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THE TOPOLOGY OF CRITICAL PROCESSES, I
(PROCESSES AND MODELS)

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Résumé. Cet article fait partie d'un sujet, la topologie algébrique dirigée, dont l'objectif général est d'inclure les processus non réversibles dans le domaine de la topologie générale et algébrique. Ici, comme une étape successive, nous voulons également couvrir les « processus critiques », c'est-à-dire indivisibles et inarrêtables.

Cette partie introductive est consacrée à la mise en place du nouveau cadre et à la représentation de processus issus de divers domaines, en faisant appel à des prérequis mathématiques minimaux. La catégorie fondamentale et la structure d'homotopie du cadre actuel seront traitées dans les prochains articles.

Abstract. This article belongs to a subject, Directed Algebraic Topology, whose general aim is including non-reversible processes in the range of topology and algebraic topology. Here, as a further step, we also want to cover 'critical processes', indivisible and unstoppable.

This introductory part is devoted to fixing the new framework and representing processes of diverse domains, with minimal mathematical prerequisites. The fundamental category and the homotopy structure of the present setting will be dealt with in a sequel.

Keywords. Directed algebraic topology, category theory for algebraic topology, hysteresis, hybrid systems.

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Introduction

0.1 Aims

Directed Algebraic Topology is a recent subject, dating from the 1990's. It is an extension of Algebraic Topology, dealing with 'spaces' – typically the *directed spaces* studied in [Gr1, Gr2] – where the paths need not be reversible, with the general aim of including the representation of *irreversible processes*.

We want to introduce a further extension, called *controlled spaces*, where the paths need not be decomposable, in order to include *critical processes*, indivisible and unstoppable, either reversible or not.

Taking into account transformations that cannot be stopped is an unfortunate aspect of our time. But there are plenty of normal events which cannot be stopped or decomposed in parts, like quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, deleting a file in a computer, the action of a siphon, the eruption of a geyser, an all-or-nothing transform in cryptography, moving in a section of an underground network, etc.

Critical processes and transport networks are often represented by graphs, in an effective way as far as they do not interact with continuous variation. We want to show that they can also be modelled by structured spaces, in a theory that includes classical topology and 'non-reversible spaces'.

Controlled spaces can thus unify aspects of continuous and discrete mathematics, interacting with sectors of Control Theory (see 0.6). The simple fact of classifying phenomena of diverse domains by mathematical models which live in the same world may have an interest: these models can be combined together, and studied with extensions of the usual tools of Algebraic Topology.

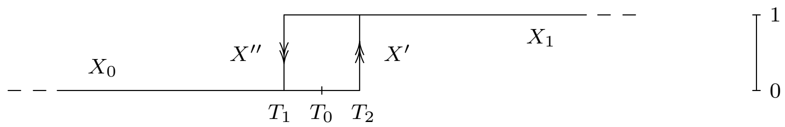
In this introductory part we fix the general framework, presenting many models and their concrete interpretations. The mathematical background is essentially restricted to elementary topology and basic category theory: limits and adjoint functors.

Part II of this series will introduce the fundamental category of controlled spaces, with suitable methods of calculation. Their homotopy theory and the breaking of the usual symmetries of topology will be dealt with in a sequel.

0.2 An example

On-off controllers are systems overseeing a certain variable. Their description, as in the usual figure below, combines classical topology, where the variable moves freely, and graph theory, where a change of state takes place. We want to model them in one framework – an enriched form of topology.

For concreteness, let us think of a cooling system, with a thermostat set at temperature T_0, and a tolerance interval $[T_1, T_2]$. In the following picture the horizontal axis measures the temperature, and the vertical axis denotes two states: at level 0 the cooling device is off, at level 1 it is on



On the left branch X_0 the system is in stand by; if the temperature reaches T_2 the cooling device goes on, jumping to state 1; from there, if the temperature cools to T_1, it goes back to state 0.

An elementary hysteresis process, or 'hysteron', behaves the same way: for instance, the change of state in a memory cell, or the change of orientation in an elementary domain of a ferromagnetic object.

We shall construct a model of this process, 'pasting' two *natural* intervals X_0, X_1 (with euclidean topology and nothing more) and two *one-jump* intervals X', X'' where the paths allowed have to jump the whole interval, in the marked direction: see 3.1(a). More complex models can be used for combined systems, like a heating and cooling device, in 3.1(c), or a system that regulates two variables, for instance temperature and pressure in an air-supported dome, in 3.2.

0.3 Directed and controlled spaces

Directed spaces, our main structure meant to cover irreversible processes, were introduced in [Gr1] and extensively studied in a book on Directed Algebraic Topology [Gr2]; they are frequently used in the theory of concurrent processes, see 0.6.

A *directed space* X, or *d-space*, is a topological space equipped with a set $X^\sharp$ of *directed paths* $[0, 1] \to X$, or *d-paths*, closed under: trivial loops, concatenation and partial increasing reparametrisation (including restrictions to subintervals). The selected paths, generally, cannot be travelled backwards but are *reflected* in the opposite d-space X^{op}.

A topological space has a *natural* structure of d-space, where all paths are selected. Directed Algebraic Topology is an extension of the classical case; in particular, the fundamental groupoid and the groups of singular homology are extended to directed versions: the fundamental category $\uparrow\Pi_1(X)$ and the preordered abelian groups $\uparrow H_n(X)$ of directed singular homology.

For all this we shall mainly refer to the book [Gr2]. The prefixes *d*- and $\uparrow$ are used to distinguish a directed notion from the corresponding 'reversible' one.

We now relax the axioms of d-spaces, to include critical processes: essentially, the selected paths are no longer required to be closed under restriction; they are called *controlled paths*, and the prefix *c*- is used to distinguish the new notions. This is still a directed setting, pertaining to Directed Algebraic Topology.

In this extension we gain models of phenomena which have no place in the previous setting, and interesting formal 'shapes', like the *one-jump interval* $\mathrm{c}\mathbb{I}$, the *one-stop circle* $\mathrm{c}\mathbb{S}^1$, the *n-stop circle* $\mathrm{c}_n\mathbb{S}^1$, or the higher controlled spheres and tori described in 2.3-2.6.

We also loose some good properties of the theory of d-spaces. The fundamental category and directed singular homology of d-spaces can be extended to c-spaces, but new methods of calculation will be needed: the van Kampen theorem and the Mayer-Vietoris sequence are both based on the subdivision of paths and homological chains, which is no longer permitted. Nevertheless, the fundamental category of 'rigid' c-spaces, as the previous ones, can be fairly simple to analyse, precisely because of the scarcity of allowed paths.

Essentially, the previous setting of d-spaces extends classical topology by breaking the symmetry of reversion: the allowed paths need no longer be reversible and the fundamental groupoid becomes a category. This further extension to c-spaces breaks a flexibility feature that d-spaces still retain: paths can no longer be subdivided, and this has drastic consequences.

0.4 The threshold effect

As another example, in the *threshold effect*, or *siphon effect*, the process is partially described by a variable v which can vary in a real interval $[v_0, v_1]$; when the variable reaches the highest value, the *threshold* v_1, it jumps down to the least value v_0, in a way that cannot be stopped within the process itself.

There are many examples of this effect in Particle Physics, Natural Sciences, Computer Science, Medicine, Economics, Sociology, etc.

Some cases are well-known:

- in Hydraulics: the empting of a basin through a siphon (see 3.3);

- in Biology: the onset of a nerve impulse (v is an electric potential);

- in Engineering: the combustion in a piston (v is the quantity of fuel);

- in Zoology and Sociology: mass migration (v is the rate of the population present in a region, with respect to the sustainable population).

The *anti-siphon effect* behaves in the opposite way: the threshold is at the lowest level v_0; reaching it, the variable goes up to the highest value. The management of stocks of a given article, in a store or at home, roughly follows this pattern.

Two models are proposed for the siphon effect, in 3.3.

0.5 An outline

In Section 1 we introduce the category cTop of controlled spaces and we recall the category dTop of directed spaces. We also examine the links among them and other domains: the categories Top of topological spaces and pTop of preordered spaces. Flexible and rigid paths, critical paths and critical points are dealt with in 1.6.

