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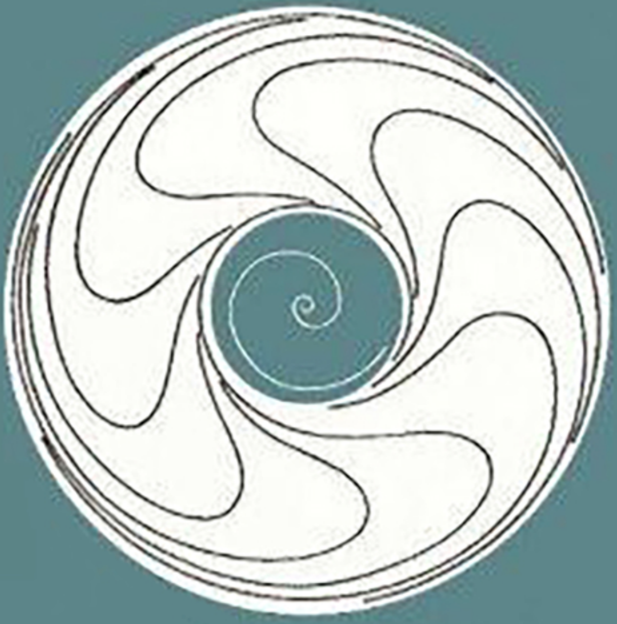

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CAHIERS DE TOPOLOGIE ET
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L'ENRICHISSEMENT ET SES DIFFÉRENTS POINTS DE VUE, II

Jacques PENON

Résumé. $\mathbb{V}$ étant une catégorie monoïdale, il y a trois façons d'enrichir une catégorie dans $\mathbb{V}$. On obtient les trois concepts de 1) catégorie $\mathbb{V}$-*enrichie* ou $\mathbb{V}$-*catégorie* (voir [3]), 2) catégorie $\mathbb{V}$-*prétensorisée* (voir [1], [5] et [4]), 3) catégorie $\mathbb{V}$-*précotensorisées*. On a montré dans la partie I comment passer des uns aux autres et les équivalences qui en résultent. Dans cette partie II on introduit le concept de catégorie *mutante* sur $\mathbb{V}$ (voir [2]) qui généralise les trois concepts précédents. On construit alors trois plongements 2-pleinement fidèles vers la 2-catégorie des catégories mutantes (sur $\mathbb{V}$).

Abstract. Let $\mathbb{V}$ be a monoidal category. For a general category, there are three manners to enrich them in $\mathbb{V}$. We get the three following concepts : 1) $\mathbb{V}$-enriched category (or $\mathbb{V}$-category- see [3]), 2) $\mathbb{V}$-pretensorised category (see [1], [5] and [4]), 3) $\mathbb{V}$-precotensorised category. In the part I, we showed how to pass from one to the other and the equivalences of 2-categories which result of them. In this part II we introduce the concept of mutant category on $\mathbb{V}$ (see [2]) for generalise the three concepts come before. We build three 2-functors full and faithfull to the 2-category of mutant categories (on $\mathbb{V}$).

Keywords. Monoidal category. Enriched category. Bicategory. Fibred category.

Mathematics Subject Classification (2020). 18D20.

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PARTIE II
UN CONCEPT RÉSUMANT L'ENSEMBLE

Sommaire

1. Catégories mutantes
2. Composition stricte
3. Saveur d'une catégorie mutante
4. Plongement dans la 2-catégorie $Cat\mu(\mathbb{V})$
5. Complément sur les saveurs

Introduction de la partie II

On se reportera à l'introduction commune aux deux parties, en tête de la partie I. On a vu, dans cette partie I, que catégories enrichies, catégories prétensorisées et catégories précotensorisées étaient des présentations différentes d'une même idée : il est temps maintenant de conceptualiser cette idée. C'est ce que nous abordons dans cette partie II avec le concept de *catégorie mutante*.

1. Catégories mutantes

Définition 1.1. :(voir [2]) Fixons une catégorie monoïdale $\mathbb{V}$. Une *catégorie mutante sur* $\mathbb{V} = (\underline{V}, \otimes, ...)$ est la donnée :
- d'une bicatégorie $\mathbb{B}$,
- d'un morphisme strict de bicatégorie $U : \mathbb{B} \to \mathbb{V}$ (où $\mathbb{V}$ est vu comme une bicatégorie à un objet $\star$. Ses flèches sont les objets de $\underline{V}$. Ses 2-cellules sont les flèches de $\underline{V}$. On a encore $Id_\star = I$).

Ces données satisfont l'axiome suivant :
(CM) Pour tout $X, Y \in |\mathbb{B}|$,
$U_{XY} : \mathbb{B}(X,Y) \to \mathbb{V}(UX, UY) = \mathbb{V}(\star, \star) = \underline{V}$ est une fibration discrète.

Dans la suite on notera $\otimes$ la composition horizontale de $\mathbb{B}$ et "." sa composition verticale (en accord avec les notations pour $\mathbb{V}$).

Remarque 1.2. : C'est René Guitart, à la suite des travaux de J.Wood sur les $\hat{\mathbb{V}}$-catégories (voir [6]), qui a introduit ce concept pour la première fois dans [2], sous le nom de *bicatégorie* $\mathbb{V}$*-graduée*.

Exemples 1.3. : 1) (voir aussi [2]) Soit $\mathcal{C}$ une catégorie enrichie dans $\mathbb{V}$. On lui associe la catégorie mutante suivante $\mu e(\mathcal{C}) = (\mathbb{B}, U)$ où :
- $|\mathbb{B}| = |\mathcal{C}|$,
- Pour $X, Y \in |\mathbb{B}|$, $\mathbb{B}(X,Y)$ est la catégorie $\underline{V}/\mathcal{C}(X,Y)$.
.. Si maintenant $(A, f)) \in |\mathbb{B}(X,Y)|$ et $(B, g) \in |\mathbb{B}(Y,Z)|$ alors leur composé est donné par :
$(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$, où $g \circ f : B \otimes A \to \mathcal{C}(X,Z)$ est donné par :

$$g \circ f = (\ B \otimes A \xrightarrow{g \otimes f} \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \xrightarrow{comp} \mathcal{C}(X,Z)\ )$$

.. la composition verticale de 2-cellules est donnée par la composition dans $\underline{V}/\mathcal{C}(X,Y)$.
.. La composition horizontale de 2-cellules : Pour le couple (b, a), où $b : (B, g) \to (B', g') : Y \to Z$ et $a : (A, f) \to (A', f') : X \to Y$,
on a $(\ (B, g) \otimes (A, f) \xrightarrow{b \otimes a} (B', g') \otimes (A', f')\ ) =$
$(\ B \otimes A, g \circ f) \xrightarrow{b \otimes a} (B' \otimes A', g' \circ f')\ )$ (c.a.d. le produit tensoriel dans $\mathbb{V}$).
.. Pour tout $X, Y \in |\mathbb{B}|, U_{XY} : \underline{V}/\mathcal{C}(X,Y) \to \underline{V}$ est le foncteur d'oubli habituel qui est clairement une fibration discrète.

2) (voir la remarque qui suit) Soit maintenant $\mathbb{E} = (\underline{E}, \wedge, s, am)$ une catégorie $\mathbb{V}$-prétensorisée. On lui associe alors la catégorie mutante suivante $\mu t(\mathbb{E}) = (\mathbb{B}, U)$ où :
- $|\mathbb{B}| = |\underline{E}|$,
- Pour $X, Y \in |\mathbb{B}|$, $|\mathbb{B}(X,Y)| = \{(A, f)/A \in |\underline{V}|, f \in \underline{E}(A \wedge X, Y)\}$. Et si $(A, f), (A', f') \in |\mathbb{B}(X,Y)|$, une flèche $a : (A, f) \to (A', f')$ dans $\mathbb{B}(X,Y)$ est une flèche $a : A \to A'$ dans $\underline{V}$ rendant le triangle suivant commutatif

dans $\underline{E}$:

$$\begin{tikzcd}
A \wedge X \arrow[rr, "a\wedge Id"] \arrow[dr, "f"'] & & A' \wedge X \arrow[dl, "f'"] \\
& Y &
\end{tikzcd}$$

.. Si maintenant $(A, f) \in |\mathbb{B}(X, Y)|$ et $(B, g) \in |\mathbb{B}(Y, Z)|$ alors leur composé est donné par :
$(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où $g \circ f$ est donné par :

$$g \circ f = (\ (B \otimes A) \wedge X \xrightarrow{am} B \wedge (A \wedge X) \xrightarrow{Id\wedge f} B \wedge Y \xrightarrow{g} Z\ )$$

.. La composition verticale de 2-cellules est la composition dans $\underline{V}$.
.. La composition horizontale du couple de 2-cellules (b, a),
où $b : (B, g) \to (B', g') : Y \to Z$ et $a : (A, f) \to (A', f') : X \to Y$ est
$b \otimes a : (B, g) \otimes (A, f) \to (B', g') \otimes (A', f')$.
.. Pour tout $X, Y \in |\mathbb{B}|, U_{XY} : \mathbb{B}(X, Y) \to \underline{V}$ est donné par $U_{XY}(A, f) = A$ et sur une flèche $U_{XY}((A, f) \overset{a}{\to} (A', f')) = (A \overset{a}{\to} A')$.

3) Soit $\mathbb{E} = (\underline{E}, H, \sigma, \alpha m)$ une catégorie $\mathbb{V}$-précotensorisée. On lui associe la catégorie mutante suivante $\mu c(\mathbb{E}) = (\mathbb{B}, U)$ où :
- $|\mathbb{B}| = |\underline{E}|$,
- Pour $X, Y \in |\mathbb{B}|, |\mathbb{B}(X, Y)| = \{(A, f)/A \in |\underline{V}|, f \in \underline{E}(X, Y^A)\}$. Et si $(A, f), (A', f') \in |\mathbb{B}(X, Y)|$, une flèche $a : (A, f) \to (A', f')$ dans $\mathbb{B}(X, Y)$ est une flèche $a : A \to A'$ dans $\underline{V}$ rendant le triangle suivant commutatif dans $\underline{E}$:

$$\begin{tikzcd}
& X \arrow[dl, "f'"'] \arrow[dr, "f"] & \\
Y^{A'} \arrow[rr, "Id^a"'] & & Y^A
\end{tikzcd}$$

.. Si maintenant $(A, f) \in |\mathbb{B}(X, Y)|$ et $(B, g) \in |\mathbb{B}(Y, Z)|$ alors leur composé est donné par :
$(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où $g \circ f$ est donné par :

$$g \circ f = (\ X \xrightarrow{f} Y^A \xrightarrow{g^{Id}} (Z^B)^A \xrightarrow{\alpha m} Z^{B \otimes A}\ ).$$

.. La composition verticale de 2-cellules est la composition dans $\underline{V}$.
.. La composition horizontale du couple de 2-cellules (b, a),

où $b : (B,g) \to (B',g') : Y \to Z$ et $a : (A,f) \to (A',f') : X \to Y$ est $b \otimes a : (B,g) \otimes (A,f) \to (B',g') \otimes (A',f')$.
.. Pour tout $X, Y \in |\mathbb{B}|, U_{XY} : \mathbb{B}(X,Y) \to \underline{V}$ est donné par $U_{XY}(A,f) = A$ et sur une flèche $U_{XY}((A,f) \overset{a}{\to} (A',f')) = (A \overset{a}{\to} A')$.

4) Soit $\mathbb{E} = (\underline{E}, U)$ une catégorie fibrée sur une catégorie $\underline{B}$ à produits fibrés. Fixons aussi un objet $B \in |\underline{B}|$. À ces données on associe la catégorie mutante sur $\underline{B}/B$ suivante : $(\mathbb{B}, U)$ où :
- $|\mathbb{B}| = |\mathbb{E}_B|$ ($\mathbb{E}_B$ étant la fibre au dessus de B).
- Pour $X, Y \in |\mathbb{B}|, |\mathbb{B}(X,Y)|$ est l'ensemble
$\{(A,a,x)/(A,a) \in |\underline{B}/B|, x \in \mathbb{E}_A(a^\star X, a^\star Y)\}$ et étant donné
$(A,a,x),(A',a',x') \in |\mathbb{B}(X,Y)|$ une flèche $(A,a,x) \to (A',a',x')$ dans $\mathbb{B}(X,Y)$ est une flèche $\alpha : (A,a) \to (A',a')$ dans $\underline{B}/B$ telle que le carré suivant commute :

$$\begin{tikzcd}
a^\star X \arrow[r, "Id"] \arrow[d, "x"'] & (a'.\alpha)^\star X \arrow[r, "can", "\sim"'] & \alpha^\star a'^\star X \arrow[d, "\alpha^\star x'"] \\
a^\star Y \arrow[r, "Id"] & (a'.\alpha)^\star Y \arrow[r, "can", "\sim"'] & \alpha^\star a'^\star Y
\end{tikzcd}$$

.. Si maintenant $(A_0, a_0, x_0) \in |\mathbb{B}(X,Y)|$ et $(A_1, a_1, x_1) \in |\mathbb{B}(Y,Z)|$ alors leur composé est donné par :
$(A_1, a_1, x_1) \otimes (A_0, a_0, x_0) = (A_1 \underset{B}{\times} A_0, a_1 \otimes a_0, x_1 \circ x_0)$ où
$(A_1 \underset{B}{\times} A_0, a_1 \otimes a_0) = (A_1, a_1) \times (A_0, a_0)$ (le produit dans $\underline{B}/B$), et

$$x_1 \circ x_0 = (\ (a_1 \otimes a_0)^\star X \xrightarrow{\bar{x_0}} (a_1 \otimes a_0)^\star Y \xrightarrow{\bar{x_1}} (a_1 \otimes a_0)^\star Z\ )$$

où $\bar{x_0}$ et $\bar{x_1}$ sont donnés par les carrés commutatifs suivants :

$$\begin{tikzcd}
(a_1 \otimes a_0)^\star X \arrow[r, "can", "\sim"'] \arrow[d, "\bar{x_0}"'] & \pi_0^\star a_0^\star X \arrow[d, "\pi_0^\star x_0"] \\
(a_1 \otimes a_0)^\star Y \arrow[r, "can", "\sim"'] & \pi_0^\star a_0^\star Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(a_1 \otimes a_0)^\star Y \arrow[r, "can", "\sim"'] \arrow[d, "\bar{x_1}"'] & \pi_1^\star a_1^\star Y \arrow[d, "\pi_1^\star x_1"] \\
(a_1 \otimes a_0)^\star Z \arrow[r, "can", "\sim"'] & \pi_1^\star a_1^\star Z
\end{tikzcd}$$

et où $a_1 \otimes a_0 = a_0.\pi_0 = a_1.\pi_1$ (π_0 et π_1 désignant les projections canoniques $A_1 \underset{B}{\times} A_0 \to A_0$ ou A_1).

.. La composition verticale de 2-cellules est la composition dans $\underline{B}/B$.
.. La composition horizontale du couple de 2-cellules (α_1, α_0) où
$\alpha_i : (A_i, a_i, x_i) \to (A'_i, a'_i, x'_i) : X_i \to Y_i$ avec $X_0 = X,\ X_1 = Y,$
$Y_0 = Y,\ Y_1 = Z$ est donné par :
$((A_1, a_1, x_1) \otimes (A_0, a_0, x_0) \xrightarrow{\alpha_1 \otimes \alpha_0} (A'_1, a'_1, x'_1) \otimes (A'_0, a'_0, x'_0)) =$
$(((A_1 \underset{B}{\times} A_0, a_1 \otimes a_0, x_1 \circ x_0) \xrightarrow{\alpha_1 \times \alpha_0} (A'_1 \underset{B}{\times} A'_0, a'_1 \otimes a'_0, x'_1 \circ x'_0))$ où
$\alpha_1 \times \alpha_0$ est le produit dans $\underline{B}/B$.
.. Pour $X, Y \in |\mathbb{B}| = |\mathbb{E}_B|, U_{XY} : \mathbb{B}(X,Y) \to \underline{V} = \underline{B}/B$ est donné sur
... les objets par $U_{XY}(A, a, x) = (A, a)$,
... les flèches par $U_{XY}(\alpha) = \alpha$.

Remarque 1.4. : L'exemple 1 précédent est déjà dans [2]. Y figure aussi l'exemple 2 dans le cas particulier où $\mathbb{E} = \mathbb{V}$.

Définition 1.5. :$(\mathbb{B}, U)$ et $(\mathbb{B}', U')$ étant deux catégories mutantes un *foncteur mutant* $(\mathbb{B}, U) \to (\mathbb{B}', U')$ est la donnée d'un morphisme strict de bicatégories $\Phi : \mathbb{B} \to \mathbb{B}'$ tel que le triangle ci-dessous commute :

$$\begin{tikzcd}
\mathbb{B} \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \mathbb{B}' \arrow[dl, "U'"] \\
& \mathbb{V} &
\end{tikzcd}$$

Remarque 1.6. : Pour le concept de transformation naturelle mutante nous avons besoin de la "composition stricte" que nous allons définir dans la prochaine section. C'est pourquoi nous renvoyons la définition des transformation naturelles mutantes à la section 2.

2. Composition stricte

• Fixons ici une catégorie mutante $\mathcal{B} = (\mathbb{B}, U)$ sur une catégorie monoïdale $\mathbb{V}$. Pour $X, Y \in |\mathbb{B}|$ et $A \in |\mathbb{V}|$, notons :

$$Hom_A(X, Y) = \{\phi \in |\mathbb{B}(X, Y)| / U_{XY}(\phi) = A\}$$

On écrira encore $X \xrightarrow[A]{\phi} Y$ pour $\phi \in Hom_A(X, Y)$. Remarquons que pour tout $X \in |\mathbb{B}|$, $Id_X \in Hom_I(X, X)$.

- Dans la situation suivante $X \xrightarrow[A]{\phi} Y \xrightarrow[I]{f} Z$, on définit $X \xrightarrow[A]{f\phi} Z$ de la façon suivante :
Comme $U(f \otimes \phi) = U(f) \otimes U(\phi) = I \otimes A,$ que l'on a l'isomorphisme $u_g : I \otimes A \to A$ et que $U_{XZ} : \mathbb{B}(X, Z) \to \underline{V}$ est une fibration discrète, on peut définir $f\phi = (u_g^{-1})^\star(f \otimes \phi)$. On a $f\phi \in Hom_A(X, Z)$.

- De même, dans la situation inverse $X \xrightarrow[I]{f} Y \xrightarrow[A]{\phi} Z$, on définit $X \xrightarrow[A]{\phi f} Z$ en posant $\phi f = (u_d^{-1})^\star(\phi \otimes f)$.

Remarque 2.1. : Lorsque $A = I$, et $X \xrightarrow[I]{g} Y \xrightarrow[I]{f} Z$, les deux possibilités de définition de gf précédentes coïncident car $u_g = u_d : I \otimes I \to I$.

Définition 2.2. :$f\phi$ et ϕf sont appelés les composés stricts de f par ϕ et ϕ par f.

• Étudions maintenant les propriétés de "la" composition stricte.

Proposition 2.3. : 1) Soit $\phi \in Hom_A(X, Y)$. Alors $\phi Id_X = \phi = Id_Y \phi$.
2) Considérons le triplet de flèches composables suivant :
$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ dans $\mathbb{B}$. Dans les trois cas suivants :

$$a)\ U(g) = U(h) = I,\ b)\ U(f) = U(h) = I,\ c)\ U(f) = U(g) = I,$$

alors on a l'associativité $(hg)f = h(gf)$.

<u>*Preuve*</u> : 1) Car $U(u_g) = u_g$ et $U(u_d) = u_d$ (U est strict).
2) a) Résulte de la commutation du triangle (T_1) suivant dans $\mathbb{V}$ (où ici $B = I$ et $A = U(f)$):

$$\begin{tikzcd}
(B \otimes I) \otimes A \arrow[rr, "ass"] \arrow[dr, "u_d \otimes Id"'] & & B \otimes (I \otimes A) \arrow[dl, "Id \otimes u_g"] \\
& B \otimes A &
\end{tikzcd}$$

b) Résulte de la commutation du triangle (T_2) suivant dans $\mathbb{V}$ (où ici $B = I$ et $A = U(g)$) et de la naturalité de u_d dans $\mathbb{V}$:

$$\begin{tikzcd}
(B \otimes A) \otimes I \arrow[rr, "ass"] \arrow[dr, "u_d"'] & & B \otimes (A \otimes I) \arrow[dl, "Id \otimes u_d"] \\
& B \otimes A &
\end{tikzcd}$$

c) Mêmes arguments qu'au (a) (où ici $U(h) = B$ et $A = I$).

Proposition 2.4. : Considérons, à nouveau, le triplet de flèches composables suivant $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ dans $\mathbb{B}$. Alors :
a) si $U(f) = I$, on a $(h \otimes g)f = h \otimes (gf)$,
b) si $U(g) = I$, on a $(hg) \otimes f = h \otimes (gf)$
c) si $U(h) = I$, on a $(hg) \otimes f = h(g \otimes f)$

Preuve : a) Résulte de la commutation dans $\mathbb{V}$ du triangle (T_2) dans la proposition précédente (où $U(g) = A, U(h) = B$). b) Résulte de la commutation dans $\mathbb{V}$ du triangle (T_1) dans la proposition précédente (où $U(f) = A, U(h) = B$). c) Résulte de la commutation suivante dans $\mathbb{V}$ (où $U(f) = A, U(g) = B$)

$$\begin{tikzcd}
(I \otimes B) \otimes A \arrow[rr, "ass"] \arrow[dr, "u_g \otimes Id"'] & & I \otimes (B \otimes A) \arrow[dl, "u_g"] \\
& B \otimes A &
\end{tikzcd}$$

Proposition 2.5. : 1) Soit $X \xrightarrow[A]{\alpha} Y$ une flèche de $\mathbb{B}$, alors on a :

$$Id_Y \otimes \alpha = u^{\star}_{g,A}(\alpha) \quad \text{et} \quad \alpha \otimes Id_X = u^{\star}_{d,A}(\alpha)$$

2) Soient $X \xrightarrow[A]{\alpha} Y \xrightarrow[B]{\beta} Z \xrightarrow[C]{\gamma} T$ trois flèches composables alors on a :

$$\gamma \otimes (\beta \otimes \alpha) = ass^{\star}_{C,B,A}((\gamma \otimes \beta) \otimes \alpha)$$

Preuve : Immédiat.

Définition 2.6. :$\mathcal{B} = (\mathbb{B}, U)$ étant une catégorie mutante sur $\mathbb{V}$, on appelle *catégorie sous-jacente* à $\mathcal{B}$ (et on la note $\underline{\mathcal{B}}$), la catégorie dont :
- les objets sont ceux de $\mathbb{B}$ (i.e. $|\underline{\mathcal{B}}| = |\mathbb{B}|$).
- les flèches sont données par $\underline{\mathcal{B}}(X, Y) = Hom_I(X, Y)$.

La composition de $\underline{\mathcal{B}}$ est la composition stricte. Les axiomes de catégorie résultent de la proposition 2.3.

• Considérons maintenant un foncteur mutant $\Phi : \mathcal{B} \to \mathcal{B}'$ où $\mathcal{B} = (\mathbb{B}, U)$ et $\mathcal{B}' = (\mathbb{B}', U')$.

Proposition 2.7. : Considérons dans $\mathcal{B}$ le couple de flèches composables suivant :

$$X \xrightarrow[A]{f} Y \xrightarrow[B]{g} Z$$

Alors, si $A = I$ ou $B = I$ on a l'identité suivante :

$$\Phi(gf) = \Phi(g)\Phi(f).$$

Preuve : Immédiat.

Définition 2.8. : $\Phi : \mathcal{B} \to \mathcal{B}'$ étant un foncteur mutant, on appelle *foncteur sous-jacent à* Φ le foncteur noté $\underline{\Phi} : \underline{\mathcal{B}} \to \underline{\mathcal{B}}'$, défini,
- sur les objets par $|\underline{\Phi}| = |\Phi|$,
- sur une flèche $f : X \to X'$ de $\underline{\mathcal{B}}$, par $\underline{\Phi}(f) = \Phi(f)$,
(Les axiomes de foncteur résultent de la proposition précédente).

• Nous sommes maintenant en mesure de définir les transformations naturelles mutantes.

Définition 2.9. :1) $\Phi, \Phi' : \mathcal{B} \to \mathcal{B}'$ étant deux foncteurs mutants, on appelle *transformation naturelle mutante* sur $\mathbb{V}$, et on note $t : \Phi \to \Phi'$, la donnée d'une famille de flèches de $\underline{\mathcal{B}}'$, $(t_X : \Phi X \to \Phi' X)_{X \in |\underline{\mathcal{B}}|}$, vérifiant la condition suivante :
Pour tout $X, Y \in |\underline{\mathcal{B}}|$ et toute flèche $\alpha : X \to Y$ de $\mathcal{B}$, on a l'identité $t_Y \Phi(\alpha) = \Phi'(\alpha) t_X$ (où ici la composition dans $\mathcal{B}'$ est stricte).
2) Clairement une transformation naturelle mutante $t : \Phi \to \Phi'$ produit canoniquement une transformation naturelle notée $\underline{t} : \underline{\Phi} \to \underline{\Phi}'$. On dit que c'est la *transformation naturelle sous-jacente à* t.

On vérifie sans difficulté que, catégories mutantes (sur $\mathbb{V}$), foncteurs mutants (sur $\mathbb{V}$) et transformation naturelles mutantes (sur $\mathbb{V}$) forment une 2-catégorie notée $Cat\mu(\mathbb{V})$. On a aussi un 2-foncteur canonique

$$U : Cat\mu(\mathbb{V}) \to \mathbb{C}at,$$

défini sur les objets par $U(\mathcal{B}) = \underline{\mathcal{B}}$, sur les flèches par $U(\Phi) = \underline{\Phi}$, sur les 2-cellules par $U(t) = \underline{t}$.

Les exemples de catégories mutantes donnés dans 1.3 se prolongent en des 2-foncteurs $\mu e : \mathbb{V}\text{-}Cat \rightarrow Cat\mu(\mathbb{V})$, $\mu t : \mathbb{V}\text{-}Pretens \rightarrow Cat\mu(\mathbb{V})$, $\mu c : \mathbb{V}\text{-}Precot \rightarrow Cat\mu(\mathbb{V})$. On les construit de la façon suivante:
• *Pour* μe :
- sur un objet $\mathcal{C} \in |\mathbb{V}\text{-}Cat|$, $\mu e(\mathcal{C})$ a été construit à l'exemple 1.3(1).
- sur une flèche $F : \mathcal{C} \rightarrow \mathcal{C}'$ de $\mathbb{V}\text{-}Cat$ (c'est donc un $\mathbb{V}$-foncteur), $\mu e(F) : \mu e(\mathcal{C}) \rightarrow \mu e(\mathcal{C}')$ est lui même donné,
.. sur les objets par $|\mu e(F)| = |F|$.
.. sur les flèches, pour $X, Y \in |\mu e(\mathcal{C})| = |\mathcal{C}|$,
$\mu e(F)_{XY} : \mu e(\mathcal{C})(X, Y) \rightarrow \mu e(\mathcal{C}')(FX, FY)$ est le foncteur défini par $\mu e(F)_{XY} = \Sigma_{F_{XY}} : \underline{V}/\mathcal{C}(X, Y) \rightarrow \underline{V}/\mathcal{C}'(FX, FY)$. On voit facilement que $\mu e(F)$ est un foncteur mutant.
- sur une 2-cellule $t : F \rightarrow F' : \mathcal{C} \rightarrow \mathcal{C}', \mu e(t) : \mu e(F) \rightarrow \mu e(F')$ est la trasformation naturelle mutante donnée par
$\mu e(t) = ((I, t_X) : FX \rightarrow F'X)_{x \in |\mathcal{C}|}$.
Avant de montrer les axiomes d'une transformation naturelle mutante on a besoin de caractériser la composition stricte dans les catégories mutantes $\mu e(\mathcal{C})$). Or, dans la situation suivante :

$$X \xrightarrow{(I,f)} Y \xrightarrow{(A,\alpha)} Z \xrightarrow{(I,g)} T$$

où (I, f) et (I, g) sont des flèches de $\underline{\mu e(\mathcal{C})}$, on a : $(A, \alpha)(I, f) = (A, \alpha/f)$ et $(I, g)(A, \alpha) = (A, g/\alpha)$ où ...

$$\alpha/f = (\ A \xrightarrow{u_d^{-1}} A \otimes I \xrightarrow{\alpha \otimes f} \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \xrightarrow{comp} \mathcal{C}(X, Z)\ )$$

$$g/\alpha = (\ A \xrightarrow{u_g^{-1}} I \otimes A \xrightarrow{g \otimes \alpha} \mathcal{C}(Z, T) \otimes \mathcal{C}(Y, Z) \xrightarrow{comp} \mathcal{C}(Y, T)\ )$$

La vérification des axiomes de transformation naturelle mutante se fait ensuite sans difficulté particulière. Enfin la vérification que μe est un 2-foncteur est longue et fastidieuse mais sans grande difficulté.

• *Pour* μt :
- Sur un objet $\mathbb{E} \in |\mathbb{V}\text{-}Pretens|$, $\mu t(\mathbb{E})$ a été construit aux exemples 1.3(2).
- Sur une flèche $(F, \phi) : \mathbb{E} \rightarrow \mathbb{E}'$, $\mu t(F, \phi) : \mu t(\mathbb{E}) \rightarrow \mu t(\mathbb{E}')$ est lui-même donné...
.. Sur les objets par $|\mu t(F, \phi)| = |F|$.

.. Sur les flèches, pour $X, Y \in |\mu t(\mathbb{E})| = |\underline{E}|$,
$\mu t(F,\phi)_{XY} : \mu t(\mathbb{E})(X,Y) \to \mu t(\mathbb{E}')(FX,FY)$ est le foncteur défini...
... sur un objet $(A,\alpha) \in |\mu t(\mathbb{E})(X,Y)|$, par $\mu t(F,\phi)_{XY}(A,\alpha) = (A,\bar{\alpha})$ où

$$\bar{\alpha} = (\ A \wedge FX \xrightarrow{\phi_{A,X}} F(A \wedge X) \xrightarrow{F(\alpha)} FY\ )$$

... sur une flèche $a : (A,\alpha) \to (A',\alpha')$ par $\mu t(F,\phi)_{XY}(a) = a$.
$\mu t(F,\phi)_{XY}$ est fonctoriel de façon évidente. On voit facilement ensuite que $\mu t(F,\phi)$ est un foncteur mutant.
- Sur une 2-cellule $t : (F,\phi) \to (F',\phi')$, $\mu t(t) : \mu t(F,\phi) \to \mu t(F',\phi')$ est la transformation naturelle mutante donnée par
$\mu t(t) = ((I,\tilde{t}_X) : FX \to F'X)_{X \in |\underline{E}|}$ où

$$\tilde{t}_X = (\ I \wedge FX \xrightarrow{s_{FX}} FX \xrightarrow{t_X} F'X\ )$$

Là encore, avant de montrer les axiomes d'une transformation naturelle mutante on caractérise la composition stricte dans les catégories mutantes $\mu t(\mathbb{E})$. Or, dans la situation suivante :

$$X \xrightarrow{(I,f)} Y \xrightarrow{(A,\alpha)} Z \xrightarrow{(I,g)} T$$

où (I,f) et (I,g) sont des flèches de $\underline{\mu t(\mathbb{E})}$, on a : $(A,\alpha)(I,f) = (A,\alpha/f)$ et $(I,g)(A,\alpha) = (A,g/\alpha)$ où...

$$\alpha/f = (\ A \wedge X \xrightarrow{u_d^{-1} \wedge Id} (A \otimes I) \wedge X \xrightarrow{am} A \wedge (I \wedge X) \xrightarrow{Id \wedge f} A \wedge Y \xrightarrow{\alpha} Z\ )$$

$$g/\alpha = (\ A \wedge Y \xrightarrow{u_g^{-1} \wedge Id} (I \otimes A) \wedge Y \xrightarrow{am} I \wedge (A \wedge Y) \xrightarrow{Id \wedge \alpha} I \wedge Z \xrightarrow{g} T\ )$$

La vérification des axiomes de transformation naturelle mutante se fait ensuite sans difficulté particulière. Comme pour μe, la preuve que μt est 2-fonctoriel est longue et fastidieuse mais sans difficulté notable.

• *Pour* μc:
Sa construction s'obtient grâce à celle de μt. Le 2-foncteur μc est défini comme étant le composé suivant :

$$\mathbb{V}\text{-}Precot \xrightarrow{Red} (\mathbb{V}^{\star}\text{-}Pretens)^{opv} \xrightarrow{\mu t^{opv}} Cat\mu(\mathbb{V}^{\star})^{opv} \xrightarrow{et^{opv}} Cat\mu(\mathbb{V})$$

où $et = (-)^\star$ et où Red est défini dans la partie I, section 4.
Le 2-foncteur μc se caractérise de la façon suivante :
- Sur un objet $\mathbb{E}$, on retrouve la construction de $\mu c(\mathbb{E})$ donnée dans l'exemple 1.3(3).
- Sur une flèche $(F, \psi) : \mathbb{E} \to \mathbb{E}'$, $\mu c(F, \psi) : \mu c(\mathbb{E}) \to \mu c(\mathbb{E}')$ est donné...
.. sur les objets par $|\mu c(F, \psi)| = |F|$,
.. sur les flèches, pour $X, Y \in |\mu c(\mathbb{E})| = |\underline{E}|$,
$\mu c(F, \psi)_{XY} : \mu c(\mathbb{E})(X, Y) \to \mu c(\mathbb{E}')(FX, FY)$ est le foncteur défini...
... sur un objet $(A, \alpha) \in |\mu c(\mathbb{E})(X, Y)|$, par $\mu c(F, \psi)_{XY}(A, \alpha) = (A, \hat{\alpha})$ où, dans $\underline{E'}$,

$$\hat{\alpha} = (\ FY \xrightarrow{F\alpha} F(X^A) \xrightarrow{\psi_{X,A}} F(X)^A\ )$$

... sur une flèche $a : (A, \alpha) \to (A', \alpha')$ par $\mu c(F, \psi)_{XY}(a) = a.$
- Sur une 2-cellule $t : (F, \psi) \to (F', \psi')$,
$\mu c(t) = ((I, \check{t}_X) : FX \to F'X)_{X \in |\underline{E}|}$ où

$$\check{t}_X = (\ FX \xrightarrow{t_X} F'X \xrightarrow{\sigma} F'(X)^I)\ .$$

3. Saveurs d'une catégorie mutante

• Fixons une catégorie mutante $\mathcal{B} = (\mathbb{B}, U)$ sur une catégorie monoïdale $\mathbb{V}$. On construit un foncteur $Tri_{\mathcal{B}} : \underline{V}^{op} \times \underline{\mathcal{B}}^{op} \times \underline{\mathcal{B}} \to \underline{\mathbb{E}ns}$ (ou simplement Tri). Il est défini,
- sur un objet (A, X, Y) par $Tri(A, X, Y) = Hom_A(X, Y)$ (voir la notation à la section 2),
- sur une flèche $(a, x, y) : (A, X, Y) \to (A', X', Y')$, $Tri(A, X, Y)$ est l'application composée :

$$Hom_A(X, Y) \overset{a^\star(-)}{\to} Hom_{A'}(X, Y) \overset{(-)x}{\to} Hom_{A'}(X', Y) \overset{y(-)}{\to} Hom_{A'}(X', Y').$$

La première application vient du fait que $U_{XY} : \mathbb{B}(X, Y) \to \underline{V}$ est une fibration discrète. Les autres applications proviennent de la composition stricte. Pour montrer la fonctorialité de Tri on utilise la commutation des carrés $(C_1), (C_2)$ et (C_3) suivants :

$$\begin{tikzcd}
Hom_A(X,Y) \arrow[r, "(-)x"] \arrow[d, "y(-)"'] & Hom_A(X',Y) \arrow[d, "y(-)"] \\
Hom_A(X,Y') \arrow[r, "(-)x"'] & Hom_A(X',Y')
\end{tikzcd}
\qquad
\begin{tikzcd}
Hom_A(X,Y) \arrow[r, "(-)x"] \arrow[d, "a^\star(-)"'] & Hom_A(X',Y) \arrow[d, "a^\star(-)"] \\
Hom_{A'}(X,Y) \arrow[r, "(-)x"'] & Hom_{A'}(X',Y)
\end{tikzcd}$$

$$\begin{tikzcd}
Hom_A(X,Y) \arrow[r, "y(-)"] \arrow[d, "a^\star(-)"'] & Hom_A(X,Y') \arrow[d, "a^\star(-)"] \\
Hom_{A'}(X,Y) \arrow[r, "y(-)"'] & Hom_{A'}(X,Y')
\end{tikzcd}$$

(Pour le premier carré cela résulte de l'associativité stricte et pour les deux autres l'axiome (CM) est appliqué à $U_{X'Y}$ et $U_{XY'}$).

Définition 3.1. : On dit que :
1) $\mathcal{B}$ est *de saveur enrichie* (ou simplement *de saveur* E) si :
Pour tout $X, Y \in |\underline{\mathcal{B}}|$, $Tri(-,X,Y) : \underline{V}^{op} \to \underline{\mathbb{E}ns}$ est représentable.
2) $\mathcal{B}$ est *de saveur tensorisée* (ou simplement *de saveur* T) si :
Pour tout $A \in |\underline{V}|$ et tout $X \in |\underline{\mathcal{B}}|$, $Tri(A,X,-) : \underline{\mathcal{B}} \to \underline{\mathbb{E}ns}$ est co-représentable.
3) $\mathcal{B}$ est *de saveur cotensorisée* (ou simplement *de saveur* C) si :
Pour tout $A \in |\underline{V}|$ et tout $Y \in |\underline{\mathcal{B}}|$, $Tri(A,-,Y) : \underline{\mathcal{B}}^{op} \to \underline{\mathbb{E}ns}$ est représentable.

- Étudions maintenant chacune des différentes saveurs :

Proposition 3.2. : On suppose que $\mathcal{B}$ est de saveur E. Alors il existe une catégorie enrichie $\mathcal{B}^e$ telle que $\mu e(\mathcal{B}^e) \simeq \mathcal{B}$.

Preuve : Pour chaque couple $(X,Y) \in |\mathcal{B}|^2$, choisissons une représentation $(\mathcal{B}^e(X,Y),\ X \xrightarrow[\mathcal{B}^e(X,Y)]{ev_{XY}} Y\ )$ du préfaisceau $Tri(-,X,Y)$.

1) À partir de cette famille de choix on construit canoniquement une catégorie enrichie (notée $\mathcal{B}^e$) où :
- $|\mathcal{B}^e| = |\mathcal{B}|$,
- Les Hom internes sont donnés par les $\mathcal{B}^e(X,Y)$,
- Pour tout $X \in |\mathcal{B}^e|, id_X : I \to \mathcal{B}^e(X,Y)$ est l'unique flèche de $\underline{V}$ telle

que $Tri(id_X, X, X)(ev_{XY}) = Id_X$ (i.e. $id^\star_X(ev_{XX}) = Id_X$).
- Pour tout $X, Y, Z \in |\mathcal{B}^e|$, $comp_{XYZ} : \mathcal{B}^e(Y,Z) \otimes \mathcal{B}^e(X,Y) \to \mathcal{B}^e(X,Z)$ est l'unique flèche de $\underline{V}$ telle que
$Tri(comp_{XYZ}, X, Z)(ev_{XZ}) = ev_{YZ} \otimes ev_{XY}$
(i.e. $comp^\star_{XYZ}(ev_{XZ}) = ev_{YZ} \otimes ev_{XY}$).
Vérifions maintenant les axiomes des catégories enrichies.
- L'unité à gauche : On montre sans difficulté l'identité suivante
(où $A = \mathcal{B}^e(X,Y)$): $(comp_{XYY}.(id_Y \otimes Id_A))^\star(ev_{XY}) = u^\star_{g,A}(ev_{XY})$.
- L'unité à droite se montre de la même façon.
- L'associativité : Cela résulte des identités suivantes (où $U = \mathcal{B}^e(Z,T), V = \mathcal{B}^e(Y,Z), W = \mathcal{B}^e(X,Y)$):
$(comp_{XZT}.(Id_U \otimes comp_{XYZ}).ass_{UVW})^\star(ev_{XT}) = (ev_{ZT} \otimes ev_{YZ}) \otimes ev_{XY} = (comp_{XYT}.(comp_{YZT} \otimes Id_W))^\star(ev_{XT})$.
2) On construit ensuite un isomorphisme $\gamma^e : \mathcal{B} \to \mu e(\mathcal{B}^e)$. Il est donné...
- sur les objets par $|\gamma^e| = Id$,
- sur une flèche $\alpha : X \to Y$ de $\mathcal{B}$ par $\gamma^e(\alpha) = (A, f)$ où $A = U_{XY}(\alpha)$ et $f : A \to \mathcal{B}^e(X,Y)$ est l'unique flèche de $\underline{V}$ telle que $f^*(ev_{XY}) = \alpha$.
- sur une 2-cellule $a : \alpha \to \alpha' : X \to Y$ par $\gamma^e(a) = U_{XY}(a)$.
On vérifie facilement que γ^e est un isomorphisme où $\gamma^{e-1}(A, f) = f^*(ev_{XY})$ et sur une 2-cellule $a : (A, f) \to (A', f')$,
$\gamma^{e-1}(a) : \gamma^{e-1}(A, f) \to \gamma^{e-1}(A', f')$ est l'unique flèche de $\mathbb{B}(X,Y)$ de but $\gamma^{e-1}(A', f')$ telle que $U_{XY}(\gamma^{e-1}(a)) = a$.

Proposition 3.3. :On suppose que $\mathcal{B}$ est de saveur T. Alors il existe une catégorie prétensorisée $\mathcal{B}^t$ telle que $\mu t(\mathcal{B}^t) \simeq \mathcal{B}$.

