not in general full functors, this is they are not surjective on morphisms, simply because the action pull back functor $\varphi_{\mathrm{grp}}^* : \mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{MF}_G$ is not a full functor unless the image of G in $\hat{G}$ is dense. In other words there will be more parallel smooth homomorphisms of associated fiber bundles than there are G–equivariant smooth maps between their model fibers unless the principal connection ω has dense holonomy group.

According to Corollary 5.2 a spin structure on an oriented pseudo–Riemannian manifold $(\, M, \, g \,)$ can be defined as a functor extending the association functor $\mathbf{MF}_{\mathbf{SO}(T)} \longrightarrow \mathbf{FB}_M^\nabla$ determined by the oriented orthonormal frame bundle of M to a functor $\mathbf{MF}_{\mathbf{Spin}(T)} \longrightarrow \mathbf{FB}_M^\nabla$ still satisfying the assumptions of Theorem 5.1, the corresponding spinor bundle $\$M$ is simply the image of the irreducible Clifford module under the extended functor. A fundamental problem in differential geometry related to spin structures is to characterize the vector and fiber bundles in the image of a given association functor. A partial answer to this problem is given in Proposition 4.6, which opens the way to an axiomatic characterization of spinor bundles and highlights the universality of principal curvatures.

This paper breaks down into five sections. Section 2 is a leisurely introduction to non–linear or Ehresmann connections on fiber bundles; we relate their curvature to the commutator of iterated covariant derivatives and discuss how non–linear connections specialize to principal and linear connections. In Section 3 we generalize objects of group type in categories with Cartesian

products to principal objects. Association functors are studied in Section 4, the universality of curvature is formulated in Proposition 4.4. Having proved Theorem 5.1 in Section 5 we define the category of gauge theory sectors and establish the equivalence of categories formulated in Corollary 5.2.

The research project described in this article was inspired by the first part of the article [N] and can be seen as a direct analogue of this work in the framework of differential instead of algebraic geometry, moreover we address the additional complications brought about by the presence of connections.

2　Fiber Bundles and Non–Linear Connections

Perhaps the single most important concept in differential geometry is the notion of connections or the closely related notion of covariant derivatives on a vector or more general on a fiber bundle over a fixed manifold M. In this section we will modify the standard category $\mathbf{FB}_M$ of fiber bundles over M to a category more useful for our study, the category $\mathbf{FB}_M^\nabla$ of fiber bundles with non–linear connections over M. Moreover we will discuss principal and linear connections in the framework of this category.

In general a fiber bundle over a manifold M with model fiber manifold $\mathscr{F}$ is a manifold $\mathscr{F}M$ endowed with a smooth projection map $\pi : \mathscr{F}M \longrightarrow M$, which is locally trivializable. The preimage of a point $p \in M$ under π is called the fiber of the bundle over p, it is a submanifold $\mathscr{F}_pM := \pi^{-1}(p) \subset \mathscr{F}M$ of the total space $\mathscr{F}M$ diffeomorphic to the model fiber $\mathscr{F}$. Fiber bundles over M are the objects in the category $\mathbf{FB}_M$, morphisms in this category are smooth maps $\varphi : \mathscr{F}M \longrightarrow \hat{\mathscr{F}}M$ between the total spaces which commute with the respective projections $\hat{\pi} \circ \varphi = \pi$ and thus map the fibers of $\mathscr{F}M$ to the fibers of $\hat{\mathscr{F}}M$. Terminal objects in the category $\mathbf{FB}_M$ correspond to diffeomorphisms $\pi : \hat{M} \longrightarrow M$ thought of as fiber bundles over M with single point fiber.

The Cartesian product of two fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ in $\mathbf{FB}_M$ is called the fibered product in differential geometry $\mathscr{F}M \times_M \hat{\mathscr{F}}M$ and it is defined as the equalizer of $\pi \circ \mathrm{pr}_L$ and $\hat{\pi} \circ \mathrm{pr}_R$ in the manifold product $\mathscr{F}M \times \hat{\mathscr{F}}M$.

In order to study connections in the context of category theory we prefer the following definition:

Definition 2.1 (Non–linear Connections on Fiber Bundles)
A non–linear connection on a fiber bundle $\mathscr{F}M$ over a manifold M is a field
$\mathbb{P}^\nabla \in \Gamma(\mathscr{F}M, \operatorname{End} T\mathscr{F}M)$ *of projections* $(\mathbb{P}^\nabla)^2 = \mathbb{P}^\nabla$ *on the tangent bundle $T\mathscr{F}M$ such that its image distribution equals the vertical foliation:*

$$\operatorname{im}\left(\mathbb{P}^\nabla_f: \; T_f\mathscr{F}M \longrightarrow T_f\mathscr{F}M\right) \; \overset{!}{=} \; \operatorname{Vert}_f\mathscr{F}M.$$

Every non–linear connection $\mathbb{P}^\nabla$ on a fiber bundle $\mathscr{F}M$ allows us to define the first order differential operator

$$D^\nabla: \; \Gamma(M, TM) \times \Gamma_{\mathrm{loc}}(M, \mathscr{F}M) \longrightarrow \Gamma_{\mathrm{loc}}(M, \operatorname{Vert}\mathscr{F}M) \quad (1)$$

such that

$$(D^\nabla_X f)_p \; := \; \left(T_pM \xrightarrow{f_{*,p}} T_{f(p)}\mathscr{F}M \xrightarrow{\mathbb{P}^\nabla_{f(p)}} \operatorname{Vert}_{f(p)}\mathscr{F}M\right) X_p,$$

which is the non–linear analogue of the classical definition of covariant derivatives on vector bundles. Somewhat annoyingly this covariant derivative $D^\nabla_X f$ contains the redundant information $f = \pi_{\mathscr{F}M} \circ D^\nabla_X f$, where $\pi_{\mathscr{F}M}$ denotes the vertical tangent bundle projection $\operatorname{Vert}\mathscr{F}M \longrightarrow \mathscr{F}M$, the simplicity of linear and principal connections stems from the fact that we can get rid of this redundancy altogether, the reduced covariant derivative $\nabla_X f$ captures only the partial derivatives of the section f.

The Nijenhuis or curvature tensor of a non–linear connection $\mathbb{P}^\nabla$ on a fiber bundle $\mathscr{F}M$ over a manifold M is the horizontal 2–form R^∇ on the total space $\mathscr{F}M$ of the fiber bundle with values in the vertical tangent bundle defined for two arbitrary vector fields X, Y on $\mathscr{F}M$ by:

$$R^\nabla(X, Y) \; = \; -\mathbb{P}^\nabla[(\operatorname{id} - \mathbb{P}^\nabla)X, (\operatorname{id} - \mathbb{P}^\nabla)Y]. \quad (2)$$

In particular the curvature R^∇ measures exactly the failure of the horizontal distribution $\ker\mathbb{P}^\nabla \subseteq T\mathscr{F}M$ associated to $\mathbb{P}^\nabla$ to be integrable. An interpretation of the curvature tensor along classical lines as a commutator of covariant derivatives is shown in [SaW]

Definition 2.2 (Parallel Homomorphisms between Fiber Bundles)
A parallel homomorphism between fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ over the

same manifold M endowed with connections $\mathbb{P}^{\nabla}$ and $\mathbb{P}^{\hat{\nabla}}$ respectively is a homomorphism $\varphi : \mathscr{F}M \longrightarrow \hat{\mathscr{F}}M$ of fiber bundles such that the following diagram commutes:

$$
\begin{array}{ccc}
T\,\mathscr{F}M & \xrightarrow{\;\varphi_*\;} & T\,\hat{\mathscr{F}}M \\
{\scriptstyle\mathbb{P}^{\nabla}}\big\downarrow & & \big\downarrow{\scriptstyle\mathbb{P}^{\hat{\nabla}}} \\
T\,\mathscr{F}M & \xrightarrow{\;\varphi_*\;} & T\,\hat{\mathscr{F}}M
\end{array}
$$

The constraint $\hat{\pi} \circ \varphi = \pi$ characterizing homomorphisms of fiber bundles in the category $\mathbf{FB}_M$ readily implies $\varphi_*(\operatorname{Vert}\mathscr{F}M) \subset \operatorname{Vert}\hat{\mathscr{F}}M$, hence the homomorphism φ of fiber bundles is parallel, if and only if φ_* maps the horizontal distribution of $\mathscr{F}M$ to the horizontal distribution of $\hat{\mathscr{F}}M$:

$$
\varphi \text{ parallel} \qquad \Longleftrightarrow \qquad \varphi_*(\ker \mathbb{P}^{\nabla}) \ \subset \ \ker \mathbb{P}^{\hat{\nabla}}.
$$

Modifying the category $\mathbf{FB}_M$ we define the category $\mathbf{FB}_M^{\nabla}$ of fiber bundles with connection over M, in this category morphisms are parallel homomorphisms of fiber bundles.

In the resulting category terminal objects are still diffeomorphisms considered as fiber bundles with single point fibers endowed with the zero connection $\mathbb{P}^{\nabla} = 0$. Besides terminal objects the category $\mathbf{FB}_M^{\nabla}$ has Cartesian products: The fibered product $\mathscr{F}M \times_M \hat{\mathscr{F}}M$ of two fiber bundles $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ over M with connections $\mathbb{P}^{\nabla}$ and $\mathbb{P}^{\hat{\nabla}}$ carries the product connection $(\mathbb{P}^{\nabla} \oplus \mathbb{P}^{\hat{\nabla}}) : T(\mathscr{F}M \times_M \hat{\mathscr{F}}M) \longrightarrow \operatorname{Vert}\mathscr{F}M \oplus \operatorname{Vert}\hat{\mathscr{F}}M$ defined by

$$
\frac{d}{dt}\Big|_0 (f_t, \hat{f}_t) \longmapsto \mathbb{P}^{\nabla}\Big(\frac{d}{dt}\Big|_0 f_t\Big) \oplus \mathbb{P}^{\hat{\nabla}}\Big(\frac{d}{dt}\Big|_0 \hat{f}_t\Big),
$$

where $t \longmapsto f_t$ and $t \longmapsto \hat{f}_t$ are smooth curves in $\mathscr{F}M$ and $\hat{\mathscr{F}}M$ subject to the fibered product constraint $\pi(f_t) = \hat{\pi}(\hat{f}_t)$ for all t. In light of all these definitions the Cartesian product with the base manifold M becomes a functor from the category $\mathbf{MF}$ of smooth manifolds to the category $\mathbf{FB}_M^{\nabla}$

$$
M\times : \ \mathbf{MF} \ \longrightarrow \ \mathbf{FB}_M^{\nabla}, \qquad \mathscr{F} \ \longmapsto \ M \times \mathscr{F}, \tag{3}
$$

because every trivial fiber bundle $M \times \mathscr{F}$ over M comes along with the trivial connection $\mathbb{P}^{\mathrm{triv}}$, namely the projection to the tangent bundle of $\mathscr{F}$:

$$
T(M \times \mathscr{F}) \ \xrightarrow{\cong} \ TM \times T\mathscr{F} \ \xrightarrow{\pi \times \mathrm{id}} \ M \times T\mathscr{F} \ \cong \ \operatorname{Vert}(M \times \mathscr{F}).
$$

Evidently the horizontal distribution $TM \times \mathscr{F} \subset T(M \times \mathscr{F})$ is an integrable foliation with leaves $M \times \{f\}$ for every trivial connection $\mathbb{P}^{\mathrm{triv}}$, in consequence $R^{\mathrm{triv}} = 0$ vanishes necessarily. The product functor $M \times$ defined in equation (3) will feature prominently in Sections 3 and 5.

Having discussed general non–linear connections on fiber bundles in some detail we now want to specialize to principal and linear connections in the second part of this section. Recall first of all that a principal bundle modeled on a Lie group G is a smooth fiber bundle GM with model fiber G endowed with a smooth right ρ, fiber preserving action of G on its total space GM. Also it is possible to define the affine product $\backslash : GM \times_M GM \longrightarrow G$.

