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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXV-4 (2024)

TOPOLOGICAL PROOFS OF CATEGORICAL COHERENCE

Pierre-Louis CURIEN and Guillaume LAPLANTE-ANFOSSI

"We shall construct KP_n, as a CW-complex, in Section 2 and show that it is an $(n-1)$-ball. This gives an instant one-step proof of Mac Lane's theorem in full generality." – Mikhail M. Kapranov

Résumé. Nous donnons une courte preuve topologique de cohérence pour les opérades non-symétriques catégorifiées en utilisant le fait que les diagrammes impliqués forment le 1-squelette de CW complexes simplement connexes. Nous obtenons également une preuve topologique "en une étape" du théorème de cohérence de Mac Lane pour les catégories monoïdales symétriques, tel que suggéré par Kapranov en 1993. Notre analyse est basée sur une notion combinatoire d'homotopie que nous étudions plus en détail dans le cas particulier des complexes polyédraux, conduisant à une seconde preuve géométrique de cohérence qui est très proche de l'argument original de Mac Lane. Nous utilisons la théorie de Morse pour montrer que cette seconde méthode est strictement moins générale que la première. Nous fournissons une analyse détaillée de la façon dont les deux méthodes nous permettent de déduire ces deux résultats de cohérence catégorielle et discutons de généralisations possibles aux catégories supérieures.

Abstract. We give a short topological proof of coherence for categorified non-symmetric operads by using the fact that the diagrams involved form the 1-skeleton of simply connected CW complexes. We also obtain a "one-step" topological proof of Mac Lane's coherence theorem for symmetric monoidal categories, as suggested by Kapranov in 1993. Our analysis is based on a notion of combinatorial homotopy, which we further study in the special case of polyhedral complexes, leading to a second geometrical proof of coherence which is very close to Mac Lane's original argument. We use Morse theory to show that this second method is strictly less general than the first. We provide a detailed analysis of how both methods allow us to deduce these two categorical coherence results and discuss possible generalizations to higher categories.

Keywords. Categorified operads, categorical coherence, Seifert–Van Kampen theorem, polytopes, Mac Lane coherence theorem, rewriting theory, Morse theory.

Mathematics Subject Classification (2010). Primary 18N20, Secondary 52B11.

Introduction

The n-dimensional permuto-associahedron, a CW-complex whose faces are in bijection with parenthesized ordered partitions of $n+1$ letters, was first introduced by M. Kapranov in his study of higher dimensional Yang–Baxter equations, through the moduli spaces of curves $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ and the solutions of the Knizhnik–Zamolodchikov equation [Kap93]. It was later realized as a convex polytope by V. Reiner and G. M. Ziegler [RZ94], and more recently through the nested braid fan by F. Castillo and F. Liu in [CL23].

The present study stems from a desire to understand the epigraph, taken from the introduction of [Kap93]: what is the precise relationship between the permuto-associahedron and Mac Lane's coherence theorem for symmetric monoidal categories? We show that the *simple connectedness* of the former implies the latter, thereby refining and proving Kapranov's claim (see Theorem 2.16).

This is done through a general "topological coherence theorem" which applies to any simply connected, regular CW complex (Theorem 1.1). Applying it to the operahedra, another family of polytopes which encodes categorified non-symmetric operads [DP15, COI19, Lap22], we obtain a "one-step" proof of the associated coherence theorem as well.

There is little price to pay, though. For both theorems, one needs to provide a precise bijective correspondence between the 1-skeleton (resp. the 2–cells) on the topological side, and canonical morphisms (resp. bifunctoriality, naturality, and applications of coherence conditions) on the categorical side (Propositions 2.13 and 2.6). Since the 2-skeleton of the permuto-associahedra corresponds to other basic canonical morphisms and coherence conditions than those of Mac Lane (hexagons and naturality of the involutive braiding on one hand versus dodecagons on the other hand), one needs to show that the two presentations are equivalent, which is non-trivial, see Remark 2.17. There is yet a third equivalent presentation (and hence another proof of coherence) due to D. Baralić, J. Ivanović and D. Petrić [BIP19], that matches the 2-skeleton of a different polytope, which unlike the permuto-associahedron is simple, see Remark 2.14.

We further investigate a topological incarnation of Mac Lane's original argument, in the spirit of rewriting theory. We study polyhedral complexes endowed with a generic orientation vector, or equivalently a Morse function in the sense of [BB97], whose 1-skeletons naturally feature terminating and confluent rewriting systems (Proposition 1.12). We focus on the family of simply connected polyhedral complexes whose outgoing links are connected. The study of directed paths on their 1-skeleton leads to a second general proof of coherence (Theorem 1.9). In particular, this second theorem can be applied to all polytopes, allowing us to give a second, "rewriting-theoretic" proof of both previously mentioned coherence results. In the case of operahedra, our rewriting proof simplifies the original proof of Došen and Petrić [DP15], see Remark 2.2.

It is worth noting that, while the above polyhedral complexes admit *abstract* rewriting systems on their 1-skeleton, the family of operahedra (which includes the associahedra, encoding non-symmetric monoidal categories) further admits *term* rewriting systems, which exhibit more structure and are the subject of a companion paper [CLA24]. In contrast, we shall argue that the abstract rewriting approach to *symmetric* monoidal categories is not informative, see Remark 2.18.

Using Morse theory on affine cell complexes [BB97], we relate our two approaches by showing that the second is (strictly) less general than the first (Proposition 1.5).

Our two general topological coherence theorems can be used to prove other categorical results where polytopes appear, such as coherence for monoidal functors between monoidal categories [Eps66], see Section 3.1. They also shed light on some statements in the literature, such as the proof of [KV94, Prop. 3.9], see Section 3.2. This all points towards further investigation of the relationship between n-categorical coherence and n-connectedness of appropriate spaces. While topological proofs of 2-categorical coherence already appeared in [Gur11], higher dimensional results have been obtained recently by S. Barkan in the context of ∞-operads [Bar22], for which the present results could well be the strict, $n = 1$ case.

1. Topological coherence

1.1 Coherence à la Van Kampen

Let X be a regular CW complex, and let X_k, $k \geq 0$ denote its k-skeleton. For an edge e of X, denote its attaching map $f_e : \mathbb{S}^0 \to X_0$. Consider the category $\mathcal{A}(X)$ with set of objects X_0, and generating morphisms $\alpha_e : f_e(-1) \to f_e(1)$ and $\alpha_e^{-1} : f_e(1) \to f_e(-1)$ for each edge $e \in X_1$. A *combinatorial path* on X is a composable sequence of α and α^{-1} morphisms (a *word* in α and α^{-1}). Two combinatorial paths $\gamma, \gamma' \in \mathcal{A}(X)(x, y)$ with the same endpoints are said to be *parallel*.

Let A be a 2-cell of X, let $f_A : \mathbb{S}^1 \to X_1$ be its attaching map, and let $x \in X_0$ be a vertex in the image of f_A. Then f_A defines a morphism $\gamma_A \in \mathcal{A}(X)(x, x)$, given by the sequence of edges $e_1, \ldots, e_n$ in its image starting at x and respecting the anti-clockwise orientation of $\mathbb{S}^1$. Here, one selects α_{e_i} if the orientation of f_A restricted to e_i agrees with the one of f_{e_i}, and $\alpha_{e_i}^{-1}$ otherwise. Two parallel combinatorial paths γ, γ' are said to be *elementary combinatorially homotopic* if they differ exactly by a relation of the form $\alpha_e \alpha_e^{-1} = \mathrm{id}_{f_e(1)}$ or $\alpha_e^{-1} \alpha_e = \mathrm{id}_{f_e(-1)}$, or of the form $\gamma_A = \mathrm{id}_x$, for some 2-cell A and vertex x as above. That is, one can rewrite γ into γ' or γ' into γ by replacing some (possibly empty) subword of γ with an equivalent subword using a relation $\gamma_A = \mathrm{id}_x$. More generally, two parallel combinatorial paths are *combinatorially homotopic* if they are related by a sequence of elementary combinatorial homotopies.

The quotient of the category $\mathcal{A}(X)$ by the relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \mathrm{id}$ is the free groupoid $\mathcal{F}(X)$ generated by the α morphisms. Let $\mathcal{C}(X)$ denote the further quotient of the groupoid $\mathcal{F}(X)$ by the relations $\gamma_A = \mathrm{id}_x$ for some choice of x, for each 2-cell A of X. In other words, $\mathcal{C}(X)$ is the quotient of $\mathcal{A}(X)$ by the combinatorial homotopy equivalence relation. Note that the definition of $\mathcal{C}(X)$ does not depend on the choice of x, for every 2-cell A. Indeed, if $x' \neq x \in A_0$ defines a relation $\gamma'_A = \mathrm{id}_{x'}$, we have $\gamma'_A = \delta\gamma_A\delta^{-1}$

in $\mathcal{F}(X)$, where δ is the morphism in $\mathcal{A}(X)(x,x')$ induced by γ_A. Thus, a path γ can be rewritten into γ' using $\gamma_A = \mathrm{id}_x$ if and only if it can be rewritten using $\gamma'_A = \mathrm{id}_{x'}$.

Let $\Pi(X)$ denote the *fundamental groupoid* of X, that is the groupoid with objects the points of X and morphisms the homotopy classes of paths between them.

Theorem 1.1. *Let X be a regular CW complex. Any two parallel combinatorial paths on X are combinatorially homotopic if and only if every path component of X is simply connected.*

Proof. For $Y \subseteq X$, let us write $\Pi(X)Y$ for the full subcategory of the fundamental groupoid of X spanned by Y. Then, we have an isomorphism of groupoids

$$\Pi(X)X_0 \cong \mathcal{C}(X) .$$

To show this, one proceeds in three steps. First, one shows that the fundamental groupoid $\Pi(X_1)X_0$ of the 1-skeleton of X is free on the homotopy classes of maps generated by the attaching maps of the 1-cells, that is, free on the α-morphisms [Bro06, 9.1.5]. Thus, one gets $\Pi(X_1)X_0 \cong \mathcal{F}(X)$. Second, one shows that the fundamental groupoid $\Pi(X_2)X_0$ of the 2-skeleton of X is the free groupoid $\Pi(X_1)X_0$ modulo the relations $\gamma_A = 1$, for A a 2-cell of X [Bro06, 9.1.6]. This is done through repeated application of the Seifert–Van Kampen theorem; one then has $\Pi(X_2)X_0 \cong \mathcal{C}(X)$. Third, one shows that the inclusion of X_2 in X induces an isomorphism of fundamental groupoids $\Pi(X_2)X_0 \cong \Pi(X)X_0$ [Bro06, 9.1.7], which concludes the proof of the isomorphism $\Pi(X)X_0 \cong \mathcal{C}(X)$. The theorem then follows, since every path component of X is simply connected if and only if its fundamental groupoid $\Pi(X)$ is trivial, which holds if and only if its full subcategory $\Pi(X)X_0$ is trivial. □

Note that any CW complex is locally path connected, and therefore is connected if and only if it is path connected. Therefore, we could have replaced in the preceding theorem "path component" by "connected component".

Let us say that X is *combinatorially connected* if there is a combinatorial path between any two vertices of X. In the course of the preceding proof, we have in particular showed the following.

Corollary 1.2. *A regular CW complex is combinatorially connected if and only if it is connected.*

1.2 Coherence à la Morse

Let $X \subset \mathbb{R}^n$ be a polyhedral complex. Let $\vec{v} \in \mathbb{R}^n$ be *generic* on the edges of X, meaning that for any pair of vertices $x, y \in X$ belonging to the same edge of X, we have $\langle \vec{v}, x\rangle \neq \langle \vec{v}, y\rangle$. Such a generic vector $\vec{v}$ induces a natural orientation on the edges of X, directed from the source vertex where the functional $\langle \vec{v}, -\rangle$ is minimal to the target vertex where it is maximal.

One of the basic, very useful facts about polyhedral complexes with a generic vector is that, for any face $F \subseteq X$ of X, there is a unique *source* vertex $\mathrm{sc}(F)$ such that all its adjacent edges $e \subseteq F$ are outgoing, and a unique *sink* vertex $\mathrm{sk}(F)$ whose adjacent edges are all incoming, see [Zie95, Thm. 3.7]. More generally a vertex whose adjacent edges $e \subseteq X$ are all incoming is called a *local sink*, and when X has only one such vertex, we call it *global sink* and denote it by $\mathrm{sk}(X)$.

Let $H := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y\rangle = 0\}$ be the linear hyperplane orthogonal to $\vec{v}$. For every vertex $x \in X$, choose $\varepsilon > 0$ such that the interval between $\langle \vec{v}, x\rangle$ and $\langle \vec{v}, x\rangle + \varepsilon$ does not contain the image of any other vertex under the "height" function $\langle \vec{v}, -\rangle$.

Definition 1.3. The *outgoing link* $\mathrm{Lk}^+(x, X)$ of a vertex $x \in X$ is the intersection $\mathcal{F} \cap (H + x + \varepsilon\vec{v})$ of the family of faces $\mathcal{F}(x, X) := \{F \subseteq X \mid \mathrm{sc}(F) = x\}$ with the affine hyperplane $H + x + \varepsilon\vec{v}$.

Recall from [Zie95, Sec. 2.1] that the *vertex figure* P/x of a polytope P at a vertex x is obtained by cutting P by a hyperplane that cuts off the single vertex x. Such a cut establishes a bijection between the $(k-1)$-faces of P/x and the k-faces of P which contain x [Zie95, Prop. 2.4].

Lemma 1.4. *Let X be a polyhedral complex with a generic vector. For any $k \geq 0$, there is a bijection between the k-faces of $\mathcal{F}(x, X)$ and the $(k-1)$-faces of $\mathrm{Lk}^+(x, X)$.*

Proof. Each maximal face of $\mathcal{F}(x, X)$ with respect to inclusion is a polytope P, for which the intersection $P \cap (H + x + \varepsilon\vec{v})$ is the vertex figure P/x of P at x. By [Zie95, Prop. 2.4], there is a bijection between the k-faces of P and the $(k-1)$-faces of P/x. Collecting these bijections for all maximal faces of $\mathcal{F}(x, X)$, and making the appropriate identifications, we get the desired global bijection. □

In this section we shall focus on polyhedral complexes whose outgoing links are connected. The following proposition gives the topological significance of this condition.

Proposition 1.5. *Let X be a polyhedral complex. If there is a generic vector such that the outgoing link of every vertex is connected, then every path component of X is simply connected.*

Proof. Let $\vec{v} \in \mathbb{R}^n$ be generic with respect to X, and suppose that the outgoing link of every vertex is connected. Since $\vec{v}$ is generic on edges, it defines a Morse function $\langle \vec{v}, - \rangle$ on X, in the sense of [BB97, Def. 2.2]. As in classical Morse theory, one can determine the homotopy type of X by considering its successive level sets. For $t \in \mathbb{R}$ denote by X_t the closed subspace of X containing points x such that $\langle x, \vec{v} \rangle$ is at least t. Let x be a vertex of X of height $h = \langle x, \vec{v} \rangle$. Observe first that $X_{h+\varepsilon}$, for some small $\varepsilon > 0$, is homotopy equivalent to $X_{h'}$ where $h' > h$ is the next greater height at which there is a vertex. That is, the homotopy type of X can only change at vertices [BB97, Lem. 2.3]. Then, one proves that X_h is homotopy equivalent to the pushout of $X_{h+\varepsilon}$ with the cone over the outgoing link of x along the outgoing link of x [BB97, Lem. 2.5]. By our assumption, the outgoing link of x is connected, and thus the cone over it is simply connected. Since the pushout of simply connected spaces over a connected space is always simply connected (this is an application of the Seifert–Van Kampen theorem), we obtain by induction that every path component of X is simply connected [BB97, Point (3) of Cor. 2.6]. □

The converse of Proposition 1.5 is not true in general: many simply connected polyhedral complexes, as the one represented in Figure 1, have disconnected outgoing links, for many (sometimes for all) choices of generic vectors.

An important class of complexes which have connected outgoing links are polytopes, which will be our main object of study in the next sections.

Proposition 1.6. *Let P be a polytope with a generic vector. The outgoing link of every vertex of P is connected.*

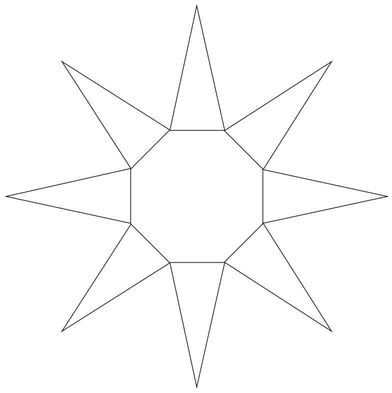

Figure 1: A simply connected polyhedral complex which admits disconnected outgoing links for every choice of generic vector.

Proof. Define the linear hyperplane $H := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle = 0\}$, and consider the two half-spaces $H^- := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle < 0\}$ and $H^+ := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle < 0\}$. Since $\vec{v}$ is not perpendicular to any edge of P, it defines a partition of the vertices of the vertex figure P/x into two connected components: the vertices that lie in H^-, which correspond to incoming edges of P at x, and the vertices that lie in H^+, which correspond to outgoing edges of P at x. Thus, the outgoing link of x is connected, and the proof is complete. □

From now on we shall suppose that the polyhedral complexes that we consider are endowed with a regular CW structure and provided with a generic vector. Combining Proposition 1.5 with Theorem 1.1, we have that any polyhedral complex X whose outgoing links are connected satisfies the property that "any two parallel combinatorial paths on X are combinatorially homotopic". We shall now derive this same result by following an alternative, more combinatorial path (indeed!), getting close to the proof of [ML63, Thm 3.1].

A combinatorial path γ on a polyhedral complex X is *directed* if for any pair (e, f) of consective edges in γ, we have that $\mathrm{sk}(e) = \mathrm{sc}(f)$. When no ambiguity arises, we will omit the adjective "combinatorial" and say only "directed path".

In the rest of this section we shall use the notion of combinatorial connectedness, which as we have seen in Corollary 1.2 is equivalent to connectedness for the spaces we consider.

Lemma 1.7. *Let X be a polyhedral complex with generic vector $\vec{v}$ such that the outgoing link of every vertex is combinatorially connected. Let e, e' be two edges of X such that $\mathrm{sc}(e) = \mathrm{sc}(e')$, and suppose that there are directed paths from $\mathrm{sk}(e)$ and $\mathrm{sk}(e')$ to local sinks s and s', respectively. Then, we have $s = s'$.*

Proof. Define the *height* $\mathfrak{h}(x)$ of a vertex x as the length of the longest directed path in X starting at x. Since the vector $\vec{v}$ is generic and X_0 is finite, this is well-defined. We proceed by induction on $\mathfrak{h}(x)$. The statement holds vacuously for vertices x such that $\mathfrak{h}(x) = 0$. Suppose that the assertion above holds for all vertices $x \in X$ such that $\mathfrak{h}(x) = n$, and consider a vertex x with $\mathfrak{h}(x) = n + 1$. Since the outgoing link $\mathrm{Lk}^+(x, X)$ is combinatorially connected, there is a combinatorial path θ in $\mathrm{Lk}^+(x, X)$ between the vertices corresponding to e and e' (Lemma 1.4). The path θ determines a sequence of edges $e_0 := e, e_1, \dots, e_k, e' =: e_{k+1}$ of X with $\mathrm{sc}(e_i) = x$ for all $0 \leq i \leq k + 1$. Moreover, each consecutive pair e_i, e_{i+1} determines a 2-face F_{i+1} of X. Now, choose for each e_i with $1 \leq i \leq k$, a directed path of maximal length starting at $\mathrm{sk}(e_i)$ and passing through $\mathrm{sk}(F_i)$. Each of these paths ends at a local sink s_i, including $s_0 := s$ and $s_{k+1} := s'$. Since we have $\mathfrak{h}(\mathrm{sk}(e_i)) < \mathfrak{h}(x)$ for all $0 \leq i \leq k + 1$, we can apply induction to the two directed paths from $\mathrm{sk}(e_i)$ to s_i and s_{i+1}, which gives $s_i = s_{i+1}$. Therefore, we have $s = s_0 = s_1 = \cdots = s_k = s_{k+1} = s'$, as desired. □

Two parallel directed paths are said to be *elementary combinatorially homotopic* if they are as undirected paths. They are *combinatorially homotopic* if they are related by a sequence of elementary combinatorial homotopies between directed paths.

The following Proposition 1.8 and its consequence Theorem 1.9 express in topological terms the original proof technique used by Mac Lane in [ML63, Thm 3.1]. Note that Proposition 1.8 involves first *directed* paths, while Theorem 1.9 treats the general, undirected case.

Proposition 1.8. *Let X be a polyhedral complex with a generic vector. Consider the following three properties:*

(i) the outgoing link of every vertex is combinatorially connected,

(ii) there is a global sink in every connected component,

(iii) any two parallel directed *combinatorial paths on X are combinatorially homotopic.*

Then, X satisfies (i) if and only if it satisfies (ii) and (iii).

Proof. First, we prove that (i) implies (ii). Suppose that there are two local sinks s_1 and s_2 in the same connected component of X. Consider a combinatorial path γ between s_1 and s_2, whose existence is garanteed by Corollary 1.2. We proceed by induction on the number of *peaks* in γ, that is the number of vertices x which are the source $\mathrm{sc}(e) = x = \mathrm{sc}(e')$ of two edges e, e' of γ. The path γ has at least a peak, otherwise s_1 and s_2 would not be both local sinks. If γ has a unique peak, Lemma 1.7 implies that $s_1 = s_2$. Now suppose that for any $k \leq n$, if γ has k peaks, then we have $s_1 = s_2$. If γ has $n + 1$ peaks, consider the first peak $x = \mathrm{sc}(e) = \mathrm{sc}(e')$ of γ. By Lemma 1.7, there is a directed path δ from $\mathrm{sk}(e')$ to s_1. Replacing the initial section of γ ending in e' by δ, we get a path with n peaks, and by the induction hypothesis we get $s_1 = s_2$, completing the proof.

