



COVERS IN THE CANONICAL GROTHENDIECK TOPOLOGY

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Résumé. Nous explorons la topologie canonique de Grothendieck dans certaines circonstances spécifiques. Tout d'abord, nous utilisons une description de la topologie canonique pour obtenir une variante du théorème de Giraud, qui indique quand une catégorie est équivalente à une catégorie de gerbes sur un site de Grothendieck. Ensuite, nous explorons la topologie canonique de Grothendieck sur les catégories d'ensembles et d'espaces topologiques. Nous donnons une base et une présentation pour la topologie canonique sur la catégorie des ensembles. De plus, puisqu'il existe plusieurs catégories qui peuvent représenter la catégorie des espaces topologiques, nous explorons deux de ces catégories : la catégorie de tous les espaces topologiques, et la catégorie des espaces de Hausdorff faiblement engendrés et compacts. Cette exploration se concentre sur les différences des topologies canoniques de Grothendieck résultantes, ainsi que sur leurs bases et présentations. Troisièmement, nous examinons les topologies canoniques de Grothendieck sur la catégorie des R -modules. Une attention particulière est accordée à la recherche de réductions et à la catégorie des groupes abéliens.

Abstract. We explore the canonical Grothendieck topology in some specific circumstances. First, we use a description of the canonical topology to get a variant of Giraud's Theorem, which indicates when a category is equivalent to a category of sheaves on a Grothendieck site. Second, we explore the canonical Grothendieck topology on the categories of sets and topological spaces. We give a basis and presentation for the canonical topology on the category of sets. Additionally, since there are several categories that can represent "the category of topological spaces," we explore two such categories:

the category of all topological spaces, and the category of compactly generated weakly Hausdorff spaces. This exploration focuses on the differences of the resulting canonical Grothendieck topologies, along with their bases and presentations. Third, we look at the canonical Grothendieck topology on the category of R -modules. A special focus is paid to finding reductions and to the category of abelian groups.

Keywords. Grothendieck topology, Giraud’s Theorem, colimit, quotient space, compactly generated weakly Hausdorff, abelian group

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1. Introduction

In SGA 4.2.2 Verdier defined the canonical Grothendieck topology as the largest Grothendieck topology where all representable presheaves are sheaves. This paper grew out of an attempt to obtain a precise description of the covers in this Grothendieck topology in the cases of some familiar categories; we investigate the question for sets, abelian groups, R -modules, topological spaces and compactly generated Hausdorff spaces. The category of sets is

simple enough that we can give a complete answer, and in the two categories of topological spaces we give a fairly precise description. The question for abelian groups and R -modules seems to be very subtle, though, and we have only been able to obtain partial results. Along the way we prove that the canonical topology has a natural appearance in Giraud's Theorem, which is the source for some of our interest in it.

Sieves will be of particular importance in this paper and so we start with a reminder of its definition; we follow the notation and terminology used by Mac Lane and Moerdijk in [S. Mac Lane and I. Moerdijk, 2012]. For any object X of a category \mathcal{C} , we call S a *sieve on X* if S is a collection of morphisms, all of whose codomains are X , that is closed under precomposition, i.e. if $f \in S$ and $f \circ g$ makes sense, then $f \circ g \in S$. In particular, we can view a sieve S on X as a full subcategory of the overcategory $(\mathcal{C} \downarrow X)$.

By work from [C. Lester, 2019], the canonical Grothendieck topology can be characterized in terms of colimits. Specifically, the canonical Grothendieck topology can be described as the collection of all universal colim sieves where:

Definition 1.1. *For a category \mathcal{C} , an object X of \mathcal{C} and sieve S on X , we call S a *colim sieve* if $\text{colim}_S U$ exists and the canonical map $\text{colim}_S U \rightarrow X$ is an isomorphism. (Alternatively, S is a *colim sieve* if X is the universal cocone under the diagram $U: S \rightarrow \mathcal{C}$.) Moreover, we call S a *universal colim sieve* if for all arrows $\alpha: Y \rightarrow X$ in \mathcal{C} , α^*S is a *colim sieve* on Y .*

One use of this presentation is the following variant of Giraud's Theorem:

Proposition 3.14. *If \mathcal{E} is a 'nice' category, then \mathcal{E} is equivalent to the category of sheaves on \mathcal{E} under the canonical topology.*

The universal-colim-sieve presentation also affords us an explicit description of the canonical Grothendieck topology's covers on the category of topological spaces:

Proposition 4.6. *In the category of all topological spaces, $\{A_\alpha \rightarrow X\}_{\alpha \in A}$ is part of a basis for the canonical topology if and only if $\alpha: \coprod_{\alpha \in A} A_\alpha \rightarrow X$ is*

a universal quotient map (i.e. α and every pullback of α is a quotient map). Additionally, a sieve S on X is a (universal) colim sieve if and only if there exists some collection $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \subset S$ such that $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a (universal) quotient map. In particular, $T = \langle \{f: Y \rightarrow X\} \rangle$ is a (universal) colim sieve if and only if f is a (universal) quotient map.

Proposition 4.7. *In the category of compactly generated weakly Hausdorff spaces, $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}}$ is part of the basis for the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a quotient map. In particular, a sieve $S = \langle \{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \rangle$ on X is in the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a quotient map. Moreover, every colim sieve is universal.*

Furthermore, this presentation allows us to more easily compute examples and non-examples in the category of topological spaces; for instance,

Example 4.14/Example 4.15. Take $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be the closed inclusion map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$ and use \mathbb{R}^∞ to denote the direct limit $\text{colim}_{n \in \mathbb{N}} \mathbb{R}^n$ with maps $\iota_n: \mathbb{R}^n \rightarrow \mathbb{R}^\infty$. Then the cover generated by $\{\iota_n\}_{n \in \mathbb{N}}$ is not in the canonical topology for the category of all topological spaces but is in the canonical topology for the category of compactly generated weakly Hausdorff spaces.

Additionally, we can use the universal-colim-sieve presentation to get a better idea of the canonical Grothendieck topology's covers on the category of R -modules. For example,

Proposition 5.6. *Let S be the cover generated by $\{f_1: M_1 \rightarrow R, f_2: M_2 \rightarrow R\}$ such that $\text{im}(f_i) = a_i R$ for $i = 1, 2$. Then S is in the canonical topology on $R\text{-Mod}$ if and only if $(a_1, a_2) = R$.*

Proposition 5.8. *Let R be an infinite principal ideal domain. Let S be the cover generated by $\{g_i: R^n \hookrightarrow R^n\}_{i=1}^M \cup \{f_i: R^{m_i} \hookrightarrow R^n \mid m_i < n\}_{i=1}^N$. If S is a cover in the canonical topology on $R\text{-Mod}$, then $g_1 \oplus \dots \oplus g_M: R^{nM} \rightarrow R^n$ is a surjection.*

Proposition 5.24. *Let S be the cover generated by $\{\mathbb{Z} \xrightarrow{\times a_i} \mathbb{Z}\}_{i=1}^N$. Then S is in the canonical topology on $\mathbb{Z}\text{-Mod}$ if and only if $\gcd(a_1, \dots, a_N) = 1$.*

Proposition 5.25. *Let S be the cover generated by $\{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N$ where A_i is a diagonal matrix with $\det(A_i) \neq 0$. Then there exists a map $\beta: \mathbb{Z} \rightarrow \mathbb{Z}^n$ such that β^*S is not a colim sieve in $\mathbb{Z}\text{-Mod}$ if and only if $\gcd(\det(A_1), \dots, \det(A_N))$ does not equal 1.*

Organization.

To start this paper we recall some results from [C. Lester, 2019] in Section 2. Then in Section 3 we review Giraud’s theorem and prove our Corollary to Giraud’s Theorem, i.e. we prove that that every category \mathcal{C} , which satisfies some hypotheses, is equivalent to the category of sheaves on \mathcal{C} with the canonical topology.

In Section 4 we briefly discuss the canonical topology on the category of sets before exploring the canonical topology on the category of topological spaces. Specifically, we look at the category of all topological spaces and the category of compactly generated weakly Hausdorff spaces. We are able to refine our description and obtain a basis for the canonical topology; this result reduces the question “Is this in the canonical topology?” to the question “Is a specific map a universal quotient map?” Since universal quotient maps have been studied in-depth (for example by Day and Kelly in [B.J. Day and G.M. Kelly, 1970]), this reduction becomes our most computationally agreeable description of the canonical topology and hence we use it to find some specific examples and non-examples.

Lastly, in Section 5 we investigate the canonical topology on the category of R -modules and the category of abelian groups, where we work towards refining our description by making some reductions and obtaining some exclusionary results. While these reductions and results lead us to some specific examples and non-examples, a basis for the canonical topology remains elusive.

General Notation.

Notation 1.2. For any subcategory S of $(\mathcal{C} \downarrow X)$, we will use U to represent the forgetful functor $S \rightarrow \mathcal{C}$. For example, for a sieve S on X , $U(f) = \text{domain } f$.

Notation 1.3. We say that a sieve S on X is *generated* by the morphisms $\{f_\alpha: A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}}$ and write $S = \langle \{f_\alpha: A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \rangle$ if each $f \in S$ factors through one of the f_α , i.e. if $f \in S$ then there exists an $\alpha \in \mathcal{A}$ and morphism g such that $f = f_\alpha \circ g$.

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2. Background

This section contains a review of the results from [C. Lester, 2019] that will be used in this paper.

Lemma 2.1. *Suppose \mathcal{C} is a category with all pullbacks.*

*Let $S = \langle \{g_\alpha: A_\alpha \rightarrow X\}_{\alpha \in \mathfrak{A}} \rangle$ be a sieve on object X of \mathcal{C} and $f: Y \rightarrow X$ be a morphism in \mathcal{C} . Then $f^*S = \langle \{A_\alpha \times_X Y \xrightarrow{\pi_2} Y\}_{\alpha \in \mathfrak{A}} \rangle$.*

Proposition 2.2. *Let \mathcal{C} be a cocomplete category. For a sieve in \mathcal{C} on X of the form $S = \langle \{f_\alpha: A_\alpha \rightarrow X\}_{\alpha \in \mathfrak{A}} \rangle$ such that $A_i \times_X A_j$ exists for all $i, j \in \mathfrak{A}$,*

$$\frac{\text{colim}_S U}{S} \cong \text{Coeq} \left(\begin{array}{c} \coprod_{(i,j) \in \mathfrak{A} \times \mathfrak{A}} A_i \times_X A_j \\ \downarrow \downarrow \\ \coprod_{k \in \mathfrak{A}} A_k \end{array} \right)$$

where the left and right vertical maps are induced from the projection morphisms $\pi_1: A_i \times_X A_j \rightarrow A_i$ and $\pi_2: A_i \times_X A_j \rightarrow A_j$.