Preuve : Pour chaque couple $(A, X) \in |\underline{V}| \times |\underline{\mathcal{B}}|$, choisissons une co-représentation $(A \wedge X,\ X \xrightarrow[A]{val_{A,X}} A \wedge X\ )$ de $Tri(A, X, -)$.
1) À partir de cette famille de choix on construit canoniquement une catégorie $\mathbb{V}$-prétensorisée notée $\mathcal{B}^t$ où $\mathcal{B}^t = (\underline{\mathcal{B}}, \wedge, s, am)$.
- $\wedge : \underline{V} \times \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ est construit sur les objets comme précédemment. Sur une flèche $(a, x) : (A, X) \to (A', X')$, on disdingue les deux cas suivants :
.. si $x = Id$, alors $a \wedge Id : A \wedge X \to A' \wedge X$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que $Tri(A', X, a \wedge Id_X)(val_{A,X})) = a^\star(val_{A',X})$
(i.e. $(a \wedge Id)val_{A,X} = a^\star(val_{A',X})$) (à gauche la composition est stricte).
.. si $a = Id$, alors $Id \wedge x : A \wedge X \to A \wedge X'$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que

$Tri(A, X, Id_A \wedge x)(val_{A,X})) = val_{A,X'}x$ (i.e.$(Id \wedge x)val_{A,X} = val_{A,X'}x$) (les compositions sont strictes).
Dans le cas général on vérifie d'abord que le carré (C) suivant est commutatif dans $\underline{\mathcal{B}}$:

$$\begin{tikzcd}
A \wedge X \arrow[r, "a \wedge Id"] \arrow[d, "Id \wedge x"'] & A' \wedge X \arrow[d, "Id \wedge x"] \\
A \wedge X' \arrow[r, "a \wedge Id"'] & A' \wedge X'
\end{tikzcd}$$

(Cela résulte des identités suivantes : $(a \wedge Id)(Id \wedge x)val_{A,X} = a^{\star}(val_{A',X'})x = (Id \wedge x)a^{\star}(val_{A',X}) = (Id \wedge x)(a \wedge Id)val_{A,X}$ où la seconde égalité résulte de (C_2) et (C_3)). On peut alors poser $a \wedge x = (Id_{A'} \wedge x)(a \wedge Id_X) = (a \wedge Id_{X'})(Id_A \wedge x)$. La fonctorialité de $\wedge$ résulte de celle de $(-) \wedge X$ et $A \wedge (-)$, pour tout $X \in |\underline{\mathcal{B}}|$ et $B \in |\underline{V}|$ (pour la première on utilise (C_3)) et de la commutation du carré (C).
- Pour chaque $X \in |\underline{\mathcal{B}}|, s_X : I \wedge X \to X$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que $s_X val_{I,X} = Id_X$. Le fait que s_X est un isomorphisme résulte du fait que $Tri(I, X, -) = \underline{\mathcal{B}}(X, -)$. La naturalité de s est sans difficulté.
- Pour chaque $A, B \in |\underline{V}|, X \in |\underline{\mathcal{B}}|$, $am_{A,B,X} : (A \otimes B) \wedge X \to A \wedge (B \wedge X)$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que $am_{A,B,X} val_{A \otimes B,X} = val_{A,B \wedge X} \otimes val_{B,X}$. La naturalité de $am_{A,B,X}$ se montre successivement en A, B et X.
.. En A, soit $a : A \to A'$ une flèche de $\underline{V}$. La commutation voulue résulte des identités suivantes : $(a \wedge Id)am\, val_{A \otimes B,X} = a^{\star}(val_{A',B \wedge X}) \otimes val_{B,X} = am((a \otimes Id) \wedge Id)val_{A \otimes B,X}$ où pour la seconde égalité on utilise (C_3).
.. En B, mêmes arguments qu'en A.
.. En X. Soit $x : X \to X'$ une flèche de $\underline{\mathcal{B}}$. La commutation voulue résulte des identités suivantes :
$(Id \wedge (Id \wedge x))am\, val_{A \otimes B,X} = (val_{A,B \wedge X'} \otimes val_{B,X'})x = am(Id \wedge x)val_{A \otimes B,X}$
- La vérification des axiomes (UG) et (UD) est sans difficulté.
- Pour l'axiome (AM) on utilise (C_3).
2) On construit ensuite un isomorphisme canonique $\gamma : \mathcal{B} \to \mu t(\mathcal{B}^t)$. Il est donné...
- sur les objets par $|\gamma| = Id$,
- sur une flèche $\alpha : X \to Y$ de $\mathbb{B}$, par $\gamma(\alpha) = (A, f)$ où $A = U_{XY}(\alpha)$ et $f : A \wedge X \to Y$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que $f val_{A,X} = \alpha$ (la composition est stricte).
- sur une 2-cellule $a : \alpha \to \alpha' : X \to Y$ par $\gamma(a) = U_{XY}(a)$.

Le fait que γ est un isomorphisme de catégorie mutante est sans difficulté.

Proposition 3.4. :On suppose que $\mathcal{B}$ est de saveur C. Alors il existe une catégorie précotensorisée $\mathcal{B}^c$ telle que $\mu t(\mathcal{B}^c) \simeq \mathcal{B}$.

Preuve : - Avant de montrer cette proposition construisons $\mathcal{B}^\star$ la catégorie mutante (sur $\mathbb{V}^\star$) duale.
Elle est définie par $\mathcal{B}^\star = (\mathbb{B}^\star, U^\star)$ où $\mathbb{B}^\star$ est la bicatégorie opposée pour la loi $\otimes$, caractérisée par :
$|\mathbb{B}^\star| = |\mathbb{B}|,\ \forall X, Y \in |\mathbb{B}^\star|,\ \mathbb{B}^\star(X,Y) = \mathbb{B}(Y,X),\ \forall X \in |\mathbb{B}^\star|,\ id^\star_X = id_X$
et $\forall X,Y,Z \in |\mathbb{B}^\star|,\ (\ \mathbb{B}^\star(Y,Z) \times \mathbb{B}^\star(X,Z) \xrightarrow{\otimes^\star_{XYZ}} \mathbb{B}^\star(X,Z)\ ) =$
$(\ \mathbb{B}(Z,Y) \times \mathbb{B}(Y,X) \xrightarrow{\sim} \mathbb{B}(Y,X) \times \mathbb{B}(Z,Y) \xrightarrow{\otimes_{ZYX}} \mathbb{B}(Z,X)\ ).$
Pour $U^\star$, on a $|U^\star| = |U|$ et $\forall X, Y \in |\mathbb{B}^\star|,\ U^\star_{XY} = U_{YX}$.
On vérifie que $\underline{\mathcal{B}^\star} = (\underline{\mathcal{B}})^{op}$ et que $Tri^\star : \underline{V}^{op} \times (\underline{\mathcal{B}^\star})^{op} \times (\underline{\mathcal{B}^\star}) \to \underline{\mathbb{E}ns}$ est donné par $Tri^\star(A,X,Y) = Tri(A,Y,X)$ et pour
$(a,x,y) : (A,X,Y) \to (A',X',Y')$ dans $\underline{V}^{op} \times (\underline{\mathcal{B}^\star})^{op} \times (\underline{\mathcal{B}^\star}) =$
$\underline{V}^{op} \times \underline{\mathcal{B}} \times \underline{\mathcal{B}}^{op},\ Tri^\star(a,x,y) = Tri(a,y,x)$.
- Si on en revient à notre proposition, on constate que $\mathcal{B}$ est de saveur cotensorisée ssi pour tout $A \in |\underline{V}|$, pour tout $Y \in |\mathbb{B}|, Tri(A,-,Y) : (\underline{\mathcal{B}})^{op} \to \underline{\mathbb{E}ns}$ est représentable et donc ssi $Tri^\star(A,Y,-) : \underline{\mathcal{B}^\star} \to \underline{\mathbb{E}ns}$ est co-représentable c.a.d. que $\mathcal{B}^\star$ est de saveur tensorisée sur $\mathbb{V}^\star$. Donc $\mathcal{B}^\star \simeq \mu t(\mathcal{B}^{\star t})$ et donc $\mathcal{B} \simeq \mu t(\mathcal{B}^{\star t})^\star$. Ainsi, si on pose $\mathcal{B}^c = Red^{-1}(\mathcal{B}^{\star t})$, $\mathcal{B}^c$ est une catégorie $\mathbb{V}$-précotensorisée et $\mu c(\mathcal{B}^c) = \mu t(Red(\mathcal{B}^c))^\star$ se vérifie sans difficulté. On a donc bien $\mu c(\mathcal{B}^c) \simeq \mathcal{B}$.

Proposition 3.5. : Fixons deux catégories mutantes $\mathcal{B}$ et $\mathcal{B}'$. On les suppose toutes deux de saveurs E.
1) $\Phi : \mathcal{B} \to \mathcal{B}'$ étant un foncteur mutant, il existe un foncteur enrichi $\Phi^e : \mathcal{B}^e \to \mathcal{B}'^e$ tel que le carré (C_e) suivant commute :

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi"] \arrow[d, "\gamma^e"'] & \mathcal{B}' \arrow[d, "\gamma'^e"] \\
\mu e(\mathcal{B}^e) \arrow[r, "\mu e(\Phi^e)"'] & \mu e(\mathcal{B}'^e)
\end{tikzcd}$$

(pour les catégories enrichies $\mathcal{B}^e$ et $\mathcal{B}'^e$ et les isomorphismes γ^e et γ'^e voir la proposition 3.2).

2) $t : \Phi \to \Phi' : \mathcal{B} \to \mathcal{B}'$ étant une transformation naturelle mutante, il existe une transformation naturelle enrichie $t^e : \Phi^e \to \Phi'^e : \mathcal{B}^e \to \mathcal{B}'^e$ telle que :

$$(\gamma'^e.\Phi \xrightarrow{Id_{\gamma'^e}.t} \gamma'^e.\Phi') = (\mu e(\phi^e).\gamma^e \xrightarrow{\mu e(t^e).Id_{\gamma^e}} \mu e(\phi'^e).\gamma^e)$$

Preuve : 1) Comme pour la construction de $\mathcal{B}^e$, choisissons pour chaque $X, Y \in |\mathcal{B}|$, une représentation $(\mathcal{B}^e(X,Y), ev_{XY})$ de $Tri_{\mathcal{B}}(-,X,Y)$ puis, de la même façon, choisissons aussi, pour chaque $X', Y' \in |\mathcal{B}'|$ une représentation $(\mathcal{B}'^e(X',Y'), ev_{X'Y'})$ du préfaisceau $Tri_{\mathcal{B}'}(-,X',Y')$. À partir de ces deux familles de choix on a construit canoniquement des catégories enrichies $\mathcal{B}^e$ et $\mathcal{B}'^e$ et des isomorphismes de catégories mutantes $\gamma^e : \mathcal{B} \to \mu e(\mathcal{B}^e)$ et $\gamma'^e : \mathcal{B}' \to \mu e(\mathcal{B}'^e)$(voir 3.2). Pour chaque $X, Y \in |\mathcal{B}|$, puisque $\Phi_{XY}(ev_{XY}) \in Tri(\mathcal{B}^e(X,Y), \Phi X, \Phi Y)$, il existe une unique flèche $\Phi^e_{XY} : \mathcal{B}^e(X,Y) \to \mathcal{B}'^e(\Phi X, \Phi Y)$ telle que

$$Tri(\Phi^e_{XY}, \Phi X, \Phi Y)(ev_{\Phi X \Phi Y}) = \Phi_{XY}(ev_{XY})$$

i.e. $\Phi^{e*}_{XY}(ev_{\Phi X \Phi Y}) = \Phi_{XY}(ev_{XY})$. On construit ainsi un foncteur enrichi $\Phi^e : \mathcal{B}^e \to \mathcal{B}'^e$ tel que $|\Phi^e| = |\Phi|$.

Dans la vérification de la fonctionnalité...
- pour la commutation avec les unités, il suffit de montrer l'identité
$(\Phi^e_{XX}.id_X)^*(ev_{\Phi X \Phi X}) = id^*_{\Phi X}(ev_{\Phi X \Phi X})$,
ce qui se fait sans difficulté.
- pour la commutation avec la composition, cela résulte des identités :
$(comp.(\Phi^e_{YZ} \otimes \Phi^e_{XY}))^*(ev_{\Phi X \Phi Z}) = \Phi_{XZ}(ev_{YZ} \otimes ev_{XY}) =$
$(\Phi^e_{XZ}.comp)^*(ev_{\Phi X \Phi Z})$.
- On montre enfin que le carré (C_e) de l'énoncé commute.

Sur les objets c'est immédiat. Sur une flèche $\alpha : X \to Y$ de $\mathcal{B}$, après avoir posé $A = U(\alpha)$, on voit que $\gamma'^e\Phi(\alpha) = (A, f')$ où $f' : A \to \mathcal{C}'(\Phi X, \Phi Y)$ est l'unique flèche de $\mathbb{V}$ telle que $f'^*(ev_{\Phi X \Phi Y}) = \Phi(\alpha)$ et $\mu e(\Phi^e)\gamma^e(\alpha) = (A, \bar{f})$ où $\bar{f} = (A \xrightarrow{f} \mathcal{C}(XY) \xrightarrow{\Phi^e_{XY}} \mathcal{C}'(\Phi X, \Phi Y))$, en notant f l'unique flèche de $\mathbb{V}$ telle que $f^*(ev_{XY}) = \alpha$. On vérifie ensuite que $f' = \bar{f}$, mais cela résulte des identités suivantes : $\bar{f}^*(ev_{\Phi X \Phi Y}) = \Phi_{XY}(\alpha) = f'^*(ev_{\Phi X \Phi Y})$. Enfin sur une 2-cellule c'est immédiat.

2) - Pour chaque $X \in |\mathcal{B}|$, comme $t_X \in Tri(I, \Phi X, \Phi' X)$, on peut considérer l'unique flèche $t^e_X : I \to \mathcal{B}'^e(\Phi X, \Phi' X)$ dans $\mathbb{V}$ telle que $t^{e*}_X(ev_{\Phi X \Phi' X}) = t_X$. Montrons qu'on construit ainsi une transformation naturelle enrichie $t^e : \Phi^e \to \Phi'^e$. Pour cela on montre que le diagramme

suivant commute, pour tout $X, Y \in |\mathcal{B}|$:

$$\begin{tikzcd}
 & \mathcal{B}^e(X,Y) \arrow[dl, "\Phi^e_{XY}"'] \arrow[dr, "\Phi'^e_{XY}"] & \\
\mathcal{B}'^e(\Phi X, \Phi Y) \arrow[d, "u_g^{-1}"'] & & \mathcal{B}'^e(\Phi' X, \Phi' Y) \arrow[d, "u_d^{-1}"] \\
I \otimes \mathcal{B}'^e(\Phi X, \Phi Y) \arrow[d, "t^e_Y \otimes Id"'] & & \mathcal{B}'^e(\Phi' X, \Phi' Y) \otimes I \arrow[d, "Id \otimes t^e_X"] \\
\mathcal{B}'^e(\Phi Y, \Phi' Y) \otimes \mathcal{B}'^e(\Phi X, \Phi Y) \arrow[dr, "comp"'] & & \mathcal{B}'^e(\Phi' X, \Phi' Y) \otimes \mathcal{B}'^e(\Phi X, \Phi' X) \arrow[dl, "comp"] \\
 & \mathcal{B}'^e(\Phi X, \Phi' Y) &
\end{tikzcd}$$

Soit $x, y : \mathcal{B}^e(X,Y) \to \mathcal{B}'^e(\Phi X, \Phi' Y)$ les deux flèches composées de gauche et de droite. Pour montrer que $x = y$ on voit que $x^*(ev_{\Phi X, \Phi' Y}) = t_Y \Phi_{XY}(ev_{XY}) = \Phi'_{XY}(ev_{XY}) t_X = y^*(ev_{\Phi X, \Phi' Y})$.

- Enfin, pour montrer l'identité de l'énoncé, on voit que pour tout $X \in |\mathcal{B}|$, $\gamma'^e(t_X) = (I, t^e_X) = \mu e(t^e)_{\gamma X}$.

Proposition 3.6. : Fixons deux catégories mutantes $\mathcal{B}$ et $\mathcal{B}'$. On les suppose toutes deux de saveurs T.

1) $\Phi : \mathcal{B} \to \mathcal{B}'$ étant un foncteur mutant sur $\mathbb{V}$. alors il existe un morphisme de catégories $\mathbb{V}$-prétensorisées $\Phi^t : \mathcal{B}^t \to \mathcal{B}'^t$ tel que le carré (C_t) suivant commute :

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi"] \arrow[d, "\gamma^t"'] & \mathcal{B}' \arrow[d, "\gamma'^t"] \\
\mu t(\mathcal{B}^t) \arrow[r, "\mu t(\Phi^t)"'] & \mu t(\mathcal{B}'^t)
\end{tikzcd}$$

où les catégories $\mathbb{V}$-prétensorisées $\mathcal{B}^t$ et $\mathcal{B}'^t$ et les morphismes de catégories mutantes γ^t et γ'^t ont été construits à la proposition 3.3.

2) $\theta : \Phi \to \Phi' : \mathcal{B} \to \mathcal{B}'$ étant une transformation naturelle mutante, il existe une 2-cellule $\theta^t : \Phi^t \to \Phi'^t : \mathcal{B}^t \to \mathcal{B}'^t$ dans $\mathbb{V}$-Pretens telle que :

$$(\ \gamma'^t.\Phi \xrightarrow{Id_{\gamma'^t}.\theta} \gamma'^t.\Phi'\ ) = (\ \mu t(\Phi^t).\gamma^t \xrightarrow{\mu t(\theta^t).Id_{\gamma^t}} \mu t(\Phi'^t).\gamma^t\ )$$

Preuve : 1)- Là encore, comme pour la construction de $\mathcal{B}^t$, choisissons, pour chaque $(A, X) \in |\underline{V}| \times |\underline{\mathcal{B}}|$ une co-représentation $(A \wedge X, val_{A,X})$ de $Tri_{\mathcal{B}}(A, X, -)$, puis de la même façon, pour chaque $(A', X') \in |\underline{V}| \times |\underline{\mathcal{B}'}|$, une co-représentation $(A' \wedge X', val_{A',X'})$ de $Tri_{\mathcal{B}'}(A', X', -)$. À partir de ces deux familles de choix, on a construit canoniquement des catégories $\mathbb{V}$-prétensorisées $\mathcal{B}^t$ et $\mathcal{B}'^t$ et des isomorphismes de catégories mutantes, soit $\gamma^t : \mathcal{B} \to \mu t(\mathcal{B}^t)$ et $\gamma'^t : \mathcal{B}' \to \mu t(\mathcal{B}'^t)$ (voir la proposition 3.3). Pour chaque $(A, X) \in |\underline{V}| \times |\underline{\mathcal{B}}|$, comme $\Phi(val_{A,X}) \in Tri(A, \Phi X, \Phi(A \wedge X))$ il existe une unique flèche $\phi_{A,X} : A \wedge \Phi X \to \Phi(A \wedge X)$ dans $\underline{\mathcal{B}'}$ telle que $\phi_{A,X} val_{A,\Phi X} = \Phi(val_{A,X})$ (à gauche la composition est stricte). On montre que $\phi_{A,X}$ est naturelle en A, X. Pour la naturalité en A, on montre que, pour toute flèche $a : A \to A'$ de $\mathbb{V}$ et $X \in |\underline{\mathcal{B}}|$ on a les identités suivantes $\Phi(a \wedge Id)\phi_{A,X} val_{A,\Phi X} = a^*\Phi(val_{A',X}) = \phi_{A',X}(a \wedge Id)val_{A,\Phi X}$. Pour la naturalité en X, on établit que , pour tout $A \in |\underline{V}|$, et toute flèche $x : X \to X'$ dans $\underline{\mathcal{B}}$ on a les identités suivantes $\Phi(Id \wedge x)\phi_{A,X} val_{A,\Phi X} = \Phi(val_{A,X'} x) = \phi_{A,X'}(Id \wedge \Phi x)val_{A,\Phi X}$. On vérifie ensuite les axiomes de morphisme entre catégories $\mathbb{V}$-prétensorisées.

- Pour l'axiome (MS) on établit les identités suivantes :
$\Phi(s_X)\phi_{I,X} val_{I,\Phi X} = Id_{\Phi X} = s_{\Phi X} val_{I,\Phi X}$.
- Pour l'axiome (Mam) on l'obtient grâce aux identités suivantes (où $A, B \in |\underline{V}|$ et $X \in |\underline{\mathcal{B}}|$):
$\phi_{A,B \wedge X}(Id_A \wedge \phi_{B,X})am_{A,B,\Phi X}\, val_{A \otimes B, \phi X} = \Phi(val_{A,B \wedge X}) \otimes \Phi(val_{B,X}) = \Phi(am_{A,B,X})\phi_{A \otimes B,X} val_{A \otimes B, \Phi X}$.

Ainsi $\Phi^t = (\underline{\Phi}, \phi) : \mathcal{B}^t \to \mathcal{B}'^t$ est un morphisme de catégories $\mathbb{V}$-prétensorisées.

- Montrons maintenant la commutation du carré (C_t). Sur les objets c'est immédiat. Sur une flèche $\alpha : X \to Y$ de $\mathbb{B}$, on voit que $\gamma'^t\Phi(\alpha) = (A, f')$ où $A = U(\alpha)$ et $f' : A \wedge \Phi X \to \Phi Y$ est l'unique flèche de $\underline{\mathcal{B}'}$ telle que $f' val_{A,\Phi X} = \Phi(\alpha)$ et $\mu t(\Phi^t)\gamma^t(\alpha) = (A, \Phi(f)\phi_{A,X})$ où $f : A \wedge X \to Y$ est l'unique flèche de $\underline{\mathcal{B}}$ telle que $f val_{A,X} = \alpha$. L'identité $f' = \Phi(f)\phi_{A,X}$ se montre sans difficulté. Enfin la commutation de (C_t) sur les 2-cellules est immédiate.

2)- Pour tout $(A, X) \in |\underline{V}| \times |\underline{\mathcal{B}}|$, montrons que le carré suivant commute dans $\underline{\mathcal{B}'}$:

$$\begin{tikzcd}
A \wedge \Phi(X) \arrow[r, "Id\wedge\theta_X"] \arrow[d, "\phi_{A,X}"'] & A \wedge \Phi'(X) \arrow[d, "\phi'_{A,X}"] \\
\Phi(A \wedge X) \arrow[r, "\theta_{A\wedge X}"'] & \Phi'(A \wedge X)
\end{tikzcd}$$

(où $\Phi^t = (\underline{\Phi}, \phi), \Phi'^t = (\underline{\Phi'}, \phi')$). En fait cela résulte des identités suivantes : $\theta_{A\wedge X}\phi_{A,X}val_{A,\Phi X} = \Phi'(val_{A,X})\theta_X = \phi'_{A,X}(Id \wedge \theta_X)val_{A,\Phi X}$.
- pour l'identité de l'énoncé on voit que pour tout $X \in |\mathcal{B}|$,
$\gamma'^t(\theta_X) = (I, \bar{\theta}_X) = \mu t(\theta^t)_{\gamma^t X}$, où $\bar{\theta}_X = (\ I \wedge \Phi X \xrightarrow{s} \Phi X \xrightarrow{\theta_X} \Phi' X\ )$
(la première identité résultant du fait que $\bar{\theta}_X val_{I,\Phi X} = \theta_X$).

Remarque 3.7. : Dans la proposition qui suit on utilise le foncteur μp (où plus exactement le foncteur $\mu p_{\mathcal{B}^e\mathcal{B}'^t}$). Mais celui-ci ne sera défini qu'à la section suivante, autour de la proposition 4.6 (pour des questions d'organisation du texte). Nous vous renvoyons donc à cette section pour cette définition avant d'aborder la proposition suivante.

Proposition 3.8. : Soient encore deux catégories mutantes $\mathcal{B}$ et $\mathcal{B}'$. On suppose maintenant que $\mathcal{B}$ est de saveur E et $\mathcal{B}'$ est de saveur T.
1) $\Phi : \mathcal{B} \to \mathcal{B}'$ étant un foncteur mutant, il existe une passerelle $\Phi^p : \mathcal{B}^e \to \mathcal{B}'^t$ telle que le carré (C_p) suivant commute :

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi"] \arrow[d, "\gamma^e"'] & \mathcal{B}' \arrow[d, "\gamma'^t"] \\
\mu e(\mathcal{B}^e) \arrow[r, "\mu p(\Phi^p)"'] & \mu t(\mathcal{B}'^t)
\end{tikzcd}$$

où $\mathcal{B}^e$ est construit à la proposition 3.2 et $\mathcal{B}'^t$ l'est à la proposition 3.3 et enfin où $\mu p = \mu p_{\mathcal{B}^e\mathcal{B}'^t}$ est défini à la section suivante, comme averti dans la remarque 3.7.
2) Soit $\theta : \phi \to \phi' : \mathcal{B} \to \mathcal{B}'$ une transformation naturelle mutante. Il existe un morphisme canonique $\theta^p : \phi^p \to \phi'^p$ dans $\mathbb{V}$-$Pass(\mathcal{B}^e, \mathcal{B}'^t)$ telle que :

$$(\ \gamma'^t.\Phi \xrightarrow{Id_{\gamma'^t}.\theta} \gamma'^t.\Phi'\ ) = (\ \mu p(\Phi^p).\gamma^e \xrightarrow{\mu p(\theta^p).Id_{\gamma^e}} \mu p(\Phi'^p).\gamma^e\ )$$

.

Preuve : 1) Comme pour la construction de $\mathcal{B}^e$, choisissons pour chaque $X, Y \in |\mathcal{B}|$, une représentation $(\mathcal{B}^e(X,Y), ev_{XY})$ du préfaisceau $Tri_{\mathcal{B}}(-, X, Y)$ puis, comme pour la construction de $\mathcal{B}'^t$, choisissons, pour chaque $(A', X') \in |\underline{V}| \times |\underline{\mathcal{B}'}|$, une co-représentation $(A' \wedge X', val_{A',X'})$ de $Tri_{\mathcal{B}'}(A', X', -)$. Soient $X, Y \in |\mathcal{B}^e| = |\mathcal{B}|$. Comme
$\Phi_{XY}(ev_{XY}) \in Tri(\mathcal{B}^e(X,Y), \Phi X, \Phi Y)$, il existe une unique flèche
$\pi_{XY} : \mathcal{B}^e(X,Y) \wedge \Phi X \to \Phi Y$ dans $\underline{\mathcal{B}'}$ telle que
$\pi_{XY} val_{\mathcal{B}^e(X,Y)\Phi X} = \Phi_{XY}(ev_{XY})$ (à gauche la composition est stricte). On note $\pi = (\pi_{XY})_{(X,Y) \in |\mathcal{B}^e|^2}$.

Montrons que le couple $(|\Phi|, \pi)$ est une $\mathbb{V}$-passerelle $\mathcal{B}^e \to \mathcal{B}'^t$.

- (PU) Résulte des identités suivantes (où pour $X \in |\mathcal{B}^e|$):
$\pi_{XX}(id_X \wedge Id_{\Phi X}) val_{I,\Phi X} = Id_{\Phi X} = s_X val_{I,\Phi X}$.

- (PC) Résulte, là encore, des identités suivantes, où pour X, Y, Z dans $|\mathcal{B}^e| = |\mathcal{B}|$ on note $U = \mathcal{B}(X,Y), V = \mathcal{B}(Y,Z)$:
$\pi_{YZ}(Id_V \wedge \pi_{XY}) am_{V,U,\Phi X} val_{V \otimes U, \Phi X} = \Phi_{YZ}(ev_{YZ}) \otimes \Phi_{XY}(ev_{XY}) =$
$\pi_{XZ}(comp_{XYZ} \wedge Id_{\Phi X}) val_{V \otimes U, \Phi X}$.

On note Φ^p cette passerelle $\mathcal{B}^e \to \mathcal{B}'^t$.

- La commutation du carré (C_p) est immédiate sur les objets. Sur une flèche $\alpha : X \to Y$ de $\mathcal{B}$ on a $\gamma'^t \Phi(\alpha) = (A, f_0)$ où $A = U_{XY}(\alpha)$ et
$f_0 : A \wedge \Phi X \to \Phi Y$ est l'unique flèche de $\underline{\mathcal{B}'}$ telle que $f_0 val_{A,\Phi X} = \Phi(\alpha)$.
Et d'un autre côté, on a $\mu p(\Phi^p).\gamma^e(\alpha) = \mu p(\Phi^p)(A, f) = (A, f_1)$ où
$f : A \to \mathcal{B}^e(X,Y)$ est l'unique flèche de $\mathbb{V}$ telle que $f^*(ev_{XY}) = \alpha$ et où f_1 est la flèche composée suivante dans $\underline{\mathcal{B}'}$:
$A \wedge \Phi X \xrightarrow{f \wedge Id} \mathcal{B}^e(X,Y) \wedge \Phi X \xrightarrow{\pi_{XY}} \Phi Y$ et où π_{XY} provient de
$\Phi^p = (|\Phi|, \pi)$. L'identité $f_0 = f_1$ se montre sans difficulté.

2) Montrons que la famille $(\theta_X : |\Phi| X \to |\Phi'| X)_{X \in |\mathcal{B}^e|}$ définit un morphisme de passerelles $\theta^p : \phi^p \to \phi'^p$ où $\phi^p = (|\Phi|, \pi), \phi'^p = (|\Phi'|, \pi')$.

- (MP) Résulte des identités suivantes, où $X, Y \in |\mathcal{B}^e| = |\mathcal{B}|$ en posant $A = \mathcal{B}^e(X,Y)$):
$\theta_Y \pi_{XY} val_{A,\Phi X} = \Phi'_{XY}(ev_{XY}) \theta_X = \pi'_{XY}(Id_A \wedge \theta_X) val_{A,\Phi X}$.

- Pour l'identité du (2), on voit que, pour tout $X \in |\mathcal{B}|$, $\gamma'^t(\theta_X) = (I, f)$, où $f : I \wedge \Phi X \to \Phi' X$ est l'unique flèche de $\underline{\mathcal{B}'}$ telle que $f val_{I,\Phi X} = \theta_X$.
D'un autre côté $\mu p(\Phi^p)_{\gamma^e X} = (I, f')$ où f' est la flèche composée suivante dans $\underline{\mathcal{B}'}$: $I \wedge \Phi X \xrightarrow{s} \Phi X \xrightarrow{\theta_X} \Phi' X$. Le fait que $f = f'$ se montre sans difficulté.

4. Plongement dans la 2-catégorie $Cat\mu(\mathbb{V})$

• Le but principal de cette section est de montrer que les 2-foncteurs μe, μt et μc sont 2-pleinement fidèles.

Proposition 4.1. : μe : $\mathbb{V}$-$Cat \to Cat\mu(\mathbb{V})$ est 2-pleinement fidèle.

Preuve : Avant de le monter, commençons par prouver le lemme suivant

Lemme 4.2. : Soit $\mathcal{C} \in |\mathbb{V}\text{-}Cat|$, alors,
1) $\mu e(\mathcal{C})$ est de saveur E. On considére, pour chaque $X, Y \in |\mathcal{C}|$, le choix de représentation $(\mathcal{C}(X,Y),\ X \xrightarrow[\mathcal{C}(X,Y)]{ev_{XY}} Y\ )$ de $Tri(-, X, Y) : \underline{V}^{op} \to \underline{Ens}$ où $ev_{XY} = (\mathcal{C}(X,Y), Id_{\mathcal{C}(X,Y)})$ (dans la suite on l'appelera *la représentation canonique*). Alors, pour cette représentation canonique, on obtient
$\mu e(\mathcal{C})^e = \mathcal{C}$.
2) Pour la même représentation, $(\mu e(\mathcal{C}) \overset{\gamma^e}{\to} \mu e(\mu e(\mathcal{C})^e)) = Id$.

Preuve du lemme:1) On remarque que $Tri(A, X, Y) =$
$\{(A, f)/A \in |\underline{V}|,\ f \in \underline{V}(A, \mathcal{C}(X,Y))\}$ et, pour toute flèche $a : A' \to A$ dans $\underline{V}$, on a $Tri(a, X, Y) : (A, f) \mapsto (A', f.a)$. On vérifie alors, très facilement, que $(\mathcal{C}(X,Y), ev_{XY})$ est une représentation de $Tri(-, X, Y)$ et on déduit, au passage, que $\mu e(\mathcal{C})$ est de saveur E. Enfin l'identité $\mu e(\mathcal{C})^e = \mathcal{C}$ s'obtient encore assez simplement.
2) - Sur les objets on a déjà $|\gamma^e| = Id$ (c'est toujours le cas).
- sur une flèche $(A, f) : X \to Y$, comme $f : (A, f) \to (\mathcal{C}(X,Y), Id) = ev_{XY}$ est une flèche de $\mu e(\mathcal{C})(X,Y)$ dont l'image par U est
$f : A \to \mathcal{C}(X,Y)$, alors $f^*(ev_{XY}) = (A, f)$ et donc $\gamma^e(A, f) = (A, f)$.
- sur une 2-cellule $a : (A, f) \to (A', f')$, $\gamma^e(a) = U_{XY}(a) = a$.

Preuve de la proposition: Soient $\mathcal{C}, \mathcal{C}' \in |\mathbb{V}\text{-}Cat|$ et $F : \mu e(\mathcal{C}) \to \mu e(\mathcal{C}')$ une flèche de $Cat\mu(\mathbb{V})$. Comme $\mu e(\mathcal{C})$ et $\mu e(\mathcal{C}')$ sont de saveur E (voir Lemme 4.2) il existe un foncteur enrichi $f : \mathcal{C} \to \mathcal{C}'$ tel que $\mu e(f) = F$ (résulte de la proposition 3.5) car $\gamma^e = \gamma'^e = Id$ (Lemme 4.2). En fait f est unique. En effet, si $f' : \mathcal{C} \to \mathcal{C}'$ est tel que $\mu e(f') = \mu e(f)$, on a :
$(\mathcal{C}(X,Y), f_{XY}) = \mu e(f)(\mathcal{C}(X,Y), Id) = \mu e(f')(\mathcal{C}(X,Y), Id) =$
$(\mathcal{C}(X,Y), f'_{XY})$ et donc $f_{XY} = f'_{XY}$, ceci pour tout $X, Y \in |\mathcal{C}|$.
Comme on a aussi $|f| = |\mu e(f)| = |\mu e(f')| = |f'|$, on obtient $f = f'$.

Si maintenant on se donne des foncteurs enrichis $f, f' : \mathcal{C} \to \mathcal{C}'$ et une transformation naturelle mutante $t : \mu e(f) \to \mu e(f')$, toujours par la proposition 3.5, il existe une transformation naturelle enrichie t^e telle que $\mu e(t^e) = t$ (car $\mu e(f)^e = f$, voir l'unicité précédente). t^e est unique car si $t, t' : f \to f'$ sont deux transformations naturelles enrichies telles que $\mu e(t) = \mu e(t')$, alors pour $X \in |\mathcal{C}|$, $(I, t_X) = (I, t'_X)$ implique $t_X = t'_X$. Finalement $t = t'$.

Proposition 4.3. : $\mu t : \mathbb{V}\text{-}Pretens \to Cat\mu(\mathbb{V})$ est 2-pleinement fidèle.

Preuve : La encore, avant de le montrer, commençons par prouver le lemme suivant :

Lemme 4.4. : Soit $\mathbb{E} = (\underline{E}, \wedge, ...) \in |\mathbb{V}\text{-}Pretens|$, alors,
1) $\mu t(\mathbb{E})$ est de saveur T. On considére pour chaque $(A, X) \in |\underline{V}| \times |\underline{E}|$, le choix de co-représentation $(A \wedge X,\ X \xrightarrow[A]{val_{A,X}} A \wedge X\ )$ de $Tri(A, X, -) : \underline{\mu t(\mathbb{E})} \to \underline{Ens}$, où $val_{A,X} = (A, Id_{A\wedge X})$ (on l'appelle *la co-représentation canonique*). Alors, pour cette co-représentation canonique, on a l'isomorphisme $(J, Id) : \mathbb{E} \to \mu t(\mathbb{E})^t$, où $J : \underline{E} \to \underline{\mu t(\mathbb{E})}$ est l'isomorphisme fonctoriel donné par $|J| = Id$ et
$J(X \overset{f}{\to} Y) = (\ I \wedge X \xrightarrow{s_X} X \xrightarrow{f} Y\ )$. Ce (J, Id) est naturel en $\mathbb{E}$.
2) Toujours pour la co-représentation canonique on a l'identité suivante :

$$(\mu t(\mathbb{E}) \overset{\gamma^t}{\to} \mu t(\mu t(\mathbb{E})^t)) = \mu t(\mathbb{E} \overset{(J,Id)}{\to} \mu t(\mathbb{E})^t).$$

Preuve du lemme:1) Ici $Tri(A, X, Y) =$
$\{(A, \alpha)/A \in |\underline{V}|, \alpha \in \underline{E}(A \wedge X, Y)\}$ et pour toute flèche $(I, f) : Y \to Y'$ de $\underline{\mu t(\mathbb{E})}$, $Tri(A, X, (I, f))(A, \alpha) = (A, \underline{f}.\alpha)$ où $\underline{f} = (\ Y \xrightarrow{s^{-1}} I \wedge Y \xrightarrow{f} Y'\ )$. On vérifie facilement que $(A \wedge X, val_{A,X})$ est une co-représentation de $Tri(A, X, -)$ et on en déduit que $\mu t(\mathbb{E})$ est de saveur T. Enfin $\mu t(\mathbb{E})^t = (\underline{\mu t(\mathbb{E})}, \wedge^t, s^t, am^t)$ où $|\underline{\mu t(\mathbb{E})}| = |\underline{E}|$ et $\underline{\mu t(\mathbb{E})}(X, Y) = \{(I, f)/f \in \mathbb{E}(I \wedge X, Y)\}$. Le composé de $X \xrightarrow{(I,f)} Y \xrightarrow{(I,g)} Z$ est $X \xrightarrow{(I,\underline{g}.f)} Z$. Sur un couple d'objets (A, X) de $|\underline{V}| \times |\underline{\mu t(\mathbb{E})}|$, $A \wedge^t X = A \wedge X$, sur un couple de flèches $(a, x) : (A, X) \to (A', X')$ de $\underline{V} \times \underline{\mu t(\mathbb{E})}$, $a \wedge^t x = J(a \wedge J^{-1}(x))$, $s^t_X = J(s_X)$ et $am^t_{A,B,X} = J(am_{A,B,X})$. Enfin la naturalité de (J, Id) se

montre sans difficulté.
2) - Sur les objets on a $|\mu t(J, Id)| = |J| = Id = |\gamma^t|$.
- Sur une flèche $(A, \alpha) : X \to Y,\ \mu t(J, Id)(A, \alpha) = (A, J(\alpha)) = \gamma^t(A, \alpha)$.

Preuve de la proposition: Soient $\mathbb{E}, \mathbb{E}' \in |\mathbb{V}\text{-}Pretens|$ et $\Phi : \mu t(\mathbb{E}) \to \mu t(\mathbb{E}')$ une flèche de $Cat\mu(\mathbb{V})$. On sait que $\mu t(\mathbb{E})$ et $\mu t(\mathbb{E}')$ sont de saveur T (voir Lemme 4.4). Considérons alors, pour chaque $(A, X) \in |\underline{V}| \times |\underline{\mu t(\mathbb{E})}|$, les co-représentations de $Tri(A, X, -)$ données dans le lemme 4.4. On sait qu'il existe un morphisme canonique $\Phi^t : \mu t(\mathbb{E})^t \to \mu t(\mathbb{E}')^t$ qui fait commuter le carré suivant dans $Cat\mu(\mathbb{V})$ (où $\gamma^t = \mu t(J, Id)$ et $\gamma'^t = \mu t(J', Id)$. Voir le Lemme 4.4 et la proposition 3.6):

$$\begin{tikzcd}
\mu t(\mathbb{E}) \arrow[r, "\Phi"] \arrow[d, "\gamma^t"'] & \mu t(\mathbb{E}') \arrow[d, "\gamma'^t"] \\
\mu t(\mu t(\mathbb{E})^t) \arrow[r, "\mu t(\Phi^t)"'] & \mu t(\mu t(\mathbb{E}')^t)
\end{tikzcd}$$

On en déduit que $\Phi = \mu t(\tilde{\Phi})$ où

$$\tilde{\Phi} = (\ \mathbb{E} \xrightarrow{(J,Id)} \mu t(\mathbb{E})^t \xrightarrow{\Phi^t} \mu t(\mathbb{E}')^t \xrightarrow{(J^{-1},Id)} \mathbb{E}'\ )$$

Montrons que $\tilde{\Phi}$ est unique. Soient $(F, \Phi), (F', \phi') : \mathbb{E} \to \mathbb{E}'$ des flèches de $\mathbb{V}$-$Pretens$. On suppose $\mu t(F, \phi) = \mu t(F', \phi')$. Alors, comme le diagramme suivant commute

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "(J,Id)"] \arrow[d, "(F,\phi)"'] & \mu t(\mathbb{E})^t \arrow[r, "Id"] \arrow[d, "\mu t(F,\phi)^t"'] & \mu t(\mathbb{E})^t \arrow[r, "(J,Id)^{-1}"] \arrow[d, "\mu t(F',\phi')^t"'] & \mathbb{E} \arrow[d, "(F',\phi')"] \\
\mathbb{E}' \arrow[r, "(J',Id)"'] & \mu t(\mathbb{E}')^t \arrow[r, "Id"'] & \mu t(\mathbb{E}')^t \arrow[r, "(J',Id)^{-1}"'] & \mathbb{E}'
\end{tikzcd}$$

on en déduit que $(F, \Phi) = (F', \phi')$.
- Soit maintenant $(F, \Phi), (F', \phi') : \mathbb{E} \to \mathbb{E}'$ deux flèches de $\mathbb{V}$-$Pretens$ et $\theta : \mu t(F, \phi) \to \mu t(F', \phi')$ une 2-cellule de $\mathbb{V}$-$Pretens$. Comme $\mu t(\mathbb{E})$ et $\mu t(\mathbb{E}')$ sont de saveur T, il existe une 2-cellule $\theta^t : \mu t(F, \phi)^t \to \mu t(F', \phi')^t$ telle que

$$(\,\mu t(\mu t(F, \phi)^t).\gamma^t \xrightarrow{\mu t(\theta^t).Id_{\gamma^t}} \mu t(\mu t(F', \phi')^t).\gamma^t\,) = (\,\gamma'^t.\mu t(F, \phi) \xrightarrow{Id_{\gamma'^t}.\theta} \gamma'^t.\mu t(F', \phi')\,)$$

(voir la proposition 3.6). Or $\gamma^t = \mu t(J, Id),\ \gamma'^t = \mu t(J', Id)$. On en déduit que $\theta = \mu t(\tilde{\theta})$ où

$$\tilde{\theta} = (\ (J^{-1}, Id).\mu t(F, \phi)^t.(J, Id) \xrightarrow{Id.\theta^t.Id} J^{-1}, Id).\mu t(F', \phi')^t.(J, Id)\ )$$

Montrons que $\tilde{\theta}$ est unique. Soient $(F_0, \phi_0), (F_1, \phi_1) \in |\mathbb{V}\text{-}Pretens(\mathbb{E}, \mathbb{E}')|$ et $\theta, \theta' : (F_0, \phi_0) \to (F_1, \phi_1)$ deux flèches parallèles de $\mathbb{V}\text{-}Pretens(\mathbb{E}, \mathbb{E}')$ telles que $\mu t(\theta) = \mu t(\theta')$. Alors, pour tout $X \in |\mathbb{E}|$, on a
$(I, J(\theta_X)) = \mu t(\theta)_X = \mu t(\theta')_X = (I, J(\theta'_X))$ et donc
$J(\theta_X) = J(\theta'_X) \Rightarrow\ \theta_X = \theta'_X$ et enfin $\theta = \theta'$.

Proposition 4.5. : $\mu c : \mathbb{V}\text{-}Precot \to Cat\mu(\mathbb{V})$ est 2-pleinement fidèle.

Preuve : Comme $\mu c = et^{opv}.\mu t^{opv}.Red$ (voir la construction de μc) où $et : Cat\mu(\mathbb{V}^*) \to Cat\mu(\mathbb{V})^{opv}$ (ainsi que et^{opv}) et $Red : \mathbb{V}\text{-}Precot \to (\mathbb{V}\text{-}Pretens)^{opv}$ sont des isomorphismes et μt est 2-pleinement fidèle de même que μt^{opv}, on en déduit que μc est lui aussi 2-pleinement fidèle.