Second, we prove that (i) implies (iii). Let us assume that X is connected, otherwise we apply the same reasoning to each connected component. From the preceding paragraph, we know that X has a global sink $\mathrm{sk}(X)$. Suppose that the outgoing link of every vertex is combinatorially connected. Let γ and γ' be two parallel directed paths between two vertices x and y. We prove that they are combinatorially homotopic. We proceed by induction on the maximal length m of a directed path between x and y in X. Without loss of generality, we can suppose that $y = \mathrm{sk}(X)$, since if $y \neq \mathrm{sk}(X)$ we can always find a directed path between y and $\mathrm{sk}(X)$. The cases when $m = 0$ and $m = 1$ are trivial. Suppose that the hypothesis holds up to $m = k-1, k \geq 2$, and consider two paths γ and γ' for which $m = k$. Let e and e' denote the edges of γ and γ' that are adjacent to x. We examine three cases.

1. If $e = e'$, we can apply the induction hypothesis to $\gamma \setminus e$ and $\gamma' \setminus e'$.

2. If $e \neq e'$ and both edges are on the same 2-face F of X, then using the induction hypothesis we have that γ and γ' are respectively combinatorially homotopic to the paths δ and δ' defined as follows: they go

from $x = \mathrm{sc}(F)$ to $\mathrm{sk}(F)$ by the unique path containing e and e', respectively, and then from $\mathrm{sk}(F)$ to y along the same arbitrary directed path. Since δ and δ' are combinatorially homotopic by definition, the conclusion follows from the transitivity of the combinatorial homotopy equivalence relation.

3. Suppose that $e \neq e'$, and that e and e' are *not* on the same 2-face of X. Since the outgoing link of x is combinatorially connected, there exists a combinatorial path θ between the vertices corresponding to e and e' in this link (Lemma 1.4). For every edge e_i of X in the path θ, choose a directed path γ_i in X from x to $y = \mathrm{sk}(X)$ going through e_i. Now apply Point (2) above to every pair of parallel directed paths (γ_i, γ_{i+1}) with e_i and e_{i+1} consecutive in θ, and conclude again by transitivity of the combinatorial homotopy equivalence relation.

Finally, we prove that (ii) and (iii) imply (i). Suppose that every pair of parallel directed combinatorial paths are combinatorially homotopic. We show that for any vertex x, its outgoing link is combinatorially connected. Indeed, take two edges e, e' of X with source x, and consider their extensions to directed paths γ, γ' from x to $\mathrm{sk}(X)$. By hypothesis, these two paths are combinatorially homotopic, that is, there is a sequence of parallel directed paths from γ to γ'. The collection of first edges in each of these paths defines a combinatorial path between e and e' in the outgoing link of x. Thus, this link is combinatorially connected. □

Theorem 1.9. *Let X be a polyhedral complex with generic vector such that the outgoing link of every vertex is combinatorially connected. Then, any two parallel combinatorial paths on X are combinatorially homotopic.*

Proof. Assume that X is connected, otherwise apply the argument to each connected component. By Proposition 1.8, the polyhedral complex X admits a global sink $\mathrm{sk}(X)$ and the conclusion holds for *directed* paths. Let us show that this implies the undirected version. Let γ be an undirected combinatorial path on X between x and y. For every vertex z along γ, one can choose a directed path δ_z from z to $\mathrm{sk}(X)$. We observe that for any edge $e : z \to z'$ of γ, the directed paths δ_z and $\delta_{z'}e$ are combinatorially homotopic by hypothesis. Going from x to y inductively one edge at a time and using transitivity of the homotopy equivalence relation, one obtains that γ is combinatorially homotopic to $\delta_y^{-1}\delta_x$. Taking another combinatorial path γ' parallel to γ, the same argument shows that γ' is combinatorial homotopic to $\delta_y^{-1}\delta_x$. Thus γ and γ' are combinatorially homotopic, which completes the proof. $\square$

As Proposition 1.5 shows, the class of polyhedral complexes to which Theorem 1.9 applies is a strict subclass of simply connected complexes. This implies that the converse of Theorem 1.9 does not hold, and thus that Mac Lane's original proof is far from reaching the full generality of Theorem 1.1. However, it will be sufficient for our purposes, since – as we have seen in Proposition 1.6 – it applies to any polytope.

Another feature of polyhedral complexes with generic vector is that their 1-skeleton defines abstract rewriting systems which are terminating and confluent, as we now show.

1.3 Rewriting systems

We refer to [BN98] for more details on rewriting systems.

Definition 1.10. An *abstract rewriting system* is a set A together with a binary relation $\to$.

We denote by $\xrightarrow{*}$ the reflexive and transitive closure of $\to$. We say that $(A, \to)$ is *locally confluent* (resp. *confluent*) if for all $a, a_1, a_2 \in A$ such that $a_1 \leftarrow a \to a_2$ (resp. $a_1 \xleftarrow{*} a \xrightarrow{*} a_2$), there exists a term b with $a_1 \xrightarrow{*} b \xleftarrow{*} a_2$. The diagram

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ a_1 \arrow[dr, "*"'] & & a_2 \arrow[dl, "*"] \\ & b & \end{tikzcd}$$

is called a *local confluence diagram*. A rewriting system is *terminating* if every reduction sequence $a \to a_1 \to a_2 \to \cdots$ eventually must terminate. An element $a \in A$ is *reducible* if there exists an $a' \in A$ such that $a \to a'$;

otherwise it is called *irreducible* – the rewriting synonymous of local sink! We say that b is a *normal form* of a if $a \xrightarrow{*} b$ and b is irreducible.

Given a polyhedral complex X and a generic vector $\vec{v}$, one can consider the abstract rewriting system defined by $\vec{v}$ on the vertices of X.

Definition 1.11. The *vertices rewriting system* is the pair $(X_0, \to)$ made of the set of vertices X_0 of X, together with the following relation: we have $x \to y$ if x and y are vertices of the same edge and $\langle v, x\rangle < \langle v, y\rangle$.

According to this definition, we have $x \xrightarrow{*} y$ if and only if there is a directed path from x to y in X_1. The hypothesis of Theorem 1.9 imposes that the rewriting system $(X_0, \to)$ is terminating and confluent.

Proposition 1.12. *Let X be a polyhedral complex and $\vec{v}$ be a generic vector. If the outgoing link of every vertex is combinatorially connected, the rewriting system $(X_0, \to)$ is terminating and confluent.*

Proof. Since $\vec{v}$ is generic, and thus strictly increasing along edges, it defines a partial order, and since the set X_0 is finite, the rewriting system $(X_0, \to)$ is terminating. By Proposition 1.8, there is a global sink in each connected component of X. Confluence then follows: given any pair of vertices x, y in the same connected component, since $\vec{v}$ is generic there are directed paths $x \xrightarrow{*} s \xleftarrow{*} y$ to the global sink s of this connected component. □

Corollary 1.13. *The abstract rewriting system on the vertices of any polytope P is terminating and confluent. Moreover, every pair of vertices admits a unique normal form* $\mathrm{sk}(P)$.

Recall that a polytope P is *simple* if each vertex of P is incident to precisely $\dim P$ edges.

Lemma 1.14. *If a polytope P is simple, then there is a bijection between the local confluence diagrams of $(P_0, \to)$ and the oriented boundaries of the 2-faces of P.*

Proof. When P is simple, the vertex figure P/x of every vertex x is a simplex [Zie95, Prop. 2.16], with each edge in P/x corresponding to a 2-face of P (Lemma 1.4). Thus every pair of edges e, e' with source $x = \mathrm{sc}(e) = \mathrm{sc}(e')$ determines a 2-face of P. □

Not much more can be said at this level of generality. For the specific familiy of operahedra that we will consider in the next section, the rewriting systems possess more structure (they are *term* rewriting system) and are studied in a companion paper [CLA24].

2. Categorical coherence

2.1 Categorified non-symmetric operad

Throughout this section we consider structures without units. Unless otherwise stated, the adjective "non-unital" will be implicitly assumed.

Definition 2.1. A *categorified non-symmetric operad* $\mathcal{P}$ is a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of small categories equipped with bifunctors

$$\circ_i \quad : \quad \mathcal{P}(n) \times \mathcal{P}(k) \longrightarrow \mathcal{P}(n+k-1) \ , \quad \text{for } 1 \leq i \leq n \ ,$$

and for each $\kappa \in \mathcal{P}(m)$, $\mu \in \mathcal{P}(n)$, $\nu \in \mathcal{P}(k)$, $1 \leq i \leq m$, $1 \leq j \leq n$ natural isomorphisms

$$\begin{aligned} \beta_{\kappa,\mu,\nu} \quad &: \quad (\kappa \circ_i \mu) \circ_{j+i-1} \nu \xrightarrow{\cong} \kappa \circ_i (\mu \circ_j \nu) \ , \\ \theta_{\kappa,\nu,\mu} \quad &: \quad (\kappa \circ_i \nu) \circ_{j+k-1} \mu \xrightarrow{\cong} (\kappa \circ_j \mu) \circ_i \nu \ , \quad \text{when } i < j \ , \end{aligned}$$

such that the following diagrams commute: the pentagonal

$$\begin{tikzcd}
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\beta_{\kappa,\tau,\mu} \circ 1_\nu"'] \arrow[dr, "\beta_{\kappa\circ\tau,\mu,\nu}"] & \\
(\kappa \circ (\tau \circ \mu) \circ \nu) \arrow[d, "\beta_{\kappa,\tau\circ\mu,\nu}"'] & & (\kappa \circ \tau) \circ (\mu \circ \nu) \arrow[ddl, "\beta_{\kappa,\tau,\mu\circ\nu}"] \\
\kappa \circ ((\tau \circ \mu) \circ \nu) \arrow[dr, "1_\kappa \circ \beta_{\tau,\mu,\nu}"'] & & \\
& \kappa \circ (\tau \circ (\mu \circ \nu)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\theta_{\kappa\circ\tau,\mu,\nu}"'] \arrow[dr, "\beta_{\kappa,\tau,\mu} \circ 1_\nu"] & \\
((\kappa \circ \tau) \circ \nu) \circ \mu \arrow[d, "\theta_{\kappa,\tau,\nu} \circ 1_\mu"'] & & (\kappa \circ (\tau \circ \mu) \circ \nu \arrow[ddl, "\theta_{\kappa,\tau\circ\mu,\nu}"] \\
((\kappa \circ \nu) \circ \tau) \circ \mu \arrow[dr, "\beta_{\kappa\circ\nu,\tau,\mu}"'] & & \\
& (\kappa \circ \nu) \circ (\tau \circ \mu) &
\end{tikzcd}$$

$$\begin{tikzcd}
& ((\kappa\circ\tau)\circ\mu)\circ\nu \arrow[dl, "\theta_{\kappa,\tau,\mu}\circ 1_\nu"'] \arrow[ddr, "\beta_{\kappa\circ\tau,\mu,\nu}"] & \\
((\kappa\circ\mu)\circ\tau)\circ\nu \arrow[dd, "\theta_{\kappa\circ\mu,\tau,\nu}"'] & & \\
& & (\kappa\circ\tau)\circ(\mu\circ\nu) \arrow[ddl, "\theta_{\kappa,\tau,\mu\circ\nu}"] \\
((\kappa\circ\mu)\circ\nu)\circ\tau \arrow[dr, "\beta_{\kappa,\mu,\nu}\circ 1_\tau"'] & & \\
& (\kappa\circ(\mu\circ\nu))\circ\tau &
\end{tikzcd}$$

and hexagonal identities

$$\begin{tikzcd}
& ((\kappa\circ\tau)\circ\mu)\circ\nu \arrow[dl, "\beta_{\kappa,\tau,\mu}\circ 1_\nu"'] \arrow[dr, "\theta_{\kappa\circ\tau,\mu,\nu}"] & \\
(\kappa\circ(\tau\circ\mu))\circ\nu \arrow[d, "\beta_{\kappa,\tau\circ\mu,\nu}"'] & & ((\kappa\circ\tau)\circ\nu)\circ\mu \arrow[d, "\beta_{\kappa,\tau,\nu}\circ 1_\mu"] \\
\kappa\circ((\tau\circ\mu)\circ\nu) \arrow[dr, "1_\kappa\circ\theta_{\tau,\mu,\nu}"'] & & (\kappa\circ(\tau\circ\nu))\circ\mu \arrow[dl, "\beta_{\kappa,\tau\circ\nu,\mu}"] \\
& \kappa\circ((\tau\circ\nu)\circ\mu) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ((\kappa\circ\tau)\circ\mu)\circ\nu \arrow[dl, "\theta_{\kappa,\tau,\mu}\circ 1_\nu"'] \arrow[dr, "\theta_{\kappa\circ\tau,\mu,\nu}"] & \\
((\kappa\circ\mu)\circ\tau)\circ\nu \arrow[d, "\theta_{\kappa\circ\mu,\tau,\nu}"'] & & ((\kappa\circ\tau)\circ\nu)\circ\mu \arrow[d, "\theta_{\kappa,\tau,\nu}\circ 1_\mu"] \\
((\kappa\circ\mu)\circ\nu)\circ\tau \arrow[dr, "\theta_{\kappa,\mu,\nu}\circ 1_\tau"'] & & ((\kappa\circ\nu)\circ\tau)\circ\mu \arrow[dl, "\theta_{\kappa\circ\nu,\tau,\mu}"] \\
& ((\kappa\circ\nu)\circ\mu)\circ\tau &
\end{tikzcd}$$

.

The diagrams above hold for all instances of composable β and θ; these depend on the indices i, j, k, which are omitted for the sake of readability. Observe that a categorified non-symmetric operad concentrated in arity 1 is a non-symmetric monoidal category.

As formalized in Proposition 2.6 below, one can picture an object $\mu \in \mathcal{P}(n)$ as a planar tree with one vertex decorated by μ, n leaves and one root (a *corolla*). The $\circ_i$ bifunctors then correspond to the operation of *grafting* a corolla on top of another. Iterated applications of the $\circ_i$ can be visualized as fully nested planar trees, with vertices decorated by objects of $\mathcal{P}$, see Figure 2. A *nesting* of a planar tree is a collection of subtrees (*nests*) which are either included in one another or disjoint. A nesting is *full* if its number of nests is maximal, equal to the number of internal edges of the tree [Lap22, Def. 2.2].

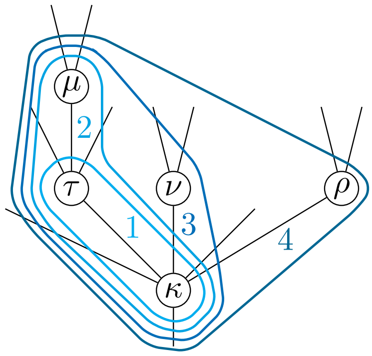

Figure 2: A fully nested planar tree.

The β and θ arrows correspond to the sequential and parallel axioms of an operad, and relate the two possible ways of fully nesting a tree with 3 vertices, see Figure 3. Moreover, there is then one coherence diagram (pentagon or hexagon) for every planar tree with 4 vertices, see Figure 4.

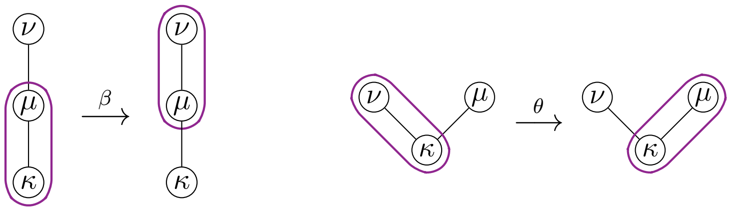

Figure 3: The β and θ isomorphisms defining a categorified non-symmetric operad.

Remark 2.2. K. Došen and Z. Petrić introduced in [DP15, Sec. 12] the notion of weak Cat-operad. Despite looking different at first sight, the two notions of categorified non-symmetric operad and weak Cat-operad are in fact equivalent. The crucial observation is the following: the θ-isomorphisms of Došen–Petrić comprise both the isomorphisms θ in Definition 2.1 and their inverses θ^{-1}. Therefore, there are only two pentagonal coherence diagrams in the definition of a weak Cat-operad, the equations ($\beta\ pent_e$) and ($\beta\theta 2_e$) of [DP15, Sec. 9]. The set of diagrams of the form ($\beta\ pent_e$) is the same as the set of diagrams which arises from the first pentagon in Definition 2.1, while

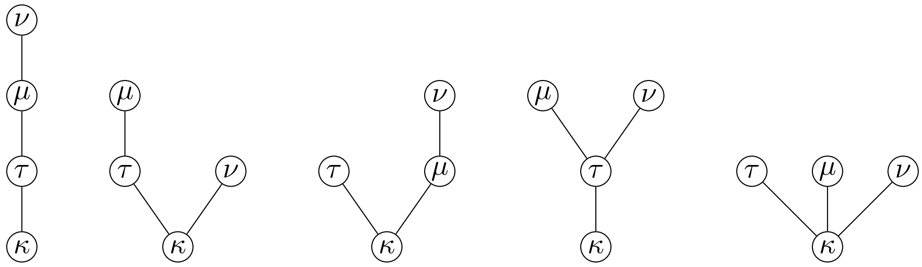

Figure 4: The five planar trees with four vertices, giving rise to the pentagonal (first three) and hexagonal (last two) identites.

the set of diagrams of the form $(\beta\theta 2_e)$ is partitioned into the sets of diagrams which arise from the second and third pentagons in Definition 2.1.

We will give in Theorem 2.5 two topological proofs of coherence for categorified non-symmetric operads. A benefit of our presentation is that, adopting the oriented approach (see the second proof of Theorem 2.5), we get a proof of coherence where β and θ are both treated as rewriting rules, in contrast with the proof in [DP15], which proceeds in two stages, much like in Mac Lane's proof of coherence for symmetric monoidal categories (see Remark 2.18): first get rid of β (rewriting), then deal with θ.

Definition 2.3. A *strong morphism* of categorified non-symmetric operads $F : \mathcal{P} \to \mathcal{Q}$ is a collection of functors $F_n : \mathcal{P}(n) \to \mathcal{Q}(n)$ together with natural isomorphisms

$$\gamma_{\kappa,\mu} : F_{m-1+n}(\kappa \circ_i \mu) \xrightarrow{\cong} F_m(\kappa) \circ_i F_n(\mu)$$

such that the following diagrams commute:

$$\begin{tikzcd}
 & F((\kappa \circ \mu) \circ \nu) \arrow[dl, "\gamma_{\kappa\circ\mu,\nu}"'] \arrow[dr, "F(\beta_{\kappa,\mu,\nu})"] & \\
F(\kappa \circ \mu) \circ F(\nu) \arrow[d, "\gamma_{\kappa\circ\mu} \circ 1_{F(\nu)}"'] & & F(\kappa \circ (\mu \circ \nu)) \arrow[d, "\gamma_{\kappa,\mu\circ\nu}"] \\
(F(\kappa) \circ F(\mu)) \circ F(\nu) \arrow[dr, "\beta_{F(\kappa),F(\mu),F(\nu)}"'] & & F(\kappa) \circ F(\mu \circ \nu) \arrow[dl, "1_{F(\kappa)} \circ \gamma_{\mu,\nu}"] \\
 & F(\kappa) \circ (F(\mu) \circ F(\nu)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F((\kappa \circ \nu) \circ \mu) \arrow[dl, "\gamma_{\kappa\circ\nu,\mu}"'] \arrow[dr, "F(\theta_{\kappa,\nu,\mu})"] & \\
F(\kappa \circ \nu) \circ F(\mu) \arrow[d, "\gamma_{\kappa,\nu} \circ 1_{F(\mu)}"'] & & F((\kappa \circ \mu) \circ \nu) \arrow[d, "\gamma_{\kappa\circ\mu,\nu}"] \\
(F(\kappa) \circ F(\nu)) \circ F(\mu) \arrow[dr, "\theta_{F(\kappa),F(\nu),F(\mu)}"'] & & F(\kappa \circ \mu) \circ F(\nu) \arrow[dl, "\gamma_{\kappa,\nu} \circ 1_{F(\nu)}"] \\
 & (F(\kappa) \circ F(\mu)) \circ F(\nu) &
\end{tikzcd}$$

It is said to be *strict* if the natural isomorphisms are identities.

Once again, the diagrams above hold for all instances of β and θ arrows, and we have omitted the (i, j, k)-indices for readability. Observe that a strong (resp. strict) morphism between categorified non-symmetric operads concentrated in arity 1 is a strong (resp. strict) monoidal functor between non-symmetric monoidal categories.

2.2 Coherence for categorified non-symmetric operads

We now aim at the coherence theorem for categorified non-symmetric operads. In order to state the theorem, we construct the free non-symmetric categorified operad on a family of sets $S = \{S_n\}_{n\geq 1}$. We define a family of categories $\mathcal{S} = \{\mathcal{S}_n\}_{n\geq 1}$ whose objects are given by the following rules:

1. if $a \in S_n$, then a is an object of $\mathcal{S}_n$;
2. if $t_1 \in \mathcal{S}_m$ and $t_2 \in \mathcal{S}_n$, then $t_1 \circ_i t_2$ is an object of $\mathcal{S}_{m-1+n}$, for any $1 \leq i \leq m$.