The automorphism group bundle of a principal bundle GM over a manifold M is the Lie group bundle $\operatorname{Aut} GM$ over M defined by

$$\operatorname{Aut} GM \quad := \quad \{ \ (p, \psi) \ \mid \ \psi : G_p M \longrightarrow G_p M \text{ is } G\text{–equivariant} \ \} \quad (4)$$

with the bundle projection $\pi_{\operatorname{Aut} GM} : \operatorname{Aut} GM \longrightarrow M, (p, \psi) \longmapsto p$. In mathematical physics the Fréchet–Lie group $\Gamma(M, \operatorname{Aut} GM)$ of all global sections of the automorphism bundle is called the gauge group of GM.

The fiber of the Lie group bundle $\operatorname{Aut} GM$ over a point $p \in M$ is a Lie group $\operatorname{Aut}_p GM$ isomorphic, although not canonically so, to the original group G, in particular its Lie algebra $\mathfrak{aut}_p GM \cong \mathfrak{g}$ is isomorphic to the Lie algebra of G. All these Lie algebras assemble into a smooth Lie algebra bundle $\mathfrak{aut} GM$, whose global sections $\Gamma(M, \mathfrak{aut} GM)$ form the Fréchet–Lie algebra of the gauge group $\Gamma(M, \operatorname{Aut} GM)$ of the principal bundle GM.

Definition 2.3 (Principal Connections)
A principal connection on a principal G–bundle GM over a manifold M is a non–linear connection $\mathbb{P}^\nabla$ on the fiber bundle GM, which is invariant under the right action of G on GM in the sense that the right translations $R_\gamma : GM \longrightarrow GM, g \longmapsto g\gamma$, are parallel automorphisms for all $\gamma \in G$.

In difference to general fiber bundles the vertical tangent bundle of a principal bundle GM is trivializable by

$$\mathrm{vtriv} : \operatorname{Vert} GM \longrightarrow GM \times \mathfrak{g}, \ \left.\frac{d}{dt}\right|_0 g_t \longmapsto \left(g_0, \left.\frac{d}{dt}\right|_0 g_0^{-1} g_t\right).$$

This allows to establish the following well–known result

Lemma 2.4 (Principal Connection Axiom)

On every principal G–bundle GM the association $\mathbb{P}^\nabla \longleftrightarrow \omega$ characterized by $\omega := \mathrm{vtriv} \circ \mathbb{P}^\nabla$ induces a bijection between principal connections in the sense of Definition 2.3 and $\mathfrak{g}$–valued 1–forms ω on GM satisfying the axiom

$$\omega_{g_0\gamma_0}\left(\left.\frac{d}{dt}\right|_0 g_t\,\gamma_t \right) \;=\; \mathrm{Ad}_{\gamma_0^{-1}}\,\omega_{g_0}\left(\left.\frac{d}{dt}\right|_0 g_t \right) \;+\; \left.\frac{d}{dt}\right|_0 \gamma_0^{-1}\gamma_t$$

for all choices of smooth curves $t \longmapsto g_t$ in GM and curves $t \longmapsto \gamma_t$ in G.

Cartan's Second Structure Equation [B] is a convenient description of the image of the composition of the curvature tensor R^∇ with the vertical trivialization vtriv in terms of the exterior derivative of the connection form

$$\Omega \;:=\; \mathrm{vtriv} \circ R^\nabla \;\overset{!}{=}\; d\omega \;+\; \frac{1}{2}\left[\,\omega \wedge \omega\,\right], \tag{5}$$

where $\frac{1}{2}[\,\omega \wedge \omega\,](X,Y) := [\,\omega(X),\,\omega(Y)\,]$.

The strategy persued for linear connections on vector bundles VM follows the model of principal connections closely. The tangent bundle of a vector space is canonically trivializable $TV \cong V \times V$ by taking actual derivatives and this becomes via $[\,\mathrm{Vert}\,VM\,]_p = T(\,V_pM\,)$ the vertical trivialization

$$\mathrm{vtriv} : \mathrm{Vert}\,VM \xrightarrow{\;\cong\;} VM \oplus VM, \quad \left.\frac{d}{dt}\right|_0 v_t \longmapsto v_0 \oplus \lim_{t\to 0}\frac{1}{t}(v_t - v_0).$$

This map can be used to project out $\nabla_X v := \mathrm{vtriv}(D_X^\nabla v)$ the redundant information from the covariant derivative $D_X^\nabla v$ of a section $v \in \Gamma(\,M, VM\,)$:

Definition 2.5 (Linear Connections on Vector Bundles)

A linear connection on a vector bundle VM on M is a non–linear connection $\mathbb{P}^\nabla$ on VM such that the reduced covariant derivative is $\mathbb{R}$–bilinear:

$$\nabla : \; \Gamma(\,M, TM\,) \;\times\; \Gamma(\,M, VM\,) \;\longrightarrow\; \Gamma(\,M, VM\,)\,.$$

In [SaW] it is showed a proof of the following lemma.

Lemma 2.6 (Characterization of Linear Connections)

A non–linear connection $\mathbb{P}^\nabla$ on a vector bundle gives rise to an $\mathbb{R}$–bilinear covariant derivative $\nabla : \Gamma(\,M, TM\,) \times \Gamma(\,M, VM\,) \longrightarrow \Gamma(\,M, VM\,)$, if and only if the multiplication by every $\lambda \in \mathbb{R}$ is a parallel endomorphism:

$$\Lambda_\lambda : \; VM \;\longrightarrow\; VM, \qquad v \;\longmapsto\; \lambda\,v\,.$$

3   Principal Objects in Categories

In every category $\mathscr{C}$ with terminal objects and Cartesian products the notion of a group so fundamental to all of mathematics can be generalized to the notion of a group like object in $\mathscr{C}$. In this section we take this beautiful idea to characterize homogeneous spaces with trivial stabilizers, generally known as principal homogeneous or affine group spaces, in terms of their structure morphisms. Moreover we apply this characterization of affine group spaces to the category $\mathbf{FB}_M^\nabla$ of fiber bundles with connections over a manifold M in order to characterize principal bundles with principal connections.

A group like object in a category $\mathscr{C}$ with terminal objects and Cartesian products is an object $G \in \mathrm{OBJ}\,\mathscr{C}$ together a choice of structure morphisms

$$ m:\ G \times G \longrightarrow G \qquad \iota:\ G \longrightarrow G \qquad \epsilon:\ * \longrightarrow G $$

in $\mathscr{C}$ called the multiplication, the inverse and the neutral element respectively with an arbitrary fixed terminal object $*$ such that the three diagrams

$$
\begin{array}{ccc}
G\times G\times G \xrightarrow{\ \mathrm{id}\times m\ } G\times G & \quad & G \xrightarrow{\ \mathrm{id}\times e\ } G\times G \\
\downarrow{\scriptstyle m\times\mathrm{id}} \qquad \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle e\times\mathrm{id}}\ \searrow^{\mathrm{id}}\ \downarrow{\scriptstyle m} \\
G\times G \xrightarrow{\quad m \quad} G & & G\times G \xrightarrow{\quad m \quad} G
\end{array}
\qquad
\begin{array}{c}
G \xrightarrow{\ \mathrm{id}\times\iota\ } G\times G \\
\downarrow{\scriptstyle \iota\times\mathrm{id}}\ \searrow^{e}\ \downarrow{\scriptstyle m} \\
G\times G \xrightarrow{\ m\ } G
\end{array}
$$

all commute, where $e = \epsilon \circ \mathrm{term}$ equals the composition of ϵ with the terminal morphism $\mathrm{term}:\ G \longrightarrow *$. In the category $\mathbf{Set}$ of sets for example the terminal objects are sets with exactly one element, hence $\epsilon:\ * \longrightarrow G$ essentially corresponds to an element of G. In turn the commutative diagrams above convert respectively into the associativity, the existence of a neutral element and the existence of inverses axiom in the definition of a group. In other words group like objects in $\mathbf{Set}$ are just plain groups.

In categories more complicated than $\mathbf{Set}$ the classification of group like objects can be simplified by the use of functors: Every covariant functor $\mathfrak{F}:\ \mathscr{C} \longrightarrow \hat{\mathscr{C}}$, which maps terminal objects to terminal objects and preserves Cartesian products, maps group like objects in the category $\mathscr{C}$ to group like objects in $\hat{\mathscr{C}}$. The standard forgetful functor $\mathbf{MF} \longrightarrow \mathbf{Set}$ from manifolds to sets for examples maps a group like object in $\mathbf{MF}$ to a group, albeit a Lie group whose multiplication and inverse are smooths maps.

In the same vein group like objects G in the category **Grp** of groups carry two different group structures, one for being an object in **Grp** and the other due to the forgetful functor **Grp** $\longrightarrow$ **Set**. It is a rather insightful exercise to verify that these two group structures actually agree so that G is necessarily abelian, because its multiplication $m : G \times G \longrightarrow G$ is a morphism in **Grp**. In consequence the fundamental group $\pi_1(G, e)$ of a topological group G is always abelian, because the functor π_1 maps terminal objects to terminal objects and preserves Cartesian products.

With these examples of the usefulness of functors in combination with a categorical definition of groups in mind we want to describe the concept of an affine group or principal homogeneous space in terms of category theory. Given a group like object G in a category $\mathscr{C}$ we define a (right) principal G–object to be an object $X \in \mathrm{OBJ}\,\mathscr{C}$ endowed with two structure morphisms

$$\rho :\ X \times G \longrightarrow X \qquad\qquad \backslash :\ X \times X \longrightarrow G \qquad (6)$$

in $\mathscr{C}$ called action and left division respectively such that the action diagrams

$$\begin{array}{ccc}
X \xrightarrow{\ \mathrm{id}\times e\ } X\times G & \qquad & X\times G\times G \xrightarrow{\ \mathrm{pr}_L\times m\ } X\times G \\
{\scriptstyle \mathrm{id}}\searrow\quad\swarrow{\scriptstyle \rho} & \qquad & {\scriptstyle \rho\times\mathrm{pr}_R}\downarrow\qquad\qquad\downarrow{\scriptstyle \rho} \\
X & \qquad & X\times G \xrightarrow{\quad \rho \quad} X
\end{array} \qquad (7)$$

and the following diagrams encoding simple transitivity all commute:

$$\begin{array}{ccc}
X\times X \xrightarrow{\ \mathrm{pr}_L\times\backslash\ } X\times G & \qquad & X\times G \xrightarrow{\ \mathrm{pr}_L\times\rho\ } X\times X \\
{\scriptstyle \mathrm{pr}_R}\searrow\quad\swarrow{\scriptstyle \rho} & \qquad & {\scriptstyle \mathrm{pr}_R}\searrow\quad\swarrow{\scriptstyle \backslash} \\
X & \qquad & G
\end{array} \qquad (8)$$

In these diagrams pr_L and pr_R denote the projections to the leftmost and rightmost factor respectively, moreover $m : G \times G \longrightarrow G$ denotes multiplication in G and $e : X \longrightarrow G$ the composition of ϵ with the terminal morphism $\mathrm{term} : X \longrightarrow *$. Left principal objects can be defined in complete analogy simply by switching left and right factors.

Intuitively, a principal object is essentially the group object itself, where we have forgotten the neutral element, in fact every group like object G in

a category $\mathscr{C}$ becomes a principal object over itself under the two structure morphisms $\rho := m$ and $\backslash := m \circ (\iota \times \mathrm{id})$. In the category **Set** of sets for example a principal object over a group G is a set X endowed with a right action $\rho : X \times G \longrightarrow X$, $(x, g) \longmapsto xg$, due to the commutative diagrams in (7) and an additional application $\backslash : X \times X \longrightarrow G$, $(x, y) \longmapsto x^{-1}y$, such that the following two axioms are met for all $x, y \in X$ and $g \in G$

$$x \, (\, x^{-1} \, y \,) \;\; = \;\; y \qquad\qquad x^{-1} \, (\, x \, g \,) \;\; = \;\; g \, ,$$

which reflect the commutative diagrams in (8). In consequence the right action of G on X is transitive with trivial stabilizers, once we have declared an arbitrary point $x \in X$ to be the neutral element a principal object $X \neq \emptyset$ becomes indiscernible from the group G. In linear algebra for example it would be appropriate to define an affine space to be a principal object $\mathscr{V} \neq \emptyset$ under the additive group underlying a vector space V over a field $\mathbb{K}$.

Lemma 3.1 (Group Like and Principal Objects in FB_M^∇)
For every Lie group G the trivial fiber bundle $M \times G$ over a manifold M endowed with the trivial connection and the obvious structure morphisms is a group like object in the category FB_M^∇ of fiber bundles with non–linear connections over M. Principal $M \times G$–objects are exactly the principal G–bundles GM over M endowed with a principal connection ω.