• Après ces trois plongements μe, μt et μc intéressons-nous maintenant aux différents foncteurs μp suivants.
Soient $\mathcal{C} \in |\mathbb{V}\text{-}Cat|$ et $\mathbb{E} \in |\mathbb{V}\text{-}Pretens|$. On construit canoniquement un foncteur $\mu p_{\mathcal{C}\mathbb{E}} : \mathbb{V}\text{-}Pass(\mathcal{C}, \mathbb{E}) \to Cat\mu(\mu e(\mathcal{C}), \mu t(\mathbb{E}))$,
- sur une passerelle $P = (|P|, \pi) \in |\mathbb{V}\text{-}Pass(\mathcal{C}, \mathbb{E})|$, par $\mu p_{\mathcal{C}\mathbb{E}}(P) = \Phi$ où Φ est lui-même obtenu...
.. sur les objets par $|\Phi| = |P|$,
.. sur une flèche $(A, f) : X \to Y$ de $\mu e(\mathcal{C})$, par $\Phi(A, f) = (A, \bar{f})$ où
$\bar{f} : A \wedge \Phi(X) \to \Phi(Y)$ est la flèche composée suivante dans $\mathbb{E}$:

$$A \wedge |P|(X) \xrightarrow{f \wedge Id} \mathcal{C}(X, Y) \wedge |P|(X) \xrightarrow{\pi_{XY}} |P|(Y)$$

.. sur une 2-cellule $a : (A, f) \to (A', f')$ de $\mu e(\mathcal{C})$ par
$\Phi(a) = (a : (A, \bar{f}) \to (A', \bar{f}'))$.
On vérifie que $\Phi : \mu e(\mathcal{C}) \to \mu t(\mathbb{E}))$ est un foncteur mutant. La partie un peu délicate est de montrer que pour le couple de flèches composables suivant dans $\mu e(\mathcal{C})$, $X \xrightarrow{(A,f)} Y \xrightarrow{(B,g)} Z$ on a
$\Phi((B, g) \otimes (A, f)) = \Phi(B, g) \otimes \Phi(A, f)$. Cela revient à montrer que
$\overline{g \circ f} = \bar{g} \circ \bar{f}$. Or, si on pose $U = \mathcal{C}(X, Y), V = \mathcal{C}(Y, Z)$, on a les identités suivantes :
$\bar{g} \circ \bar{f} = \pi_{YZ}.(g \wedge Id_{PY}).(Id_B \wedge \pi_{XY}).(Id_B \wedge (f \wedge Id_{PX})).am_{B,A,PX} = \pi_{YZ}.(Id_V \wedge \pi_{XY}).am_{V,U,PX}.(g \otimes f) \wedge Id_{PX} =$
$\pi_{XZ}.(comp_{X,Y,Z} \wedge Id_{PX}).((g \otimes f) \wedge Id_{PX}) = \overline{g \circ f}$
- sur un morphisme de passerelles $t : P \to P'$, on construit $\theta = \mu p_{\mathcal{C}\mathbb{E}}(t)$,
.. sur un objet $X \in |\mu e(\mathcal{C})| = |\mathcal{C}|$, en posant
$\theta_X = ((I, J(t_X)) : |P|X \to |P'|X)$ (θ_X est une flèche de $\mu t(\mathbb{E})$). Le fait que $\theta = (\theta_X)_{X \in |\mu e(\mathcal{C})|}$ est une 2-cellule $\mu p_{\mathcal{C}\mathbb{E}}(P) \to \mu p_{\mathcal{C}\mathbb{E}}(P')$ se montre sans difficulté.
Enfin, la fonctorialité de $\mu p_{\mathcal{C}\mathbb{E}}$ est très simple à vérifier.

Proposition 4.6. :Soient $\mathcal{C} \in |\mathbb{V}\text{-}Cat|$ et $\mathbb{E} \in |\mathbb{V}\text{-}Pretens|$. Le foncteur $\mu p_{\mathcal{C}\mathbb{E}}$ est un isomorphisme.

Preuve : Notons déjà $M = \mu p_{\mathcal{C}\mathbb{E}}$ et soit $\Phi \in |Cat\mu(\mu e(\mathcal{C}), \mu t(\mathbb{E}))|$. On sait que $\mu e(\mathcal{C})$ est de saveur E et $\mu t(\mathbb{E})$ de saveur T (voir les lemmes 4.2 et 4.4). Choisissons pour eux leur représentation et co-représentation canoniques. Pour la première écrivons là $(\mathcal{C}(X,Y), ev_{XY})$ (pour chaque $X, Y \in |\mathcal{C}|$) et pour la seconde $(A \wedge X, val_{A,X})$ (pour chaque (A, X) de $|\underline{V}| \times |\underline{E}|$). On a alors vu, à la proposition 3.8, qu'il existe une passerelle $\Phi^p : \mu e(\mathcal{C})^e \to \mu t(\mathbb{E})^t$ telle que le carré suivant commute dans $Cat\mu$:

$$\begin{tikzcd}
\mu e(\mathcal{C}) \arrow[r, "\Phi"] \arrow[d, "\gamma^e"'] & \mu t(\mathbb{E}) \arrow[d, "\gamma^t"] \\
\mu e(\mu e(\mathcal{C})^e) \arrow[r, "\mu p(\Phi^p)"'] & \mu t(\mu t(\mathbb{E})^t)
\end{tikzcd}$$

Mais, à cause des choix canoniques, on a $\gamma^e = Id$ (voir le lemme 4.2)
et $\gamma^t = \mu t(J, Id)$ (voir le lemme 4.4). Considérons alors la passerelle composée P suivante : $\mathcal{C} \xrightarrow{\Phi^p} \mu t(\mathbb{E})^t \xrightarrow{J^{-1}} \mathbb{E}$ (ce qui a un sens - voir la partie I section 5). On vérifie immédiatement que $M(P) = \Phi$. Pour l'unicité de l'antécédent, soit $P, P' \in |\mathbb{V}\text{-}Pass(\mathcal{C}, \mathbb{E})|$ tels que $M(P) = M(P')$. Écrivons $P = (|P|, \pi)$ et $P' = (|P'|, \pi')$. Alors clairement $|P| = |P'|$ et pour $X, Y \in |\mathcal{C}|$, $(\mathcal{C}(X,Y), \pi_{XY}) = M(P)(\mathcal{C}(X,Y), Id) =$
$M(P')(\mathcal{C}(X,Y), Id) = (\mathcal{C}(X,Y), \pi'_{XY})$ et donc $\pi_{XY} = \pi'_{XY}$. Alors $\pi = \pi'$ et donc $P = P'$.
- Soit maintenant $\theta : \Phi \to \Phi'$ une flèche de $Cat\mu(\mu e(\mathcal{C}), \mu t(\mathbb{E}))$. Comme $\mu e(\mathcal{C})$ est de saveur E et $\mu t(\mathbb{E})$ de saveur T, il existe un morphisme canonique de passerelle $\theta^p : \Phi^p \to \Phi'^p : \mu e(\mathcal{C})^e \to \mu t(\mathbb{E})^t$ tel que
$Id_{\gamma^t}.\theta = \mu p(\theta^p).Id_{\gamma^e}$ (voir la proposition 3.8). Alors, posons $\tau = Id_{J^{-1}}.\theta^p$. C'est une flèche $J^{-1}.\Phi^p \to J^{-1}.\Phi'^p$ dans $\mathbb{V}\text{-}Pass(\mathcal{C}, \mathbb{E})$ telle que $M(\tau) = \theta$. Pour l'unicité de l'antécédent, si on considère $\tau, \tau' : P \to P'$ deux flèches de $\mathbb{V}\text{-}Pass(\mathcal{C}, \mathbb{E})$ telles que $M(\tau) = M(\tau')$. Alors, pour tout $X \in |\mathcal{C}|$,
$(I, J(\tau_X)) = M(\tau)_X = M(\tau')_X = (I, J(\tau'_X)) \Rightarrow J(\tau_X) = J(\tau'_X) \Rightarrow$
$\tau_X = \tau'_X$ et finalement $\tau = \tau'$.

5. Complément sur les saveurs

Proposition 5.1. :Soit $\mathcal{C} \in |\mathbb{V}\text{-}Cat|$. Alors,
1) $\mathcal{C}$ est à tenseurs ssi $\mu e(\mathcal{C})$ est de saveur T.
2) $\mathcal{C}$ est à co-tenseurs ssi $\mu e(\mathcal{C})$ est de saveur C.

Preuve : 1) Dire que $\mathcal{C}$ est à tenseurs c'est dire que, pour tout $A \in |\underline{V}|$ et $X \in |\underline{\mathcal{C}}|$, il y a un objet libre $(A \wedge X, \eta^X_A : A \to \mathcal{C}(X, A \wedge X))$ associé à A pour $\underline{y}^X$. Cela signifie que pour tout $Y \in |\underline{\mathcal{C}}|$ et toute flèche $f : A \to \mathcal{C}(X,Y)$ il existe une unique flèche $\bar{f} : A \wedge X \to Y$ dans $\underline{\mathcal{C}}$ telle que $\underline{y}^X(\bar{f}).\eta_A^X = f$ (dans $\underline{V}$) ce qui s'exprime encore en disant que le diagramme (D_{et}) suivant commute :

$$\begin{tikzcd}
& A \arrow[dl, "\eta_A^X"'] \arrow[dr, "f"] & \\
\mathcal{C}(X, A\wedge X) \arrow[d, "u_g^{-1}"'] & & \mathcal{C}(X,Y) \\
I \otimes \mathcal{C}(X, A \wedge X) \arrow[rr, "\bar{f}\otimes Id"'] & & \mathcal{C}(A\wedge X, Y) \otimes \mathcal{C}(X, A\wedge X) \arrow[u, "comp"']
\end{tikzcd}$$

D'un autre coté, dire que $\mu e(\mathcal{C})$ est de saveur T c'est dire que, pour tout $A \in |\underline{V}|$ et $X \in |\underline{\mu e(\mathcal{C})}| = |\mathcal{C}|$, il y a une co-représentation $(A \wedge X, val_{A,X})$ de $Tri(A,X,-)$ où $val_{A,X} \in Tri(A,X,A\wedge X)$. Ce $val_{A,X}$ s'écrit donc (A, η_A^X) où $\eta_A^X : A \to \mathcal{C}(X, A \wedge X)$ est une flèche de $\underline{V}$. Cette co-représentation signifie que pour tout $Y \in |\underline{\mu e(\mathcal{C})}|$ et tout $(A,f) \in Tri(A,X,Y)$ (c.a.d. que $f : A \to \mathcal{C}(X,Y)$ est une flèche de $\underline{V}$) il existe une unique flèche $\tilde{f} = (I, \bar{f}) : A \wedge X \to Y$ dans $\underline{\mu e(\mathcal{C})}$ telle que $\tilde{f} val_{A,X} = (A,f)$ ce qui s'écrit encore en disant que f est le composé (I_{et}) suivant dans $\underline{V}$:

$$A \xrightarrow{u_g^{-1}} I \otimes A \xrightarrow{\bar{f}\otimes\eta_A^X} \mathcal{C}(A\wedge X, Y) \otimes \mathcal{C}(X, A \wedge X) \xrightarrow{comp} \mathcal{C}(X,Y)$$

On constate que la commutation de (D_{et}) et l'identité (I_{et}) se déduisent l'une de l'autre et que l'unicité de $\bar{f}$ pour la commutation de (D_{et}) ou pour l'identité (I_{et}) s'entrainent l'une de l'autre. D'où l'équivalence voulue.

2) Dire que $\mathcal{C}$ est à co- tenseurs c'est dire que, pour tout $A \in |\underline{V}|$ et $Y \in |\mathcal{C}|$ il y a un objet libre $(H(Y,A), \varepsilon_{Y,A})$ associé à A pour $\underline{y}_Y$. Cela signifie que pour tout $X \in |\mathcal{C}|$ et toute flèche $f : A \to \mathcal{C}(X,Y)$ il existe une unique flèche $\bar{f} : X \to H(Y,A)$ dans $\underline{\mathcal{C}}$ telle que $\underline{y}_Y(\bar{f}).\varepsilon_{Y,A} = f$ (dans $\underline{V}$) ce qui s'exprime encore en disant que le diagramme (D_{ec}) suivant commute :

$$\begin{tikzcd}
& A \arrow[dl, "\varepsilon_{Y,A}"'] \arrow[dr, "f"] & \\
\mathcal{C}(H(Y,A), Y) \arrow[d, "u_d^{-1}"'] & & \mathcal{C}(X,Y) \\
\mathcal{C}(H(Y,A), Y) \otimes I \arrow[rr, "Id\otimes\bar{f}"'] & & \mathcal{C}(H(Y,A), Y) \otimes \mathcal{C}(X, H(Y,A)) \arrow[u, "comp"']
\end{tikzcd}$$

D'un autre coté, dire que $\mu e(\mathcal{C})$ est de saveur C c'est dire que, pour tout $A \in |\underline{V}|$ et $Y \in |\underline{\mu e(\mathcal{C})}| = |\mathcal{C}|$, il y a une représentation $(H(Y,A), cov_{Y,A})$ de $Tri(A,-,Y)$ où $cov_{Y,A} \in$

$Tri(A, H(Y, A), Y)$. Il s'écrit donc $cov_{Y,A} = (A, \varepsilon_{Y,A})$
où $\varepsilon_{Y,A} : A \to \mathcal{C}(H(Y, A), Y)$ est une flèche de $\underline{V}$. Cette représentation signifie que pour tout $X \in |\underline{\mu e(\mathcal{C})}|$ et tout $(A, f) \in Tri(A, X, Y)$ (c.a.d. que $f : A \to \mathcal{C}(X, Y)$ est une flèche de $\underline{V}$), il existe une unique flèche $\tilde{f} = (I, \bar{f}) : X \to H(Y, A)$ dans $\underline{\mu e(\mathcal{C})}$ telle que $cov_{Y,A}\tilde{f} = (A, f)$ ce qui s'écrit encore en disant que f est le composé (I_{ec}) suivant dans $\underline{V}$:

$$I \xrightarrow{u_d^{-1}} A \otimes I \xrightarrow{\varepsilon_{Y,A} \otimes \bar{f}} \mathcal{C}(H(Y, A), Y) \otimes \mathcal{C}(X, H(Y, A)) \xrightarrow{comp} \mathcal{C}(X, Y)$$

On constate que la commutation de (D_{ec}) et l'identité (I_{ec}) se déduisent l'une de l'autre et que l'unicité de $\bar{f}$ pour la commutation de (D_{ec}) ou pour l'identité (I_{ec}) s'entrainent l'une de l'autre. D'où l'équivalence voulue.

Proposition 5.2. :Soit $\mathbb{E} \in |\mathbb{V}\text{-}Pretens|$. Alors :
1) $\mathbb{E}$ est enrichissable (voir partie I, section 2) ssi $\mu t(\mathbb{E})$ est de saveur E.
2) $\mathbb{E}$ est cotensorisable (voir partie I, section 3) ssi $\mu t(\mathbb{E})$ est de saveur C.

Preuve : 1) Dire que $\mathbb{E}$ est enrichissable c'est dire que, pour tout $X, Y \in |\mathbb{E}|$ il y a un objet co-libre $(\mathcal{C}(X, Y), Ev_Y^X)$ associé à Y pour le foncteur $(-) \wedge X : \underline{V} \to \underline{E}$. Cela signifie que pour tout $A \in |\underline{V}|$ et toute flèche $f : A \wedge X \to Y$ il existe une unique flèche $\tilde{f} : A \to \mathcal{C}(X, Y)$ dans $\underline{V}$ telle que $f \underset{J_{te}}{=} Ev_Y^X.(\tilde{f} \wedge Id)$ (dans $\underline{E}$).
D'un autre coté, dire que $\mu t(\mathbb{E})$ est de saveur E c'est dire que, pour tout $X, Y \in |\underline{\mu t(\mathbb{E})}| = |\mathbb{E}|$, il y a une représentation $(\mathcal{C}(X, Y), ev_{XY})$ de $Tri(-, X, Y)$ où
$ev_{XY} \in Tri(\mathcal{C}(X, Y), X, Y)$. Il s'écrit donc $ev_{XY} = (\mathcal{C}(X, Y), Ev_Y^X)$ où
$Ev_Y^X : \mathcal{C}(X, Y) \wedge X \to Y$ est une flèche de $\mathbb{E}$. Cette représentation signifie que pour tout $(A, f) \in Tri(A, X, Y)$ il existe une unique flèche $\tilde{f} : A \to \mathcal{C}(X, Y)$ dans $\underline{V}$ telle que $\tilde{f}^*(ev_{XY}) \underset{I_{te}}{=} (A, f)$. On constate que l'identité (J_{te}) et l'identité (I_{te}) coïncident. D'où l'équivalence voulue.
2) Dire que $\mathbb{E}$ est cotensorisable c'est dire que, pour tout $A \in |\underline{V}|$ et tout $Y \in |\mathbb{E}|$ il y a un objet co-libre $(H(Y, A), cv_Y^A)$ associé à Y pour le foncteur $A \wedge (-) : \underline{E} \to \underline{E}$. Cela signifie que pour tout $X \in |\underline{E}|$ et toute flèche $f : A \wedge X \to Y$ de $\underline{E}$ il existe une unique flèche $\tilde{f} : X \to H(Y, A)$ dans $\underline{E}$ telle que $cv_Y^A.(Id \wedge \tilde{f}) \underset{I_{tc}}{=} f$ (dans $\underline{E}$) .
D'un autre coté, dire que $\mu t(\mathbb{E})$ est de saveur C c'est dire que, pour tout $A \in |\underline{V}|$ et tout $Y \in |\mu t(\mathbb{E})| = |\mathbb{E}|$, il y a une représentation $(H(Y, A), cov_Y^A)$ de $Tri(A, -, Y)$ où $cov_Y^A \in Tri(A, H(Y, A), Y)$. Il s'écrit donc $cov_Y^A = (A, cv_Y^A)$ où $cv_Y^A : A \wedge H(Y, A) \to Y$ est une flèche de $\underline{E}$. Cette représentation signifie que pour toute flèche $f : A \wedge X \to Y$ de $\underline{E}$ il existe une unique flèche $\bar{f} : I \wedge X \to H(Y, A)$ dans $\underline{E}$ telle que $cov_Y^A(I, \bar{f}) = (A, f)$ ce

qui s'écrit encore en disant que le diagramme (D_{tc}) suivant commute :

$$\begin{tikzcd}
 & & A \wedge X \arrow[dll, "u_d^{-1} \wedge Id"'] \arrow[drr, "f"] & & \\
(A \otimes I) \wedge X \arrow[dr, "am"'] & & & & Y \\
 & A \wedge (I \wedge X) \arrow[rr, "Id \wedge \bar{f}"'] & & A \wedge H(Y, A) \arrow[ur, "cv_Y^A"'] &
\end{tikzcd}$$

Clairement $\bar{f}$ et $\tilde{f}$ se définissent l'un par rapport à l'autre par la relation $\tilde{f}.s_X = \bar{f}$ et, au passage, on constate que la commutation de (D_{tc}) et l'identité (I_{tc}) se déduisent l'une de l'autre. Même chose pour l'unicité de $\bar{f}$ et de$\tilde{f}$. D'où l'équivalence voulue.

Proposition 5.3. :Soit $\mathbb{E} \in |\mathbb{V}\text{-}Precot|$. Alors :
1) $\mathbb{E}$ est enrichissable (voir partie I, section 4) ssi $\mu c(\mathbb{E})$ est de saveur E.
2) $\mathbb{E}$ est tensorisable (voir partie I, section 4) ssi $\mu c(\mathbb{E})$ est de saveur T.

Preuve : 1) Dire que $\mathbb{E}$ est enrichissable c'est dire que, pour tout $X, Y \in |\mathbb{E}|$ il y a un objet libre $(\mathcal{C}(X,Y), ve_Y^X)$ associé à X pour le foncteur $Y^{(-)} : \underline{V}^{op} \to \underline{E}$. Cela signifie que pour tout $A \in |\underline{V}|$ et toute flèche $f : X \to Y^A$ de $\underline{E}$, il existe une unique flèche $\tilde{f} : A \to \mathcal{C}(X,Y)$ dans $\underline{V}$ telle que $Y^{\tilde{f}}.ve_Y^X \underset{J_{ce}}{=} f$ (dans $\underline{E}$).
D'un autre coté, dire que $\mu c(\mathbb{E})$ est de saveur E c'est dire que, pour tout $X, Y \in |\underline{\mu c(\mathbb{E})}| = |\mathbb{E}|$, il y a une représentation $(\mathcal{C}(X,Y), ev_Y^X)$ de $Tri(-, X, Y)$ où
$ev_Y^X \in Tri(\mathcal{C}(X,Y), X, Y)$. Il s'écrit donc $ev_Y^X = (\mathcal{C}(X,Y), ve_Y^X)$ où $ve_Y^X : X \to Y^{\mathcal{C}(X,Y)}$ est une flèche de $\mathbb{E}$.
Cette représentation signifie que pour tout $(A, f) \in Tri(A, X, Y)$ il existe une unique flèche $\tilde{f} : A \to \mathcal{C}(X,Y)$ dans $\underline{V}$ telle que $\tilde{f}^*(ev_Y^X) \underset{I_{ce}}{=} (A, f)$. On constate que l'identité (J_{ce}) et l'identité (I_{ce}) coïncident. D'où l'équivalence voulue.
2) Dire que $\mathbb{E}$ est tensorisable c'est dire que, pour tout $A \in |\underline{V}|$ et tout $X \in |\mathbb{E}|$ il y a un objet libre $(A \wedge X, Vl_X^A)$ associé à X pour le foncteur $(-)^A : \underline{E} \to \underline{E}$. Cela signifie que pour tout $Y \in |\underline{E}|$ et toute flèche $f : X \to Y^A$ de $\underline{E}$ il existe une unique flèche $\underline{f} : A \wedge X \to Y$ dans $\underline{E}$ telle que $\underline{f}^A.Vl_X^A \underset{I_{ct}}{=} f$ (dans $\underline{E}$) .
D'un autre coté, dire que $\mu c(\mathbb{E})$ est de saveur T c'est dire que, pour tout $A \in |\underline{V}|$ et tout $X \in |\mathbb{E}|$, il y a une co-représentation $(A \wedge X, val_X^A)$ de $Tri(A, X, -) : \underline{\mu c(\mathbb{E})} \to \underline{Ens}$ où $val_X^A \in Tri(A, X, A \wedge X)$. Il s'écrit donc $val_X^A = (A, Vl_X^A)$ où $Vl_X^A : X \to (A \wedge X)^A$ est une flèche de $\underline{E}$. Cette co-représentation signifie que pour toute flèche $f : X \to Y^A$ de $\underline{E}$ il existe une unique flèche $(I, \bar{f}) : A \wedge X \to Y$ dans $\underline{\mu c(\mathbb{E})}$ telle que $(I, \bar{f}) val_X^A = (A, f)$

ce qui s'écrit encore en disant que le diagramme (D_{ct}) suivant commute :

$$\begin{tikzcd}
 & & X \arrow[dll, "Vl^A_X"'] \arrow[drr, "f"] & & \\
(A \wedge X)^A \arrow[dr, "\bar{f}^A"'] & & & & Y^A \\
 & (Y^I)^A \arrow[rr, "\alpha m"'] & & Y^{I \otimes A} \arrow[ur, "Y^{u_g^{-1}}"'] &
\end{tikzcd}$$

Clairement $\bar{f}$ et $\underline{f}$ se définissent l'un par rapport à l'autre par la relation $\bar{f} = \sigma_Y.\underline{f}$ et, au passage, on constate que la commutation de (D_{ct}) et l'identité (I_{ct}) se déduisent l'une de l'autre. Même chose pour l'unicité de $\bar{f}$ et de $\underline{f}$. D'où l'équivalence voulue.

• Intéressons nous maintenant aux exemples de catégories mutantes que sont les catégories fibrées (voir l'exemple 4 de 1.3).
Soit $\underline{B}$ une catégorie à produits fibrés et $\mathbb{E} = (\underline{E}, U)$ une catégorie fibrée sur $\underline{B}$. Commençons par considérer la catégorie mutante $\mathcal{B}_B$ associée à $\mathbb{E}$ en B (voir l'exemple 4 de 1.3).

Proposition 5.4. $\mathbb{E}$ est localement petite ssi pour tout $B \in |\underline{B}|$, $\mathcal{B}_B$ est de saveur E.

Preuve : Rappelons qu'on dit que $\mathbb{E}$ est localement petite si pour tout $B \in |\underline{B}|$ et tout $X, X' \in |\mathbb{E}_B|$, le préfaiceau $Q_{XX'}$ sur $\underline{B}/B$ est représentable, où $Q_{XX'} : (\underline{B}/B)^{op} \to \underline{Ens}$ est défini,
- sur un objet $(A, a) \in |(\underline{B}/B)^{op}|$, par $Q_{XX'}(A, a) = \mathbb{E}_A(a^*X, a^*X')$,
- sur une flèche $\alpha : (A', a') \to (A, a)$,
$Q_{XX'}(\alpha) : \mathbb{E}_A(a^*X, a^*X') \to \mathbb{E}_{A'}(a'^*X, a'^*X')$ est défini par
$Q_{XX'}(\alpha)(x) = (\ a'^*X \xrightarrow{can} \alpha^*a^*X \xrightarrow{\alpha^*x} \alpha^*a^*X' \xrightarrow{can^{-1}} a'^*X'\ ).$
L'équivalence proposée va alors résulter immédiatement du lemme suivant :

Lemme 5.5. : $B \in |\underline{B}|$ étant fixé, alors pour tout $X, X' \in |\mathbb{E}_B|$,
$Q_{XX'} \simeq Tri(-, X, X')$.

Preuve du lemme: L'isomorphisme $\gamma : Q_{XX'} \to Tri(-, X, X')$ se construit en posant, pour chaque $(A, a) \in |\underline{B}/B|$, $\gamma_{(A,a)}(x) = (A, a, x)$.

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STRICT MONADIC TOPOLOGY II: DESCENT FOR CLOSURE SPACES

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Résumé. Par espace de fermeture nous entendons une paire $(A, \mathcal{C})$, dans laquelle A est un ensemble et $\mathcal{C}$ est un ensemble de sous-ensembles de A fermé sous les intersections arbitraires. Le but de cet article est d'initier un développement de la théorie de la descente des espaces de fermeture, nos principaux résultats étant les suivants: (a) caractérisation des morphismes de descente des espaces de fermeture; (b) dans la catégorie des espaces de fermeture finis, tout morphisme de descente est un morphism de descente effectif; (c) chaque morphisme surjectif fermé et chaque morphisme surjectif ouvert d'espaces de fermeture est un morphisme de descente effectif.

Abstract. By a closure space we will mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections. The purpose of this paper is to initiate a development of descent theory of closure spaces, with our main results being: (a) characterization of descent morphisms of closure spaces; (b) in the category of finite closure spaces every descent morphism is an effective descent morphism; (c) every surjective closed map and every surjective open map of closure spaces is an effective descent morphism.

Keywords. closure space, descent morphism, effective descent morphism, closed map, open map.
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1. Introduction

By a *closure space* we will mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections; we will also write informally $\mathcal{C} = \mathcal{C}_A$ and $A = (A, \mathcal{C}) = (A, \mathcal{C}_A)$. A closure space structure $\mathcal{C}$ on a set A can be equivalently described as a closure operator on the power set $\mathrm{P}(A)$ of A written as $X \mapsto \overline{X}$ (or, more precisely, as $X \mapsto \overline{X}^A$) and satisfying

$$X \subseteq X' \Rightarrow \overline{X} \subseteq \overline{X'},\ \ X \subseteq \overline{X},\ \ \overline{\overline{X}} = \overline{X}.$$

The relationship between these two types of structures is given by

$$\overline{X} = \bigcap_{X \subseteq A' \in \mathcal{C}} A' \ \text{ and } \ X \in \mathcal{C} \Leftrightarrow X = \overline{X}.$$

Our reason of using this notion comes from what we called *strict monadic topology* in [3]:

Indeed, for a monad T on the category of sets and a T-algebra A, we can make A a closure space by taking $\mathcal{C}_A$ to be set of all T-subalgebras of A – and then, conversely, every closure space is of this form for a suitably chosen monad.

The purpose of this paper is to initiate a development of descent theory of closure spaces, specifically to:

- characterize descent morphisms (= pullback stable regular epimorphisms) of closure spaces (Proposition 2.10);
- prove that in the category of finite closure spaces every descent morphism is an effective descent morphism (Theorem 4.3);
- compare the above-mentioned result with what happens with finite topological spaces;
- prove that surjective closed maps and surjective open maps of closure spaces are always effective descent morphisms (Theorem 6.5).

The paper is organized as follows: we begin with (mostly known, maybe in slightly different contexts) auxiliary results on closure spaces in Section 2 and on general descent theory in Section 3, except that Section 2 also includes the above-mentioned Proposition 2.10; Sections 4-6 are devoted to other main results, and Section 7 to some additional remarks and open questions.

2. Closure spaces

We will consider the category **CLS** of closure spaces, where a morphism $\alpha : A \to B$ is a map α from A to B with

$$B' \in \mathcal{C}_B \Rightarrow \alpha^{-1}(B') \in \mathcal{C}_A.$$

It is easy to see that the underlying set functor $U : \mathbf{CLS} \to \mathbf{Sets}$ is *topological* in the sense of categorical topology, which then easily gives the Propositions 2.1 and 2.2 below:

Proposition 2.1. *A diagram in* CLS *of the form*

$$\begin{tikzcd} D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a pullback diagram in **CLS** *if and only if its U-image is a pullback diagram in* Sets *and* $\mathcal{C}_D = \{\pi_1^{-1}(E') \cap \pi_2^{-1}(A') \mid E' \in \mathcal{C}_E \,\&\, A' \in \mathcal{C}_A\}$. □

We will, however, present the diagram above as

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

informally identifying $E \times_B A$ with $\{(e, a) \in E \times A \mid p(e) = \alpha(a)\}$, and write

$$\mathcal{C}_{E\times_B A} = \{E' \times_B A' = \pi_1^{-1}(E') \cap \pi_2^{-1}(A') \mid E' \in \mathcal{C}_E \,\&\, A' \in \mathcal{C}_A\}.$$

We will refer to this diagram as the pullback diagram for (p, α).

Proposition 2.2. *A diagram in* $\mathbf{CLS}$ *of the form*

$$\begin{tikzcd} F \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \arrow[r, "p"] & B \end{tikzcd}$$

is a coequalizer diagram in $\mathbf{CLS}$ *if and only if its* U*-image is a coequalizer diagram in* Sets *and* $\mathcal{C}_B = \{B' \subseteq B \mid p^{-1}(B') \in \mathcal{C}_E\}$. ☐

Corollary 2.3. *A morphism* $p : E \to B$ *in* $\mathbf{CLS}$ *is a regular epimorphism if and only if* p *is a surjective map with* $\mathcal{C}_B = \{B' \subseteq B \mid p^{-1}(B') \in \mathcal{C}_E\}$. ☐

Most of what we present in the rest of this section either automatically extends what is known for topological spaces, or known itself, possibly as ‘folklore’, or is presented in some form in [6]:

Proposition 2.4. *For closure spaces* E *and* B*, and a map* $p : E \to B$*, the following conditions are equivalent:*

(a) $p : E \to B$ *is a morphism in* $\mathbf{CLS}$*;*

(b) $\overline{p^{-1}(X)} \subseteq p^{-1}(\overline{X})$ *for every* $X \subseteq B$*;*

(c) $p(\overline{p^{-1}(X)}) \subseteq \overline{X}$ *for every* $X \subseteq B$*;*

(d) $p(\overline{Y}) \subseteq \overline{p(Y)}$ *for every* $Y \subseteq E$*;*

(e) $\overline{Y} \subseteq p^{-1}(\overline{p(Y)})$ *for every* $Y \subseteq E$*.*

Proof. (a)⇒(b): Since $X \subseteq \overline{X}$, we have $p^{-1}(X) \subseteq p^{-1}(\overline{X})$, and then $\overline{p^{-1}(X)} \subseteq \overline{p^{-1}(\overline{X})}$, but $\overline{p^{-1}(\overline{X})} = p^{-1}(\overline{X})$ by (a), since $\overline{X} \in \mathcal{C}_B$.

(b)⇒(a): If $B' \in \mathcal{C}_B$, then $\overline{B'} = B'$ and (b) gives $\overline{p^{-1}(B')} \subseteq p^{-1}(B')$, making $\overline{p^{-1}(B')} = p^{-1}(B')$ and so making $p^{-1}(B') \in \mathcal{C}_E$.

(b)⇔(c) and (d)⇔(e) are obvious.

(b)⇒(e): Since $Y \subseteq p^{-1}(p(Y))$, we have that $\overline{Y} \subseteq \overline{p^{-1}(p(Y))}$, but $\overline{p^{-1}(p(Y))} \subseteq p^{-1}(\overline{p(Y)})$ by (b).

(d)⇒(c): Since $p(p^{-1}(X)) \subseteq X$, we have that $\overline{p(p^{-1}(X))} \subseteq \overline{X}$, but $p(\overline{p^{-1}(X)}) \subseteq \overline{p(p^{-1}(X))}$ by (d). ☐

Proposition 2.5. *The following conditions on a morphism* $p : E \to B$ *in* $\mathbf{CLS}$ *are equivalent:*

(a) p is closed, that is, $Y \in \mathcal{C}_E \Rightarrow p(Y) \in \mathcal{C}_B$;

(b) $p(\overline{Y}) \supseteq \overline{p(Y)}$ for every $Y \subseteq E$;

(c) $p(\overline{Y}) = \overline{p(Y)}$ for every $Y \subseteq E$.

Proof. (a)⇒(b): Since $Y \subseteq \overline{Y}$, we have $p(Y) \subseteq p(\overline{Y})$ and then $\overline{p(Y)} \subseteq \overline{p(\overline{Y})}$, but $\overline{p(\overline{Y})} = p(\overline{Y})$ by (a), since $\overline{Y} \in \mathcal{C}_E$.

(b)⇒(c) follows from the implication (a)⇒(d) of Proposition 2.4.

(c) ⇒(a): If $Y = \overline{Y}$, then $p(Y) = \overline{p(Y)}$ by (c). □

Proposition 2.6. *The following conditions on a morphism $p : E \to B$ in* **CLS** *are equivalent:*

(a) p is open, that is, $-Y \in \mathcal{C}_E \Rightarrow -p(Y) \in \mathcal{C}_B$;

(b) $\overline{X} \subseteq -p(-\overline{p^{-1}(X)})$ for every $X \subseteq B$;

(c) $\overline{p^{-1}(X)} \supseteq p^{-1}(\overline{X})$ for every $X \subseteq B$;

(d) $\overline{p^{-1}(X)} = p^{-1}(\overline{X})$ for every $X \subseteq B$.

Proof. (a)⇒(b): Since $\overline{p^{-1}(X)} \in \mathcal{C}_E$, we have $-p(-\overline{p^{-1}(X)}) \in \mathcal{C}_B$ by (a). Therefore to deduce (b) is to show that $X \subseteq -p(-\overline{p^{-1}(X)})$, but we have

$$X \subseteq -p(-\overline{p^{-1}(X)}) \Leftrightarrow p(-\overline{p^{-1}(X)}) \subseteq -X \Leftrightarrow -\overline{p^{-1}(X)} \subseteq p^{-1}(-X)$$

$$\Leftrightarrow -\overline{p^{-1}(X)} \subseteq -p^{-1}(X) \Leftrightarrow p^{-1}(X) \subseteq \overline{p^{-1}(X)}.$$

(b)⇒(a): Applying (b) to $X = -p(Y)$, we obtain the first inclusion in

$$\overline{-p(Y)} \subseteq -p(-\overline{p^{-1}(-p(Y))}) = -p(-\overline{-p^{-1}(p(Y))}) \subseteq -p(-\overline{-Y}),$$

and for $-Y \in \mathcal{C}_E$ this gives $\overline{-p(Y)} \subseteq -p(- - Y) = -p(Y)$, which means that $-p(Y) \in \mathcal{C}_B$.

(b)⇔(c): We have

$$\overline{X} \subseteq -p(-\overline{p^{-1}(X)}) \Leftrightarrow p(-\overline{p^{-1}(X)}) \subseteq -\overline{X} \Leftrightarrow -\overline{p^{-1}(X)} \subseteq p^{-1}(-\overline{X})$$

$$\Leftrightarrow -\overline{p^{-1}(X)} \subseteq -p^{-1}(\overline{X}) \Leftrightarrow \overline{p^{-1}(X)} \supseteq p^{-1}(\overline{X}).$$

(c)⇔(d) follows from the implication (a)⇒(b) of Proposition 2.4. □

For a morphism $p : E \to B$ in **CLS** and $X \subseteq B$, let us define $p_\infty(X)$ by transfinite induction as follows:

$$p_0(X) = X,\ \ p_{\lambda+1}(X) = p(\overline{p^{-1}(p_\lambda(X))}) = p_1(p_\lambda(X)),$$

$$p_\mu(X) = \bigcup_{\lambda<\mu} p_\lambda(X)\,(\text{for a limit ordinal}\ \mu),\ \ p_\infty(X) = \bigcup_{\lambda} p_\lambda(X).$$

Note that, $p_1(X) \subseteq p(E)$ and, using transfinite induction, we conclude that $p_\infty(X) \subseteq p(E)$ for every $X \subseteq B$. Furthermore, when p is surjective, we have $X \subseteq p_1(X)$, and so

$$\lambda \leqslant \mu \Rightarrow p_\lambda(X) \subseteq p_\mu(X)\,(\subseteq p_\infty(X)).$$

Proposition 2.7. *The following conditions on a morphism* $p : E \to B$ *in* **CLS** *are equivalent:*

(a) p *is a regular epimorphism;*

(b) $\overline{X} \subseteq p_\infty(X)$ *for every* $X \subseteq B$*;*

(c) $\overline{X} = p_\infty(X)$ *for every* $X \subseteq B$*.*

Proof. (a)⇒(c): Suppose p is a regular epimorphism, and so

$$\mathcal{C}_B = \{B' \subseteq B \mid p^{-1}(B') \in \mathcal{C}_E\}$$

by Corollary 2.3. Let C be the closure space whose underlying set is the same as for B and whose closure operator is p_∞, that is, it is defined by $\overline{X} = p_\infty(X)$ (all required conditions for a closure operator are obviously satisfied). We have

$$X \in \mathcal{C}_C \Leftrightarrow X = p_\infty(X) \Leftrightarrow X = p_1(X) \Leftrightarrow X = p(\overline{p^{-1}(X)}) \Leftrightarrow p(\overline{p^{-1}(X)})$$

$$\subseteq X \Leftrightarrow \overline{p^{-1}(X)} \subseteq p^{-1}(X) \Leftrightarrow \overline{p^{-1}(X)} = p^{-1}(X) \Leftrightarrow p^{-1}(X) \in \mathcal{C}_E,$$

which means that $C = B$ as closure spaces. That is, (c) holds.

(c)⇒(b) is trivial.

(b)⇒(a): Suppose $\overline{X} \subseteq p_\infty(X)$ for every $X \subseteq B$. First of all we have

$$B = \overline{B} \subseteq p_\infty(B) \subseteq p(E),$$

and so p surjective. Next, take any $X \subset B$ with $p^{-1}(X) \in \mathcal{C}_E$; we have

$$p_1(X) = p(\overline{p^{-1}(X)}) = p(p^{-1}(X)) = X,$$

and then $p_\infty(X) = X$ by transfinite induction. Hence $\overline{X} \subseteq X$. That is, $X \in \mathcal{C}_B$ whenever $p^{-1}(X) \in \mathcal{C}_E$ and p is a regular epimorphism by Corollary 2.3. $\square$

Consider again the pullback diagram for (p, α):

Proposition 2.8. *For $Z \subseteq E \times_B A$ one has $\overline{Z} = \pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)})$*

Proof. Since $\pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)}) \in \mathcal{C}_{E\times_B A}$, we only need to prove that if $Z \subseteq \pi_1^{-1}(E') \cap \pi_2^{-1}(A')$ for $E' \in \mathcal{C}_E$ and $A' \in \mathcal{C}_A$, then

$$\pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)}) \subseteq \pi_1^{-1}(E') \cap \pi_2^{-1}(A').$$

We have

$$Z \subseteq \pi_1^{-1}(E') \cap \pi_2^{-1}(A') \Rightarrow Z \subseteq \pi_1^{-1}(E') \Rightarrow \pi_1(Z) \subseteq E' \Rightarrow \overline{\pi_1(Z)} \subseteq E',$$

where the last implication follows from $E' \in \mathcal{C}_E$. That is, we can write $\overline{\pi_1(Z)} \subseteq E'$; similarly $\overline{\pi_2(Z)} \subseteq A'$. Now, for $(e, a) \in \pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)})$, we have

$$e = \pi_1(e, a) \in \overline{\pi_1(Z)} \subseteq E' \text{ and } a = \pi_2(e, a) \in \overline{\pi_2(Z)} \subseteq A',$$

and so $(e, a) \in \pi_1^{-1}(E') \cap \pi_2^{-1}(A')$, as desired. $\square$

Let S be a subset of (the underlying of) a closure space B, and $\iota : S \to B$ the inclusion map. This makes S a closure space, which we will denote by S_B, and which has

$$\mathcal{C}_{S_B} = \{S \cap B' \mid B' \in C_B\} \text{ and } \overline{U}^{S_B} = S \cap \overline{U}^B$$

for every $U \subseteq S$. From Proposition 2.8, or directly, we easily obtain

Proposition 2.9. *For a morphism $p : E \to B$ in* **CLS** *and a subset S of B, the diagram*

$$\begin{tikzcd} p^{-1}(S)_E \arrow[r, "p'"] \arrow[d, "\kappa"'] & S_B \arrow[d, "\iota"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

in which ι and κ are the inclusion maps, is a pullback diagram in **CLS**. $\square$

Proposition 2.10. *The following conditions on a morphism $p : E \to B$ in* **CLS** *are equivalent:*

(a) p is a pullback stable regular epimorphism;

(b) $\overline{X} \subseteq p(\overline{p^{-1}(X)})$ for every $X \subseteq B$;

(c) $\overline{X} = p(\overline{p^{-1}(X)})$ for every $X \subseteq B$;

(d) $p(\overline{p^{-1}(X)})$ is closed for every $X \subseteq B$

Proof. (a)⇒(b): Given $X \subseteq B$, consider the pullback diagram of Proposition 2.9 with $X \subseteq S \subseteq B$. Assuming (a), p' must be a regular epimorphism, and, in particular,

$$S \cap \overline{X}^B = \overline{X}^{S_B} = p'_\infty(X),$$

where the second equality follows from the implication (a)⇒(c) of Proposition 2.7. We take $S = X \cup -p(Y)$ with $Y = \overline{p^{-1}(X)}^E$ and calculate:

$$p'_1(X) = p'(\overline{p'^{-1}(X)}^{p^{-1}(S)_E}) = p(\overline{p^{-1}(X)}^{p^{-1}(S)_E}) = p(p^{-1}(S) \cap \overline{p^{-1}(X)}^E)$$
$$= p(p^{-1}(S) \cap Y) = p(p^{-1}(X \cup -p(Y)) \cap Y)$$
$$= p((p^{-1}(X) \cap Y) \cup (-p^{-1}(p(Y)) \cap Y)) = X,$$

where the last equality follows from $p^{-1}(X) \subseteq \overline{p^{-1}(X)}^E = Y$, $Y \subseteq p^{-1}(p(Y))$, and $p(p^{-1}(X)) = X$. Since $p'_1(X) = X$, using transfinite induction we also obtain $p'_\infty(X) = X$. This gives

$$(X \cup -p(\overline{p^{-1}(X)}^E)) \cap \overline{X}^B = S \cap \overline{X}^B = p'_\infty(X) = X,$$

which implies that $-p(\overline{p^{-1}(X)}^E) \cap \overline{X}^B \subseteq X$. Since $X = p(p^{-1}(X)) \subseteq p(\overline{p^{-1}(X)}^E)$, it follows that $\overline{X}^B \subseteq p(\overline{p^{-1}(X)}^E)$, as desired.