If an object t_1 is in $\mathcal{S}_n$, we say that t_1 has *arity* n. Now we define a set M of *basic morphisms* $\beta : (t_1 \circ_i t_2) \circ_{j+i-1} t_3 \leftrightarrow t_1 \circ_i (t_2 \circ_j t_3) : \beta^{-1}$ for every $t_1 \in \mathcal{S}_m, t_2 \in \mathcal{S}_n, t_3 \in \mathcal{S}_k$, $1 \leq i \leq m$ and $1 \leq j \leq n$, and $\theta : (t_1 \circ_i t_3) \circ_{j-1+k} t_2 \leftrightarrow (t_1 \circ_j t_2) \circ_i t_3 : \theta^{-1}$ whenever $i < j$. We then define the *generating morphisms* of the family $\mathcal{S}$ by the following rules:

1. if $\phi \in M$, then ϕ is a generating morphism of $\mathcal{S}$;
2. if $\phi : t_1 \to t_2$ is a generating morphism in $\mathcal{S}$, and $t_3 \in \mathcal{S}$, then $\phi \circ_i \mathrm{id} : t_1 \circ_i t_3 \to t_2 \circ_i t_3$ and $\mathrm{id} \circ_j \phi : t_3 \circ_j t_1 \to t_3 \circ_j t_2$ are generating morphisms, for any i (resp. j) between 1 and the arity of t_1 (resp. t_3).

Note that by construction, for every morphism $\phi : t_1 \to t_2$, the objects t_1 and t_2 have the same arity, and we say that ϕ has this *arity*. We then define $\mathcal{S}_n$ as the free category over all generating morphisms of arity n. This finishes the construction of our family $\mathcal{S}$ of categories.

Definition 2.4. We denote by $\mathcal{F}(S)$ the quotient of the family of categories $\mathcal{S}$ by localization (inverting the β and θ morphisms), the axioms of bifunctors for the $\circ_i$, the naturality conditions for β and θ, and the coherence diagrams (pentagons and hexagons) defining a categorified non-symmetric operad.

We obtain that $\mathcal{F}(S)$ is the free categorified non-symmetric operad on S. That is, for any categorified non-symmetric operad $\mathcal{P}$, and for any family of functions $\rho_n : S_n \to \mathrm{Ob}(\mathcal{P}(n))$, there is a unique *strict* morphism of non-symmetric categorified operads $\mathcal{F}(S) \to \mathcal{P}$ which extends $\rho := \{\rho_n\}_{n\geq 1}$. By precomposing it with the quotient map $\mathcal{S} \to \mathcal{F}(S)$, we get a levelwise functor $[[-]] : \mathcal{S} \to \mathcal{P}$.

Theorem 2.5 (Coherence theorem)**.** *For any categorified non-symmetric operad $\mathcal{P}$, for any family of functions $\rho : S \to \mathrm{Ob}(\mathcal{P})$, and for any two parallel morphisms $\phi_1, \phi_2 : t_1 \to t_2$ in $\mathcal{S}$, we have $[[\phi_1]] = [[\phi_2]]$.*

In order to prove this Theorem 2.5, we need to first recall the construction of the *operahedra*, a family of polytopes whose faces are in bijection with the set of all nestings of a planar tree. We refer to [Lap22, Sec. 2] for details, see also [DP15, Sec. 13] and [COI19]. Given a planar tree t with n internal edges, and a full nesting $\mathcal{N}$ of t, one associates a point $M(t, \mathcal{N}) \in \mathbb{R}^n$ via a simple algorithm which is due to J.-L. Loday [Lap22, Sec. 2.2]. The *operahedron* $P_t \subset \mathbb{R}^n$ is the convex hull of the points $M(t, \mathcal{N})$, for all maximal nestings $\mathcal{N}$ of t. It has dimension $n - 1$. One then shows that the poset of nestings of t, ordered by reverse inclusion, is isomorphic to the poset of faces of P_t [Lap22, Prop. 2.15]. The dimension of a face is given by n minus the number of nests in the corresponding nesting of t.

Reading a planar tree t from the leaves to the root defines a family of *incoming edges* and one *outgoing edge* at each vertex of t. Given the family of sets S and a planar tree t, we say that a decoration of the vertices of t by elements of S is *admissible* if at every vertex the number of incoming edges is equal to the arity of the element of S decorating it. Now, let us consider the collection $\mathcal{O}(S)$ of polytopes with one copy of the operahedron P_t for each admissible decoration of the planar tree t by elements of S.

Proposition 2.6. *There are bijections between*

1. *objects of $\mathcal{S}$ and vertices of the operahedra in $\mathcal{O}(S)$,*
2. *generating morphisms of $\mathcal{S}$ and edges of the operahedra in $\mathcal{O}(S)$,*
3. *bifunctoriality, naturality and coherence diagrams and 2-faces of the operahedra in $\mathcal{O}(S)$.*

Proof. To each element a of S_n, we associate a planar corolla with n leaves and vertex decorated by a. Then, we identify $a \circ_i b$ with the planar tree obtained from grafting the corolla decorated by b at the ith leaf of the corolla decorated by a. Continuing in this fashion, and remembering the order in which we graft the corollas, we obtain all possible fully nested planar trees with vertices decorated by elements of S (Figure 2). A generating morphism f in $\mathcal{S}$ is an application of one of the associativity rules β or θ to a fully nested tree t, moving only one nest (Figure 3). If t has n internal edges, forgetting the nest that has been moved gives a nesting of t with $n-1$ nests. We associate to f the edge of the operahedron P_t in $\mathcal{O}(S)$ labeled by this nesting, see [Lap22, Def. 2.8 & Prop. 3.11]. It remains to consider all the possible diagrams one can obtain by applying two generating morphisms to a given fully nested tree t with n internal edges. These arise from moving two different nests in the same fully nested tree. Starting by moving one or the other of these 2 nests, one faces two types of situations:

(A1) If the two nests are disjoint, one obtains a bifunctoriality square,

(A2) If the two nests are nested, but do not share the same root, one obtains a naturality square,

(B) If the two nests are nested and share the same root, one obtains either a pentagon or a hexagon as in Definition 2.1.

To such a diagram, we associate the 2-face of the operahedron P_t in $\mathcal{O}(S)$ corresponding to the nesting of t obtained by forgetting the two nests that have been moved along the edges. We refer to [CLA24, Sec. 2] for a more detailed analysis of the 2-faces. □

Remark 2.7. The fact that every possible choice of initial moves gives rise to a 2-face amounts to the fact that the operahedron P_t is a *simple* polytope [DP11, Sec. 9]. As Lemma 1.14 shows, this property garantees the correspondence between geometric and rewriting-theoretic proofs of coherence, see [CLA24] for more details on the latter.

The conceptual origin of the bijections of Proposition 2.6 is the fact that the combinatorics of the faces of the operahedra correspond exactly to the monad of trees [LV12, Sec. 5.6.1]. Or, said differently, it lies in the fact that the operahedra encode (via the cellular chains functor) the minimal resolution of the colored symmetric operad whose algebras are non-unital non-symmetric operads, see [VdL03] and [Lap22, Sec. 4.1].

We are now ready to prove Theorem 2.5, using either our non-oriented or oriented topological coherence results for polytopes.

Proof of Theorem 2.5. From Point (2) in Proposition 2.6, we have that the morphisms of $\mathcal{S}$ are in bijection with combinatorial paths on the operahedra of $\mathcal{O}(S)$. Two parallel morphisms in $\mathcal{S}$ thus define two parallel combinatorial paths on some operahedron P_t in $\mathcal{O}(S)$. Since an operahedron P_t is simply connected, Theorem 1.1 implies that these two combinatorial paths are combinatorially homotopic. By Point (3) in Proposition 2.6 the 2-faces of the operahedra are exactly either a bifunctoriality or naturality square, a pentagon or a hexagon (witnessing a coherence condition) as in Definition 2.1. Therefore, two parallel morphisms ϕ_1, ϕ_2 in $\mathcal{S}$ are equal in the quotient $\mathcal{F}(S)$, and thus their images $[[\phi_1]], [[\phi_2]]$ are also equal in $\mathcal{P}$. $\square$

Second proof of Theorem 2.5. Alternatively, since the operahedra are polytopes, one can use Proposition 1.6 and Theorem 1.9. As shown in [Lap22, Prop. 3.11], choosing a generic vector $\vec{v}$ which has strictly decreasing coordinates gives the orientations of the diagrams given in Definition 2.1 on the 2-faces. One then obtains a topological proof of coherence which follows closely the original proof of Mac Lane [ML63, Thm. 3.1], suitably generalized to categorified operads. $\square$

Following Remark 2.2, we have that Theorem 2.5 gives an alternative, more economical proof of coherence for weak Cat-operads [DP15, Prop. 14.2]. Incidentally, it gives an alternative input to the proof of coherence for cyclic symmetric categorified operads [CO20].

Restricting the theorem above to non-symmetric operads concentrated in arity 1, the category $\mathcal{F}(S)$ becomes the free non-symmetric monoidal category on S, and we get the following corollary.

Corollary 2.8 (Mac Lane coherence theorem for non-symmetric monoidal categories)**.** *For any non-symmetric monoidal category $\mathcal{C}$, for any function $\rho : S \to \mathrm{Ob}(\mathcal{C})$, and for any two parallel morphisms $\phi_1, \phi_2 : t_1 \to t_2$ in $\mathcal{S}$, we have* $[[\phi_1]] = [[\phi_2]]$.

Remark 2.9. As mentioned at the end of Section 1.3, the rewriting systems obtained on the vertices of the operahedra by choosing a generic vector with strictly decreasing coordinates are in fact *term* rewriting systems. The faces of type (B) in Point (3) of Proposition 2.6 (the coherence conditions) correspond precisely to the *critical pairs* of these rewriting systems, see [CLA24, Sec. 3.4]. Moreover, the associated posets on fully nested planar trees have recently been shown to be lattices [DS24].

2.3 Symmetric monoidal categories

We now formulate and prove Mac Lane's coherence theorem for *symmetric* monoidal categories in the same style as above. Recall that in a symmetric monoidal category $\mathcal{C}$, in addition to the natural isomorphisms β, with components $\beta_{\kappa,\mu,\nu} : (\kappa \otimes \mu) \otimes \nu \to \kappa \otimes (\mu \otimes \nu)$, there are involutive natural transformations τ, with components $\tau_{\mu,\nu} : \mu \otimes \nu \to \nu \otimes \mu$. Here, we use $\kappa, \mu, \nu, \ldots$ to range over the objects of the category, consistently with the notation used in Sections 2.1 and 2.2. In addition to the pentagons, obtained from the first pentagon in Definition 2.1 by replacing $\circ$ with $\otimes$, there are hexagons

$$\begin{tikzcd}
 & (\kappa \otimes \mu) \otimes \nu \arrow[dl, "\beta"'] \arrow[dr, "\tau \otimes 1"] & \\
\kappa \otimes (\mu \otimes \nu) \arrow[d, "\tau"'] & & (\mu \otimes \kappa) \otimes \nu \arrow[d, "\beta"] \\
(\mu \otimes \nu) \otimes \kappa \arrow[dr, "\beta"'] & & \mu \otimes (\kappa \otimes \nu) \arrow[dl, "1 \otimes \tau"] \\
 & \mu \otimes (\nu \otimes \kappa) &
\end{tikzcd}$$

for all objects κ, μ, ν in $\mathcal{C}$.

In order to state the coherence theorem, we construct a free category on a set S of *generating objects*. We define a small category $\mathcal{S}^{\mathrm{ML}}$ whose set of objects

$$\mathcal{T}_S = \bigcup \{\mathcal{T}_A \mid A \text{ is a non-empty finite subset of } S\}$$

is defined as follows:

1. if $a \in S$, then $a \in \mathcal{T}_{\{a\}}$;
2. if $t_1 \in \mathcal{T}_A$ and $t_2 \in \mathcal{T}_B$, and if $A \cap B = \emptyset$, then $t_1 \otimes t_2 \in \mathcal{T}_{A \cup B}$.

We can see the objects of $\mathcal{S}^{\mathrm{ML}}$ as fully parenthesized words over S where letters are not repeated. We then define a set M^{ML} of *basic morphisms* $\beta : (t_1 \otimes t_2) \otimes t_3 \leftrightarrow t_1 \otimes (t_2 \otimes t_3) : \beta^{-1}$ and $\tau : t_1 \otimes t_2 \leftrightarrow t_2 \otimes t_1$, for every $t_1, t_2, t_3 \in \mathcal{T}_S$. We then define the *generating morphisms* of $\mathcal{S}^{\mathrm{ML}}$ by the following rules:

1. if $\phi \in M^{\mathrm{ML}}$, then ϕ is a generating morphism;
2. if $\phi : t_1 \to t_2$ is a generating morphism and $t_3 \in \mathcal{T}_S$, then $\phi \otimes \mathrm{id} : t_1 \otimes t_3 \to t_2 \otimes t_3$ and $\mathrm{id} \otimes \phi : t_3 \otimes t_1 \to t_3 \otimes t_2$ are generating morphisms.

We then define $\mathcal{S}^{\mathrm{ML}}$ as the free category over all generating morphisms. This finishes the construction of the category $\mathcal{S}^{\mathrm{ML}}$.

Definition 2.10. We denote by $\mathcal{F}(S)$ the quotient of $\mathcal{S}^{\mathrm{ML}}$ by localization (inverting the β morphisms), by the axioms $\tau_{t_1,t_2} \circ \tau_{t_2,t_1} = 1$, by the axioms of bifunctors, by the naturality conditions for β and τ, and by the coherence conditions of symmetric monoidal categories.

By freeness, we have that for any symmetric monoidal category $\mathcal{C}$, and for any function $\rho : S \to \mathrm{Ob}(\mathcal{C})$, there is a unique functor $[[-]]^{\mathrm{ML}} : \mathcal{S}^{\mathrm{ML}} \to \mathcal{C}$ which extends ρ and sends the formal basic morphisms to the actual canonical morphisms of $\mathcal{C}$. This functor factorizes through the quotient map $[-]^{\mathrm{ML}} : \mathcal{S}^{\mathrm{ML}} \to \mathcal{F}(S)$.

It turns out that Kapranov's topological proof is not based on the above presentation of $\mathcal{F}(S)$, but on another presentation of this category, that is made explicit in [BIP19, Sec. 2]. Let us recall this presentation. We define another category $\mathcal{S}^{\mathrm{K}}$ as follows. Its objects are the same as those of $\mathcal{S}^{\mathrm{ML}}$. We define a set M^{K} of *basic morphisms* $\beta : (t_1 \otimes t_2) \otimes t_3 \leftrightarrow t_1 \otimes (t_2 \otimes t_3) : \beta^{-1}$ for every $t_1, t_2, t_3 \in \mathcal{T}_S$, and $\tau : a \otimes b \leftrightarrow b \otimes a$ for every $a, b \in S$, i.e., we *limit τ to generating objects*. *Generating morphisms* are defined in the same way as for $\mathcal{S}^{\mathrm{ML}}$. We note that by construction $\mathcal{S}^{\mathrm{K}}$ is a wide subcategory of $\mathcal{S}^{\mathrm{ML}}$.

Definition 2.11. We denote by $\mathcal{F}(S)^{\mathrm{K}}$ the quotient of $\mathcal{S}^{\mathrm{K}}$ by localization (inverting the β morphisms), by the axioms $\tau_{a,b} \circ \tau_{b,a} = 1$, by the axioms of bifunctors, by the naturality conditions for β, by the coherence conditions of monoidal categories, and by the axioms in dodecagonal form given by the solid arrows in Figure 5 (left), for a, b, c ranging over S only.

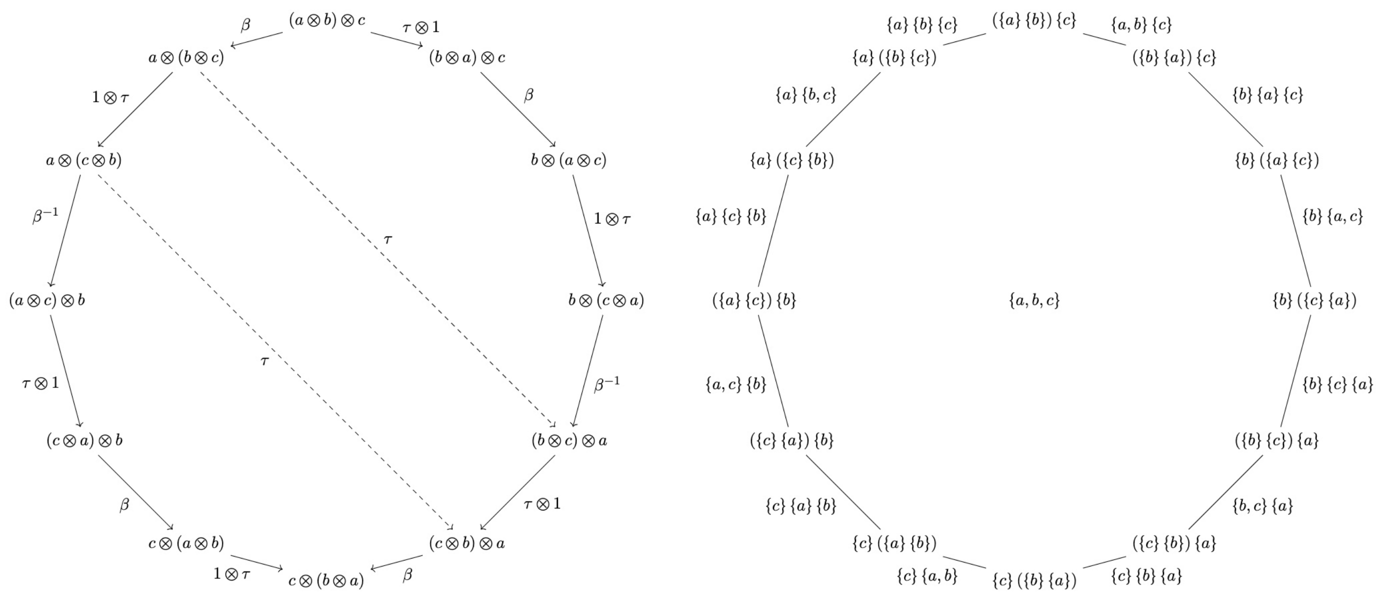

Figure 5: Kapranov dodecagons.

We pause here to reflect on the difference between the two presentations. In the second one, we have less generators, and we have lost hexagons. For an intuition, here is how Mac Lane himself motivated his hexagonal axioms (verbatim, just changing the notation to fit with ours) in [ML63]:

> The instance $\tau_{\kappa\otimes\mu,\nu}$ interchanges the block $\kappa\mu$ with the single letter ν; the hexagon condition states that this interchange may be replaced by two instances of τ which interchange single letters with ν. Repeated such replacement using instances of the hexagon shows that any interchange of successive blocks may be replaced by interchanges of successive letters.

In other words, hexagons are now taken as definitions rather than axioms. But how do we guarantee that the general τ morphisms defined in this way define a natural transformation? This is what the dodecagons are for.

Let $\mathcal{C}$ be a symmetric monoidal category. By freeness again, any function $\rho : S \to \mathrm{Ob}(\mathcal{C})$ extends uniquely to a functor $[[-]]^{\mathrm{K}} : \mathcal{S}^{\mathrm{K}} \to \mathcal{C}$. This functor is the restriction of $[[-]]^{\mathrm{ML}}$ to $\mathcal{S}^{\mathrm{K}}$, and factorizes through the quotient functor $[-]^{\mathrm{K}} : \mathcal{S}^{\mathrm{K}} \to \mathcal{F}(S)^{\mathrm{K}}$.

Theorem 2.12 (Kapranov coherence theorem for symmetric monoidal categories)**.** *For any two parallel morphisms $\phi_1, \phi_2 : t_1 \to t_2$ in $\mathcal{S}^{\mathrm{K}}$, we have $[\phi_1]^{\mathrm{K}} = [\phi_2]^{\mathrm{K}}$.*

In order to prove this "Kapranov style" coherence, we need to first recall the construction of the *permuto-associahedra*, a family of polytopes whose faces are in bijection with parenthesized ordered partitions of a finite set. We refer to [Zie95, Sec. 9.3] for details, see also [Kap93] and [RZ94]. Given a finite set A of cardinal n and a parenthesized permutation σ of its elements, one associates a section γ^σ of the projection from the n-cube to the cyclic polygon with $n+1$ vertices [Zie95, Ex. 9.14], whose integral over the base gives a point $M(\sigma)$ in $\mathbb{R}^n$. The *permuto-associahedron* P_A is the convex hull of the points $M(\sigma)$, for all parenthesized permutations σ of the elements of A. It has dimension $n-1$. One then shows that the poset of parenthesized ordered partitions of A, ordered according to the rules below, is isomorphic to the poset of faces of P_A [Zie95, Thm. 9.15].

Parenthesized ordered partitions of A can be drawn as planar trees whose leaves are decorated with the parts of a partition of A. The subface relation $\prec$ is defined by two clauses: one can contract an edge of the tree, or remove a node all of whose incoming edges are leaves and decorate its outcoming edge – now a leaf – with the union of the decorations of those incoming edges. The maximal face is A. For example, with $A = \{a_1, \ldots, a_7\}$, the following is a face:

$$(\{a_1\}\,\{a_4\}\,\{a_2, a_6\})\,\{a_3, a_5, a_7\}$$

which is covered by the following two elements.

$$\begin{array}{lcl} (\{a_1\}\,\{a_4\}\,\{a_2, a_6\})\,\{a_3, a_5, a_7\} & \prec & \{a_1\}\,\{a_4\}\,\{a_2, a_6\}\,\{a_3, a_5, a_7\} \\ (\{a_1\}\,\{a_4\}\,\{a_2, a_6\})\,\{a_3, a_5, a_7\} & \prec & \{a_1, a_2, a_4, a_6\}\,\{a_3, a_5, a_7\}\,. \end{array}$$

Given the set S, let us consider the collection $\mathcal{P}(S)$ of polytopes with one copy of the permuto-associahedron P_A for each finite subset $A \subseteq S$.