Section 2 begins with limits and colimits for c-spaces and d-spaces. Then we describe diverse c-structures on the interval $[0, 1]$, on the spheres, on the square $[0, 1]^2$, etc.

Finally, Section 3 explores less elementary processes and how they can be modelled: on-off controllers in 3.1 and 3.2; the threshold effect in 3.3; transport networks in 3.4.

0.6 Literature

The framework of d-spaces, its fundamental category and singular homology are used by various authors working in the theory of concurrency by methods of Directed Algebraic Topology.

This topic is covered in a recent book by L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram and M. Raussen [FjGHMR], and many articles among which [CaGM, FjR, Gb, GbM, MeR, Ra1, Ra2]. The present setting of c-spaces is closely related to the 'multipointed d-spaces' introduced in [Ga].

In a different perspective, there are various approaches to what we are calling critical processes, generally more concrete than the present one. A comprehensive study of hysteretic processes, in the form of operators turning an input function into an output function, can be found in the book [BrS].

A combined analysis of continuous behaviours (possibly ruled by differential equations) and 'jumps' between them (possibly controlled by a state machine) is also present in the theory of switched systems [Fl, Li], hybrid control systems [Bra], hybrid automata [He] and networked control systems [BeHJ]. The author is indebted to the Referee and M. Raussen for suggesting these links. Some work will be required to explore the relationship between these approaches and the present one: for instance, differential equations can be used to select smooth paths, which would then generate part of the c-structure.

Hopefully the controlled spaces proposed here might form a common ground for diverse more specific frameworks.

0.7 Notation

The symbol $\subset$ denotes weak inclusion. A continuous mapping between topological spaces, possibly structured, is called a *map*. Open and semiopen intervals of the real line are always denoted by square brackets, like $]0,1[$, $[0,1[$ etc. Marginal remarks are written in small characters.

A *preorder* relation, generally written as $x \prec y$, is assumed to be reflexive and transitive; an *order* relation, often written as $x \leqslant y$, is also assumed to be anti-symmetric. A mapping which preserves (resp. reverses) preorders is said to be *increasing* (resp. *decreasing*), always meant in the weak sense.

1. Spaces with selected paths

We introduce the category cTop of controlled spaces, or c-spaces, an extension of the category dTop of directed spaces studied in [Gr1, Gr2], and we examine the links between them. Both structures are based on topological spaces with 'selected paths' satisfying some axioms, more general for the new structure.

1.1 Spaces and preordered spaces

Top is the category of topological spaces and continuous mappings, or *maps*.

A *preordered topological space* is just a space equipped with a preorder relation $x \prec x'$ (reflexive and transitive), without assuming any relationship between these structures. They form the category pTop of preordered topological spaces, with the increasing (i.e. preorder preserving) continuous mappings.

A preordered topological space X is a 'directed notion', which can be reversed: the object X^{op} has the opposite preorder $x \prec^{\mathrm{op}} x'$ (defined by $x' \prec x$). This gives a (covariant) involutive endofunctor, called *reversor*

$$R\colon \mathrm{p}\mathsf{Top} \to \mathrm{p}\mathsf{Top}, \qquad RX = X^{\mathrm{op}}. \tag{1}$$

(The category Cat of small categories has a similar reversor.)

$\mathbb{R}$ will denote the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0,1]$. Similarly $\mathbb{R}^n$ and $\mathbb{I}^n$ are euclidean spaces. $\mathbb{S}^n$ is the n-dimensional sphere.

On the other hand, $\uparrow\mathbb{R}$ and $\uparrow\mathbb{I}$ are ordered topological spaces, with their natural (total) order; $\uparrow\mathbb{R}^n$ and $\uparrow\mathbb{I}^n$ are cartesian powers in pTop, with the product order: $(x_i) \leqslant (y_i)$ if and only if, for all i, $x_i \leqslant y_i$.

Homotopy theory in Top is parametrised on $\mathbb{I}$. In pTop it is parametrised on the ordered interval $\uparrow\mathbb{I}$, yielding an elementary form of directed homotopy (cf. [Gr2], 1.1.3-5).

1.2 The terminology of paths

In a topological space X, a (continuous) map $a\colon \mathbb{I} \to X$ is called a *path* in X, from $a(0)$ to $a(1)$ – its endpoints. It is a *loop* at x if $a(0) = x = a(1)$.

We begin by listing the (rather standard) terminology that we shall use for paths.

(a) *Concatenation.* The concatenation of paths will be written as $a' * a''$; the constant (or trivial, or degenerate) loop at the point x is written as e_x; the *reversed* path $t \mapsto a(1-t)$ as $a^\sharp$.

We recall that the (standard, or regular) concatenation $a = a' * a''$ of two consecutive paths a', a'' (with $a'(1) = a''(0)$) is defined as

$$a(t) = \begin{cases} a'(2t), & \text{for } 0 \leqslant t \leqslant 1/2, \\ a''(2t-1), & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases} \qquad (2)$$

As an important feature of topological spaces, called here the *path splitting property*, every path a has a unique decomposition $a = a' * a''$, with:

$$a'(t) = a(t/2), \qquad a''(t) = a((t+1)/2)) \qquad (t \in \mathbb{I}). \qquad (3)$$

The operation of concatenation is not associative, the constant loops do not behave as identities, and the reversed paths are not inverses – except in trivial cases (e.g. in discrete spaces). But this works up to homotopy with fixed endpoints, which allows us to define the fundamental groupoid $\Pi_1(X)$ of a space, and the fundamental group $\pi_1(X, x_0)$ of a pointed space.

(b) *Regular concatenation.* The *regular n-ary concatenation* $a = a_1 * ... * a_n$ of consecutive paths is based on the regular partition $0 < 1/n < 2/n < ... < 1$ of the standard interval, and is again uniquely determined (it is understood that $i = 1, ..., n$):

$$\begin{aligned} &a(t) = a_i(nt - i + 1), && \text{for } t \in [(i-1)/n,\, i/n], \\ &a_i(t) = a((t + i - 1)/n), && \text{for } t \in \mathbb{I}. \end{aligned} \tag{4}$$

(c) *General concatenation.* More generally, $a = C((a_i), (t_i))$ will denote a *general concatenation* of n consecutive paths $a_1, ..., a_n$, based on an arbitrary partition $0 = t_0 < t_1 < ... < t_n = 1$ of $\mathbb{I}$

$$\begin{aligned} &a(t) = a_i((t - t_{i-1})/\tau_i), && \text{for } t \in [t_{i-1}, t_i], \\ &a_i(t) = a(\tau_i t + t_{i-1}), && \text{for } t \in \mathbb{I} \qquad (\tau_i = t_i - t_{i-1}). \end{aligned} \tag{5}$$

(d) *Reparametrisation.* We are interested in reparametrising the path a as $a\rho\colon \mathbb{I} \to X$, where the *reparametrisation* $\rho\colon \mathbb{I} \to \mathbb{I}$ is any increasing map. We speak of a *global reparametrisation* if ρ is surjective, that is $\rho(0) = 0$ and $\rho(1) = 1$. We speak of an *invertible reparametrisation* if ρ is an increasing homeomorphism, or equivalently an automorphism $\mathbb{I} \to \mathbb{I}$ of ordered sets (or of ordered topological spaces).

Plainly, all n-ary concatenations are equivalent, up to invertible reparametrisation.

Of course, a non-surjective reparametrisation 'restricts' a path: for instance, if $\rho(t) = t/2$ (as in formula (3)), the path $a\rho$ covers the first half of a; let us note that it is still parametrised on $\mathbb{I}$. More drastically, if ρ is constant $a\rho$ is a constant loop.

A *restriction* will be an affine, non-degenerate (i.e. non-constant), increasing map:

$$\rho\colon \mathbb{I} \to \mathbb{I} \qquad \rho(t) = (t_2 - t_1)t + t_1 \qquad (0 \leqslant t_1 < t_2 \leqslant 1). \tag{6}$$

By the usual pleonastic terminology, a reparametrisation will also be called a *partial reparametrisation* when we want to stress that it is not assumed to be global (although it might be).