(b)⇔(c) follows from the implication (a)⇒(c) of Proposition 2.4, and (c)⇔(d) follows from Proposition 2.7.

(c)⇒(a): Suppose (c) holds. We have to prove that, for every pullback diagram as in Proposition 2.8, π_2 is a regular epimorphism. Thanks to the implication (b)⇒(a) of Proposition 2.7, it suffices to prove that

$$\overline{U} \subseteq \pi_2(\overline{\pi_2^{-1}(U)})$$

for every $U \subseteq A$. We have:

$$\overline{U} \subseteq \alpha^{-1}(\overline{\alpha(U)}) \cap \overline{U} \ \ \text{(since } \alpha(\overline{U}) \subseteq \overline{\alpha(U)}\text{) gives } \overline{U} \subseteq \alpha^{-1}(\overline{\alpha(U)})\text{)}$$
$$= \alpha^{-1}(p(\overline{p^{-1}(\alpha(U))})) \cap \overline{U} \ \ \text{(by (c))}$$
$$= \pi_2(\pi_1^{-1}(\overline{\pi_1(\pi_2^{-1}(U))})) \cap \overline{U} \ \ \text{(Beck–Chevalley Condition used twice)}$$
$$= \pi_2(\pi_1^{-1}(\overline{\pi_1(\pi_2^{-1}(U))}) \cap \pi_2^{-1}(\overline{U})) \ \ \text{(another Beck–Chevalley Condition)}$$
$$= \pi_2(\pi_1^{-1}(\overline{\pi_1(\pi_2^{-1}(U))}) \cap \pi_2^{-1}(\overline{\pi_2(\pi_2^{-1}(U))})) = \pi_2(\overline{\pi_2^{-1}(U)}),$$

as desired. □

3. General remarks on descent

In this section **C** denotes a category with pullbacks and coequalizers of equivalence relations. All pullback projections will denoted by π's with suitable indices.

We will list notions and results of general descent theory in the form convenient for our purposes, not repeating any motivations and further explanations that can be found in [5] or in [4]; we will also use a particular result from [7].

Definition 3.1. Let $p : E \to B$ be a morphism in **C**. Then:

(a) A *descent data* for p is a triple (C, γ, ξ) as in the diagram

$$\begin{tikzcd}
E \times_B (E \times_B C) \arrow[r, "E \times_B \xi"] \arrow[d, "E \times_B \pi_2"'] & E \times_B C \arrow[d, "\xi"'] & C \arrow[l, "\langle \gamma, 1_C \rangle"'] \arrow[dl, equal] \\
E \times_B C \arrow[r, "\xi"] \arrow[d, "\pi_1"'] & C \arrow[dl, "\gamma"] & \\
E & &
\end{tikzcd}$$

(in obvious notation), which is required to commute. The category of all such triples will be denoted by $\mathrm{Des}(p)$.

(b) The functor $K^p : (\mathbf{C} \downarrow B) \to \mathrm{Des}(p)$, defined by

$$K^p(A,\alpha) = (E \times_B (E \times_B A) \xrightarrow{E\times_B\pi_2} E \times_B A \xrightarrow{\pi_1} E)$$

is called the *comparison functor* (for p).

(c) The morphism p is said to be a *descent morphism* if the functor K^p is fully faithful.

(d) The morphism p is said to be an *effective* E*-descent morphism* if the functor K^p is a category equivalence.

Remark 3.2. Each of the following statements is either well known or immediately follows from well-known facts:

(a) If (C, γ, ξ) is a descent data for $p : E \to B$, then

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "E\times_B\gamma"'] & C \arrow[d, "\gamma"] \\
E \times_B E \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & E
\end{tikzcd}$$

is a discrete fibration of equivalence relations. Moreover, sending (C, γ, ξ) to this discrete fibration determines a category equivalence

$$\mathrm{Des}(p) \to \mathrm{DFib}(\mathrm{Eq}(p)),$$

where $\mathrm{DFib}(\mathrm{Eq}(p))$ is the category of discrete fibrations of equivalence relations whose codomain is

$$\mathrm{Eq}(p) = (E \times_B E \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} E).$$

(b) Suppose p is a regular epimorphism, and so we can assume that B (equipped with p) is the coequalizer of the bottom equivalence relation in (a). Then sending (C, γ, ξ) to the morphism of the coequalizers of equivalence relations in (a) determines a left adjoint L^p of K^p.

(c) As follows from (a) and (b), p is an effective descent morphism if and only if it is a descent morphism and the functor L^p reflects isomorphisms, or, equivalently, the coequalizer functor

$$\mathrm{DFib}(\mathrm{Eq}(p)) \to (\mathbf{C} \downarrow B)$$

does so.

(d) A morphism in $\mathbf{C}$ is a descent morphism if and only if it is a pullback stable regular epimorphism.

(e) As easily follows from previous observations, every descent morphism in $\mathbf{C}$ is an effective descent morphism if and only if for every descent morphism $p : E \to B$ and every diagram of the form

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "E\times_B\gamma"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & A \arrow[d, "\alpha"] \\
E \times_B E \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

where (C, γ, ξ) is a descent data for p, the top row is a coequalizer diagram, the right-hand square commutes, and α is an isomorphism, γ also is an isomorphism. More generally, if $\mathbf{D}$ is a pullback stable class of morphisms containing the class of descent morphisms and satisfying the condition above (with p is in $\mathbf{D}$), then $\mathbf{D}$ is contained in the class of effective descent morphisms.

(f) A regular epimorphism p in $\mathbf{CLS}$ is an effective descent morphism if and only if, for each descent data (C, γ, ξ) for p, the coequalizer of

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\pi_2"'] & C
\end{tikzcd}$$

is a pullback stable regular epimorphism. This follows from the observation in [7] made immediately after Corollary 2.8 there.

4. Descent for closure spaces

In this section we go back to the category **CLS** of closure spaces and $p : E \to B$ will denote a fixed morphism there, which is a surjective map. We will also use a closure space E', which has the same underlying set as E, and, for $X \subseteq B$, $Y \subseteq E$, and $Z \subseteq E \times_B E'$, write

$$\overline{X} = \overline{X}^B,\ \overline{Y} = \overline{Y}^E,\ \overline{Y}' = \overline{Y}^{E'},\ \text{and}\ \overline{Z} = \overline{Z}^{E\times_B E'}.$$

Lemma 4.1. *Suppose the identity map $1_E : E' \to E$ is a morphism in* **CLS**, *that is, $\overline{Y}' \subseteq \overline{Y}$ for all $Y \subseteq E$. Then the following conditions are equivalent:*

- *(a) there exists a descent data for p of the form $(E', 1_E, \xi)$;*
- *(b) there exists a unique descent data for p of the form $(E', 1_E, \xi)$;*
- *(c) the triple $(E', 1_E, \pi_1)$, where $\pi_1 : E \times_B E' \to E'$ is defined as the first projection, that is, by $\pi_1(e, e') = e$, is a descent data for p;*
- *(d) the first projection $\pi_1 : E \times_B E' \to E'$ is a morphism in* **CLS**;
- *(e) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}'))\subseteq \overline{Y}'$ for all $Y \subseteq E$;*
- *(f) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) = \overline{Y}'$ for all $Y \subseteq E$;*
- *(g) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) \subseteq Y$ for all $Y \in \mathcal{C}_{E'}$;*
- *(h) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) = Y$ for all $Y \in \mathcal{C}_{E'}$.*

Proof. The implications (a)⇔(b)⇒(c)⇒(d) follow from the commutativity of the bottom triangle of the diagram in 3.2(a), where γ becomes the map $1_E : E' \to E$ in the present case. The implication (d)⇒(c) can be checked with a straightforward calculation and the implication (c)⇒(a) is trivial. Hence conditions (a)-(d) are all equivalent to each other.

(d)⇔(e): As follows from the equivalence (a)⇔(c) of Proposition 2.4, condition (d) holds if and only if

$$\pi_1(\overline{\pi_1^{-1}(Y)}) \subseteq \overline{Y}'$$

for all $Y \subseteq E$. Using Proposition 2.8, we obtain:

$$\pi_1(\overline{\pi_1^{-1}(Y)}) = \pi_1(\overline{Y \times_B E})$$

$$= \pi_1(\pi_1^{-1}(\overline{\pi_1(Y \times_B E)}) \cap \pi_2^{-1}(\overline{\pi_2(Y \times_B E)}'))$$

$$= \overline{\pi_1(Y \times_B E)} \cap \pi_1(\pi_2^{-1}(\overline{\pi_2(Y \times_B E)}')) = \overline{Y} \cap \pi_1(\pi_2^{-1}(\overline{p^{-1}(p(Y))}'))$$

$$= \overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')),$$

and so (d) is indeed equivalent to (e).

Since $\overline{Y}' \subseteq \overline{Y}$ and $\overline{Y}' \subseteq p^{-1}(p(\overline{p^{-1}(p(Y))}'))$, we have (e)⇔(f); similarly, we have (g)⇔(h). (e)⇔(g) is also straightforward. □

Lemma 4.2. *Suppose the equivalent conditions of Lemma 4.1 are satisfied and let us write p' for p considered as a morphism from E' to B. If both p and p' are regular epimorphisms, then, for every $Y \in \mathcal{C}_{E'} \setminus \mathcal{C}_E$, there exists $Y^* \in \mathcal{C}_{E'} \setminus \mathcal{C}_E$ with $Y \subset Y^*$. In particular, if $\mathcal{C}_{E'} \neq \mathcal{C}_E$, then E is infinite.*

Proof. For $Y \in \mathcal{C}_{E'} \setminus \mathcal{C}_E$, we have $Y \subset p^{-1}(p(Y))$. Indeed, since p and p' are regular epimorphisms, the equality $Y = p^{-1}(p(Y))$ would imply

$$Y \in \mathcal{C}_{E'} \Leftrightarrow p(Y) \in \mathcal{C}_B \Leftrightarrow Y \in \mathcal{C}_E$$

(by Corollary 2.3), which is a contradiction.

Let us take

$$Y^* = \overline{p^{-1}(p(Y))}'.$$

We have $Y \subset Y^*$ and $Y^* \in \mathcal{C}_{E'}$. Therefore it remains to show that Y^* does not belong to $\mathcal{C}_E$. Suppose it does. Then, since it contains Y as a subset, we have $\overline{Y} \subseteq Y^*$. This gives

$$\overline{Y} = \overline{Y} \cap Y^* = \overline{Y} \cap \overline{p^{-1}(p(Y))}' \subseteq \overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) = Y$$

(the last equality here is condition (h) of Lemma 4.1), which is a contradiction since Y does not belong to $\mathcal{C}_E$. □

Let **FCLS** be the category of finite closure spaces, that is, the full subcategory of **FCLS** with objects all closure spaces whose underlying sets are finite. From Remark 3.3(e) and Lemma 4.2 we obtain:

Theorem 4.3. *Every descent morphism in the category* **FCLS** *is an effective descent morphism.* □

5. Preorders as closure spaces

There are full inclusions

$$\mathbf{Preord} \to \mathbf{Top} \to \mathbf{CLS},$$

where $\mathbf{Preord}$ is the category of preorders (=preordered sets) and $\mathbf{Top}$ is the category of topological spaces. Considering a preorder B as either a topological space or a closure space, for any $X \subseteq B$, we have

$$\overline{X} = \uparrow X = \{b \in B \mid \exists_{x \in X}\, x \leqslant b\}.$$

As mentioned in Remark 2.4(b) of [1], not every descent morphism in $\mathbf{Preord}$ is a descent morphism in $\mathbf{Top}$; nevertheless we have:

Proposition 5.1. *A morphism in* $\mathbf{Preord}$ *is a descent morphism in* $\mathbf{Preord}$ *if and only if it is a descent morphism in* $\mathbf{CLS}$.

Proof. Let $p : E \to B$ be a morphism in $\mathbf{Preord}$. As shown in [2], p is a descent morphism in $\mathbf{Preord}$ if and only if for all $b \leqslant b'$ in B there exist $e \leqslant e'$ in E with $p(e) = b$ and $p(e') = b'$. This, in turn, is easily equivalent to

$$p(\overline{p^{-1}(X)}) = p(\uparrow p^{-1}(X)) \supseteq \uparrow X = \overline{X},$$

and it remains to apply Proposition 2.10 and Remark 3.2(d). □

On the other hand, the result similar to Theorem 4.3 does not hold in $\mathbf{Preord}$, and not even in the category $\mathbf{FPreord}$ of finite preorders [2]. In order to clarify the phenomenon behind this, consider the following example, the simplest one in a sense:

Let $p : E \to B$ be the morphism in $\mathbf{FPreord}$, and $\alpha : A \to B$ be the morphism in the category $\mathbf{FRR}$ of finite reflexive relations (=sets equipped with a reflexive relation) defined as follows:

- $B = \{b_1, b_2, b_3\}$ is the ordered set with $b_1 < b_2 < b_3$.
- $E = \{e_1, e_{2-}, e_{2+}, e_3\}$ is the ordered set with $e_1 < e_{2-}$, $e_{2+} < e_3$, $e_1 < e_3$, and no other strict inequalities.
- $p(e_1) = b_1$, $p(e_{2-}) = b_2 = p(e_{2+})$, and $p(e_3) = b_3$.

- $A = B$ but with the pair (b_1, b_3) removed from the relation.
- α is the identity map of B considered as a morphism from A to B.

The pullback $E' = E \times_B A$ of p and α can be identified with the ordered set $E = \{e_1, e_{2-}, e_{2+}, e_3\}$ with $e_1 < e_{2-}$, $e_{2+} < e_3$, and no other strict inequalities. And after that the pullback $E \times_B E'$ can be presented as the diagram

$$\begin{tikzcd}
 & & & (e_1, e_1) \arrow[dlll] & & & \\
(e_{2-}, e_{2-}) & & (e_{2-}, e_{2+}) & & (e_{2+}, e_{2-}) & & (e_{2+}, e_{2+}) \arrow[dlll] \\
 & & & (e_3, e_3) & & &
\end{tikzcd}$$

whose vertexes are its elements and whose arrows represent strict inequalities. We observe:

(a) Although A is not a preorder, E' is. This tells us that $(E', 1_E, \pi_1)$ is a descent data for p in $\mathbf{FPreord}$. Comparing it with $(E, 1_E, \pi_1)$ is a simple way to show that p is not an effective descent morphism in $\mathbf{FPreord}$.

(b) The set $Y = \{e_1, e_{2,-}\}$ is closed in E' and its inverse image

$$Z = \pi_1^{-1}(Y) = \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2-}, e_{2+})\}$$

is closed of course in the pullback $E \times_B E'$ displayed above.

(c) However, if we define $E \times_B E'$ as the pullback in $\mathbf{FCLS}$, then

$$\overline{Z} = \pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)}')$$

$$= \pi_1^{-1}(\overline{\{e_1, e_{2-}\}}) \cap \pi_2^{-1}(\overline{\{e_1, e_{2-}, e_{2+}\}}')$$

$$= \pi_1^{-1}(\{e_1, e_{2-}, e_3\}) \cap \pi_2^{-1}(\{e_1, e_{2-}, e_{2+}, e_3\}) = \pi_1^{-1}(\{e_1, e_{2-}, e_3\})$$

$$= \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2-}, e_{2+}), (e_3, e_3)\} \neq Z,$$

and so Z will not be closed anymore.

(d) As follows from (c), for the pullback $E \times_B E'$ defined as in **FCLS**, the map $\pi_1 : E \times_B E' \to E'$ is not a morphism in **FCLS**. Therefore there is no 'bad' descent data $(E', 1_E, \pi_1)$ in **FCLS**, to prevent p from being an effective descent morphism.

Of course this is only an example of one preorder argument that does not hold for closure spaces and it cannot replace the proof of Theorem 4.3, but it shows a crucial difference between the descent stories of preorders and of closure spaces.

Furthermore, in the pullback $E \times_B E'$ defined as in **FCLS**, putting $Z = U \cup V$ with $U = \{(e_1, e_1), (e_{2-}, e_{2-})\}$ and $V = \{(e_{2-}, e_{2+})\}$, we calculate

$$\overline{U} = \pi_1^{-1}(\overline{\pi_1(U)}) \cap \pi_2^{-1}(\overline{\pi_2(U)}') = \pi_1^{-1}(\overline{\{e_1, e_{2-}\}}) \cap \pi_2^{-1}(\overline{\{e_1, e_{2-}\}}')$$

$$= \pi_1^{-1}(\{e_1, e_{2-}, e_3\}) \cap \pi_2^{-1}(\{e_1, e_{2-}\})$$

$$= \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2-}, e_{2+}), (e_3, e_3)\} \cap \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2+}, e_{2-})\}$$

$$= \{(e_1, e_1), (e_{2-}, e_{2-})\} = U;$$

$$\overline{V} = \pi_1^{-1}(\overline{\pi_1(V)}) \cap \pi_2^{-1}(\overline{\pi_2(V)}')$$

$$= \pi_1^{-1}(\overline{\pi_1(\{(e_{2-}, e_{2+})\})}) \cap \pi_2^{-1}(\overline{\pi_2(\{(e_{2-}, e_{2+})\})}')$$

$$= \pi_1^{-1}(\overline{\{e_{2-}\}}) \cap \pi_2^{-1}(\overline{\{e_{2+}\}}') = \pi_1^{-1}(\{e_{2-}\}) \cap \pi_2^{-1}(\{e_{2+}, e_3\})$$

$$= \{(e_{2-}, e_{2-}), (e_{2-}, e_{2+})\} \cap \{(e_{2-}, e_{2+}), (e_{2+}, e_{2+}), (e_3, e_3)\}$$

$$= \{(e_{2-}, e_{2+})\} = V.$$

That is,

$$\overline{U} = U \text{ and } \overline{V} = V, \text{ while } \overline{U \cup V} \neq U \cup V$$

in $E \times_B E'$ defined as the pullback in **FCLS**, which is what could not happen in a preorder (since it could not happen in a topological space in general).

6. Surjective closed and open maps are effective descent morphisms

Returning to the context of Section 3 and using a result of [7], we easily obtain:

Theorem 6.1. *Let* $U : \mathbf{C} \to \mathsf{Sets}$ *be a faithful functor between categories with pullbacks and coequalizers of equivalence relations that preserves these constructions, and let* $\mathcal{P}$ *be a class of regular epimorphisms in* $\mathbf{C}$ *satisfying the following conditions:*

(a) $\mathcal{P}$ *is pullback stable;*

(b) if

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

is a coequalizer diagram in $\mathbf{C}$ *whose* U*-image is exact, that is, it is a coequalizer diagram that is also a kernel pair diagram, then* $f, g \in \mathcal{P} \Rightarrow h \in \mathcal{P}$.

Then $\mathcal{P}$ *is contained in the class of effective descent morphisms in* $\mathbf{C}$.

Proof. As follows from (a) and the fact that $\mathcal{P}$ is a class of regular epimorphisms in $\mathbf{C}$, $\mathcal{P}$ is a class of pullback stable regular epimorphisms in $\mathbf{C}$. Note also that, for every descent data (C, γ, ξ) over a given $p : E \to B$ in $\mathcal{P}$, we have

- since $U(p)$ being a regular epimorphism is an effective descent morphism in Sets, the U-image of the coequalizer diagram

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\pi_2"'] & C \arrow[r, "q"] & A \end{tikzcd}$$

  is exact;

- as follows from (a), the morphisms ξ and π_2 in that diagram belong to $\mathcal{P}$.

After that all we need is to apply the categorical counterpart of Corollary 2.8 in [7], as the next sentence (after Corollary 2.8) in [7] shows. □

By a *closed map* we mean a morphism **CLS** that is closed, or, equivalently, satisfies the equivalent conditions of Proposition 2.5. Similarly, by an *open map* we mean a morphism **CLS** that is open, or, equivalently, satisfies the equivalent conditions of Proposition 2.6. In the rest of this section we will show that Theorem 6.1 applies to the classes of surjective closed maps and of surjective open maps in **CLS**.

Proposition 6.2. *The class of closed maps is pullback stable. In particular, so is the class of surjective closed maps.*

Proof. Consider the pullback for (p, α) with closed p. We have to prove that the map $\pi_2 : E \times_B A \to A$ is closed. However, this follows from Proposition 2.1 and the fact that we have

$$\pi_2(\pi_1^{-1}(E') \cap \pi_2^{-1}(A')) = \alpha^{-1}(p(E')) \cap A'$$

for all $E' \subseteq E$ and $A' \subseteq A$. Indeed, if E' is closed in E and A' is closed in A, then $\alpha^{-1}(p(E')) \cap A'$ is closed in A since p is a closed map. □

Proposition 6.3. *The class of surjective open maps is pullback stable.*

Proof. Consider the pullback for (p, α) with open p. We have to prove that the map $\pi_2 : E \times_B A \to A$ is open. For $U \subseteq A$, we have

$$\overline{\pi_2^{-1}(U)} = \pi_1^{-1}(\overline{\pi_1(\pi_2^{-1}(U))}) \cap \pi_2^{-1}(\overline{\pi_2(\pi_2^{-1}(U))})$$

$$= \pi_1^{-1}(\overline{p^{-1}(\alpha(U))}) \cap \pi_2^{-1}(\overline{U}) = \pi_1^{-1}(p^{-1}(\overline{\alpha(U)})) \cap \pi_2^{-1}(\overline{U})$$

$$= \pi_2^{-1}(\alpha^{-1}(\overline{\alpha(U)})) \cap \pi_2^{-1}(\overline{U}) = \pi_2^{-1}(\alpha^{-1}(\overline{\alpha(U)}) \cap \overline{U})$$

and since

$$\overline{U} \subseteq \alpha^{-1}(\alpha(\overline{U})) \subseteq \alpha^{-1}(\overline{\alpha(U)}),$$

this gives $\overline{\pi_2^{-1}(U)} = \pi_2^{-1}(\overline{U})$. Therefore π_2 is open by Proposition 2.6. □

Proposition 6.4. *The classes of surjective closed maps and of surjective open maps both satisfy condition (b) of Theorem 6.1 for U being the forgetful functor* **CLS** $\to$ Sets.

Proof. Consider the diagram of 6.1(b). At the level of underlying sets, the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ Y \arrow[r, "h"'] & Z \end{tikzcd}$$

is a pullback, and so, for each subset Y' of Y, we have $h^{-1}(h(Y')) = f(g^{-1}(Y'))$. Since h is a regular epimorphism, for closed f this gives:

$$Y' \text{ is closed} \Rightarrow g^{-1}(Y') \text{ is closed} \Rightarrow f(g^{-1}(Y')) \text{ is closed}$$
$$h^{-1}(h(Y')) \text{ is closed} \Rightarrow h(Y') \text{ is closed,}$$

and, similarly, for open f:

$$Y' \text{ is open} \Rightarrow g^{-1}(Y') \text{ is open} \Rightarrow f(g^{-1}(Y')) \text{ is open}$$
$$h^{-1}(h(Y')) \text{ is open} \Rightarrow h(Y') \text{ is open,}$$

as desired. □

From Theorem 6.1 and these three propositions, as promised, we obtain:

Theorem 6.5. *Every surjective closed map and every surjective open map in* **CLS** *is an effective descent morphism.* □

7. Final remarks

7.1. For a morphism $p : E \to B$ in **CLS**, which is surjective, let us call a subset Y of E *saturated* if it is of the form $Y = p^{-1}(X)$ for some $X \subseteq B$, or, equivalenly, if $Y = p^{-1}(p(Y))$. Consider the following conditions on p:

(a) $p(Y)$ is closed whenever Y is saturated and closed, or, equivalently (by Corollary 2.3), p is a regular epimorphism in **CLS**;

(b) $p(Y)$ is closed whenever Y is the closure of a saturated subset, or, equivalently (by Proposition 2.10), p is a pullback stable regular epimorphism (=descent morphism) in **CLS**;

(c) p is an effective descent morphism in **CLS**.

(d) $p(Y)$ is closed whenever so is Y.

We have (d)⇒(c) (Theorem 6.5) and trivial implications (c)⇒(b)⇒(a). It seems that none of the opposite implications holds. In fact it is very easy to construct counterexamples for (a)⇒(b) and, using Theorem 6.5 for (c)⇒(d), but we have no counterexamples for (b)⇒(c).

7.2. For a monad T on the category of sets, consider the forgetful functor

$$U : \mathsf{Alg}(T) \to \mathbf{CLS}.$$

The category $\mathsf{Alg}(T)$ is Barr exact and, for a morphism p in it, we have

$$p \text{ in an effective descent morphism} \Leftrightarrow p \text{ is a surjective map},$$

and the functor U sends all morphisms of $\mathsf{Alg}(T)$ to closed maps; in particular it preserves regular epimorphisms, descent morphisms, and effective descent morphisms. However, it obviously does not preserve kernel pairs of non-injective maps.

7.3. Let $\mathbf{E}$ be one of the following three classes of morphisms in $\mathbf{CLS}$: (i) of closed maps; (ii) of surjective closed maps; (iii) of surjective open maps. As follows from Propositions 6.2 and 6.3 (and simple arguments used in the proof of Proposition 6.4), every effective descent morphism in $\mathbf{CLS}$ is also an effective $\mathbf{E}$-descent morphism. And it is obvious that every descent morphism in $\mathbf{CLS}$ is also an $\mathbf{E}$-descent morphism. However, none of these assertions is true for the class of (all) open maps. Indeed, consider the pullback diagram

$$\begin{tikzcd}
\{-1,1\} \arrow[r, "q"] \arrow[d, "\beta"'] & \{1\} \arrow[d, "\alpha"] \\
\{-2,-1,1,2\} \arrow[r, "p"'] & \{1,2\}
\end{tikzcd}$$

in which:

- $\{-2,-1,1,2\}$ has five closed subsets; apart from itself and the empty set they are $\{-2,2\}$, $\{1,2\}$, and $\{2\}$.

- $\{1,2\}$ has three closed subsets; apart from itself and the empty set it is just the set $\{2\}$.

- p is defined by $p(k) = |k|$.

- α and β are the inclusion maps, q is induced by p, and the closure space structures on the top are induced by the bottom ones; that is,

  $$\mathcal{C}_{\{1\}} = \{\emptyset, \{1\}\}, \;\; \mathcal{C}_{\{-1,1\}} = \{\emptyset, \{1\}, \{-1,1\}\}$$

  (this makes $\{-1,1\}$ isomorphic to $\{1,2\}$, but that is not relevant for our purposes).

It is easy to check that p and α are open maps; furthermore, since p is surjective, it is an effective descent morphism. On the other hand, β is not open since $\{-1\}$ is open in $\{-1,1\}$ but not in $\{-2,-1,1,2\}$, and so the pullback functor along p is not even well defined for the class of all open maps.

In spite of all this, a complete characterization of effective $\mathbf{E}$-descent morphisms remains an open question for $\mathbf{E}$ being any of the four classes of morphisms that appear in this subsection. Of course in the 'forth case', that is, when $\mathbf{E}$ is the class of open maps, one should suitably reformulate the problem first characterizing those $p : E \to B$ in $\mathbf{CLS}$ for which the above-mentioned pullback functor *is* well defined.

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CARTESIAN EXPONENTIATION AND MONADICITY

Emily RIEHL and Dominic VERITY

Résumé. Un résultat important de la théorie des quasi-catégories dû à Lurie est que les fibrations cocartésiennes sont *exponentiables*, dans le sens où le produit fibré le long d'une fibration cocartésienne admet un adjoint à droite de Quillen qui préserve de plus les fibrations cartésiennes; il en est de même pour le cas où le rôle des fibrations cartésiennes et cocartésiennes est interchangé. Pour expliquer ce résultat classique, on prouve que le produit fibré le long d'une fibration cocartésienne entre quasi-catégories est la colimite oplax de sa "rigidification," un diagramme homotopiquement cohérent à valeurs dans les quasi-catégories; on retrouve ainsi un résultat déjà observé par Gepner, Haugseng, et Nikolaus. Comme application de l'opération d'exponentiation d'une fibration cartésienne par une fibration cocartésienne, on utilise le lemme de Yoneda pour construire des adjoints à gauche et à droite du foncteur oubli qui envoie une fibration cartésienne au-dessus de B vers sa famille de fibres indéxée par $\mathrm{ob}\,\mathsf{B}$, et on prouve que ce foncteur oubli est monadique et comonadique. Ce résultat de monadicité est ensuite appliqué pour construire la réflexion d'une fibration cartésienne en une fibration cartésienne *groupoïdale*, dont les fibres sont des complexes de Kan plutôt que des quasi-catégories.

Abstract. An important result in quasi-category theory due to Lurie is that the cocartesian fibrations are *exponentiable*, in the sense that pullback along a cocartesian fibration admits a right Quillen right adjoint that moreover preserves cartesian fibrations; the same is true with the cartesian and cocartesian fibrations interchanged. To explicate this classical result, we prove that the pullback along a cocartesian fibration between quasi-categories forms the oplax colimit of its "straightening," a homotopy coherent diagram valued in quasi-categories, recovering a result first observed by Gepner, Haugseng, and Nikolaus. As an application of the exponentiation operation of a cartesian fibration by a cocartesian one, we use the Yoneda lemma to construct left and right adjoints to the forgetful functor that carries a cartesian fibration over B to its $\mathrm{ob}\,\mathsf{B}$-indexed family of fibers, and prove that this forgetful functor is monadic and comonadic. This monadicity is then applied to construct the reflection of a cartesian fibration into a *groupoidal* cartesian fibration, whose fibers are Kan complexes rather than quasi-categories.

Keywords. infinity category, cartesian fibration, oplax colimit, monadic adjunction.

Mathematics Subject Classification (2020). Primary 18N60, 55U10, 55U35; Secondary 18D20, 18N45, 55U40

1. Introduction

Famously the category Cat of small categories is not a topos because, among other things, it fails to be locally cartesian closed. A finitely complete category $\mathcal{E}$ is *locally cartesian closed* just when each slice category $\mathcal{E}_{/B}$ is cartesian closed, or equivalently, which the pullback functor associated to any morphism $f\colon A \to B$ admits a right adjoint (as well as a left adjoint given by composition with f):

$$\begin{tikzcd}
\mathcal{E}_{/B} \arrow[r, "f^*" description] \arrow[r, phantom, bend left=25, "\perp"] \arrow[r, phantom, bend right=25, "\perp"] & \mathcal{E}_{/A} \arrow[l, bend right=50, "\Sigma_f"'] \arrow[l, bend left=50, "\Pi_f"]
\end{tikzcd}$$

In the case $\mathcal{E} = \underline{\text{Cat}}$, those functors f for which the pullback functor f^* does admit a right adjoint Π_f are called *exponentiable* and have been characterized by Conduché [4]. Famously,

(i) All cartesian and cocartesian fibrations $p\colon E \to B$ of 1-categories are exponentiable.

(ii) If $p\colon E \to B$ is a cocartesian fibration and $q\colon F \to E$ is a cartesian fibration then the pushforward $\Pi_p(q\colon F \to E)$ is also a cartesian fibration, and the dual result holds when the cartesian and cocartesian fibrations are interchanged.

In [13] Lurie established ∞-categorical analogues of these results for quasi-categories.[1] Subsequent authors, for instance Barwick and Shah [2], have stressed the importance of these results to the theory and practice of ∞-categories, and we have further applications in mind. In [25, §12.3], we use this result to prove that *modules* between quasi-categories admit all right and left extensions. It follows that the question of existence of pointwise right and left Kan extensions can be reduced to the existence of certain limits and colimits. For those results, it is useful to have a somewhat more refined version of these results than is easily found in the literature—see especially Theorem 4.2.9 and Corollary 4.2.10—which is the motivation for the present exposition

A *cocartesian fibration* of quasi-categories is an isofibration[2] $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ whose fibers depend covariantly functorially on B. In the simplest non-trivial case, when $\mathsf{B} = \Delta^1$, the data is given by a pair of quasi-categories E_0 and E_1 together with a functor $\mathsf{E}_0 \to \mathsf{E}_1$. In general, the *comprehension construction* of [21] "straightens" $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ into a simplicial functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$ that sends each vertex $b \in \mathsf{B}$ to the fiber E_b. The domain category appearing here is

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[1] A general characterization of the exponentiable functors between quasi-categories, while not the focus of our interest here, can be found in [13, §B.3] or [1].

[2] In the Joyal model structure on simplicial sets, we refer to the fibrations between fibrant objects (the quasi-categories) as *isofibrations* because they have a lifting property for isomorphisms analogous to that for the isofibrations in classical category theory.

the *homotopy coherent realization* of the quasi-category B, a cofibrant simplicial category[3] that indexes B-shaped homotopy coherent diagrams. At the level of objects and 1-arrows $f\colon a \to b$ in B, the comprehension construction is defined by lifting the 1-arrow f to a *p-cocartesian 1-arrow* with codomain E:

$$\begin{tikzcd}
\mathsf{E}_a \arrow[rrr, bend left=20, "\ell^{\mathsf{E}}_a", ""{name=U, below}] \arrow[dr, dotted, hook, "\mathsf{E}_f"'] \arrow[rrr, dotted, bend right=10] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads, "p_a"'] & & & \mathsf{E} \arrow[dd, two heads, "p"] \\
& \mathsf{E}_b \arrow[urr, "\ell^{\mathsf{E}}_b"', ""{name=D, above}] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads, "p_b"'] & & \\
1 \arrow[rrr, bend left=20, "a", ""{name=A, below}] \arrow[dr, equal] & & & \mathsf{B} \\
& 1 \arrow[urr, bend right=10, "b"', ""{name=C, above}] & &
\arrow[Rightarrow, from=U, to=D, "\ell^{\mathsf{E}}_f"]
\arrow[Rightarrow, from=A, to=C, "f"]
\end{tikzcd}$$

Together, this data defines a lax cocone ℓ^{E} under the comprehension functor c_p with nadir E the data of which is given by a functor $\ell^{\mathsf{E}}\colon \mathfrak{C}[\mathsf{B} \star \Delta^0] \to \mathcal{Q}\mathit{Cat}$ that restricts along $\mathfrak{C}\mathsf{B} \hookrightarrow \mathfrak{C}[\mathsf{B} \star \Delta^0]$ to c_p. In fact, ℓ^{E} is a colimit cocone:

Corollary 3.2.8. *The domain of a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is equivalent to the oplax colimit of the associated comprehension functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$, with colimit cocone:*

$$\begin{tikzcd}
& \mathbb{1} + \mathfrak{C}\mathsf{B} \arrow[dl, hook'] \arrow[dr, "{\langle \mathsf{E}, c_p \rangle}"] & \\
\mathfrak{C}[\mathsf{B} \star \Delta^0] \arrow[rr, "\ell^{\mathsf{E}}"'] & & \mathcal{Q}\mathit{Cat}
\end{tikzcd}$$

In particular the domain E of the cocartesian fibration p can be recovered up to equivalence as the oplax colimit of a the comprehension functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$. Gepner, Haugseng, and Nikolaus, who obtain a similar result to Corollary 3.2.8 as one of the main theorems of [8], interpret this result as a proof that "Lurie's unstraightening functor is a model for the ∞-categorical analogue of the Grothendieck construction."[4] Their methodology is quite different from ours, constructing oplax colimits directly at the quasi-categorical level, whereas our comprehension construction enables us to work at the level of simplicial categories and functors. The comprehension functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$ can be used to define a "straightening" of the pullback of p along any generalized element $b\colon X \to \mathsf{B}$, even in the case where X is not a quasi-category simply by restricting the comprehension functor (and its lax cocone) along b. We derive Corollary 3.2.8 as a special case of our first main theorem, which proves that the fiber E_b is equivalent to the oplax colimit of this straightened diagram.

---

[3] The simplicial categories that are cofibrant in the Bergner model structure are precisely the *simplicial computads* that are freely generated by their non-degenerate "atomic" n-arrows for each $n \geq 0$, admitting no non-trivial factorizations; see Definition 2.1.10.

[4] Unfortunately, the assignment of the terms "oplax colimit" and "lax colimit" given in [8, 2.8] is opposite to the one used here. The standard convention in 2-category theory is that the 2-cell component of an oplax natural transformation is parallel to its 1-cell components, while these 2-cells are reversed in a lax natural transformation. A lax cocone is then a lax natural transformation whose codomain is a constant diagram. Confusingly, due to the principle that a W-weighted colimit in an enriched category coincides with a W-weighted limit in the opposite category, oplax colimits represent lax cocones under a diagram.

Theorem 3.1.3. *For any cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ and any $b\colon X \to \mathsf{B}$, the comprehension cocone induces a canonical map over E from the oplax colimit of the diagram*

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{QC}\!at$$

to the fiber

$$\begin{tikzcd}
E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\
X \arrow[r, "b"'] & \mathsf{B}
\end{tikzcd}$$

and this map is a natural weak equivalence in the Joyal model structure.

The canonical natural transformation of Theorem 3.1.3 defines a natural Joyal equivalence relating the pullback functor p^* to a functor $\tilde{p}^*$ defined by forming oplax colimits of restrictions of the comprehension cocone:

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[r, bend left, "\tilde{p}^*"{name=U}] \arrow[r, bend right, "p^*"'{name=D}] & \underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[from=U, to=D, phantom, "\Downarrow\gamma"]
\end{tikzcd}$$

Both functors p^* and $\tilde{p}^*$ are left Quillen with respect to the sliced Joyal model structures, admitting right Quillen adjoints:

Proposition 4.2.5. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration, the adjunctions*

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "p^*"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "\tilde{p}^*"']
\end{tikzcd}$$

are Quillen with respect to the sliced Joyal model structures.

By taking mates, there is a canonical natural transformation $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$ whose component at any isofibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ is an equivalence. In this way we obtain an alternate model $\tilde{\Pi}_p$ for the pushforward functor that is more easily understood: at an isofibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$, $\tilde{\Pi}_p q$ is the pullback along the comprehension cocone of the induced map between lax slices induced by whiskering with q:[5]

$$\begin{tikzcd}
\tilde{\Pi}_p(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{E}) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ-"] \\
\mathsf{B} \arrow[r, "\ell^{\mathsf{E}}"'] & \mathsf{qCat}_{2/\!\!/\mathsf{E}}
\end{tikzcd}$$

---

[5] The precise meaning of this notation, involving slices of the homotopy coherent nerve of $\mathcal{QC}\!at$ regarded as a 2-complicial set, is explained in Lemma 4.2.1.

To prove Proposition 4.2.5, we show that the "whiskering with q" map is an isofibration. This establishes the quasi-categorical analogue of desiderata (i) above. We then show further that if $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ is a cartesian fibration, then the "whiskering with q" map has a certain right horn lifting property, thereby proving the quasi-categorical analogue of desiderata (ii):

Corollary 4.2.8. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ is a cartesian fibration between quasi-categories, then the pushforward*

$$\Pi_p(q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}) \twoheadrightarrow \mathsf{B}$$

is a cartesian fibration between quasi-categories.

We show also that the pullback and pushforward functors along a cocartesian fibration preserve the accompanying class of *cartesian functors* between cartesian fibrations. These results are summarized in the following theorem:

Theorem 4.2.9. *For any cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ between quasi-categories, the pullback-pushforward adjunction restricts to define an adjunction*

$$\begin{tikzcd}
\mathcal{QC}at_{/\mathsf{E}} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{QC}at_{/\mathsf{B}} \arrow[l, bend right, "p^*"'] \\
\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{E}} \arrow[u, hook] \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \arrow[u, hook] \arrow[l, bend right, "p^*"']
\end{tikzcd}$$

As an immediate corollary, we construct "exponentials" whose exponents are either cartesian or cocartesian, justifying the appelation "exponentiable" for these maps, and prove:

Proposition 4.3.3. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ is a cartesian fibration, then*

$$(q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})^{p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}}$$

is a cartesian fibration.

The final two sections of this paper supply some first applications of these results. As the comprehension construction reveals, cartesian fibrations over B encode functors $\mathsf{B}^{\mathrm{op}} \to \mathsf{qCat}$ valued in the (large) quasi-category of small quasi-categories. In ordinary category theory it is well-known that for any small category B and complete and cocomplete category C, the forgetful functor $C^B \to C^{\mathrm{ob}\,B}$ that carries a diagram to its $\mathrm{ob}\,B$-indexed family of objects admits both left and right adjoints, given by left and right Kan extension, and is moreover *monadic* and *comonadic*. Informally, this means that B-indexed diagrams can be understood as "algebras" or as "coalgebras" for a monad or comonad acting on the category of $\mathrm{ob}\,B$-indexed families of objects.

The corresponding result for quasi-categories will be proven in a sequel to this paper, but here we demonstrate that the analogous result holds for cartesian fibrations, using a version of Beck's monadicity theorem for quasi-categories proven in [18]. Writing $\mathsf{Cart}_{/\mathsf{B}}$ for the large quasi-category of cartesian fibrations and cartesian functors over B, we prove:

Theorem 5.2.6. *The forgetful functor*

$$u\colon \mathsf{Cart}_{/\mathsf{B}} \longrightarrow \mathsf{Cart}_{/\operatorname{ob}\mathsf{B}} \cong \prod_{\operatorname{ob}\mathsf{B}} \mathsf{qCat}$$

is comonadic and hence also monadic.

In a further sequel, we will use this monadicity to establish an equivalence between $\mathsf{Cart}_{/\mathsf{B}}$ and $\mathsf{qCat}^{\mathsf{B}^{\mathrm{op}}}$, both quasi-categories being monadic over $\prod_{\operatorname{ob}\mathsf{B}}\mathsf{qCat}$.

Here we include another application of the monadicity of Theorem 5.2.6. Using our analysis of limits and colimits in quasi-categories defined as homotopy coherent nerves in [22], we prove:

Theorem 6.1.2. *The inclusion* $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$ *admits both left and right adjoints*

$$\begin{tikzcd} \mathsf{Kan} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{qCat} \arrow[l, bend right=40, "\text{invert}"'] \arrow[l, bend left=40, "\text{core}"] \end{tikzcd}$$

and is monadic and comonadic.

The right adjoint here is the familiar functor that takes a quasi-category to its maximal Kan complex core, while the left adjoint is a somewhat more delicate “groupoidal reflection” functor. Our final result establishes an analogous groupoidal reflection for cartesian fibrations into the subcategory of *groupoidal cartesian fibrations*, whose fibers are Kan complexes rather than quasi-categories.

Theorem 6.3.6. *There is a left adjoint to the inclusion*

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[r, bend right=40] \arrow[r, phantom, "\perp"] & \mathsf{Cart}_{/\mathsf{B}} \arrow[l, bend right=40, "\text{invert}"'] \end{tikzcd}$$

defining the reflection of a cartesian fibration into a groupoidal cartesian fibration.

All of the results mentioned above have duals with cocartesian and cartesian fibrations interchanged. The comprehension functor associated to a cartesian fibration is contravariant and its domain is recovered as the lax colimit of this diagram. It is to avoid this contravariance that we choose to focus the bulk of our presentation on the case of cocartesian fibrations.

This paper is organized as follows. In §2, we provide background material on oplax colimits, cocartesian fibrations, and the comprehension construction from [21]. Then in §3, we prove that pullback along a cocartesian fibration can be modeled as a oplax colimit of a restriction of the comprehension functor.

The corresponding results for the pushforward functor, including in particular (i) and (ii), are then proven in §4. The oplax colimits defining the functor $\tilde{p}^*$ in §3 are properly understood as a variety of $(\infty, 2)$-categorical colimits. Consequently, the description of the corresponding right adjoint $\tilde{\Pi}_p$ involves an $(\infty, 2)$-categorical cocone construction, instantiated by forming the slice of 2-complicial set over a vertex. As we explain in §4.1, 2-complicial sets are simplicial sets in which certain simplices are marked as "thin." This notion is not as unfamiliar as it may seem at first: Kan complexes are precisely the 0-complicial sets while quasi-categories correspond to 1-complicial sets.