Proof: The product functor $M \times : \mathbf{MF} \longrightarrow \mathbf{FB}_M^\nabla$, $\mathscr{F} \longmapsto M \times \mathscr{F}$, maps of course terminal objects in **MF** to terminal objects in $\mathbf{FB}_M^\nabla$ and preserves Cartesian products, hence it maps the Lie group G, a group like object in the category **MF**, to the group like object $M \times G$ in the category $\mathbf{FB}_M^\nabla$. Consider now a principal $M \times G$–object in $\mathbf{FB}_M^\nabla$, this is a fiber bundle GM over M endowed with a non–linear connection ∇ and structure homomorphisms:

$$\rho : \; GM \; \times_M (M \times G) \; \longrightarrow \; GM \qquad \backslash : \; GM \; \times_M GM \; \longrightarrow \; M \times G \, .$$

The obvious diffeomorphism $GM \times_M (M \times G) \cong GM \times G$ of fiber bundles provides GM with a fiber preserving right action $\rho : GM \times G \longrightarrow GM$ such that each fiber $G_p M$ becomes a principal G–object in the category **Set**, this is to say that the action ρ is simply transitive on fibers. For every $\gamma \in G$

the element morphism $\gamma : \{*\} \longrightarrow G$ in the category $\mathbf{MF}$ induces moreover a parallel homomorphism in the category $\mathbf{FB}_M^\nabla$ of fiber bundles

$$ GM \ \overset{\mathrm{id}\times\mathrm{term}}{\longrightarrow} \ GM \ \times_M (\, M \times \{*\} \,) \ \overset{\mathrm{id}\times\gamma}{\longrightarrow} \ GM \ \times_M (\, M \times G \,) \ \overset{\rho}{\longrightarrow} \ GM \ , $$

which is just the right multiplication $R_\gamma : \ GM \ \longrightarrow \ GM, \ g \ \longmapsto \ g\gamma$. In turn the non–linear connection $\mathbb{P}^\nabla$ present on the object GM in $\mathbf{FB}_M^\nabla$ arises from a principal connection ω in the sense of Definition 2.3. $\blacksquare$

A general group like object in the category $\mathbf{FB}_M$ of fiber bundles over a manifold M is just a bundle of Lie groups over M, a fiber bundle GM endowed with the structure of a Lie group on each fiber such that the multiplication $m : \ GM \times_M GM \ \longrightarrow \ GM$, the inverse $\iota : \ GM \ \longrightarrow \ GM$ and the neutral element section $\epsilon : \ M \ \longrightarrow \ GM$ are smooth. Somewhat stronger is the concept of a Lie group bundle: A bundle GM of Lie groups, which can be trivialized locally by group isomorphisms. Evidently this stronger condition is necessary and sufficient for the existence of a non–linear connection $\mathbb{P}^\nabla$, under which GM becomes a group like object in the category $\mathbf{FB}_M^\nabla$.

4 Association Functors and Principal Bundles

Principal bundles are in a sense universal fiber bundles, every given principal bundle induces myriad fiber bundles with a large variety of model fibers over the same base manifold. The construction of all these fiber bundles is functorial in nature and best thought of as a functor, the association functor Ass_{GM}^ω, from a suitably defined category $\mathbf{MF}_G$ of model fibers to the category $\mathbf{FB}_M^\nabla$ of fiber bundles with connections over a manifold M. In this section we study the more important properties of association functors, the universality of principal connections and their curvature and characterize all vector bundles in the image of a fixed association functor.

Besides the categories $\mathbf{FB}_M$ and $\mathbf{FB}_M^\nabla$ of fiber bundles we are interested in the category $\mathbf{MF}_G$ of manifolds $\mathscr{F}$ acted upon by a fixed Lie group G under smooth G–equivariant maps $\varphi : \ \mathscr{F} \ \longrightarrow \ \hat{\mathscr{F}}$ as morphisms. Terminal objects are one point manifolds $\{*\}$ and Cartesian products in the category $\mathbf{MF}_G$ see G acting diagonally on the Cartesian product $\mathscr{F} \times \hat{\mathscr{F}}$ of the manifolds

underlying two objects $\mathscr{F}$ and $\hat{\mathscr{F}}$. Interestingly the category $\mathbf{MF}_G$ comes along with a canonical endofunctor, the tangent bundle endofunctor

$$T: \ \mathbf{MF}_G \ \longrightarrow \ \mathbf{MF}_G, \qquad \mathscr{F} \ \longrightarrow \ T\mathscr{F}\,,$$

which sends an object $\mathscr{F} \in \mathrm{OBJ}\,\mathbf{MF}_G$ to the tangent bundle of its underlying manifold considered as a manifold $T\mathscr{F}$ in its own right, on which the Lie group G acts by the differential of its characteristic action $\star$ on $\mathscr{F}$:

$$\star_{T\mathscr{F}}: \ G \ \times \ T\mathscr{F} \ \longrightarrow \ T\mathscr{F}, \qquad \left(\gamma, \ \frac{d}{dt}\Big|_0 f_t\,\right) \ \longmapsto \ \frac{d}{dt}\Big|_0 \gamma \star f_t\,.$$

In order to define the tangent bundle functor on morphisms we observe that the differential $\varphi_*: T\mathscr{F} \longrightarrow T\hat{\mathscr{F}}$ of a G–equivariant map $\varphi: \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ is again G–equivariant and this observation suggests $T\varphi := \varphi_*$. It should be noted that the Lie group G provides a distinguished object in the category $\mathbf{MF}_G$, namely its Lie algebra $\mathfrak{g} := T_e G$ considered just as a manifold endowed with the adjoint representation $\mathrm{Ad}: G \times \mathfrak{g} \longrightarrow \mathfrak{g}$. The infinitesimal action links this distinguished object to the tangent bundle endofunctor:

Definition 4.1 (Infinitesimal Action)
Consider a smooth left action $\star: G \times \mathscr{F} \longrightarrow \mathscr{F}, (\gamma, f) \longmapsto \gamma \star f$, of a Lie group G on a smooth manifold $\mathscr{F}$. The infinitesimal action of the Lie algebra $\mathfrak{g}$ of the group G associated to this smooth action $\star$ is defined by

$$\star_{\mathrm{inf}}: \ \mathfrak{g} \ \times \ \mathscr{F} \ \longrightarrow \ T\mathscr{F}, \qquad \left(\frac{d}{dt}\Big|_0 \gamma_t, \ f\right) \ \longmapsto \ \frac{d}{dt}\Big|_0 \gamma_t \star f\,,$$

where $t \longmapsto \gamma_t$ with $\gamma_0 = e$ represents the tangent vector $\frac{d}{dt}\big|_0 \gamma_t \in \mathfrak{g}$.

En nuce the infinitesimal action is a natural transformation from the endofunctor $\mathfrak{g}\times$ to the tangent bundle endofunctor. In fact $\star_{\mathrm{inf}}: \mathfrak{g} \times \mathscr{F} \longrightarrow T\mathscr{F}$ is G–equivariant and thus a morphism in $\mathbf{MF}_G$ for all objects $\mathscr{F}$ due to

$$\gamma \star_{T\mathscr{F}} (X \star_{\mathrm{inf}} f) \ = \ \frac{d}{dt}\Big|_0 (\gamma\,\gamma_t\,\gamma^{-1}) \star (\gamma \star f) \ = \ (\mathrm{Ad}_\gamma X) \star_{\mathrm{inf}} (\gamma \star f)$$

for all $f \in \mathscr{F}$ and all tangent vectors $X = \frac{d}{dt}\big|_0 \gamma_t$ at $\gamma_0 = e$, moreover $\star_{\mathrm{inf}}$ intertwines with the differential φ_* of every G–equivariant smooth map $\varphi: \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ in the identity $\varphi_*(X \star_{\mathrm{inf}} f) = X \star_{\mathrm{inf}} \varphi(f)$.

Definition 4.2 (Association Functor)

Consider a Lie group G and a principal G–bundle GM over a manifold M. Every smooth action $\star : G \times \mathscr{F} \longrightarrow \mathscr{F}$ of the group G on a manifold $\mathscr{F}$ extends to a free and smooth right action of the group G on the Cartesian product $GM \times \mathscr{F}$ via $(g, f) \star \gamma := (g\gamma, \gamma^{-1} \star f)$. The quotient of $GM \times \mathscr{F}$ by this free action is a fiber bundle over M with model fiber $\mathscr{F}$

$$\mathrm{Ass}_{GM}(\mathscr{F}) = GM \times_G \mathscr{F} := (GM \times \mathscr{F})/_G$$

called the fiber bundle associated to GM and $\mathscr{F} \in \mathrm{OBJ}\ \mathbf{MF}$. Every G–equivariant map $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ induces a homomorphism of fiber bundles

$$\mathrm{Ass}_{GM}(\varphi) : GM \times_G \mathscr{F} \longrightarrow GM \times_G \hat{\mathscr{F}}, \qquad [g, f] \longmapsto [g, \varphi(f)],$$

which is well–defined in terms of representatives (g, f) of the equivalence class $[g, f]$. In other words $\mathrm{Ass}_{GM} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M, \mathscr{F} \longmapsto GM \times_G \mathscr{F}$, is a functor from $\mathbf{MF}_G$ to the category $\mathbf{FB}_M$ of fiber bundles over M.

Recall now that the each of the categories $\mathbf{MF}_G$ and $\mathbf{FB}_M$ has a canonical endofunctor associated with it, namely the tangent bundle endofunctor T for the category $\mathbf{MF}_G$ of manifolds with G–action and the vertical tangent bundle functor Vert for the category $\mathbf{FB}_M$. Considered as a fiber bundle over M the vertical tangent bundle has fiber $[\mathrm{Vert}\ \mathscr{F}M]_p = T[\mathscr{F}_pM]$ over every $p \in M$ and so we may suspect that the following diagram commutes

$$
\begin{array}{ccc}
\mathbf{MF}_G & \xrightarrow{\ \mathrm{Ass}_{GM}\ } & \mathbf{FB}_M \\
T \downarrow & & \downarrow \mathrm{Vert} \\
\mathbf{MF}_G & \xrightarrow{\ \mathrm{Ass}_{GM}\ } & \mathbf{FB}_M\ .
\end{array}
\tag{9}
$$

up to a natural isomorphism $\mathrm{Vert}(GM \times_G \mathscr{F}) \xrightarrow{\cong} GM \times_G T\mathscr{F}$ given by:

$$\left.\frac{d}{dt}\right|_0 [\,g_t,\ f_t\,] \longmapsto \left[\,g_0,\ \left.\frac{d}{dt}\right|_0 (g_0^{-1} g_t) \star f_t\,\right]. \tag{10}$$

Of course this isomorphism is motivated by $[\,g_t, f_t\,] = [\,g_0, (g_0^{-1}g_t) \star f_t\,]$, whenever the representative curve $t \longmapsto [\,g_t,\ f_t\,]$ for a vertical tangent vector to $GM \times_G \mathscr{F}$ has been chosen such that g_t stays in the fiber of g_0 for all t.

Remark 4.3 (Action of Automorphism Group Bundle)
The automorphism group bundle of a principal bundle GM acts naturally $\star:\ \mathrm{Aut}\, GM \times_M \mathscr{F}M \longrightarrow \mathscr{F}M$ on every fiber bundle $\mathscr{F}M := GM \times_G \mathscr{F}$ associated to GM and an object $\mathscr{F} \in \mathrm{OBJ}\,\mathbf{MF}_G$ by means of

$$(p, \psi) \star [g, f] \ :=\ [\psi(g), f]$$

for all $(p, \psi) \in \mathrm{Aut}_p GM$ and $[g, f] \in \mathscr{F}_{\pi(g)}M$ in the fibers of $\mathrm{Aut}\, GM$ and $\mathscr{F}M$ over the same point $p = \pi(g)$ of the base manifold M.