Lemma 2.3. *Let \mathcal{C} be a category. Then S is a colim sieve on X if and only if f^*S is a colim sieve for any isomorphism $f: Y \rightarrow X$.*

Recall that a morphism $f: Y \rightarrow X$ is called an *effective epimorphism* provided $Y \times_X Y$ exists, f is an epimorphism and $c: \text{Coeq}(Y \times_X Y \rightrightarrows Y) \rightarrow X$ is an isomorphism. Note that this third condition actually implies the second because $f = c \circ g$ where $g: Y \rightarrow \text{Coeq}(Y \times_X Y \rightrightarrows Y)$ is the canonical map. Indeed, g is an epimorphism by an easy exercise and c is an epimorphism since it is an isomorphism.

Additionally, $f: Y \rightarrow X$ is called a *universal effective epimorphism* if f is an effective epimorphism with the additional property that for every pullback diagram

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \pi_g \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & X \end{array}$$

π_g is also an effective epimorphism.

Corollary 2.4. *Let \mathcal{C} be a cocomplete category with pullbacks. If*

$$S = \langle \{f: Y \rightarrow X\} \rangle$$

is a sieve on X , then S is a colim sieve if and only if f is an effective epimorphism. Moreover, S is a universal colim sieve if and only if f is a universal effective epimorphism.

Theorem 2.5. *Let \mathcal{C} be any category. The collection of all universal colim sieves on \mathcal{C} forms a Grothendieck topology.*

Theorem 2.6. *For any (locally small) category \mathcal{C} , the collection of all universal colim sieves on \mathcal{C} is the canonical topology.*

Proposition 2.7. *Let \mathcal{C} be a cocomplete category with pullbacks. Further assume that coproducts and pullbacks commute in \mathcal{C} . Then a sieve of the form $S = \langle \{f_\alpha: A_\alpha \rightarrow X\}_{\alpha \in A} \rangle$ is a (universal) colim sieve if and only if the sieve $T = \langle \{\coprod f_\alpha: \coprod_{\alpha \in A} A_\alpha \rightarrow X\} \rangle$ is a (universal) colim sieve.*

Theorem 2.8. *Let \mathcal{C} be a cocomplete category with pullbacks whose coproducts and pullbacks commute. A sieve S on X is a (universal) colim sieve of \mathcal{C} if and only if there exists some $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \subset S$ where $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a (universal) effective epimorphism.*

Theorem 2.9. *Let \mathcal{C} be a cocomplete category with stable and disjoint coproducts and all pullbacks. For each X in \mathcal{C} , define $K(X)$ by*

$$\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \in K(X) \iff \coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X \text{ is a universal effective epimorphism.}$$

Then K is a Grothendieck basis and generates the canonical topology on \mathcal{C} .

3. Giraud's Theorem and the Canonical Topology

Giraud's Theorem shows that categories with certain nice properties can be written as sheaves on a Grothendieck site. We show that in fact, modulo universe considerations, one may take this site to be the original category with the canonical topology.

We will use the version of Giraud's Theorem from [S. Mac Lane and I. Moerdijk, 2012]. In fact, the appendix of [S. Mac Lane and I. Moerdijk, 2012] has a thorough discussion of Giraud's theorem and all of the terminology used in it; we will include the basics of this discussion for completeness. We will begin by recalling the definitions used in Mac Lane and Moerdijk's version of Giraud's Theorem.

Throughout this section, let \mathcal{E} be a category with small hom-sets and all finite limits.

DISJOINT AND STABLE COPRODUCTS

Let E_α be a family of objects in \mathcal{E} and $E = \coprod_\alpha E_\alpha$.

Definition 3.1. *The coproduct E is called disjoint if every coproduct inclusion $i_\alpha: E_\alpha \rightarrow E$ is a monomorphism and, whenever $\alpha \neq \beta$, $E_\alpha \times_E E_\beta$ is the initial object in \mathcal{E} .*

Definition 3.2. *The coproduct E is called stable (under pullback) if for every $f: D \rightarrow E$ in \mathcal{E} , the morphisms j_α obtained from the pullback diagrams*

$$\begin{array}{ccc}
D \times_E E_\alpha & \longrightarrow & E_\alpha \\
j_\alpha \downarrow & & \downarrow i_\alpha \\
D & \xrightarrow{f} & E
\end{array}$$

induce an isomorphism $\coprod_\alpha (D \times_E E_\alpha) \cong D$.

Remark 3.3. If every coproduct in \mathcal{E} is stable, then the pullback operation $- \times_E D$ “commutes” with coproducts, i.e. $(\coprod_\alpha B_\alpha) \times_E D \cong \coprod_\alpha (B_\alpha \times_E D)$.

COEQUALIZER MORPHISMS AND KERNEL PAIRS

Definition 3.4. We call a morphism $f: Y \rightarrow Z$ in \mathcal{E} a coequalizer if there exists some object X and morphisms $\partial_0, \partial_1: X \rightarrow Y$ such that

$$X \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} Y \xrightarrow{f} Z$$

is a coequalizer diagram.

We remark that every coequalizing morphism is an epimorphism but the converse of this statement is not guaranteed.

Definition 3.5. The pair of morphisms $\partial_0, \partial_1: X \rightarrow Y$ are called a kernel pair for $f: Y \rightarrow Z$ if the following is a pullback diagram

$$\begin{array}{ccc}
X & \xrightarrow{\partial_1} & Y \\
\partial_0 \downarrow & & \downarrow f \\
Y & \xrightarrow{f} & Z
\end{array}$$

EQUIVALENCE RELATIONS AND QUOTIENTS

Definition 3.6. An equivalence relation on the object E of \mathcal{E} is a subobject R of $E \times E$, represented by the monomorphism $(\partial_0, \partial_1): R \rightarrow E \times E$, satisfying the following axioms

1. (reflexive) the diagonal $\Delta: E \rightarrow E \times E$ factors through (∂_0, ∂_1) ,
2. (symmetric) the map $(\partial_1, \partial_0): R \rightarrow E \times E$ factors through (∂_0, ∂_1) ,

3. (transitivity) if $R \times_E R$ is the pullback

$$\begin{array}{ccc} R \times_E R & \xrightarrow{\pi_1} & R \\ \pi_0 \downarrow & & \downarrow \partial_0 \\ R & \xrightarrow{\partial_1} & E \end{array}$$

then $(\partial_1 \pi_1, \partial_0 \pi_0): R \times_E R \rightarrow E \times E$ factors through R .

Definition 3.7. If E is an object of \mathcal{E} with equivalence relation R , then the quotient is denoted E/R and is defined to be

$$\text{Coeq} \left(R \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} E \right)$$

provided that this coequalizer exists.

STABLY EXACT FORKS

A diagram is called a *fork* if it is of the form

$$X \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} Y \xrightarrow{q} Z. \quad (1)$$

Definition 3.8. The fork (1) is called *exact* if ∂_0 and ∂_1 are the kernel pair for q , and q is the coequalizer of ∂_0 and ∂_1 .

Definition 3.9. The fork (1) is called *stably exact* if the pullback of (1) along any morphism in \mathcal{E} yields an exact fork, i.e. if for any $Z' \rightarrow Z$ in \mathcal{E} ,

$$X \times_Z Z' \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} Y \times_Z Z' \xrightarrow{q \times 1} Z \times_Z Z'$$

is an exact fork.

GENERATING SETS

Definition 3.10. A set of objects $\{A_i \mid i \in I\}$ of \mathcal{E} is said to *generate* \mathcal{E} if for every object E of \mathcal{E} , $W = \{A_i \rightarrow E \mid i \in I\}$ is an *epimorphic family* (in the sense that for any two parallel arrows $u, v: E \rightarrow E'$, if every $w \in W$ yields the identity $uw = vw$, then $u = v$).

GIRAUD'S THEOREM

Theorem 3.11 (Giraud, [S. Mac Lane and I. Moerdijk, 2012]). *A category \mathcal{E} with small hom-sets and all finite limits is a Grothendieck topos if and only if it has the following properties (which we will refer to as Giraud's axioms):*

- (i) \mathcal{E} has small coproducts which are disjoint and stable under pullback,
- (ii) every epimorphism in \mathcal{E} is a coequalizer,
- (iii) every equivalence relation $R \rightrightarrows E$ in \mathcal{E} is a kernel pair and has a quotient,
- (iv) every exact fork $R \rightrightarrows E \rightarrow Q$ is stably exact,
- (v) there is a small set of objects of \mathcal{E} which generate \mathcal{E} .

Discussion 3.12. Taken together, Giraud's axioms (ii) and (iv) imply that for each epimorphism $B \xrightarrow{f} A$, the fork $B \times_A B \rightrightarrows B \rightarrow A$ is stably exact. The exactness implies f is an effective epimorphism and the stability implies f is a universal effective epimorphism.

Notation 3.13. We use $Sh(\mathcal{E}, J)$ to represent the category of sheaves on the category \mathcal{E} under the topology J .

Suppose the category \mathcal{E} has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is \mathcal{C} . In [S. Mac Lane and I. Moerdijk, 2012] Mac Lane and Moerdijk specifically prove $\mathcal{E} \cong Sh(\mathcal{C}, J)$ where J is the Grothendieck topology on \mathcal{C} defined by:

$$S \in J(X) \text{ if and only if } \coprod_{(g: D \rightarrow X) \in S} D \rightarrow X \text{ is an epimorphism in } \mathcal{E}.$$

(In particular, Mac Lane and Moerdijk prove that J is a Grothendieck topology.)

Proposition 3.14. *Suppose the category \mathcal{E} has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is \mathcal{C} . Then \mathcal{E} is equivalent to $Sh(\mathcal{C}, C)$ where C is the canonical topology on \mathcal{C} .*

Proof. Let J be the topology defined above. Additionally, the above discussion implies that it suffices to show that J is the canonical topology. By Theorem 2.6, we will instead show that every universal colim sieve is in J and that every sieve in J is a universal colim sieve.

By Remark 3.3, coproducts and pullbacks commute and hence for any collection of morphisms $\{A_i \rightarrow X\}_{i \in I}$ in \mathcal{E} , the diagrams

$$\begin{array}{ccc} \coprod_{I^2} (A_i \times_X A_j) & & (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow \downarrow & \text{and} & \downarrow \downarrow \\ \coprod_I A_k & & \coprod_I A_k \end{array}$$

are isomorphic. Note: in both diagrams, the two maps down are the obvious ones induced/obtained from a pullback diagram. Thus

$$\text{Coeq} \left(\begin{array}{c} \coprod_{I^2} (A_i \times_X A_j) \\ \downarrow \downarrow \\ \coprod_I A_k \end{array} \right) \cong \text{Coeq} \left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow \downarrow \\ \coprod_I A_k \end{array} \right).$$

But by Proposition 2.2 (which is usable since \mathcal{E} is cocomplete),

$$\text{Coeq} \left(\begin{array}{c} \coprod_{I^2} (A_i \times_X A_j) \\ \downarrow \downarrow \\ \coprod_I A_k \end{array} \right) \cong \frac{\text{colim}}{S} U \quad \text{where } S = \langle \{A_i \rightarrow X\}_{i \in I} \rangle$$

and

$$\text{Coeq} \left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow \downarrow \\ \coprod_I A_k \end{array} \right) \cong \frac{\text{colim}}{T_S} U \quad \text{where } T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \rightarrow X \right\} \right\rangle.$$

Hence

$$\frac{\text{colim}}{S} U \cong \frac{\text{colim}}{T_S} U$$

where $S = \langle \{A_i \rightarrow X\}_{i \in I} \rangle$ and $T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \rightarrow X \right\} \right\rangle$ (2)

for any generating set $\{A_i \rightarrow X\}_{i \in I}$ of S .