Proposition 2.13. *There are bijections between*

1. *objects of $\mathcal{S}^{\mathrm{K}}$ and vertices of the permuto-associahedra in $\mathcal{P}(S)$,*
2. *generating morphisms of $\mathcal{S}^{\mathrm{K}}$ and edges of the permuto-associahedra in $\mathcal{P}(S)$,*
3. *bifunctoriality, naturality and coherence diagrams and 2-faces of the permuto-associahedra in $\mathcal{P}(S)$.*

Proof. The 0-dimensional faces of P_A are fully parenthesized words whose letters are singletons, and are in obvious bijective correspondence with the elements of $\mathcal{T}_A$. The 1-dimensional faces are

- either fully parenthesized words whose letters are singletons but for one letter which is a two-element set $\{a_i, a_j\}$ and feature an application of the basic morphism τ_{a_i,a_j},
- or an "almost" fully parenthesized word of singletons, with just one parenthesis removed, yielding a subword $(\{a_i\}\,\{a_j\}\,\{a_k\})$, featuring an application of the basic morphism β_{a_i,a_j,a_k} or $\beta^{-1}_{a_i,a_j,a_k}$ – the orientation of the edge being "decided" by the shape of its end vertices.

Finally, the 2-dimensional faces can be analyzed much in the same way as in Proposition 2.6, and seen to correspond to bifunctoriality, naturality of β, and to the pentagons and dodecagons. We have pictured the poset view of the latter in Figure 5 (right). The reader can also convince himself on this figure how the orientation of the β arrows on the left can be reconstructed from the non-oriented dodecagon on the right. $\square$

Proof of Theorem 2.12. Having Proposition 2.13 in hand, the proof is similar to the proof of Theorem 2.5, using either the Van Kampen (Theorem 1.1) or the Morse (Proposition 1.6 and Theorem 1.9) technique. $\square$

Remark 2.14. Alternatively, one could use the same strategy with the *simple permutoassociahedra* from [BIP19], involving yet another equivalent presentation of symmetric monoidal categories.

The following proposition establishes relations between the Mac Lane and Kapranov presentations of symmetric monoidal categories.

Proposition 2.15 (Kapranov–Mac Lane comparison)**.**

1. *Let* $\phi_1, \phi_2 : t_1 \to t_2$ *be parallel morphisms of* $\mathcal{S}^{\mathrm{K}}$*. If we have* $[\phi_1]^{\mathrm{K}} = [\phi_2]^{\mathrm{K}}$*, then we have* $[\phi_1]^{\mathrm{ML}} = [\phi_2]^{\mathrm{ML}}$*.*

2. *For any morphism* ϕ *of* $\mathcal{S}^{\mathrm{ML}}$*, there is a morphism* ψ *of* $\mathcal{S}^{\mathrm{K}}$ *such that* $[\phi]^{\mathrm{ML}} = [\psi]^{\mathrm{ML}}$*.*

Proof. The proof of Point (1) is visualized in Figure 5 (left). The two dotted lines delimit two Mac Lane hexagons on the top and at the bottom and a naturality square in the middle. Explicitly, the two dotted τ-morphisms are $\tau_{a,b\otimes c}$ and $\tau_{a,c\otimes b}$. As for Point (2), we observe that a morphism ψ as in the statement can be obtained by repeatedly applying the procedure described by Mac Lane in the quotation which follows Definition 2.11 above. □

Theorem 2.16 (Mac Lane coherence theorem for symmetric monoidal categories)**.** *For any symmetric monoidal category* $\mathcal{C}$*, for any function* $\rho : S \to \mathrm{Ob}(\mathcal{C})$*, and for any two parallel morphisms* $\phi_1, \phi_2 : t_1 \to t_2$ *in* $\mathcal{S}^{\mathrm{ML}}$*, we have* $[[\phi_1]]^{\mathrm{ML}} = [[\phi_2]]^{\mathrm{ML}}$*.*

Proof. Since the functor $[[-]]^{\mathrm{ML}}$ factorizes through the functor $[-]^{\mathrm{ML}}$, it is enough to prove that $[\phi_1]^{\mathrm{ML}} = [\phi_2]^{\mathrm{ML}}$. By Point (2) of Proposition 2.15, there exist ψ_1 and ψ_2 in $\mathcal{S}^{\mathrm{K}}$ such that $[\psi_1]^{\mathrm{ML}} = [\phi_1]^{\mathrm{ML}}$ and $[\psi_2]^{\mathrm{ML}} = [\phi_2]^{\mathrm{ML}}$. In particular ψ_1 and ψ_2 are parallel, so by Theorem 2.12 we get $[\psi_1]^{\mathrm{K}} = [\psi_2]^{\mathrm{K}}$, and by Point (1) of Proposition 2.15 we have $[\psi_1]^{\mathrm{ML}} = [\psi_2]^{\mathrm{ML}}$. Thus, we have $[\phi_1]^{\mathrm{ML}} = [\psi_1]^{\mathrm{ML}} = [\psi_2]^{\mathrm{ML}} = [\phi_2]^{\mathrm{ML}}$, which concludes the proof. □

Remark 2.17. One can see easily that this proof also shows that the categories $\mathcal{F}(S)^{\mathrm{K}}$ and $\mathcal{F}(S)$ are isomorphic. The statement of this fact is unrelated to coherence issues, but its proof relies on Kapranov style coherence. In other words, the proof that Kapranov's conditions imply Mac Lane's conditions is non-trivial, in contrast to the converse direction (cf. Proposition 2.15); a result of the magic of polytopes!

Remark 2.18. Note that contrary to the case of the operahedra, there does not seem to exist a generic vector whose induced orientation on the edges of the permuto-associahedra coincides with a consistent orientation of the β and τ arrows *based on conventions independent of the orientation vector*. This follows from the observation that the dodecagon (Figure 5, left) involves β^{-1} arrows. The same remarks apply to the simple permuto-associahedra of [BIP19]. As for the original presentation of Mac Lane (for which no polytopal correspondence is known), one could still hope to have an associated term rewriting system. But instead Mac Lane's proof (rightly!) proceeds in two stages: first using rewriting for the monoidal part (β only), and then dealing with the symmetric part using Coxeter's presentation of the symmetric groups. It seems that one cannot do better. Indeed, even if Mac Lane's hexagon does not involve β^{-1} arrows, the latter would pop up when taking the combinatorics of orientation of the τ arrows into account. As an illustration, suppose that we decide to move parentheses to the right for β, fix a total order on S and split the involutive τ into τ and τ^{-1} according to where the maximum lies. Then, for $\mu < \kappa < \nu$ the hexagon becomes

$$\begin{tikzcd}
 & (\kappa \otimes \mu) \otimes \nu \arrow[dl, "\beta"'] \arrow[dr, "\tau \otimes 1"] & \\
\kappa \otimes (\mu \otimes \nu) & & (\mu \otimes \kappa) \otimes \nu \arrow[d, "\beta"] \\
(\mu \otimes \nu) \otimes \kappa \arrow[u, "\tau"] \arrow[dr, "\beta"'] & & \mu \otimes (\kappa \otimes \nu) \\
 & \mu \otimes (\nu \otimes \kappa) \arrow[ur, "1 \otimes \tau"'] &
\end{tikzcd}$$

and a local confluence diagram for the pair of rewritings out of $(\mu \otimes \nu) \otimes \kappa$ cannot be completed without inverting β arrows.

3. Perspectives

3.1 Further applications

One can also use the same strategy to prove coherence for *unital* non-symmetric monoidal categories, using the unital associahedra of F. Muro and A. Tonks [MT14].

It is natural to ask if the construction of unital associahedra could be extended to the permutoassociahedra, in such a way as to provide a topological proof of coherence for unital symmetric monoidal categories. The question of the existence of these constructions at the operadic level (i.e. does there exist unital operahedra, symmetric operahedra, and unital symmetric operahedra?) is, to our knowledge, still open as well.

Another immediate application of Theorem 1.1 is the coherence of strong non-symmetric monoidal functors between non-symmetric monoidal categories [Eps66]. The corresponding topological objects are in this case the family of multiplihedra [Sta70, For08]. The generalization to strong morphisms between non-symmetric categorified operads also goes through, involving this time the family of multiploperahedra described at the end of the introduction in [LM23].

In the same spirit as in Theorem 2.5, one could obtain coherence results for categorifications of many operad-like structures, for instance the ones described in [BMO23]: categorified modular operads, wheeled properads, and permutads (shuffle algebras), among others. In order to treat cyclic and symmetric structures, one could take inspiration from the reduction process followed in [CO20] for the case of cyclic symmetric categorified operads.

3.2 Higher categories

Theorem 1.1 shows the precise relationship between coherence and connectedness. In addition to Kapranov's claim [Kap93], it clarifies other statements in the literature, such as the proof of [KV94, Prop. 3.9]. There, the incipit "since P_n is a convex polytope" could be replaced by a more precise "since P_n is simply connected".

In the case of (symmetric) monoidal categories, Theorem 1.1 demonstrates that coherence is equivalent to the vanishing of the first homotopy groups of the (permuto-)associahedra. Since the (permuto-)associahedra are contractible, and therefore all their homotopy groups vanish, one could hope for a topological proof of higher dimensional coherence theorems.

One dimension higher, N. Gurski has shown in [Gur11, Thms. 22 & 23] that coherence for (braided) monoidal bicategories is equivalent to the vanishing of fundamental 2-groupoids of braid groups. Recent results of S. Barkan provide evidence for higher dimensional statements, relating coherence diagrams of ∞-operads to the connectivity of certain operadic partition complexes [Bar22]. It seems likely that the present results could be interpreted as a strict version and a special case of [Bar22, Thm. B]. It would be interesting to see how the permuto-associahedra arise in the strictification process, and how they are related to operadic partition complexes.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXV-4 (2024)

NOTES ON LIMITS OF ACCESSIBLE CATEGORIES

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Résumé. Soient κ un cardinal régulier, $\lambda < \kappa$ un cardinal infini plus petit, et K une catégorie κ-accessible qui admet les colimites de chaînes indexées par λ. Nous démontrons que diverses constructions catégoriques appliquées à K, comme les équifiers et inserters produisent de nouvelles catégories κ-accessibles E, et que les objets κ-présentables de E admettent une caractérisation naturelle. En particulier, si C est une catégorie κ-petite, alors la catégorie des foncteurs $\mathsf{C} \longrightarrow \mathsf{K}$ est aussi κ-accessible et ses objets κ-présentables sont exactement les foncteurs à valeurs dans la sous-catégorie des objets κ-présentables de K. Nous discutons aussi la préservation de la κ-accessibilité par les pseudo-limites coniques, les limites lax et oplax et les pseudo-limites à poids. Une partie de ces résultats peuvent se retrouver dans une note non-publiée de Ulmer de 1977. Ce travail est motivé par la théorie des modules plats et des faisceaux quasi-cohérents.

Abstract. Let κ be a regular cardinal, $\lambda < \kappa$ be a smaller infinite cardinal, and K be a κ-accessible category where colimits of λ-indexed chains exist. We show that various category-theoretic constructions applied to K, such as the inserter and the equifier, produce κ-accessible categories E again, and the most obvious expected description of the full subcategory of κ-presentable objects in E in terms of κ-presentable objects in K holds true. In particular, if C is a κ-small category, then the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$ is κ-accessible, and its κ-presentable objects are precisely all the functors from C to the κ-presentable objects of K. We proceed to discuss the preservation of κ-accessibility by conical pseudolimits, lax and oplax limits, and weighted pseudolimits. The results of this paper go back to an unpublished 1977 preprint of Ulmer. Our motivation comes from the theory of flat modules and flat quasi-coherent sheaves.

Keywords. κ-directed colimits, κ-presentable objects, κ-accessible categories, limits of categories, products, inserters, equifiers, lax and oplax limits, weighted pseudolimits, diagram categories, flat modules, diagrams and complexes of modules.

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Introduction

Let κ be a cardinal and K be a category such that all the objects of K are κ-filtered colimits of (suitably defined) "objects of small size relative to κ". Suppose E is the category of objects from K or collections of objects from K with a certain additional structure and/or some equations imposed. Is every object of E a κ-filtered colimit of objects whose underlying objects from K have small size relative to κ?

To specify the context of the discussion, let κ be a regular cardinal and K be a κ-accessible category (in the sense of [19, §2.1] or [1, Chapter 2]). Let C be a κ-small category, and let $\mathsf{E} = \mathsf{Fun}(\mathsf{C}, \mathsf{K})$ be the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$. Ideally, one may wish to claim that the category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is κ-accessible and its κ-presentable objects are precisely all the functors $\mathsf{C} \longrightarrow \mathsf{K}_{<\kappa}$, where $\mathsf{K}_{<\kappa}$ is the full subcategory of κ-presentable objects in K. But is it true?

The "ideal" state of affairs described in the previous paragraph was claimed as a general result in a 1988 paper [18, Lemma 5.1]. A general outline of a proof of the lemma was presented in [18]; the details were declared to be "direct calculations" and omitted. A refutation came in the recent preprint [12, Theorem 1.3]. The ideal state of affairs does not hold in general.

The assertions of [12, Theorem 1.3] provide a complete characterization of all small categories C such that the "ideal" statement holds *for all κ-accessible categories* K. All such categories C are essentially κ-small, but being essentially κ-small is *not* enough. The category C needs to be also *well-founded* in the sense of the definition in [12].

But are there some κ-accessible categories K that are better behaved than some other ones, with respect to the question at hand? Another theorem from [12] tells that there are. According to [12, Theorem 1.2], if the category C is κ-small and the category K is locally κ-presentable (in the sense of [10] or [1, Chapter 1]), then the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is locally κ-presentable and its full subcategory of κ-presentable objects is $\mathsf{Fun}(\mathsf{C}, \mathsf{K}_{<\kappa}) \subset \mathsf{Fun}(\mathsf{C}, \mathsf{K})$.

Are there any better behaved κ-accessible categories beyond the locally κ-presentable ones? The present paper purports to answer this question by generalizing the result of [12, Theorem 1.2].

We show that the following much weaker version of local presentability is sufficient to guarantee the "ideal state of affairs": it is enough to assume existence of an infinite cardinal $\lambda < \kappa$ such that colimits of all λ-indexed chains of objects and morphisms exist in K. If this is the case and K is κ-accessible, then for any κ-small category C the category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is also κ-accessible, and the κ-presentable objects of $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ are precisely all the functors $\mathsf{C} \longrightarrow \mathsf{K}_{<\kappa}$. This is the result of our Theorem 6.1.

Let us mention that the idea of our condition on a category K involving a pair of cardinals $\lambda < \kappa$ is certainly not new. It appeared in the discussion of *pseudopullbacks* in [6, Proposition 3.1] and [28, Theorem 2.2] (and our arguments in this paper bear some similarity to the one in [6]). The fact that this condition is sufficient for the "ideal" result on accessibility of diagram categories $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ (our Theorem 6.1) seems to be if not quite new, then a "well-forgotten old" discovery, however.

The discussion in the beginning of this introduction suggests that we are also interested in other category-theoretic constructions beyond the categories of functors or diagrams; and indeed we are.

Limits of accessible categories are mentioned in the title of this paper. There are many relevant concepts of limits of categories, the most general ones being the weighted pseudolimits or weighted bilimits [19, §5.1], [13], [5]. All of them can be built from certain elementary building blocks.

We discuss the *Cartesian product* (easy), the *equifier* (a representative case for our techniques), the *inserter* (difficult), and the *pseudopullback* (for which our result is already known in relatively recent literature [6, 28]), as well as the nonadditive and the additive/k-linear diagram categories. The pseudopullbacks and the diagram categories are built from the products, the inserters, and the equifiers.

In fact, all weighted pseudolimits and weighted bilimits can be built from products, inserters, and equifiers, up to category equivalence [13, 5]. Hence the importance of our detailed discussion of the products, the inserters, and the equifiers in the general context of limits of accessible categories.

In all the settings (with the exception of the trivial case of the Cartesian products), our results are very similar. The main assumptions are that κ is a regular cardinal and $\lambda < \kappa$ is a smaller infinite cardinal (so the case of finitely accessible categories, $\kappa = \aleph_0$, is excluded). The category K is assumed to be κ-accessible with colimits of λ-indexed chains. If this is the case, then the category E of (collections of) objects from K with an additional structure satisfying some equations is also κ-accessible (again with colimits of λ-indexed chains), and the κ-presentable objects of E are precisely those whose underlying objects are κ-presentable in K.

We do not dare to speculate on what the author of the paper [18] might have in mind back in 1988, but the proofs of our results seem to follow the general outline suggested in [18, proof of Lemma 5.1]. They are, indeed, "direct calculations" (which, however, get complicated at times).

In fact, our results go back all the way to late 1970s, to an unpublished 1977 preprint of Ulmer [29]. The very concept and terminology of an *accessible category* was only introduced by Makkai and Paré in their 1989 book [19]. Accordingly, the exposition in [29] was written mainly in the generality of locally presentable categories (which had been known since the 1971 book of Gabriel and Ulmer [10]).

The main results of [29] relevant in our context are [29, Theorem 3.8 and Corollary 3.9]. These are stated for locally presentable categories, followed by a remark [29, Remark 3.11(II)] explaining that the assertions are actually valid for some (what we would now call) accessible categories. This work of Ulmer was subsequently taken up and developed in the 1984 dissertation of Bird [4], which was also written in the generality of locally presentable categories. Ulmer's remark [29, Remark 3.11(II)] was not taken up, and apparently remained almost forgotten.

The topic of limits of accessible categories was studied by Makkai and Paré [19, §5.1] using methods which seem to be quite different from those of Ulmer. The Limit Theorem of Makkai and Paré [19, Theorem 5.1.6] claimed that all weighted bilimits of accessible categories are accessible, but offered no cardinality estimate on the accessibility rank. The fact that a tight estimate can be obtained from Ulmer's results was not realized. See our Corollary 9.2.

The present author learned about the existence of Ulmer's preprint from [28, paragraph after Pseudopullback Theorem 2.2], where the knowledge about Ulmer's work is attributed to Porst. Still, no traces of such knowledge can be found in Porst's own earlier paper [22] (cf. [23, Remark 3.2] and [26]). I only got hold of my copy of Ulmer's preprint after the first version of the present paper, with my own detailed proofs of the main results, was already available on the `arXiv`.

Let us explain our motivation now. In terms of the intended applications, we are primarily interested in the "minimal cardinality" case $\kappa = \aleph_1$ and $\lambda = \aleph_0$. The examples we care about arise from flat modules over rings, flat quasi-coherent sheaves over schemes, flat comodules, and flat contramodules.

It is shown in the preprint [27, Theorem 2.4] that the category $X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$ of flat quasi-coherent sheaves on a quasi-compact quasi-separated scheme X is $\aleph_1$-accessible. More genenerally, the same holds for any countably quasi-compact, countably quasi-separated scheme [27, Theorem 3.5]. The $\aleph_1$-presentable objects of $X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$ are the locally countably presentable flat quasi-coherent sheaves, i. e., the quasi-coherent sheaves $\mathcal{F}$ on X such that the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is flat and countably presented for all affine open subschemes $U \subset X$ (equivalently, for the affine open subschemes U_α appearing in some fixed affine open covering $X = \bigcup_\alpha U_\alpha$ of the scheme X). Obviously, all directed colimits, and in particular directed colimits of $\aleph_0$-indexed chains, exist in $\mathsf{K} = X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$. So the results of this paper are applicable to this category.

The results of [27] were extended to certain noncommuative stacks and noncommutative ind-affine ind-schemes in the preprint [23]. Specifically, let $\mathcal{C}$ be a (coassociative, counital) coring over a noncommutative ring A. According to [23, Theorem 3.1], the category of A-flat left $\mathcal{C}$-comodules $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$ is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of of $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$ are the A-countably presentable A-flat left $\mathcal{C}$-comodules. Once again, it is obvious that all directed colimits exist in $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$; so the results of the present paper can be applied. There is also a version for flat contramodules over certain topological rings [23, Theorem 10.1], where the results of the present paper are applicable as well.

Some results about constructing A-pure acyclic complexes of A-flat $\mathcal{C}$-comodules as $\aleph_1$-filtered colimits of A-pure acyclic complexes of A-countably presentable A-flat $\mathcal{C}$-comodules are discussed in [23, Section 4]. A contramodule version can be found in [23, Section 11]. The techniques developed in the present paper are used throughout the current (new) versions of the papers [27] and [23]. The same methods are also used in the preprint [25], where accessibility of categories of modules of finite flat dimension and two-sided/F-totally acyclic flat resolutions is discussed, and in the preprint [26], where we discuss local presentability and accessibility ranks of the categories of corings and coalgebras over rings.

In the present paper, we do not go into any details on sheaves, comodules, or contramodules, restricting ourselves to "toy examples" of diagrams and complexes of modules over a noncommutative ring R. It is easy to see that the category of flat left R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is κ-accessible for any regular cardinal κ; the κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ are those flat R-modules that are κ-presentable in the category of arbitrary R-modules $R\text{–}\mathsf{Mod}$. Applying the results of this paper, we obtain descriptions of diagrams of flat modules and pure acyclic complexes of flat modules as directed colimits (recovering, in particular, a weaker version of a result from the papers [9, 21] with very general category-theoretic methods).

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I am grateful to Jiří Rosický for bringing the preprint [12] to my attention, as well as for providing me with a copy of Ulmer's preprint [29]. Simon Henry suggested to include some general, abstract material on lax limits and weighted pseudolimits; I followed this suggestion by writing Sections 7, 8, and 9. The idea to include Corollary 9.2 is also due to Simon. I want to thank Jan Šťovíček for helpful conversations. The author is supported by the GAČR project 23-05148S and the Czech Academy of Sciences (RVO 67985840).