In concrete examples the automorphism group bundle $\mathrm{Aut}\, GM$ is usually more readily identified than the principal bundle GM itself due to its omnipresent action on associated fiber bundles. Consider the orthonormal frame bundle of a pseudo–Riemannian manifold (M, g) for example

$$\mathbf{O}(M, g) \ :=\ \{\ (p, F) \mid p \in M \text{ and } F:\ T \longrightarrow T_p M \text{ isometry } \},$$

where T is a pseudo–euclidean model vector space of the correct signature and $\mathbf{O}(T)$ acts from the right by precomposition $(p, F)\,\gamma = (p, F \circ \gamma)$. The automorphism group bundle of the orthonormal frame bundle $\mathbf{O}(M, g)$ equals the Lie group bundle of all infinitesimal isometries of tangent spaces

$$\mathbf{O}(TM, g) \ :=\ \{\ (p, \psi) \mid \psi:\ T_p M \longrightarrow T_p M \text{ isometry } \}$$

acting by postcomposition $(p, \psi) \star (p, F) = (p, \psi \circ F)$; it just as well acts on the tangent bundle TM and all kinds of the tensor bundles etc.

For a general principal bundle GM we can use the same idea to identify the automorphism group bundle $\mathrm{Aut}\, GM$ as a Lie group bundle over M with the image of a group object in the category $\mathbf{MF}_G$. Letting G act on itself by conjugation $\star:\ G \times G \longrightarrow G,\ (\gamma, g) \longmapsto \gamma g \gamma^{-1}$, we obtain in fact a group object $G^{\mathrm{ad}} \in \mathrm{OBJ}\,\mathbf{MF}_G$, whose image under the association functor is a Lie group bundle $\mathrm{Ass}_{GM}(G^{\mathrm{ad}})$ over M acting G–equivariantly on GM by

$$\mathrm{Ass}_{GM}(G^{\mathrm{ad}}) \times_M GM \ \longrightarrow\ GM, \quad ([g, \gamma], \hat{g}) \ \longmapsto\ g\gamma(g^{-1}\hat{g}) \quad (11)$$

for all $\gamma \in G$ and all $g, \hat{g} \in GM$ in the same fiber. In particular $\mathrm{Aut}_p GM$ is isomorphic, but not naturally so, to the Lie group G in every $p \in M$.

Under this identification $\mathrm{Ass}_{GM}(\,G^{\mathrm{ad}}\,)\ =\ \mathrm{Aut}\,GM$ of Lie group bundles the natural action of $\mathrm{Aut}\,GM$ on associated fiber bundles $GM\times_G\mathscr{F}$ pointed out in Remark 4.3 becomes the functorial extension of the original action $\star$ considered as a G–equivariant smooth map $\star:\ G^{\mathrm{ad}}\times\mathscr{F}\ \longrightarrow\ \mathscr{F}$. In the same vein the functor Ass_{GM} converts the infinitesimal action of Definition 4.1 considered as a G–equivariant map $\star_{\mathrm{inf}}:\ \mathfrak{g}\times\mathscr{F}\ \longrightarrow\ T\mathscr{F}$ into

$$\mathrm{Ass}_{GM}(\,\star_{\mathrm{inf}}\,):\ \ (\,GM\times_G\mathfrak{g}\,)\ \times_M\ (\,GM\times_G\mathscr{F}\,)\ \longrightarrow\ (\,GM\times_G T\mathscr{F}\,),$$

which in turn becomes the infinitesimal action associated to Remark 4.3:

$$\star_{\mathrm{inf}}:\ \ \mathfrak{aut}\,GM\ \times_M\ (\,GM\times_G\mathscr{F}\,)\ \longrightarrow\ \mathrm{Vert}(\,GM\times_G\mathscr{F}\,).$$

Before we proceed to prove the universality of principal connections and their curvature we want to digress a little to discuss the gauge principle, a fundamental principle in the study of principal bundles allowing us to translate calculations on GM to statements about M. In its most basic formulation the gauge principle is the assertion that we have a canonical bijection

$$[\,\Omega^{\bullet}_{\mathrm{hor}}(\,GM,\,V\,)\,]^G\ \xrightarrow{\ \cong\ }\ \Omega^{\bullet}(\,M,\,GM\times_G V\,),\qquad \eta\ \longmapsto\ \mathrm{GP}[\,\eta\,]\quad(12)$$

between the horizontal differential forms $\eta\ \in\ \Omega^{\bullet}_{\mathrm{hor}}(\,GM,\,V\,)$ on GM with values in some representation V of G satisfying $R_{\gamma}^*\eta\ =\ \gamma\star\eta$ for all $\gamma\ \in\ G$ and general differential forms on the base manifold M with values in the associated vector bundle $GM\times_G V$. Explicitly this gauge principle reads

$$\mathrm{GP}[\,\eta\,]_p(\,X_1,\,\ldots,\,X_r\,)\ :=\ [\,g,\,\eta_g(\,\tilde{X}_1,\,\ldots,\,\tilde{X}_r\,)\,]$$

for arbitrary lifts $\tilde{X}_1,\,\ldots,\,\tilde{X}_r\ \in\ T_gGM$ of the argument tangent vectors $X_1,\,\ldots,\,X_r\ \in\ T_pM$ to an arbitrary point $g\ \in\ G_pM$ in the fiber over $p\ \in\ M$. Due to horizontality the resulting differential form $\mathrm{GP}[\,\eta\,]$ does not depend on the choice of lifts and the assumption $R_{\gamma}^*\eta\ =\ \gamma\star\eta$ ensures that $\mathrm{GP}[\,\eta\,]$ does not depend on the choice of $g\ \in\ G_pM$ either. The gauge principle converts the curvature 2–form $\Omega\ \in\ \Omega^2_{\mathrm{hor}}(\,GM,\,\mathfrak{g}\,)$ of Cartan's Second Structure Equation (5) into a 2–form on M with values in $\mathfrak{aut}\,GM$:

$$R^{\omega}\ :=\ \mathrm{GP}[\,\Omega\,]\ \in\ \Omega^2(\,M,\,\mathfrak{aut}\,GM\,)\qquad\qquad(13)$$

Proposition 4.4 (Universality of Principal Curvature)
Every choice of a principal connection ω on a principal G–bundle GM allows us to promote the association functor $\mathrm{Ass}_{GM} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M$ to a functor to the category of fiber bundles over M with non–linear connections:

$$\mathrm{Ass}^{\omega}_{GM} : \quad \mathbf{MF}_G \longrightarrow \mathbf{FB}^{\nabla}_M, \qquad \mathscr{F} \longmapsto GM \times_G \mathscr{F} \;.$$

In other words ω induces a natural connection ∇ on $GM \times_G \mathscr{F}$ for every G–manifold $\mathscr{F} \in \mathbf{OBJ}\,\mathbf{MF}_G$. The curvature R^{∇} of this induced connection is determined by the infinitesimal action of the Lie algebra bundle $\mathfrak{aut}\,GM$

$$\star_{\mathrm{inf}} : \quad \mathfrak{aut}\,GM \times_M (GM \times_G \mathscr{F}) \longrightarrow \mathrm{Vert}(GM \times_G \mathscr{F})$$

and the 2–form $R^{\omega} \in \Omega^2(M, \mathfrak{aut}\,GM)$. More precisely for all local sections $f \in \Gamma_{\mathrm{loc}}(M, GM \times_G \mathscr{F})$ and all $X, Y \in \Gamma(M, TM)$ we find:

$$R^{\nabla}_{X,Y} f \;=\; R^{\omega}(X, Y) \star_{\mathrm{inf}} f \;.$$

Proof: By definition $GM \times_G \mathscr{F}$ is the quotient of the Cartesian product $GM \times \mathscr{F}$ by a free right action of the Lie group G. In turn the canonical projection $\mathrm{pr} : GM \times \mathscr{F} \longrightarrow GM \times_G \mathscr{F}$ defines a tower of fiber bundles

$$
\begin{array}{c}
GM \times \mathscr{F} \\
\downarrow {\scriptstyle \mathrm{pr}} \\
GM \times_G \mathscr{F} \\
\downarrow {\scriptstyle \pi} \\
M
\end{array}
\qquad (14)
$$

over M, which becomes $U \times (G \times \mathscr{F}) \xrightarrow{\mathrm{pr}} U \times \mathscr{F} \xrightarrow{\pi} U$ in a local equivariant trivialization of GM. The central idea of the proof is to choose the connection $\mathbb{P}^{\nabla}$ on $GM \times_G \mathscr{F}$ such that pr is parallel with respect to the product $\mathbb{P}^{\omega} \times \mathbb{P}^{\mathrm{triv}}$ of the principal connection ω on GM and the trivial connection $\mathbb{P}^{\mathrm{triv}}$ on $M \times \mathscr{F}$.

For this purpose we consider a curve $t \longmapsto (g_t, f_t)$ in $GM \times \mathscr{F}$ and choose a curve $t \longmapsto \gamma_t$ in G with $\gamma_0 = e$ representing the tangent vector $\frac{d}{dt}\big|_0 \gamma_t = \omega_{g_0}(\frac{d}{dt}\big|_0 g_t) \in \mathfrak{g}$. The Principal Connection Axiom 2.4 ensures

$$\omega_{g_0 e}\left(\left.\frac{d}{dt}\right|_0 g_t \gamma_t^{-1}\right) \;=\; \mathrm{Ad}_e^{-1}\,\omega_{g_0}\left(\left.\frac{d}{dt}\right|_0 g_t\right) + \left.\frac{d}{dt}\right|_0 e^{-1} \gamma_t^{-1} \;=\; 0$$

and so $t \longmapsto g_t \gamma_t^{-1}$ represents a horizontal tangent vector. In turn

$$
\begin{aligned}
(\, \mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}} \,) & \Big(\; \frac{d}{dt} \Big|_0 \, (\, g_t, \, f_t \,) \; \Big) \\
&= \; (\, \mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}} \,) \Big(\; \frac{d}{dt} \Big|_0 \, (\, g_t \gamma_t^{-1} \gamma_0, \, f_0 \,) \; + \; \frac{d}{dt} \Big|_0 \, (\, g_0 \, \gamma_0^{-1} \, \gamma_t, \, f_t \,) \; \Big) \\
&= \; \frac{d}{dt} \Big|_0 \, (\, g_0 \, \gamma_t, \, f_t \,) \, ,
\end{aligned}
$$

because the first summand is horizontal and the second vertical in $GM \times \mathscr{F}$. Projecting this identity to equivalence classes in $GM \times_G \mathscr{F}$ we find

$$
\begin{aligned}
\mathbb{P}^\nabla \Big(\; \frac{d}{dt} \Big|_0 \, [\, g_t, \, f_t \,] \; \Big) \; &:= \; \frac{d}{dt} \Big|_0 \, [\, g_0, \, \gamma_t \star f_t \,] \\
&= \; \Big[\; g_0, \; \frac{d}{dt} \Big|_0 \, f_t \; + \; \omega_{g_0} \Big(\frac{d}{dt} \Big|_0 \, g_t \Big) \star_{\mathrm{inf}} f_0 \; \Big]
\end{aligned}
$$

due to the Definition 4.1 of the infinitesimal action and the choice of the curve $t \longmapsto \gamma_t$. In light of the isomorphism (10) the right hand side denotes a vertical tangent vector to $GM \times_G \mathscr{F}$ and so the latter formula defines a non–linear connection $\mathbb{P}^\nabla$ on the fiber bundle $GM \times_G \mathscr{F}$.