Suppose S is a universal colim sieve. Since S has the some generating set, then by the definition of colim sieve and (2),

$$X \cong \frac{\text{colim}}{S} U \cong \frac{\text{colim}}{T_S} U.$$

This implies that T_S is a colim sieve. Hence $\left(\coprod_{(g: D \rightarrow X) \in S} D \right) \rightarrow X$ is an effective epimorphism by Corollary 2.4 and so $S \in J(X)$.

For the converse, suppose that $S \in J(X)$. Thus $p_s: \left(\coprod_{(g: D \rightarrow X) \in S} D \right) \rightarrow X$ is an epimorphism, which by Discussion 3.12 is a universal effective epimorphism. Hence by Corollary 2.4, p_s generates a universal colim sieve called T_S . Then by the definition of colim sieve and (2),

$$X \cong \frac{\text{colim}}{T_S} U \cong \frac{\text{colim}}{S} U.$$

Therefore S is a colim sieve.

Similar to the last paragraph, we can use (2) to show that f^*S is a colim sieve for any morphism f in \mathcal{E} if we know that T_{f^*S} is a colim sieve. So to finish the proof we will use the fact that T_S is a universal colim sieve to show that T_{f^*S} is a colim sieve. Let $f: Y \rightarrow X$ be any morphism in \mathcal{E} . Then by using S as a generating collection for itself and Lemma 2.1, $f^*S = \langle \{A \times_X Y \rightarrow Y \mid A \rightarrow X \in S\} \rangle$. Similarly, using Lemma 2.1, $f^*T_S = \left\langle \left\{ \left(\coprod_{(A \rightarrow X) \in S} A \right) \times_X Y \rightarrow Y \right\} \right\rangle$. Then by Remark 3.3

$$\coprod_{(A \rightarrow X) \in S} (A \times_X Y) \cong \left(\coprod_{(A \rightarrow X) \in S} A \right) \times_X Y$$

over Y . Therefore,

$$\frac{\text{colim}}{T_{f^*S}} U \cong \frac{\text{colim}}{f^*T_S} U \cong Y$$

where the first isomorphism is due to the previous few sentences and the second isomorphism is due to the fact that T_S is a universal colim sieve. Thus T_{f^*S} is a colim sieve. \square

4. Universal Colim Sieves in the Categories of Sets and Topological Spaces

In this section we examine the canonical topology on the categories of sets, all topological spaces and compactly generated weakly Hausdorff spaces.

Notation 4.1. We will use **Sets** to denote the category of sets. We will use **Top** to denote the category of all topological spaces, **CG** to denote the category of compactly generated spaces, and **CGWH** to denote the category of compactly generated weakly Hausdorff spaces. When we want to talk about the category of topological spaces without differentiating between **Top** and **CGWH**, then we will use **Spaces**; all results about **Spaces** will hold for both **Top** and **CGWH**.

We will begin with a few reminders about the category of compactly generated weakly Hausdorff spaces based on the references [N.P. Strickland, 2009] and [J.P. May, 1999]. Specifically, there are functors $k: \mathbf{Top} \rightarrow \mathbf{CG}$ and $h: \mathbf{CG} \rightarrow \mathbf{CGWH}$ such that

- For a topological space X with topology τ , a subset Y of X is called k -closed if $u^{-1}(Y)$ is closed in K for every continuous map $u: K \rightarrow X$ and compact Hausdorff space K . The collection of all k -closed subsets, called $k(\tau)$, is a topology.
- The functor k takes X with topology τ to the set X with topology $k(\tau)$.
- k is right adjoint to the inclusion functor $\iota: \mathbf{CG} \rightarrow \mathbf{Top}$.
- $h(X)$ is X/E where E is the smallest equivalence relation on X closed in $X \times X$.

- h is left adjoint to the inclusion functor $\iota': \mathbf{CGWH} \rightarrow \mathbf{CG}$.
- A limit in \mathbf{CGWH} is k applied to the limit taken in \mathbf{Top} , i.e. for a diagram $F: I \rightarrow \mathbf{CGWH}$, the limit of F is $k(\lim_I \iota' F)$.
- A colimit in \mathbf{CGWH} is h applied to the colimit taken in \mathbf{Top} , i.e. for a diagram $F: I \rightarrow \mathbf{CGWH}$, the colimit of F is $h(\underline{\text{colim}}_I \iota' F)$.

Proposition 4.2. *Let S be a sieve on X in either \mathbf{Sets} or \mathbf{Top} . Let C be $\underline{\text{colim}}_S U$. Then the natural map $\varphi: C \rightarrow X$ is an injection.*

Proof. Suppose $\tilde{y}, \tilde{z} \in C$ and $\varphi(\tilde{y}) = x = \varphi(\tilde{z})$. We can pick a $(Y \rightarrow X) \in S$ and a $y \in Y$ that represents \tilde{y} , i.e. where $y \mapsto \tilde{y}$ under the natural map $Y \rightarrow C$; similarly, we can pick a $(Z \rightarrow X) \in S$ and a $z \in Z$ representing \tilde{z} . Then the inclusion $i: \{x\} \hookrightarrow X$ factors through both Y and Z by $x \mapsto y$ and $x \mapsto z$ respectively. Thus $i \in S$. Hence $\tilde{y} = \tilde{z}$ in C . \square

Corollary 4.3. *Let S be a sieve on X in \mathbf{CGWH} . Then the colimit over S taken in \mathbf{Top} is in \mathbf{CGWH} , i.e. $h(\underline{\text{colim}}_I \iota' U) = \underline{\text{colim}}_I \iota' U$. Moreover, the natural map $\varphi: \underline{\text{colim}}_S U \rightarrow X$ is an injection.*

Proof. We will make use of the following Proposition from [N.P. Strickland, 2009]: if Z is in \mathbf{CG} , then Z is weakly Hausdorff if and only if the diagonal subspace Δ_Z is closed in $Z \times Z$. Additionally, we remark that colimits of compactly generated spaces computed in \mathbf{Top} are automatically compactly generated.

Let $C = \underline{\text{colim}}_S \iota' U$, i.e. C is the colimit over S taken in \mathbf{Top} . By Proposition 4.2, the natural map $\varphi: C \rightarrow X$ is an injection; we remark that it is not the statement of Proposition 4.2 that gives this observation since S is not a sieve in \mathbf{Top} , instead the proof of Proposition 4.2 holds in this situation since $\{x\}$ is in \mathbf{CGWH} . Since X is \mathbf{CGWH} , then Δ_X is closed in $X \times X$. Since φ is a continuous injection, then $(\varphi \times \varphi)^{-1}(\Delta_X) = \Delta_C$ is closed in $C \times C$. \square

4.1 Basis and Presentation

The categories \mathbf{Sets} , \mathbf{Top} and \mathbf{CGWH} all satisfy the hypotheses of Theorems 2.9 and 2.8. Thus we have the following corollaries of Theorems 2.9 and 2.8 based on what the universal effective epimorphisms are in each category.

Proposition 4.4. *In **Sets**, $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}}$ is part of a basis for the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a surjection. In particular, a sieve of the form $S = \langle \{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \rangle$ on X is in the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a surjection. Moreover, every colim sieve is universal.*

Proof. It is easy to see in **Sets** that the effective epimorphisms are precisely the surjections. Since pulling back a surjection yields a surjection, then the universal effective epimorphisms in the category of sets are also the surjections. Lastly, this implies, by Theorem 2.8, that every colim sieve is universal. \square

Remark 4.5. Since **Sets** is a Grothendieck topos, we can compare Proposition 4.4 to the proof of Proposition 3.14. Specifically, Proposition 4.4 allows us to determine if a sieve is in the canonical topology by looking only at the sieve's generating set whereas the proof of Proposition 3.14 along with the Grothendieck topology J require us to look at the entire sieve.

Recall that a quotient map f is called *universal* if every pullback of f along a map yields a quotient map.

Proposition 4.6. *In **Top**, $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}}$ is part of a basis for the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a universal quotient map. Additionally, a sieve S on X is a (universal) colim sieve if and only if there exists some collection $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \subset S$ such that $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a (universal) quotient map. In particular, $T = \langle \{f: Y \rightarrow X\} \rangle$ is a (universal) colim sieve if and only if f is a (universal) quotient map.*

Proof. It is a well-known fact that in **Top** the effective epimorphisms are precisely the quotient maps. \square

Proposition 4.7. *In **CGWH**, $\{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}}$ is part of the basis for the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a quotient map. In particular, a sieve $S = \langle \{A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \rangle$ on X is in the canonical topology if and only if $\coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ is a quotient map. Moreover, every colim sieve is universal.*

Proof. This is a consequence of Corollary 2.4, Corollary 4.3, the fact that the universal effective epimorphisms in **Top** are precisely the universal quotient maps, and Proposition 2.36 from [N.P. Strickland, 2009], which states that every quotient map in **CGWH** is universal. \square

4.2 Examples in the category of Spaces

In this section we will use our basis to talk about some specific examples; including a special circumstance (when a sieve is generated by one function) and how the canonical topology on the categories **CGWH** and **Top** can differ in this situation.

Definition 4.8. For a category D , we call $\mathfrak{A} \subset \text{ob}(D)$ a weakly terminal set of D if for every object X in D , there exists some $A \in \mathfrak{A}$ and morphism $X \rightarrow A$ in D .

Additionally, if $F: D \rightarrow C$ is a functor and D has a weakly terminal set \mathfrak{A} , then we call $\{F(A)\}_{A \in \mathfrak{A}}$ a weakly terminal set of F .

For example, if $S = \langle \{A_\alpha \rightarrow X\}_{\alpha \in \mathfrak{A}} \rangle$ is a sieve on X then $\{A_\alpha\}_{\alpha \in \mathfrak{A}}$ is the weakly terminal set of U . Or as another example, $\{Y\}$ is the weakly terminal set of the diagram $Y \times_X Y \rightrightarrows Y$. One easy consequence of this in **Top** is a reduction of the colimit topology: V is open in the colimit if and only if the preimage of V is open in each member of the weakly terminal set.