In §5, we consider the forgetful functor $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \to \mathsf{Cart}(\mathcal{QC}at)_{/\,\mathrm{ob}\,\mathsf{B}}$ and construct left and right *biadjoints*: quasi-categorically enriched functors equipped with a natural equivalence of function complexes encoding the adjoint transpose relation. Such data descends to an adjunction between the quasi-categorical cores of these quasi-categorically enriched categories. We then review the monadicity theorem from [18] and apply it to prove that this forgetful functor is monadic and comonad as a map between large quasi-categories.

To say that the functor $\mathsf{Cart}_{/\mathsf{B}} \to \mathsf{Cart}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ is monadic is to say that $\mathsf{Cart}_{/\mathsf{B}}$ may be recovered as the quasi-category of *homotopy coherent algebras* for a homotopy coherent monad acting on $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$. In §6, we show that $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}}$ is similarly the quasi-category of homotopy coherent algebras for the restriction of this homotopy coherent monad along the inclusion $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \hookrightarrow \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$. We then show that this characterization allows us to construct the groupoidal reflection functor as a lift of the groupoidal reflection functor $\mathsf{qCat} \to \mathsf{Kan}$.

This paper is a continuation of a series of papers that redevelop the foundations of $(\infty, 1)$-category theory [16, 18, 17, 19, 20, 21, 23, 22], the results of which are referenced as I.x.x.x, . . . , and VIII.x.x.x respectively. However, we deploy relatively few of the tools developed in our previous work to prove the theorems appearing here, and when we do reference prior results, we typically restate them in considerably less generality. Many of the results from previous work recalled here — for instance Theorem 2.3.9 — are proven in the more abstract setting of any ∞-*cosmos*, while in the present manuscript we consider only a single example: the quasi-categorically enriched category $\mathcal{QC}at$ of quasi-categories. As we do not need this notion, we do not recall any specifics here.[6] While this paper was in press, the book [25] was published, so in the final version of the present manuscript we have cut a few proofs and instead refer to the corresponding results X.x.x.x that now appear there.

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[6] This said, however, we cannot resist appealing to ∞-cosmic techniques in a handful of our proofs.

2. Background

In §2.1, we introduce oplax colimits through the general mechanism of weighted colimits. We prove that oplax weights are *flexible*, which implies that the oplax weighted colimit functor is equivalence-invariant. We also review the collage construction, which allows us to construct flexible weights by instead specifying the shape of their corresponding cocones. In particular, a lax cocone of shape X is indexed by the homotopy coherent realization of the join $X \star \Delta^0$.

In §2.2, we review some basic aspects of the theory of cocartesian fibrations between quasi-categories. We introduce a quasi-categorical version of the collage construction and prove that the quasi-categorical collage of a functor $f\colon \mathsf{A} \to \mathsf{B}$ defines a cocartesian fibration over Δ^1 that models the oplax colimit of f.

In §2.3, we review the comprehension construction, devoting somewhat more attention to the lax cocones that are the focus of much of the work here.

2.1 Oplax colimits in simplicial categories

Our aim in this section is to define the *oplax colimit* of a homotopy coherent diagram $\mathfrak{C}X \to \mathcal{S}\mathit{Set}$ indexed by the homotopy coherent realization of a simplicial set X. Oplax colimits are introduced as particular *weighted colimits*, where the weights in question are simplicial functors that describe the shape of lax cocones. Some of this material was previously discussed in §VII.4, where the "oplax" weights were called "pseudo" weights. See Remark 2.1.15 for an explanation of this contrast in nomenclature.

In a simplicially enriched category, the appropriately general notion of colimit allows for the specification of any particular "shape" of cone under the diagrams being considered. This specification is given by a simplicial functor referred to as a *weight* for the colimit.

Definition 2.1.1 (weights for simplicial colimits)**.** Suppose $\mathcal{D}$ is a small simplicial category, which we think of as a diagram shape. Then a *weight* on $\mathcal{D}$ is a simplicial functor $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$. For any diagram $F\colon \mathcal{D} \to \mathcal{K}$ valued in a simplicial category $\mathcal{K}$, a *W-cocone* with *nadir* an object $e \in \mathcal{K}$ is a simplicial natural transformation $\iota\colon W \to \mathsf{Fun}_{\mathcal{K}}(F(-), e)$. We say that the W-cocone ι *displays e as a W-colimit of F* if and only if for all objects $e' \in \mathcal{K}$ the simplicial map

$$\mathsf{Fun}_{\mathcal{K}}(e, e') \xrightarrow{\cong} \mathsf{Fun}_{\mathcal{S}\mathit{Set}^{\mathcal{D}^{\mathrm{op}}}}(W, \mathsf{Fun}_{\mathcal{K}}(F(-), e'))$$

given by pre-composition with ι is an isomorphism.

Many notations are common for the nadir of a weighted colimit cone; here we write $\operatorname{colim}^W F$ for the colimit of F weighted by W. When these exist for all weights and diagrams in $\mathcal{K}$ then colim defines a simplicial bifunctor that is cocontinuous in both variables:

$$\mathcal{S}\mathit{Set}^{\mathcal{D}^{\mathrm{op}}} \times \mathcal{K}^{\mathcal{D}} \xrightarrow{\operatorname{colim}} \mathcal{K}$$

A simplicial functor $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ may otherwise be described as comprising a family of simplicial sets $\{Wd\}_{d\in\mathrm{obj}(\mathcal{D})}$ along with right actions of the hom-spaces of $\mathcal{D}$

$$Wd' \times \mathsf{Fun}_{\mathcal{D}}(d, d') \xrightarrow{\quad * \quad} Wd \tag{2.1.2}$$

which satisfy axioms with respect to the identities and composition of $\mathcal{D}$. This description leads us to define a simplicially enriched category $\mathrm{coll}(W)$, called the *collage* of W.

Definition 2.1.3 (collages)**.** For any weight $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathsf{Set}$, the *collage* of W is a simplicial category $\mathrm{coll}(W)$ that contains $\mathcal{D}$ as a full simplicial subcategory along with precisely one extra object $\top$ whose endomorphism space is the point. The function complexes $\mathsf{Fun}_{\mathrm{coll}(W)}(\top, d)$ are all taken to be empty and we define:

$$\mathsf{Fun}_{\mathrm{coll}(W)}(d, \top) := Wd \qquad \text{for objects } d \in \mathcal{D}.$$

The composition operations between hom-spaces in $\mathcal{D}$ and those with codomain $\top$ are given by the actions depicted in (2.1.2).

In the statement of the following result, $\underline{\mathrm{sSet}}^{\mathcal{D}^{\mathrm{op}}}$ denotes the underlying category of the simplicially enriched category $\mathcal{S}\mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}$.

Proposition 2.1.4 (collage adjunction, VII.5.2.3)**.**

(i) The collage construction defines a fully faithful functor

$$\underline{\mathrm{sSet}}^{\mathcal{D}^{\mathrm{op}}} \xrightarrow{\quad \mathrm{coll} \quad} {}^{\mathbb{1}+\mathcal{D}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$$

from the category of $\mathcal{D}$-indexed weights to the category of simplicial categories under $\mathbb{1}+\mathcal{D}$ whose essential image is comprised of those $\langle e, F\rangle\colon \mathbb{1}+\mathcal{D} \to \mathcal{K}$ that are bijective on objects, fully faithful when restricted to $\mathcal{D}$ and $\mathbb{1}$, and have the property that there are no maps in $\mathcal{K}$ from e to the image of F.

(ii) The collage functor admits a right adjoint, which carries a pair $\langle e, F\rangle\colon \mathbb{1}+\mathcal{D} \to \mathcal{K}$ to the weight $\mathsf{Fun}_{\mathcal{K}}(F(-), e)\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathsf{Set}$.

$$\begin{tikzcd} {}^{\mathbb{1}+\mathcal{D}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}} \arrow[r, bend right, "\mathrm{wgt}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}^{\mathcal{D}^{\mathrm{op}}} \arrow[l, bend right, "\mathrm{coll}"'] \end{tikzcd} \qquad \square$$

This adjunction has a useful and important interpretation:

Corollary 2.1.5 (VII.5.2.4)**.** *The collage $\mathrm{coll}(W)$ of a weight realises the shape of W-cocones, in the sense that simplicial functors*

$$G\colon \mathrm{coll}(W) \longrightarrow \mathcal{K}$$

stand in bijection to W-cocones under the diagram $G|_{\mathcal{D}}$ with nadir $G(\top)$. $\square$

We record some basic properties about weighted colimits, collages, and left Kan extensions for later use. For proof see [24, §2.1].

Lemma 2.1.6. *For any simplicial functor $I\colon \mathcal{D} \to \mathcal{C}$ and weight $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathsf{Set}$:*

(i) *For any diagram $G\colon \mathcal{C} \to \mathcal{K}$, we have an isomorphism*

$$\mathrm{colim}^W GI \cong \mathrm{colim}^{\mathrm{lan}_I W} G$$

where the colimit on one side exists if and only if the one on the other does.

(ii) *Left Kan extension of W along I gives rise to a pushout square in the category of simplicial categories and simplicial functors:*

$$\begin{tikzcd} \mathbb{1} + \mathcal{D} \arrow[r, "\mathbb{1}+I"] \arrow[d, hook] & \mathbb{1} + \mathcal{C} \arrow[d, hook] \arrow[dl, phantom, very near start, "\ulcorner"] \\ \mathrm{coll}(W) \arrow[r] & \mathrm{coll}(\mathrm{lan}_I W) \end{tikzcd}$$ □

In ordinary unenriched category theory, the colimit cone under a $\mathcal{D}$-shaped diagram may be formed as the left Kan extension along the inclusion $\mathcal{D} \hookrightarrow \mathcal{D} \star \mathbb{1}$ into the category $\mathcal{D} \star \mathbb{1}$ formed by freely adjoining a terminal object "$\top$" to $\mathcal{D}$. The following lemma reveals that the collage plays the roll of the category $\mathcal{D} \star \mathbb{1}$ for weighted colimits.

Lemma 2.1.7. *The pointwise left Kan extension of any simplicial functor $F\colon \mathcal{D} \to \mathcal{K}$ along $I\colon \mathcal{D} \hookrightarrow \mathrm{coll}(W)$ exists if and only if the colimit $\mathrm{colim}^W F$ exists in $\mathcal{K}$, in which case $\mathrm{lan}_I F(\top) \cong \mathrm{colim}^W F$.* □

In order to understand the sense in which certain weighted colimits, including in particular the oplax colimits to be introduced below, are homotopically well behaved, we recall some facts about weights and simplicial computads from §II.5.3:

Definition 2.1.8 (flexible weights and projective cell complexes)**.** For a simplicial category $\mathcal{D}$, the *projective n-cell* associated with $[n] \in \mathbb{\Delta}$ and $d \in \mathcal{D}$ is the simplicial natural transformation

$$\partial\Delta^n \times \mathsf{Fun}_{\mathcal{D}}(-, d) \hookrightarrow \Delta^n \times \mathsf{Fun}_{\mathcal{D}}(-, d).$$

A natural transformation $\alpha\colon W \to V$ in $\mathcal{S}\mathit{Set}^{\mathcal{D}^{\mathrm{op}}}$ is a *relative projective cell complex* if it factors as a countable composite of pushouts of coproducts of projective cells. A weight W in $\mathcal{S}\mathit{Set}^{\mathcal{D}^{\mathrm{op}}}$ is a *flexible weight* if the map $!\colon \emptyset \to W$ is a relative projective cell complex.

The following result extends without change to pointwise cofibrant diagrams valued in any model category enriched over the Joyal model structure on simplicial sets.

Proposition 2.1.9 (II.5.2.6, VII.4.1.5)**.**

(i) *For a flexible weight $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ and any diagram $F\colon \mathcal{D} \to \mathcal{S}\mathit{Set}$, $\mathrm{colim}^W F$ may be expressed as a countable composite of pushouts of coproducts of maps*

$$\partial\Delta^n \times Fd \hookrightarrow \Delta^n \times Fd.$$

(ii) If $\alpha\colon F \to G$ is a simplicial natural transformation between two such diagrams whose components are weak equivalences in the Joyal model structure, then for any flexible weight W the map

$$\operatorname{colim}^W \alpha\colon \operatorname{colim}^W F \to \operatorname{colim}^W G$$

is a weak equivalence in the Joyal model structure. □

The collage construction defines a correspondence between flexible weights and *simplicial computads*, a class of "freely generated" simplicial categories that define precisely the cofibrant objects [14, §16.2] in the model structure due to Bergner [3].

Definition 2.1.10 (simplicial computad)**.** A simplicial category $\mathcal{A}$, regarded as a simplicial object $[n] \mapsto \mathcal{A}_n$ in the category of categories with a common set of objects and identity-on-objects functors, is a *simplicial computad* if and only if:

- each category $\mathcal{A}_n$ of *n-arrows* is freely generated by the reflexive directed graph of *atomic* n-arrows, these being those arrows that admit no non-trivial factorizations, and
- if f is an atomic n-arrow in $\mathcal{A}_n$ and $\alpha\colon [m] \to [n]$ is a degeneracy operator in $\mathbb{\Delta}$ then the degenerated m-arrow $f \cdot \alpha$ is atomic in $\mathcal{A}_m$.

We have the following recognition principle for flexible weights on simplicial computads, a mild variant of Proposition II.5.3.5, proven in §VII.5.2.

Proposition 2.1.11 (relating flexible weights and simplicial computads, VII.5.2.6)**.** *Suppose that $\mathcal{D}$ is a simplicial computad. Then a weight $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ is flexible if and only if its collage* $\mathrm{coll}(W)$ *is a simplicial computad.* □

By the next result, the left adjoint to the homotopy coherent nerve

$$\begin{tikzcd} \underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}} \arrow[r, bend right, "N"', ""{name=B}] & \underline{\mathrm{sSet}} \arrow[l, bend right, "\mathfrak{C}"', ""{name=A}] \arrow[phantom, from=A, to=B, "\perp"] \end{tikzcd}$$

the *homotopy coherent realization* functor, provides a source of flexible weights. See §VI.4 for a more leisurely presentation with considerably more details.

Proposition 2.1.12 (VI.4.4.7)**.** *For any simplicial set X, the* homotopy coherent realization $\mathfrak{C}X$ *is a simplicial computad.* □

Recall 2.1.13. For any simplicial set X, there is a canonical inclusion $X \hookrightarrow X \star \Delta^0$ into its join with the point. The join $X \star \Delta^0$ has a single vertex of $X \star \Delta^0$ that is not also a vertex of its subset X, which we shall denote by "$\top$." Now for each non-degenerate n-simplex $x \in X$ the join $X \star \Delta^0$ has two corresponding non-degenerate simplices:

- a simplex of dimension n identified with x itself and
- a simplex $(x, \top)$ of dimension $n+1$,

and these two cases enumerate all of the non-degenerate simplices of $X \star \Delta^0$ with the exception of $\top$.

Oplax colimits represent particular cones under a homotopy coherent diagram $\mathfrak{C}X \to \mathcal{K}$ indexed by a simplicial set X. In this context, the homotopy coherent realization of of the joint $X \star \Delta^0$ defines a collage that presents the weight for oplax colimits.

Definition 2.1.14 (weights for oplax colimits)**.** Applying homotopy coherent realisation to the canonical inclusion $X \hookrightarrow X \star \Delta^0$ for any simplicial set X, yields a simplicial subcomputad $I_X \colon \mathfrak{C}X \hookrightarrow \mathfrak{C}[X \star \Delta^0]$ so that the conditions discussed in Proposition 2.1.4(i) hold for the inclusion $\langle \top, I_X \rangle \colon \mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}[X \star \Delta^0]$. Hence, via the counit isomorphism of the collage adjunction, this simplicial category is isomorphic to the collage of the corresponding weight, defining the *weight for oplax colimits* of diagrams of shape $\mathfrak{C}X$.

$$\mathfrak{C}X^{\mathrm{op}} \xrightarrow{L_X} \mathcal{S}\mathit{Set} \qquad \text{given by} \qquad L_X(x) := \mathsf{Fun}_{\mathfrak{C}[X \star \Delta^0]}(x, \top).$$

When $F \colon \mathfrak{C}X \to \mathcal{K}$ is a homotopy coherent diagram of shape X, then its *oplax colimit* is defined to be the weighted colimit

$$\operatorname{colim}^{\mathrm{oplax}} F := \operatorname{colim}^{L_X} F.$$

Remark 2.1.15. The oplax weights being defined here are precisely the "pseudo" weights introduced in Definition VII.5.2.8. The reason for the difference in nomenclature is that in that paper the diagrams considered in [23] are valued in Kan complex enriched categories, whereas here the diagrams are valued in quasi-categorically (or simplicially) enriched categories. In a Kan complex, the 1-simplex Δ^1 represents an invertible morphism, while in a quasi-category it models a non-invertible morphism.

Immediately from Proposition 2.1.11:

Lemma 2.1.16 (VII.5.2.9)**.** *For all simplicial sets X the weight $L_X \colon \mathfrak{C}X^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ for oplax colimits of diagrams of shape $\mathfrak{C}X$ is a flexible weight.* □

2.2 Cocartesian fibrations and quasi-categorical collages

In this section, we construct an explicit example of an oplax colimit of diagram of quasi-categories via the *quasi-categorical collage construction*. In an important special case, the quasi-categorical collage defines a cocartesian fibration over the 1-simplex, so we first introduce the quasi-categorically enriched category of cocartesian fibrations and cartesian functors.

Of the many equivalent definitions of cocartesian fibration (see §IV.4 and §VI.3), the following will be the most convenient for this paper:

Definition 2.2.1 ([12, 2.4.1.8,2.4.2.1], IV.4.1.24)**.** Let $p \colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ be an isofibration between quasi-categories.

(i) A 1-arrow $\chi\colon e \to e'$ of E is *p-cocartesian* if and only if any lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\chi"] & \Lambda^{n,0} \arrow[r] \arrow[d] & \mathsf{E} \arrow[d, two heads, "p"] \\ & \Delta^{n} \arrow[r] \arrow[ur, dashed] & \mathsf{B} \end{tikzcd}$$

has a solution.

(ii) An isofibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a *cocartesian* fibration of quasi-categories precisely when any arrow $\alpha\colon pe \to b$ in B admits a lift to an arrow $\chi\colon e \to e'$ in E which enjoys the lifting property of (i).

If p and q are cocartesian fibrations over B then a functor

$$\begin{tikzcd} \mathsf{E} \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & \mathsf{F} \arrow[dl, two heads, "q"] \\ & \mathsf{B} & \end{tikzcd}$$

is a *cartesian functor* just when it carries p-cocartesian 1-arrows to q-cocartesian 1-arrows. As one illustration of the importance of this notion:

Proposition 2.2.2 (VIII.5.1.3)**.** *A cartesian functor between cocartesian fibrations of quasi-categories is an equivalence if and only if it is a fiberwise equivalence:*

$$\begin{tikzcd} \mathsf{E} \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & \mathsf{F} \arrow[dl, two heads, "q"] \\ & \mathsf{B} & \end{tikzcd}$$

i.e., for each $b \in \mathrm{ob}\,\mathsf{B}$, the induced functor $g_b\colon \mathsf{E}_b \to \mathsf{F}_b$ is an equivalence. ☐

If B is a quasi-category, then we adopt the notation $\mathcal{QC}at_{/\mathsf{B}}$ for the quasi-categoricaly enriched category of isofibrations over B defined as follows.

Definition 2.2.3. For a quasi-category B, let $\mathcal{QC}at_{/\mathsf{B}}$ denote the category whose:

- objects are isofibrations $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ with codomain B and
- whose function complexes $\mathsf{Fun}_{\mathsf{B}}(p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}, q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})$ are defined by the pullbacks

$$\begin{tikzcd} \mathsf{Fun}_{\mathsf{B}}(p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}, q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(\mathsf{E}, \mathsf{F}) \arrow[d, "q\circ-"] \\ \Delta^0 \arrow[r, "p"] & \mathsf{Fun}(\mathsf{E}, \mathsf{B}) \end{tikzcd}$$

where $\mathsf{Fun}(\mathsf{E},\mathsf{F}) \cong \mathsf{F}^{\mathsf{E}}$ denotes the usual internal hom in $\mathcal{Q}\mathit{Cat}$.

Definition 2.2.4. For a quasi-category B, let $\mathsf{coCart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}}$ denote the category whose:

- objects are cocartesian fibrations $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ with codomain B and
- whose function complexes $\mathsf{Fun}^c_{\mathsf{B}}(p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}, q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})$ are defined to be the full sub quasi-categories of the function complexes $\mathsf{Fun}_{\mathsf{B}}(p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}, q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})$ of $\mathcal{Q}\mathit{Cat}_{/\mathsf{B}}$ defined by restricting the 0-arrows to be cartesian functors over B.

The quasi-categorically enriched category $\mathsf{Cart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}}$ of cartesian fibrations and cartesian functors is defined similarly.

Proposition IV.5.2.1 proves that the pullback of a cocartesian fibration is a cocartesian fibration

$$\begin{tikzcd} \mathsf{F} \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\ \mathsf{A} \arrow[r, "f"'] & \mathsf{B} \end{tikzcd}$$

in which an arrow χ is q-cocartesian if and only $g\chi$ is p-cocartesian. It follows that pullback also preserves cartesian functors. Hence:

Proposition 2.2.5. *Pullback along any* $f\colon \mathsf{A} \to \mathsf{B}$ *defines a quasi-categorically enriched functor*

$$\begin{tikzcd} \mathsf{coCart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}} \arrow[r, "f^*"] \arrow[d, phantom, "\cap"] & \mathsf{coCart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{A}} \arrow[d, phantom, "\cap"] \\ \mathcal{Q}\mathit{Cat}_{/\mathsf{B}} \arrow[r, "f^*"] & \mathcal{Q}\mathit{Cat}_{/\mathsf{A}} \end{tikzcd} \qquad \Box$$

We now argue that the pullback functor preserves simplicial tensors. This will be used in §4 to show that its right adjoint is simplicially enriched, when this functor exists.

Observation 2.2.6 (tensors and pullback). Let $X \in \underline{\mathrm{sSet}}$ be a simplicial set. The tensor of an isofibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ with X is the right-hand vertical composite, which pulls back to the right-hand vertical composite

$$\begin{tikzcd} \mathsf{F} \times X \arrow[r] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \times X \arrow[d, two heads, "\pi_1"] \\ \mathsf{F} \arrow[r] \arrow[d, two heads, "f^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\ \mathsf{A} \arrow[r, "f"'] & \mathsf{B} \end{tikzcd}$$

which defines the tensor of $f^*(p)\colon \mathsf{F} \twoheadrightarrow \mathsf{A}$ with X.

The following lemma tells us that this tensor construction respects cartesian functors.

Lemma 2.2.7. *For any simplicial set X and cocartesian fibrations $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$, the isomorphism $\mathsf{Fun}_{\mathsf{B}}(\mathsf{E} \times X, \mathsf{F}) \cong \mathsf{Fun}_{\mathsf{B}}(\mathsf{E}, \mathsf{F})^X$ restricts to an isomorphism*

$$\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E} \times X, \mathsf{F}) \cong \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E}, \mathsf{F})^X.$$

Proof. We make use of Theorem IV.5.1.4 which provides the following characterization of the sub quasi-category $\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E}, \mathsf{F}) \subset \mathsf{Fun}_{\mathsf{B}}(\mathsf{E}, \mathsf{F})$. Any functor $f\colon \mathsf{E} \to \mathsf{F}$ over B induces a commutative square over B

$$\begin{tikzcd} \mathsf{E} \arrow[r, "f"] \arrow[d] \arrow[d, phantom, shift right=3, "\dashv"] & \mathsf{F} \arrow[d] \arrow[d, phantom, shift left=3, "\vdash"] \\ p \downarrow \mathsf{B} \arrow[r, "{(f, \mathrm{id}_{\mathsf{B}})}"'] \arrow[u, bend left=50, "\ell"] & q \downarrow \mathsf{B} \arrow[u, bend right=50, "\ell"'] \end{tikzcd}$$

whose vertical functors are the canonical ones induced by $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$. Because p and q are cocartesian, Theorem IV.4.1.10 proves the vertical functors admit left adjoints over B. Theorem IV.5.1.4 proves that f is cartesian if and only if the mate of this canonical isomorphism is an isomorphism.

The mate that detects whether f is a cartesian functor lives as a 1-simplex in the simplicial set

$$\mathsf{Sq}_{\mathsf{B}}(p \downarrow \mathsf{B} \to \mathsf{E}, q \downarrow \mathsf{B} \to \mathsf{F}) := \mathsf{Fun}_{\mathsf{B}}(\mathsf{E}, \mathsf{F}) \times_{\mathsf{Fun}_{\mathsf{B}}(p \downarrow \mathsf{B}, \mathsf{F})} \mathsf{Fun}_{\mathsf{B}}(p \downarrow \mathsf{B}, q \downarrow \mathsf{B}).$$

of commutative squares from $\ell\colon p \downarrow \mathsf{B} \to \mathsf{E}$ to $\ell\colon q \downarrow \mathsf{B} \to \mathsf{F}$. The adjunction over B associated to the cocartesian fibration $\mathsf{E} \times X \xrightarrow{\pi} \mathsf{E} \xrightarrow{p} \mathsf{B}$ is

$$\begin{tikzcd} \mathsf{E} \times X \arrow[rr] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, "p\pi"'] & & p \downarrow \mathsf{B} \times X \arrow[ll, bend right=20, "\ell"'] \arrow[dl, "p_0\pi"] \\ & \mathsf{B} & \end{tikzcd}$$

the product of the adjunction for p with X. In particular,

$$\mathsf{Sq}_{\mathsf{B}}(p \downarrow \mathsf{B} \times X \to \mathsf{E} \times X, q \downarrow \mathsf{B} \to \mathsf{F}) \cong \mathsf{Sq}_{\mathsf{B}}(p \downarrow \mathsf{B} \to \mathsf{E}, q \downarrow \mathsf{B} \to \mathsf{F})^X.$$

Now a 1-simplex C^X is an isomorphism if and only if it is a pointwise isomorphism, which proves that $\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E} \times X, \mathsf{F}) \cong \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E}, \mathsf{F})^X$. □

We conclude this section with an example of an oplax colimit. When $X = \Delta^1$ a homotopy coherent diagram $\mathfrak{C}\Delta^1 \to \mathcal{QC}\mathit{at}$ is just a functor $f\colon \mathsf{A} \to \mathsf{B}$ between quasi-categories. The oplax colimit in simplicial sets is given by the pushout

$$\begin{tikzcd} \mathsf{A} \arrow[r, "f"] \arrow[d, "\mathrm{id} \times \delta^0"'] & \mathsf{B} \arrow[d] \\ \mathsf{A} \times \Delta^1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{colim}^{\mathrm{oplax}} f \end{tikzcd}$$

Up to equivalence, this oplax colimit is modeled by the *quasi-categorical collage construction* that we now introduce.

Definition 2.2.8 (the quasi-categorical collage construction X.F.5.2)**.** For any cospan $f\colon \mathsf{A} \to \mathsf{C}$ and $g\colon \mathsf{B} \to \mathsf{C}$, with A, B, and C all quasi-categories, define a new simplicial set $\mathrm{coll}(f, g)$ by declaring that

$$\mathrm{coll}(f,g)_n = \left\{ \left(\Delta^i \xrightarrow{a} \mathsf{A}, \Delta^j \xrightarrow{b} \mathsf{B}, \Delta^n \xrightarrow{c} \mathsf{C} \right) \;\middle|\; \begin{array}{lll} c|_{\{0,\ldots,i\}} & = f(a), & i, j \geq -1, \\ c|_{\{n-j,\ldots,n\}} & = g(b), & i + j = n - 1. \end{array} \right\}$$

with the convention that conditions indexed by Δ^{-1} are empty (or that each simplicial set is terminally augmented). There are simplicial maps

$$\begin{tikzcd}
B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{coll}(f,g) \arrow[d, "\rho"] & A \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\{1\} \arrow[r, hook] & \Delta^1 & \{0\} \arrow[l, hook']
\end{tikzcd}$$

where the map ρ sends an n-simplex $(a\colon \Delta^i \to \mathsf{A}, b\colon \Delta^j \to \mathsf{B}, c\colon \Delta^n \to \mathsf{C})$ to the n-simplex $[n] \to [1]$ that carries $0, \ldots, i$ to 0 and $i + 1, \ldots, n$ to 1. Note that the fiber of ρ over 0 is isomorphic to A while the fiber of ρ over 1 is isomorphic to B.

Lemma 2.2.9 (X.F.5.3)**.** *The map* $\rho\colon \mathrm{coll}(f, g) \to \Delta^1$ *is an inner fibration. In particular, the simplicial set* $\mathrm{coll}(f, g)$ *is a quasi-category.* ☐

We write $\mathrm{coll}(f, \mathsf{B})$ for the collage of $f\colon \mathsf{A} \to \mathsf{B}$ with the identity on B.

Lemma 2.2.10 (X.F.5.4)**.** *For any* $f\colon \mathsf{A} \to \mathsf{B}$*, the map* $\rho\colon \mathrm{coll}(f, \mathsf{B}) \to \Delta^1$ *is a cocartesian fibration.* ☐

Proposition 2.2.11 (X.F.5.5)**.** *For any* $f\colon \mathsf{A} \to \mathsf{B}$ *between quasi-categories, the collage* $\mathrm{coll}(f, \mathsf{B})$ *defines the oplax colimit of* f *in* $\mathcal{Q}\mathit{Cat}$*. That is* $\mathrm{coll}(f, \mathsf{B})$ *defines a cone under the pushout diagram*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "f"] \arrow[d, hook, "\mathrm{id} \times \delta^0"'] & \mathsf{B} \arrow[d, hook] \arrow[ddr, hook, bend left] \arrow[dl, phantom, "\ulcorner", very near end] & \\
\mathsf{A} \times \Delta^1 \arrow[r] \arrow[drr, bend right, "h"'] & P \arrow[dr, dashed, "k"] & \\
& & \mathrm{coll}(f, \mathsf{B})
\end{tikzcd}$$

so that the induced map k *is inner anodyne, and in particular a weak equivalence in the Joyal model structure.* ☐

Corollary 2.2.12 (X.F.5.6)**.** *Consider a pair of functors between quasi-categories* $f\colon \mathsf{A} \to \mathsf{B}$ *and* $u\colon \mathsf{B} \to \mathsf{A}$*. Then* f *is left adjoint to* u *if and only if the collages* $\mathrm{coll}(f, \mathsf{B})$ *and* $\mathrm{coll}(\mathsf{A}, u)$ *are equivalent under* $\mathsf{B} + \mathsf{A}$ *and over* Δ^1. ☐

2.3 The comprehension construction

In this section we review the *comprehension construction* from [21]. It constructs, for any cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ of quasi-categories, a "straightening," which has the form of a simplicial functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{QC}at$ that sends each vertex $b \in \mathsf{B}$ to the fiber E_b. It also constructs a canonical lax cocone $\ell^{\mathsf{E}}\colon \mathfrak{C}[\mathsf{B} \star \Delta^0] \to \mathcal{QC}at$ of shape B under this diagram with nadir E.

Corollary 2.1.5 tells us that the collage of a weight W realizes the shape of W-cocones. Applying this result to the weights for oplax colimits introduced in Definition 2.1.14, we obtain the following definition of a *lax cocone*.

Definition 2.3.1 (lax cocones VI.5.2.4)**.** Suppose that X is a simplicial set. Then a *lax cocone of shape* X in $\mathcal{SS}et$ is defined to be a simplicial functor $\ell^B\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{SS}et$

$$\begin{tikzcd}
& \mathbb{1} + \mathfrak{C}X \arrow[dl, hook'] \arrow[dr, "\langle B, B_\bullet\rangle"] & \\
\mathfrak{C}[X \star \Delta^0] \arrow[rr, "\ell^B"'] & & \mathcal{SS}et
\end{tikzcd}$$

The restriction of a lax cocone $\ell^B\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{K}$ to a functor $B_\bullet\colon \mathfrak{C}X \to \mathcal{SS}et$ is called its *base*. We say that ℓ^B is a lax cocone *under the diagram* $B_\bullet$; the object $B \in \underline{\mathrm{sSet}}$ obtained by evaluating ℓ^B at the object $\top$ is called the *nadir* of that lax cocone.

Example 2.3.2 (canonical lax cocones VI.6.1.6)**.** For any simplicial set X, there exists a lax cocone

$$\begin{tikzcd}
& \mathbb{1} + \mathfrak{C}X \arrow[dl, hook'] \arrow[dr, "\langle X, 1\rangle"] & \\
\mathfrak{C}[X \star \Delta^0] \arrow[rr, "k^X"'] & & \mathcal{SS}et
\end{tikzcd}$$

whose base is constant at the terminal quasi-category 1 and whose nadir is X that we refer to as the *canonical* X*-shaped lax cocone*.

Observation 2.3.3 (whiskering lax cocones VI.5.2.6). Let $\ell^A\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{SS}et$ be a lax cocone with base diagram $A_\bullet\colon \mathfrak{C}X \to \mathcal{SS}et$ and nadir $\ell^A_\top = A$, and let $f\colon A \to B$ be any map of simplicial sets. Then there is a *whiskered lax cocone* $f \cdot \ell^A\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{SS}et$ with the same base diagram $A_\bullet\colon \mathfrak{C}X \to \mathcal{SS}et$ and with nadir B, whose components from a vertex $x \in X$ to $\top$ are defined by whiskering with f:

$$\mathsf{Fun}_{\mathfrak{C}[X\star\Delta^0]}(x, \top) \xrightarrow{\ell^X_{x,\top}} \mathsf{Fun}(A_x, A) \xrightarrow{f\circ -} \mathsf{Fun}(A_x, B)$$

Lemma 2.3.4. *For any map of simplicial sets $f\colon Y \to X$, the canonical lax cocone of shape X restricts along $\mathfrak{C}[f \star \mathrm{id}]\colon \mathfrak{C}[Y \star \Delta^0] \to \mathfrak{C}[X \star \Delta^0]$ to the whiskered composite*

$$\begin{tikzcd}
& \mathbb{1} + \mathfrak{C}Y \arrow[rr, "\mathbb{1}+\mathfrak{C}f"] \arrow[dl] & & \mathbb{1} + \mathfrak{C}X \arrow[dl] \arrow[dr, "\langle X, 1\rangle"] & & = & & \mathbb{1} + \mathfrak{C}Y \arrow[dl] \arrow[dr, "\langle X, 1\rangle"] & \\
\mathfrak{C}[Y \star \Delta^0] \arrow[rr, "\mathfrak{C}[f\star\mathrm{id}]"'] & & \mathfrak{C}[X \star \Delta^0] \arrow[rr, "k^X"'] & & \mathcal{SS}et & & \mathfrak{C}[Y \star \Delta^0] \arrow[rr, "f\cdot k^Y"'] & & \mathcal{SS}et
\end{tikzcd}$$

of the canonical lax cocone of shape Y with $f\colon Y \to X$.

Proof. By direct verification from Lemma VI.6.1.6 and Observation 2.3.3. □

§VI.5 introduces a mechanism for producing new lax cocones from given ones: namely as domain components of cocartesian cocones over a given codomain lax cocone.

Definition 2.3.5 (cocartesian cocones VI.5.3.1)**.** Suppose we are given a simplicial set X and lax cocones $\ell^E, \ell^B\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{S}\mathit{Set}$ of shape X with bases $E_\bullet$ and $B_\bullet$ respectively. Suppose also that we are given a simplicial natural transformation

$$\begin{tikzcd} \mathfrak{C}[X \star \Delta^0] \arrow[r, bend left, "\ell^E"{name=U}] \arrow[r, bend right, "\ell^B"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow p"] & \mathcal{S}\mathit{Set}. \end{tikzcd}$$

Then we say that the triple (ℓ^E, ℓ^B, p) is a *cocartesian cocone* if and only if

(i) the nadir of the natural transformation p, that being its component $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ at the object $\top$, is a cocartesian fibration between quasi-categories

(ii) for all 0-simplices $x \in X$ the naturality square is a pullback, and

$$\begin{tikzcd} E_x \arrow[r, "\ell^E_x"] \arrow[d, two heads, "p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\ B_x \arrow[r, "\ell^B_x"'] & \mathsf{B} \end{tikzcd}$$

(iii) for all non-degenerate 1-simplices $f\colon x \to y \in X$ the 1-arrow is p-cocartesian.

$$\begin{tikzcd} E_x \arrow[dr, "\ell^E_x"] \arrow[dd, "E_f"'] & \\ \Downarrow \ell^E_f & \mathsf{E} \\ E_y \arrow[ur, "\ell^E_y"'] & \end{tikzcd}$$

Lemma 2.3.6 (pullbacks of cocartesian cocones VI.5.3.3)**.** *Suppose given:*

- *a pullback diagram of quasi-categories in which p and q are cocartesian fibrations;*

$$\begin{tikzcd} \mathsf{F} \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\ \mathsf{A} \arrow[r, "f"'] & \mathsf{B} \end{tikzcd} \tag{2.3.7}$$

- *a lax cocone $\ell^A\colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{K}$ with nadir A; and*

- *a cocartesian cocone* (ℓ^E, ℓ^B, p) *whose nadir is* $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ *and whose codomain cocone* $\ell^B = f \cdot \ell^A$ *is obtained from the lax cocone* ℓ^A *by whiskering with* $f\colon \mathsf{A} \to \mathsf{B}$.

Then there is a cocartesian cone (ℓ^F, ℓ^A, q) *whose codomain is* ℓ^A*, whose nadir is* $q\colon \mathsf{F} \twoheadrightarrow \mathsf{A}$*, and whose domain component is a lax cocone* ℓ^F *that whiskers with* g *to the lax cone* $\ell^E = g \cdot \ell^F$. ☐

Conversely, a cocartesian cocone (ℓ^F, ℓ^A, q) with nadir $q\colon \mathsf{F} \twoheadrightarrow \mathsf{A}$ can be whiskered with a pullback square (2.3.7) to define a cocartesian cocone $(g \cdot \ell^F, f \cdot \ell^A, p)$ with nadir $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ and whose domain and codomain are whiskered lax cocones as defined in Observation 2.3.3.

Remark 2.3.8. If the map f of Lemma 2.3.6 is replaced by any map of simplicial sets $f\colon X \to \mathsf{B}$, whose domain is not necessarily a quasi-category, it is still possible to pull back the data of a cocartesian cocone (ℓ^E, ℓ^B, p) whose codomain lax cocone $\ell^B = f \cdot \ell^X$ is obtained by whiskering a lax cocone with nadir X. This constructs a simplicial natural transformation (ℓ^F, ℓ^X, q) whose nadir is the pullback $q\colon F \to X$ of p along f. Since this is not a map between quasi-categories, it does not really make sense to call it a cocartesian fibration. Nonetheless, this construction produces a lax cocone ℓ^F of shape X, which will have some utility. See Remark 2.3.12.

A cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ between quasi-categories has a "straightening" called the *comprehension functor* $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$, a homotopy coherent diagram of shape B that sends each vertex b to the fiber E_b of p over b. This arises as the base diagram of the domain of a cartesian cocone over the canonical B-shaped lax cocone.

Theorem 2.3.9 (VI.6.1.7)**.** *For any cocartesian fibration* $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ *of quasi-categories, there is a cocartesian cocone*

$$\begin{tikzcd}
\mathfrak{C}[\mathsf{B} \star \Delta^0] \arrow[r, bend left, "\ell^{\mathsf{E}}"] \arrow[r, bend right, "k^{\mathsf{B}}"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{Q}\mathit{Cat}.
\end{tikzcd}$$

of shape B *in* $\mathcal{Q}\mathit{Cat}$ *with nadir* $p\colon \mathsf{E} \to \mathsf{B}$ *over the canonical lax cocone* k^{B}*. The base of the domain component defines the comprehension functor* c_p*, which acts on an object* $b\colon 1 \to \mathsf{B}$ *of* $\mathfrak{C}\mathsf{B}$ *by forming the pullback*

$$\begin{tikzcd}
\mathsf{E}_b \arrow[r, "\ell^{\mathsf{B}}_b"] \arrow[d, two heads, "p_b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\
1 \arrow[r, "b"'] & \mathsf{B}
\end{tikzcd}$$

and acts on 1-arrows $f\colon a \to b$ *of* B *by factoring the codomain of a* p*-cocartesian lift* ℓ^{E}_f *of* f *through the pullback at the front of the diagram:*

$$\begin{tikzcd}
\mathsf{E}_a \arrow[rrr, bend left, "\ell^{\mathsf{E}}_a"] \arrow[dr, dotted, "\mathsf{E}_f"'] \arrow[rrr, dotted, "\ell^{\mathsf{E}}_f"] \arrow[dd, two heads, "p_a"'] & & & \mathsf{E} \arrow[dd, two heads, "p"] \\
& \mathsf{E}_b \arrow[urr, "\ell^{\mathsf{E}}_b"'] \arrow[dd, two heads, "p_b"'] & \Swarrow & \\
1 \arrow[rrr, bend left, "a"] \arrow[dr, equals] & & & \mathsf{B} \\
& 1 \arrow[urr, bend right, "b"'] & \Swarrow f &
\end{tikzcd} \tag{2.3.10}$$

These cocartesian lifts define components of the lax cocone

$$\begin{tikzcd}
 & \mathbb{1}+\mathfrak{C}\mathsf{B} \arrow[dl, hook'] \arrow[dr, "{\langle \mathsf{E}, c_p\rangle}"] & \\
\mathfrak{C}[\mathsf{B}\star\Delta^0] \arrow[rr, "\ell^{\mathsf{E}}"'] & & \mathcal{QC}at
\end{tikzcd}$$

with nadir E *under the comprehension functor.* □

By Observation VI.6.1.9, any pair of lax cocones that arise as the domain of a cocartesian cocone over a common codomain define vertices in a contractible Kan complex and are in particular equivalent as diagrams. This is used to prove the following result relating comprehension functors with pullbacks.

Proposition 2.3.11 (comprehension and change of base VI.6.1.11)**.** *Suppose that we are given a pullback*

$$\begin{tikzcd}
\mathsf{F} \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\
\mathsf{A} \arrow[r, "f"'] & \mathsf{B}
\end{tikzcd}$$

of quasi-categories in which p *and thus* q *are cocartesian fibrations. Then the diagrams*

$$\mathfrak{C}\mathsf{A} \xrightarrow{c_q} \mathcal{QC}at \qquad \text{and} \qquad \mathfrak{C}\mathsf{A} \xrightarrow{\mathfrak{C}f} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{QC}at$$

are connected by a homotopy coherent natural isomorphism. □

Remark 2.3.12. If A is not a quasi-category, it is not possible to directly construct the comprehension functor for the pullback of p along f. However, by Lemmas 2.3.4 and Remark 2.3.8, the cocartesian cocone over the canonical B-shaped lax cocone can be pulled back along any map of simplicial sets $f\colon X \to \mathsf{B}$ to define a cocartesian cocone over the canonical X-shaped lax cocone. Thus, a posteriori, we can think of the base of the lax cocone

$$\mathfrak{C}[X\star\Delta^0] \xrightarrow{\mathfrak{C}[f\star\mathrm{id}]} \mathfrak{C}[\mathsf{B}\star\Delta^0] \xrightarrow{\ell^{\mathsf{E}}} \mathcal{QC}at$$

as defining a comprehension functor for the pullback of $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ along $f\colon X \to \mathsf{B}$.