With respect to this non–linear connection $\mathbb{P}^\nabla$ the canonical projection $\mathrm{pr} \, : \, GM \times \mathscr{F} \longrightarrow GM \times_G \mathscr{F}$ is parallel, because it maps horizontal tangent vectors $\frac{d}{dt} \big|_0 \, [\, g_t, f_0 \,]$ with $\omega_{g_0} (\frac{d}{dt} \big|_0 \, g_t) = 0$ to horizontal vectors. The construction of $\mathbb{P}^\nabla$ is natural in the category $\mathbf{MF}_G$ as well: The functorial extension $\mathrm{Ass}_{GM}(\, \varphi \,) : \, GM \times_G \mathscr{F} \longrightarrow GM \times_G \hat{\mathscr{F}}$, $[\, g, f \,] \longmapsto [\, g, \varphi(f) \,]$, of every G–equivariant smooth map $\varphi : \, \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ is parallel

$$
\begin{aligned}
\mathbb{P}^{\hat\nabla} \Big(\; \frac{d}{dt} \Big|_0 \, [\, g_t, \varphi(f_t) \,] \; \Big) \; &= \; \Big[\; g_0, \; \frac{d}{dt} \Big|_0 \, \varphi(f_t) \; + \; \omega_{g_0} \Big(\frac{d}{dt} \Big|_0 \, g_t \Big) \star_{\mathrm{inf}} \varphi(f_0) \; \Big] \\
&= \; \mathrm{Ass}_{GM}(\, \varphi_* \,) \, \mathbb{P}^\nabla \Big(\; \frac{d}{dt} \Big|_0 \, [\, g_t, \, f_t \,] \; \Big)
\end{aligned}
$$

due to the infinitesimal equivariance $X \star_{\mathrm{inf}} \varphi(f) = \varphi_*(X \star_{\mathrm{inf}} f)$. In order to calculate the curvature of the connection $\mathbb{P}^\nabla$ we use the fact that in a tower of fiber bundles like (14) with a parallel submersion pr the curvature of the image connection $\mathbb{P}^\nabla$ is just the image of the preimage connection $\mathbb{P}^\omega \times \mathbb{P}^{\mathrm{triv}}$

under the differential pr_*. Using arbitrary lifts $\tilde{X},\ \tilde{Y}\ \in\ T_g GM$ of tangent vectors $X,\ Y\ \in\ T_p M$ to a point $g\ \in\ G_p M$ we calculate in this way

$$
\begin{aligned}
R^\nabla_{[g,f]}(\tilde{X},\tilde{Y}) \ &= \ \mathrm{pr}_{*,(g,f)} R^{\omega\times\mathrm{triv}}_{(g,f)}(\tilde{X},\tilde{Y}) \\
&= \ \frac{d}{dt}\Big|_0\ \Big[\,g\,\exp\Big(t\,\Omega_g(\tilde{X},\tilde{Y})\Big),\ f\,\Big] \\
&= \ [\,g,\,\Omega_g(\tilde{X},\tilde{Y})\,]\ \star_{\mathrm{inf}}\ [\,g,f\,] \ = \ R^\omega_p(X,Y)\ \star_{\mathrm{inf}}\ [\,g,f\,]\,,
\end{aligned}
$$

where $R^\omega\ :=\ \mathrm{GP}[\,\Omega\,]\ \in\ \Omega^2(\,M,\mathfrak{aut}\,GM\,)$ is the 2–form with values in $\mathfrak{aut}\,GM$ the gauge principle (12) associates to $\Omega\ :=\ d\omega\ +\ \frac{1}{2}[\,\omega\wedge\omega\,]$. Formulated in terms of local sections $f\ \in\ \Gamma_{\mathrm{loc}}(\,M,\,GM\times_G\,\mathscr{F}\,)$ the latter identity becomes $R^\nabla_{X,Y}f\ =\ R^\omega(X,Y)\ \star_{\mathrm{inf}}\,f$.   ∎

One of the most important properties of association functors is that they intertwine the actions of smooth functors on the categories $\mathbf{Rep}_G$ and $\mathbf{VB}_M$. A smooth functor is an endofunctor $\mathbb{S}:\ \mathbf{Vect}^\times_\mathbb{R}\ \longrightarrow\ \mathbf{Vect}^\times_\mathbb{R}$ of the category of finite dimensional vector spaces under linear isomorphisms such that

$$
\mathrm{MOR}_{\mathbf{Vect}^\times_\mathbb{R}}(\,V,\,V\,)\ \longrightarrow\ \mathrm{MOR}_{\mathbf{Vect}^\times_\mathbb{R}}(\,\mathbb{S}\,V,\,\mathbb{S}\,V\,),\qquad \varphi\ \longmapsto\ \mathbb{S}(\varphi),
$$

is a smooth map between the smooth manifolds $\mathrm{MOR}_{\mathbf{Vect}^\times_\mathbb{R}}(V,V)\ =\ \mathbf{GL}\,V$ and $\mathrm{MOR}_{\mathbf{Vect}^\times_\mathbb{R}}(\mathbb{S}\,V,\mathbb{S}\,V)$ for every finite dimensional vector space V over $\mathbb{R}$. Smooth functors extend naturally to endofunctors of the category $\mathbf{Rep}_G$ of representations V of a Lie group G by letting G act on $\mathbb{S}\,V$ via:

$$
\star:\ G\ \times\ \mathbb{S}V\ \longrightarrow\ \mathbb{S}V,\qquad (\gamma,s)\ \longmapsto\ \mathbb{S}(\,\gamma\star:\ V\ \xrightarrow{\ \cong\ }\ V\,)\,s\,.
$$

This extension to representations makes the classification of smooth functors an exercise in the representation theory of general linear groups: Every smooth functor is naturally isomorphic $\mathbb{S}\ \cong\ \mathbb{S}_1\oplus\ldots\oplus\mathbb{S}_r$ to a finite direct sum of Schur functors $\mathbb{S}_1,\ \ldots,\ \mathbb{S}_r$ twisted by density lines [FH].

In the same vein every smooth functor $\mathbb{S}$ extends naturally to an endofunctor of the category $\mathbf{VB}^\nabla_M$ of vector bundles with connections over a manifold M. The smoothness of $\mathbb{S}$ allows us to define a differentiable structure on the disjoint union of vector spaces obtained by applying $\mathbb{S}$ fiberwise

$$
\mathbb{S}\,VM\ :=\ \bigcup_{p\,\in\,M}\,\mathbb{S}\,V_p M
$$

to obtain a new vector bundle $\mathbb{S}VM$ over M; every connection ∇ on the original vector bundle VM extends naturally to a connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}VM$ by the requirement that parallel transport with respect to this connection along an arbitrary curve $t \longmapsto p_t$ in the manifold M is simply the image

$$\left(\mathbf{PT}_t^{\nabla^{\mathbb{S}}} : \ \mathbb{S}\,V_{p_0}M \ \xrightarrow{\cong} \ \mathbb{S}\,V_{p_t}M \right) \ = \ \mathbb{S}\left(\mathbf{PT}_t^{\nabla} : \ V_{p_0}M \ \xrightarrow{\cong} \ V_{p_t}M \right)$$

of parallel transport with respect to ∇ under the functor $\mathbb{S}$. Because parallel transport in associated vector bundles is essentially the image of parallel transport in the principal bundle GM itself, every association functor $\mathrm{Ass}_{GM}^{\omega}$ intertwines the two extensions of a smooth functor $\mathbb{S}$ to the categories $\mathbf{Rep}_G$ of representations and $\mathbf{VB}_M^{\nabla}$ of vector bundles with connections:

$$
\begin{array}{ccc}
\mathbf{Rep}_G & \xrightarrow{\ \mathrm{Ass}_{GM}^{\omega}\ } & \mathbf{VB}_M^{\nabla} \\
\mathbb{S}\ \downarrow & & \downarrow\ \mathbb{S} \\
\mathbf{Rep}_G & \xrightarrow{\ \mathrm{Ass}_{GM}^{\omega}\ } & \mathbf{VB}_M^{\nabla}\ .
\end{array}
\tag{15}
$$

Classically the vector bundles of the form $\mathbb{S}TM$ on a manifold M with a smooth functor $\mathbb{S}$ are called pseudotensor bundles, their sections pseudotensors, and they comprise exactly the natural vector bundles of order one. Some modern authors however seem to confuse the classical concept of tensors with the property of having a value defined at every point.

Lemma 4.5 (Properties of Association Functors)
Consider a principal G–bundle GM over a manifold M endowed with a principal connection ω and the corresponding association functor from the category $\mathbf{MF}_G$ of manifolds endowed with smooth G–actions to the category $\mathbf{FB}_M^{\nabla}$ of fiber bundles over M endowed with non–linear connections:

  1. *The association functor $\mathrm{Ass}_{GM}^{\omega}$ preserves Cartesian products:*

  $$GM \ \times_G \ (\,\mathscr{F} \times \hat{\mathscr{F}}\,) \ = \ (\,GM \times_G \mathscr{F}\,) \times_M (\,GM \times_G \hat{\mathscr{F}}\,)\,.$$

  2. *On the full subcategory $\mathbf{MF} \subset \mathbf{MF}_G$ of manifolds with trivial G–action the association functor $\mathrm{Ass}_{GM}^{\omega}$ agrees with the product functor:*

  $$\mathrm{Ass}_{GM}^{\omega}|_{\mathbf{MF}} : \ \mathbf{MF} \ \longrightarrow \ \mathbf{FB}_M^{\nabla}, \qquad \mathscr{F} \ \longmapsto \ M \times \mathscr{F}\,.$$

3. *Restricted to the subcategory* $\mathbf{Rep}_G \subset \mathbf{MF}_G$ *of finite dimensional smooth representations of the Lie group G under G–equivariant linear maps the association functor* Ass^ω_{GM} *takes values in the subcategory* $\mathbf{VB}^\nabla_M$ *of vector bundles over M endowed with linear connections:*

$$\mathrm{Ass}^\omega_{GM}|_{\mathbf{Rep}_G} : \quad \mathbf{Rep}_G \longrightarrow \mathbf{VB}^\nabla_M, \qquad V \longmapsto GM \times_G V \; .$$

Proof: Of course all three statements of this lemma are easily proved directly by unwrapping all the definitions made above; the second statement for example is an elaborate description of the trivial fiber bundle isomorphism

$$GM \times_G \mathscr{F} \xrightarrow{\;=\;} GM/_G \times \mathscr{F} \xrightarrow{\;\cong\;} M \times \mathscr{F} \; ,$$

whenever G acts trivially on $\mathscr{F}$ and thus effectively only on the first factor of $GM \times \mathscr{F}$ in the construction of the quotient $GM \times_G \mathscr{F}$. This fiber bundle isomorphism is evidently natural, it is compatible with all the fiber bundle homomorphisms induced by smooth maps $\varphi : \mathscr{F} \longrightarrow \hat{\mathscr{F}}$ between manifolds $\mathscr{F}$ and $\hat{\mathscr{F}}$ with trivial G–action.

Nevertheless we think the lemma is quite interesting, because the third is actually a consequence of the first two statements. Combining the existence of additive inverses and the unity axiom $\forall v : \; 1 \cdot v = v$ into the axiom $\forall v : \; v + (-1) \cdot v = 0$ we see that only three structure maps are needed to formulate all axioms for a vector space object V in a category $\mathscr{C}$ in terms of commutative diagrams provided we have specified a field object $\mathbb{K}$:

$$\cdot : \; \mathbb{K} \times V \longrightarrow V \qquad + : \; V \times V \longrightarrow V \qquad 0 : \; \{*\} \longrightarrow V \; .$$

In the category $\mathbf{MF}_G$ for example we may take the manifold $\mathbb{R}$ with the trivial G–action as the field object $\mathbb{K} = \mathbb{R}^{\mathrm{triv}}$, the corresponding vector space objects are smooth representations of the Lie group G over $\mathbb{R}$.

On the other hand the first and second statement of the lemma assert that the association functor Ass^ω_{GM} preserves Cartesian products and agrees with the product functor $M \times$ on the full subcategory $\mathbf{MF} \subset \mathbf{MF}_G$. In consequence Ass^ω_{GM} sends terminal objects in $\mathbf{MF}_G$ to terminal objects in $\mathbf{FB}^\nabla_M$ and a representation V to a fiber bundle $VM := GM \times_G V$ with three parallel structure maps, the zero section $0 : M \longrightarrow VM$ and:

$$\cdot : \; \mathbb{R} \times VM \longrightarrow VM \qquad + : \; VM \times_M VM \longrightarrow VM \; .$$

According to Lemma 2.6 the parallelity of the scalar multiplication map alone suffices to force the non–linear connection $\mathbb{P}^\nabla$ on $VM \in \mathrm{OBJ}\ \mathbf{FB}_M^\nabla$ to be a linear connection in the sense of Definition 2.5. ∎

Historically the concept of principal bundles and principal connections arose from Cartan's beautiful idea of moving frames, which asserts that every vector bundle VM with connection ∇ lies in the image of the association functor Ass_{GM}^ω for some principal bundle with connection. A suitable choice for the principal bundle GM is the frame bundle with model vector space V

$$\mathbf{GL}(M, VM) \ := \ \{ \ (p, F) \ | \ p \in M \text{ and } F : V \xrightarrow{\cong} V_pM \ \},$$

which is a principal $\mathbf{GL}\,V$–bundle over M with right multiplication given by precomposition $(p, F)\,\gamma = (p, F \circ \gamma)$. The tautological diffeomorphism

$$\mathbf{GL}(M, VM) \times_{\mathbf{GL}V} V \xrightarrow{\cong} VM, \qquad [\,(p, F),\, v\,] \longmapsto Fv,$$

is a parallel isomorphism for the principal connection on $\mathbf{GL}(M, VM)$

$$\omega\Big(\ \frac{d}{dt}\Big|_0 (p_t, F_t) \ \Big) \ := \ \frac{d}{dt}\Big|_0 F_0^{-1} \circ (\mathbf{PT}_t^\nabla)^{-1} \circ F_t \ \in \ \mathrm{End}\ V$$

constructed from the parallel transport $\mathbf{PT}_t^\nabla : V_{p_0}M \longrightarrow V_{p_t}M$ with respect to ∇ along the curve $t \longmapsto p_t$; the principal connection axiom of Lemma 2.4 is particularly easy to verify using this definition for ω.