Proposition 4.9. Let $F: D \rightarrow \mathbf{Spaces}$ be a functor where D has a weakly terminal set \mathfrak{A} . Suppose $f_A: F(A) \rightarrow X$ is an open map for all $A \in \mathfrak{A}$, then the induced map $\varphi: \text{colim}_D F \rightarrow X$ is an open map. Similarly, if the f_A are all closed and \mathfrak{A} is a finite set, then φ is a closed map.

Proof. Let $C = \text{colim}_D F$ and $i_A: F(A) \rightarrow C$ be the natural maps. Both results follow from the easy set equality below for $B \subset C$

$$\varphi(B) = \bigcup_{A \in \mathfrak{A}} f_A(i_A^{-1}(B))$$

since i_A^{-1} , f_A and unions respect open/closed sets in their respective scenarios. \square

Corollary 4.10. *Let $S = \langle \{f_\alpha: A_\alpha \rightarrow X\}_{\alpha \in \mathcal{A}} \rangle$ be a sieve on X in **Spaces** with the induced map $\eta: \coprod_{\alpha \in \mathcal{A}} A_\alpha \rightarrow X$ a surjection. If all of the f_α are open maps or if \mathcal{A} is a finite collection and all of the f_α are closed maps, then S is a colim sieve.*

Proof. Let $\varphi: \underset{S}{\operatorname{colim}} U \rightarrow X$ be the natural map. By Proposition 4.2, Corollary 4.3, and the surjectivity of η , φ is a continuous bijection. Then Proposition 4.9 implies that φ is open or closed, depending on the case, and hence an isomorphism. \square

This corollary leads us to some nice examples of sieves we would hope are in the canonical topology and actually are!

Example 4.11. Let X be any space and let $\{U_i\}_{i \in I}$ be an open cover of X . Then the inclusion maps $U_i \hookrightarrow X$ generate a universal colim sieve, call it S . Indeed, by Corollary 4.10, S is a colim sieve. Universality is obvious, as the preimage of an open cover is an open cover.

Example 4.12. Let X be any space and let K_1, \dots, K_n be a closed cover of X . For the exact same reasons as the previous example, the inclusions $K_i \hookrightarrow X$ generate a sieve in the canonical topology.

Before we give our next example, we provide a rephrasing of Theorem 1 from [B.J. Day and G.M. Kelly, 1970], which completely characterizes universal quotient maps in **Top**:

Theorem 4.13 (Day and Kelly, 1970). *Let $f: Y \rightarrow X$ be a quotient map. Then f is a universal quotient map if and only if for every $x \in X$ and cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of $f^{-1}(x)$ by opens in Y , there is a finite set $\{\alpha_1, \dots, \alpha_n\} \subset \Lambda$ such that $fG_{\alpha_1} \cup \dots \cup fG_{\alpha_n}$ is a neighborhood of x .*

Example 4.14. Consider the diagram $B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \dots$ and the direct limit $B = \underset{\rightarrow}{\operatorname{colim}} B_n$ in **Top**. Let $S = \langle \{\iota_n: B_n \rightarrow B \mid n \in \mathbb{N}\} \rangle$ where ι_n are the natural maps into the colimit. By Proposition 4.6, S is a colim sieve because $\coprod_{n \in \mathbb{N}} B_n \rightarrow B$ is obviously a quotient map. However, S is not necessarily in the canonical topology – we can use Proposition 4.6 on specific examples to see when S is and is not in the canonical topology.

For example, suppose there exists an N such that $B_m = B_N$ whenever $m > N$. Then $B = B_N$. Hence it is easy to see by Day and Kelly's condition that the map $\coprod_{n \in \mathbb{N}} B_n \rightarrow B$ is a universal quotient map. Therefore, the S from this example is in the canonical topology.

As another example, take $B_n = \mathbb{R}^n$ and let $B_n \rightarrow B_{n+1}$ be the closed inclusion map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. Use \mathbb{R}^∞ to denote the direct limit. We claim that $\coprod_{n \in \mathbb{N}} \mathbb{R}^n \rightarrow \mathbb{R}^\infty$ is not a universal quotient map. Indeed, consider Day and Kelly's condition; take $x = 0 \in \mathbb{R}^\infty$ and the open cover in $\coprod_{n \in \mathbb{N}} \mathbb{R}^n$ consisting of open disks $D^n \subset \mathbb{R}^n$ centered at the origin with fixed radius $\epsilon > 0$. Pick any finite collection D^{n_1}, \dots, D^{n_k} with $n_1 < \dots < n_k$. Then for $i = 1, \dots, k$ we can view D^{n_i} as a subset of \mathbb{R}^{n_k} . Hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i})$ is $\cup_{i=1}^k \iota_{n_i}(D^{n_i}) \subset \iota_{n_k}(\mathbb{R}^{n_k})$. However, by dimensional considerations, we can see that for all $b \in \mathbb{N}$, $\iota_b(\mathbb{R}^b)$ contains no open sets of \mathbb{R}^∞ and hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i})$ cannot be a neighborhood of x in \mathbb{R}^∞ . Remark: To see that $\iota_b(\mathbb{R}^b)$ contains no open sets, suppose to the contrary and call the open set V . Then $\iota_{b+1}^{-1}(V)$ is open in \mathbb{R}^{b+1} and in particular, contains an open ball of dimension $b+1$. Thus dimensional considerations imply that $\iota_{b+1}^{-1}(V)$ is not contained in the image of \mathbb{R}^b in \mathbb{R}^{b+1} . Since each ι_n is an inclusion map, then $\iota_{b+1}^{-1}(V) \not\subset \iota_{b+1}(\mathbb{R}^b)$ and so V is not contained in $\iota_b(\mathbb{R}^b)$, which is our contradiction. Therefore, the S from this example is not in the canonical topology.

Example 4.15. Consider the diagram $B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \dots$ and the direct limit $B = \operatorname{colim} B_n$ in **CGWH**. Let $S = \langle \{\iota_n: B_n \rightarrow B \mid n \in \mathbb{N}\} \rangle$ where ι_n are the natural maps into the colimit. Then by Proposition 4.7, S is a universal colim sieve because $\coprod_{n \in \mathbb{N}} B_n \rightarrow B$ is a quotient map.

Now we shift our focus to sieves that can be generated by one map, called *monogenic sieves*. There are many reasons one could focus on these kinds of sieves, however by Proposition 2.7, if we fully comprehend when monogenic sieves are in the canonical topology, then we can (in some sense) completely understand the canonical topology. From this point onward, this section will be about monogenic sieves; in other words, by Proposition 4.6 and Proposition 4.7, we will be focusing on (universal) quotient maps.

Remark 4.16. Some examples will talk about the space \mathbb{R}/\mathbb{Z} . In this section, this space is not a group quotient but instead is the squashing of the subspace \mathbb{Z} to a point.

Example 4.17. Consider the quotient maps $f: S^n \rightarrow \mathbb{R}P^n$ and $g: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. There is some subtlety, which will depend on the category we are in, in determining if f or g generate universal colim sieves. Throughout the rest of this section we will continue to explore this particular example.

MONOGENIC SIEVES IN **CGWH**

By Proposition 4.7, if X and Y are in **CGWH** and $h: Y \rightarrow X$, then $\langle\{h\}\rangle$ is in the canonical topology if and only if h is a quotient map. Therefore, we immediately get the following examples:

Example 4.18. Topological manifolds are in **CGWH**. Thus S^n and $\mathbb{R}P^n$ are in **CGWH**. Hence $\langle\{f: S^n \rightarrow \mathbb{R}P^n\}\rangle$ is in the canonical topology.

Example 4.19. Every CW-complex is in **CGWH**. Thus \mathbb{R} and \mathbb{R}/\mathbb{Z} are in **CGWH**. Hence $\langle\{g: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}\}\rangle$ is in the canonical topology.

MONOGENIC SIEVES IN **TOP**

This section will heavily rely on Theorem 4.13 (the Theorem by Day and Kelly characterizing universal quotient maps in **Top**) because a monogenic sieve generated by f is in the canonical topology if and only if f is a universal quotient map.

Example 4.20. Day and Kelly's theorem implies that every open quotient map is a universal quotient map. Therefore, the quotient map $f: S^n \rightarrow \mathbb{R}P^n$ is a universal quotient map and $\langle\{f: S^n \rightarrow \mathbb{R}P^n\}\rangle$ is in the canonical topology.

Example 4.21. The quotient map $g: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is not universal. We will demonstrate this in two ways, first by using Day and Kelly's theorem and second by directly showing g is not universal. Note: many sets of \mathbb{R}/\mathbb{Z} will be written as if they are in \mathbb{R} for ease of presentation.

(i) We will look at Day and Kelly's condition for $\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ with the open cover (in \mathbb{R}) $\{G_i := (i - m, i + m)\}_{i \in \mathbb{Z}}$ for a fixed $m \in (0, \frac{1}{2})$. For any open set U of \mathbb{R}/\mathbb{Z} containing \mathbb{Z} , the quotient topology tells us that $g^{-1}(U)$ is an open neighborhood of $\mathbb{Z} \subset \mathbb{R}$. But for any n , $g^{-1}(\bigcup_{k=1}^n gG_{i_k}) =$

$\mathbb{Z} \cup (\bigcup_{k=1}^n (i_k - m, i_k + m))$ is not a neighborhood of $\mathbb{Z} \subset \mathbb{R}$. So there cannot be any open set of \mathbb{R}/\mathbb{Z} containing \mathbb{Z} that is contained in $\bigcup_{k=1}^n gG_{i_k}$ for any finite collection of the cover.

(ii) To directly show that g is not universal we need to come up with a space and map to \mathbb{R}/\mathbb{Z} where g pulledback along this map is not a quotient map. Our candidate is the following: Let $t(\mathbb{R}/\mathbb{Z})$ be the set \mathbb{R}/\mathbb{Z} with the topology where U (written as if it is in \mathbb{R}) is said to be open if (a) $\mathbb{Z} \not\subset U$ or (b) U contains \mathbb{Z} and is a neighborhood (in the typical topology) of $(\mathbb{Z} - \{\text{finitely many or no points}\})$. Remark: this topology was used in Day and Kelly's paper (in the proof of their theorem), however they defined the topology using a filter and we have merely rephrased it for convenience.

Define $\kappa: t(\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$ by the set identity map; this is a continuous map. As a set, the pullback of $\text{domain}(g)$ along κ is \mathbb{R} but since it now has the limit topology, we denote the pullback as $t(\mathbb{R})$; in particular, $t(\mathbb{R})$ is \mathbb{R} with the discrete topology. Denote the projection maps as $g': t(\mathbb{R}) \rightarrow t(\mathbb{R}/\mathbb{Z})$ and $\kappa': t(\mathbb{R}) \rightarrow \mathbb{R}$.

We claim that g' is not a quotient map, i.e. there is some non-open set B in $t(\mathbb{R}/\mathbb{Z})$ with $(g')^{-1}(B)$ open in $t(\mathbb{R})$. Since every $(g')^{-1}(B)$ is open in $t(\mathbb{R})$, then we merely need to find a B that is not open in $t(\mathbb{R}/\mathbb{Z})$; $B = \{\mathbb{Z}\}$ obviously works.