3. Pullback along a cocartesian fibration as an oplax colimit

Our aim in this section is to provide an equivalent model of the pullback functor

$$p^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \longrightarrow \underline{\mathrm{sSet}}_{/\mathsf{E}}$$

along a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ between quasi-categories. In the next section, we will use this to construct an equivalent model of its right adjoint, the pushforward Π_p, whose homotopical properties are more easily established. Before commencing with our work, we briefly sketch the connection between the pullback and pushforward.

The category of simplicial sets, as a presheaf topos, is locally cartesian closed, so pullback along any map $p\colon E \to B$ admits a right adjoint:

$$\begin{tikzcd} \underline{\mathrm{sSet}}_{/E} \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/B} \arrow[l, bend right, "p^*"'] \arrow[l, phantom] \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{sSet}}_{/E} \arrow[rr, bend right, "\Pi_p"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{sSet}}_{/B} \arrow[ll, bend right, "p^*"'] \end{tikzcd}$$

By Observation 2.2.6, pullback along p is a simplicially enriched functor that preserves tensors with simplicial sets, so by [11, 4.85] it follows that the adjunction $p^* \dashv \Pi_p$ is simplicially enriched.

The right adjoint may be described explicitly:

Lemma 3.0.1. *The n-simplices of the pushforward $\Pi_p q\colon \Pi_p F \to B$ of $q\colon F \to E$ correspond to pair comprised of an n-simplex $b\colon \Delta^n \to B$ together with a map $E_b \to F$ in $\underline{\mathrm{sSet}}_{/E}$, whose domain is defined by the pullback*

$$\begin{tikzcd} E_b \arrow[r, "e_b"] \arrow[d, "p_b"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"'] & B \end{tikzcd}$$

Moreover, a simplicial operator $\alpha\colon [m] \to [n]$ acts on an n-simplex by pre-composition with

$$\begin{tikzcd} E_{b\cdot\alpha} \arrow[r, "E_\alpha"] \arrow[d, "p_{b\cdot\alpha}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_b \arrow[d, "p_b"] \\ \Delta^m \arrow[r, "\alpha"'] & \Delta^n \end{tikzcd} \qquad \square$$

To study the pushforward construction along a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$, we will replace the test objects $e_b\colon \mathsf{E}_b \to \mathsf{E}$ involved the description of Π_p given in Lemma 3.0.1 by weakly equivalent test objects in the Joyal model structure. These new test objects $\tilde{e}_b\colon \tilde{E}_b \to \mathsf{E}$ will arise as certain weighted colimits of a fixed diagram $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{QC}at \subseteq \mathcal{SS}et$, namely the straightening of the cocartesian fibration p defined using the comprehension construction.

The construction of the replacement to the pullback functor is given in §3.1, and the proof that the pullback replacement is equivalent to the strict pullback is given in §3.2.

3.1 A replacement for pullback along a cocartesian fibration

Notation 3.1.1. For the remainder of this section shall fix a cocartesian fibration of quasi-categories $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ as well as a corresponding comprehension functor

$$\mathfrak{C}\mathsf{B} \xrightarrow{\quad c_p \quad} \mathcal{QC}at,$$

the "straightening" of the cocartesian fibration p. We also fix the associated lifted lax cocone with nadir E described in Theorem 2.3.9:

$$\begin{tikzcd}
 & \mathbb{1} + \mathfrak{C}\mathsf{B} \arrow[dl, hook'] \arrow[dr, "{\langle \mathsf{E}, c_p \rangle}"] & \\
\mathfrak{C}[\mathsf{B} \star \Delta^0] \arrow[rr, "\ell^{\mathsf{E}}"'] & & \mathcal{QC}at
\end{tikzcd} \tag{3.1.2}$$

Our aim is to prove that this lax cocone is a colimit cocone. We will achieve this as the $b = \mathrm{id}_{\mathsf{B}}$ special case of our first main theorem:

Theorem 3.1.3. *For any cocartesian fibration* $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ *and any* $b\colon X \to \mathsf{B}$*, the comprehension cocone induces a canonical map over* E *from the oplax colimit of the diagram*

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{QC}at$$

to the fiber

$$\begin{tikzcd}
E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\
X \arrow[r, "b"'] & \mathsf{B}
\end{tikzcd}$$

and this map is a natural weak equivalence in the Joyal model structure.

Before proving this result, we tighten up its statement. As we explain presently, there is a functor $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ that acts on objects by carrying a generalized element $b\colon X \to \mathsf{B}$ to a canonical map $\mathrm{colim}^{\mathrm{oplax}}(c_p \circ \mathfrak{C}b) \to \mathsf{E}$. After defining this more formally, we construct a comparison natural transformation

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[r, bend left, "\tilde{p}^*"] \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\Downarrow\gamma"] & \underline{\mathrm{sSet}}_{/\mathsf{E}}
\end{tikzcd} \tag{3.1.4}$$

Theorem 3.1.3 asserts that this map is a componentwise Joyal equivalence. We first describe the action of the functor $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ on objects before establishing the functoriality of this construction. Recall from Remark 2.3.12 that the comprehension functor $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{QC}at$ can be used to define a "straightening" of its pullbacks:

$$\mathfrak{C}X \xrightarrow{\quad \mathfrak{C}b \quad} \mathfrak{C}\mathsf{B} \xrightarrow{\quad c_p \quad} \mathcal{SS}et$$

even in the case where X is not a quasi-category.

Definition 3.1.5. Given an generalized element $b\colon X \to \mathsf{B}$ in $\underline{\mathrm{sSet}}_{/\mathsf{B}}$, define a simplicial set

$$\tilde{E}_b := \mathrm{colim}^{\mathrm{oplax}} \left(\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{SS}et \right).$$

The oplax colimit $\tilde{\mathsf{E}}_b$ is the nadir of the universal lax cocone under the diagram $c_p \circ \mathfrak{C}X$. This is the weighted colimit weighted by the weight for oplax colimits L_X introduced in Definition 2.1.14.

The simplicial functor

$$\begin{tikzcd}
 & \mathbb{1} + \mathfrak{C}X \arrow[rr, "\mathbb{1}+\mathfrak{C}b"] \arrow[dl, hook'] & & \mathbb{1} + \mathfrak{C}\mathsf{B} \arrow[dl, hook'] \arrow[dr, "\langle E{,}\mathsf{E}\rangle"] & \\
\mathfrak{C}[X \star \Delta^0] \arrow[rr, "\mathfrak{C}[b\star\mathrm{id}]"'] & & \mathfrak{C}[\mathsf{B} \star \Delta^0] \arrow[rr, "\ell^{\mathsf{E}}"'] & & \mathcal{S}\mathit{Set}
\end{tikzcd} \tag{3.1.6}$$

defines a lax cocone $\ell^{\mathsf{E}}|_b \colon \mathfrak{C}[X \star \Delta^0] \to \mathcal{S}\mathit{Set}$ under the diagram $c_p \circ \mathfrak{C}b$ with nadir E, inducing a unique simplicial map $\tilde{e}_b \colon \tilde{E}_b \to \mathsf{E}$ from the oplax colimit. This constructs an object of $\mathcal{S}\mathit{Set}_{/\mathsf{E}}$.

To establish the functoriality of this construction, it will be convenient to re-express the oplax colimits of Definition 3.1.5.

Lemma 3.1.7. *For any simplicial map $b \colon X \to B$ define a weight $L_b \colon \mathfrak{C}\mathsf{B}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ by taking the left Kan extension along $\mathfrak{C}b \colon \mathfrak{C}X^{\mathrm{op}} \to \mathfrak{C}\mathsf{B}^{\mathrm{op}}$ of the weight for oplax colimits.*

(i) Then for any diagram $F \colon \mathfrak{C}\mathsf{B} \to \mathcal{S}\mathit{Set}$, there is an isomorphism

$$\mathrm{colim}^{\mathrm{oplax}}\,(F \circ \mathfrak{C}b) := \mathrm{colim}^{L_X}\,(F \circ \mathfrak{C}b) \cong \mathrm{colim}^{L_b}\,F.$$

(ii) The weight $L_b \colon \mathfrak{C}\mathsf{B}^{\mathrm{op}} \to \mathcal{S}\mathit{Set}$ is flexible and its collage is given by the pushout

$$\begin{tikzcd}
\mathbb{1} + \mathfrak{C}X \arrow[r, "\mathbb{1}+\mathfrak{C}b"] \arrow[d, hook] & \mathbb{1} + \mathfrak{C}\mathsf{B} \arrow[d, hook] \arrow[dl, phantom, "\ulcorner" very near end] \\
\mathfrak{C}[X \star \Delta^0] \arrow[r] & \mathfrak{C}((X \star \Delta^0) \underset{X+\Delta^0}{\cup} (B + \Delta^0)) \cong \mathrm{coll}\,L_b
\end{tikzcd}$$

Proof. Statement (i) and the second part of (ii) follow from Lemma 2.1.6. Since $\mathrm{coll}(L_b)$ is the homotopy coherent realization of the pushout of simplicial sets, Proposition 2.1.12 tells us that it is a simplicial computad and thus, by Proposition 2.1.11, L_b is a flexible weight. □

Observation 3.1.8. The utility of Lemma 3.1.7 is as follows. Suppose now that we have a map

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "b"'] & & Y \arrow[dl, "c"] \\
 & \mathsf{B} &
\end{tikzcd}$$

in the slice category $\underline{\mathrm{sSet}}_{/\mathsf{B}}$. This gives rise to a commutative diagram of simplicial computads

$$\begin{tikzcd}
\mathfrak{C}[X \star \Delta^0] \arrow[d, "\mathfrak{C}u^{\triangleright}"'] & \mathbb{1} + \mathfrak{C}X \arrow[l, hook'] \arrow[r, "\mathbb{1}+\mathfrak{C}b"] \arrow[d, "\mathbb{1}+\mathfrak{C}u"'] & \mathbb{1} + \mathfrak{C}\mathsf{B} \arrow[d, equal] \\
\mathfrak{C}[Y \star \Delta^0] & \mathbb{1} + \mathfrak{C}Y \arrow[l, hook'] \arrow[r, "\mathbb{1}+\mathfrak{C}c"'] & \mathbb{1} + \mathfrak{C}\mathsf{B}
\end{tikzcd} \tag{3.1.9}$$

inducing a simplicial computad morphism $\mathrm{coll}(L_b) \to \mathrm{coll}(L_c)$ in the category ${}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cptd}}$.

This construction is functorial, defining the horizontal functor in the following square

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[r, "\mathrm{coll}\, L_\bullet"] \arrow[d, dashed, "L_\bullet"'] & {}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cptd}} \arrow[d, hook] \\
\underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}} \arrow[r, hook, "\mathrm{coll}"] & {}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}
\end{tikzcd}$$

By the description of the essential image of the collage functor given in Proposition 2.1.4, we see that $\mathrm{coll}\, L_\bullet$ factors as indicated defining a functor $L_\bullet \colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to {}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$.

Finally note that the collage $\mathrm{coll}\, L_{\mathsf{B}^{\mathrm{op}}} \cong \mathfrak{C}\mathsf{B} \star \Delta^0$ for the weight for oplax colimits of shape B defines a cone under the pushout diagram of Lemma 3.1.7(ii). Thus the codomain of the functor $\mathrm{coll}(L_\bullet)$ lifts to the slice category

$$\underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{\mathrm{coll}\, L_\bullet} \left({}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cptd}}\right)_{/\mathfrak{C}\mathsf{B}\star\Delta^0}$$

Correspondingly, by the fully faithfulness of the collage construction, we can equally regard $L_\bullet$ as a functor

$$\underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{L_\bullet} \left(\underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}}\right)_{/L_{\mathsf{B}}}$$

landing in the full subcategory spanned by the flexible weights.

Observation 3.1.8 allows us to extend Definition 3.1.5 to a functor.

Definition 3.1.10. Define $\tilde{p}^* \colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ to be the composite functor

$$\tilde{p}^* := \quad \underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{L_\bullet} \left(\underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}}\right)_{/L_{\mathsf{B}^{\mathrm{op}}}} \xrightarrow{\mathrm{colim}^{-} c_p} \underline{\mathrm{sSet}}_{/\tilde{\mathsf{E}}_{\mathsf{B}}} \xrightarrow{\tilde{e}_{\mathsf{B}}} \underline{\mathrm{sSet}}_{/\mathsf{E}}$$

where $\tilde{\mathsf{E}}_{\mathsf{B}} := \mathrm{colim}^{\mathrm{oplax}}\, c_p$ and $\tilde{e}_{\mathsf{B}} \colon \tilde{\mathsf{E}}_{\mathsf{B}} \to \mathsf{E}$ is the map induced by the lax cocone (3.1.2).

For later use, we record a few properties of the functor just constructed.

Lemma 3.1.11. *The functor* $\tilde{p}^* \colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ *preserves colimits.*

Proof. In Definition 3.1.10 the functor under consideration is defined as a composite of three functors, the latter two of which manifestly preserve colimits. Since colimits in a slice category over an object are created by the forgetful functor, it remains only to prove that the functor $L_\bullet \colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}}$ preserves colimits. Since Proposition 2.1.4 demonstrates that the inclusion $\underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}} \hookrightarrow {}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$ is full and coreflective, to show that this functor preserves colimits, it suffices to show that

$$\underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{\mathrm{coll}\, L_\bullet} {}^{\mathbb{1}+\mathfrak{C}\mathsf{B}/}\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$$

preserves them.

By Observation 3.1.8, the action of this functor on objects and morphisms is defined by the pushout of Lemma 3.1.7(ii), which we regard as a diagram in $^{\mathbb{1}}/\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$. The functors

$$\underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{\mathbb{1}+\mathfrak{C}(-)} {}^{\mathbb{1}}/\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}} \qquad \text{and} \qquad \underline{\mathrm{sSet}}_{/\mathsf{B}} \xrightarrow{\mathfrak{C}(-)\star\Delta^0} {}^{\mathbb{1}}/\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$$

both preserve colimits. Thus, the functor from $\underline{\mathrm{sSet}}_{/\mathsf{B}}$ to the category of pushout diagrams in $^{\mathbb{1}}/\underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$ with one vertex fixed at $\mathbb{1} + \mathfrak{C}\mathsf{B}$ preserves colimits. The pushout preserves colimits as well so we conclude that $\mathrm{coll}(L_\bullet)$ and hence $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ preserves colimits, as desired. ☐

Lemma 3.1.12. *The functor* $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ *preserves monomorphisms.*

Proof. From the Definition 3.1.10, to see that $\tilde{p}^*$ preserves monomorphisms

$$\begin{tikzcd} X \arrow[rr, hook, "u"] \arrow[dr, "b"'] & & Y \arrow[dl, "c"] \\ & \mathsf{B} & \end{tikzcd}$$

over B it suffices to show that the composite of the functors $L_\bullet$ and $\mathrm{colim}_{\mathfrak{C}\mathsf{B}}(-, c_p)$ preserve monomorphisms. To do so, we'll prove that the comparison functor $L_u\colon L_b \to L_c$ between weights in $\underline{\mathrm{sSet}}^{\mathfrak{C}\mathsf{B}^{\mathrm{op}}}$ is a relative projective cell complex, as defined in 2.1.8. A theorem of Gambino [7] implies that $\mathrm{colim}_{\mathfrak{C}\mathsf{B}}(-, c_p)$ carries relative projective cell complexes to monomorphisms in simplicial sets; see also [14, 11.5.1]

Recall that the weight L_b is constructed as a collage defined by a pushout, which is the homotopy coherent realization of a pushout of simplicial sets. The natural transformation L_u is encoded by the map between collages constructed as the pushout (3.1.9); again this map is the homotopy coherent realization of a map of simplicial sets. Since the left-hand horizontal inclusions are also simplicial subcomputad inclusions, it follows from the standard argument that the induced map $\mathrm{coll}(L_u)\colon \mathrm{coll}(L_b) \hookrightarrow \mathrm{coll}(L_c)$ between the pushouts is a simplicial subcomputad inclusion and by the relative analogue Proposition II.5.3.5 of Proposition 2.1.11, $L_u\colon L_b \to L_c$ is a relative projective cell complex, as desired. ☐

3.2 Comparison with the strict pullback

Now that we've precisely defined a functor $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ that carries a generalized element to the oplax colimit of the restricted comprehension functor, our next task is to define the natural transformation (3.1.4) alluded to in the statement of Theorem 3.1.3. To explain the existence of the natural map $\gamma_b\colon \tilde{E}_b \to E_b$ for $b\colon X \to \mathsf{B}$, recall that the p-cocartesian lifts with codomain E used to define the action of $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{QC}at$ on arrows in the image of $\mathfrak{C}b\colon \mathfrak{C}X \to \mathfrak{C}\mathsf{B}$ lie over arrows with codomain B which have a given factorisation through $b\colon X \to \mathsf{B}$. This is depicted in the following diagram by the arrow $b\gamma$ and its p-cocartesian lift

$\ell^{\mathsf{E}}_{b\gamma}$:

$$\begin{tikzcd}
\mathsf{E}_{bx} \arrow[dr, "e_{b\gamma}"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddd, two heads, "p_{bx}"'] \arrow[ddrrr, bend left, "\ell^{\mathsf{E}}_{bx}", name=L1] \arrow[ddrr, dotted, bend left, name=D1] & & & & \\
& \mathsf{E}_{bx'} \arrow[drrr, "\ell^{\mathsf{E}}_{bx'}", name=L2] \arrow[dr, dotted, name=D2] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dddd, two heads, "p_{bx'}"'] & & & \\
& & E_b \arrow[rr, "\ell^{\mathsf{E}}_b"'] \arrow[rr, phantom, very near start, "\lrcorner"] \arrow[dd, two heads, "p_b"] & & \mathsf{E} \arrow[dd, two heads, "p"] \\
\Delta^0 \arrow[drr, "x", name=X1] \arrow[ddr, equal] & & & & \\
& & X \arrow[rr, "b"'] & & \mathsf{B} \\
& \Delta^0 \arrow[ur, "x'"', name=X2] & & &
\arrow[Rightarrow, from=L1, to=L2, "\ell^{\mathsf{E}}_{b\gamma}"]
\arrow[Rightarrow, dotted, from=D1, to=D2]
\arrow[Rightarrow, from=X1, to=X2, "\gamma"]
\end{tikzcd}$$

From the diagram, it is clear that such p-cocartesian arrows factor through $e_b\colon E_b \to \mathsf{E}$ to give the dotted arrows with codomain E_b as drawn; this is the main component of the proof of Lemma 2.3.6. This idea is formalized as follows:

Lemma 3.2.1. *For any $b\colon X \to \mathsf{B}$, the diagram $c_p \circ \mathfrak{C}b$ is the base of a lax cocone with nadir E_b*

$$\begin{tikzcd}
& \mathbb{1} + \mathfrak{C}X \arrow[dl, hook'] \arrow[dr, "{\langle E_b, c_p \circ \mathfrak{C}b\rangle}"] & \\
\mathfrak{C}[X \star \Delta^0] \arrow[rr, "\ell^E"'] & & \mathcal{S}\mathit{Set}
\end{tikzcd}$$

Hence, the universal property of the oplax colimit defines a natural map $\tilde{E}_b \to E_b$ over E.

Proof. Apply Lemma 2.3.6 to the cocartesian cone of Theorem 2.3.9 as described in Remark 2.3.8. ☐

To prove Theorem 3.1.3 we must verify that (3.1.4) is a componentwise Joyal weak equivalence. We first demonstrate this for generalized elements $b\colon \Delta^n \to B$ whose domains are simplices and then use the results of Lemmas 3.1.11 and 3.1.12 to extend these results to the general case.

Example 3.2.2. By definition $\tilde{p}^*(b\colon \Delta^0 \to \mathsf{B})$ is the oplax colimit of the diagram

$$\mathfrak{C}\Delta^0 \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{Q}\mathit{Cat}$$

that sends the unique object to the fiber E_b of $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ over $b\colon \Delta^0 \to \mathsf{B}$. The weight for lax cocones of shape Δ^0 is the terminal weight so the weighted colimit is just the ordinary colimit of this one object diagram. Thus $\tilde{p}^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ sends $b\colon \Delta^0 \to \mathsf{B}$ to $\mathsf{E}_b \to \mathsf{E}$, which is isomorphic to the strict pullback $p^*(b\colon \Delta^0 \to \mathsf{B})$.

For $b\colon \Delta^1 \to \mathsf{B}$, $\tilde{p}^*(b\colon \Delta^1 \to \mathsf{B})$ is the oplax colimit of the diagram

$$\mathfrak{C}\Delta^1 \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{Q}\mathit{Cat}$$

whose image is diagram $e_b \colon \mathsf{E}_{b_0} \to \mathsf{E}_{b_1}$ of quasi-categories constructed in (3.2.4). In this case, the oplax colimit has a simple description: it is given by an "mapping cylinder" formed by attaching E_{b_1} along the codomain edge of the cylinder $\mathsf{E}_{b_0} \times \Delta^1$ via the map e_b. We now show this simplicial set is Joyal weak equivalent to the strict fiber E_b.

Proposition 3.2.3. *The data formed by applying the comprehension construction*

$$\begin{tikzcd}
\mathsf{E}_{b_0} \arrow[drr, bend left, "\ell_0^{\mathsf{E}}"] \arrow[ddr, dotted, "e_b"'] \arrow[ddd, "p_0"'] & & & \\
& & \mathsf{E}_b \arrow[r] \arrow[dd, two heads, "p_b"] \arrow[ddr, phantom, "\lrcorner" very near start] & \mathsf{E} \arrow[dd, two heads, "p"] \\
& \mathsf{E}_{b_1} \arrow[ur, "\ell_1^{\mathsf{E}}"'] \arrow[dd, two heads, "p_1"'] \arrow[ddr, phantom, "\lrcorner" very near start] \arrow[u, phantom, "\overset{\chi}{\Leftarrow}"] & & \\
1 \arrow[rr, "0"] \arrow[dr, equal] & & \Delta^1 \arrow[r, "b"] & \mathsf{B} \\
& 1 \arrow[ur, "1"'] \arrow[u, phantom, "\overset{\kappa}{\Leftarrow}"] & &
\end{tikzcd} \tag{3.2.4}$$

to the pullback $p_b \colon \mathsf{E}_b \to \Delta^1$ of $p \colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ along $b \colon \Delta^1 \to \mathsf{B}$ induces a Joyal weak equivalence

$$\begin{tikzcd}
\mathsf{E}_{b_0} \arrow[r, "e_b"] \arrow[d, hook, "\mathrm{id} \times \delta^0"'] & \mathsf{E}_{b_1} \arrow[d, hook] \arrow[ddr, bend left, "\ell_1^{\mathsf{E}}"] & \\
\mathsf{E}_{b_0} \times \Delta^1 \arrow[r] \arrow[drr, bend right, "\chi"'] & \tilde{\mathsf{E}}_b \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "\gamma_b", "\sim"'] & \\
& & \mathsf{E}_b
\end{tikzcd}$$

Proof. By Proposition 2.2.11, the oplax colimit $\tilde{\mathsf{E}}_b$ is weakly equivalent to the quasi-categorical collage $\mathrm{coll}(e_b, \mathsf{E}_{b_1})$, introduced in Definition 2.2.8. Moreover, Proposition 2.2.11 demonstrates that the equivalence $k \colon \tilde{\mathsf{E}}_b \xrightarrow{\sim} \mathrm{coll}(e_b, \mathsf{E}_{b_1})$ is inner anodyne. In particular, there exists a lift

$$\begin{tikzcd}
\tilde{\mathsf{E}}_b \arrow[r, "\gamma_b"] \arrow[d, tail, "k"', "\wr"] & \mathsf{E}_b \arrow[d, two heads, "p_b"] \\
\mathrm{coll}(e_b, \mathsf{E}_{b_1}) \arrow[r, "\rho"'] \arrow[ur, dashed, "\ell"'] & \Delta^1
\end{tikzcd}$$

defining a direct comparison map $\ell \colon \mathrm{coll}(e_b, \mathsf{E}_{b_1}) \to \mathsf{E}_b$ over Δ^1.

To prove that ℓ is an equivalence, observe by Lemma 2.2.10 and Proposition 2.2.5 that ρ and p_b are both cocartesian fibrations. Hence, Proposition 2.2.2 tells us that if ℓ is a cartesian functor, then to demonstrate that ℓ is an equivalence, we need only show that it restricts to an equivalence on the fibers over 0 and 1. Indeed, ℓ is an isomorphism on both fibers, so now our only remaining task is to demonstrate that it is a cartesian functor.

The proof of Lemma 2.2.10 reveals that the non-degenerate cocartesian edges of $\mathrm{coll}(e_b, \mathsf{E}_{b_1})$ are those represented by the degenerate edge of some vertex lying over the non-degenerate 1-simplex in Δ^1. Such edges lie in the image of the functor $\mathsf{E}_{b_0} \times \Delta^1 \to \mathsf{E}_b$ used to define the map $\tilde{E}_b \to \mathsf{E}_b$, and this functor in turn is defined to be a representative for the cocartesian lift of the 1-arrow between the two objects of Δ^1 to a map with domain $\mathsf{E}_{b_0} \to \mathsf{E}_b$. In particular, it defines a cocartesian cylinder in the sense of Lemma VI.3.2.4, which tells us that its components indexed by vertices of E_{b_0} are p-cocartesian 1-arrows. This proves that ℓ carries ρ-cocartesian arrows to p-cocartesian arrows, and thus ℓ is an equivalence. □

We argue inductively that $\gamma_b \colon \tilde{\mathsf{E}}_b \to \mathsf{E}_b$ is an equivalence for any n-simplex $b \colon \Delta^n \to \mathsf{B}$ under the assumption that this is true for simplices of lower dimension. Our strategy mirrors that adopted for the 1-simplex: we construct a quasi-categorical model for the oplax colimit of a homotopy coherent diagram $c_p \circ \mathfrak{C}b \colon \mathfrak{C}\Delta^n \to \mathcal{QC}at$, i.e., a quasi-category equivalent to the simplicial set $\tilde{\mathsf{E}}_b$ defined as the oplax colimit of $c_p \circ \mathfrak{C}b$, and then show that this is equivalent to the strict pullback E_b. The inductive step makes use of the following weights.

Notation 3.2.5 (weights for the inductive comparison)**.** To compare the weights $L_{\Delta^{n-1}}$ and L_{Δ^n} for oplax colimits of a homotopy coherent $n-1$-simplex and n-simplex, we left Kan extend the former along the inclusion $\delta^n \colon (\mathfrak{C}\Delta^{n-1})^{\mathrm{op}} \hookrightarrow (\mathfrak{C}\Delta^n)^{\mathrm{op}}$, writing $L_{\Delta^{n-1}} \colon (\mathfrak{C}\Delta^n)^{\mathrm{op}} \to \underline{\mathrm{sSet}}$ for the left Kan extension of $L_{\Delta^{n-1}}$. Explicitly, this weight is defined by

$$\begin{array}{ccc} L_{\Delta^{n-1}} \colon (\mathfrak{C}\Delta^n)^{\mathrm{op}} & \longrightarrow & \mathcal{S}Set \\ i & \mapsto & \begin{cases} \mathsf{Fun}_{\mathfrak{C}\Delta^n}(i,n) & i < n \\ \emptyset & i = n \end{cases} \end{array}$$

Let Y^n denote the representable weight

$$\begin{array}{ccc} Y^n \colon (\mathfrak{C}\Delta^n)^{\mathrm{op}} & \longrightarrow & \mathcal{S}Set \\ i & \mapsto & \mathsf{Fun}_{\mathfrak{C}\Delta^n}(i,n) \end{array}$$

Note there is a natural inclusion $L_{\Delta^{n-1}} \hookrightarrow Y^n$ that is the identity in all components except the one indexed by the object $n \in \mathfrak{C}\Delta^n$.

Lemma 3.2.6.

(i) The following diagram defines a pushout of weights in $\mathcal{S}Set^{(\mathfrak{C}\Delta^n)^{\mathrm{op}}}$:

$$\begin{tikzcd} L_{\Delta^{n-1}} \arrow[r, hook] \arrow[d, "\mathrm{id} \times \delta^0"'] & Y^n \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\ L_{\Delta^{n-1}} \times \Delta^1 \arrow[r] & L_{\Delta^n} \end{tikzcd}$$

(ii) Let $F\colon \mathfrak{C}\Delta^n \to \mathcal{SSet}$ be a homotopy coherent diagram whose $L_{\Delta^{n-1}}$-weighted colimit is Joyal weakly equivalent to the simplicial set E_{n-1}. Then the oplax colimit of F is Joyal weakly equivalent to the pushout along a canonical map ι_n induced by the diagram F.

$$\begin{tikzcd} \mathsf{E}_{n-1} \arrow[r, "\iota_n"] \arrow[d, "\mathrm{id}\times\delta^0"'] & F_n \arrow[d] \\ \mathsf{E}_{n-1}\times\Delta^1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{E}_n \end{tikzcd}$$

Proof. The pushout in (i) can be verified componentwise at each $i \in \mathfrak{C}\Delta^n$ at which point this relationship is evident from the definitions.

The pushout of (ii) follows. If E_{n-1} is isomorphic to the $L_{\Delta^{n-1}}$-weighted colimit of F, then the pushout diagram of (ii) is obtained by applying the cocontinuous functor $\mathrm{colim}^{-}F$ to the pushout diagram of (i). In this case, the map ι_n has a natural explicit description. By Lemma 2.1.6, the $L_{\Delta^{n-1}}$-weighted colimit of F coincides with the oplax colimit of the restricted diagram $F \circ \mathfrak{C}(\delta^n)^{\mathrm{op}}$. The functor F itself defines a canonical lax cocone under this restricted diagram with nadir F_n. Hence there is a natural comparison ι_n from the $L_{\Delta^{n-1}}$-weighted colimit to F_n.

Observe from Proposition 2.1.11 and Lemma 2.1.16 that all of the weights appearing in (i) are flexible. Proposition 2.1.9 then demonstrates that the pushout being constructed is equivalence-invariant. □

This lemma provides the inductive step in the following computation:

Proposition 3.2.7. *For any simplex $b\colon \Delta^n \to \mathsf{B}$, the component $\gamma_b\colon \tilde{\mathsf{E}}_b \to \mathsf{E}_b$ from the oplax colimit to the strict pullback is a Joyal weak equivalence.*

Proof. The base cases for $n = 0$ and $n = 1$ appear as Example 3.2.2 and Proposition 3.2.3. For the induction step, suppose we have shown this is a componentwise weak equivalence for all $n-1$-simplices in B. By Lemma 2.1.6 and Notation 3.2.5, the $L_{\Delta^{n-1}}$-weighted colimit of the diagram $\mathfrak{C}\Delta^n \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \underline{\mathrm{sSet}}$ is isomorphic to the oplax weighted colimit of the restricted diagram

$$\mathfrak{C}\Delta^{n-1} \xrightarrow{\mathfrak{C}\delta^n} \mathfrak{C}\Delta^n \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \underline{\mathrm{sSet}}.$$

By the inductive hypothesis, this weighted colimit $\tilde{\mathsf{E}}_{b\cdot\delta^n}$ is weakly equivalent to the pullback $\mathsf{E}_{b\cdot\delta^n}$. By Lemma 3.2.6, the diagram

$$\begin{tikzcd} \mathsf{E}_{b\cdot\delta^n} \arrow[r, "\iota_n"] \arrow[d, "\mathrm{id}\times\delta^0"'] & \mathsf{E}_{b_n} \arrow[d] \\ \mathsf{E}_{b\cdot\delta^n}\times\Delta^1 \arrow[r] & \tilde{\mathsf{E}}_b \end{tikzcd}$$

is then a pushout up to Joyal weak equivalence. So it follows from Proposition 2.2.11 that $\tilde{\mathsf{E}}_b$ is equivalent to the quasi-categorical collage $\mathrm{coll}(\iota_n, \mathsf{E}_{b_n})$, and as in the proof of Proposition 3.2.3, the map γ_b factors to define a map

$$\begin{tikzcd}
\mathrm{coll}(\iota_n, \mathsf{E}_{b_n}) \arrow[rr, dashed, "\ell"] \arrow[dr, two heads, "\rho"'] & & \mathsf{E}_b \arrow[dl, "\pi_\ell \cdot b"] \\
& \Delta^1 &
\end{tikzcd}$$

in this case involving the map $\pi_\ell \colon \Delta^n \to \Delta^1$ that carries every element but the last one to 0. Observe that π_ℓ a cocartesian fibration, and indeed a bifibration, as it is covariantly represented by the functor $! \colon [n-1] \to [0]$, which admits both left and right adjoints; see Corollary 2.2.12.

Our task, again, is to show that ℓ is an equivalence. By Lemma 2.2.10 and the fact that cocartesian fibrations compose, it is a functor between cocartesian fibrations. Moreover, ℓ is bijective on the fibers over $0, 1 \in \Delta^1$, the latter being E_{b_n} in both cases and the former being $\mathsf{E}_{b \cdot \delta^n}$. As in the proof of Proposition 3.2.3, ℓ is a cartesian functor, so Proposition 2.2.2 implies that ℓ is an equivalence, as desired. □

Combining the work in this section, we can finally prove our main result.

Proof of Theorem 3.1.3. Our task is to demonstrate that the canonical natural transformation

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[r, bend left, "\tilde{p}^*"{name=U}] \arrow[r, bend right, "p^*"'{name=D}] & \underline{\mathrm{sSet}}_{/\mathsf{E}}
\arrow[from=U, to=D, phantom, "\Downarrow\gamma"]
\end{tikzcd}$$

is a componentwise Joyal weak equivalence using the result of Proposition 3.2.7, which demonstrates that this is the case for the simplices $b \colon \Delta^n \to \mathsf{B}$ of B.

The category $\underline{\mathrm{sSet}}_{/\mathsf{B}}$ is equivalent to the category $\underline{\mathrm{sSet}}^{\mathrm{el}\,\mathsf{B}^{\mathrm{op}}}$ of presheaves indexed by the category $\mathrm{el}\,\mathsf{B}$ of simplices of B; its objects are simplices $b \colon \Delta^n \to \mathsf{B}$ and a morphism from b to $c \colon \Delta^m \to \mathsf{B}$ is a simplicial operator $\alpha \colon \Delta^n \to \Delta^m$ so that $c \cdot \alpha = b$. The representable presheaves generate $\underline{\mathrm{sSet}}^{\mathrm{el}\,\mathsf{B}^{\mathrm{op}}}$ under colimits, and such colimits are preserved by both of the functors p^* and $\tilde{p}^*$, the former case because of the right adjoint Π_p that exists in the locally cartesian closed category $\underline{\mathrm{sSet}}$ and the latter case by Lemma 3.1.11. Under the equivalence $\underline{\mathrm{sSet}}^{\mathrm{el}\,\mathsf{B}^{\mathrm{op}}} \cong \underline{\mathrm{sSet}}_{/\mathsf{B}}$, these representables correspond to the objects $b \colon \Delta^n \to \mathsf{B}$ whose domain is a simplex. Proposition 3.2.7 verifies that the components of γ indexed by such objects are equivalences, which is the moral reason why γ is an equivalence at all objects.

To demonstrate this, note that $b \colon X \to \mathsf{B}$ is a colimit indexed by the category $\mathrm{el}\,X$ of its simplices $\Delta^n \xrightarrow{x} X \xrightarrow{b} \mathsf{B}$, i.e.,

$$(X \xrightarrow{b} \mathsf{B}) \cong \operatorname*{colim}_{\mathrm{el}\,X} (\Delta^n \to \mathsf{B}).$$

The map γ_b factors as

$$\gamma_b \colon \tilde{p}^*(\operatorname*{colim}_{\mathrm{el}\,X} \Delta^n \to \mathsf{B}) \cong \operatorname*{colim}_{\mathrm{el}\,X} \tilde{p}^*(\Delta^n \to \mathsf{B}) \longrightarrow \operatorname*{colim}_{\mathrm{el}\,X} p^*(\Delta^n \to \mathsf{B}) \cong p^*(\operatorname*{colim}_{\mathrm{el}\,X} \Delta^n \to \mathsf{B}),$$

so it remains only to show that this middle map, the colimit of the equivalences γ_{bx} indexed by the simplices of X, is itself an equivalence. The indexing category $\mathrm{el}\,X$ is a Reedy category, so if we can show that the two $\mathrm{el}\,X$-indexed diagrams are Reedy cofibrant and that the category $\mathrm{el}\,X$ has fibrant constants, then the pointwise equivalence between the diagrams will induce the desired equivalence between their colimits. To say that the Reedy category $\mathrm{el}\,X$ has fibrant constants means that for each element $x\colon \Delta^n \to X$, the category of elements of the covariant representable boundary functor $\partial\,\mathrm{el}\,X_x$ is either empty or connected. This category is empty just when x is non-degenerate and has a terminal object, and is connected in particular, when x is degenerate. So $\mathrm{el}\,X$ has fibrant constants and the colimit functor $(\underline{\mathrm{sSet}}_{/\mathsf{B}})^{\mathrm{el}\,X} \to \underline{\mathrm{sSet}}_{/\mathsf{B}}$ carries pointwise weak equivalences between Reedy cofibrant diagrams to weak equivalences.

To verify this Reedy cofibrancy, it suffices to show

(i) the canonical diagram $\mathrm{el}\,X \to \underline{\mathrm{sSet}}_{/\mathsf{B}}$ is Reedy cofibrant

(ii) $\tilde{p}^*$ and p^* preserve Reedy cofibrant objects.

Since $\tilde{p}^*$ and p^* preserve colimits, they in particular preserve latching objects, so for this second item it suffices to show that both functors also preserve monomorphisms. Here, the fact that the pullback functor p^* preserves monomorphisms is standard, and the fact that its replacement $\tilde{p}^*$ preserves monomorphisms was proven in Lemma 3.1.12.

So it remains only to prove (i), that is, to argue that the functor

$$\begin{array}{ccc} \mathrm{el}\,X & \longrightarrow & \underline{\mathrm{sSet}}_{/\mathsf{B}} \\ \\ x\colon \Delta^n \to X & \mapsto & bx\colon \Delta^n \to \mathsf{B} \end{array}$$

is Reedy cofibrant. The latching object associated to $x\colon \Delta^n \to X$ is the composite $\partial\Delta^n \hookrightarrow \Delta^n \xrightarrow{x} X \xrightarrow{b} \mathsf{B}$ and the latching map is the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ over B, which is obviously a monomorphism. This completes the proof. $\square$

Specializing Theorem 3.1.3 to the identity morphism on B, we have

Corollary 3.2.8. *The domain of a cocartesian fibration* $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ *is equivalent to the oplax colimit of the associated comprehension functor* $c_p\colon \mathfrak{C}\mathsf{B} \to \mathcal{Q}\mathit{Cat}$*, with colimit cocone:*

$$\begin{tikzcd} & \mathbb{1}+\mathfrak{C}\mathsf{B} \arrow[dl, hook'] \arrow[dr, "{\langle \mathsf{E}, c_p\rangle}"] & \\ \mathfrak{C}[\mathsf{B}\star\Delta^0] \arrow[rr, "\ell^{\mathsf{E}}"'] & & \mathcal{Q}\mathit{Cat} \end{tikzcd} \qquad \square$$

4. Pushforward along a cocartesian fibration

In this section, we shall fix a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ of quasi-categories and prove that the pushforward functor

$$\underline{\mathrm{sSet}}_{/\mathsf{E}} \xrightarrow{\Pi_p} \underline{\mathrm{sSet}}_{/\mathsf{B}}$$

has two properties that are relevant to the development of the category of theory of quasi-categories:

(i) The pushforward functor preserves *isofibrations*. In model categorical terminology, this implies that the adjunction

$$\begin{tikzcd} \underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\Pi_p"'] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

is Quillen with respect to slices of the Joyal model structure.

Moreover:

(ii) The pushforward functor preserves *cartesian fibrations* and *cartesian functors* between them.

In fact, both pullback and pushforward along p define *cosmological functors*, a property we briefly note for use in future work.

Both of the properties (i) and (ii) are more easily established for an alternate model of the pushforward functor defined as a right adjoint to the functor $\tilde{p}^* \colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$ introduced in §3. Theorem 3.1.3 demonstrates that the pullback $E_b \to \mathsf{E}$ of a functor $b \colon X \to \mathsf{B}$ along a cocartesian fibration $p \colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is computed, up to equivalence, as the oplax colimit of a particular diagram

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}\mathsf{B} \xrightarrow{c_p} \mathcal{Q}\mathcal{C}at$$

When the oplax colimit is defined strictly as a simplicial set it enjoys the universal property of Definition 2.1.1: maps $\mathrm{colim}^{\mathrm{oplax}}(c_p \circ \mathfrak{C}b) \to \mathsf{F}$ correspond to lax cocones under $c_p \circ \mathfrak{C}b$ with nadir F. This correspondence defines a right adjoint

$$\begin{tikzcd} \underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\tilde{\Pi}_p"'] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "\tilde{p}^*"'] \end{tikzcd}$$

characterized on an object $q \colon \mathsf{F} \to \mathsf{E}$ by the bijection

$$\begin{tikzcd} \Delta^n \arrow[rr] \arrow[dr, "b"'] & & \tilde{\Pi}_p\mathsf{F} \arrow[dl, "\tilde{\Pi}_p q"] & \arrow[r, phantom, "\leftrightsquigarrow"] & & \mathrm{colim}^{\mathrm{oplax}}(c_p \circ \mathfrak{C}b) \arrow[rr] \arrow[dr, "\tilde{p}^*(b) := \ell^E|_b"'] & & \mathsf{F} \arrow[dl, "q"] \\ & \mathsf{B} & & & & & \mathsf{E} & \end{tikzcd}$$

That is, n-simplices in $\tilde{\Pi}_p\mathsf{F}$ over $b \colon \Delta^n \to \mathsf{B}$ correspond to lax cocones under the homotopy coherent n-simplex $c_p \circ \mathfrak{C}b$ with nadir F whose whiskered composite with q recovers the restriction $\ell^{\mathsf{E}}|_b$ of the lax cocone produced by the comprehension construction of Theorem 2.3.9.

To make this simplex level construction of $\tilde{\Pi}_p q$ precise, we require a simplicial set whose n-simplices correspond to lax cocones under a homotopy coherent n-simplex in $\mathcal{QC}at$ with nadir F. One might think that the slice quasi-category $\mathsf{qCat}_{/\mathsf{F}}$ provides just such a gadget, where qCat is the quasi-category of quasi-categories defined by passing to the maximal Kan complex enriched core and then applying the homotopy coherent nerve, but this isn't quite correct: since we've passed to the $(\infty, 1)$-categorical core of $\mathcal{QC}at$ before taking the homotopy coherent nerve, simplices in $\mathsf{qCat}_{/\mathsf{F}}$ correspond to *pseudo* cocones rather than lax cocones. The solution is to drop the core functor, in which case the homotopy coherent nerve $\mathsf{qCat}_2 := N\mathcal{QC}at$ is not a quasi-category but rather a *2-complicial set*, a type of marked simplicial set which is introduced in §4.1. Definition 4.1.3 introduces a slice construction for marked simplicial sets, which does *not* commute with the functor that forgets the markings, but this is a good thing. The marked slice $\mathsf{qCat}_{2/\!\!/\mathsf{F}}$ has exactly the property we desire, in that its n-simplices correspond to lax cocones under a homotopy coherent n-simplex in $\mathcal{QC}at$ with nadir F.