1. Preliminaries

We use the book [1] as the main background reference source on the foundations of the theory of accessible categories.

Let κ be a regular cardinal. We refer to [1, Definition 1.4, Theorem 1.5, Definition 1.13(1), and Remark 1.21] for the discussion of *κ-directed posets* vs. *κ-filtered categories* and, accordingly, *κ-directed* vs. *κ-filtered diagrams* and their colimits.

Let K be a category in which all κ-directed (equivalently, κ-filtered) colimits exist. An object $S \in \mathsf{K}$ is said to be *κ-presentable* [1, Definitions 1.1 and 1.13(2)] if the functor $\mathrm{Hom}_{\mathsf{K}}(S,-)\colon \mathsf{K} \longrightarrow \mathsf{Sets}$ preserves κ-directed colimits. We will denote by $\mathsf{K}_{<\kappa} \subset \mathsf{K}$ the full subcategory of κ-presentable objects in K.

A category K with κ-directed colimits is called *κ-accessible* [1, Definition 2.1] if there is a set S of κ-presentable objects in K such that every object of K is a κ-directed colimit of objects from S. In any κ-accessible category, there is only a set of isomorphism classes of κ-presentable objects; in fact, the κ-presentable objects of K are precisely the retracts of the objects from S [1, Remarks 1.9 and 2.2(4)].

Let K be a category and $\mathsf{S} \subset \mathsf{K}$ be a set of objects. For any object $K \in \mathsf{K}$, the *canonical diagram* [1, Definition 0.4] of morphisms from objects from S into K is indexed by the small indexing category $\Delta = \Delta_{\mathsf{S},K}$ whose objects $v \in \Delta$ are morphisms $v\colon D_v \longrightarrow K$ into K from objects $D_v \in \mathsf{S}$. A morphism $a\colon v \longrightarrow w$ in Δ is a morphism $a\colon D_v \longrightarrow D_w$ in K making the triangular diagram $D_v \longrightarrow D_w \longrightarrow K$ commutative in K. The canonical diagram $D = D_{\mathsf{S},K}\colon \Delta \longrightarrow K$ takes an object $v \in \Delta$ to the object $D_v \in \mathsf{K}$, and acts on the morphisms in the obvious way.

Lemma 1.1. *Let K be a κ-accessible category and S be a set of representatives of isomorphism classes of κ-presentable objects in K. Then, for every object $K \in \mathsf{K}$, the canonical diagram $D = D_{\mathsf{S},K}$ of morphisms from objects from S into K (or in other words, its indexing category $\Delta = \Delta_{\mathsf{S},K}$) is κ-filtered. The natural morphism $\varinjlim_{v\in\Delta} D_v \longrightarrow K$ is an isomorphism in K.*

Proof. This is [1, Definition 1.23 and Proposition 2.8(i–ii)]. ☐

Let K be a category with κ-directed colimits and $\mathsf{A} \subset \mathsf{K}$ be a class of objects (full subcategory). Then we denote by $\varinjlim_{(\kappa)} \mathsf{A} \subset \mathsf{K}$ the class of all objects of K that can be obtained as κ-directed colimits of objects from A.

The following proposition is also essentially well-known. In the particular case of finitely accessible ($\kappa = \aleph_0$) additive categories, it was discussed in [17, Proposition 2.1], [8, Section 4.1], and [15, Proposition 5.11]. (The terminology "finitely presented categories" was used in [8, 15] for what are called finitely accessible categories in [1].)

Proposition 1.2. *Let* K *be a* κ*-accessible category and* $\mathsf{S} \subset \mathsf{K}_{<\kappa}$ *be a set of* κ*-presentable objects in* K*. Then the full subcategory* $\varinjlim_{(\kappa)} \mathsf{S} \subset \mathsf{K}$ *is closed under* κ*-directed colimits in* K*. The category* $\varinjlim_{(\kappa)} \mathsf{S}$ *is* κ*-accessible; the full subcategory of all* κ*-presentable objects of* $\varinjlim_{(\kappa)} \mathsf{S}$ *consists of all the retracts of objects from* S *in* K*. An object* $E \in \mathsf{K}$ *belongs to* $\varinjlim_{(\kappa)} \mathsf{S}$ *if and only if, for every* κ*-presentable object* $T \in \mathsf{K}_{<\kappa}$*, every morphism* $T \longrightarrow E$ *in* K *factorizes through an object from* S*.*

Proof. The key assertion is that if an object $E \in \mathsf{K}$ has the property that every morphism $T \longrightarrow K$ into K from an object $T \in \mathsf{K}_{<\kappa}$ factorizes through some object from S, then $E \in \varinjlim_{(\kappa)} \mathsf{S}$. (All the other assertions follow easily from this one.)

Indeed, let T denote a representative set of κ-presentable objects in K. Consider the canonical diagram $C\colon \Delta_{\mathsf{S}} \longrightarrow \mathsf{K}$ of morphisms into E from objects of S and the canonical diagram $D\colon \Delta_{\mathsf{T}} \longrightarrow \mathsf{K}$ of morphisms into E from objects of T. Then we have $E = \varinjlim_{w \in \Delta_{\mathsf{T}}} D_w$ by Lemma 1.1, and we need to show that $E = \varinjlim_{v \in \Delta_{\mathsf{S}}} C_v$. So it remains to check that the natural functor between the index categories $\delta\colon \Delta_{\mathsf{S}} \longrightarrow \Delta_{\mathsf{T}}$ is cofinal in the sense of [1, Section 0.11].

Let $w\colon D_w \longrightarrow E$ be an object of Δ_{T}. Then $D_w \in \mathsf{T}$, and by assumption the morphism v factorizes as $D_w \xrightarrow{a} S \xrightarrow{v} E$ with $S \in \mathsf{S}$. So $v\colon C_v = S \longrightarrow E$ is an object of Δ_{S}, and we have a morphism $a\colon w \longrightarrow \delta(v)$ in Δ_{T}. This proves condition (a) from [1, Section 0.11]. Since the category Δ_{T} is κ-filtered and the functor δ is fully faithful, condition (b) follows automatically. $\square$

Any cardinal λ can be considered as a totally ordered set, which is a particular case of a poset; and any poset I can be viewed as a category (with the elements of I being the objects, and a unique morphism $i \longrightarrow j$ for every pair of objects $i \le j \in I$). A λ*-indexed chain* (of objects and morphisms) in a category K is a functor $\lambda \longrightarrow \mathsf{K}$, where λ is viewed as a category as explained above.

2. Product

The result of this short section is easy and straightforward; it is only included here for the sake of completeness of the exposition. It is essentially a trivial particular case of [12, Theorem 1.3], and also the correct particular case of an erroneous (generally speaking) argument in [1, proof of Proposition 2.67].

Proposition 2.1. *Let κ be a regular cardinal and $(\mathsf{K}_i)_{i\in I}$ be a family of κ-accessible categories. Assume that the cardinality of the indexing set I is smaller than κ. Then the Cartesian product $\mathsf{K} = \prod_{i\in I}\mathsf{K}_i$ is also a κ-accessible category. An object $S\in\mathsf{K}$, $S=(S_i\in\mathsf{K}_i)_{i\in I}$ is κ-presentable in K if and only if all its components S_i are κ-presentable in K_i.*

Proof. The condition that the cardinality of I is smaller than κ (which is missing in [1, proof of Proposition 2.67]) is needed in order to claim that an object $S\in\mathsf{K}$ is κ-presentable whenever its components $S_i\in\mathsf{K}_i$ are κ-presentable for all i. Essentially, this holds because κ-directed colimits commute with κ-small products in the category of sets (cf. [12, Proposition 2.1]). Once this is established, it remains to observe that every object of K is a κ-directed colimit of such objects S, just as [1, proof of Proposition 2.67] tells. Indeed, let $K=(K_i)_{i\in I}\in\mathsf{K}$ be an object and $(\Xi_i)_{i\in I}$ be nonempty κ-filtered categories such that $K_i=\varinjlim_{\xi_i\in\Xi_i}S_{i,\xi_i}$ in K_i for all $i\in I$ with $S_{i,\xi_i}\in(\mathsf{K}_i)_{<\kappa}$. Then $\Xi=\prod_{i\in I}\Xi_i$ is a κ-filtered category and $K=\varinjlim_{\xi\in\Xi}S_\xi$, where $S_\xi=(S_{i,\xi_i})_{i\in I}$ whenever $\xi=(\xi_i)_{i\in I}\in\prod_{i\in I}\Xi_i$. One also needs to use the fact that any retract of an object $S\in\mathsf{K}$ with κ-presentable components S_i is again an object with κ-presentable components. $\square$

3. Equifier

Let κ be a regular cardinal and λ be a smaller infinite cardinal, i. e., $\lambda<\kappa$. Let K and L be κ-accessible categories in which all λ-indexed chains (of objects and morphisms) have colimits. Let $F, G\colon \mathsf{K}\rightrightarrows\mathsf{L}$ be two parallel functors preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functor F takes κ-presentable objects to κ-presentable objects. Let $\phi,\psi\colon F\rightrightarrows G$ be two parallel natural transformations of functors.

Let $\mathsf{E} \subset \mathsf{K}$ be the full subcategory consisting of all objects $E \in \mathsf{K}$ such that $\phi_E = \psi_E$. This construction of the category E is known as the *equifier* [13, Section 4], [5, Section 1], [1, Lemma 2.76].

The aim of this section is to prove the following theorem going back to the unpublished preprint [29, Theorem 3.8, Corollary 3.9, and Remark 3.11(II)].

Theorem 3.1. *In the assumptions above, the equifier category* E *is* κ*-accessible. The* κ*-presentable objects of* E *are precisely all the objects of* E *that are* κ*-presentable as objects of* K.

We start with the obvious observations that κ-directed colimits (as well as colimits of λ-indexed chains) exist in E and are preserved by the inclusion functor $\mathsf{E} \longrightarrow \mathsf{K}$ (because such colimits exist in K and are preserved by the functor F). It follows immediately that any object of E that is κ-presentable in K is also κ-presentable in E. The proof of the theorem is based the following proposition.

Proposition 3.2. *Let* $E \in \mathsf{E}$ *be an object and* $S \in \mathsf{K}_{<\kappa}$ *be a* κ*-presentable object. Then any morphism* $S \longrightarrow E$ *in* K *factorizes through an object* $U \in \mathsf{E} \cap \mathsf{K}_{<\kappa}$.

Proof. Let $E = \varinjlim_{\xi\in\Xi} T_\xi$ be a representation of the object E as a κ-filtered colimit of κ-presentable objects in the category K. Then we have $G(E) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ in L and $F(S)$, $F(T_\xi) \in \mathsf{L}_{<\kappa}$. There exists an index $\xi_0 \in \Xi$ such that the morphism $S \longrightarrow E$ factorizes through the morphism $T_{\xi_0} \longrightarrow E$ in K.

Since $E \in \mathsf{E}$, we have $\phi_E = \psi_E \colon F(E) \longrightarrow G(E)$. Hence the two compositions

$$F(T_{\xi_0}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_0}) \longrightarrow G(E)$$

are equal to each other in L. Since $G(E) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ and $F(T_{\xi_0}) \in \mathsf{L}_{<\kappa}$, it follows that there exists an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the two compositions

$$F(T_{\xi_0}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_0}) \longrightarrow G(T_{\xi_1})$$

are equal to each other in L.

Similarly, there exists an index $\xi_2 \in \Xi$ together with an arrow $\xi_1 \longrightarrow \xi_2$ in Ξ such that the two compositions

$$F(T_{\xi_1}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_1}) \longrightarrow G(T_{\xi_2})$$

are equal to each other, etc.

Proceeding in this way, we construct a λ-indexed chain of indices $\xi_i \in \Xi$ and arrows $\xi_i \longrightarrow \xi_j$ in Ξ for all $0 \le i < j < \lambda$ such that, for all ordinals $0 \le i < \lambda$, the two compositions

$$F(T_{\xi_i}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$$

are equal to each other in L. Specifically, for a limit ordinal $k < \lambda$, we just pick an index $\xi_k \in \Xi$ and arrows $\xi_i \longrightarrow \xi_k$ in Ξ for all $i < k$ making the triangles $\xi_i \longrightarrow \xi_j \longrightarrow \xi_k$ commutative in Ξ for all $i < j < k$. This can be done, because $k < \kappa$ and the index category Ξ is κ-filtered. For a successor ordinal $k = i + 1 < \lambda$, the same argument as above in this proof provides the desired arrow $\xi_i \longrightarrow \xi_{i+1}$.

After the construction is finished, it remains to put $U = \varinjlim_{i<\lambda} T_{\xi_i}$. We have $U \in \mathsf{K}_{<\kappa}$, since $\lambda < \kappa$ and the class of all κ-presentable objects in a category with κ-directed colimits is closed under those κ-small colimits that exist in the category [1, Proposition 1.16]. We also have $\phi_U = \psi_U$ by construction, since $F(U) = \varinjlim_{i<\lambda} F(T_{\xi_i})$; so $U \in \mathsf{E}$. $\square$

Proof of Theorem 3.1. Combine Propositions 1.2 and 3.2. $\square$

Remark 3.3. In applications of Theorem 3.1, one may be interested in the *joint equifier* of a family of pairs of natural transformations (cf. [1, Remark 2.76]). Let K be a κ-accessible category and $(\mathsf{L}_i)_{i\in I}$ be a family of κ-accessible categories. Let $F_i, G_i \colon \mathsf{K} \rightrightarrows \mathsf{L}_i$ be a family of pairs of parallel functors, all of them preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functors F_i take κ-presentable objects to κ-presentable objects, and that the cardinality of the indexing set I is smaller than κ. Let $\phi_i, \psi_i \colon F_i \rightrightarrows G_i$ be a family of pairs of parallel natural transformations.

Consider the full subcategory $\mathsf{E} \subset \mathsf{K}$ consisting of all objects $E \in \mathsf{K}$ such that $\phi_{i,E} = \psi_{i,E}$ for all $i \in I$. Then the category E is κ-accessible, and the κ-presentable objects of E are precisely all the objects of E that are κ-presentable as objects of K. This assertion can be deduced from Proposition 2.1 and Theorem 3.1 by passing to the Cartesian product category $\mathsf{L} = \prod_{i\in I} \mathsf{L}_i$. The family of functors $F_i\colon \mathsf{K} \longrightarrow \mathsf{L}_i$ defines a functor $F\colon \mathsf{K} \longrightarrow \mathsf{L}$, the family of functors $G_i\colon \mathsf{K} \longrightarrow \mathsf{L}_i$ defines a functor $G\colon \mathsf{K} \longrightarrow \mathsf{L}$, and the family of pairs of natural transformations ϕ_i, $\psi_i\colon F_i \rightrightarrows G_i$ defines a pair of natural transformations ϕ, $\psi\colon F \rightrightarrows G$. It follows from Proposition 2.1 that all the assumptions of Theorem 3.1 are satisfied by the category L and the pair of functors F, G.

4. Inserter

As in Section 3, we consider a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$. Let K and L be κ-accessible categories in which all λ-indexed chains have colimits. Let $F, G\colon \mathsf{K} \rightrightarrows \mathsf{L}$ be two parallel functors preserving κ-directed colimits and colimits of λ-indexed chains; assume further that the functor F takes κ-presentable objects to κ-presentable objects.

Let E be the category of pairs (K, ϕ), where $K \in \mathsf{K}$ is an object and $\phi\colon F(K) \longrightarrow G(K)$ is a morphism in L. This construction of the category E is known as the *inserter* [13, Section 4], [5, Section 1], [19, Section 5.1.1], [1, Section 2.71].

The aim of this section is to prove the following theorem, which also goes back to the unpublished preprint [29, Theorem 3.8, Corollary 3.9, and Remark 3.11(II)].

Theorem 4.1. *In the assumptions above, the inserter category* E *is* κ*-accessible. The* κ*-presentable objects of* E *are precisely all the pairs* (S, ψ) *where* S *is a* κ*-presentable object of* K.

We start with the obvious observations that κ-directed colimits (as well as colimits of λ-indexed chains) exist in E and are preserved by the forgetful functor $\mathsf{E} \longrightarrow \mathsf{K}$ (because such colimits exists in K and are preserved by the functor F).

The proof of the theorem is based on three propositions. It uses the same idea as the proof of Theorem 3.1 above, but the details are much more complicated in the case of Theorem 4.1.

Proposition 4.2. *Let* $(S, \psi) \in \mathsf{E}$ *be an object such that* $S \in \mathsf{K}_{<\kappa}$. *Then* $(S, \psi) \in \mathsf{E}_{<\kappa}$.

Proof. The assumptions concerning cardinal λ are not needed for this proposition. Essentially, the assertion holds because κ-directed colimits commute with finite limits in the category of sets (cf. [12, Proposition 2.1]). To be more specific, it helps to observe that, given an object (K, ϕ) in E, the set of morphisms $\mathrm{Hom}_{\mathsf{E}}((S, \psi), (K, \phi))$ is computed as the equalizer of the natural pair of maps

$$\mathrm{Hom}_{\mathsf{K}}(S, K) \underset{f \mapsto G(f) \circ \psi}{\overset{f \mapsto \phi \circ F(f)}{\rightrightarrows}} \mathrm{Hom}_{\mathsf{L}}(F(S), G(K)).$$

Then one needs to use the assumptions that the functor G preserves κ-directed colimits and the functor F takes κ-presentable objects to κ-presentable objects. $\square$

Denote by $\mathsf{E}'_{<\kappa} \subset \mathsf{E}$ the full subcategory formed by all the pairs $(S, \psi) \in \mathsf{E}$ with $S \in \mathsf{K}_{<\kappa}$. By Proposition 4.2, we have $\mathsf{E}'_{<\kappa} \subset \mathsf{E}_{<\kappa}$.

Proposition 4.3. *Let* $E = (K, \phi) \in \mathsf{E}$ *be an object. Consider the canonical diagram* $C = D_E$ *of morphisms into* E *from (representatives of isomorphism classes of) objects* $B = (S, \psi) \in \mathsf{E}'_{<\kappa}$, *with the indexing category* $\Delta = \Delta_E$. *Then the indexing category* Δ *is* κ*-filtered.*

Proposition 4.4. *In the context of Proposition 4.3, consider also the canonical diagram* $D = D_K$ *of morphisms into* K *from (representatives of isomorphism classes of) objects* $S \in \mathsf{K}_{<\kappa}$, *with the indexing category* Δ_K. *Then the natural functor between the indexing categories* $\Delta_E \longrightarrow \Delta_K$ *is cofinal (in the sense of [1, Section 0.11]).*

The proofs of Propositions 4.3 and 4.4 are based on the following lemma.

Lemma 4.5. *Let* $E = (K, \phi) \in \mathsf{E}$ *be an object, let* S, $T \in \mathsf{K}_{<\kappa}$ *be* κ*-presentable objects, and let* $\sigma \colon F(S) \longrightarrow G(T)$ *be a morphism in* L. *Let*

$S \longrightarrow T$ and $T \longrightarrow K$ be morphisms in K. Assume that the pentagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T) \arrow[rr] & & G(K) \end{tikzcd}$$

is commutative in L. Then there exists an object $B = (U, \psi) \in \mathsf{E}'_{<\kappa}$ together with a morphism $(U, \psi) \longrightarrow (K, \phi)$ in E and a morphism $T \longrightarrow U$ in K such that the pentagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(U) \arrow[d, "\psi"] \\ G(T) \arrow[rr] & & G(U) \end{tikzcd}$$

is commutative in L and the triangular diagram $T \longrightarrow U \longrightarrow K$ is commutative in K.

Proof. Let $K = \varinjlim_{\xi \in \Xi} T_\xi$ be a representation of the object K as a κ-filtered colimit of κ-presentable objects in the category K. Then we have $G(K) = \varinjlim_{\xi \in \Xi} G(T_\xi)$ in L and $F(S)$, $F(T_\xi) \in \mathsf{L}_{<\kappa}$. There exists an index $\xi_0 \in \Xi$ such that the morphism $T \longrightarrow K$ factorizes through the morphism $T_{\xi_0} \longrightarrow K$ in K. Then the heptagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(T_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T) \arrow[rr] & & G(T_{\xi_0}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L.

Since $G(K) = \varinjlim_{\xi \in \Xi} G(T_\xi)$ and $F(T_{\xi_0}) \in \mathsf{L}_{<\kappa}$, there exists an index $\xi_1 \in \Xi$ such that the composition $F(T_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_1}) \longrightarrow G(K)$ in L:

$$\begin{tikzcd} F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

Moreover, since $G(K) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ and $F(S) \in \mathsf{L}_{<\kappa}$, one can choose the index ξ_1 together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the hexagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(T_{\xi_0}) \arrow[d, "\psi_0"] \\ G(T) \arrow[r] & G(T_{\xi_0}) \arrow[r] & G(T_{\xi_1}) \end{tikzcd}$$

is commutative in L. Notice that the pentagonal diagram

$$\begin{tikzcd} F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(T_{\xi_1}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_1}) \arrow[rr] & & G(K) \end{tikzcd}$$

is also commutative in L.

Hence one can choose an index $\xi_2 \in \Xi$ together with an arrrow $\xi_1 \longrightarrow \xi_2$ in Ξ such that the composition $F(T_{\xi_1}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_2}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(T_{\xi_1}) \arrow[r] \arrow[d, "\psi_1"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_2}) \arrow[r] & G(K) \end{tikzcd}$$

and the square diagram

$$\begin{tikzcd} F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(T_{\xi_1}) \arrow[d, "\psi_1"] \\ G(T_{\xi_1}) \arrow[r] & G(T_{\xi_2}) \end{tikzcd}$$

is commutative in L. Then the pentagonal diagram

$$\begin{tikzcd} F(T_{\xi_1}) \arrow[r] \arrow[d, "\psi_1"'] & F(T_{\xi_2}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_2}) \arrow[rr] & & G(K) \end{tikzcd}$$

is commutative in L.