1.3 Main definitions, I

A *controlled space* X, or *c-space*, will be a topological space equipped with a set $X^\sharp$ of (continuous) maps $a\colon \mathbb{I} \to X$, called *controlled paths*, or *c-paths*, that satisfies three axioms:

(csp.0) (*constant paths*) the trivial loops at the endpoints of a controlled path are controlled,

(csp.1) (*concatenation*) the controlled paths are closed under path concatenation: if the consecutive paths a, b are controlled, their concatenation $a * b$ is also,

(csp.2) (*global reparametrisation*) the controlled paths are closed under pre-composition with every surjective increasing map $\rho\colon \mathbb{I} \to \mathbb{I}$: if a is a controlled path, $a\rho$ is also.

As a consequence, the c-paths are also closed under general concatenation. The underlying topological space is written as $U(X)$, or $|X|$, and called the *support* of X.

A *map* of c-spaces $f\colon X \to Y$, or *c-map*, is a continuous mapping between c-spaces which preserves the selected paths. Their category will be written as $\mathrm{c}\mathsf{Top}$.

A c-space X is a directed notion. Reversing c-paths, by the involution $r(t) = 1 - t$, yields the *opposite* c-space $RX = X^{\mathrm{op}}$, where $a \in (X^{\mathrm{op}})^\sharp$ if and only if ar belongs to $X^\sharp$. This defines the *reversor* endofunctor

$$R\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad RX = X^{\mathrm{op}}. \tag{7}$$

A c-path a of X is *reversible* if ar is also controlled. The c-space itself is *reversible* if $X = X^{\mathrm{op}}$, that is if all its c-paths are reversible. More generally, it is *reversive* if it is isomorphic to X^{op}.

1.4 Main definitions, II

Controlled spaces extend a structure introduced in [Gr1], also studied in [Gr2] and elsewhere (see 0.6).

A *directed space* X, or *d-space*, is equipped with a set $X^\sharp$ of maps $a\colon \mathbb{I} \to X$, called *directed paths*, or *d-paths*, that satisfies three axioms:

(dsp.0) (*constant paths*) every trivial loop is directed,

(dsp.1) (*concatenation*) if the consecutive paths a, b are directed, their concatenation $a * b$ is also,

(dsp.2) (*partial reparametrisation*) if $\rho\colon \mathbb{I} \to \mathbb{I}$ is an increasing map and a is a directed path, $a\rho$ is also.

The second axiom is the same of c-spaces (up to terminology), the others are stronger; every d-space is a c-space, and the notation $X^\sharp$ for the set of selected paths is consistent. (In [Gr1, Gr2] this set is written as dX, a notation which has no good extension here.)

A *map* of d-spaces, or *directed map*, or *d-map*, is a continuous mapping which preserves the directed paths. Their category dTop is a full subcategory of cTop. The reversor endofunctor works in the same way.

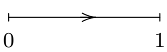
A 'multipointed d-space', introduced by P. Gaucher [Ga] in 2009, is more general: (dsp.0) is not assumed and (dsp.2) is only required for invertible reparametrisations. It also generalises a c-space.

1.5 Standard intervals

The difference between these settings shows clearly in two structures of the euclidean interval $[0, 1]$.

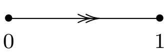
(a) In dTop the *standard d-interval* $\uparrow\!\mathbb{I}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{I}$. It plays the role of the standard interval in the category dTop, because the directed paths of any d-space X coincide with the d-maps $\uparrow\!\mathbb{I} \to X$.

It may be viewed as the essential model of a non-reversible process, or a one-way route in transport networks. It will be represented as



(8)

(b) In cTop the *standard c-interval* c$\mathbb{I}$, or *one-jump interval*, has the same support, with controlled paths the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at 0 or 1. The controlled paths of any c-space X coincide with the c-maps c$\mathbb{I} \to X$. It models a *non-reversible unstoppable process*, or a *one-way no-stop route*



(9)

1.6 Flexible paths and critical points

(a) *Flexible paths.* In a c-space X, a point x will be said to be *flexible* if its trivial loop e_x is controlled; the *flexible support* $|X|_0$ is the subspace of these points. In a diagram, an isolated flexible point will be marked by a bullet, as in figure (9) above.

We say that a controlled path a is *splittable* if its halves a', a'' (cf. (3)) are also controlled, so that the decomposition $a = a' * a''$ stays within c-paths; we say that a is *flexible* if all its restrictions are controlled (see (6)), or equivalently all its decompositions in general concatenations give raise to c-paths. Each controlled trivial loop is flexible. A c-map preserves all these properties.

The c-space itself is *flexible* if every point and every c-path is flexible. A c-space is a d-space if and only if it is flexible, if and only if every trivial loop is controlled and all its controlled paths are splittable.

A c-path a is *rigid* if in each general concatenation of a by controlled paths, precisely one of them is not constant. A c-space is *rigid* if every non-trivial path is a general concatenation of rigid paths. The interval $\mathrm{c}\mathbb{I}$ is rigid, as well as many c-spaces introduced in the next section.

(b) *Critical paths and critical points.* In a c-space X, a controlled path is *critical* if it is not flexible.

A point x is:

- *critical*, if every non-trivial c-path a through x (i.e. $x \in \mathrm{Im}\, a$) is critical, and there is some,

- *future critical*, if every non-trivial c-path starting there is critical, and there is some,

- *past critical*, if every non-trivial c-path arriving there is critical, and there is some.

A future or past critical point x is always flexible, a critical point need not. A d-space has no critical points.

In the interval $\mathrm{c}\mathbb{I}$ all points are critical, the point 0 is also future critical, while 1 is also past critical. There are c-spaces where these three kinds are disjoint: see 2.3(e).

1.7 Reshaping and generated structures

The c-structures on a topological space X are closed under arbitrary intersection (as subsets of $\mathsf{Top}(\mathbb{I}, X)$), and form a complete lattice for the inclusion: we say that the structure X_1 is *finer* than X_2 if $X_1^\sharp \subset X_2^\sharp$, or equivalently if the identity map of X gives a map $X_1 \to X_2$; this map is called a *reshaping*.

(a) Every set S of paths in the space X generates a c-structure, the finest, or smallest, containing it. It is obtained by adding all the constant loops at the endpoints of the paths of S, and stabilising the latter under global reparametrisation and general concatenation.

(b) Similarly, the d-structures on a topological space X form a complete lattice. Every set of paths of X generates a d-structure.

(c) If we start from a c-space X, the d-structure generated by the c-paths can be obtained stabilising them under constant paths, restriction and general concatenation.

(d) The forgetful functor $U\colon \mathrm{d}\mathsf{Top} \to \mathsf{Top}$ takes a d-space to its support, the underlying topological space $|X|$. It has a left and a right adjoint

$$U\colon \mathrm{d}\mathsf{Top} \leftrightarrows \mathsf{Top} : D, D' \qquad\qquad D \dashv U \dashv D'. \tag{10}$$

For a topological space T, the d-space DT is the same space with the *discrete d-structure* (the finest, or smallest), with directed paths all the trivial loops. $D'T$ has the *indiscrete d-structure* (the largest, or coarsest), where all paths are directed.

(e) The category $\mathrm{c}\mathsf{Top}$ has two forgetful functors to topological spaces

$$U\colon \mathrm{c}\mathsf{Top} \to \mathsf{Top}, \qquad\qquad U_0\colon \mathrm{c}\mathsf{Top} \to \mathsf{Top}, \tag{11}$$

where $U(X) = |X|$ is the topological support and $U_0(X) = |X|_0$ is the flexible support. U has both adjoints, U_0 has only the left one

$$D_c \dashv U \dashv D', \qquad\qquad D \dashv U_0. \tag{12}$$

For a topological space T, the c-space D_cT is the same space with the *discrete c-structure*: no path is controlled. $D'T$ has the *indiscrete* *c-structure*, where all paths are controlled. In DT all trivial loops are controlled. The functors D and D' take values in dTop, and are denoted as previously.

A topological space will be viewed as a c-space (and a d-space) by its *natural* structure $D'T$, so that all its paths are selected.