The above example shows us that quotient maps of the form $X \rightarrow X/A$ may not generate universal colim sieves. So let's understand these special quotient maps a little better. Specifically, using Day and Kelly's theorem, we can completely state what kinds of subspaces A yield universal quotient maps $X \rightarrow X/A$:

Corollary 4.22. *The quotient map $\pi: X \rightarrow X/A$ is universal if and only if both of the following properties hold:*

1. *If A is not open, then for every open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of $(\partial A) \cap A$ in X there is a finite collection $\{\alpha_1, \dots, \alpha_n\} \subset \Lambda$ with $A \cup G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ open in X .*
2. *If A is not closed, then for every open U in X such that $U \cap (\overline{A} - A) \neq \emptyset$, $U \cup A$ is open in X .*

Proof. We will be using Theorem 4.13 in two ways: first by finding the necessary conditions for π to be a universal quotient map (i.e. proving the

forward direction) and then second by checking the sufficient conditions in the three cases (i) $x = A$, (ii) $x \in X - \bar{A}$, and (iii) $x \in \bar{A} - A$ (i.e. proving the backward direction).

First suppose that π is a universal quotient map. To see that the first property is necessary, assume that $(\partial A) \cap A \neq \emptyset$, i.e. A is not open, and we have an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of $(\partial A) \cap A$. Then we can expand this cover to an open cover of A by adding $\text{Int}(A)$ to $\{G_\alpha\}_{\alpha \in \Lambda}$. Now by assumption (using the point A in X/A) there is a finite subcollection $G_{\alpha_1}, \dots, G_{\alpha_n}, \text{Int}(A)$ such that $\pi G_{\alpha_1} \cup \dots \cup \pi G_{\alpha_n} \cup \pi \text{Int}(A)$ is a neighborhood of A in X/A . But $\pi \text{Int}(A) \subset \pi G_\alpha$ since $G_\alpha \cap A \neq \emptyset$ and so $\text{Int}(A)$ is not necessary in our finite subcollection. Thus $\pi G_{\alpha_1} \cup \dots \cup \pi G_{\alpha_n}$ is a neighborhood of A ; let U be an open subset of $\pi G_{\alpha_1} \cup \dots \cup \pi G_{\alpha_n}$ containing A . Now by looking at the preimages of U and $\bigcup_{i=1}^n \pi G_{\alpha_i}$ in X , we get that

$$A \subset \pi^{-1}(U) \subset \pi^{-1}\left(\bigcup_{i=1}^n \pi G_{\alpha_i}\right) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup A.$$

Since $\pi^{-1}(U)$ is open, then the above expression implies $A \subset \text{Int}(G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup A)$. But since all of the G_α are open, then $G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup A$ is open. Therefore, the first property is necessary.

To see that the second property is necessary, assume that A is not closed and U is any open neighborhood of a fixed $x \in \bar{A} - A$ in X . Since U is an open cover of $\pi^{-1}(\pi(x)) = x$, then by Theorem 4.13, πU is a neighborhood of x ; let V be an open subset of πU that contains x . Then by looking at the preimages of V and πU , we see (using that U intersects A nontrivially) that

$$A \subset \pi^{-1}(V) \subset \pi^{-1}(\pi U) = U \cup A.$$

But since $\pi^{-1}(V)$ is open, then $A \subset \text{Int}(U \cup A)$, i.e. $U \cup A$ is open. Therefore, the second condition is necessary.

Second let's assume the two conditions hold. We will show π is a universal quotient map by checking that the conditions of Theorem 4.13 hold in all three locations in X/A (i.e. for (i) $x = A$, (ii) $x \in X - \bar{A}$, and (iii) $x \in \bar{A} - A$).

(i) For $A \in X/A$, take any open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of A in X . If A is open in X , then $\{A\}$ is open in X/A and hence every πG_α is a neighborhood. If A is not open, let Γ be the finite portion of Λ that property 1 guarantees exists,

i.e. $A \cup \left(\bigcup_{i \in \Gamma} G_{\alpha_i}\right)$ is open in X and each G_{α_i} intersects A nontrivially. This implies that $\bigcup_{i \in \Gamma} \pi G_{\alpha_i}$ is an open neighborhood of A in X/A (since its preimage is $A \cup \left(\bigcup_{i \in \Gamma} G_{\alpha_i}\right)$).

(ii) Any $x \in X - \overline{A}$ has an open neighborhood $U_x \subset X - \overline{A}$. Notice that π is a homeomorphism on $X - \overline{A}$. Thus for any such x and any open cover W of $\pi^{-1}(x) = x$ in X , πW is a neighborhood of x because the open neighborhood (in X/A) $U_x \cap W$ is contained in πW .

(iii) If A is closed, then this is trivial so assume that A is not closed and let $x \in \overline{A} - A$. For any open cover W of $\pi^{-1}(x) = x$ in X , $\pi^{-1}(\pi W) = W \cup A$, which is open in X by condition 2. Thus πW is an open neighborhood of x in X/A .

Therefore, our two conditions ensure that π satisfies Day and Kelly's universal quotient map condition. \square

Corollary 4.22 now gives us a way to produce more examples of sieves in the canonical topology:

Example 4.23. Every quotient of a Hausdorff space by a compact subspace is universal. For example, $\pi: D^n \rightarrow S^n$ (where $S^n = D^n / \partial D^n$) generates a universal colim sieve.

Example 4.24. If A is closed, then $S = \langle \{X \rightarrow X/A\} \rangle$ is always a colim sieve. Moreover, it is universal if and only if ∂A is compact. For example, this tells us $\langle \{\mathbb{R} \rightarrow \mathbb{R}/[0, \infty)\} \rangle$ is in the canonical topology and reaffirms that $\langle \{\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}\} \rangle$ is not.

5. Universal Colim Sieves in the Category of R -modules

The category of R -modules does not satisfy the assumptions of Theorem 2.8 or Theorem 2.9. Indeed, coproducts and pullbacks of R -modules do not commute (for example, let $\mathbb{Z}_{(a,b)}$ denote the domain of $\mathbb{Z} \rightarrow \mathbb{Z}^2$, $1 \mapsto (a, b)$, then we see that $(\mathbb{Z}_{(1,0)} \oplus \mathbb{Z}_{(0,1)}) \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)} \cong \mathbb{Z}$ but $(\mathbb{Z}_{(1,0)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \oplus (\mathbb{Z}_{(0,1)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \cong 0$). Thus we do not have basis and presentation results. Instead, we have some smaller results, reductions and examples.

Notation 5.1. Let R be a commutative ring with identity. We will use $R\text{-Mod}$ for the category of R -modules and \mathbf{Ab} for the category of abelian groups.

We start with some basic results.

Corollary 5.2. *Any sieve containing a universal effective epimorphism (e.g. a surjection in $R\text{-Mod}$ or in \mathbf{Sets}) is a universal colim sieve.*

Proof. This is an immediate consequence of Theorem 2.5 and Corollary 2.4. \square

Lemma 5.3. *In $R\text{-Mod}$, if a sieve S on X can be generated by at most two morphisms, then the canonical map $c: \underset{S}{\text{colim}} U \rightarrow X$ is an injection.*

Proof. Suppose $S = \langle \{f: Y \rightarrow X, g: Z \rightarrow X\} \rangle$ and $c(x) = 0$. Since every map in S either factors through f or g , then x , as an element of $\bigoplus_{A \rightarrow X \in S} A$, is really an element $(y, z) \in Y \oplus Z$ in the colimit. So $c(x) = 0$ implies that $y + z = 0$ in X , i.e. $(y, -z) \in Y \times_X Z$. Thus $y \in Y$ gets identified with $-z \in Z$ in the colimit; hence $(y, z) = (0, z - z) = 0$ in the colimit. Therefore, $x = 0$ in the colimit and the map c is an injection.

Using the fact that $\langle \{A_i \rightarrow X\}_\alpha \rangle = \langle \{A_i \rightarrow X\}_\alpha \cup \{Z \xrightarrow{0} X\} \rangle$, we can say that any sieve generated by one morphism is also generated by two morphisms. This completes the proof. \square

Proposition 5.4. *In $R\text{-Mod}$, let*

$$S = \langle \{f: Y \rightarrow X\} \rangle \quad \text{and} \quad T = \langle \{g: U \rightarrow X, h: V \rightarrow X\} \rangle$$

be sieves on X . Then

1. *S is a universal colim sieve if and only if f is a surjection.*
2. *T is a colim sieve if and only if $g \oplus h: U \oplus V \rightarrow X$ is a surjection.*

Proof. For part 2, Lemma 5.3 tells us that we only need to worry about the surjectivity of $\underset{T}{\text{colim}} U \rightarrow X$ but this is exactly what the above condition is.

For part 1, Lemma 5.3 and Lemma 2.1 tell us that we only need worry about the surjectivity of $A \times_X Y \xrightarrow{\pi_1} A$ (the generator of k^*S) for every map $k: A \rightarrow X$. But $A \times_X Y = \{(a, y) \in A \times Y \mid k(a) = f(y)\}$. Hence π_1 is a surjection for every map k if and only if f is a surjection. \square

Lemma 5.5. *In $R\text{-Mod}$, suppose $S = \langle \{f_i: M_i \rightarrow R\}_{i \in I} \rangle$ is a sieve on R such that for every $i \in I$ there exists an $a_i \in R$ with $\text{im}(f_i) = a_i R$. If the ideal $(a_i \mid i \in I)$ equals R , then for every R -module homomorphism $g: N \rightarrow R$, the natural map $\text{colim}_{\rightarrow g^*S} U \rightarrow N$ is a surjection.*

Proof. By Proposition 2.2 it suffices to show that $\eta: \bigoplus_i M_i \times_R N \rightarrow N$ is a surjection. Let $\pi_i: M_i \times_R N \rightarrow N$ be the natural map. Fix $x \in N$. Then $a_i g(x) \in a_i R = \text{im}(f_i)$ and $a_i g(x) \in \text{im}(g)$. Thus $a_i \cdot x \in \text{im}(\pi_i) \subset N$ for all $i \in I$. Therefore, $x = 1_R \cdot x$ is in $\bigoplus_i \text{im}(\pi_i) = \text{im}(\eta)$ since R is a unital ring and $(a_i \mid i \in I) = R$. \square

Proposition 5.6. *Suppose $S = \langle \{f_1: M_1 \rightarrow R, f_2: M_2 \rightarrow R\} \rangle$ is a sieve on R such that $\text{im}(f_i) = a_i R$ for $i = 1, 2$. Then S is in the canonical topology on $R\text{-Mod}$ if and only if $(a_1, a_2) = R$.*

Proof. If S is in the canonical topology, then S is a colim sieve and hence by Proposition 5.4, $a_1 R \oplus a_2 R = R$.