Proceeding in this way, we construct a λ-indexed chain of indices $\xi_i \in \Xi$ and arrows $\xi_i \longrightarrow \xi_j$ in Ξ for all $0 \le i < j < \lambda$ together with morphisms $\psi_i \colon F(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$ in L such that, for all ordinals $0 \le i < \lambda$, the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L and, for all ordinals $0 \le i < j < \lambda$, the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[d, "\psi_j"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(T_{\xi_{j+1}}) \end{tikzcd}$$

is commutative in L.

Specifically, similarly to the proof of Proposition 3.2, for a limit ordinal $k < \lambda$, we just pick an index $\xi_k \in \Xi$ and arrows $\xi_i \longrightarrow \xi_k$ in Ξ for all $i < k$ making the triangles $\xi_i \longrightarrow \xi_j \longrightarrow \xi_k$ commutative in Ξ for all $i < j < k$. For a successor ordinal $k = j + 1 < \lambda$, we choose an index $\xi_{j+1} \in \Xi$ together with an arrow $\xi_j \longrightarrow \xi_{j+1}$ in Ξ such that the composition $F(T_{\xi_j}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_{j+1}}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(T_{\xi_j}) \arrow[r] \arrow[d, "\psi_j"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{j+1}}) \arrow[r] & G(K) \end{tikzcd}$$

and the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[d, "\psi_j"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(T_{\xi_{j+1}}) \end{tikzcd}$$

is commutative in L for all $i < j$. The latter condition can be satisfied because the pentagonal diagrams

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{i+1}}) \arrow[rr] & & G(K) \end{tikzcd}$$

are commutative in L for all $i < j$ and the index category Ξ is κ-filtered.

After the construction is finished, it remains to put $U = \varinjlim_{i<\lambda} T_{\xi_i}$, and define $\psi\colon F(U) \longrightarrow G(U)$ to be the colimit of the morphisms $\psi_i\colon F(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$. It is important here that $F(U) = \varinjlim_{i<\lambda} F(T_{\xi_i})$. We have $U \in \mathsf{K}_{<\kappa}$ for the reason explained in the proof of Proposition 3.2. $\square$

Proof of Proposition 4.3. Firstly, let $v_a\colon (S_a, \psi_a) \longrightarrow (K, \phi)$ be a family of morphisms into (K, ϕ) from objects $(S_a, \psi_a) \in \mathsf{E}'_{<\kappa}$, with the set of indices a having cardinality smaller than κ. We need to show that there is a morphism $u\colon (T, \tau) \longrightarrow (K, \phi)$ into (K, ϕ) from an object $(T, \tau) \in \mathsf{E}'_{<\kappa}$ such that all the morphisms v_a factorize through u. For this purpose, choose a representation $K = \varinjlim_{\xi\in\Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$.

Then there exists an index $\xi_0 \in \Xi$ such that all the morphisms $v_a\colon S_a \longrightarrow K$ factorize through the morphism $S_{\xi_0} \longrightarrow K$ in K. The hexagonal diagram

$$\begin{tikzcd} F(S_a) \arrow[r] \arrow[d, "\psi_a"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(S_a) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L for all indices a. Therefore, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(S_{\xi_1}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\ G(S_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

and the pentagonal diagrams

$$\begin{tikzcd}
F(S_a) \arrow[rr] \arrow[d, "\psi_a"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] \\
G(S_a) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1})
\end{tikzcd}$$

are commutative in L for all a. Then the pentagonal diagram

$$\begin{tikzcd}
F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(S_{\xi_1}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(S_{\xi_1}) \arrow[rr] & & G(K)
\end{tikzcd}$$

is also commutative in L. It remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and use Lemma 4.5.

Secondly, let $v\colon (P,\pi) \longrightarrow (K,\phi)$ be a morphism into (K,ϕ) from an object $(P,\pi) \in \mathsf{E}'_{<\kappa}$, and let $w_a\colon (R,\rho) \longrightarrow (P,\pi)$ be a family of parallel morphisms into (P,π) from an object $(R,\rho) \in \mathsf{E}'_{<\kappa}$, with the set of indices a having cardinality smaller than κ. Assume that all the morphisms $vw_a\colon (R,\rho) \longrightarrow (K,\phi)$ are equal to each other. We need to show that the morphism $v\colon (P,\pi) \longrightarrow (K,\phi)$ can be factorized as $(P,\pi) \xrightarrow{u} (U,\psi) \longrightarrow (K,\phi)$ in such a way that $(U,\psi) \in \mathsf{E}'_{<\kappa}$ and all the morphisms $uw_a\colon (R,\rho) \longrightarrow (U,\psi)$ are equal to each other.

For this purpose, choose a representation $K = \varinjlim_{\xi\in\Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_0 \in \Xi$ such that the morphism $v\colon P \longrightarrow K$ factorizes through the morphism $S_{\xi_0} \longrightarrow K$ and all the compositions $R \xrightarrow{w_a} P \longrightarrow S_{\xi_0}$ are equal to each other. The hexagonal diagram

$$\begin{tikzcd}
F(P) \arrow[r] \arrow[d, "\pi"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(P) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K)
\end{tikzcd}$$

is commutative in L.

Therefore, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(S_{\xi_1}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\ G(S_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

and the pentagonal diagram

$$\begin{tikzcd} F(P) \arrow[rr] \arrow[d, "\pi"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] \\ G(P) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1}) \end{tikzcd}$$

is commutative in L. Once again, it remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and refer to Lemma 4.5. $\square$

Proof of Proposition 4.4. Firstly, let $P \longrightarrow K$ be a morphism into K from an object $P \in \mathsf{K}_{<\kappa}$. We need to show that there exists an object $(U, \psi) \in \mathsf{E}'_{<\kappa}$ together with a morphism $(U, \psi) \longrightarrow (K, \phi)$ in E and a morphism $P \longrightarrow U$ in K such that the triangular diagram $P \longrightarrow U \longrightarrow K$ is commutative in K.

For this purpose, choose a representation $K = \varinjlim_{\xi \in \Xi} T_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $T_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_1 \in \Xi$ such that the morphism $P \longrightarrow K$ factorizes through the morphism $T_{\xi_1} \longrightarrow K$ in K and the composition $F(P) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_1}) \longrightarrow G(K)$ in L:

$$\begin{tikzcd} F(P) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

It remains to put $S = P$ and $T = T_{\xi_1}$, and refer to Lemma 4.5.

Secondly, let (R', ρ') and (R'', ρ'') be two objects of $\mathsf{E}'_{<\kappa}$, let

$$(R', \rho') \longrightarrow (K, \phi) \longleftarrow (R'', \rho'')$$

be two morphisms in E, and let $R' \longleftarrow P \longrightarrow R''$ be two morphisms in K such that the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & K & R'' \arrow[l] \end{tikzcd}$$

is commutative in K. We need to show that there exists an object $(U, \psi) \in \mathsf{E}_{<\kappa}$ together with two morphisms $(R', \rho') \longrightarrow (U, \psi) \longleftarrow (R'', \rho'')$ and a morphism $(U, \psi) \longrightarrow (K, \phi)$ in E such that the two triangular diagrams

$$\begin{tikzcd} (R', \rho') \arrow[r] \arrow[dr] & (U, \psi) \arrow[d] & (R'', \rho'') \arrow[l] \arrow[dl] \\ & (K, \phi) & \end{tikzcd}$$

are commutative in E and the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & U & R'' \arrow[l] \end{tikzcd}$$

is commutative in K.

For this purpose, choose a representation $K = \varinjlim_{\xi \in \Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_0 \in \Xi$ such that both the morphisms $R' \longrightarrow K$ and $R'' \longrightarrow K$ factorize through the morphism $S_{\xi_0} \longrightarrow K$ in K and the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & S_{\xi_0} & R'' \arrow[l] \end{tikzcd}$$

is commutative in K. So the whole diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] \arrow[dr] & S_{\xi_0} \arrow[d] & R'' \arrow[l] \arrow[dl] \\ & K & \end{tikzcd}$$

is commutative. Then the two hexagonal diagrams

$$\begin{tikzcd} F(R') \arrow[r] \arrow[d, "\rho'"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] & F(S_{\xi_0}) \arrow[l] & F(R'') \arrow[l] \arrow[d, "\rho''"] \\ G(R') \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K) & G(S_{\xi_0}) \arrow[l] & G(R'') \arrow[l] \end{tikzcd}$$

are commutative in L.

Hence one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(\xi_1) \longrightarrow G(K)$:

$$\begin{tikzcd} F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\ G(S_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

and the two pentagonal diagrams

$$\begin{tikzcd} F(R') \arrow[rr] \arrow[d, "\rho'"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] & & F(R'') \arrow[ll] \arrow[d, "\rho''"] \\ G(R') \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1}) & G(S_{\xi_0}) \arrow[l] & G(R'') \arrow[l] \end{tikzcd}$$

are commutative in L. Then it remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and refer to Lemma 4.5. $\square$

Finally, we are ready to prove the theorem.

Proof of Theorem 4.1. By Proposition 4.2, all the pairs $(S,\psi)\in\mathsf{E}$ with $S\in\mathsf{K}_{<\kappa}$ are κ-presentable in E. It is also clear that the full subcategory $\mathsf{E}'_{<\kappa}$ of all such pairs (S,ψ) is closed under retracts in E (since the full subcategory $\mathsf{K}_{<\kappa}$ is closed under retracts in K). Let $\mathsf{S}\subset\mathsf{E}$ be a set of representatives of isomorphism classes of objects from $\mathsf{E}'_{<\kappa}$. In view of [1, Remarks 1.9 and 2.2(4)] (see the discussion in Section 1), it suffices to prove that, for every object $E\in\mathsf{E}$, the indexing category $\Delta=\Delta_E$ of the canonical diagram $C=D_E$ of morphisms into E from objects of S is κ-filtered, and that $E=\varinjlim_{v\in\Delta}C_v$.

The former assertion is the result of Proposition 4.3. To prove the latter one, notice that by Lemma 1.1 we have $K=\varinjlim_{w\in\Delta_K}D_w$ in K, where $D\colon\Delta_K\longrightarrow\mathsf{K}$ is the canonical diagram of morphisms into K from representatives of isomorphisms classes of objects from $K_{<\kappa}$. Since the natural functor $\delta\colon\Delta\longrightarrow\Delta_K$ between the indexing categories is cofinal by Proposition 4.4, it follows that $K=\varinjlim_{v\in\Delta_E}D_{\delta(v)}$ in K. As the forgetful functor $\mathsf{E}\longrightarrow\mathsf{K}$ is conservative and preserves κ-filtered colimits, we can conclude that $E=\varinjlim_{v\in\Delta_E}C_v$ in E. $\square$

Remark 4.6. In applications of Theorem 4.1, one may be interested in the *joint inserter* of a family of pairs of functors. Let K be a κ-accessible category and $(\mathsf{L}_i)_{i\in I}$ be a family of κ-accessible categories. Let F_i, $G_i\colon\mathsf{K}\rightrightarrows\mathsf{L}_i$ be a family of pairs of parallel functors, all of them preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functors F_i take κ-presentable objects to κ-presentable objects, and that the cardinality of the indexing set I is smaller than κ.

Let E be the category of pairs (K,ϕ), where $K\in\mathsf{K}$ is an object and $\phi=(\phi_i)_{i\in I}$ is a family of morphisms $\phi_i\colon F_i(K)\longrightarrow G_i(K)$ in L_i. Then the category E is κ-accessible, and the κ-presentable objects of E are precisely all the pairs (S,ψ) where S is a κ-presentable object of K. This assertion can be deduced from Proposition 2.1 and Theorem 4.1 by passing to the Cartesian product category $\mathsf{L}=\prod_{i\in I}\mathsf{L}_i$. The family of functors $F_i\colon\mathsf{K}\longrightarrow\mathsf{L}_i$ defines a functor $F\colon\mathsf{K}\longrightarrow\mathsf{L}$, and the family of functors $G_i\colon\mathsf{K}\longrightarrow\mathsf{L}_i$ defines a functor $G\colon\mathsf{K}\longrightarrow\mathsf{L}$. It follows from Proposition 2.1 that all the assumptions of Theorem 4.1 are satisfied by the category L and the pair of functors F, G.

5. Pseudopullback

As in Sections 3 and 4, we consider a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$. Let A, B, and C be κ-accessible categories in which all λ-indexed chains (of objects and morphisms) have colimits. Let $\Theta_{\mathsf{A}} : \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_{\mathsf{B}} \colon \mathsf{B} \longrightarrow \mathsf{C}$ be two functors preserving κ-directed colimits and colimits of λ-indexed chains, and taking κ-presentable objects to κ-presentable objects.

Let D be the category of triples (A, B, θ), where $A \in \mathsf{A}$ and $B \in \mathsf{B}$ are objects and $\theta \colon \Theta_{\mathsf{A}}(A) \simeq \Theta_{\mathsf{B}}(B)$ is an isomorphism in C. This construction of the category D is known as the *pseudopullback* [6, Proposition 3.1], [28, Section 2]. The aim of this section is to deduce the following corollary of Theorems 3.1 and 4.1.

Corollary 5.1. *In the assumptions above, the category D is κ-accessible. The κ-presentable objects of D are precisely all the triples (A, B, θ), where A is a κ-presentable object of A and B is a κ-presentable object of B.*

Proof. This result, going back to [29, Remark 3.2(I), Theorem 3.8, Corollary 3.9, and Remark 3.11(II)], appears in the recent literature as [6, Proposition 3.1], [28, Pseudopullback Theorem 2.2]. So we include this proof for the sake of completeness of the exposition and for illustrative purposes.

The point is that the pseudopullback can be constructed as a combination of products, inserters, and equifiers. Put $\mathsf{K} = \mathsf{A} \times \mathsf{B}$ and $\mathsf{L} = \mathsf{C} \times \mathsf{C}$, and consider the following pair of parallel functors $F, G \colon \mathsf{K} \longrightarrow \mathsf{L}$. The functor F takes a pair of objects $(A, B) \in \mathsf{A} \times \mathsf{B}$ to the pair of objects $(\Theta_{\mathsf{A}}(A), \Theta_{\mathsf{B}}(B)) \in \mathsf{C} \times \mathsf{C}$. The functor G takes a pair of objects $(A, B) \in \mathsf{A} \times \mathsf{B}$ to the pair of objects $(\Theta_{\mathsf{B}}(B), \Theta_{\mathsf{A}}(A)) \in \mathsf{C} \times \mathsf{C}$. Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of quadruples $(A, B, \theta', \theta'')$, where $A \in \mathsf{A}$ and $B \in \mathsf{B}$ are objects, while $\theta' \colon \Theta_{\mathsf{A}}(A) \longrightarrow \Theta_{\mathsf{B}}(B)$ and $\theta'' \colon \Theta_{\mathsf{B}}(B) \longrightarrow \Theta_{\mathsf{A}}(A)$ are arbitrary morphisms.

Theorem 4.1 together with Proposition 2.1 tell that the category E is κ-presentable, and the κ-presentable objects of E are precisely all the quadruples $(A, B, \theta', \theta'')$ such that A is a κ-presentable object of A and B is a κ-presentable object of B.

It remains to apply the joint equifier construction of Section 3 and Remark 3.3 to the family of two pairs of parallel natural transformations $(\mathrm{id}, \theta' \circ$

θ'') and $(\mathrm{id},\ \theta''\circ\theta')$ of functors $\mathsf{E}\longrightarrow\mathsf{C}$ in order to produce the full subcategory $\mathsf{D}\subset\mathsf{E}$ of all quadruples (A,B,θ',θ'') such that θ' and θ'' are mutually inverse isomorphisms $\Theta_{\mathsf{A}}(A)\simeq\Theta_{\mathsf{B}}(B)$. Then Theorem 3.1 tells that the category D is κ-accessible and describes its full subcategory of κ-presentable objects, as desired. □

Remark 5.2. Alternatively, one can consider what we would call the *isomorpher* construction for two parallel functors between two categories $P,\ Q\colon \mathsf{H}\rightrightarrows\mathsf{G}$. (It appears in the literature under the name of the "iso-inserter" [13, Section 4], [5, Section 1].) The isomorpher category D consists of all pairs (H,θ), where $H\in\mathsf{H}$ is an object and $\theta\colon P(H)\simeq Q(H)$ is an isomorphism in G.

One can observe that the pseudopullback and the isomorpher constructions are actually equivalent, in the sense that they can be reduced to one another. Given a pair of functors $\Theta_{\mathsf{A}}\colon\mathsf{A}\longrightarrow\mathsf{C}$ and $\Theta_{\mathsf{B}}\colon\mathsf{B}\longrightarrow\mathsf{C}$, one can put $\mathsf{H}=\mathsf{A}\times\mathsf{B}$ and $\mathsf{G}=\mathsf{C}$, and denote by $P\colon\mathsf{H}\longrightarrow\mathsf{G}$ and $Q\colon\mathsf{H}\longrightarrow\mathsf{G}$ the compositions $\mathsf{A}\times\mathsf{B}\longrightarrow\mathsf{A}\longrightarrow\mathsf{C}$ and $\mathsf{A}\times\mathsf{B}\longrightarrow\mathsf{B}\longrightarrow\mathsf{C}$. In this context, the two constructions of the category D agree.

Conversely, given a pair of parallel functors $P,Q\colon\mathsf{H}\rightrightarrows\mathsf{G}$, put $\mathsf{A}=\mathsf{B}=\mathsf{H}$ and $\mathsf{C}=\mathsf{H}\times\mathsf{G}$. Let the functor $\Theta_{\mathsf{A}}\colon\mathsf{A}\longrightarrow\mathsf{C}$ take an object $H'\in\mathsf{H}$ to the pair $(H',P(H'))\in\mathsf{H}\times\mathsf{G}$ and the functor $\Theta_{\mathsf{B}}\colon\mathsf{B}\longrightarrow\mathsf{C}$ take an object $H''\in\mathsf{H}$ to the pair $(H'',Q(H''))\in\mathsf{H}\times\mathsf{G}$. Then an isomorphism $\Theta_{\mathsf{A}}(H')\simeq\Theta_{\mathsf{B}}(H'')$ in C means a pair of isomorphisms $H'\simeq H''$ in H and $P(H')\simeq Q(H'')$ in G. Up to a category equivalence, the datum of two objects H', $H''\in\mathsf{H}$ endowed with such two isomorphisms is the same thing as a single object $H\in\mathsf{H}$ together with an isomorphism $P(H)\simeq Q(H)$ in G. Thus, in this context, the two constructions of the category D agree as well.

Assume that the categories H and G are κ-accessible with colimits of λ-indexed chains (for a regular cardinal κ and a smaller infinite cardinal $\lambda<\kappa$). Assume further that the functors F and G preserve κ-directed colimits and colimits of λ-indexed chains, and that they take κ-presentable objects to κ-presentable objects. Then it follows from Proposition 2.1 and Corollary 5.1 that the isomorpher category D is κ-accessible, and the κ-presentable objects of D are precisely all the pairs (H,θ) with $H\in\mathsf{H}_{<\kappa}$.

6. Diagram categories

In this section, we discuss two constructions: the category of functors $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ and the category of k-linear functors $\mathsf{Fun}_k(\mathsf{A}, \mathsf{K})$. The former one is of interest to the general category theory, while the latter one is relevant for additive category theory, module theory, complexes in additive categories, etc.

Let us start with the nonadditive case. Given a small category C and a category K, we denote by $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$.

Recall that a category K is called *locally κ-presentable* [1, Definitions 1.9 and 1.17] if K is κ-accessible and all colimits exist in K. The following theorem is a generalization of [12, Theorem 1.2] from the case of locally κ-presentable categories to the case of κ-accessible categories with colimits of λ-indexed chains (for some fixed infinite cardinal $\lambda < \kappa$). It is also a correct version of [18, Lemma 5.1] (which was shown to be erroneous in full generality in [12, Theorem 1.3]).

A category C is said to be *κ-small* if the cardinality of the set of all objects and morphisms in C is smaller than κ.

Theorem 6.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let C be a κ-small category. Let K be a κ-accessible category in which all λ-indexed chains (of objects and morphisms) have colimits. Then the category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is κ-accessible. The full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{K}_{<\kappa})$ is precisely the full subcategory of all κ-presentable objects in $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$.*

Proof. Similarly to the proof Corollary 5.1, the point is that the diagram category can be constructed as a combination of products, inserters, and equifiers. Let $\mathsf{K}' = \prod_{c\in\mathsf{C}} \mathsf{K}$ be the Cartesian product of copies of the category K indexed by the objects of the category C, and let $\mathsf{L}' = \prod_{(c\to d)\in\mathsf{C}} \mathsf{K}$ be the similar product of copies of K indexed by the morphisms of the category C. Proposition 2.1 tells that the categories K' and L' are κ-accessible, and describes their full subcategories of κ-presentable objects.