(f) The singleton has two structures in cTop: the *c-discrete* singleton $D_c\{*\}$ and the *flexible singleton* $\{*\}$ $(= D\{*\} = D'\{*\})$, having a controlled loop e_*; the flexible singleton is by far more important, as it is the terminal object and the unit of the cartesian product (see 2.1).

A c-map $x\colon D_c\{*\} \to X$ is 'the same' as a point of X, while a c-map $x\colon \{*\} \to X$ is a flexible point. In other words, $D_c\{*\}$ represents the functor $U\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}$, while $\{*\}$ represents $U_0\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}$.

(g) All the c-spaces DT are trivially flexible and rigid.

1.8 Comparing directed structures

We are considering three ways of enriching topological spaces by a directed structure (in a general sense), encoded in the categories pTop, dTop and cTop. We now examine their interplay.

A preordered topological space X (in the sense recalled in 1.1) will always be viewed as a d-space (and a c-space) by selecting the increasing (continuous) paths $\uparrow\mathbb{I} \to X$, where $\uparrow\mathbb{I}$ denotes the ordered euclidean interval $[0, 1]$.

This defines a functor $\mathsf{d}\colon \mathrm{p}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$, and our categories are linked by three obvious functors

$$\mathsf{d}\colon \mathrm{p}\mathsf{Top} \to \mathrm{d}\mathsf{Top}, \qquad \mathrm{d}\mathsf{Top} \subset \mathrm{c}\mathsf{Top}, \qquad \mathsf{d}\colon \mathrm{p}\mathsf{Top} \to \mathrm{c}\mathsf{Top}. \tag{13}$$

Let us note that d *is not an embedding*: trivially, all preorders on a discrete topological space give the same selected paths, namely the trivial loops. (One can find more interesting examples in [Gr2], 1.4.5.)

(a) There is an adjunction

$$\mathsf{d}\colon \mathrm{p}\mathsf{Top} \leftrightarrows \mathrm{d}\mathsf{Top} : \mathsf{p}, \qquad \mathsf{p} \dashv \mathsf{d}, \tag{14}$$

where the left adjoint p provides a d-space with the *path-preorder* $x \preceq x'$, meaning that there exists a d-path from x to x'. The counit on a preordered space X is the preorder-reshaping $\mathsf{pd}X \to X$: if $x \preceq x'$ there exists a d-path from x to x' in $\mathsf{d}X$, whence $x \prec x'$ in X.

Both functors p, d are faithful. A d-space is said to be *of (pre)order type* if it can be obtained, as above, from a topological space with such a structure. Thus $\uparrow\mathbb{R}^n$ and $\uparrow\mathbb{I}^n$ are of order type; $\mathbb{R}^n$, $\mathbb{I}^n$ and $\mathbb{S}^n$ are of chaotic-preorder type. The directed sphere $\uparrow\mathbb{S}^n$ described in 2.5 is not of preorder type (for $n > 0$).

(b) The embedding $\mathrm{d}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ has a left and a right adjoint:

$$\begin{aligned} \hat{}\colon \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top} \qquad & \text{(the \textit{reflector})}, \\ \mathsf{Fl} : \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top} \qquad & \text{(the \textit{coreflector})}. \end{aligned} \tag{15}$$

For a c-space X, the *generated d-space* $\hat{X}$ has the same underlying topological space with the d-structure generated by the c-paths. The unit of the adjunction is the reshaping $X \to \hat{X}$, the counit is the identity $\hat{Y} = Y$ for a d-space Y.

In the second construction the *flexible part* $\mathsf{Fl}\,X$ is the flexible support $|X|_0$ with the d-structure of the flexible c-paths. The counit is the inclusion $\mathsf{Fl}\,X \to X$, the unit is the identity $Y = \mathsf{Fl}\,Y$ for a d-space Y.

The full subcategory of reversible c-spaces has a similar reflector and coreflector: the *generated reversible c-space* and the *reversible part*.

(c) Composing the adjunction (14) with the previous reflection

$$\mathrm{p}\mathsf{Top} \underset{\mathsf{d}}{\overset{\mathsf{p}}{\rightleftarrows}} \mathrm{d}\mathsf{Top} \underset{\subset}{\overset{\hat{}}{\rightleftarrows}} \mathrm{c}\mathsf{Top} \qquad\qquad \mathrm{p}\mathsf{Top} \underset{\mathsf{d}}{\overset{\hat{\mathsf{p}}}{\rightleftarrows}} \mathrm{c}\mathsf{Top} \tag{16}$$

we get the adjunction $\hat{\mathsf{p}} \dashv \mathsf{d}$, where d still equips a preordered space X with the increasing maps $\uparrow\mathbb{I} \to X$ as c-paths (producing a d-space), while $\hat{\mathsf{p}}(X) = \mathsf{p}(\hat{X})$ provides a c-space with the *generated-path preorder* $x \preceq x'$, depending on the d-paths of $\hat{X}$. (The c-paths of X give a preorder on the flexible support $|X|_0$, not used here.)

2. Limits, colimits and structural models

Limits and colimits, for c-spaces and d-spaces, are easily obtained as topological limits and colimits with the initial or terminal structure determined by the structural maps.

Then we describe diverse c-structures on the interval, the spheres and the square; they can represent elementary events and will be used as bricks to form models of more complex processes.

2.1 Limits and colimits

We already remarked that the c-structures on a topological space T form a complete lattice. Therefore every family of maps $f_i\colon T \to X_i$ with values in c-spaces defines an initial c-structure on the space T: a path a is controlled if and only if all composites $f_i a$ are. Dually, every family of maps $f_i\colon X_i \to T$ defined on c-spaces gives raise to a final c-structure on the space T: the controlled paths in T are generated by all the paths $f_i a$, where $a \in X_i^\sharp$ for some index i.

A (controlled) *subspace* $X' \subset X$ of a c-space X has the initial structure of the embedding, which selects those paths in X' that are controlled in X. A (controlled) *quotient* X/R has the quotient structure, that is the final one for the projection $p\colon X \to X/R$; it is generated by the projected c-paths through general concatenation (see 1.7(a)).

The category cTop has all limits and colimits, constructed as in Top and equipped with the initial or final c-structure for the structural maps. For instance a path $\mathbb{I} \to \prod_i X_i$ with values in a product of c-spaces is controlled if and only if all its components $\mathbb{I} \to X_i$ are, while a path $\mathbb{I} \to \sum_i X_i$ with values in a sum is controlled if and only if it is in some summand X_i. Equalisers and coequalisers are realised as subspaces or quotients, in the sense described above.

We already described the terminal $\{*\}$, which is the unit of the cartesian product. On the other hand, $X \times D_c\{*\}$ is the discrete c-structure $D_c|X|$ on the underlying space.

If X is a c-space and $A \subset |X|$ is a *non-empty* subset, X/A will denote the c-quotient of X which identifies all points of A.

All this works in the same way in pTop and dTop. The embedding d$\mathsf{Top} \subset$ cTop preserves all limits and colimits, as it has both adjoints (see (15)). On the other hand, the canonical functors $\mathsf{d}\colon$ p$\mathsf{Top} \to$ dTop and $\mathsf{d}\colon$ p$\mathsf{Top} \to$ cTop of (13) preserve limits (as right adjoints) and sums (obviously), but do not preserve coequalisers.

In fact, in pTop the coequaliser of the endpoints $\{*\} \rightrightarrows {\uparrow}\mathbb{I}$ is the circle $\mathbb{S}^1$ with the indiscrete preorder. In dTop (and cTop) we get a non-trivial d-structure, the directed circle $\uparrow\mathbb{S}^1$, described below in 2.5(a). Essentially, this is 'why' directed homotopy is simple but very elementary in pTop.

(The standard c-circle c$\mathbb{S}^1$, described in 2.6(a), is the coequaliser of the endpoints in c$\mathbb{I}$.)

2.2 Controlled actions

Let G be a group, in additive notation (although not necessarily commutative). A *controlled G-space* is a c-space X equipped with a (right) *action* of G: this is an action on the underlying topological space such that, for each $g \in G$, the induced map

$$X \to X, \qquad\qquad x \mapsto x + g, \tag{17}$$

is a map of c-spaces (and therefore an isomorphism of cTop). Directed G-spaces are a particular case.