In consequence of this moving frames argument it does not make too much sense to ask, whether or not a vector bundle with connection is in the image of *some* association functor. The appropriate answer to this question for an association functor fixed in advance is definitely more interesting and was given in the master thesis of one of the authors. A closely related concept is the concept of geometric vector bundles defined in [SW]:

Proposition 4.6 (Images of Association Functors)
Let G be a simply connected Lie group and let GM be a principal G–bundle over a simply connected manifold M endowed with a principal connection ω. A vector bundle VM with a linear connection $\mathbb{P}^\nabla$ over M is isomorphic in the vector bundle category $\mathbf{VB}_M^\nabla$ to a vector bundle in the image of the association functor Ass_{GM}^ω, if and only if there exists a parallel bilinear map

$$\star_{\mathrm{inf}} \ : \ \mathfrak{aut}\,GM \ \times_M VM \longrightarrow VM, \qquad (X, v) \longmapsto X \star v,$$

which is a representation of the Lie algebra $\mathfrak{aut}_p GM$ *at every* $p \in M$

$$(\star_{\mathrm{inf}})_p : \quad \mathfrak{aut}_p GM \; \times \; V_p M \; \longrightarrow \; V_p M$$

with the additional property that the curvature of the given connection ∇ *agrees with the pointwise action of the curvature* $R^\omega \in \Omega^2(M, \mathfrak{aut}\,GM)$:

$$R^\nabla_{X,Y} v \;=\; R^\omega_{X,Y} \star_{\mathrm{inf}} v \,.$$

Proof: Consider to begin with the vector bundle $VM := GM \times_G V$ associated to a representation V of the Lie group G. According to our discussion of the infinitesimal action following Definition 4.1 the composition

$$\star_{\mathrm{inf}} : \quad \mathfrak{g} \; \times \; V \; \xrightarrow{\star_{\mathrm{inf}}} \; TV \; \xrightarrow{\cong} \; V \times V \; \xrightarrow{\mathrm{pr}_R} \; V$$

is G–equivariant and thus gives rise to a parallel $\mathbb{R}$–bilinear map, which is a representation $(\star_{\mathrm{inf}})_p$ of the Lie algebra $\mathfrak{aut}_p GM$ on $V_p M$ in every point:

$$\star_{\mathrm{inf}} : \quad \mathfrak{aut}\,GM \; \times_M \; VM \; \longrightarrow \; VM \,.$$

Conversely assume that $\star_{\mathrm{inf}} : \mathfrak{aut}\,GM \times_M VM \longrightarrow VM$ is a parallel representation of the Lie algebra bundle $\mathfrak{aut}\,GM$ on a vector bundle VM with a linear connection $\mathbb{P}^\nabla$. According to equation (11) the fiber Lie group $\mathrm{Aut}_p GM$ is isomorphic to G in every point $p \in M$ and so simply connected, in consequence the infinitesimal action $(\star_{\mathrm{inf}})_p$ of its Lie algebra $\mathfrak{aut}_p GM$ integrates to a representation of the Lie group $\mathrm{Aut}_p GM$ on the vector space $V_p M$. Though slightly technical it is straightforward to prove that the integrated representation depends smoothly on the point $p \in M$

$$\star : \quad \mathrm{Aut}\,GM \; \times_M \; VM \; \longrightarrow \; VM \,, \tag{16}$$

the details of this argument are left to the reader. In addition to the vector bundle VM with its connection $\mathbb{P}^\nabla$ we consider the vector bundle $GM \times_G V$ associated to some representation V of G endowed with the linear connection $\mathbb{P}^\omega$ induced by the principal connection ω in Proposition 4.4. The two connections determine a linear connection $\mathbb{P}^{(\omega,\nabla)}$ on the vector bundle $\mathrm{Hom}(GM \times_G V, VM)$ characterized by the fact that its parallel transport

$$\mathbf{PT}^{(\omega,\nabla)}_t : \quad \mathrm{Hom}(G_{p_0}M \times_G V, V_{p_0}M) \; \longrightarrow \; \mathrm{Hom}(G_{p_t}M \times_G V, V_{p_t}M)$$

along an arbitrary curve $t \longmapsto p_t$ makes the following diagram commute

$$
\begin{array}{ccc}
G_{p_0} M \times_G V & \xrightarrow{\quad F \quad} & V_{p_0} M \\
\mathbf{PT}_t^\omega \Big\downarrow & & \Big\downarrow \mathbf{PT}_t^\nabla \\
G_{p_t} M \times_G V & \xrightarrow{\quad \mathbf{PT}_t^{(\omega,\nabla)} F \quad} & V_{p_t} M
\end{array}
\qquad (17)
$$

for all linear maps $F : G_{p_0} M \times_G V \longrightarrow V_{p_0} M$, where $\mathbf{PT}_t^\omega$ and $\mathbf{PT}_t^\nabla$ are the parallel transports along the same curve with respect to $\mathbb{P}^\omega$ and $\mathbb{P}^\nabla$.

The principal idea of the proof is now to construct a parallel and actually flat vector subbundle of the vector bundle $\mathrm{Hom}(GM \times_G V, VM)$ over M. For this purpose we consider the family of vector subspaces of the fibers

$$
\Big[\mathrm{Hom}_{\mathrm{Aut}\, GM}(GM \times_G V, VM) \Big]_p
$$
$$
:= \ \{ \ F : G_p M \times_G V \longrightarrow V_p M \ \mid \ \text{linear and } \mathrm{Aut}_p GM \text{ equivariant} \ \}
$$

of the vector bundle $\mathrm{Hom}(GM \times_G V, VM)$ in each point $p \in M$. In order to show that this family of subspaces is the family of fibers of a vector subbundle of $\mathrm{Hom}(GM \times_G V, VM)$ we observe that the parallel transport

$$
\mathbf{PT}_t^\omega : \ G_{p_0} M \ \times_G V \ \xrightarrow{\ \cong\ } G_{p_t} M \ \times_G V \qquad\qquad \mathbf{PT}_t^\nabla : \ V_{p_0} M \ \xrightarrow{\ \cong\ } V_{p_t} M
$$

in both vector bundles $GM \times_G V$ and VM along a curve $t \longmapsto p_t$ is equivariant over the parallel transport with respect to the Lie group connection $\mathbb{P}^\omega$ on the automorphism bundle $\mathrm{Aut}\, GM$ induced by ω. More precisely we find

$$
\mathbf{PT}_t^\nabla \big(\, (p_0, \, \psi) \, \star \, v \, \big) \ = \ \mathbf{PT}_t^\omega(p_0, \, \psi) \, \star \, \mathbf{PT}_t^\nabla v
$$

for the vector bundle VM, because $\star_{\mathrm{inf}} : \mathfrak{aut}\, GM \times_M VM \longrightarrow VM$ is parallel by assumption. In consequence the parallel transport $\mathbf{PT}^{(\omega,\nabla)}$ with respect to the linear connection $\mathbb{P}^{(\omega,\nabla)}$ specified in diagram (17) induces for all $t \in \mathbb{R}$ vector space isomorphisms $F \longmapsto \mathbf{PT}_t^\nabla \circ F \circ (\mathbf{PT}_t^\omega)^{-1}$ between:

$$
\Big[\mathrm{Hom}_{\mathrm{Aut}\, GM}(GM \times_G V, VM) \Big]_{p_0} \xrightarrow{\ \cong\ } \Big[\mathrm{Hom}_{\mathrm{Aut}\, GM}(GM \times_G V, VM) \Big]_{p_t} .
$$

By assumption the underlying manifold M is (simply) connected, and hence all vector subspaces $\big[\mathrm{Hom}_{\mathrm{Aut}\, GM}(GM \times_G V, VM) \big]_p$ have the same dimension. With parallel transport depending smoothly on the curve we conclude that $\mathrm{Hom}_{\mathrm{Aut}\, GM}(GM \times_G V, VM)$ is a genuine vector subbundle of

$\mathrm{Hom}(\,GM \times_G V,\, VM\,)$, moreover it is a parallel subbundle as it is invariant under parallel transport along arbitrary curves.

On the other hand the curvature of the linear connection $\mathbb{P}^{(\omega,\nabla)}$ on the vector bundle $\mathrm{Hom}(\,GM \times_G V,\, VM\,)$ is determined by the universality of principal curvature discussed in Proposition 4.4, namely it holds true that

$$R_{X,Y}^{(\omega,\nabla)} F \;=\; R_{X,Y}^{\nabla} \circ F \;-\; F \circ (\, R_{X,Y}^{\omega} \star_{\mathrm{inf}}\,)$$

for all tangent vectors $X,\, Y \,\in\, T_p M$ and $F \,\in\, \mathrm{Hom}_p(\,GM \times_G V,\, VM\,)$. Due to equivariance the curvature of the connection $\mathbb{P}^{(\omega,\nabla)}$ restricted to the parallel vector subbbundle $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ vanishes identically, put differently $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ is a flat vector bundle over M under the restriction of the connection $\mathbb{P}^{(\omega,\nabla)}$.

In the argument presented so far the actual choice of the representation V did not play any role. In order to make a diligent choice we fix a frame $g \,\in\, G_p M$ over a point $p \,\in\, M$ and consider the Lie group isomorphism

$$\Phi:\; G \;\xrightarrow{\;\cong\;}\; \mathrm{Aut}_p GM, \qquad \gamma \;\longmapsto\; \big(\,p,\, \hat{g} \;\longmapsto\; g\gamma(g^{-1}\hat{g})\,\big)\,,$$

which is essentially the Lie group bundle isomorphism (11) restricted to the fiber of p. This Lie group isomorphism allows us to pull back the integrated representation (16) of $\mathrm{Aut}_p GM$ on the vector space $V \,:=\, V_p M$ to a smooth representation $\star:\, G \times V \,\longrightarrow\, V$ enjoying the critical property that

$$\overline{\Phi}:\; G_p M \,\times_G V \;\xrightarrow{\;\cong\;}\; V \;\xrightarrow{\;=\;}\; V_p M, \qquad [\,\hat{g},\, v\,] \;\longmapsto\; \Phi(\,g^{-1}\hat{g}\,) \star v$$

is an equivariant vector space isomorphism under $\mathrm{Aut}_p GM$ in the sense:

$$\begin{aligned}
\overline{\Phi}\big(\,\Phi(\,\gamma\,)\,[\,\hat{g},\, v\,]\,\big) \;&=\; \overline{\Phi}\big(\,[\,g\gamma(g^{-1}\hat{g}),\, v\,]\,\big) \\
&=\; \Phi\big(\,g^{-1}g\gamma(g^{-1}\hat{g})\,\big) \,\star\, v \;=\; \Phi(\,\gamma\,)\,\overline{\Phi}\big(\,[\,\hat{g},\, v\,]\,\big)\,.
\end{aligned}$$

In consequence the fiber of the vector bundle $\mathrm{Hom}_{\mathrm{Aut}\,GM}(\,GM \times_G V,\, VM\,)$ over the chosen point $p \,\in\, M$ contains the vector space isomorphism $\overline{\Phi}$, which translates under parallel transport along arbitrary curves with respect to the flat connection $\mathbb{P}^{(\omega,\nabla)}$ into a parallel, globally defined section $\overline{\Phi}$ on the simply connected manifold M. Evaluation of this parallel section in the points of M converts it into a parallel isomorphism of vector bundles:

$$\overline{\Phi}:\; GM \,\times_G V \;\xrightarrow{\;\cong\;}\; VM, \qquad [\,\hat{g},\, v\,] \;\longmapsto\; \overline{\Phi}_{\pi(\hat{g})}[\,\hat{g},\, v\,]\,. \qquad\blacksquare$$

5   The Category of Gauge Theory Sectors

Every association functor is in a sense a reproducing functor, there exists in its source category an object, whose image in its target category is isomorphic to the principal bundle defining the association functor in the first place. Based on this simple observation we characterize the association functors among all functors from $\mathbf{MF}_G$ to $\mathbf{FB}_M^\nabla$ in this section, moreover we establish an equivalence of categories between the category of principal bundles and a suitably defined category of functors called gauge theory sectors.