Define a pair of parallel functors $F, G\colon \mathsf{K}' \longrightarrow \mathsf{L}'$ as follows. The functor F assigns to a collection of objects $(K_c \in \mathsf{K})_{c\in\mathsf{C}} \in \mathsf{K}'$ the collection of objects $(L_{c\to d})_{(c\to d)\in\mathsf{C}} \in \mathsf{L}'$ given by the rules $L_{c\to d} = K_c$ for any morphism $c \longrightarrow d$ in C. Similarly, the functor G assigns to a collection of objects $(K_c \in \mathsf{K})_{c\in\mathsf{C}} \in \mathsf{K}'$ the collection of objects $(L_{c\to d})_{(c\to d)\in\mathsf{C}} \in \mathsf{L}'$ given by the rules $L_{c\to d} = K_d$ for any morphism $c \longrightarrow d$ in C.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of all "nonmultiplicative functors" $\mathsf{C} \longrightarrow \mathsf{K}$. An object $E \in \mathsf{E}$ is a rule assigning to every object $c \in \mathsf{C}$ an object $E_c \in \mathsf{K}$ and to every morphism $c \longrightarrow d$ in C a morphism $E_c \longrightarrow E_d$ in K. The conditions of compatibility with the compositions of morphisms and with the identity morphisms are *not* imposed. Morphisms of "nonmultiplicative functors" (i. e., the morphisms in E) are similar to the usual morphisms of functors; so the desired functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is a full subcategory in E.

Theorem 4.1 tells that the category E is κ-accessible, and describes its full subcategory of κ-presentable objects. Now the desired full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{K}) \subset \mathsf{E}$ can be produced as a joint equifier category, as in Section 3 and Remark 3.3. There are two kinds of pairs of parallel natural transformations to be equified.

Firstly, for every composable pair of morphisms $b \longrightarrow c \longrightarrow d$ in C, we have a pair of parallel functors $F_{b\to c\to d}$, $G_{b\to c\to d}\colon \mathsf{E} \rightrightarrows \mathsf{K}$ and a pair of parallel natural transformations $\phi_{b\to c\to d}$, $\psi_{b\to c\to d}\colon F_{b\to c\to d} \rightrightarrows G_{b\to c\to d}$. The functor $F_{b\to c\to d}$ takes an object $E \in \mathsf{E}$ to the object $E_b \in \mathsf{K}$, and the functor $G_{b\to c\to d}$ takes an object $E \in \mathsf{E}$ to the object $E_d \in \mathsf{K}$. The natural transformation $\phi_{b\to c\to d}$ acts by the composition of the morphisms $E_b \longrightarrow E_c \longrightarrow E_d$ in K assigned to the morphisms $b \longrightarrow c$ and $c \longrightarrow d$ by the datum of the object E. The natural transformation $\psi_{b\to c\to d}$ acts by the morphism $E_b \longrightarrow E_d$ assigned to the composition of the morphisms $b \longrightarrow c \longrightarrow d$ in C by the datum of the object E.

Secondly, for every object $c \in \mathsf{C}$, we have a pair of parallel functors $F_c = G_c\colon \mathsf{E} \longrightarrow \mathsf{K}$ and a pair of parallel natural transformations $\phi_c, \psi_c\colon F_c \rightrightarrows G_c$. The functor $F_c = G_c$ takes an object $E \in \mathsf{E}$ to the object $E_c \in \mathsf{K}$. The natural transformation ϕ_c acts by the morphism $E_c \longrightarrow E_c$ in K assigned to the identity morphism id_c in C by the datum of the object E; while ψ_c is the identity natural transformation.

The resulting joint equifier is the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. $\square$

Now let k be a commutative ring. A *k-linear category* A is a category enriched in k-modules. This means that, for any two objects a and $b \in \mathsf{A}$, the set of morphisms $\mathrm{Hom}_{\mathsf{A}}(a, b)$ is a k-module, and the composition maps

$\mathrm{Hom}_{\mathsf{A}}(b,c) \times \mathrm{Hom}_{\mathsf{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(a,c)$ are k-bilinear.

Suppose given a set of objects a and, for every pair of objects a, b, a *generating set* of morphisms $\mathrm{Gen}(a,b)$. Then one can construct the k-linear category B on the given set of objects *freely generated* by the given generating sets of morphisms. For every pair of objects a, b, the free k-module $\mathrm{Hom}_{\mathsf{B}}(a,b)$ has a basis consisting of all the formal compositions $g_n \cdots g_1$, $n \geq 0$, where $g_i \in \mathrm{Gen}(c_i, c_{i+1}),\ c_1 = a,\ c_{n+1} = b$.

Furthermore, suppose given a *set of defining relations* $\mathrm{Rel}(a,b) \subset \mathrm{Hom}_{\mathsf{B}}(a,b)$ for every pair of objects a, b. Then one can construct the two-sided ideal of morphisms $\mathsf{J} \subset \mathsf{B}$ generated by all the relations, and pass to the k-linear quotient category $\mathsf{A} = \mathsf{B}/\mathsf{J}$ by the ideal J.

Abusing terminology, we will say that a k-linear category A is *κ-presented* if it has the form $\mathsf{A} = \mathsf{B}/\mathsf{J}$ as per the construction above, where the set of objects $\{a\}$, the set of all generators $\coprod_{a,b} \mathrm{Gen}(a,b)$, and the set of all relations $\coprod_{a,b} \mathrm{Rel}(a,b)$ all have the cardinalities smaller than κ. In another terminology, one could say that A is “the path category of a κ-small quiver with a κ-small set of relations”.

A k-linear category K is said to be *κ-accessible* if it is κ-accessible as an abstract category. Given a small k-linear category A and a k-linear category K, we denote by $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ the (k-linear) category of k-linear functors $\mathsf{A} \longrightarrow \mathsf{K}$. The following theorem is a k-linear version of Theorem 6.1.

Theorem 6.2. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let k be a commutative ring, let A be a κ-presented k-linear category, and let K be a κ-accessible k-linear category in which all λ-indexed chains have colimits. Then the category $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ is κ-accessible. The full subcategory $\mathsf{Fun}_k(\mathsf{A},\mathsf{K}_{<\kappa})$ is precisely the full subcategory of all κ-presentable objects in $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$.*

Proof. The argument is similar to the proof of Theorem 6.1, with the only difference that one works with the generating morphisms and defining relations in A instead of all morphisms and all compositions in C. Let $\mathsf{K}' = \prod_{a\in\mathsf{A}} \mathsf{K}$ be the Cartesian product of copies of the category K indexed by the objects of the category A, and let $\mathsf{L}' = \prod_{a,b\in\mathsf{A}} \prod_{(a\to b)\in\mathrm{Gen}(a,b)} \mathsf{K}$ be the similar product of copies of K indexed by the set of generating morphisms $\coprod_{a,b} \mathrm{Gen}(a,b)$. Proposition 2.1 tells that the categories K' and L' are κ-accessible, and describes their full subcategories of κ-presentable objects.

Define a pair of parallel functors $F, G\colon \mathsf{K}' \longrightarrow \mathsf{L}'$ as follows. The functor F assigns to a collection of objects $(K_a \in \mathsf{K})_{a\in\mathsf{A}} \in \mathsf{K}'$ the collection of objects $(L_{a\to b})_{(a\to b)\in\mathrm{Gen}(a,b),\,a,b\in\mathsf{A}} \in \mathsf{L}'$ given by the rules $L_{a\to b} = K_a$ for any generating morphism $(a \to b) \in \mathrm{Gen}(a,b)$. Similarly, the functor G assigns to a collection of objects $(K_a \in \mathsf{K})_{a\in\mathsf{A}} \in \mathsf{K}'$ the collection of objects $(L_{a\to b})_{(a\to b)\in\mathrm{Gen}(a,b),\,a,b\in\mathsf{A}} \in \mathsf{L}'$ given by the rules $L_{a\to b} = K_b$ for any generating morphism $(a \to b) \in \mathrm{Gen}(a,b)$.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is naturally equivalent to the category $\mathsf{Fun}_k(\mathsf{B},\mathsf{K})$, where B is the "path category of the quiver without relations" constructed in the discussion preceding the formulation of the theorem. Theorem 4.1 tells that the category E is κ-accessible, and defines its full subcategory of κ-presentable objects. The category $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ we are interested in is a full subcategory in $\mathsf{E} = \mathsf{Fun}_k(\mathsf{B},\mathsf{K})$ consisting of all the "quiver representations in K for which the relations are satisfied". The full subcategory $\mathsf{Fun}_k(\mathsf{A},\mathsf{K}) \subset \mathsf{Fun}_k(\mathsf{B},\mathsf{K})$ can be produced as a joint equifier category, as in Section 3 and Remark 3.3.

The pairs of parallel natural transformations to be equified are indexed by elements of the set of defining relations $\coprod_{a,b}\mathrm{Rel}(a,b)$. Given a defining relation $r \in \mathrm{Rel}(a,b)$, we have a pair of parallel functors $F_r, G_r\colon \mathsf{E} \rightrightarrows \mathsf{K}$ and a pair of natural transformations $\phi_r, \psi_r\colon F_r \rightrightarrows G_r$. The functor $F_r\colon \mathsf{Fun}_k(\mathsf{B},\mathsf{K}) \longrightarrow \mathsf{K}$ takes a functor $E\colon \mathsf{B} \longrightarrow \mathsf{K}$ to the object $E(a) \in \mathsf{K}$, and the functor G_r takes the functor E to the object $E(b) \in \mathsf{K}$. The natural transformation ϕ_r acts by the morphism $E(r)\colon E(a) \longrightarrow E(b)$. The natural transformation ψ_r acts by the zero morphism $0\colon E(a) \longrightarrow E(b)$ in the k-linear category K.

The resulting joint equifier is the category of k-linear functors $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. $\square$

7. Brief preliminaries on 2-categories

The aim of this section is to provide a very brief and mostly terminological preliminary discussion for the purposes of the next two Sections 8–9. The reader can find the details by following the references.

Throughout the three sections, for the most part we adopt the policy of benign neglect with respect to set-theoretical issues of size (i. e., the distinction between sets and classes). When specific restrictions on the size matter, we mention them.

In the terminology of higher category theory, the prefix “2-” means strict concepts, while the prefix “bi-” refers to relaxed ones. So 2-categories are strict, while bicategories are relaxed [3].

A 2-*category* is a category enriched in the category of categories Cat (with the monoidal structure on Cat given by the Cartesian product) [14]. In particular, there is the important 2-*category of categories* $\mathbf{Cat}$: categories are the objects, functors are the 1-cells, natural transformations are the 2-cells.

In the terminology of the bicategory theory, one speaks of *morphisms of bicategories* (which are multiplicative and unital on 1-cells up to coherent families of 2-cells) or *homomorphisms of bicategories* (which are multiplicative and unital on 1-cells up to coherent families of *invertible* 2-cells) [3, Section 4]. Even when one is only interested in 2-categories, the notion of a 2-functor may be too strict, and one may want to relax it by considering morphisms of 2-categories (known as *lax functors*), or homomorphisms of 2-categories (known as *pseudofunctors*).

Let Γ and Δ be two 2-categories. Then 2-functors $\Gamma \longrightarrow \Delta$ form a 2-category $[\Gamma, \Delta]$. The objects of $[\Gamma, \Delta]$ are the 2-functors $\Gamma \longrightarrow \Delta$, the 1-cells of $[\Gamma, \Delta]$ are the 2-natural transformations, and the 2-cells of $[\Gamma, \Delta]$ are called *modifications* [14, Section 1.4]. A 2-functor $\Gamma \longrightarrow \Delta$ is a rule assigning to every object of Γ an object of Δ, to every 1-cell of Γ a 1-cell of Δ, and to every 2-cell of Γ a 2-cell of Δ. A 2-natural transformation is a rule assiging to every object of Γ a 1-cell in Δ. A modification is a rule assigning to every object of Γ a 2-cell in Δ. 2-categories and 2-functors form the 3-*category of* 2-*categories*: 2-categories are the objects, 2-functors are the 1-cells, 2-natural transformations are the 2-cells, and modifications are the 3-cells.

Even when one is only interested in 2-functors rather than the more relaxed concepts of lax functors or pseudofunctors, the notion of a 2-natural transformation may be too strict, and one may want to relax it. Then one can consider *lax natural transformations* (compatible with the action of the 2-functors on 1-cells in Γ up to a coherent family of 2-cells in Δ) or *pseudonatural transformations* (compatible with the action of the 2-functors on the 1-cells in Γ up to a coherent family of *invertible* 2-cells in Δ). In the terminology of [19, §4.1], lax natural transformations are called "transformations", pseudonatural transformations are called "strong transformations", and 2-natural transformations are called "strict transformations". The 2-category of 2-functors $\Gamma \longrightarrow \Delta$, pseudonatural transformations, and modifications is denoted by $\mathsf{Psd}[\Gamma, \Delta]$ in [4], [13, Section 5], [5, Section 2].

In connection with the "lax" notions, the choice of the direction of the (possibly noninvertible) 2-cells providing the relaxed compatibility becomes important. When the direction is reversed, the correspoding notions are called "oplax". For "pseudo" notions, the compatibility 2-cells are assumed to be invertible, and so the choice of the direction in which they act no longer matters.

8. Conical pseudolimits, lax limits, and oplax limits

We denote by $\mathbf{Cat}$ the 2-category of small categories and by $\mathbf{CAT}$ the 2-category of locally small categories (i. e., large categories in which morphisms between any fixed pair of objects form a set). So the categories of morphisms in $\mathbf{CAT}$ need not be even locally small; this will present no problem for our constructions.

Let Γ be a small 2-category and $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be a 2-functor. The (*conical*) *lax limit* of H is a category L whose objects are the following sets of data:

i. for every object $\gamma \in \Gamma$, an object $L_\gamma \in H(\gamma)$ of the category $H(\gamma)$ is given;

ii. for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, a morphism $l_a\colon H(a)(L_\gamma) \longrightarrow L_\delta$ in the category $H(\delta)$ is given.

Here $H(a)\colon H(\gamma) \longrightarrow H(\delta)$ is the functor assigned to the 1-cell $a\colon \gamma \longrightarrow \delta$ by the 2-functor H.

The set of data (i–ii) must satisfy the following conditions:

iii. for every identity 1-cell $a = \mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ, one has $l_{\mathrm{id}_\gamma} = \mathrm{id}_{L_\gamma}\colon L_\gamma \longrightarrow L_\gamma$;

iv. for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, one has $l_{ba} = l_b \circ H(b)(l_a)$ in the category $H(\epsilon)$;

v. for every 2-cell $t\colon a \longrightarrow b$, where $a, b\colon \gamma \rightrightarrows \delta$ is a pair of parallel 1-cells in Γ, the triangular diagram

$$\begin{tikzcd} H(a)(L_\gamma) \arrow[dd, "H(t)_{L_\gamma}"'] \arrow[dr, "l_a"] & \\ & L_\delta \\ H(b)(L_\gamma) \arrow[ur, "l_b"'] & \end{tikzcd}$$

is commutative in the category $H(\delta)$.

Here $H(t)\colon H(a) \longrightarrow H(b)$ is the morphism of functors from the category $H(\gamma)$ to the category $H(\delta)$ assigned to the 2-cell $t\colon a \longrightarrow b$ by the 2-functor H.

A morphism $L \longrightarrow M$ in the category L is the datum of a morphism $L_\gamma \longrightarrow M_\gamma$ in the category $H(\gamma)$ for every object $\gamma \in \Gamma$, satisfying the obvious compatibility condition with the data (ii) for the objects L and M.

The (*conical*) *oplax limit* of the 2-functor H is the category M whose objects are the following sets of data:

i*. for every object $\gamma \in \Gamma$, an object $M_\gamma \in H(\gamma)$ of the category $H(\gamma)$ is given;

ii*. for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, a morphism $m_a\colon M_\delta \longrightarrow H(a)(M_\gamma)$ in the category $H(\delta)$ is given.

The set of data (i*–ii*) must satisfy the following conditions:

iii*. for every identity 1-cell $a = \mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ, one has $m_{\mathrm{id}_\gamma} = \mathrm{id}_{M_\gamma}\colon M_\gamma \longrightarrow M_\gamma$;

iv*. for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, one has $m_{ba} = H(b)(m_a) \circ m_b$ in the category $H(\epsilon)$;

v*. for every 2-cell $t\colon a \longrightarrow b$, where $a, b\colon \gamma \rightrightarrows \delta$ is a pair of parallel

1-cells in Γ, the triangular diagram

$$\begin{tikzcd} & H(a)(M_\gamma) \arrow[dd, "H(t)_{M_\gamma}"] \\ M_\delta \arrow[ur, "m_a"] \arrow[dr, "m_b"'] & \\ & H(b)(M_\gamma) \end{tikzcd}$$

is commutative in the category $H(\delta)$.

A morphism $L \longrightarrow M$ in the category M is the datum of a morphism $L_\gamma \longrightarrow M_\gamma$ in the category $H(\gamma)$ for every object $\gamma \in \Gamma$, satisfying the obvious compatibility condition with the data (ii*) for the objects L and M.

The *pseudolimit* of the 2-functor H is the full subcategory $\mathsf{E} \subset \mathsf{L}$ consisting of all the objects $E \in \mathsf{L}$ such that the morphism $e_a\colon H(a)(E_\gamma) \longrightarrow E_\delta$ in (ii) is an isomorphism in $H(\delta)$ for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ. Equivalently, the pseudolimit E can be defined as the full subcategory $\mathsf{E} \subset \mathsf{M}$ consisting of all the objects $E \in \mathsf{M}$ such that the morphism $e_a\colon E_\delta \longrightarrow H(a)(E_\gamma)$ in (ii*) is an isomorphism in $H(\delta)$ for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ.

Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Denote by $\mathbf{ACC}_{\lambda,\kappa} \subset \mathbf{CAT}$ the following 2-subcategory in $\mathbf{CAT}$. The objects of $\mathbf{ACC}_{\lambda,\kappa}$ are all the κ-accessible categories with colimits of λ-indexed chains. The 1-cells of $\mathbf{ACC}_{\lambda,\kappa}$ are the functors preserving κ-directed colimits and colimits of λ-indexed chains. The 2-cells of $\mathbf{ACC}_{\lambda,\kappa}$ are the (arbitrary) natural transformations.

As usual, we will say that a 2-category is *κ-small* if it has less than κ objects, less than κ 1-cells, and less than κ 2-cells.

Theorem 8.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}_{\lambda,\kappa}$ be a 2-functor. Then the oplax limit M of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{M} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}_{\lambda,\kappa}$. An object $S \in \mathsf{M}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similarly to the proofs of Corollary 5.1 and Theorems 6.1–6.2, one constructs the oplax limit M as a combination of products, inserters, and equifiers.

Let $\mathsf{K} = \prod_{\gamma\in\Gamma} H(\gamma)$ be the Cartesian product of the categories $H(\gamma)$ taken over all objects $\gamma \in \Gamma$, and let $\mathsf{L} = \prod_{(a:\gamma\to\delta)\in\Gamma} H(\delta)$ be the Cartesian product of the categories $H(\delta)$ taken over all the 1-cells $a\colon \gamma \longrightarrow \delta$ in Γ. Consider the following pair of parallel functors F, $G\colon \mathsf{K} \longrightarrow \mathsf{L}$. The functor F takes a collection of objects $(M_\gamma \in H(\gamma))_{\gamma\in\Gamma} \in \mathsf{K}$ to the collection of objects $(M_\delta \in H(\delta))_{(a:\gamma\to\delta)} \in \mathsf{L}$. The functor G takes the same collection of objects $(M_\gamma \in H(\gamma))_{\gamma\in\Gamma} \in \mathsf{K}$ to the collection of objects $(F(a)(M_\gamma) \in H(\delta))_{(a:\gamma\to\delta)} \in \mathsf{L}$.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of all sets of data (i*–ii*) from the definition of the oplax limit above. The conditions (iii*–v*) have not been imposed yet.

Theorem 4.1 tells that E is a κ-accessible category and describes its full subcategory of κ-presentable objects. The desired oplax limit M is a full subcategory $\mathsf{M} \subset \mathsf{E}$ which can be produced as a joint equifier category, as in Section 3 and Remark 3.3. There are three kinds of pairs of parallel natural transformations to be equified, corresponding to the three conditions (iii*–v*).

Firstly, for every object $\gamma \in \Gamma$, we have a pair of parallel functors $F_\gamma = G_\gamma\colon \mathsf{E} \longrightarrow H(\gamma)$ and a pair of parallel natural transformations ϕ_γ, $\psi_\gamma\colon F_\gamma \longrightarrow G_\gamma$. The functor $F_\gamma = G_\gamma$ takes an object $E \in \mathsf{E}$ to the object $E_\gamma \in H(\gamma)$. The natural transformation ϕ_γ acts by the morphism $e_{\mathrm{id}_\gamma}\colon E_\gamma \longrightarrow E_\gamma$ assigned to the identity 1-cell $\mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ by the datum (ii*) for the object $E \in \mathsf{E}$; while ψ_γ is the identity natural transformation.

Secondly, for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, we have a pair of parallel functors $F_{a,b}$, $G_{a,b}\colon \mathsf{E} \rightrightarrows H(\epsilon)$ and a pair of parallel natural transformations $\phi_{a,b}$, $\psi_{a,b}\colon F_{a,b} \rightrightarrows G_{a,b}$. The functor $F_{a,b}$ takes an object $E \in \mathsf{E}$ to the object $E_\epsilon \in H(\epsilon)$. The functor $G_{a,b}$ takes an object $E \in \mathsf{E}$ to the object $H(ba)(E_\gamma) \in H(\epsilon)$. The natural transformation $\phi_{a,b}$ acts by the morphism $e_{ba}\colon E_\epsilon \longrightarrow H(ba)(E_\gamma)$. The natural transformation $\psi_{a,b}$ acts by the composition of morphisms $H(b)(e_a) \circ e_b\colon E_\epsilon \longrightarrow H(b)(E_\delta) \longrightarrow H(ba)(E_\gamma)$.