The c-space *of orbits* X/G is the quotient c-space, modulo the equivalence relation which collapses each orbit to a point. Its c-paths are simply the projections of the directed paths of X, as verified below. The same holds for d-spaces.

We have to prove that these projections are closed under global (resp. partial) reparametrisation and binary concatenation. The first fact is obvious. As to the second, let $a, b\colon \mathbb{I} \to X$ be two selected paths whose projections are consecutive in X/G: there is some $g \in G$ such that $a(1) = b(0) + g$. Then the path $b'(t) = b(t) + g$ is selected in X, and $a * b'$ is also. Finally, writing as $p\colon X \to X/G$ the canonical projection, $pa * pb = p(a * b')$ is the projection of a selected path.

2.3 Elementary models

(a) The euclidean interval $\mathbb{I}$ and the euclidean line $\mathbb{R}$ have the natural d-structure, where all paths are selected. The same holds for their cartesian powers $\mathbb{I}^n$ and $\mathbb{R}^n$, and for all spheres $\mathbb{S}^n$. $\mathbb{I}$ will be called the *natural* interval.

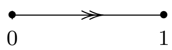
(b) The ordered euclidean interval $\uparrow\mathbb{I}$ and the ordered euclidean line $\uparrow\mathbb{R}$ have the d-structure given by the increasing paths (already recalled for the former). The same holds for their cartesian powers $\uparrow\mathbb{I}^n$ and $\uparrow\mathbb{R}^n$. They are not reversible (for $n > 0$), yet reversive, i.e. isomorphic to the opposite structure.

$\uparrow\mathbb{I}$ is the standard ordered interval, and also the standard d-interval, as already said. Its d-structure is generated by the identity map $\mathbb{I} \to \mathbb{I}$; a d-map $\uparrow\mathbb{I} \to X$ is the same as a directed path of X.

But $\uparrow\mathbb{I}$ will also be important in cTop, as the *flexible interval*. Indeed, for a c-space X, the c-maps $\uparrow\mathbb{I} \to X$ are the flexible paths of X.

(c) We already introduced the standard controlled interval $\mathrm{c}\mathbb{I}$, with the c-structure generated by the identity map $\mathbb{I} \to \mathbb{I}$: the c-paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at the endpoints. The c-maps $\mathrm{c}\mathbb{I} \to X$ are the selected paths of the c-space X (possibly a d-space).

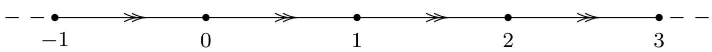
The c-space $\mathrm{c}\mathbb{I}$ will also be called the *quantum interval*, or the *one-jump interval*



$\mathrm{c}\mathbb{I}$ (18)

The generated d-space is $(\mathrm{c}\mathbb{I})\hat{} = \uparrow\mathbb{I}$, while the flexible part $\mathsf{Fl}\,(\mathrm{c}\mathbb{I}) = D\{0, 1\}$ is the discrete boundary $\partial\mathbb{I}$ of the interval, with its trivial loops.

(d) The *line with integral stops* $\mathrm{c}\mathbb{R}$, or *integral jumps*, is equipped with the c-structure generated by the family of embeddings $\mathbb{I} \to \mathbb{R}$, $t \mapsto t + k$ ($k \in \mathbb{Z}$). Now the c-paths are the increasing maps $\mathbb{I} \to \mathbb{R}$ whose image is precisely an interval $[k, k']$ with integral endpoints (possibly the same)



(19)

The line $\mathrm{c}\mathbb{R}$ is a controlled $\mathbb{Z}$-space, with respect to the action of the group $\mathbb{Z}$ by translations. The interval $\mathrm{c}\mathbb{I}$ is a subspace of $\mathrm{c}\mathbb{R}$, and the latter is the controlled $\mathbb{Z}$-space generated by the embedding of $\mathrm{c}\mathbb{I}$.

The line $\mathrm{c}\mathbb{R}$ is a rigid c-space (see 1.6): the rigid paths are those of length 1, and every non-trivial c-path is a concatenation of them, on a suitable partition. All points of $\mathrm{c}\mathbb{R}$ are critical; the integral numbers are also past and future critical. The generated d-space $(\mathrm{c}\mathbb{R})\hat{} = {\uparrow}\mathbb{R}$ is of order type; the flexible part $\mathsf{Fl}\,(\mathrm{c}\mathbb{I}) = D\mathbb{Z}$ is the discrete integral line with the discrete d-structure.

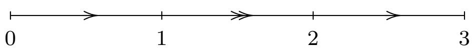
(e) Let X be the euclidean ordered interval $[0, 3]$, with controlled paths given by the increasing maps $\mathbb{I} \to X$ whose image either contains the open subinterval $]1, 2[$ or does not meet it.

Loosely speaking, we are modelling a process measured on the interval $[0, 3]$, which

- can only proceed 'forward',

- passing point 1, is obliged to go on to point 2, at least,

or a one-way route with a no-stop section, or a stream with rapids



(20)

The point 1 is future critical; all the points of $]1, 2[$ are not flexible and critical; 2 is past critical. The generated d-space $\hat{X} = {\uparrow}[0, 3]$ is the ordered structure, while $\mathsf{Fl}\,X$ is the ordered structure on the flexible support $[0, 1] \cup [2, 3]$.

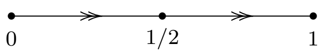
2.4 Other structures on the interval

We have already seen three c-structures on the euclidean interval $[0, 1]$: the natural structure $\mathbb{I}$, where all paths are controlled; the ordered structure ${\uparrow}\mathbb{I}$, with the increasing paths; the one-jump structure $\mathrm{c}\mathbb{I}$, with the surjective increasing paths and the trivial loops at the endpoints.

There are many others, that can be used as bricks of modelisation. We list here some of them; two 'siphon structures' can be found in 3.3.

(a) The *two-jump interval* $\mathrm{c}\mathbb{J}$ has the c-structure generated by the restrictions to the first or second half

$$c^-(t) = t/2, \qquad c^+(t) = (t+1)/2 \qquad (t \in \mathbb{I}), \tag{21}$$



The non-trivial c-paths are the increasing maps $\mathbb{I} \to \mathbb{I}$ whose image is either $[0, 1/2]$, or $[1/2, 1]$, or $[0, 1]$. This c-space is isomorphic to the subspace $\mathrm{c}[0, 2] \subset \mathrm{c}\mathbb{R}$, and can model a *non-reversible two-stage process*. Formally, it parametrises the concatenation of two c-paths (see Part II).

(b) *Delayed intervals*. The *past-delayed c-interval* $\mathrm{c}_{-}\mathbb{I}$ will be the euclidean interval $[0, 1]$ with the c-structure generated by the past-delayed reparametrisation $\rho \colon \mathbb{I} \to \mathbb{I}$

$$\rho(t) = 0 \vee (2t - 1), \qquad\qquad \sigma(t) = 2t \wedge 1, \tag{22}$$

while the *future-delayed c-interval* $\mathrm{c}_{+}\mathbb{I}$ has the c-structure generated by the future-delayed reparametrisation $\sigma \colon \mathbb{I} \to \mathbb{I}$.

In $\mathrm{c}_{-}\mathbb{I}$ the non-trivial controlled paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ which are constant on some non-degenerate interval $[0, t_1]$. For a c-space X, a c-map $\mathrm{c}_{-}\mathbb{I} \to X$ is the same as a *past-delayed c-path* (constant as above).

These c-spaces are not reversive, but anti-isomorphic to each other, by reversion

$$r \colon \mathrm{c}_{-}\mathbb{I} \to (\mathrm{c}_{+}\mathbb{I})^{\mathrm{op}}, \qquad\qquad r(t) = 1 - t. \tag{23}$$

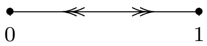
Their structures are rigid and finer than $\mathrm{c}\mathbb{I}$, because their generators are surjective increasing maps, i.e. controlled paths in $\mathrm{c}\mathbb{I}$.

There are many delayed structures on the interval. They can model *irreversible non-stoppable processes with inertia*, or *inductance*.