If $(a_1, a_2) = R$, then by Proposition 5.4, S is a colim sieve. The universality of S follows immediately from Lemma 2.1, Proposition 5.4 and Lemma 5.5. \square

Next we include two results that can help us identify when a sieve is not in the canonical topology.

Proposition 5.7. *Let R be any nonzero ring. Let $S = \langle \{f_i: A_i \rightarrow X\}_{i \in I} \rangle$ be any sieve on X for any nonzero R -module X . If there exists a nonzero $b \in X$ such that $\text{span}_R(b) \subset (X - \bigcup_I \text{Im}(f_i)) \cup \{0\}$, then S is not a universal colim sieve.*

Proof. Suppose such a $b \in X$ exists. Define $g: R \rightarrow X$ by $1 \rightarrow b$. Then $\text{Im}(g) \cap \text{Im}(f_i) = \{0\}$ for all i . Thus for all i , the pullback $R \times_X A_i = \ker(g) \times \ker(f_i)$ and the image of the natural map $R \times_X A_i \rightarrow R$ is $\ker(g)$. In particular, $\text{Im}(\bigoplus_i R \times_X A_i \rightarrow R) = \ker(g)$, which by construction is not R . Therefore, $\text{colim}_{\rightarrow g^*S} U \rightarrow R$ is not surjective and so g^*S not a colim sieve on R . \square

Proposition 5.8. *Let R be an infinite principal ideal domain. Let*

$$S = \langle \{g_i: R^n \hookrightarrow R^n\}_{i=1}^M \cup \{f_i: R^{m_i} \hookrightarrow R^n \mid m_i < n\}_{i=1}^N \rangle$$

be a sieve on R^n . If S is a universal colim sieve, then $g_1 \oplus \cdots \oplus g_M: R^{nM} \rightarrow R^n$ is a surjection.

Proof. Let $G = g_1 \oplus \cdots \oplus g_M$. Suppose that G is not a surjection. We will produce a map ϕ that shows S is not universal.

By a change of basis (which is allowable by Lemma 2.3) we may assume that $G = \text{diag}(d_1, d_2, \dots, d_n)$ with $d_i | d_{i+1}$. Because G is not surjective, then d_n is not a unit. Indeed, if d_n was a unit, then all of the d_i 's would also be units and thus G would be surjective. By Lemma 5.9 below, there exists an $x \in R^{n-1}$ so that $\text{span}_R\{(x, 1)\} \cap \text{Im}(f_i) = \{0\}$ for all $i = 1, \dots, N$. Additionally, since d_n is not a unit, then $(x, 1) \notin \text{Im}(G)$.

Define $\phi: R \rightarrow R^n$ by $1 \mapsto (x, 1)$. We will show that ϕ^*S is not a colim sieve. First we will simplify the generating set of ϕ^*S . By the choice of x , the pullback module of R^{m_i} along ϕ is $\{0\}$ for all $i = 1, \dots, N$. Therefore, we can write ϕ^*S as $\phi^*S = \langle \{\pi_i: R^n \times_{R^n} R \rightarrow R\}_{i=1}^M \rangle$ where the π_i are the pullbacks of the g_i along ϕ . Since $(x, 1) \notin \text{Im}(G)$ and we have the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^M R_i^n \times_{R^n} R & \xrightarrow{\bigoplus_{i=1}^M \pi_i} & R \\ \downarrow & & \downarrow \phi \\ \bigoplus_{i=1}^M R_i^n & \xrightarrow{G} & R^n \end{array}$$

then $1 \notin \text{Im}(\pi_1 \oplus \cdots \oplus \pi_M)$. Therefore, $\eta: \underset{\phi^*S}{\text{colim}} U \rightarrow R$ is not surjective;

hence ϕ^*S is not a colim sieve. \square

Lastly, for completeness we include the linear algebra result referenced in Proposition 5.8.

Lemma 5.9. *Let R be an infinite principal ideal domain. For any finite collection V_1, \dots, V_N of submodules of R^n with $\dim(V_i) < n$, there exists an $x \in R^{n-1}$ such that $\text{span}_R\{(x, 1)\} \cap V_i = \{0\}$ for all i .*

Proof. Let F be the quotient field of R . Let

$$W_i = V_i \otimes_R F.$$

We will use F^{n-1} to refer to the subspace $\{(a_1, \dots, a_{n-1}, 0) \mid a_i \in F\}$ in F^n . For each $V_i \not\subseteq F^{n-1}$, fix an element $\nu_i \in V_i$ such that $\nu_i \notin F^{n-1}$ and

write $\nu_i = (v_{i1}, \dots, v_{in})$. Let $\nu_i^0 = (v_{i1}, \dots, v_{i(n-1)}, 0)$. Lastly, for each $V_i \not\subset F^{n-1}$, define a vector space map $\phi_i: W_i \rightarrow F^{n-1}$ by $w = (w_1, \dots, w_n) \mapsto w - \frac{w_n}{v_{in}}\nu_i$

Ideally, we will find an x such that $(x, 1) \notin W_i$ for all i . So first, let's see what kinds of $(z, 1)$ are in W_i by computing $\phi_i(z, 1)$.

$$\begin{aligned}\phi_i(z, 1) &= (z, 1) - \frac{1}{v_{in}}\nu_i \\ &= z - \frac{1}{v_{in}}\nu_i^0\end{aligned}$$

Thus

$$z = \phi_i(z, 1) + \frac{1}{v_{in}}\nu_i^0.$$

Therefore, if $(z, 1) \in W_i$, then $z = \phi_i(z, 1) + \frac{1}{v_{in}}\nu_i^0$. Based on this result, define $\Gamma_i = \text{im}(\phi_i) \oplus \text{span}_F\{\nu_i^0\}$. So $(z, 1) \in W_i$ implies $z \in \Gamma_i$.

For each index i exactly one of the following is true:

1. $W_i \subset F^{n-1}$,
2. $W_i \not\subset F^{n-1}$ and $\dim_F(\Gamma_i) < n - 1$,
3. $W_i \not\subset F^{n-1}$ and $\Gamma_i = F^{n-1}$.

For every index j in collection 1, every $x \in R^{n-1}$ satisfies the equation $\text{span}_R\{(x, 1)\} \cap V_j = \{0\}$. Thus when picking our x , we only need to consider the indices in collections 2 and 3.

For each index i in collection 2, Γ_i is a proper subspace of F^{n-1} . Since there are only finitely many Γ_i and F is an infinite field, then there exists a $y = (y_1, \dots, y_{n-1})$ such that $y \neq 0$ and $\text{span}_F\{(y, 0)\} \cap \Gamma_i = \{0\}$ for all i in collection 2. By multiplying y by an appropriate $s \in F$ we can clear denominators and so we may assume that $y \in R^{n-1}$. In particular, for all $r \in R$, $ry \notin \Gamma_i$, which implies that $(ry, 1) \notin W_i$. Therefore, for all $r \in R$, $\text{span}_R\{(ry, 1)\} \cap V_i = \{0\}$ for all indices in collection 2.

Continuing with the y from the previous paragraph, we now consider the indices k in collection 3 and their corresponding Γ_k . In this situation, $(y, 0) \in \Gamma_k$, i.e. $y = \phi_k(z) + u_k\nu_k^0$ for some $z \in W_k$ and $u_k \in F$. Since R is an infinite ring and collection 3 contains finitely many indices k , we can

pick a nonzero $\rho \in R$ such that for all k , $\rho u_k \in R$ and $\rho u_k \neq \frac{1}{v_{kn}}$. Thus $\rho y \neq \phi_k(a) + \frac{1}{v_{kn}} \nu_k^0$ for any $a \in W_k$, which implies that $(\rho y, 1) \notin W_k$. Therefore, $\text{span}_R\{(\rho y, 1)\} \cap V_k = \{0\}$ for all indices in collection 3.

We can take $x = \rho y$. \square

EXAMPLES

Here we include a few examples and non-examples of sieves in the canonical topology for various rings R .

Example 5.10. In the category of R -modules every surjective map generates a universal colim sieve (see Proposition 5.4). As more specific examples, the sieve $\langle\{\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \mid 1 \mapsto 1\}\rangle$ is in the canonical topology on \mathbf{Ab} , and in $R\text{-Mod}$ the sieve $\langle\{R^n \rightarrow R \mid (a_1, \dots, a_n) \mapsto a_1\}\rangle$ is in the canonical topology.

Example 5.11. By Proposition 5.6, $\langle\{R \xrightarrow{a} R, R \xrightarrow{b} R\}\rangle$ is in the canonical topology if and only if $(a, b) = R$. As more specific examples, the sieve $\langle\{\mathbb{Z} \xrightarrow{2} \mathbb{Z}, \mathbb{Z} \xrightarrow{3} \mathbb{Z}\}\rangle$ is in the canonical topology on \mathbf{Ab} ; when the function $\cdot g(x): C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ is the map $f(x) \mapsto (g \cdot f)(x)$, then the sieve $\langle\{C^\infty(\mathbb{R}) \xrightarrow{x} C^\infty(\mathbb{R}), C^\infty(\mathbb{R}) \xrightarrow{\sin(x)} C^\infty(\mathbb{R})\}\rangle$ is not in the canonical topology on $C^\infty(\mathbb{R})$ -modules.

Example 5.12. The sieve $S = \langle\{R \xrightarrow{i_1} R^2, R \xrightarrow{i_2} R^2\}\rangle$ where $i_1(1) = (1, 0)$ and $i_2(1) = (0, 1)$ (in the category of R -modules for nontrivial R) is not in the canonical topology. By Proposition 5.4, S is clearly a colim sieve so to see that S is not universal consider the map $\Delta: R \rightarrow R^2, 1 \mapsto (1, 1)$. Then for $k = 1, 2$, i_k pulled back along Δ yields the zero map $z: 0 \rightarrow R$. Hence Lemma 2.1 says $\Delta^*S = \langle\{z: 0 \rightarrow R\}\rangle$, which is clearly not a colim sieve.

Similarly $\langle\{R \xrightarrow{i_k} R^n \mid k = 1, \dots, n\}\rangle$ is a colim sieve but is not in the canonical topology. (This is also a consequence of Proposition 5.7.)

Example 5.13. Let $S = \langle\{f_k: \mathbb{Q} \rightarrow \mathbb{Q}[t] \mid f_k(1) = 1+t+\dots+t^k\}_{k=1}^\infty\rangle$ in the category of rational vector spaces. This S is not in the canonical topology. (This is a direct consequence of Proposition 5.7 using $b = t$.)

Example 5.14. Let F be an infinite field. In the category of F vector spaces, a sieve of the form $S = \langle \{F^{m_i} \hookrightarrow F^n \mid m_i \leq n\}_{i=1}^M \rangle$ is in the canonical topology if and only if $m_i = n$ for some i if and only if S contains an isomorphism. (This is a consequence of Proposition 5.8.)