Consider the smooth action of a given Lie group G on its underlying manifold by left multiplication $\star: G \times G \longrightarrow G$, $(\gamma, g) \longmapsto \gamma g$, which defines an object $G^{\text{left}} \in \mathbf{OBJ}\, \mathbf{MF}_G$ in the category of G–manifolds. The image of G^{left} under the functor Ass_{GM}^ω is isomorphic as a fiber bundle to GM

$$\mathrm{Ass}_{GM}^\omega(\,G^{\text{left}}\,) \;\overset{\cong}{\longrightarrow}\; GM, \qquad [\,g, \gamma\,] \;\longmapsto\; g\,\gamma\,, \tag{18}$$

and the inverse isomorphism $g \longmapsto [g, e]$ is easily verified to be parallel with

$$\mathbb{P}^\nabla\Big(\; \frac{d}{dt}\Big|_0 [\,g_t, e\,] \;\Big) \;=\; \Big[\; g_0,\; \frac{d}{dt}\Big|_0 e \;+\; \omega\Big(\frac{d}{dt}\Big|_0 g_t\Big) \star_{\text{inf}} e \;\Big] \;=\; 0$$

whenever $\frac{d}{dt}\big|_0 g_t$ is horizontal in the sense $\omega(\frac{d}{dt}\big|_0 g_t) = 0$. This reproducing property of Ass_{GM}^ω lies at the heart of the proof of the following theorem:

Theorem 5.1 (Characterization of Association Functors)
Consider a covariant functor $\mathfrak{F}: \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ from the category of G–manifolds to the category of fiber bundles with connection over M. If the functor $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor

$$M \times: \; \mathbf{MF} \; \longrightarrow \; \mathbf{FB}_M^\nabla, \qquad \mathscr{F} \; \longmapsto \; M \times \mathscr{F}\,,$$

on the full subcategory $\mathbf{MF} \subset \mathbf{MF}_G$ of manifolds with trivial G–action, then $\mathfrak{F}$ is naturally isomorphic to the association functor corresponding to some principal G–bundle GM endowed with a principal connection ω.

Proof: Consider a functor $\mathfrak{F}: \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ from the category of G–manifolds to the category of fiber bundles over M endowed with non–linear

connections, which preserves Cartesian products and agrees with the product functor $M\times\ :\ \mathbf{MF}\ \longrightarrow\ \mathbf{FB}_M^\nabla$ on the full subcategory of trivial G–manifolds. At least three different objects in the domain category $\mathbf{MF}_G$ of the functor $\mathfrak{F}$ have underlying manifold equal to the Lie group G:

$$G^{\mathrm{left}} \qquad\qquad G^{\mathrm{ad}} \qquad\qquad G^{\mathrm{triv}}\ .$$

The difference between these three objects in $\mathbf{MF}_G$ resides in their actions, which is by left multiplication $\gamma \star g\ :=\ \gamma g$ and conjugation $\gamma \star g\ :=\ \gamma g \gamma^{-1}$ respectively for G^{left} and G^{ad}, whereas G acts trivially on G^{triv}. Every terminal object in the category $\mathbf{MF}_G$ is a zero–dimensional manifold point $\{*\}$ with necessarily trivial G–action, hence $\mathfrak{F}$ maps it to the terminal object $M \times \{*\}$ in the category $\mathbf{FB}_M^\nabla$. In other words the functor $\mathfrak{F}$ maps terminal objects to terminal objects and preserves Cartesian products and in consequence turns group like and principal objects in the category $\mathbf{MF}_G$ into group like and principal objects in the category $\mathbf{FB}_M^\nabla$.

With G acting by automorphisms on both G^{ad} and G^{triv} both objects are group like objects in the category $\mathbf{MF}_G$ under the multiplication and inverse inherited from G. The significance of the group like object $\mathfrak{F}(\,G^{\mathrm{ad}}\,)$ in the category $\mathbf{FB}_M^\nabla$ may be somewhat obscure at this point, the group like object $\mathfrak{F}(\,G^{\mathrm{triv}}\,)\ =\ M \times G$ however is just the trivial G–bundle over M endowed with the trivial connection. Moreover the original Lie group multiplication defines G–equivariant structure maps in analogy to definition (6)

$$\rho:\ G^{\mathrm{left}}\ \times\ G^{\mathrm{triv}}\ \longrightarrow\ G^{\mathrm{left}} \qquad\qquad \backslash:\ G^{\mathrm{left}}\ \times\ G^{\mathrm{left}}\ \longrightarrow\ G^{\mathrm{triv}}$$

by means of $\rho(g,\hat{g})\ :=\ g\hat{g}$ and $\backslash(g,\hat{g})\ :=\ g^{-1}\hat{g}$, which naturally enough turn G^{left} into a G^{triv}–principal object in the category $\mathbf{MF}_G$. According to Lemma 3.1 the image of G^{left} is a principal G–bundle $GM\ :=\ \mathfrak{F}(\,G^{\mathrm{left}}\,)$ over the manifold M endowed with a principal connection ω. In passing we observe that the group like object G^{ad} acts G–equivariantly on G^{left} via

$$\star:\ G^{\mathrm{ad}}\ \times\ G^{\mathrm{left}}\ \longrightarrow\ G^{\mathrm{left}}, \qquad (\gamma,\,g)\ \longmapsto\ \gamma\,g\,,$$

and this action identifies the group like object $\mathfrak{F}(\,G^{\mathrm{ad}}\,)$ in the category $\mathbf{FB}_M^\nabla$ with the gauge group bundle $\mathrm{Aut}\,GM$ of GM by means of the action:

$$\mathfrak{F}(\star):\ \mathfrak{F}(\,G^{\mathrm{ad}}\,)\ \times_M\ GM\ \longrightarrow\ GM\,.$$

It remains to show that the original functor $\mathfrak{F}$ is naturally isomorphic to the association functor Ass^ω_{GM}. For this purpose we consider a general object $\mathscr{F} \in \mathrm{OBJ}\ \mathbf{MF}_G$; replacing its G–action by the trivial G–action on the same underlying manifold we project it to an object $\mathscr{F}^{\mathrm{triv}} \in \mathrm{OBJ}\ \mathbf{MF}$ in the subcategory of manifolds with trivial G–action. The G–equivariant map

$$\Psi:\ G^{\mathrm{left}} \times \mathscr{F} \ \xrightarrow{\ \cong\ }\ G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}}, \qquad (\,g,\,f\,) \longmapsto (\,g,\,g^{-1} \star f\,)$$

is actually an isomorphism in $\mathbf{MF}_G$ with inverse $(\,g,\,f\,) \longmapsto (\,g,\,g \star f\,)$, which fits for an arbitrary element $\gamma \in G$ into the commutative diagram

$$
\begin{array}{ccc}
G^{\mathrm{left}} \times \mathscr{F} & \xrightarrow{\ \ \Psi\ \ } & G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}} \\
\Big\downarrow{\scriptstyle \rho_\gamma \times \mathrm{id}} & \mathscr{F} & \Big\downarrow{\scriptstyle \rho_\gamma \times (\gamma^{-1}\star)} \\
G^{\mathrm{left}} \times \mathscr{F} & \xrightarrow{\ \ \Psi\ \ } & G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}}
\end{array}
\tag{19}
$$

in the category $\mathbf{MF}_G$, where $\rho_\gamma : G^{\mathrm{left}} \longrightarrow G^{\mathrm{left}}$, $g \longmapsto g\gamma$, denotes the right multiplication by γ and $\star$ the original G–action characterizing the object $\mathscr{F}$ thought of as a G–equivariant (sic!) map $\star:\ G^{\mathrm{left}} \times \mathscr{F}^{\mathrm{triv}} \longrightarrow \mathscr{F}$. Writing the right multiplication ρ_γ in the category $\mathbf{MF}_G$ as a composition

$$G^{\mathrm{left}} \xrightarrow{\ \mathrm{id}\times\mathrm{term}\ } G^{\mathrm{left}} \times \{*\} \xrightarrow{\ \mathrm{id}\times\gamma\ } G^{\mathrm{left}} \times G^{\mathrm{triv}} \xrightarrow{\ \rho\ } G^{\mathrm{left}}$$

factorizing over the element morphism $\gamma:\ \{*\} \longrightarrow G^{\mathrm{triv}}$ in the subcategory $\mathbf{MF} \subset \mathbf{MF}_G$ we conclude that $\mathfrak{F}(\,\rho_\gamma\,):\ GM \longrightarrow GM$ agrees with the right multiplication $R_\gamma:\ GM \longrightarrow GM$, $g \longmapsto g\gamma$, in the principal bundle GM induced by $\mathfrak{F}(\,\rho\,):\ GM \times G \longrightarrow GM$, because $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor $M\times$ on the trivial G–manifolds $\{*\}$ and G^{triv}. In consequence the commutative diagram (19) translates under the functor $\mathfrak{F}$ into the following commutative diagram

$$
\begin{array}{ccc}
GM \times_M \mathscr{F}M & \xrightarrow{\ \ \Psi\ \ } & GM \times \mathscr{F} \\
\Big\downarrow{\scriptstyle R_\gamma \times \mathrm{id}} & \mathscr{F}M & \Big\downarrow{\scriptstyle R_\gamma \times (\gamma^{-1}\star)} \\
GM \times_M \mathscr{F}M & \xrightarrow{\ \ \Psi\ \ } & GM \times \mathscr{F}
\end{array}
\tag{20}
$$

in the category $\mathbf{FB}_M^\nabla$ with $\mathscr{F}M := \mathfrak{F}(\mathscr{F})$, because $\mathfrak{F}$ preserves Cartesian products, hence preserves projections and agrees on manifolds with trivial G–action like $\mathscr{F}^{\mathrm{triv}}$ with the product functor $M\times$. The parallel homomorphism $\mathfrak{F}(\star) : GM \times \mathscr{F} \longrightarrow \mathscr{F}M$ thus descends to the quotient

$$\overline{\mathfrak{F}(\star)} : \quad GM \times_G \mathscr{F} \quad \longrightarrow \quad \mathscr{F}M$$

of $GM \times \mathscr{F}$ by the right G–action defining the associated fiber bundle $GM \times_G \mathscr{F}$, which lets $\gamma \in G$ act by $R_\gamma \times (\gamma^{-1}\star)$. It goes without saying that the projection $\mathrm{pr}_R : GM \times_M \mathscr{F}M \longrightarrow \mathscr{F}M$ factors through the quotient of $GM \times_M \mathscr{F}M$ by the right G–action on the principal bundle GM, the commutative diagram (20) ensures moreover that the quotient diagram

$$
\begin{array}{ccc}
(GM/_G) \times_M \mathscr{F}M & \xrightarrow{\ \overline{\Psi}\ } & GM \times_G \mathscr{F} \\
& \searrow\rule{0pt}{0pt}\mathrm{pr}_R \quad\quad \swarrow \overline{\mathfrak{F}(\star)} & \\
& \mathscr{F}M &
\end{array}
$$

still commutes. With $\mathrm{pr}_R : M \times_M \mathscr{F}M \xrightarrow{\ \cong\ } \mathscr{F}M$ and $\overline{\Psi}$ being parallel diffeomorphisms of fiber bundles with connections over M we conclude that

$$\overline{\mathfrak{F}(\star)} : \quad GM \times_G \mathscr{F} \quad \xrightarrow{\ \cong\ } \quad \mathscr{F}M$$

is actually an isomorphism in the category $\mathbf{FB}_M^\nabla$, moreover the construction of this parallel fiber bundle isomorphism $\overline{\mathfrak{F}(\star)} : \mathrm{Ass}_{GM}^\omega \mathscr{F} \longrightarrow \mathfrak{F}(\mathscr{F})$ for a given object $\mathscr{F} \in \mathrm{OBJ}\ \mathbf{MF}_G$ is natural under morphisms in $\mathbf{MF}_G$ and comprises a natural isomorphism $\overline{\mathfrak{F}(\cdot)} : \mathrm{Ass}_{GM}^\omega \longrightarrow \mathfrak{F}$ of functors. ∎

In order to press the point of Theorem 5.1 home let us define two rather special categories associated to a smooth manifold M. Objects in the category $\mathbf{PB}_M^\nabla$ of principal bundles with connections over M are triples (G, GM, ω) formed by a Lie group G and a principal G–bundle GM over M endowed with a principal connection ω. Every morphism between two such objects

$$(\varphi_{\mathrm{grp}}, \varphi) : \quad (G, GM, \omega) \quad \longrightarrow \quad (\hat{G}, \hat{G}M, \hat{\omega})$$

consists of a parallel homomorphism $\varphi : GM \longrightarrow \hat{G}M$ of fiber bundles which is G–equivariant over the Lie group homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$.