(c) The *reversible d-interval* $\mathbb{I}^{\sim}$ is the d-space generated by identity and reversion $\mathrm{id}, r \colon \mathbb{I} \to \mathbb{I}$; its directed paths are the piecewise monotone maps $\mathbb{I} \to \mathbb{I}$. The reversible directed paths of a d-space X coincide with the d-maps $\mathbb{I}^{\sim} \to X$. $\mathbb{I}^{\sim}$ is strictly finer than the natural interval $\mathbb{I}$; it can model a shock absorber.

(d) The *one-jump reversible interval* $\mathrm{c}\mathbb{I}^{\sim}$ has the c-structure generated by identity and reversion $\mathrm{id}, r \colon \mathbb{I} \to \mathbb{I}$. The reversible c-paths of a c-space X coincide with the c-maps $\mathrm{c}\mathbb{I}^{\sim} \to X$.

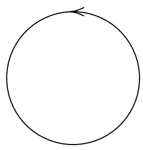
Every c-path in $\mathrm{c}\mathbb{I}^{\sim}$ has an integral length and those of length 1 are rigid; $\mathrm{c}\mathbb{I}^{\sim}$ is rigid and the generated d-space is $\mathbb{I}^{\sim}$. The interval $\mathrm{c}\mathbb{I}^{\sim}$ can (basically) model a *reversible non-stoppable process*, like the change of state in a memory cell, or a *two-way no-stop route*, or the flights of an airplane between two airports



$$\mathrm{c}\mathbb{I}^{\sim} \tag{24}$$

2.5 Directed spheres and tori

(a) The *standard d-circle* $\uparrow\mathbb{S}^1$ is the standard circle with the *anticlockwise structure*, where the directed paths $a\colon \mathbb{I} \to \mathbb{S}^1$ move this way, in the oriented plane $\mathbb{R}^2$: $a(t) = (\cos\vartheta(t), \sin\vartheta(t))$, with an increasing continuous argument $\vartheta\colon \mathbb{I} \to \mathbb{R}$



$$\uparrow\mathbb{S}^1 \tag{25}$$

The directed circle can be described as an orbit space

$$\uparrow\mathbb{S}^1 = (\uparrow\mathbb{R})/\mathbb{Z}, \tag{26}$$

with respect to the action of the group of integers on the directed line $\uparrow\mathbb{R}$, by translations: the directed paths of $\uparrow\mathbb{S}^1$ are simply the projections of the increasing paths in the line.

The c-space $\uparrow\mathbb{S}^1$ can also be obtained as the coequaliser in dTop of the following pair of maps

$$\partial^-, \partial^+\colon \{*\} \rightrightarrows \uparrow\mathbb{I}, \qquad \partial^-(*) = 0, \qquad \partial^+(*) = 1. \tag{27}$$

Indeed, this coequaliser is the quotient $\uparrow\mathbb{I}/\partial\mathbb{I}$, which identifies the endpoints; the d-structure of the quotient, generated by the projected paths, is what we want: it is sufficient to concatenate a finite number of projected paths, which are already stable under partial reparametrisation.

(b) The *standard directed n-sphere* is defined, for $n > 0$, as the quotient of the directed cube $\uparrow\mathbb{I}^n$ modulo its (ordinary) boundary $\partial\mathbb{I}^n$

$$\uparrow\mathbb{S}^n = (\uparrow\mathbb{I}^n)/(\partial\mathbb{I}^n), \qquad \uparrow\mathbb{S}^0 = \mathbb{S}^0 = \{-1, 1\} \qquad (n > 0), \tag{28}$$

while $\uparrow\mathbb{S}^0$ has the discrete topology and the natural d-structure (obviously discrete).

All directed spheres are reversive; their d-structure can be described by an asymmetric distance (see [Gr2], 6.1.5). The pointed suspension of $\mathbb{S}^0$ in the category of pointed d-spaces gives $\uparrow\mathbb{S}^1$ and, by iteration, all higher $\uparrow\mathbb{S}^n$ ([Gr2], 1.7.4, 1.7.5). The unpointed suspension gives different d-spaces, less interesting.

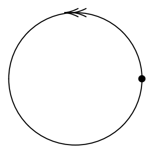
(c) The *standard directed n-torus* is a cartesian power of $\uparrow\mathbb{S}^1$

$$\uparrow\mathbb{T}^n = (\uparrow\mathbb{S}^1)^n. \qquad (29)$$

Equivalently, it is the orbit d-space $(\uparrow\mathbb{R}^n)/\mathbb{Z}^n$, for the action of the additive group $\mathbb{Z}^n$ by translations.

2.6 Controlled spheres and tori

(a) The *standard c-circle* $\mathrm{c}\mathbb{S}^1$, or *one-stop circle*, is now defined as the orbit c-space of the line $\mathrm{c}\mathbb{R}$ for the action of the group $\mathbb{Z}$, by translations



$$\mathrm{c}\mathbb{S}^1 = (\mathrm{c}\mathbb{R})/\mathbb{Z} \qquad (30)$$

The controlled paths of $\mathrm{c}\mathbb{S}^1$ are the projections of the controlled paths in the line: here this means an anticlockwise path (as in 2.5(a)) which is a loop at $[0]$, the only flexible point (corresponding to $(1,0)$ in the plane). The simple loops are rigid, and so is $\mathrm{c}\mathbb{S}^1$.

The circle $\mathrm{c}\mathbb{S}^1$ can also be obtained as the coequaliser in $\mathrm{c}\mathsf{Top}$ of the endpoints of $\mathrm{c}\mathbb{I}$

$$\partial^-, \partial^+ \colon \{*\} \rightrightarrows \mathrm{c}\mathbb{I}, \qquad \partial^-(*) = 0, \qquad \partial^+(*) = 1. \qquad (31)$$

All points are critical; the flexible point is also past and future critical. The generated d-space is $(\mathrm{c}\mathbb{S}^1)\hat{} = \uparrow\mathbb{S}^1$, while $\mathsf{Fl}\,(\mathrm{c}\mathbb{S}^1)$ is the flexible point with its trivial loop.

(b) More generally, the *n-stop c-circle* $\mathrm{c}_n\mathbb{S}^1$ $(n > 0)$ is the orbit space

$$\mathrm{c}_n\mathbb{S}^1 = (\mathrm{c}_n\mathbb{R})/\mathbb{Z} \qquad (\mathrm{c}_1\mathbb{S}^1 = \mathrm{c}\mathbb{S}^1), \qquad (32)$$

where the c-paths of $\mathrm{c}_n\mathbb{R}$ are the increasing paths whose image is an interval $[k/n, k'/n]$, for integers $k \leqslant k'$.

In $\mathrm{c}_n\mathbb{S}^1$ a c-path is an anticlockwise path between two points $[k/n]$ and $[k'/n]$ of the circle. The 'minimal generators' have length $1/n$ of the circle and are rigid; the c-space itself is also.

Rotating motions can follow this pattern, with mandatory direction and stops: for instance, the second hand of a watch, a washing machine dial, a panoramic wheel with n cabins. The mode dial of a photocamera and a railway turntable can be modelled by the reversible c-space generated by $\mathrm{c}_n\mathbb{S}^1$.

(c) The *standard c-sphere* $\mathrm{c}\mathbb{S}^n$ is defined as a quotient of $\mathrm{c}\mathbb{I}^n$ (for $n > 0$)

$$\mathrm{c}\mathbb{S}^n = (\mathrm{c}\mathbb{I}^n)/(\partial\mathbb{I}^n), \qquad \mathrm{c}\mathbb{S}^0 = \mathbb{S}^0 = \{-1, 1\} \qquad (n > 0), \tag{33}$$

and will be examined in Part III.

(d) The *standard controlled n-torus* is a cartesian power of $\mathrm{c}\mathbb{S}^1$

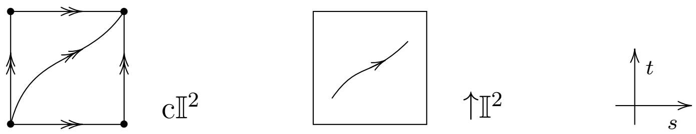
$$\mathrm{c}\mathbb{T}^n = (\mathrm{c}\mathbb{S}^1)^n, \tag{34}$$

and can also be obtained as the orbit c-space $(\mathrm{c}\mathbb{R}^n)/\mathbb{Z}^n$.