Proposition 5.15. Consider the diagram $B_1 \hookrightarrow B_2 \hookrightarrow B_3 \hookrightarrow \dots$ made with only injective maps and the direct limit $B := \operatorname{colim} B_n$ in $R\text{-mod}$. Let the maps $\iota_n: B_n \rightarrow B$ be the natural maps into the colimit. Then the sieve $\langle \{\iota_n \mid n \in \mathbb{N}\} \rangle$ is a universal colim sieve.

Proof. Let $\Gamma: \mathbb{N} \rightarrow S$ by $n \mapsto \iota_n$. Notice that Γ is a final functor; this is easy to see since the injectivity of ι_n and the maps in our diagram imply that $B_i \times_B B_j \cong B_{\min(i,j)}$. Thus $\operatorname{colim}_S U$ exists and $\operatorname{colim}_S U \cong \operatorname{colim}_{\mathbb{N}} U\Gamma \cong B$. Therefore, S is a colim sieve.

To see that S is universal, let $f: X \rightarrow B$ and set $X_i := X \times_B B_i$. For each $n \in \mathbb{N}$, ι_n and $B_n \rightarrow B_{n+1}$ are both injective maps; this implies that the natural maps $X_n \rightarrow X_{n+1}$ and $X_n \rightarrow X$ are also injective maps since the pullback of an injection in $R\text{-Mod}$ is an injection and $X_i \cong X_{i+1} \times_{B_{i+1}} B_i$. Additionally, it is an easy exercise to see that the direct limit $\operatorname{colim} X_i$ is isomorphic to X . In other words, f^*S is the type of sieve described in the assumptions of this proposition and proved to be a colim sieve in the previous paragraph. \square

Example 5.16. Take $B_n = \mathbb{R}^n$ and let $B_n \rightarrow B_{n+1}$ be the inclusion map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. Use \mathbb{R}^∞ to denote the direct limit. Then the above proposition shows that $\langle \{\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty\}_{n \in \mathbb{N}} \rangle$ is in the canonical topology on the category of \mathbb{R} vector spaces. (Compare this to Example 4.14.)

REDUCTIONS

In this part we prove some reductions that allow us to limit our view (of sieve generating sets and the maps universality must be checked over) to the non-full subcategory of free modules with injective maps when R is ‘nice.’ The first reduction will be reducing the types of sieves we need to look at:

Proposition 5.17 (Reduction 1). *In $R\text{-Mod}$, let S be a sieve on X . Then the following are equivalent*

1. S is a universal colim sieve

2. f^*S is a universal colim sieve for every surjection $f: Y \rightarrow X$
3. f^*S is a universal colim sieve for some surjection $f: Y \rightarrow X$

Proof. It is obvious that 1 implies 2 and 2 implies 3, so it suffices to show 3 implies 1.

Assume f^*S is a universal colim sieve for some fixed surjection $f: Y \rightarrow X$. Set $T = \langle \{f: Y \rightarrow X\} \rangle$. By Proposition 5.4, T is a universal colim sieve since f is a surjection. We will now use T together with the Grothendieck topology's transitivity axiom to show that S is a universal colim sieve. Notice that S satisfies the hypotheses of this axiom with respect to T . Indeed, since every $g \in T$ factors as $f \circ k$ for some k , then $g^*S = (fk)^*S = k^*(f^*S)$, which implies that g^*S is a universal colim sieve (as f^*S is universal) for every $g \in T$. Therefore, by the transitivity axiom of a Grothendieck topology, S is a universal colim sieve. \square

To rephrase our first reduction: S is a universal colim sieve on X if and only if f^*S is a universal colim on R^n where $f: R^n \rightarrow X$ is a surjection (note that n is not necessarily assumed to be finite). This reduction means that we can restrict our view to free modules (not necessarily finitely generated). Specifically, we only need to look at sieves on free modules and check the universality condition on free modules. Indeed, S is a universal colim sieve on X if and only if for all $g: Y \rightarrow X$, g^*S is a universal colim sieve on Y if and only if for all $g: Y \rightarrow X$, $(gf)^*S$ is a universal colim sieve on R^n for some surjection $f: R^n \rightarrow Y$.

Proposition 5.18 (Reduction 2). *In $R\text{-Mod}$ when R is a principal ideal domain, every sieve on R^n equals a sieve of the form*

$$\langle \{g_i: R^{m_i} \hookrightarrow R^n: m_i \leq n\}_{i \in I} \rangle$$

where the g_i are injections.

Proof. Let $S = \langle \{f_i: A_i \rightarrow R^n\}_{i \in I} \rangle$ be a sieve on R^n . Set

$$T = \langle \{g_i: \text{Im}(f_i) \rightarrow R^n\}_{i \in I} \rangle$$

where the g_i 's are inclusion maps. Since R is a PID and $\text{Im}(f_i)$ is a submodule of R^n , then $\text{Im}(f_i) \cong R^{m_i}$ for some $m_i \leq n$. Thus T is of the desired

form and we will show that $S = T$. First notice that $S \subset T$. To get that T is a subcollection of S , notice that $\tilde{f}_i: A_i \rightarrow \text{Im}(f_i)$ (i.e. f_i with a different codomain) is split because \tilde{f}_i is a surjective map onto a projective module; call the splitting χ_i . Hence $g_i = g_i \circ \tilde{f}_i \circ \chi_i = f_i \circ \chi_i$ implies that $T \subset S$ and completes the proof. \square

To rephrase our second reduction: when talking about sieves on R^n , we only need to talk about sieves generated by injections of free modules. Thus we can restrict our view of sieve generating sets to the non-full subcategory of free modules with injective morphisms.

Our next reduction will also assume R is a principal ideal domain. In particular, fix n and a map $f: X \rightarrow R^n$ for some R -module X . Then since R is a PID, we may write

$$X \cong R^m \oplus K \quad \text{for some } m \leq n, \text{ where}$$

$$R^m \cong \text{Im}(f), \quad K = \ker(f), \quad f = g + z \quad \text{with}$$

$$g: R^m \rightarrow R^n \text{ an injection and } z: K \rightarrow R^n \text{ the zero map.}$$

Proposition 5.19 (Reduction 3). *Let R be a principal ideal domain, S be a sieve on R^n in $R\text{-Mod}$ and $f: X \rightarrow R^n$. Then, using the set-up described in the previous paragraph,*

$$\frac{\text{colim}}{f^*S} U \cong \left(\frac{\text{colim}}{g^*S} U \right) \oplus \left(\frac{\text{colim}}{z^*S} U \right).$$

Moreover, z^*S is a universal colim sieve; hence f^*S is a colim sieve if and only if g^*S is a colim sieve.

Sketch of Proof. By Proposition 5.18, we may assume that S can be written in the form $S = \langle \{\eta_i: R^{p_i} \hookrightarrow R^n: p_i \leq n\}_{i \in I} \rangle$. Consider the diagrams \mathcal{X} , \mathcal{R} and \mathcal{K} defined as:

$$\mathcal{X} = \left(\begin{array}{c} \bigoplus_{i \in I} (R^{p_i} \times_{R^n} X) \times_X (R^{p_i} \times_{R^n} X) \\ \downarrow \downarrow \\ \bigoplus_{i \in I} (R^{p_i} \times_{R^n} X) \end{array} \right),$$

$$\mathcal{R} = \left(\begin{array}{c} \bigoplus_{i \in I} (R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_i} \times_{R^n} R^m) \\ \downarrow \downarrow \\ \bigoplus_{i \in I} (R^{p_i} \times_{R^n} R^m) \end{array} \right), \text{ and}$$

$$\mathcal{K} = \left(\begin{array}{c} \bigoplus_{i \in I} (R^{p_i} \times_{R^n} K) \times_K (R^{p_i} \times_{R^n} K) \\ \downarrow \downarrow \\ \bigoplus_{i \in I} (R^{p_i} \times_{R^n} K) \end{array} \right)$$

First we look at the objects of \mathcal{X} . Since each η_i is injective, then for all i

$$R^{p_i} \times_{R^n} X \cong (R^{p_i} \times_{R^n} R^m) \oplus (R^{p_i} \times_{R^n} K)$$

and for all i, q

$$\begin{aligned} (R^{p_i} \times_{R^n} X) \times_X (R^{p_q} \times_{R^n} X) \\ \cong ((R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_q} \times_{R^n} R^m)) \oplus ((R^{p_i} \times_{R^n} K) \times_K (R^{p_q} \times_{R^n} K)). \end{aligned}$$

In other words, $\mathcal{X} \cong \mathcal{R} \oplus \mathcal{K}$. But since colimits “commute” with colimits, then $\text{Coeq}(\mathcal{X}) \cong \text{Coeq}(\mathcal{R}) \oplus \text{Coeq}(\mathcal{K})$. Now by Lemma 2.1 and Proposition 2.2, the first part has been proven, i.e.

$$\frac{\text{colim}}{f^*S} U \cong \left(\frac{\text{colim}}{g^*S} U \right) \oplus \left(\frac{\text{colim}}{z^*S} U \right).$$

Next we notice that z^*S is a universal colim sieve. Indeed, since η_i is an injection and z is the zero map, it easily follows that $z^*S = \langle \{id: K \rightarrow K\} \rangle$.

To complete the proof, notice that we have the following commutative diagram

$$\begin{array}{ccc} \text{Coeq}(\mathcal{X}) \cong \text{Coeq}(\mathcal{R}) \oplus \text{Coeq}(\mathcal{K}) & & \\ \left(\begin{array}{c} \chi \quad \downarrow \rho \\ X \cong R^m \oplus K \end{array} \right) \kappa & & \end{array}$$

where the vertical maps are the obvious canonical maps. This $\chi = \rho \oplus \kappa$ is an isomorphism if and only if both ρ and κ are isomorphisms. We have

already shown that κ is an isomorphism (as z^*S is a universal colim sieve), thus this diagram implies that χ is an isomorphism if and only if ρ is; hence f^*S is colim sieve if and only if g^*S is a colim sieve. \square

Lastly, we rephrase our third reduction:

Corollary 5.20. *When R is a PID, a sieve on R^n is a universal colim sieve if and only if f^*S is a colim sieve for every injection $f: R^m \rightarrow R^n$.*

All together our reductions basically allow us to work in the subcategory of free modules with injective morphisms instead of in $R\text{-Mod}$.

5.1 The Category of Abelian Groups

This section will be primarily made up of examples. Additionally, we include a characterization of sieves on \mathbb{Z} and one result for sieves on larger free abelian groups.

Example 5.21. By Corollary 5.6, $\langle \{\mathbb{Z} \xrightarrow{\times a} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times b} \mathbb{Z}\} \rangle$ is a universal colim sieve if and only if a and b are relatively prime.

Example 5.22. The sieve $S = \langle \{\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}\} \rangle$ is a universal colim sieve on $\mathbb{Z}/4\mathbb{Z}$ by Corollary 5.2. Additionally, S is not monogenic, i.e. it cannot be written as a sieve generated by one morphism.