Objects in the category $\mathbf{GTS}_M^\nabla$ of gauge theory sectors on M with connections are on the other hand tuples $(G, \mathfrak{F})$ formed by a Lie group G and a covariant functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M^\nabla$ which preserves Cartesian products and agrees with the product functor on the full subcategory $\mathbf{MF} \subset \mathbf{MF}_G$ of manifolds with trivial G–action. In $\mathbf{GTS}_M^\nabla$ morphisms are again tuples

$$(\varphi_{\mathrm{grp}}, \Phi): (G, \mathfrak{F}) \longrightarrow (\hat{G}, \hat{\mathfrak{F}})$$

consisting of a group homomorphism $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$ between the two Lie groups and a natural transformation $\Phi : \mathfrak{F} \circ \varphi_{\mathrm{grp}}^* \longrightarrow \hat{\mathfrak{F}}$ between the two functors $\mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{FB}_M^\nabla$ involved, where the action pull back functor

$$\varphi_{\mathrm{grp}}^* : \mathbf{MF}_{\hat{G}} \longrightarrow \mathbf{MF}_G, \qquad (\hat{\mathscr{F}}, \star_{\hat{G}}) \longrightarrow (\hat{\mathscr{F}}, \star_G)$$

induced by φ_{grp} lets G act via $g \star_G f := \varphi_{\mathrm{grp}}(g) \star_{\hat{G}} f$ on a $\hat{G}$–manifold $\hat{\mathscr{F}}$. We want to interpret the construction of the association functor as a functor

$$\mathrm{Ass}: \mathbf{PB}_M^\nabla \longrightarrow \mathbf{GTS}_M^\nabla$$

with $(G, GM, \omega) \longmapsto (G, \mathrm{Ass}_{GM}^\omega)$ on objects, hence we still have to specify Ass on morphisms: Every morphism in the source category $\mathbf{PB}_M^\nabla$ is a parallel fiber bundle homomorphism $\varphi : GM \longrightarrow \hat{G}M$ equivariant over $\varphi_{\mathrm{grp}} : G \longrightarrow \hat{G}$, in the the target category $\mathbf{GTS}_M^\nabla$ such a morphism becomes the natural transformation Φ_φ defined for $\hat{\mathscr{F}} \in \mathrm{OBJ}\, \mathbf{MF}_{\hat{G}}$ by:

$$\Phi_\varphi(\hat{\mathscr{F}}): GM \times_G \hat{\mathscr{F}} \longrightarrow \hat{G}M \times_{\hat{G}} \hat{\mathscr{F}}, \qquad [g, \hat{f}] \longmapsto [\varphi(g), \hat{f}].$$

Corollary 5.2 (Association Functor as Equivalence of Categories)
For every smooth manifold M the association functor Ass *provides an equivalence of categories from the category* $\mathbf{PB}_M^\nabla$ *of principal bundles to the category* $\mathbf{GTS}_M^\nabla$ *of gauge theory sectors over M with connections:*

$$\mathrm{Ass}: \mathbf{PB}_M^\nabla \stackrel{\simeq}{\longrightarrow} \mathbf{GTS}_M^\nabla, \qquad (G, GM, \omega) \longmapsto (G, \mathrm{Ass}_{GM}^\omega).$$

In particular two principal G–bundles endowed with principal connections on M are isomorphic via a parallel, G–equivariant homomorphism of fiber bundles, if and only if their association functors are naturally isomorphic.

Proof: According to Theorem 5.1 every gauge theory sector with connection $(G, \mathfrak{F})$ is isomorphic in the category $\mathbf{GTS}_M^\nabla$ to an association functor Ass_{GM}^ω for a suitable principal G–bundle GM with a principal connection ω. In order to prove Corollary 5.2 we thus need to show that the association functor Ass induces for two arbitrary objects in $\mathbf{PB}_M^\nabla$ a bijection of sets:

$$\mathrm{Ass}: \ \mathrm{MOR}_{\mathbf{PB}_M^\nabla}\Big((G, GM, \omega), (\hat{G}, \hat{G}M, \hat{\omega}) \Big)$$
$$\stackrel{\cong}{\longrightarrow} \ \mathrm{MOR}_{\mathbf{GTS}_M^\nabla}\Big((G, \mathrm{Ass}_{GM}^\omega), (\hat{G}, \mathrm{Ass}_{\hat{G}M}^{\hat{\omega}}) \Big)$$

Consider for this purpose a morphism $(\varphi_{\mathrm{grp}}, \Phi)$ in the category $\mathbf{GTS}_M^\nabla$ from the image object $(G, \mathrm{Ass}_{GM}^\omega)$ to the image object $(\hat{G}, \mathrm{Ass}_{\hat{G}M}^{\hat{\omega}})$. The natural transformation Φ applies to every object in $\mathbf{MF}_{\hat{G}}$, specifically for the object $\hat{G}^{\mathrm{left}}$ describing the action of $\hat{G}$ on itself by left multiplication the natural transformation Φ provides a parallel homomorphism of fiber bundles

$$\Phi(\hat{G}^{\mathrm{left}}): \ GM \times_G \hat{G} \ \longrightarrow \ \hat{G}M \times_{\hat{G}} \hat{G} \,,$$

which we may use to define $\varphi: GM \longrightarrow \hat{G}M$ as the composition:

$$\begin{array}{ccccccccc}
\varphi: & GM & \longrightarrow & GM \times_G \hat{G} & \stackrel{\Phi(\hat{G}^{\mathrm{left}})}{\longrightarrow} & \hat{G}M \times_{\hat{G}} \hat{G} & \stackrel{\cong}{\longrightarrow} & \hat{G}M \\
& g & \longmapsto & [g, \hat{e}] & & [\hat{g}, \hat{\gamma}] & \longmapsto & \hat{g}\hat{\gamma} \,.
\end{array} \tag{21}$$

The argument we used in equation (18) to show that the right hand side isomorphism $\hat{G}M_{\hat{G}}\hat{G} \longrightarrow \hat{G}M$ is parallel implies that $GM \longrightarrow GM \times_G \hat{G}$ is parallel as well, in consequence $\varphi: GM \longrightarrow \hat{G}M$ is a parallel homomorphism of fiber bundles.

In order to show that φ is equivariant over the group homomorphism $\varphi_{\mathrm{grp}}: G \longrightarrow \hat{G}$ we use the characteristic property of natural transformations like Φ for the right multiplication morphism $\rho_{\hat{\gamma}}: \hat{G}^{\mathrm{left}} \longrightarrow \hat{G}^{\mathrm{left}}, \ \hat{g} \longmapsto \hat{g}\hat{\gamma}$:

$$\begin{CD}
GM \times_G \hat{G} @>{\Phi(\hat{G}^{\mathrm{left}})}>> \hat{G}M \times_{\hat{G}} \hat{G} \\
@V{(\mathrm{Ass}_{GM}^\omega \circ \varphi_{\mathrm{grp}}^*)(\rho_{\hat{\gamma}})}VV @VV{\mathrm{Ass}_{\hat{G}M}^{\hat{\omega}}(\rho_{\hat{\gamma}})}V \\
GM \times_G \hat{G} @>{\Phi(\hat{G}^{\mathrm{left}})}>> \hat{G}M \times_{\hat{G}} \hat{G}
\end{CD}$$

Of course the association functors $\mathrm{Ass}^{\omega}_{GM} \circ \varphi^*_{\mathrm{grp}}$ and $\mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}$ are explicitly specified on morphisms in Definition 4.2 and both vertical arrows turn out to be the right multiplication $[\,g,\,\hat{\mathrm{r}}\,] \longmapsto [\,g,\,\hat{\mathrm{r}}\hat{\gamma}\,]$ by $\hat{\gamma} \in \hat{G}$. In turn we find

$$\varphi(g\gamma) \;=\; \Phi(\hat{G}^{\mathrm{left}})\,[\,g\gamma,\hat{e}\,] \;=\; \Phi(\hat{G}^{\mathrm{left}})\,[\,g,\varphi_{\mathrm{grp}}(\gamma)\,] \;=\; \varphi(g)\,\varphi_{\mathrm{grp}}(\gamma)$$

for all $g \in GM$ and $\gamma \in G$ and conclude that φ is equivariant over φ_{grp}. Eventually we consider for an arbitrary object $\hat{\mathscr{F}} \in \mathbf{OBJ}\,\mathbf{MF}_{\hat{G}}$ the orbit map $\mathrm{orb}_{\hat{f}} : \hat{G}^{\mathrm{left}} \longrightarrow \hat{\mathscr{F}},\ \hat{\gamma} \longmapsto \hat{\gamma} \star \hat{f}$, associated to an element $\hat{f} \in \hat{\mathscr{F}}$ as a morphism in the category $\mathbf{MF}_{\hat{G}}$ with associated commutative diagram:

$$
\begin{array}{ccc}
GM \times_G \hat{G} & \xrightarrow{\ \Phi(\hat{G}^{\mathrm{left}})\ } & \hat{G}M \times_{\hat{G}} \hat{G} \\
{\scriptstyle (\mathrm{Ass}^{\omega}_{GM} \circ \varphi^*_{\mathrm{grp}})(\,\mathrm{orb}_{\hat{f}}\,)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Ass}^{\hat{\omega}}_{\hat{G}M}(\,\mathrm{orb}_{\hat{f}}\,)} \\
GM \times_G \hat{\mathscr{F}} & \xrightarrow[\ \Phi(\hat{\mathscr{F}})\ ]{} & \hat{G}M \times_{\hat{G}} \hat{\mathscr{F}}\,.
\end{array}
$$

Definition 4.2 provides again an explicit description of the two vertical arrows and the top arrow reads $[\,g,\,\hat{\gamma}\,] \longmapsto [\,\varphi(g),\,\hat{\gamma}\,]$, the commutativity of the diagram thus implies that $\Phi(\hat{\mathscr{F}})$ is given by $[\,g,\,\hat{f}\,] \longmapsto [\,\varphi(g),\,\hat{f}\,]$. In other words the two natural transforms Φ and Φ_φ agree on arbitrary objects and so the functor Ass is full, this is surjective on morphisms. In order to show that Ass is injective on morphisms or faithful the reader may simply verify that the equivariant map $GM \longrightarrow \hat{G}M$ defined in equation (21) equals φ in case we start with the natural transformation $\Phi = \Phi_\varphi$. ∎

Mutatis mutandis the arguments presented in this section work without taking connections into account: A functor $\mathfrak{F} : \mathbf{MF}_G \longrightarrow \mathbf{FB}_M$ is naturally isomorphic to the association functor Ass_{GM} for some principal bundle GM, if and only if $\mathfrak{F}$ preserves Cartesian products and agrees with the product functor $M\times : \mathbf{MF} \longrightarrow \mathbf{FB}_M$ on the full subcategory of trivial G–manifolds. Suitably defined categories of principal bundles and gauge theory sectors then turn the association functor into an equivalence of categories:

$$\mathrm{Ass} :\ \mathbf{PB}_M \xrightarrow{\ \simeq\ } \mathbf{GTS}_M, \qquad (\,G,\,GM\,) \longmapsto (\,G,\,\mathrm{Ass}_{GM}\,)\,.$$

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