2.7 Controlled squares and cubes

(a) We have already seen the ordered square $\uparrow\!\mathbb{I}^2$, also called the *flexible square* when viewed in cTop.

(b) The *standard c-square* $\mathrm{c}\mathbb{I}^2$, represented in the left figure below, has the structure of a cartesian power: a path $\mathbb{I} \to \mathbb{I}^2$ is controlled if and only if it is increasing and each of its projections covers $[0, 1]$, or is constant at 0 or 1



(35)

There are four flexible points, the vertices of the square. The c-paths of $\mathrm{c}\mathbb{I}^2$ have five kinds of generators: two horizontal paths $s \mapsto (s, \alpha)$ (for

$\alpha = 0, 1)$, two vertical paths $t \mapsto (\alpha, t)$, and all increasing paths from $(0, 0)$ to $(1, 1)$ in the ordered square, as exemplified in the left picture above.

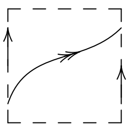
There is no finite set of generators, but we shall see that the fundamental category has only five non-trivial arrows (and four identities at the flexible points).

The space is rigid. The generated d-space is the ordered square $\uparrow\mathbb{I}^2$, whose d-paths are the increasing maps $\mathbb{I} \to \mathbb{I}^2$; one of them is drawn in the second picture above.

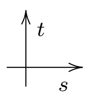
Similarly, in the *standard c-cube* $\mathrm{c}\mathbb{I}^n$ a path is controlled if and only if it is increasing and each of its projections covers $[0, 1]$, or is constant at 0 or 1. Again, $(\mathrm{c}\mathbb{I}^n)\hat{} = \uparrow\mathbb{I}^n$.

For a product, the structure $(X{\times}Y)\hat{}$ is always finer than $\hat{X}{\times}\hat{Y}$, and can be strictly finer. For instance one can take the empty structure $X = D_c\{*\}$ and $Y = \mathrm{c}\mathbb{I}$ or $\uparrow\mathbb{I}$, so that $(X{\times}Y)\hat{} = (D_c|\mathbb{I}|)\hat{} = D|\mathbb{I}|$, but $\hat{X}{\times}\hat{Y} = \{*\}{\times}\uparrow\mathbb{I} = \uparrow\mathbb{I}$. This issue will be studied in a Part II.

(c) The *hybrid square* $\mathrm{c}\mathbb{I} \times \uparrow\mathbb{I}$ will be important in the construction of the fundamental category. Here a path $\mathbb{I} \to \mathbb{I}^2$ is controlled if and only if it is increasing and its first projection is either surjective or constant at 0 or 1



$\mathrm{c}\mathbb{I} \times \uparrow\mathbb{I}$

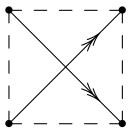


(36)

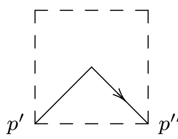
(All the horizontal paths $s \mapsto (s, t_0)$ are controlled, but already belong to the family of increasing paths whose first projection is surjective.)

The generated d-space is again $\uparrow\mathbb{I}^2$; the flexible part $D\{0, 1\}{\times}\uparrow\mathbb{I}$ only allows increasing paths in the vertical edges.

(d) The following example shows a sharp distinction between a c-structure X of the square and the generated d-space



X



$\hat{X}$

(37)

The c-paths of X are generated by two diagonal paths, $t \mapsto (t, t)$ and $t \mapsto (t, 1-t)$; the flexible points are the four vertices of the square, and it is easy to guess that the fundamental category $\uparrow\Pi_1(X)$ will only have two non-trivial arrows, the marked ones. But the fundamental category $\uparrow\Pi_1(\hat{X})$ of the generated d-space has two new arrows between the vertices, like the arrow $p' \to p''$ displayed in the right figure above.

Clearly, one cannot model a crossing of railways by a d-space. Within c-spaces, one can make X reversible adding the reversed c-paths.

3. Critical processes and models

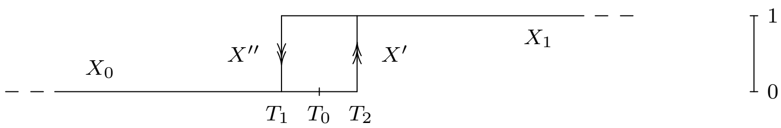
We now start from 'critical processes', trying to model them by c-spaces built with the previous ones, by limits and colimits.

3.1 On-off controllers and elementary hysteresis

We consider a system meant to regulate a certain variable, either opposing its rising, or helping it, or working both ways. An elementary hysteresis process, or 'hysteron', generally behaves in the first way – counteracting the effect.

(a) *Reacting controller.* We begin by considering a cooling device counteracting the rising of temperature, with a thermostat set at temperature T_0 and a tolerance interval $[T_1, T_2]$.

In the following picture the horizontal axis represents the temperature, and the vertical axis denotes two states, 0 and 1



(38)

On the left branch X_0 the system is off; if the temperature grows to T_2 the device jumps to state 1; then, if the temperature cools to T_1, it goes back to state 0.

The support $|X|$ of our model is a one-dimensional subspace of $\mathbb{R}^2$, the union of the supports of the following c-spaces

$$X_0 = [0, T_2] \times \{0\}, \qquad X_1 = [T_1, +\infty[\, \times \{1\},$$
$$X' = \{T_2\} \times \mathrm{c}\mathbb{I}, \qquad X'' = \{T_1\} \times \mathrm{c}\mathbb{I}^{\mathrm{op}}.$$

The c-structure of X is generated by the c-structures of:

- X_0, X_1, natural intervals, where the temperature can vary,

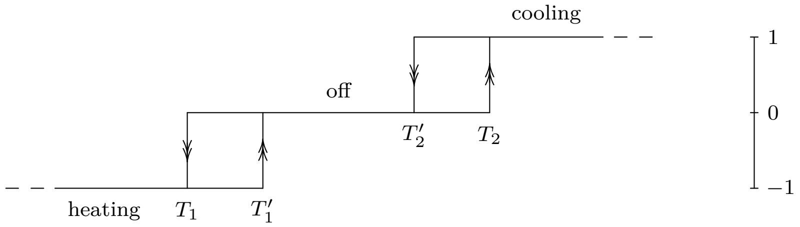
- X', X'', one-jump c-intervals, where the state of the system varies.

One could also use the plane with the terminal c-structure $\mathrm{c}_f\mathbb{R}^2$ produced by the topological embedding $f\colon X \to \mathbb{R}^2$; the c-paths are those of X.

A hysteretic process is generally studied as a functional operator that turns a piecewise monotone input function into an output function (of time, in both cases): see [BrS], Chapter 2. Here the input is the temperature function, while the output values are the states 0, 1. This analysis presents some indetermination and failure of continuity at the critical temperatures T_1, T_2, as discussed in [BrS], Example 2.1.1.

(b) *Cooperating controller.* A heating system supports the rising of temperature. It can be modelled by the opposite c-space: in state 0 the system is on; see the lower half of the following diagram.