In §4.2, we describe $\tilde{\Pi}_p q$ explicitly as the pullback of a map between lax slices of the homotopy coherent nerve of $\mathcal{QC}at$ defined by "whiskering with q." After establishing the properties (i) and (ii) for $\tilde{\Pi}_p$, we use the natural Joyal equivalence $\gamma\colon \tilde{p}^* \Rightarrow p^*$ to transfer these properties to the pushforward functor Π_p. Having established that pullback and pushforward along a cocartesian fibration both preserve cartesian fibrations, in §4.3, we construct a closely related exponentiation operation $(q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})^{p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}}$ of a cartesian fibration q by a cocartesian fibration p with the same codomain. These exponentials are used in §5 to establish the comonadicity and monadicity of the quasi-category of cartesian fibrations over B over the quasi-category of $\mathrm{ob}\,\mathsf{B}$-indexed families of quasi-categories.

4.1 2-complicial sets

We know from Cordier and Porter [5, 6] that the homotopy coherent nerve of a Kan complex enriched category is itself a quasi-category. But when we apply the homotopy coherent nerve to a quasi-category enriched category, such as $\mathcal{QC}at$ itself, it is *not* the case that these nerves are quasi-categories. Since the hom-spaces of a quasi-category enriched category contain 1-simplices that are not invertible, its homotopy coherent nerve contains 2-simplices which are not invertible. A homotopy coherent nerve of this kind is most naturally regarded as possessing the structure of a 2-*complicial set*.

We leave the precise definition to the original sources [26, 27] or to more recent expository accounts such as [15] or §X.D.1 and instead present an overview of the main ideas. Extending the terminology used by Lurie in [12, §3.1], a *marked simplicial set* is a simplicial set equipped with a chosen subset of *marked simplices*, which must be positive-dimensional and contain all degeneracies. Maps of marked simplicial sets preserve the markings. A *complicial set* is a marked simplicial set with the right lifting property with respect to certain marked horn inclusions—including both inner and outer marked horns—as well as certain marking extensions. A complicial set is *saturated* if "all n-equivalences are marked," where the notion of n-equivalence is defined relative to the collection of marked $(n+1)$-simplices. An n-*complicial set* is a saturated complicial set in which all simplices above dimension n are marked.[7] We refer to maps between

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[7] One might think of the n-complicial sets as being a model for the theory of (∞, n)-categories, although we will

complicial sets of any of the varieties just introduced that have the right lifting property with respect to the marked horn inclusions as *isofibrations*.

For our purposes here, we note the following examples.

Example 4.1.1 (Kan complexes and quasi-categories as complicial sets)**.** The 0-complicial sets are precisely the Kan complexes with all positive-dimensional simplices marked and an isofibration between such is just a Kan fibration.

The 1-complicial sets are precisely the quasi-categories with their *natural markings*—in which all 1-dimensional isomorphisms and all higher simplices are marked—and the isofibrations between such coincide with the isofibrations between quasi-categories. In this setting, the outer horn lifting property of 1-complicial sets and isofibrations between them is typically referred to as "special outer horn" lifting; see [9, 1.3] or Proposition X.D.4.6.

Example 4.1.2. Suppose that $\mathcal{K}$ is a quasi-category enriched category. Its homotopy coherent nerve $N\mathcal{K}$ has:

- 0**-simplices** corresponding to the objects a of $\mathcal{K}$,
- 1**-simplices** corresponding to 0-arrows $f\colon a_0 \to a_1$,
- 2**-simplices** corresponding to diagrams

$$\begin{tikzcd} a_0 \arrow[rr, "f_{02}"] \arrow[dr, "f_{01}"'] & {} \arrow[d, phantom, "\Downarrow\alpha"] & a_2 \\ & a_1 \arrow[ur, "f_{12}"'] & \end{tikzcd}$$

where α is a 1-arrow in the hom-space $\mathsf{Fun}_{\mathcal{K}}(a_0, a_2)$ with source f_{02} and target $f_{12} \circ f_{01}$.

Now we define the *natural marking* of the homotopy coherent nerve by marking:

(i) all n-simplices with $n > 2$,

(ii) those 2-simplices, as depicted above, for which α is an invertible arrow in the quasi-category $\mathsf{Fun}_{\mathcal{K}}(a_0, a_2)$, and

(iii) each 1-simplex $f\colon a_0 \to a_1$ which is an *equivalence*, in the sense that it possesses an *equivalence inverse* $f'\colon a_1 \to a_0$ witnessed by a pair of invertible 1-arrows

$$\begin{tikzcd} a_0 \arrow[rr, equal, "\mathrm{id}"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\sim"] & a_0 & a_1 \arrow[rr, equal, "\mathrm{id}"] \arrow[dr, "f'"'] & {} \arrow[d, phantom, "\Downarrow\sim"] & a_1 \\ & a_1 \arrow[ur, "f'"'] & & & a_0 \arrow[ur, "f"'] & \end{tikzcd}$$

in the quasi-categories $\mathsf{Fun}_{\mathcal{K}}(a_0, a_0)$ and $\mathsf{Fun}_{\mathcal{K}}(a_2, a_2)$ respectively.

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not pursue that intuition here.

By [26, Theorem 40], the naturally marked homotopy coherent nerve $\mathsf{K}_2 := N\mathcal{K}$ is a 2-complicial set.

Definition 4.1.3 (joins and slices of marked simplicial sets)**.** The join operation extends to marked simplicial sets as follows. Concretely, the join $X \star Y$ of two marked augmented simplicial sets X and Y has as its simplices pairs (x, y) with $x \in X$ and $y \in Y$ of arbitrary dimension with $\dim(x, y) = \dim(x) + \dim(y) + 1$, where the convention is to augment a marked simplicial set with a single -1-simplex. We declare that a simplex $(x, y) \in X \star Y$ is marked if x is marked in X or y is marked in Y.

Now consider a map of marked simplicial sets $f\colon X \to Y$. The slice $Y_{/\!\!/ f}$ is the simplical set of whose n-simplices are maps $g\colon \Delta^n \star X \to Y$ which restrict on $X \subseteq \Delta^n \star X$ to the fixed map $f\colon X \to Y$. Such a simplex $g\colon \Delta^n \star X \to Y$ is marked if and only if it extends along the inclusion $\Delta^n \star X \subseteq \sharp\Delta^n \star X$—where $\sharp\Delta^n$ extends the minimally marked n-simplex Δ^n by marking the non-degenerate n-simplex—and this happens exactly when g maps every simplex $(\mathrm{id}_{[n]}, x)$ for $x \in X$ to a marked simplex in Y. A dual construction defines ${}^{f/\!\!/}Y$.

Suppose that A is a complicial set and that $f\colon X \to \mathsf{A}$ is any map of marked simplicial sets. As shown in [27], it is then the case that ${}^{f/\!\!/}\mathsf{A}$ and $\mathsf{A}_{/\!\!/ f}$ are also complicial sets and that the projections $r^f\colon {}^{f/\!\!/}\mathsf{A} \to \mathsf{A}$ and $r_f\colon \mathsf{A}_{/\!\!/ f} \to \mathsf{A}$ are isofibrations of such.

4.2 The right adjoint to pullback

We now have all the tools we require to construct an alternate model of the pushforward functor along a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ whose value at any isofibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ will be equivalent to those of the strict pushforward. The alternate model for the pushforward

$$\tilde{\Pi}_p q\colon \tilde{\Pi}_p \mathsf{F} \to \mathsf{B}$$

of an isofibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ along a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is defined as a pullback of a whiskering map for slices of homotopy coherent nerves that we now introduce.. Let $\mathcal{K}$ be a quasi-category enriched category such as $\mathcal{QC}at$, and write $\mathsf{K}_2 := N\mathcal{K}$ for its naturally marked homotopy coherent nerve, a 2-complicial set.

Lemma 4.2.1. *Let $q\colon F \to E$ be a 0-arrow in a quasi-category enriched category $\mathcal{K}$.*

(i) There is a functor of slice 2-complicial sets

$$\mathsf{K}_{2/\!\!/\mathsf{F}} \xrightarrow{q\circ-} \mathsf{K}_{2/\!\!/\mathsf{E}}$$

induced from the whiskering operation for lax cocones.

(ii) If $q\colon F \twoheadrightarrow E$ is a representably-defined isofibration, then $q \circ -\colon \mathsf{K}_{2/\!\!/\mathsf{F}} \twoheadrightarrow \mathsf{K}_{2/\!\!/\mathsf{E}}$ is a isofibration of complicial sets.

Proof. By the Yoneda lemma and the natural isomorphisms arising from the slice and homotopy coherent nerve adjunctions

$$\underline{\mathrm{sSet}}(X, \mathsf{K}_{2/\!\!/F}) \cong \underline{\mathrm{sSet}}_{\top\mapsto F}(X \star \Delta^0, \mathsf{K}_2) \cong \underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}_{\top\mapsto F}(\mathfrak{C}[X \star \Delta^0], \mathcal{K}),$$

to define the map in (i), it suffices to provide a natural operation that converts a lax cocone of shape X with nadir F into a lax cocone with shape X and nadir E. The whiskering operation for lax cocones described in Observation 2.3.3 defines such a natural transformation. Since whiskering preserves fibered equivalences and isomorphisms, which correspond to marked 1- and 2-simplices in $\mathsf{K}_{2/\!\!/F}$, this defines the desired map of 2-complicial sets.

For (ii), we must show that the map between the sliced complicial sets has the right lifting property with respect to the marked horn inclusions of [27, 15], which define lifting problems of underlying simplicial sets of the form

$$\begin{tikzcd}
\Lambda^{n,k} \arrow[r] \arrow[d, hook] & \mathsf{K}_{2/\!\!/F} \arrow[d, "q\circ-"] \\
\Delta^{n,k} \arrow[r] \arrow[ur, dashed] & \mathsf{K}_{2/\!\!/E}
\end{tikzcd}$$

for $n \geq 1$ and $0 \leq k \leq n$ with additional marking constraints that we describe below. By [27, Corollary 49], it suffices to consider the case $0 < k \leq n$. Here the bottom horizontal functor is given by a homotopy coherent $n+1$-simplex

$$\mathfrak{C}\Delta^{n+1} \to \mathcal{K}$$

that sends the first $n+1$ objects to $E_0, \ldots, E_n$ and the final object to $E \in \mathcal{K}$ and satisfies one additional condition forced by the markings on $\Delta^{n,k}$ and $\mathsf{K}_{2/\!\!/E}$. If $k < n$, then this functor must be defined so that the 1-simplex $\alpha \in \mathsf{Fun}(E_{k-1}, E_{k+1})$ is invertible.

$$\begin{tikzcd}
E_{k-1} \arrow[rr, "f_{k-1,k+1}"] \arrow[dr, "f_{k-1,k}"'] & \Downarrow\alpha & E_{k+1} \\
& E_k \arrow[ur, "f_{k,k+1}"'] &
\end{tikzcd}$$

If $k = n$, then the 1-simplex $\alpha \in \mathsf{Fun}(E_{n-2}, E_n)$ must be invertible and $f_{n-1,n} \colon E_{n-1} \to E_n$ must admit an equivalence inverse.

$$\begin{tikzcd}
E_{n-2} \arrow[rr, "f_{n-2,n}"] \arrow[dr, "f_{n-2,n-1}"'] & \Downarrow\alpha & E_n \\
& E_{n-1} \arrow[ur, "f_{n-1,n}"'] &
\end{tikzcd}$$

The δ^{n+1}-face of this homotopy coherent simplex and the top horizontal together define a simplicial functor

$$\mathfrak{C}\Lambda^{n+1,k} \to \mathcal{K}$$

that carries the $n+1$ objects to $E_0, \ldots, E_n, F$, respectively, and has the property that for each $0 \leq j \leq n$ the diagram of function complexes commutes:

$$\begin{tikzcd}
\mathsf{Fun}_{\mathfrak{C}\Lambda^{n+1,k}}(j, n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_j, F) \arrow[d, two heads, "q\circ-"] \\
\mathsf{Fun}_{\mathfrak{C}\Delta^{n+1}}(j, n+1) \arrow[r] & \mathsf{Fun}(E_j, E)
\end{tikzcd}$$

Because $0 < k < n+1$, by a calculation of homotopy coherent realizations in Example VI.4.4.3, to solve the original lifting problem, it remains only to construct a single lift

$$\begin{tikzcd}
\sqcap^{n,k}_1 \cong \mathsf{Fun}_{\mathfrak{C}\Lambda^{n+1,k}}(0,n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_0,F) \arrow[d, two heads, "q\circ -"] \\
\square^n \cong \mathsf{Fun}_{\mathfrak{C}\Delta^{n+1}}(0,n+1) \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E_0,E)
\end{tikzcd}$$

the other inclusions being full. The left-hand side is a cubical horn and the right-hand side is an isofibration of quasi-categories. As noted in Example 4.1.1, isofibrations of quasi-categories admit lifts against "special outer horns" — those in which the image of the final edge is invertible. Such extensions solve this lifting problem. ☐

Proposition 4.2.2. *There is a right adjoint*

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "\tilde{p}^*"']
\end{tikzcd}$$

to the oplax colimit functor defined at $q\colon F \to E$ *by the pullback*

$$\begin{tikzcd}
\tilde{\Pi}_p F \arrow[r] \arrow[d, two heads, "\tilde{\Pi}_p q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ -"] \\
\mathsf{B} \arrow[r, "\ell^E"] & \mathsf{qCat}_{2/\!\!/\mathsf{E}}
\end{tikzcd}$$

Moreover, when $q\colon F \twoheadrightarrow E$ *is an isofibration,* $\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow \mathsf{B}$ *is an isofibration between quasi-categories.*

Proof. Recall from Lemma 4.2.1 that n-simplices in $\mathsf{qCat}_{2/\!\!/\mathsf{F}}$ correspond to lax cocones under homotopy coherent simplices with nadir F, and observe that the whiskering functor $q \circ -\colon \mathsf{qCat}_{2/\!\!/\mathsf{F}} \to \mathsf{qCat}_{2/\!\!/\mathsf{E}}$ does not change the underlying homotopy coherent diagram. By the defining universal property, an n-simplex in the pullback over $b\colon \Delta^n \to \mathsf{B}$ corresponds to lax cocone under the homotopy coherent n-simplex $c_p \circ \mathfrak{C}b\colon \mathfrak{C}\Delta^n \to \mathcal{QC}at$ with nadir F that whiskers with q to the lax cocone of (3.1.2). This recovers the characterization of the right adjoint $\tilde{\Pi}_p$ given above and Lemma 3.1.11 demonstrates that this adjoint correspondence extends to all elements of $\underline{\mathrm{sSet}}_{/\mathsf{B}}$.

The action of $\tilde{\Pi}_p$ on morphisms $u\colon G \to F$ over E is given similarly by the pullback

$$\begin{tikzcd}
\tilde{\Pi}_p G \arrow[r] \arrow[d, dashed, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{qCat}_{2/\!\!/\mathsf{G}} \arrow[d, two heads, "u\circ-"] \\
\tilde{\Pi}_p F \arrow[r] \arrow[d, two heads, "\tilde{\Pi}_p q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ-"] \\
\mathsf{B} \arrow[r, "\ell^E"] & \mathsf{qCat}_{2/\!\!/\mathsf{E}}
\end{tikzcd}$$

By Lemma 4.2.1 and Example 4.1.1, it is immediate from the fact that B is a quasi-category and qCat_2 is a 2-complicial set that $\tilde{\Pi}_p\mathsf{F}$ is a 2-complicial set. We argue that in fact all 2-simplices are marked: by the defining universal property, a 2-simplex in $\tilde{\Pi}_p\mathsf{F}$ corresponds to a pair comprised of a 2-simplex in B and a 2-simplex in $\mathsf{qCat}_{2/\!\!/\mathsf{F}}$ and both of these 2-simplices are marked. Thus, $\tilde{\Pi}_p\mathsf{F}$ is a 1-complicial set, which Example 4.1.1 tells us is the same thing as a quasi-category, and now the isofibration of complicial sets $\tilde{\Pi}_p q$ becomes an isofibration between quasi-categories. ☐

Corollary 4.2.3. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration.:*

(i) The functor $\tilde{\Pi}_p\colon \underline{\mathrm{sSet}}_{/\mathsf{E}} \to \underline{\mathrm{sSet}}_{/\mathsf{B}}$ carries isofibrations over E to isofibrations over B, restricting to define a functor

$$\mathcal{QC}\!\mathit{at}_{/\mathsf{E}} \xrightarrow{\tilde{\Pi}_p} \mathcal{QC}\!\mathit{at}_{/\mathsf{B}}. \tag{4.2.4}$$

(ii) The functor (4.2.4) preserves isofibrations, now considered as morphisms in these slice categories.

Proof. Proposition 4.2.2 demonstrates that $\tilde{\Pi}_p$ carries isofibrations to isofibrations, restricting to define a functor $\tilde{\Pi}_p\colon \mathcal{QC}\!\mathit{at}_{/\mathsf{E}} \to \mathcal{QC}\!\mathit{at}_{/\mathsf{B}}$. Moreover this functor preserves isofibrations, now considered as morphisms in these slice categories, since the action of $\tilde{\Pi}_p$ on an isofibration $u\colon \mathsf{G} \twoheadrightarrow \mathsf{F}$ over E is defined by pulling back the isofibration of complicial sets $u\circ-\colon \mathsf{qCat}_{2/\!\!/\mathsf{G}} \twoheadrightarrow \mathsf{qCat}_{2/\!\!/\mathsf{F}}$. ☐

We now transfer the properties of the functor $\tilde{\Pi}_p$ to the right adjoint $\Pi_p\colon \underline{\mathrm{sSet}}_{/\mathsf{E}} \to \underline{\mathrm{sSet}}_{/\mathsf{B}}$ to the strict pullback functor $p^*\colon \underline{\mathrm{sSet}}_{/\mathsf{B}} \to \underline{\mathrm{sSet}}_{/\mathsf{E}}$.

Proposition 4.2.5. *If $p\colon E \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration, then the adjunctions*

$$\begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "p^*"']
\end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd}
\underline{\mathrm{sSet}}_{/\mathsf{E}} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}_{/\mathsf{B}} \arrow[l, bend right, "\tilde{p}^*"']
\end{tikzcd}$$

are Quillen with respect to the sliced Joyal model structure.

In particular, Π_p preserves both fibrant objects and the fibrations between, and thus has the properties enumerated for $\tilde{\Pi}_p$ in Corollary 4.2.3. Consequently, the natural Joyal equivalence $\gamma\colon \tilde{p}^* \Rightarrow p$ of Theorem 3.1.3, which defines a natural isomorphism of total left derived functors, transposes to a natural equivalence $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$, which defines a natural isomorphism of total right derived functors.

Proof. By an observation of Joyal and Tierney [10, 7.15], to show that $\tilde{p}^* \dashv \tilde{\Pi}_p$ is Quillen it suffices to show that the left adjoint preserves cofibrations and the right adjoint preserves fibrations between fibrant objects. Lemma 3.1.12 demonstrates the first of these and Corollary 4.2.3(ii) proves the second.

To prove that $p^* \dashv \Pi_p$ is Quillen, we prove that p^* is left Quillen. Thus functor preserves cofibrations because pullbacks preserve monomorphisms. By Theorem 3.1.3, p^* is naturally weakly equivalent to the left Quillen functor $\tilde{p}^*$. Since all objects in $\underline{\mathrm{sSet}}_{/\mathsf{E}}$ are cofibrant, the left Quillen functor $\tilde{p}^*$ preserves all Joyal weak equivalences, and hence by the 2-of-3 property p^* does as well. ☐

We now consider the actions of the pushforward functors $\tilde{\Pi}_p$ and Π_p along a cocartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ when applied to a cartesian fibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$. As before, we demonstrate directly that $\tilde{\Pi}_p q\colon \tilde{\Pi}_p \mathsf{F} \twoheadrightarrow \mathsf{B}$ is a then a cartesian fibration and then use Theorem 3.1.3 to conclude the same for Π_p.

Lemma 4.2.6. *Let* $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ *between quasi-categories. Then the corresponding map* $q \circ -\colon \mathsf{qCat}_{2/\!\!/\mathsf{F}} \twoheadrightarrow \mathsf{qCat}_{2/\!\!/\mathsf{E}}$ *of 2-complicial sets has the right lifting property with respect to any outer horn inclusion*

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r] \arrow[rr, bend left, "\chi"] & \Lambda^{n,n} \arrow[r] \arrow[d, hook] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ-"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \mathsf{qCat}_{2/\!\!/\mathsf{E}}
\end{tikzcd}$$

whose final edge defines a cartesian 1-arrow

$$\begin{tikzcd}
E_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, "f_{n-1,n-2}"'] & \Downarrow\chi & F \\
& E_n \arrow[ur, "f_n"'] &
\end{tikzcd}$$

for the cartesian fibration $q \circ -\colon \mathsf{Fun}(E_{n-1}, F) \to \mathsf{Fun}(E_{n-1}, E)$.

Proof. As in the proof of Lemma 4.2.1, the bottom horizontal functor is given by a homotopy coherent $n+1$-simplex $\mathfrak{C}\Delta^{n+1} \to \mathsf{QCat}$ that sends the first $n+1$ objects to $E_0, \ldots E_n$ and the final object to E, while the δ^{n+1}-face of this homotopy coherent simplex and the top horizontal functor together define a simplicial functor $\mathfrak{C}\Lambda^{n+1,n} \to \mathsf{QCat}$ that caries the $n+2$ objects to $E_0, \ldots, E_n, F$ and has the property that for each $0 \leq j \leq n$ the diagram of function complexes commutes:

$$\begin{tikzcd}
\mathfrak{C}\Lambda^{n+1,n}(j,n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_j,E) \arrow[d, two heads, "q\circ-"] \\
\mathfrak{C}\Delta^{n+1}(j,n+1) \arrow[r] & \mathsf{Fun}(E_j,F)
\end{tikzcd}$$

By Example VI.4.4.3, to solve the original lifting problem, we need only construct a single lift

$$\begin{tikzcd}
\sqcap^{n,n}_1 \cong \mathfrak{C}\Lambda^{n+1,n}(0,n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_0,E) \arrow[d, two heads, "q\circ-"] \\
\square^n \cong \mathfrak{C}\Delta^{n+1}(0,n+1) \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E_0,F)
\end{tikzcd}$$

This extension problem can be solved by filling inner horns and "special outer horns" $\Lambda^{m,m} \to \Delta^m$, those whose final edges are complies of the 1-simplex $\chi \in \mathsf{Fun}(E_{n-1},F)$ pre-composed with some functor $E_0 \to E_{n-1}$. Such 1-simplices represent $(q \circ -)$-cartesian cells so these "special outer horn" lifting problems also admit solutions by Lemma VI.3.2.5. $\square$

Proposition 4.2.7. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ is a cartesian fibration between quasi-categories, then*

$$\tilde{\Pi}_p q\colon \tilde{\Pi}_p\mathsf{F} \twoheadrightarrow \mathsf{B}$$

is a cartesian fibration between quasi-categories. Moreover, $\tilde{\Pi}_p$ preserves cartesian functors, restricting to define a functor

$$\tilde{\Pi}_p\colon \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{E}} \to \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}}.$$

Proof. By Proposition 4.2.2, $\tilde{\Pi}_p\mathsf{F} \twoheadrightarrow \mathsf{B}$ defines an isofibration between quasi-categories. Lemma 4.2.6 identifies a class of cartesian 1-arrows in $\tilde{\Pi}_p\mathsf{F}$ in the sense of Definition 2.2.1, which we now describe explicitly. Recall from the construction of Proposition 4.2.2, that a 1-simplex $\chi\colon \Delta^1 \to \tilde{\Pi}_p\mathsf{F}$ in the fiber over $b\colon \Delta^1 \to \mathsf{B}$ corresponds to a 1-arrow

$$\begin{tikzcd}
E_0 \arrow[rr, "f_0"] \arrow[dr, "e_b"'] & \Downarrow\chi & \mathsf{F} \\
& E_1 \arrow[ur, "f_1"'] &
\end{tikzcd}$$

in $\mathsf{Fun}(E_0,\mathsf{F})$ that whiskers with $q\colon \mathsf{F} \to E$ to define the lax cocone that restricts the lax cocone associated to the comprehension construction along b. As observed previously, this compatibility condition tells us that the quasi-categories E_0 and E_1 are the fibers of $p\colon E \twoheadrightarrow \mathsf{B}$ over the vertices in b and the functor $e_b\colon E_0 \to E_1$ is the comprehension of b. To form such a lift with codomain $f_1\colon E_1 \to \mathsf{F}$, start by lifting b to the lax cocone

$$\begin{tikzcd}
E_0 \arrow[rr, "\ell^E_0"] \arrow[dr, "e_b"'] & \Downarrow\epsilon & E \\
& E_1 \arrow[ur, "\ell^E_1"'] &
\end{tikzcd}$$

under $e_b\colon E_0 \to E_1$ with nadir E associated with the comprehension construction, as displayed in (3.2.4). Since f_1 is in the fiber of $\tilde{\Pi}_p q\colon \tilde{\Pi}_p\mathsf{F} \twoheadrightarrow \mathsf{B}$ over the codomain of b, we must have $qf_1 = \ell_1^E$. Now we can lift ϵ along the cartesian fibration q to a q-cartesian cell with codomain $f \circ e_b$. This defines the $(q \circ -)$-cartesian cell χ.

Now if $u\colon G \to \mathsf{F}$ is a cartesian functor from $r\colon G \twoheadrightarrow E$ to $q\colon \mathsf{F} \twoheadrightarrow E$, then u is representably cartesian in the sense that $u \circ -\colon \mathsf{Fun}(X, G) \to \mathsf{Fun}(X, \mathsf{F})$ carries $(r \circ -)$-cartesian 1-arrows to $(q \circ -)$-cartesian 1-arrows. Since $u \circ -\colon \mathsf{qCat}_{2/\!\!/\mathsf{G}} \to \mathsf{qCat}_{2/\!\!/\mathsf{F}}$ preserves the cartesian 1-arrows just identified, proving that $\tilde{\Pi}_p$ carries this map to a cartesian functor between cartesian fibrations over B. ☐

Corollary 4.2.8. *If* $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ *is a cocartesian fibration and* $q\colon \mathsf{F} \twoheadrightarrow E$ *is a cartesian fibration between quasi-categories, then*

$$\Pi_p q\colon \Pi_p\mathsf{F} \twoheadrightarrow \mathsf{B}$$

is a cartesian fibration between quasi-categories. Moreover, Π_p *preserves cartesian functors, restricting to define a functor*

$$\mathsf{Cart}(\mathcal{QC}\mathit{at})_{/\mathsf{E}} \xrightarrow{\Pi_p} \mathsf{Cart}(\mathcal{QC}\mathit{at})_{/\mathsf{B}}.$$

Proof. By Proposition 4.2.5, the components

$$\begin{tikzcd} \Pi_p\mathsf{F} \arrow[rr, "\hat{\gamma}_q", "\sim"'] \arrow[dr, two heads, "\Pi_p q"'] & & \tilde{\Pi}_p\mathsf{F} \arrow[dl, two heads, "\tilde{\Pi}_p q"] \\ & \mathsf{B} & \end{tikzcd}$$

at an isofibration $q\colon \mathsf{F} \twoheadrightarrow E$ of the transpose $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$ of the natural weak equivalence of Theorem 3.1.3 are equivalences of isofibrations over B. If q is a cartesian fibration, then Proposition 4.2.7 proves that $\tilde{\Pi}_p q$ is a cartesian fibration, and since the notion of cartesian fibration is equivalence-invariant, $\Pi_p q$ must be as well. ☐

Theorem 4.2.9. *For a cocartesian fibration* $p\colon E \twoheadrightarrow \mathsf{B}$ *between quasi-categories, the pullback-pushforward adjunction restricts to define an adjunction*

$$\begin{tikzcd} \mathcal{QC}\mathit{at}_{/\mathsf{E}} \arrow[rr, bend right, "\Pi_p"'] \arrow[rr, phantom, "\perp"] & & \mathcal{QC}\mathit{at}_{/\mathsf{B}} \arrow[ll, bend right, "p^*"'] \\ \mathsf{Cart}(\mathcal{QC}\mathit{at})_{/\mathsf{E}} \arrow[u, hook] \arrow[rr, bend right, "\Pi_p"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Cart}(\mathcal{QC}\mathit{at})_{/\mathsf{B}} \arrow[u, hook] \arrow[ll, bend right, "p^*"'] \end{tikzcd}$$

Proof. By Proposition 4.2.5, the adjoint functors $p^* \dashv \Pi_p$ define an adjunction

$$\begin{tikzcd} \mathcal{QC}\mathit{at}_{/\mathsf{E}} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{QC}\mathit{at}_{/\mathsf{B}} \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

By Proposition 2.2.5, the left adjoint restricts to a define a functor

$$p^* \colon \mathsf{Cart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}} \to \underline{\mathrm{Cat}}(\mathcal{Q}\mathit{Cat})_{/\mathsf{E}}.$$

By Corollary 4.2.8, the right adjoint also restricts to a functor $\Pi_p \colon \mathsf{Cart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{E}} \to \mathsf{Cart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}}$. Since the inclusion $\mathsf{Cart}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}} \hookrightarrow \mathcal{Q}\mathit{Cat}_{/\mathsf{B}}$ is not full, this is not quite enough to demonstrate adjointness of the restricted adjunction: it remains to argue that the adjoint transpose of a cartesian functor is a cartesian functor.

To that end, let $q \colon \mathsf{F} \twoheadrightarrow E$ and $r \colon G \twoheadrightarrow \mathsf{B}$ be cartesian fibrations. A functor $f \colon G \to \tilde{\Pi}_p q$ over B is cartesian if and only if the square

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, two heads, "r"'] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ-"] \\
\mathsf{B} \arrow[r, "\ell^E"'] & \mathsf{qCat}_{2/\!\!/\mathsf{E}}
\end{tikzcd}$$

carries r-cartesian arrows to representably q-cartesian arrows in $\mathsf{qCat}_{2/\!\!/\mathsf{F}}$, as described in Lemma 4.2.6. Fixing an 1-arrow arrow $\zeta \colon \Delta^1 \to G$ over $b \colon \Delta^1 \to \mathsf{B}$ as below-left, the arrow $f\zeta$ transposes to the functor over E displayed below-right

$$\begin{tikzcd}
\Delta^1 \arrow[r, "f\zeta"] \arrow[d, "b"'] & \mathsf{qCat}_{2/\!\!/\mathsf{F}} \arrow[d, two heads, "q\circ-"] & & \mathrm{colim}^{\mathrm{oplax}}\,(c_p \circ \mathfrak{C}b) \arrow[rr, "\widehat{f\zeta}"] \arrow[dr, "\ell^E|_b"'] & & \mathsf{F} \arrow[dl, two heads, "q"] \\
\mathsf{B} \arrow[r, "\ell^E"'] & \mathsf{qCat}_{2/\!\!/\mathsf{E}} \arrow[r, phantom, "\leftrightsquigarrow"] & {} & & E &
\end{tikzcd}$$

By Proposition 3.2.3, the oplax colimit is equivalent to the fiber E_b and the functor $\widehat{f\zeta}$ represents the whiskered lax cocone

$$\begin{tikzcd}
E_{b_0} \arrow[drrr, bend left, "\ell_0^E"] \arrow[ddd, "p_0"'] \arrow[ddr, dotted, "e_b"'] & & {} \arrow[dd, Rightarrow, dotted, "\chi"] & & \mathsf{F} \arrow[dr, two heads, "q"] & \\
& & & E_b \arrow[r] \arrow[ur, "\widehat{f\zeta}"] \arrow[dddd, two heads, "p_b"] \arrow[dr, phantom, "\lrcorner", very near start] & E_r \arrow[r] \arrow[u, "\hat{f}"'] \arrow[dddd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[dddd, two heads, "p"] \\
& E_{b_1} \arrow[urr, bend right, "\ell_1^E"'] \arrow[dddd, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} & & & \\
1 \arrow[ddrrr, bend left, "0"] \arrow[dddr, equal] & & {} & & & \\
& {} \arrow[r, Rightarrow, "\kappa"] & {} & & & \\
& & & \Delta^1 \arrow[r, "\zeta"] \arrow[rr, bend right, "b"'] & G \arrow[r, two heads, "r"] & \mathsf{B} \\
& 1 \arrow[urr, bend right, "1"'] & & & &
\end{tikzcd}$$

Now f is a cartesian functor if and only if the whiskered composite $\widehat{f\zeta}\chi$ is q-cartesian whenever ζ is r-cartesian. Since Proposition 2.2.5 demonstrates that cartesian arrows are created by pullbacks, this proves that f is a cartesian functor if and only if the transposed functor is cartesian:

$$\begin{tikzcd}
E_r \arrow[rr, "\hat{f}"] \arrow[dr, two heads, "p^*r"'] & & \mathsf{F} \arrow[dl, two heads, "q"] \\
& E &
\end{tikzcd}$$

□

For use in sequels to this work, we note that the pushforward is a cosmological functor between the ∞-cosmoi established in Proposition VIII.3.2.18.

Corollary 4.2.10. *Let $p\colon \mathsf{E} \to \mathsf{B}$ be a cocartesian fibration. Then the pushforward construction defines cosmological functors*

$$\Pi_p \colon \mathcal{Q}\mathsf{Cat}_{/\mathsf{E}} \to \mathcal{Q}\mathsf{Cat}_{/\mathsf{B}} \qquad \text{and} \qquad \Pi_p \colon \mathsf{Cart}(\mathcal{Q}\mathsf{Cat})_{/\mathsf{E}} \to \mathsf{Cart}(\mathcal{Q}\mathsf{Cat})_{/\mathsf{B}},$$

which is to say that they are simplically enriched, preserve all simplicially enriched limits with flexible weights, and preserve the isofibrations, considered as morphisms in the slice category.

Proof. The functor $\Pi_p \colon \mathcal{Q}\mathsf{Cat}_{/\mathsf{E}} \to \mathcal{Q}\mathsf{Cat}_{/\mathsf{B}}$ is the restriction of a Quillen right adjoint $\Pi_p \colon \underline{\mathrm{sSet}}_{/\mathsf{E}} \to \underline{\mathrm{sSet}}_{/\mathsf{B}}$. To prove that this defines a cosmological, it remains only to show that the adjunction $p^* \dashv \Pi_p$ is simplicially enriched. This follows from Observation 2.2.6, which notes that the left adjoint preserves tensors with simplicial sets. Lemma 2.2.7 observes that the simplicial enrichment descends to the subcosmoi of cartesian fibrations. $\square$

The argument given in the proof of Theorem 4.2.9 provides a characterization of the $\Pi_p q$-cartesian 1-arrows in the cartesian fibration constructed from a cocartesian fibration $p\colon E \twoheadrightarrow \mathsf{B}$ and a cartesian fibration $q\colon \mathsf{F} \twoheadrightarrow E$ between quasi-categories that lift a specified arrow $\beta\colon \Delta^1 \to \mathsf{B}$.

Lemma 4.2.11. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{E}$ is a cartesian fibration between quasi-categories, then the cartesian 1-arrows χ in $\Pi_p q\colon \Pi_p\mathsf{F} \twoheadrightarrow \mathsf{B}$ are those maps that transpose to define functors that carry p-cocartesian lifts of β to q-cartesian lifts.*

$$\begin{tikzcd}
\Delta^1 \arrow[rr, "\chi"] \arrow[dr, "\beta"'] & & \Pi_p\mathsf{F} \arrow[dl, "\Pi_p q"] \\
& \mathsf{B} &
\end{tikzcd}
\qquad \leftrightsquigarrow \qquad
\begin{tikzcd}
& \mathsf{F} \arrow[d, two heads, "q"] \\
\mathsf{E}_\beta \arrow[ur, "\hat{\chi}"] \arrow[r, "\ell_\beta"] \arrow[d, two heads, "p_\beta"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{E} \arrow[d, two heads, "p"] \\
\Delta^1 \arrow[r, "\beta"'] & \mathsf{B}
\end{tikzcd}$$

Proof. By Theorem 3.1.3, E_β may be identified with the oplax colimit of the canonical lax cocone formed by taking a p-cocartesian lift of β. From this perspective, the transposed functor $\hat{\chi}\colon \mathsf{E}_\beta \to \mathsf{F}$ acts by whiskering this p-cocartesian arrow. By the construction in the proof of Theorem 4.2.9, the $\Pi_p q$-cartesian lifts of β are those arrows for which this whiskered composite is q-cartesian, as claimed. $\square$

4.3 Exponentiation

As is familiar in any locally cartesian closed category, the pullback and pushforward functors can be used to construct *exponentials* in $\mathsf{Cart}(\mathcal{Q}\mathsf{Cat})_{/\mathsf{B}}$, where the exponent is given by a cocartesian fibration.

Definition 4.3.1 (exponentials)**.** For $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ either a cartesian or cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ an isofibration, define

$$(q\colon \mathsf{F} \twoheadrightarrow \mathsf{B})^{p\colon \mathsf{E}\twoheadrightarrow\mathsf{B}} \qquad \in \mathcal{QC}\mathsf{at}_{/\mathsf{B}} \tag{4.3.2}$$

to be the image of q under the composite functor

$$\mathcal{QC}\mathsf{at}_{/\mathsf{B}} \xrightarrow{p^*} \mathcal{QC}\mathsf{at}_{/\mathsf{E}} \xrightarrow{\Pi_p} \mathcal{QC}\mathsf{at}_{/\mathsf{B}}$$

Note that by the adjunctions $\Sigma_p \dashv p^* \dashv \Pi_p$, if $r\colon \mathsf{G} \twoheadrightarrow \mathsf{B}$ is also a cartesian or cocartesian fibration, there are natural isomorphisms

$$\mathsf{Fun}_{\mathsf{B}}(\mathsf{G} \twoheadrightarrow \mathsf{B}, (\mathsf{F} \twoheadrightarrow \mathsf{B})^{\mathsf{E}\twoheadrightarrow\mathsf{B}}) \cong \mathsf{Fun}_{\mathsf{B}}(\mathsf{E} \times_{\mathsf{B}} \mathsf{G} \twoheadrightarrow \mathsf{B}, \mathsf{F} \twoheadrightarrow \mathsf{B}) \cong \mathsf{Fun}_{\mathsf{B}}(\mathsf{E} \twoheadrightarrow \mathsf{B}, (\mathsf{F} \twoheadrightarrow \mathsf{B})^{\mathsf{G}\twoheadrightarrow\mathsf{B}}),$$

and the left-hand isomorphism still holds in the case where $r\colon \mathsf{G} \to \mathsf{B}$ is a mere functor, whose domain need not even be a quasi-category.

Proposition 4.3.3. *If $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is a cocartesian fibration and $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ is a cartesian fibration, then* (4.3.2) *is a cartesian fibration whose cartesian 1-arrows are those maps that transpose to define functors that carry p-cocartesian lifts of b to q-cartesian lifts of b.*

$$\begin{tikzcd}
\Delta^1 \arrow[rr, "\chi"] \arrow[ddr, "b"'] & & (\mathsf{F} \xrightarrow{q} \mathsf{B})^{\mathsf{E}\xrightarrow{p}\mathsf{B}} \arrow[ddl, two heads] & & \mathsf{E}_b \arrow[r] \arrow[rr, bend left, "\widehat{\chi}"] \arrow[dd, two heads, "p_b"'] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[dd, two heads, "p"] & \mathsf{F} \arrow[ddl, two heads, "q"] & & \mathsf{E}_b \arrow[rr, "\widehat{\chi}"] \arrow[ddr, two heads, "p_b"'] & & \mathsf{F}_b \arrow[ddl, two heads, "q_b"] \\
& & & \leftrightsquigarrow & & & & \leftrightsquigarrow & & & \\
& \mathsf{B} & & & \Delta^1 \arrow[r, "b"'] & \mathsf{B} & & & & \Delta^1 &
\end{tikzcd}$$

Proof. The first statement follows from Corollary 4.2.8, while the characterization of cartesian cells is given in Lemma 4.2.11. □

Recall the function complexes constructed in Definitions 2.2.3 and 2.2.4.

Lemma 4.3.4. *Let $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ be a cartesian fibration, let $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ be a cocartesian fibration, and let $\pi\colon \mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B}$ denote the projection, a bifibration. Then the isomorphism*

$$\mathsf{Fun}_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, (\mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{E}\overset{p}{\twoheadrightarrow}\mathsf{B}}) \cong \mathsf{Fun}_{\mathsf{B}}(\mathsf{E} \overset{p}{\twoheadrightarrow} \mathsf{B}, (\mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F}\overset{q}{\twoheadrightarrow}\mathsf{B}})$$

restricts to an isomorphism

$$\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, (\mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{E}\overset{p}{\twoheadrightarrow}\mathsf{B}}) \cong \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E} \overset{p}{\twoheadrightarrow} \mathsf{B}, (\mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F}\overset{q}{\twoheadrightarrow}\mathsf{B}})$$

between the function complexes in the quasi-category enriched categories $\mathsf{Cart}(\mathcal{QC}\mathsf{at})_{/\mathsf{B}}$ and $\mathsf{coCart}(\mathcal{QC}\mathsf{at})_{/\mathsf{B}}$, which is to say, cartesian functors between the cartesian fibrations on the left transpose to define cartesian functors between the cocartesian fibrations on the right.

Proof. By adjunction, the data of a map over B

$$\begin{tikzcd} \mathsf{F} \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & (\mathsf{A} \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{E} \xrightarrow{p} \mathsf{B}} \arrow[dl, two heads, "\pi^p"] \\ & \mathsf{B} & \end{tikzcd}$$

is given by a single functor $\hat{f}\colon \mathsf{F} \times_{\mathsf{B}} \mathsf{E} \to \mathsf{A}$. By Proposition 4.3.3, f is a cartesian functor if and only if for each q-cocartesian cell $\chi\colon \Delta^1 \to \mathsf{F}$ over $b\colon \Delta^1 \to \mathsf{B}$, the induced functor

$$\begin{tikzcd} \mathsf{E}_b \arrow[r, "\hat{\chi}"] \arrow[d, two heads, "p_b"'] & \mathsf{A} \times \mathsf{B} \arrow[d, two heads, "\pi"] \\ \Delta^1 \arrow[r, "b"'] & \mathsf{B} \end{tikzcd}$$

carries p-cartesian cells $\gamma\colon \Delta^1 \to \mathsf{E}$ over b to π-cocartesian ones, these being those maps $\Delta^1 \to \mathsf{A} \times \mathsf{B}$ whose component along the other projection $\mathsf{A} \times \mathsf{B} \to \mathsf{A}$ is invertible. In summary, the functor f is cartesian if and only if for every cocartesian lift χ and cartesian lift γ of b, the composite morphism

$$\Delta^1 \xrightarrow{\langle \chi, \gamma \rangle} \mathsf{F}_b \times_{\Delta^1} \mathsf{E}_b \hookrightarrow \mathsf{F} \times_{\mathsf{B}} \mathsf{E} \xrightarrow{\hat{f}} \mathsf{A} \qquad \square$$

5. Monadicity and comonadicity of cartesian fibrations

The 0-skeleton of a quasi-category B defines the "underlying set of objects" $\operatorname{ob}\mathsf{B}$, together with a canonical inclusion $\operatorname{ob}\mathsf{B} \hookrightarrow \mathsf{B}$. By Proposition 2.2.5, pulling back along the inclusion $\operatorname{ob}\mathsf{B} \hookrightarrow \mathsf{B}$ induces a forgetful functor

$$\begin{array}{ccc} \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} & \longrightarrow & \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \cong \prod_{\operatorname{ob}\mathsf{B}} \mathcal{QC}at \\ (\mathsf{E} \xrightarrow{p} \mathsf{B}) & \mapsto & (\mathsf{E}_b)_{b \in \operatorname{ob}\mathsf{B}} \end{array}$$

whose codomain is isomorphic to the product of the quasi-categorically enriched categories of quasi-categories, a cartesian fibration over a set being simply an indexed family of quasi-categories. Our aim in this section is to construct left and right adjoints and prove that this functor is monadic and comonadic in a suitable sense.