Example 5.23. Let $S = \langle \{g: \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n\} \cup \{f_i: \mathbb{Z}^{m_i} \hookrightarrow \mathbb{Z}^n \mid m_i < n\}_{i=1}^N \rangle$ be a sieve on \mathbb{Z}^n . Then S is a universal colim sieve if and only if g is a surjection, i.e. g is an isomorphism. (This is a direct corollary of Proposition 5.8 and Corollary 5.2.)

Ideally, we would like to know a ‘nice’ basis for the canonical topology on \mathbf{Ab} , like the bases in Section 4.1; to start moving towards this ideal, we look at the simplest free group, \mathbb{Z} . In Example 5.21 we see that a relative prime pair of numbers will generate a universal colim sieve; this is actually true in general, specifically:

Proposition 5.24. *Let $S = \langle \{\mathbb{Z} \xrightarrow{\times a_i} \mathbb{Z}\}_{i=1}^N \rangle$ be a sieve on \mathbb{Z} . Then S is a universal colim sieve if and only if $\gcd(a_1, \dots, a_N) = 1$.*

Proof. First assume that S is a universal colim sieve. In particular, the map $\text{colim}_S U \rightarrow \mathbb{Z}$ is a surjection, i.e. $\mathbb{Z}^N \rightarrow \mathbb{Z}$, $(x_1, \dots, x_N) \mapsto a_1x_1 + \dots + a_Nx_N$ is a surjection. Therefore, $(a_1, \dots, a_N) = \mathbb{Z}$ and this proves the forward direction.

Now assume that $\gcd(a_1, \dots, a_N) = 1$. We will break the proof that S is a universal colim sieve up into several pieces. First we will reduce the proof to showing that S is a colim sieve. By the reductions (Propositions 5.17, 5.18 and 5.19), universality only needs to be checked along maps of the form $f: \mathbb{Z} \xrightarrow{\times k} \mathbb{Z}$ where $k \neq 0$. Fix $k \neq 0$, i.e. fix f , and write \mathbb{Z}_b for the domain of $\mathbb{Z} \xrightarrow{\times b} \mathbb{Z}$. By Lemma 2.1, $f^*S = \langle \{\pi_i: \mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \rightarrow \mathbb{Z}_k\}_{i=1}^N \rangle$. Moreover, it is easy to see that the pullback $\mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \cong \mathbb{Z}$ and π_i must be multiplication by $\frac{a_i}{\gcd(a_i, k)}$. Since $\gcd(a_1, \dots, a_N)$ equals 1, then $\gcd\left(\frac{a_1}{\gcd(a_1, k)}, \dots, \frac{a_N}{\gcd(a_N, k)}\right) = 1$ and hence f^*S has the same form as S . Specifically, any argument showing that S is a colim sieve will similarly show that f^*S is a colim sieve. Therefore, it suffices to show that S is a colim sieve.

To see that S is a colim sieve, i.e. to see that the map $\text{colim}_S U \rightarrow \mathbb{Z}$ induced by a_1, \dots, a_N is an isomorphism, let $\alpha = \frac{N(N-1)}{2}$ and notice that

$$\begin{aligned} \text{colim}_S U &\cong \text{Coeq} \left(\begin{array}{c} \bigoplus_{i=1}^{\alpha} \mathbb{Z} \\ \downarrow \downarrow \\ \bigoplus_{i=1}^N \mathbb{Z} \end{array} \right) \\ &\cong \text{Cokernel} (\phi: \mathbb{Z}^{\alpha} \rightarrow \mathbb{Z}^N) \end{aligned}$$

for some map ϕ where the first isomorphism comes from Lemma 2.2 and the last isomorphism comes from the fact that we are working in an abelian category. Now this map ϕ happens to be the third map in the Taylor resolution of \mathbb{Z} , i.e. ϕ_1 in [J. Mermini, 2012]. We make two remarks about this previous sentence: (1) we will not prove that our ϕ is [J. Mermini, 2012]'s ϕ_1 , although this is easy to observe, and (2) the Taylor resolution in [J. Mermini, 2012] is specifically for polynomial rings, not \mathbb{Z} , however, both the definition of the Taylor resolution and the proof that it is in fact a free resolution are analo-

gous. Here is the end of the Taylor resolution:

$$\dots \rightarrow \mathbb{Z}^\alpha \xrightarrow{\phi} \mathbb{Z}^N \xrightarrow{(a_1 \dots a_N)} \mathbb{Z} \rightarrow \mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \rightarrow 0$$

Since $\gcd(a_1, \dots, a_N) = 1$, then it follows that $(a_1 \dots a_N)$ is a surjection and $\mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \cong 0$. Thus we obtain $0 \rightarrow \text{Im}(\phi) \rightarrow \mathbb{Z}^N \rightarrow \mathbb{Z} \rightarrow 0$, which is an exact sequence and hence implies that the cokernel of ϕ is \mathbb{Z} . Additionally, since $(a_1 \dots a_N)$ induced our map $\text{colim}_S U \rightarrow \mathbb{Z}$, then this short exact sequence also says that S is a colim sieve. \square

Because of Proposition 5.24, we can now easily determine when a sieve on \mathbb{Z} is in the canonical topology and we can easily come up with examples; for example, $\langle \{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 10} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 12} \mathbb{Z}\} \rangle$ is in the canonical topology whereas the sieve $\langle \{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 50} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 20} \mathbb{Z}\} \rangle$ is not. One may hope for a similar outcome for sieves on \mathbb{Z}^n when $n \geq 2$, however, the Taylor resolution used in the proof of Proposition 5.24 does not seem to generalize in a suitable manner. Instead, we have a proposition that may tell us when a potential sieve is not in the canonical topology.

Proposition 5.25. *Let $S = \langle \{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N \rangle$ where A_i is a diagonal matrix with $\det(A_i) \neq 0$. Then there exists a map $\beta: \mathbb{Z} \rightarrow \mathbb{Z}^n$ such that β^*S is not a colim sieve if and only if $\gcd(\det(A_1), \dots, \det(A_N)) \neq 1$.*

Proof. First we set up some notation: Let $A_i = \text{diag}(a_{1i}, \dots, a_{ni})$ and \mathbb{Z}_i^n be the domain of A_i .

To prove the backward direction, suppose that $\gcd(\det(A_1), \dots, \det(A_N))$ does not equal 1. We can rephrase the assumptions as $a_{ik} \neq 0$ for all k and there exists a prime q such that q divides the product $a_{1i} \dots a_{ni}$ for all i . Set β equal to the diagonal embedding, i.e. $1 \mapsto (1, \dots, 1)$. Then by Lemma 2.1, $\beta^*S = \langle \{f_i: \mathbb{Z}_i^n \times_{\mathbb{Z}^n} \mathbb{Z} \rightarrow \mathbb{Z}\}_{i=1}^N \rangle$. Let $k_i = \text{lcm}(a_{1i}, \dots, a_{ni})$ and $\chi_i: \mathbb{Z} \rightarrow \mathbb{Z}^n, 1 \mapsto \left(\frac{k_i}{a_{1i}}, \dots, \frac{k_i}{a_{ni}}\right)$, then

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\chi_i} & \mathbb{Z}^n \\ k_i \downarrow & & \downarrow A_i \\ \mathbb{Z} & \xrightarrow{\beta} & \mathbb{Z}^n \end{array}$$

is a pullback diagram. Moreover, the prime q divides k_i for all i since it divides $a_{1i} \dots a_{ni}$ for all i . Thus $\gcd(k_1, \dots, k_N) \neq 1$. Now by Proposition 5.24, we can see that $\beta^*S = \langle \{\mathbb{Z} \xrightarrow{\times k_i} \mathbb{Z}\}_{i=1}^N \rangle$ is not a universal colim sieve. In particular, the first part of the proof of Proposition 5.24 shows that β^*S is not a colim sieve.

To prove the forward direction, we will prove the contrapositive statement. So suppose that $\gcd(\det(A_1), \dots, \det(A_N)) = 1$. Let $\beta: \mathbb{Z} \rightarrow \mathbb{Z}^n$ be

given as the matrix $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. To see that $\beta^*S = \langle \{f_i: \mathbb{Z}_i^n \times_{\mathbb{Z}^n} \mathbb{Z} \rightarrow \mathbb{Z}\}_{i=1}^N \rangle$ is

a colim sieve, notice that we have the pullback diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^n \\ k_i \downarrow & & \downarrow A_i \\ \mathbb{Z} & \xrightarrow{\beta} & \mathbb{Z}^n \end{array}$$

where $k_i = \text{lcm}\left(\frac{a_{1i}}{\gcd(a_{1i}, b_1)}, \dots, \frac{a_{ni}}{\gcd(a_{ni}, b_n)}\right)$. Hence, k_i divides $\det(A_i)$. This implies that $\gcd(k_1, \dots, k_n)$ divides $\gcd(\det(A_1), \dots, \det(A_N))$ and hence equals 1. Now by Proposition 5.24, we can see that $\beta^*S = \langle \{\mathbb{Z} \xrightarrow{\times k_i} \mathbb{Z}\}_{i=1}^N \rangle$ is a universal colim sieve. \square

Example 5.26. Based on Proposition 5.25 we can automatically say that the sieve $\left\langle \left\{ \begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 49 \end{pmatrix} \right\} \right\rangle$ on \mathbb{Z}^2 is not in the canonical topology because each matrix has a multiple of 7 somewhere on its diagonal.

Suppose, like in Proposition 5.25, $S = \langle \{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N \rangle$ where each A_i is a diagonal matrix and $\gcd(\det(A_1), \dots, \det(A_N)) = 1$. In order to determine if S is a universal colim sieve, we (only) need to check if f^*S is a colim sieve for all $f: \mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$, $2 \leq m \leq n$. However, this is still a fair amount of work and it would be nice if this process could be simplified further.

Now we finish this section with a few more examples. Note: we will not prove any assertions in these examples, however, they are all basic computations that can be checked using undergraduate linear algebra.

Example 5.27. The sieve $S_1 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 18 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on \mathbb{Z}^2 is not in the canonical topology although it is a colim sieve. In particular, S_1 is not universal because f^*S_1 is not a colim sieve for $f: \mathbb{Z} \rightarrow \mathbb{Z}^2$, $f(1) = (1, 0)$.

If we take the generating set of S_1 and change the 1 in the first matrix to a 0, then we get the following example:

Example 5.28. The sieve $S_2 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 18 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on \mathbb{Z}^2 is not a colim sieve since $\text{colim}_S U \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore, S_2 is also not in the canonical topology.

Finally, if take the generating set of S_2 and change the 18 in the second matrix to a 9, then we get:

Example 5.29. The sieve $S_3 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on \mathbb{Z}^2 is a colim sieve, however, whether or not this sieve is in the canonical topology is unknown.

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