(c) *Dual controller.* Combining both models we can represent a heating and cooling system, like a heat pump. The system is meant to keep the temperature in an interval $[T_1, T_2]$, with a lower tolerance $[T_1, T_1']$ and an upper tolerance $[T_2', T_2]$ (disjoint intervals); the vertical axis denotes now three states: at level 0 the system is off, at level -1 the heating is on, at level 1 the cooling device is on

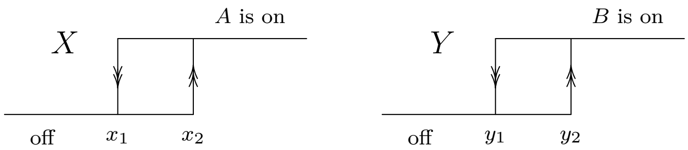


(39)

3.2 Controlling two variables

We deal now with a pair of on-off controllers, acting on two independent variables

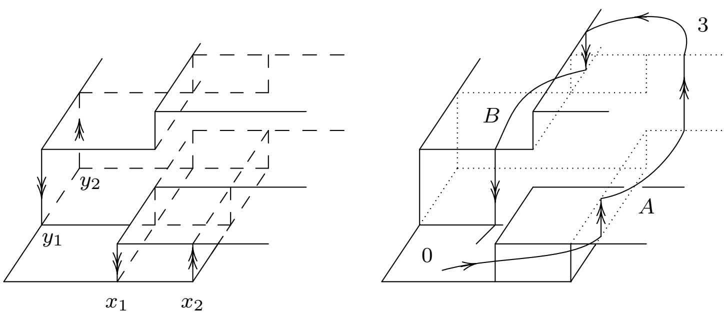
(a) *Two reacting controllers.* We start from two copies X, Y of the c-space drawn in (38)



(40)

In X a device A controls the variable x, countering its rising; in Y the device B acts similarly on the variable y.

The cartesian product $X \times Y$ models the combined system. Its support is a subspace of $\mathbb{R}^4$, but we draw it in $\mathbb{R}^3$, with four states on the vertical axis: at 0 both devices are off, at 1 only A is on, at 2 only B is on, at 3 both A and B are on



(41)

The new c-space can be obtained as follows. We put on $\mathbb{R}^3$ the terminal c-structure for the topological map

$$f\colon X \times Y \to \mathbb{R}^3, \qquad f(x, s, y, t) = (x, y, s + 2t), \tag{42}$$

and we use this c-space $\mathrm{c}_f\mathbb{R}^3$. Equivalently, we can use the c-subspace $Z = \mathrm{Im}\, f$ of the previous structure: they have the same c-paths. Z is contained in four parallel planes, at level 0, 1, 2, 3.

The right figure above represents a c-path in Z. It starts at level 0, with both variables below the active thresholds x_2, y_2, and both increasing; when the variable x reaches x_2, system A jumps on, counteracting it. Both variables are still growing; when y attains y_2, device B also activates and the process is in state 3. Then the variable x decreases below x_1 and A goes off, while B is still on, in state 2. Finally also the variable y is brought below y_1 and both devices are off, at level 0.

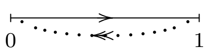
(b) The opposite case of two cooperating controllers is modelled by the opposite c-space Z^{op}. The mixed case, with A cooperating with variable x and B counteracting variable y, is also of interest.

An air-supported dome can give examples of both cases. In winter time, the compressor would rise pressure and the heating would rise temperature; in summer time, the compressor would work the same way while air-conditioning would reduce temperature.

3.3 The threshold effect and siphon structures.

In the threshold effect a variable v can vary in the interval $[v_0, v_1]$; reaching the *threshold* v_1 it jumps down to its least value v_0. Various processes of this type are listed in the Introduction, Subsection 0.4. Here we consider two structures on the interval $\mathbb{I}$ that can model such a process.

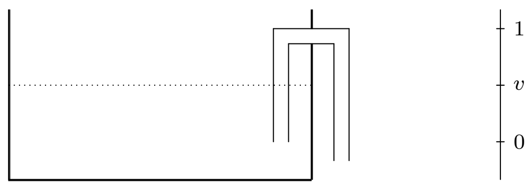
(a) *The growing siphon.* We denote as $\mathrm{c}_S\mathbb{I}$ the standard interval with the c-structure generated by all the increasing maps $\mathbb{I} \to \mathbb{I}$ and the reversion $r(t) = 1 - t$



$\mathrm{c}_S\mathbb{I}$ (43)

A controlled path can only increase between 0 and 1; reaching 1, either it stays there or jumps down to 0. Point 1 is future critical, point 0 is past critical, and there are no (bilateral) critical points. The generated d-structure $(\mathrm{c}_S\mathbb{I})\hat{}$ has two generators, $\mathrm{id}\,\mathbb{I}$ and r: it is the reversible d-interval $\mathbb{I}^{\sim}$ of 2.4(c); the flexible part is $\uparrow\mathbb{I}$.

For concreteness, we refer to a hydraulic system consisting of a water basin filled by a source; water can only get out by a siphon tube, as in the figure below, so that its level v in the basin will grow up to the upper part of the tube, marked 1, and then flow out until the level reaches the lower opening of the tube in the basin, marked 0 (the diameter of the tube is overlooked)



(44)

(b) *The oscillating siphon.* A more complex model $c'_S\mathbb{I}$ allows the c-paths to decrease in the semiopen interval $[0, 1[$. There are three kinds of generators of the c-paths:

- the increasing maps $\mathbb{I} \to \mathbb{I}$,
- the decreasing maps $a\colon \mathbb{I} \to \mathbb{I}$ with image in $[0, 1[$, i.e. $a(0) < 1$,
- the reversion $r(t) = 1 - t$.

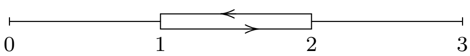
Here a controlled path is piecewise monotone; whenever it reaches 1, either it stays there or it jumps to 0 (and possibly goes on). Point 1 is still future critical, but 0 is no more past critical. The generated d-structure is again the reversible d-interval $\mathbb{I}^\sim$, the flexible part allows the piecewise monotone paths which can only reach 1 as the terminal endpoint.

In the hydraulic system previously described, the basin can now let out water by other openings or evaporation. Other processes considered in 0.4 are also better fitted by this model; for instance the electric potential at a neural membrane can increase and decrease; reaching the threshold value the impulse is emitted.

3.4 Transport networks and labelled graphs

Transport networks are usually modelled in graph theory. They can also be modelled by c-spaces, as we have already seen in various examples; this would allow to combine them with planar or three-dimensional regions.

(a) The following is a model of a road that contains a dual-carriage section (for left-hand drive one would turn the picture upside-down)



(45)

The d-space X is the quotient Y/R of the sum $Y = X_1 + ... + X_4$

$$X_1 = [0,1], \qquad X_2 = \uparrow[1,2],$$
$$X_3 = \uparrow[1,2]^{\mathrm{op}}, \qquad X_4 = [2,3],$$

modulo the equivalence relation that identifies the three points 1 (of X_1, X_2, X_3) and – separately – the three points 2 (of X_2, X_3, X_4). A path in X is directed if and only if it is a general concatenation of projections of d-paths in the various X_i. For instance, to go from 0 to the point $(1/2)_3$ of X_3 we must (at least) reach 2 along X_1 and X_2 and then come back along X_3; there are infinitely many longer paths.

(b) Similarly, one can construct a one-dimensional c-space X as a realisation of a *labelled graph*, in the sense of a multigraph whose edges are labelled with additional information on direction and critical properties. We have already drawn many examples in 3.1.

As above, the c-space X can be obtained as a quotient of a sum of intervals with the appropriate c-structure: natural intervals $\mathbb{I}$ when unlabelled, standard d-intervals $\uparrow\mathbb{I}$ when labelled by a single arrow, standard c-intervals $\mathrm{c}\mathbb{I}$ when labelled by a double arrow, etc.

This can represent a transport networks, where some routes are one-way and others are no-stop – as in an underground section, or a motorway tunnel, or an airline route. The model can be further enriched, adding delays (for a stop sign), etc. Or higher dimensional regions, as we were suggesting.

3.5 Point-like variations

In the examples of this section one can often form a 'slightly' different model, using a general procedure: if X is a c-space and $A \subset |X|$, one builds a finer c-space on $|X|$ excluding all the previous c-paths that have an endpoint in A.

Thus, in the model of the heat controller described in 3.1(a), one can omit the paths that start or end at $(0, T_2)$ or $(1, T_1)$. Similarly, in the siphon-interval 3.3(a) one can rule out the paths starting or ending at 1. In both cases we are forcing the jump at these points, which become critical and non-flexible. This can be appreciated, but the new models are more complicated and their fundamental category will also be.

In our opinion the choice between such variations is merely a theoretical issue, that should be based on the results one can obtain. In the same way as, if we model a thin rod by the interval $[0, 1]$, it is Mathematics rather than experience that leads us to use an interval of the real line instead of the rational line: the classical results on continuous functions and differential equations only hold in the former.

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