The adjoint functors are constructed as what we refer to as *biadjoint* functors of quasi-categorically enriched categories: that is, we construct quasi-categorically enriched functors

$$L, R\colon \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \longrightarrow \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}}$$

together with natural equivalences of function complexes that encode the adjoint transpose correspondence. The right adjoint makes use of the exponentiation construction of §4.3 and the Yoneda lemma is used to prove biadjointness. These tasks occupy §5.1.

Any quasi-categorically enriched category $\mathcal{K}$ has a (typically large) *quasi-categorical core* $\mathsf{K} := Ng_*\mathcal{K}$ defined by passing to the maximal Kan complex enriched core and then applying the homotopy coherent nerve. For example, the quasi-category qCat of quasi-categories and functors is the quasi-categorical core of $\mathcal{QC}at$. In §5.1 we also prove that biadjoint functors of quasi-categorically enriched categories descend to adjoint functors between their quasi-categorical cores.

In particular, this implies that the map of large quasi-categories of cartesian fibrations and cartesian functors

$$\mathsf{Cart}_{/\mathsf{B}} \longrightarrow \mathsf{Cart}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$$

admits both left and right adjoints. In §5.2, we prove first that this forgetful functor is comonadic and then use comonadicity to prove that it is also monadic. To do so, we appeal to the comonadicity theorem proven in §II.7 and recalled as Theorem 5.2.1 below. The monadicity of this forgetful functor will be used in §6 to construct a "groupoidal reflection" functor for cartesian fibrations.

5.1 Adjoint functors

A functor $U : \mathcal{K} \to \mathcal{L}$ between quasi-categorically enriched categories gives rise to a functor between the large quasi-categories defined by passing to the Kan complex enriched cores of $\mathcal{K}$ and $\mathcal{L}$ and applying the homotopy coherent nerve construction. We frequently find it convenient to construct adjoints to this functor of quasi-categories "at the point-set level" by producing the structures axiomatized in the following definition:

Definition 5.1.1. A *biadjunction of quasi-categorically enriched categories* consists of:

- a pair of quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{L}$;
- a pair of simplicial functors $F : \mathcal{L} \to \mathcal{K}$ and $U : \mathcal{K} \to \mathcal{L}$; and
- a simplicially-enriched natural equivalence

$$\mathsf{Fun}_{\mathcal{K}}(FL, K) \simeq \mathsf{Fun}_{\mathcal{L}}(L, UK)$$

of function complexes .

Proposition 5.1.2. *If $F : \mathcal{L} \to \mathcal{K}$ and $U : \mathcal{K} \to \mathcal{L}$ define a biadjunction of quasi-categorically enriched categories, then the induced functors between the quasi-categorical cores $\mathsf{K} := Ng_*\mathcal{K}$ and $\mathsf{L} := Ng_*\mathcal{L}$ define an adjunction:*

$$\begin{tikzcd} \mathsf{K} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{L} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Proof. Reprising a construction from the proof of Theorem I.6.2.1, we define simplicial categories $\mathrm{coll}(F, \mathcal{K})$ and $\mathrm{coll}(\mathcal{L}, U)$ whose objects are $\mathrm{ob}\,\mathcal{K} + \mathrm{ob}\,\mathcal{L}$ and which include $\mathcal{K}$ and $\mathcal{L}$ as full subcategories. Each of the function complexes from an object of $\mathcal{K}$ to an object of $\mathcal{L}$ are empty, while for $L \in \mathcal{L}$ and $K \in \mathcal{K}$, we define

$$\mathsf{Fun}_{\mathrm{coll}(F,\mathcal{K})}(L, K) := \mathsf{Fun}_{\mathcal{K}}(FL, K) \qquad \text{and} \qquad \mathsf{Fun}_{\mathrm{coll}(\mathcal{L},U)}(L, K) := \mathsf{Fun}_{\mathcal{L}}(L, UK).$$

The natural equivalence $\mathsf{Fun}_{\mathcal{K}}(FL, K) \xrightarrow{\sim} \mathsf{Fun}_{\mathcal{L}}(L, UK)$ of the biadjunction gives rise to a simplicial functor $\mathrm{coll}(F, \mathcal{K}) \to \mathrm{coll}(\mathcal{L}, U)$ under $\mathcal{K} + \mathcal{L}$ that is bijective on objects and a local equivalence of quasi-categories.

Passing to Kan complex enriched cores, this functor defines a Dwyer-Kan equivalence, and thus yields an equivalence of quasi-categories upon passing to homotopy coherent nerves. Note that the homotopy coherent nerve of the groupoid core of $\mathrm{coll}(F, \mathcal{K})$ is isomorphic to the quasi-categorical collage of the underlying functor $F \colon \mathsf{L} \to \mathsf{K}$ constructed in Definition 2.2.8. Now Corollary 2.2.12 demonstrates that $F \dashv U$ as functors between the quasi-categories K and L. □

Proposition 5.1.3. *The functor* $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \longrightarrow \mathsf{Cart}(\mathcal{QC}at)_{/\,\mathrm{ob}\,\mathsf{B}}$ *admits a quasi-categorically enriched right biadjoint defined by*

$$\begin{array}{ccc} \mathsf{Cart}(\mathcal{QC}at)_{/\,\mathrm{ob}\,\mathsf{B}} & \longrightarrow & \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \\[1em] (\mathsf{E}_b)_{b \in \mathrm{ob}\,\mathsf{B}} & \mapsto & \prod_b (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}} \end{array}$$

that is, an $\mathrm{ob}\,\mathsf{B}$*-indexed family of quasi-categories* $(\mathsf{E}_b)_{b \in \mathrm{ob}\,\mathsf{B}}$ *is sent to the product in* $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}}$ *of the cartesian fibrations* $(\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}$.

Proof. Here $b \downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}$ is the (groupoidal) cocartesian fibration represented by the vertex $b \in \mathsf{B}$, sending an arrow in B with domain b to its codomain; see Example IV.5.2.3. The fiber over a vertex $x \in \mathsf{B}$, is the Kan complex $b \downarrow x$ of maps from b to x in B. Since the product projection $\pi \colon \mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B}$ is a bifibration, both cartesian and cocartesian, Proposition 4.3.3 implies that $(\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}$ is a cartesian fibration.

For another cartesian fibration $q \colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ with fibers $(\mathsf{F}_b)_{b \in \mathrm{ob}\,\mathsf{B}}$, we will define a natural equivalence of function complexes

$$\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, \prod_{b \in \mathrm{ob}\,\mathsf{B}} (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}) \xrightarrow{\sim} \prod_{b \in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}(\mathsf{F}_b, \mathsf{E}_b)$$

and so establish the claimed adjoint correspondence.

To begin, the universal properties of the product and exponential provide isomorphisms

$$\begin{aligned} \mathsf{Fun}_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, \prod_{b \in \mathrm{ob}\,\mathsf{B}} (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}) &\cong \prod_{b \in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}) \\ &\cong \prod_{b \in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}_{\mathsf{B}}(b \downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}}). \end{aligned}$$

By Lemma 4.3.4, these isomorphisms restrict to the full sub quasi-categories spanned by the cartesian functors. Hence,

$$\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}, \prod_{b \in \mathrm{ob}\,\mathsf{B}} (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}}) \cong \prod_{b \in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}^c_{\mathsf{B}}(b \downarrow \mathsf{B} \twoheadrightarrow \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}}).$$

By the dual of the Yoneda lemma, proven as Theorem IV.6.0.1, restriction along the element $1 \to b \downarrow \mathsf{B}$ corresponding to the identity at $b\colon 1 \to \mathsf{B}$ defines an equivalence

$$\mathsf{Fun}^c_{\mathsf{B}}(b\downarrow\mathsf{B} \twoheadrightarrow \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F}\xrightarrow{q}\mathsf{B}}) \xrightarrow{\sim} \mathsf{Fun}_{\mathsf{B}}(1 \xrightarrow{b} \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F}\xrightarrow{q}\mathsf{B}}).$$

The proof is finished by the isomorphisms

$$\mathsf{Fun}_{\mathsf{B}}(1 \xrightarrow{b} \mathsf{B}, (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{\mathsf{F}\xrightarrow{q}\mathsf{B}}) \cong \mathsf{Fun}_{\mathsf{B}}(\mathsf{F}_b \to 1 \xrightarrow{b} \mathsf{B}, \mathsf{E}_b \times \mathsf{B} \to \mathsf{B}) \cong \mathsf{Fun}(\mathsf{F}_b, \mathsf{E}_b). \quad \square$$

The construction of the left adjoint to $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \longrightarrow \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}}$ will make use of the tensor of a cartesian fibration, namely $\mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}$, with a quasi-category, namely E_b, described in Observation 2.2.6 and Lemma 2.2.7.

Proposition 5.1.4. *The functor* $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \longrightarrow \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}}$ *admits a quasi-categorically enriched left biadjoint defined by*

$$\begin{tikzcd} \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \arrow[r] & \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \\ (\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}} \arrow[r, mapsto] & \coprod_b \mathsf{E}_b \times \mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B} \end{tikzcd}$$

that is, an $\operatorname{ob}\mathsf{B}$*-indexed family of quasi-categories* $(\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}}$ *is sent to the coproduct in* $\mathsf{Cart}_{/\mathsf{B}}$ *of the cartesian fibrations* $\mathsf{E}_b \times \mathsf{B}\downarrow b \xrightarrow{\pi} \mathsf{B}\downarrow b \xrightarrow{p_0} \mathsf{B}$.

Proof. To make sense of this construction, note that the coproduct of cartesian fibrations over B is again a cartesian fibration over B: since the horns are connected, each lifting problem of Definition 2.2.1 is supported in a single component. It follows that a functor out of the coproduct of cartesian fibrations is cartesian if and only if each of its legs is a cartesian functor.

For another cartesian fibration $q\colon \mathsf{F} \twoheadrightarrow \mathsf{B}$ with fibers $(\mathsf{F}_b)_{b\in\operatorname{ob}\mathsf{B}}$, we will define a natural equivalence of function complexes

$$\mathsf{Fun}^c_{\mathsf{B}}(\coprod_{b\in\operatorname{ob}\mathsf{B}} \mathsf{E}_b \times \mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B}) \xrightarrow{\sim} \prod_{b\in\operatorname{ob}\mathsf{B}} \mathsf{Fun}(\mathsf{E}_b, \mathsf{F}_b)$$

and so establish the claimed adjoint correspondence.

To begin, the universal property of the coproduct provides the first isomorphism, while the universal property of the tensor proven as Lemma 2.2.7 provides the second

$$\begin{aligned}\mathsf{Fun}^c_{\mathsf{B}}(\coprod_{b\in\operatorname{ob}\mathsf{B}} \mathsf{E}_b \times \mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B}) &\cong \prod_{b\in\operatorname{ob}\mathsf{B}} \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{E}_b \times \mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B}) \\ &\cong \prod_{b\in\operatorname{ob}\mathsf{B}} \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B})^{E_b}.\end{aligned}$$

By the Yoneda lemma, proven as Theorem IV.6.0.1, restriction along the element $1 \to \mathsf{B}\downarrow b$ corresponding to the identity at $b\colon 1 \to \mathsf{B}$ defines an equivalence

$$\mathsf{Fun}^c_{\mathsf{B}}(\mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B}) \xrightarrow{\sim} \mathsf{Fun}_{\mathsf{B}}(1 \xrightarrow{b} \mathsf{B}, \mathsf{F} \xrightarrow{q} \mathsf{B}) \cong \mathsf{F}_b.$$

This equivalence is respected by the cotensor $(-)^{\mathsf{E}_b}$ and the product, so we have the desired equivalence

$$\mathsf{Fun}^c_{\mathsf{B}}(\coprod_{b\in \mathrm{ob}\,\mathsf{B}} \mathsf{E}_b \times \mathsf{B}{\downarrow}b \twoheadrightarrow \mathsf{B}, \mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B}) \cong \prod_{b\in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}^c_{\mathsf{B}}(\mathsf{B}{\downarrow}b \twoheadrightarrow \mathsf{B}, \mathsf{F} \overset{q}{\twoheadrightarrow} \mathsf{B})^{E_b} \simeq \prod_{b\in \mathrm{ob}\,\mathsf{B}} \mathsf{Fun}(\mathsf{E}_b, \mathsf{F}_b).$$

□

5.2 Monadicity and comonadicity

A functor $u\colon \mathsf{A} \to \mathsf{B}$ between quasi-categories is *comonadic* if it is the left adjoint part of a comonadic adjunction. This means that A is equivalent to the *quasi-category of coalgebras* for the *homotopy coherent comonad* on B underlying the corresponding *homotopy coherent adjunction* derived from u and its right adjoint. The quasi-category of coalgebras is defined as a particular flexible weighted limit of the homotopy coherent comonad. A few specific details of this construction are needed in the proofs in §6.3 and these are reviewed there. To avoid an unnecessary digression, we refer the reader §II.7 for the definition of these notions and omit them from the current presentation.

Recall Theorem II.7.2.7, presented here in the dual:

Theorem 5.2.1 (comonadicity II.7.2.7)**.** *A functor $u\colon \mathsf{A} \to \mathsf{B}$ between quasi-categories is comonadic if and only if:*

- *(i) u admits a right adjoint,*
- *(ii) A admits and u preserves limits of u-split cosimplicial objects, and*
- *(iii) u is conservative, reflecting isomorphisms.*

Conservative functors between quasi-categories might arise as follows:

Lemma 5.2.2. *Let $U\colon \mathcal{K} \to \mathcal{L}$ be a functor of quasi-categorically enriched categories that reflects equivalences in the sense that any 0-arrow $f\colon A \to B$ in $\mathcal{K}$ whose image is an equivalence in $\mathcal{L}$ is an equivalence in $\mathcal{K}$. Then the corresponding functor $U\colon \mathsf{K} \to \mathsf{L}$ between quasi-categorical cores is conservative.*

Proof. We show that an equivalence $f\colon A \to B$ in a quasi-categorically enriched category $\mathcal{K}$ corresponds to an isomorphism in its quasi-categorical core. An equivalence in $\mathcal{K}$ is comprised of the data enumerated in Example 4.1.2(iii), which is contained in the subcategory $g_*\mathcal{K} \subset \mathcal{K}$. In the homotopy coherent nerve $\mathsf{K} = Ng_*\mathcal{K}$ this data gives rise to a pair of objects A and B, a pair of 1-simplices $f\colon A \to B$ and $f'\colon B \to A$, and a pair of 2-simplices witnessing that f and f' compose to identities. This is the data that defines an isomorphism in a quasi-category. □

Comonadic functors have the following property:

Theorem 5.2.3 (comonadicity and colimit creation III.5.7)**.** *Let $u\colon \mathsf{A} \to \mathsf{B}$ be a comonadic functor between quasi-categories. Then u creates any colimits that B admits.* □

In particular any quasi-category admits colimits of split simplicial objects[8]: a simplicial object $\Delta^{\mathrm{op}} \to \mathsf{B}$ is *split* if it extends along the inclusion $\Delta^{\mathrm{op}} \hookrightarrow \Delta_+^{\mathrm{op}} \hookrightarrow \Delta_\infty$ that augments it with an terminal object — note that $\Delta_+^{\mathrm{op}} \cong \Delta^{\mathrm{op}} \star \mathbb{1}$ — and then adds an "extra degeneracy" map in each dimension (see §I.5.3).

Theorem 5.2.4 (split simplicial objects define colimits I.5.3.1)**.** *Any split simplicial object $\Delta^{\mathrm{op}} \to \mathsf{B}$ admits a colimit, whose colimit cone is given by the augmented diagram $\Delta_+^{\mathrm{op}} \to \mathsf{B}$:*

$$\begin{tikzcd}
& \Delta^{\mathrm{op}} \arrow[dl, hook] \arrow[d, hook] \arrow[dr] & \\
\Delta_+^{\mathrm{op}} \arrow[r, hook] & \Delta_\infty \arrow[r, dashed] & \mathsf{B}
\end{tikzcd}$$

Combining these results, it follows that if a functor admits both left and right adjoints, its monadicity can be leveraged to help establish its comonadicity, or conversely:

Proposition 5.2.5. *A functor that admits both left and right adjoints is monadic if and only if it is comonadic.*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "u" description] & \mathsf{B} \arrow[l, bend right=50, "\ell"', ""{name=L, below}] \arrow[l, bend left=50, "r", ""{name=R, above}]
\end{tikzcd}$$

with $\perp$ between ℓ and u and between u and r.

Proof. If u is comonadic, then u is conservative, verifying condition (iii) of the dual Monadicity Theorem 5.2.1. We have already assumed that the left adjoint required by (i) exists. Finally, Theorem 5.2.4 implies that B admits colimits of u-split simplicial objects, and then comonadicity of u together with Theorem 5.2.3 then implies that A admits them as well and these are preserved by u. This verifies (ii), and Theorem 5.2.1 then implies that u is also monadic. A dual argument proves the converse implication. ☐

Theorem 5.2.6. *The forgetful functor*

$$u\colon \mathsf{Cart}(\mathsf{qCat})_{/\mathsf{B}} \longrightarrow \mathsf{Cart}(\mathsf{qCat})_{/\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$$

is comonadic and hence also monadic.

Proof. We use Theorem 5.2.1 to prove comonadicity and then deduce monadicity from Proposition 5.1.4 and Proposition 5.2.5. The right adjoint to u is constructed in Proposition 5.1.3 proving (i). Example VIII.6.1.7 and Remark VIII.6.1.9 combine to prove that $\mathsf{Cart}_{/\mathsf{B}}$ admits and $\mathsf{Cart}_{/\mathsf{B}} \hookrightarrow \mathsf{qCat}_{/\mathsf{B}}$ preserves all limits. The functor u is the composite of this inclusion with the projection functor $\mathsf{qCat}_{/\mathsf{B}} \to \mathsf{qCat}_{/\mathrm{ob}\,\mathsf{B}}$, which also preserves all limits, being the homotopy coherent nerve of a functor of ∞-cosmoi $\mathcal{QC}at_{/\mathsf{B}} \to \mathcal{QC}at_{/\mathrm{ob}\,\mathsf{B}}$. This proves (ii). Proposition 2.2.2 and Lemma 5.2.2 assert that $u\colon \mathsf{Cart}_{/\mathsf{B}} \to \mathsf{Cart}_{/\mathrm{ob}\,\mathsf{B}}$ is conservative, proving (iii). ☐

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[8] Furthermore, such colimits are *absolute*, that is preserved by any functor.

6. Groupoidal reflection

In this section, we give a first application of the monadicity and comonadicity results of the previous section. A cartesian fibration between quasi-categories is *groupoidal* if its fibers are Kan complexes, rather than quasi-categories. In this section we construct a reflection to the fully faithful inclusion of the quasi-category of groupoidal cartesian fibrations into the quasi-category of cartesian fibrations:

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Cart}_{/\mathsf{B}} \arrow[l, bend right, "\mathrm{invert}"'] \end{tikzcd}$$

A different proof of this result will be appear in a sequel, making use of explicit fiberwise coinverters.

In §6.1, we study the relationship between the quasi-categorically enriched category of cartesian fibrations and its subcategory of groupoidal cartesian fibrations and establish a groupoidal reflection functor in the "base case," reflecting quasi-categories into Kan complexes. In §6.2, we prove that the monadic and comonadic adjunctions of Theorem 5.2.6 restrict to define analogous monadic and comonadic adjunctions for groupoidal cartesian fibrations. It follows that the large quasi-categories $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}}$ and $\mathsf{Cart}_{/\mathsf{B}}$ can be understood as quasi-categories of algebras for closely related homotopy coherent monads acting on $\prod_{b\in\mathrm{ob}\,\mathsf{B}}\mathsf{Kan}$ and $\prod_{b\in\mathrm{ob}\,\mathsf{B}}\mathsf{qCat}$ respectively. In §6.3, we exploit this presentation to construct an adjunction defining the reflection of a cartesian fibration into a groupoidal cartesian fibration:

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Cart}_{/\mathsf{B}} \arrow[l, bend right, "\mathrm{invert}"'] \end{tikzcd}$$

6.1 Reflecting quasi-categories into Kan complexes

Before we begin, we note that the notion of groupoidal cartesian fibration of quasi-categories just defined agrees with the definition given in §IV.4.2, which declares that a cartesian fibration $p\colon \mathsf{E} \twoheadrightarrow \mathsf{B}$ is *groupoidal* just when the quasi-category of functors from any $f\colon X \to \mathsf{B}$ to p is a Kan complex; see X.12.2.3.

At the level of simplicially-enriched categories, the subcategory of groupoidal cartesian fibrations is defined by the pullback:

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}(\mathcal{QC}at)_{/\mathsf{B}} \arrow[rr, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \arrow[d] \\ \mathsf{Cart}^{\mathrm{gr}}(\mathcal{QC}at)_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathcal{K}an \arrow[r, hook] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathcal{QC}at \cong \mathsf{Cart}(\mathcal{QC}at)_{/\,\mathrm{ob}\,\mathsf{B}} \end{tikzcd} \tag{6.1.1}$$

In §6.3, we construct a *groupoidal reflection functor*, by which we mean a left adjoint to the inclusion

$$\mathsf{Cart}^{\mathrm{gr}}(\mathsf{qCat})_{/\mathsf{B}} \hookrightarrow \mathsf{Cart}(\mathsf{qCat})_{/\mathsf{B}}$$

as a functor between large quasi-categories. We begin by describing groupoidal reflection in the case where $\mathsf{B} = 1$.

Theorem 6.1.2. *The inclusion* $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$ *admits both left and right adjoints*

$$\begin{tikzcd} \mathsf{Kan} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{qCat} \arrow[l, bend right=40, "\text{invert}"'] \arrow[l, bend left=40, "\text{core}"] \end{tikzcd}$$

and is monadic and comonadic.

Proof. The inclusion $\mathcal{K}\mathit{an} \hookrightarrow \mathcal{Q}\mathit{Cat}$ is left adjoint to a functor $\mathrm{core}\colon \mathcal{Q}\mathit{Cat} \to \mathcal{K}\mathit{an}$ that carries a quasi-category to the maximal sub Kan complex spanned by the isomorphisms. This adjunction is simplicial with respect to the Kan complex enrichments of both $\mathcal{K}\mathit{an}$ and $\mathcal{Q}\mathit{Cat}$, the latter obtained by applying the core functor to the function complexes, so this simplicially enriched adjunction descends to provide a right adjoint to $\mathcal{K}\mathit{an} \hookrightarrow \mathcal{Q}\mathit{Cat}$.

The left adjoint can also be modeled at the point-set level. The quasi-category qCat is isomorphic to the homotopy coherent nerve of the Kan-complex enriched category of naturally marked quasi-categories in the sense of Example 4.1.1. There is a simplicial Quillen adjunction

$$\begin{tikzcd} \underline{\mathrm{sSet}} \arrow[r, bend right=15, "(-)^\sharp"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}_\bullet} \arrow[l, bend right=15, "U"'] \end{tikzcd}$$

connecting this simplicial model structure for quasi-categories to the Quillen model structure for Kan complexes on simplicial sets. Applying Theorem I.6.2.1, this provides the left adjoint to the inclusion $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$. From this vantage point, we may apply Proposition VII.2.2.3 to see that $\mathcal{K}\mathit{an}$ is closed in $\mathcal{Q}\mathit{Cat}$ under flexible weighted limits, so we conclude that $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$ creates all limits.

The functor $\mathcal{K}\mathit{an} \hookrightarrow \mathcal{Q}\mathit{Cat}$ reflects equivalences, so by Lemma 5.2.2 the inclusion $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$ is conservative. Now Theorem 5.2.1 implies that $\mathsf{Kan} \to \mathsf{qCat}$ is comonadic, and Proposition 5.2.5 then implies that $\mathsf{Kan} \to \mathsf{qCat}$ is also monadic. □

6.2 (Co)monadicity of groupoidal cartesian fibrations

We now argue that

$$\begin{array}{ccc} \mathsf{Cart}^{\mathrm{gr}}(\mathcal{Q}\mathit{Cat})_{/\mathsf{B}} & \longrightarrow & \mathsf{Cart}^{\mathrm{gr}}(\mathcal{Q}\mathit{Cat})_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathcal{K}\mathit{an} \\ (\mathsf{E} \xrightarrow{p} \mathsf{B}) & \mapsto & (\mathsf{E}_b)_{b \in \mathrm{ob}\,\mathsf{B}} \end{array}$$

admits left and right quasi-categorically enriched biadjoints, given by restricting those from the non-groupoidal case, and that moreover the restricted adjunction is both monadic and comonadic at the level of functors between underlying quasi-categories.

Theorem 6.2.1. *The quasi-categorical biadjoints to* $\mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \to \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}}$ *restrict to groupoidal cartesian fibrations*

$$\begin{tikzcd}
\mathsf{Cart}^{\mathrm{gr}}(\mathcal{QC}at)_{/\mathsf{B}} \arrow[r, hook] \arrow[d] & \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}} \arrow[d] \\
\mathsf{Cart}^{\mathrm{gr}}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \arrow[r, hook] \arrow[u, bend left, "L"] \arrow[u, bend right, "R"'] \arrow[u, phantom, shift left=2, "\dashv"] \arrow[u, phantom, shift right=2, "\dashv"] & \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \arrow[u, bend left, "L"] \arrow[u, bend right, "R"'] \arrow[u, phantom, shift left=2, "\dashv"] \arrow[u, phantom, shift right=2, "\dashv"]
\end{tikzcd}$$

and moreover these restricted adjunctions display the functor between the quasi-categorical cores $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \to \mathsf{Cart}^{\mathrm{gr}}_{/\operatorname{ob}\mathsf{B}} \cong \prod_{/\operatorname{ob}\mathsf{B}} \mathsf{Kan}$ *as both monadic and comonadic.*

Proof. Proposition 5.1.4 defines $L\colon \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \to \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}}$ to be the functor that carries a family $(\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}}$ to

$$L((\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}}) := \coprod_b \mathsf{E}_b \times \mathsf{B}\downarrow b \twoheadrightarrow \mathsf{B}.$$

The fiber over $x \in \operatorname{ob}\mathsf{B}$ is $\coprod_b \mathsf{E}_b \times x\downarrow b$. Since $x\downarrow b$ is a Kan complex, it is clear that this fiber is groupoidal if each E_b is a Kan complex. Thus, we see immediately that L restricts to groupoidal cartesian fibrations.

Proposition 5.1.3 defines $R\colon \mathsf{Cart}(\mathcal{QC}at)_{/\operatorname{ob}\mathsf{B}} \to \mathsf{Cart}(\mathcal{QC}at)_{/\mathsf{B}}$ to be the functor that carries a family $(\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}}$ to

$$R((\mathsf{E}_b)_{b\in\operatorname{ob}\mathsf{B}}) := \prod_b (\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow\mathsf{B}\twoheadrightarrow\mathsf{B}}.$$

The fiber over $x \in \operatorname{ob}\mathsf{B}$ of this product of fibrations is isomorphic to the product of the fibers of each $(\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow\mathsf{B}\twoheadrightarrow\mathsf{B}}$ over x, so it suffices to show that each of these fibers is a Kan complex if E_b is a Kan complex. By the bijection of Definition 4.3.1, a 1-simplex in the fiber of $(\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow\mathsf{B}\twoheadrightarrow\mathsf{B}}$ over $x\colon 1 \to \mathsf{B}$ corresponds to the displayed dashed map

$$\begin{tikzcd}
\Delta^1 \times b\downarrow x \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rrr, dashed, bend left] & b\downarrow x \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & b\downarrow\mathsf{B} \arrow[d, two heads] & \mathsf{E}_b \times \mathsf{B} \arrow[dl, two heads, "\pi"] \\
\Delta^1 \arrow[r] & \Delta^0 \arrow[r, "x"'] & \mathsf{B} &
\end{tikzcd}$$

i.e., to a map $\Delta^1 \times b\downarrow x \to \mathsf{E}_b$. If each E_b is a Kan complex, this map can be extended along the inclusion $\Delta^1 \hookrightarrow \mathbb{I}$ from the 1-simplex into the free-living isomorphism. This proves that every 1-simplex in the fiber of $(\mathsf{E}_b \times \mathsf{B} \twoheadrightarrow \mathsf{B})^{b\downarrow\mathsf{B}\twoheadrightarrow\mathsf{B}}$ is an isomorphism, which tells us that R restricts to groupoidal cartesian fibrations.

By what is now a familiar line of argument, we apply Theorem 5.2.1 to prove that the restricted adjunctions are comonadic, and then deduce monadicity from Proposition 5.2.5. The required adjoints have already been constructed and Proposition 2.2.2 and Lemma 5.2.2 imply that $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \to \mathsf{Cart}^{\mathrm{gr}}_{/\operatorname{ob}\mathsf{B}}$ is conservative, so it remains only to establish condition (ii) of Theorem 5.2.1.

The pullback of simplicial categories (6.1.1) is preserved by passing to the level of quasi-categories:

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cart}_{/\mathsf{B}} \arrow[d] \\ \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[r, hook] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \end{tikzcd}$$

By the monadicity established in Theorems 5.2.6 and 6.1.2 and the dual of Theorem 5.2.3, both the left-hand vertical and lower horizontal functors create all limits present in $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$, which is to say all limits, since by Proposition VII.6.2.1 qCat is complete. The lower inclusion is replete up to isomorphism, so $\mathsf{Kan} \hookrightarrow \mathsf{qCat}$ defines an isofibration of large quasi-categories. Thus, Lemma X.6.3.17 applies to tell us that $\mathsf{Cart}^{\mathrm{gr}}(\mathsf{qCat})_{/\mathsf{B}}$ is also complete and all limits are preserved by the left-hand vertical functor.

Now Theorem 5.2.1 implies that $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \to \mathsf{Cart}^{\mathrm{gr}}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}$ is comonadic, and monadicity follows from Proposition 5.2.5. □

6.3 Groupoidal reflection

Our monadicity results, Theorems 6.2.1 and 5.2.6, tell us that the quasi-categories $\mathsf{Cart}_{/\mathsf{B}}$ and $\mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}}$ are equivalent to the quasi-categories of algebras associated to closely related homotopy coherent monads acting on $\mathsf{Cart}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ and $\mathsf{Cart}^{\mathrm{gr}}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}$ respectively. In this section, we will use this result to lift the reflection functor $\mathrm{invert}\colon \mathsf{qCat} \to \mathsf{Kan}$ from quasi-categories to Kan complexes to a groupoidal reflection functor $\mathrm{invert}\colon \mathsf{Cart}_{/\mathsf{B}} \to \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}}$ that is left adjoint to the inclusion.

To do this we make use of a convenient representation for adjoint functors that can be expressed in any 2-category, dual to the more familiar representation of the unit of an adjunction as an absolute left extension diagram:

Lemma 6.3.1 (I.5.0.4)**.** *To define a left adjoint to a functor $u\colon A \to B$ is to define an absolute left lifting of id_B along u, in which case $f \dashv u$ with unit $\eta\colon \mathrm{id}_B \Rightarrow uf$.*

$$\begin{tikzcd} & A \arrow[d, "u"] \\ B \arrow[ur, dashed, "f"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & B \end{tikzcd}$$
□

Let $\lrcorner$ denote the category indexing a cospan and write $\mathcal{QC}\!at^{\lrcorner}$ for the simplicially enriched category of cospans of quasi-categories, whose objects are cospans and whose 0-arrows are natural transformations

$$\begin{tikzcd} & \mathsf{B} \arrow[d, "f"'] \arrow[dr, "v"] & \\ \mathsf{C} \arrow[r, "g"] \arrow[dr, "w"'] & \mathsf{A} \arrow[dr, "u"] & \mathsf{B}' \arrow[d, "f'"] \\ & \mathsf{C}' \arrow[r, "g'"'] & \mathsf{A}' \end{tikzcd} \tag{6.3.2}$$

Definition 6.3.3. Transformations of the kind depicted in (6.3.2) between diagrams which admit absolute left liftings give rise to the following diagram

$$\begin{tikzcd}
 & \mathsf{B} \arrow[d, "f"] \arrow[dr, "v"] & & & & \mathsf{B} \arrow[dr, "v"] & \\
\mathsf{C} \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[dr, "w"'] & \mathsf{A} \arrow[dr, "u"] & \mathsf{B}' \arrow[d, "f'"] & = & \mathsf{C} \arrow[ur, "\ell"] \arrow[dr, "w"'] & \Uparrow\tau & \mathsf{B}' \arrow[d, "f'"] \\
 & \mathsf{C}' \arrow[r, "g'"'] & \mathsf{A}' & & & \mathsf{C}' \arrow[ur, "\ell'"] \arrow[r, "g'"'] & \mathsf{A}'
\end{tikzcd} \tag{6.3.4}$$

in which the triangles are absolute left liftings and the 2-cell τ is induced by the universal property of the triangle on the right. We say that the transformation (6.3.4) is *left exact* if and only if the induced 2-cell τ is an isomorphism. This left exactness condition holds if and only if, in the diagram on the left, the whiskered 2-cell $u\lambda$ displays $v\ell$ as the absolute left lifting of $g'w$ through f'.

Our interest in these notions is on account of the following result:

Proposition 6.3.5 (III.4.9). *Consider any simplicial functor $T\colon \mathcal{A} \to \mathcal{QC}at^{\lrcorner}$ and any flexible weight $W\colon \mathcal{A} \to \mathcal{SS}et$. If each of the objects in the image of T admits an absolute left lifting and each of the 0-arrows in the image of T is left exact, then the weighted limit $\lim^W T \in \mathcal{QC}at^{\lrcorner}$ admits an absolute left lifting and the legs of the limit cone are left exact transformations.* ☐

By Lemma II.6.1.8, the quasi-category of algebras construction, introduced in Definition II.6.1.7, is an instance of a flexible weighted limit. We will use Proposition 6.3.5 applied in a larger Grothendieck universe to the quasi-categorically enriched category $\mathcal{QCAT}$ of large quasi-categories to lift the absolute left lifting diagram

$$\begin{tikzcd}
 & \mathsf{Kan} \arrow[d, hook] \\
\mathsf{qCat} \arrow[ur, "\text{invert}"] \arrow[r, equal] & \mathsf{qCat} \arrow[ul, phantom, "\Uparrow\eta"]
\end{tikzcd}$$

whose left adjoint part defines the groupoidal reflection associated to the inclusion

$$\mathsf{Cart}^{\mathrm{gr}}_{/\,\mathrm{ob}\,\mathsf{B}} \cong \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \hookrightarrow \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \cong \mathsf{Cart}_{/\,\mathrm{ob}\,\mathsf{B}}$$

to these flexible weighted limits, defining an absolute left lifting diagram

$$\begin{tikzcd}
 & \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[d, hook] \\
\mathsf{Cart}_{/\mathsf{B}} \arrow[ur, dashed, "\text{invert}"] \arrow[r, equal] & \mathsf{Cart}_{/\mathsf{B}} \arrow[ul, phantom, "\Uparrow\eta"]
\end{tikzcd}$$

the left adjoint being the desired groupoidal reflection functor.

Theorem 6.3.6. *There is a left adjoint to the inclusion*

$$\begin{tikzcd} \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Cart}_{/\mathsf{B}} \arrow[l, bend right, "\text{invert}"'] \end{tikzcd}$$

defining the reflection of a cartesian fibration into a groupoidal cartesian fibration.

Proof. By Theorem II.4.3.9, the adjunction

$$\begin{tikzcd} \mathsf{Cart}_{/\mathsf{B}} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[l, bend right, "L"'] \end{tikzcd} \tag{6.3.7}$$

defined in Proposition 5.1.4 extends to a *homotopy coherent adjunction*: a simplicial functor $\underline{\mathrm{Adj}} \to \mathcal{QCAT}$ valued in the quasi-categorically enriched category of large quasi-categories whose domain is the simplicial computad obtained by applying the nerve functor to the hom-categories of the free 2-category containing an adjunction; see §II.3. The two objects of $\underline{\mathrm{Adj}}$, called "$-$" and "$+$" are mapped to the quasi-categories $\mathsf{Cart}_{/\mathsf{B}}$ and $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ respectively. The full subcategory $\underline{\mathrm{Mnd}} \hookrightarrow \underline{\mathrm{Adj}}$ on the object "$+$" defines the free *homotopy coherent monad*. In this case, the data of the underlying homotopy coherent monad $T\colon \underline{\mathrm{Mnd}} \hookrightarrow \underline{\mathrm{Adj}} \to \mathcal{QCAT}$ on the object $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ is given by a the map of objects $+ \mapsto \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ and the map of function complexes

$$T\colon \mathbb{\Delta}_+ = \mathsf{Fun}_{\underline{\mathrm{Mnd}}}(+,+) \to \mathsf{Fun}_{\mathcal{QCAT}}(\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}, \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}),$$

satisfying an appropriate simplicial functoriality condition. This functoriality condition implies that the 0-arrows of $\underline{\mathrm{Mnd}}$, are indexed by the objects $[n] \in \mathbb{\Delta}_+$, are all finite composites of the object $[0]$, whose image

$$t := T[0]\colon \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \to \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$$

is the monad endofunctor defined by composing the left and right adjoints of (6.3.7).

To apply Proposition 6.3.5 we must extend the homotopy coherent monad T to a homotopy coherent monad on $\mathcal{QCAT}^{\lrcorner}$. To do so, we argue that this homotopy coherent monad restricts along $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \hookrightarrow \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$ to define a homotopy coherent monad $T^{\mathrm{gr}}\colon \underline{\mathrm{Mnd}} \to \mathcal{QCAT}$ on $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}$ in such a way that this map will define the component of a simplicial natural transformation $T^{\mathrm{gr}} \Rightarrow T$. To see this, note that $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}$, the nerve of the simplicially enriched category spanned by $\mathrm{ob}\,\mathsf{B}$-indexed families of Kan complexes, is a full sub quasi-category of $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$, the nerve of the simplicially enriched category spanned by $\mathrm{ob}\,\mathsf{B}$-indexed families of quasi-categories, in the sense that it contains all of the n-simplices whose vertices are Kan complexes, not mere quasi-categories. So to check that the data of the homotopy coherent monad restricts to define a simplicial functor given by $+ \mapsto \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}$ and

$$T^{\mathrm{gr}}\colon \mathbb{\Delta}_+ = \mathsf{Fun}_{\underline{\mathrm{Mnd}}}(+,+) \to \mathsf{Fun}_{\mathcal{QCAT}}(\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}, \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan}),$$

it suffices to check this at the level of vertices $[n] \in \Delta_+$, which amounts to checking that the monad t of (6.3.7) restricts to define a monad t^{gr} on groupoidal cartesian fibrations; this was done in Theorem 6.2.1.

In this way we obtain a simplicial functor $\underline{\mathrm{Mnd}} \to \mathcal{QCAT}^{\mathbb{2}}$ sending the object "+" to the arrow $\prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \hookrightarrow \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat}$. Pairing this with the identity simplicial natural transformation we obtain a simplicial functor $\underline{\mathrm{Mnd}} \to \mathcal{QCAT}^{\lrcorner}$ sending the object "+" to the cospan displayed below-left

$$\begin{tikzcd} & \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[d, hook] \\ \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[r, equal] & \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \end{tikzcd} \qquad \begin{tikzcd} & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[d, hook] \\ \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[r, equal] \arrow[ur, "\prod_{\mathrm{ob}\,\mathsf{B}} \mathrm{invert}"] \arrow[ur, phantom, shift right=3ex, "\Uparrow\eta"] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \end{tikzcd} \tag{6.3.8}$$

This cospan admits an absolute left lifting displayed above right, defining the left adjoint and unit of an adjunction whose counit is invertible. In fact the entire diagram $\underline{\mathrm{Mnd}} \to \mathcal{QCAT}^{\lrcorner}$ restricts to the subcategory spanned by those cospans that admit absolute left liftings and those 0-arrows that define left exact transformations between them. To see this, we need only argue that the generating 0-arrow $[0]$ in $\underline{\mathrm{Mnd}}$, the endofunctor of the monad, defines a left exact transformation. That is, we must show that the endotransformation of (6.3.8) whose components are the functors

$$\begin{tikzcd} \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[r, "t^{\mathrm{gr}}"] & \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \\ (\mathsf{E}_b)_{b\in\mathsf{B}} \arrow[r, phantom, "\mapsto"] & \left(\coprod\limits_{b\in \mathrm{ob}\,\mathsf{B}} \mathsf{E}_b \times x \downarrow b\right)_{x\in \mathrm{ob}\,\mathsf{B}} \end{tikzcd} \qquad \begin{tikzcd} \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[r, "t"] & \prod_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \\ (\mathsf{E}_b)_{b\in\mathsf{B}} \arrow[r, phantom, "\mapsto"] & \left(\coprod\limits_{b\in \mathrm{ob}\,\mathsf{B}} \mathsf{E}_b \times x \downarrow b\right)_{x\in \mathrm{ob}\,\mathsf{B}} \end{tikzcd}$$

is left exact. This amounts to showing that the whiskered 2-cell

$$\begin{tikzcd} & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[r, "t^{\mathrm{gr}}"] \arrow[d, hook] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{Kan} \arrow[d, hook] \\ \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[r, equal] \arrow[ur, "\prod_{\mathrm{ob}\,\mathsf{B}} \mathrm{invert}"] \arrow[ur, phantom, shift right=3ex, "\Uparrow\eta"] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \arrow[r, "t"'] & \prod\limits_{\mathrm{ob}\,\mathsf{B}} \mathsf{qCat} \end{tikzcd}$$

is invertible. This is the case because the process of freely inverting a family of quasi-categories commutes up to equivalence with forming the product with the Kan complex $x \downarrow b$ and with the coproduct $\coprod_{b\in\mathrm{ob}\,\mathsf{B}}$.

In this way we obtain a homotopy coherent monad $\underline{\mathrm{Mnd}} \to \mathcal{QCAT}^{\lrcorner}$ valued in the subcategory of cospans admitting absolute left liftings and left exact transformations between them. There is a flexible weight $W_- \colon \underline{\mathrm{Mnd}} \to \mathcal{SSet}$ introduced in Definition II.6.1.7 — the precise details of which are not relevant here — so that the W_--weighted limit of a homotopy coherent monad define its quasi-category of algebras, as characterized up to equivalence by the Monadicity Theorem 5.2.1. By Theorems 6.2.1 and 5.2.6 the W_--weighted limit of the composite diagram $\underline{\mathrm{Mnd}} \to \mathcal{QCAT}^{\lrcorner}$ defines the cospan displayed below and by Proposition 6.3.5 it therefore admits an absolute left lifting:

$$\begin{tikzcd}
 & \mathsf{Cart}^{\mathrm{gr}}_{/\mathsf{B}} \arrow[d, hook] \\
\mathsf{Cart}_{/\mathsf{B}} \arrow[r, equal] \arrow[ur, dashed, "\mathrm{invert}"] \arrow[ur, phantom, "\Uparrow\eta", shift left=4ex, near end] & \mathsf{Cart}_{/\mathsf{B}}
\end{tikzcd}$$

By Lemma 6.3.1, this absolute left lifting diagram defines the adjunction that constructs the groupoidal reflection of a cartesian fibration. □

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