Thirdly, for every 2-cell $t\colon a \longrightarrow b$, where a, $b\colon \gamma \rightrightarrows \delta$ is a pair of parallel 1-cells in Γ, we have a pair of parallel functors F_t, $G_t\colon \mathsf{E} \rightrightarrows H(\delta)$ and a pair of parallel natural transformations ϕ_t, $\psi_t\colon F_t \rightrightarrows G_t$. The functor F_t takes an object $E \in \mathsf{E}$ to the object $E_\delta \in H(\delta)$. The functor G_t takes an object $E \in \mathsf{E}$ to the object $H(b)(E_\gamma) \in H(\delta)$. The natural transformation ϕ_t acts by the composition of morphisms $H(t)_{E_\gamma} \circ e_a\colon E_\delta \longrightarrow H(a)(E_\gamma) \longrightarrow H(b)(E_\gamma)$. The natural transformation ψ_t acts by the morphism $e_b\colon E_\delta \longrightarrow H(b)(E_\gamma)$.

The resulting joint equifier is the oplax limit M. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. This proves the first and the third assertions of the theorem, while the second assertion is easy. $\square$

Denote by $\mathbf{ACC}^\kappa_{\lambda,\kappa} \subset \mathbf{ACC}_{\lambda,\kappa}$ the following 2-subcategory in $\mathbf{CAT}$. The objects of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the same as the objects of $\mathbf{ACC}_{\lambda,\kappa}$, i. e., all the κ-accessible categories with colimits of λ-indexed chains. The 1-cells of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the functors preserving κ-directed colimits and colimits of λ-indexed chains, and taking κ-presentable objects to κ-presentable objects. The 2-cells of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the (arbitrary) natural transformations.

Theorem 8.2. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}^\kappa_{\lambda,\kappa}$ be a 2-functor. Then the lax limit L of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{L} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. An object $S \in \mathsf{L}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similar to the proof of Theorem 8.1, with the directions of some arrows suitably reversed as needed. $\square$

Theorem 8.3. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}^\kappa_{\lambda,\kappa}$ be a 2-functor. Then the pseudolimit E of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{E} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. An object $S \in \mathsf{E}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similar to the proofs of Theorems 8.1 and 8.2, with the only difference that it is convenient to use the isomorpher construction of Remark 5.2 instead of the inserter construction of Theorem 4.1. The equifier construction of Theorem 3.1 still needs to be used. (Cf. [13, Propositions 4.4 and 5.1] and [5, Proposition 2.1].) □

Remark 8.4. The notions of (op)lax limit and pseudolimit are somewhat relaxed. The related strict notion is the 2-*limit* of categories. 2-limits of categories are *not* well-behaved in connection with accessible categories, generally speaking [19, paragraph after Proposition 5.1.1], [1, Example 2.68]. The well-behaved ones among the (weighted) 2-limits are called *flexible limits* in [5]. Still, the (op)lax limits and pseudolimits are strict enough to be defined *up to isomorphism of categories* (as per the constructions above) rather than just up to category equivalence.

The case of the pseudopullback is instructive. Let Γ be the following small 2-category. The 2-category Γ has three objects A, B, and C, and two nonidentity 1-cells $a\colon A \longrightarrow C$ and $b\colon B \longrightarrow C$. There are no nonidentity 2-cells in Γ. Hence a 2-functor $H\colon \Gamma \longrightarrow \mathbf{CAT}$ is the same thing as a triple of categories A, B, and C together with a pair of functors $\Theta_\mathsf{A}\colon \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_\mathsf{B}\colon \mathsf{B} \longrightarrow \mathsf{C}$, as in Section 5. Then [19, paragraph after Proposition 5.1.1] explains that the 2-pullbacks, i. e., the 2-limits of 2-functors $H\colon \Gamma \longrightarrow \mathbf{CAT}$, do *not* preserve accessibility of categories.

The (op)lax limits and pseudolimits are better behaved and preserve accessibility, as per the theorems above in this section; but one has to be careful. Looking into these constructions, one can observe that the definition of the pseudopullback in Section 5 was, strictly speaking, an abuse of terminology. The pseudolimit E of a 2-functor $H\colon \Gamma \longrightarrow \mathbf{CAT}$ is the category of all quintuples $(A, B, C, \theta_a, \theta_b)$, where $A \in \mathsf{A}$, $B \in \mathsf{B}$, and $C \in \mathsf{C}$ are three objects and $\theta_a\colon \Theta_\mathsf{A}(A) \simeq C$, $\theta_b\colon \Theta_\mathsf{B}(B) \simeq C$ are two isomorphisms (cf. [6, Proposition 3.1], [28, Pseudopullback Theorem 2.2]). The pseudopullback D as defined in Section 5 is *naturally equivalent* to the pseudolimit E of the 2-functor H, but *not* isomorphic to it.

The even more relaxed notion of a limit of categories defined up to a category equivalence is called the *bilimit* [19, Section 5.1.1], [13, Section 6]. In the terminology of [19, Section 5.1.1], the pseudolimits are called *strong bilimits*.

9. Weighted pseudolimits

Let Γ be a small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor (so the category $W(\gamma)$ is small for every $\gamma \in \Gamma$). The 2-functor W is called a *weight*.

Let $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be another 2-functor. The *weighted pseudolimit* $\{W, H\}_p$ [5, Sections 1–2] (called "indexed pseudolimit" in the terminology of [13, Sections 2 and 5] or "strong weighted bilimit" in the terminology of [19, Section 5.1.1]) can be simply constructed as the category of 1-cells $W \longrightarrow H$ in the 2-category of pseudonatural transformations $\mathsf{Psd}[\Gamma, \mathbf{CAT}]$ (mentioned in Section 7). So $\{W, H\}_p = \mathsf{Psd}[\Gamma, \mathbf{CAT}](W, H)$ [13, formula (5.5)].

The strict version of the same construction is the *weighted* 2*-limit* $\{W, H\}$, which can be defined as the category of 1-cells $W \longrightarrow H$ in the 2-category of 2-natural transformations $[\Gamma, \mathbf{CAT}]$; so $\{W, H\} = [\Gamma, \mathbf{CAT}](W, H)$ [13, formula (2.5)]. It is explained in [13, Section 4] or [5, Section 1] how to obtain the inserters, equifiers, and isomorphers (iso-inserters) as particular cases of weighted 2-limits. Up to category equivalence, they are also particular cases of weighted pseudolimits.

Taking Γ to be the 2-category with a single object, a single 1-cell, and a single 2-cell, one obtains the construction of the diagram category (as in Theorem 6.1), called the "cotensor product" in [13, Section 3], [5, Section 1], as the particular case of the weighted 2-limit or weighted pseudolimit.

Taking W to be the 2-functor assigning to every object $\gamma \in \Gamma$ the category with a single object and a single morphism, one obtains the construction of the pseudolimit from Section 8 as a particular case of weighted pseudolimit. To distinguish them from the more general weighted pseudolimits, the pseudolimits from Section 8 are called *conical pseudolimits* [13, Sections 3 and 5], [5, Sections 1–2].

The notation $\mathbf{ACC}^{\kappa}_{\lambda,\kappa} \subset \mathbf{ACC}_{\lambda,\kappa} \subset \mathbf{CAT}$ was introduced in Section 8.

Theorem 9.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor such that the category $W(\gamma)$ is κ-small for every object $\gamma \in \Gamma$. Let $H\colon \Gamma \longrightarrow \mathbf{ACC}^{\kappa}_{\lambda,\kappa}$ be a 2-functor. Then the weighted pseudolimit $\{W, H\}_p$ (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^{\kappa}_{\lambda,\kappa}$.*

Proof. The point is that all weighted pseudolimits can be constructed in terms of products, inserters, and equifiers [13, Proposition 5.2], [5, Proposition 2.1]; so the assertion follows from Proposition 2.1, Theorem 3.1, and Theorem 4.1. The same argument applies also to all weighted bilimits [13, Section 6] and all flexible weighted 2-limits [5, Theorem 4.9 and Remark 7.6]. $\square$

Corollary 9.2. *Let λ and κ be infinite regular cardinals such that $\lambda \triangleleft \kappa$ in the sense of [19, §2.3] or [1, Definition 2.12]. Let Γ be a κ-small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor such that the category $W(\gamma)$ is κ-small for every object $\gamma \in \Gamma$. Let $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be a 2-functor such that, for every object $\gamma \in \Gamma$, the category $H(\gamma)$ is λ-accessible, and for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, the functor $H(a)\colon H(\gamma) \longrightarrow H(\delta)$ preserves λ-directed colimits and takes κ-presentable objects to κ-presentable objects. Then the weighted pseudolimit $\{W, H\}_p$ (computed in $\mathbf{CAT}$, as per the construction above) is a κ-accessible category.*

Proof. Follows immediately from Theorem 9.1. $\square$

Remark 9.3. The assertion of Theorem 9.1 captures many, but not all the aspects of the preceding results in this paper. In particular, Theorems 3.1 and 4.1 are *not* particular cases of Theorem 9.1, if only because the assumptions of Theorems 3.1–4.1 are more general. Indeed, in the assumptions of Theorems 3.1–4.1 the functor F is required to belong to $\mathbf{ACC}^{\kappa}_{\lambda,\kappa}$, while the functor G may belong to the wider 2-category $\mathbf{ACC}_{\lambda,\kappa}$. In other words, the functor G *need not* take κ-presentable objects to κ-presentable objects. This subtlety, which was emphasized already in [29, Section 3], manifests itself in the related difference between the formulations of Theorem 8.1, on the one hand, and Theorems 8.2–8.3, on the other hand. It plays an important role in the application to comodules over corings worked out in [23, Theorem 3.1 and Remark 3.2] and in the application to corings in [26, Theorem 4.2].

10. Toy examples

The examples in this section aim to illustrate the main results of this paper in the context of additive categories, modules categories, and flat modules, which served as the main motivation for the present research. We refer to the papers [27, 23, 25, 26] for more substantial applications to flat quasi-coherent sheaves, flat comodules and contramodules, arbitrary and flat coalgebras and corings, and flat/injective (co)resolutions. This section also serves as a reference source for [27, 23, 25, 26], as it contains some results that are useful as building blocks for the more complicated constructions.

10.1 Modules and flat modules

Let R be an associative ring. We denote by $R\text{–}\mathsf{Mod}$ the abelian category of left R-modules and by $R\text{–}\mathsf{Mod}_{\mathsf{fl}} \subset R\text{–}\mathsf{Mod}$ the full subcategory of flat left R-modules.

The following two propositions are fairly standard.

Proposition 10.1. *For any ring R and any regular cardinal κ, the category of R-modules $R\text{–}\mathsf{Mod}$ is locally κ-presentable. The κ-presentable objects of $R\text{–}\mathsf{Mod}$ are precisely all the left R-modules that can be constructed as the cokernel of a morphism of free left R-modules with less than κ generators.* □

Proposition 10.2. *For any ring R and any regular cardinal κ, the category of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is κ-accessible. All directed colimits exist in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ and agree with the ones in $R\text{–}\mathsf{Mod}$. The κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ are precisely all those flat left R-modules that are κ-presentable as objects of $R\text{–}\mathsf{Mod}$.*

Proof. The connection between the present proposition and the previous one fits into the setting described in Proposition 1.2. The assertions for $\kappa = \aleph_0$ are corollaries of the classical Govorov–Lazard theorem [11, 16] characterizing the flat R-modules as the directed colimits of finitely generated projective (or free) R-modules. The general case of an arbitrary regular cardinal κ can be deduced by applying [1, Theorem 2.11 and Example 2.13(1)]. □

For a version of Proposition 10.2 for modules of bounded flat dimension, see [25, Corollary 5.2].

10.2 Diagrams of flat modules

The following two corollaries are our “toy applications” of Theorem 6.2.

Corollary 10.3. *Let k be a commutative ring and R be an associative, unital k-algebra. Let κ be an uncountable regular cardinal and A be a κ-presented k-linear category (in the sense of Section 6). Then any k-linear functor $\mathsf{A} \longrightarrow R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is a κ-directed colimit of k-linear functors $\mathsf{A} \longrightarrow R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}$ into the category of κ-presentable flat left R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}$.*

Proof. By Proposition 10.2 and Theorem 6.2 (with $\lambda = \aleph_0$), the k-linear functor/diagram category $\mathsf{Fun}_k(\mathsf{A}, R\text{–}\mathsf{Mod}_{\mathsf{fl}})$ is κ-accessible, and $\mathsf{Fun}_k(\mathsf{A}, R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})$ is its full subcategory of κ-presentable objects. □

Corollary 10.4. *Let R be an associative ring and κ be an uncountable regular cardinal. Then any cochain complex of flat R-modules is a κ-directed colimit of complexes of κ-presentable flat R-modules.*

Proof. This is the particular case of Corollary 10.3 for the ring $k = \mathbb{Z}$ and the suitable choice of additive category A describing cochain complexes. The objects of A are the integers $n \in \mathbb{Z}$, the set of generating morphisms is the singleton $\mathrm{Gen}(n, m) = \{d_n\}$ for $m = n+1$ and the empty set otherwise, and the set of defining relations is the singleton $\mathrm{Rel}(n, m) = \{d_{n+1}d_n\}$ for $m = n + 2$ and the empty set otherwise. □

For a quasi-coherent sheaf, a comodule, and a contramodule version of Corollary 10.4, see [27, Theorem 4.1] and [23, Propositions 3.3 and 10.2].

Remark 10.5. For an uncountable regular cardinal κ, the complexes of κ-presentable R-modules are precisely all the κ-presentable objects of the locally finitely presentable (hence locally κ-presentable) abelian category of complexes of R-modules. For $\kappa = \aleph_0$, the finitely presentable objects of the category of complexes of R-modules are the *bounded* complexes of finitely presentable R-modules.

Notice that *not* every complex of flat R-modules is a directed colimit of bounded complexes of finitely presentable flat (i. e., finitely generated projective) R-modules. In fact, the directed colimits of bounded complexes of finitely generated projective R-modules are the *homotopy flat* complexes of flat R-modules [7, Theorem 1.1].

Using the argument from [1, proof of Theorem 2.11 (iv) $\Rightarrow$ (i)] (for $\lambda = \aleph_0$ and $\mu = \kappa$), one can deduce the assertion that any homotopy flat complex of flat R-modules is a κ-directed colimit of homotopy flat complexes of κ-presentable flat R-modules, for any uncountable regular cardinal κ. A quasi-coherent sheaf version of this observation can be found in [27, Theorem 4.5].

10.3 Categories of epimorphisms

For any category K, we denote by $\mathsf{K}^{\to}$ the category of morphisms in K (with commutative squares in K as morphisms in $\mathsf{K}^{\to}$). The following lemma is not difficult.

Lemma 10.6. *For any regular cardinal κ and κ-accessible category K, the category of morphisms $\mathsf{K}^{\to}$ is κ-accessible. The full subcategory of κ-presentable objects in $\mathsf{K}^{\to}$ is the category $(\mathsf{K}_{<\kappa})^{\to}$ of morphisms of κ-presentable objects in K.*

Proof. One has $\mathsf{K}^{\to} = \mathsf{Fun}(\mathsf{C}, \mathsf{K})$ for the obvious finite category C with no nonidentity endomorphisms; so the result of [2, Exposé I, Proposition 8.8.5], [20, page 55], or [12, Theorem 1.3] is applicable. ☐

For any category K, let us denote by $\mathsf{K}^{\mathsf{epi}} \subset \mathsf{K}^{\to}$ the full subcategory whose objects are all the epimorphisms in K.

Lemma 10.7. *For any regular cardinal κ and any locally κ-presentable abelian category K, the category of epimorphisms $\mathsf{K}^{\mathsf{epi}}$ is locally κ-presentable. The full subcategory of κ-presentable objects in $\mathsf{K}^{\mathsf{epi}}$ is the category $(\mathsf{K}_{<\kappa})^{\mathsf{epi}}$ of epimorphisms between κ-presentable objects in K.*

Proof. Notice first of all that a morphism in $\mathsf{K}_{<\kappa}$ is an epimorphism in $\mathsf{K}_{<\kappa}$ if and only if it is an epimorphism in K (because the full subcategory $\mathsf{K}_{<\kappa}$ is closed under cokernels in K [1, Proposition 1.16]). Furthermore, the full subcategory $\mathsf{K}^{\mathsf{epi}}$ is closed under colimits in the locally presentable abelian category $\mathsf{K}^{\to}$; so all colimits exist in $\mathsf{K}^{\mathsf{epi}}$. In view of Lemma 10.6 and according to Proposition 1.2, in order to prove the lemma it suffices to check that any morphism from an object of $(\mathsf{K}_{<\kappa})^{\to}$ to an object of $\mathsf{K}^{\mathsf{epi}}$ factorizes through an object of $(\mathsf{K}_{<\kappa})^{\mathsf{epi}}$ in $\mathsf{K}^{\to}$.

Indeed, consider a commutative square diagram in K

$$\begin{tikzcd} S \arrow[r] \arrow[d] & K \arrow[d, two heads] \\ T \arrow[r] & L \end{tikzcd}$$

with an epimorphism $K \twoheadrightarrow L$ and objects S, $T \in \mathsf{K}_{<\kappa}$. Let M be the pullback of the pair of morphisms $K \longrightarrow L$ and $T \longrightarrow L$ in K; then $M \longrightarrow T$ is also an epimorphism (since the category K is assumed to be abelian).

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, two heads] & K \arrow[d, two heads] \\ T \arrow[r, equal] & T \arrow[r] & L \end{tikzcd}$$

Let $M = \varinjlim_{\xi\in\Xi} U_\xi$ be a representation of M as a κ-filtered colimit of κ-presentable objects U_ξ in K, and let V_ξ denote the images of the compositions $U_\xi \longrightarrow M \longrightarrow T$. The κ-filtered colimits are exact functors in K [1, Proposition 1.59]; hence we have $T = \varinjlim_{\xi\in\Xi} V_\xi$. Since $T \in \mathsf{K}_{<\kappa}$, it follows that there exists $\xi_0 \in \Xi$ such that the morphism $V_{\xi_0} \longrightarrow T$ is a retraction (as $V_{\xi_0} \longrightarrow T$ is a monomorphism by construction, this means that $V_{\xi_0} \longrightarrow T$ is actually an isomorphism). Hence the composition $U_{\xi_0} \longrightarrow M \longrightarrow T$ is an epimorphism. Since $S \in \mathsf{K}_{<\kappa}$, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the morphism $S \longrightarrow M$ factorizes through the morphism $U_{\xi_1} \longrightarrow M$. Hence we arrive to the desired factorization

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U_{\xi_1} \arrow[r] \arrow[d, two heads] & K \arrow[d, two heads] \\ T \arrow[r, equal] & T \arrow[r] & L \end{tikzcd}$$

through an object $(U_{\xi_1} \to T) \in (\mathsf{K}_{<\kappa})^{\mathsf{epi}}$. $\square$

Remark 10.8. It follows immediately from the first assertion of Lemma 10.7 that the category $\mathsf{K}^{\mathsf{mono}}$ of monomorphisms in K is also locally κ-presentable. In fact, the categories $\mathsf{K}^{\mathsf{epi}}$ and $\mathsf{K}^{\mathsf{mono}}$ are naturally equivalent; the functors of the kernel of an epimorphism and the cokernel of a monomorphism provide the equivalence. However, the direct analogue of the second assertion of Lemma 10.7 *fails* for monomorphisms (even though the full subcategory $\mathsf{K}^{\mathsf{mono}} \subset \mathsf{K}^{\to}$ is closed under κ-directed colimits by [1, Proposition 1.59]). In fact, a monomorphism i in K is a κ-directed colimit of monomorphisms between κ-presentable objects if and only if i is an admissible monomorphism in the *maximal locally κ-coherent exact structure on* K [24, Corollary 3.3]. In particular, if R is an associative ring that is not left coherent, then any monomorphism $i\colon N \longrightarrow M$ from a finitely generated but not finitely presentable left R-module N to a finitely presentable left R-module M is *not* a directed colimit of monomorphisms of finitely presentable modules in $R\text{–}\mathsf{Mod}$.

Given a ring R and a full subcategory $\mathsf{L} \subset R\text{–}\mathsf{Mod}$, we denote by $\mathsf{L}^{\mathsf{surj}} \subset \mathsf{L}^{\to}$ the full subcategory whose objects are all the surjective morphisms between objects of L.

Lemma 10.9. *For any associative ring R and any regular cardinal κ, the category of surjective morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ is κ-accessible. The κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are the surjective morphisms of κ-presentable flat R-modules.*

Proof. The argument is similar to the proof of Lemma 10.7. In view of Proposition 10.2, Lemma 10.6 is applicable to $\mathsf{K} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}$; so the category of morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$ is κ-accessible and the category of morphisms of κ-presentable flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}^{\to}$ is the full subcategory of κ-presentable objects in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$. According to Proposition 1.2, in order to prove the lemma it suffices to check that any morphism from an object of $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}^{\to}$ to an object of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ factorizes through an object of $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{surj}}$.

Following the proof of Lemma 10.7, one needs to observe that if $K \twoheadrightarrow L$ is a surjective morphism of flat R-modules and $T \longrightarrow L$ is a morphism of flat R-modules, then the pullback M (computed in the category $R\text{–}\mathsf{Mod}$) is a flat R-module. Indeed, the kernel F of the morphism $K \longrightarrow L$ is a flat R-module, so the short exact sequence $0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0$ shows that M is a flat R-module, too. The images V_ξ of the morphisms $U_\xi \longrightarrow T$ can be taken in the ambient abelian category $R\text{–}\mathsf{Mod}$. Otherwise, the argument is the same, except that one considers surjective morphisms in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ rather than epimorphisms in K. $\square$

Remark 10.10. Alternatively, one can drop the assumption that the category K is abelian in Lemma 10.7, requiring it only to be additive; but assume the cardinal κ to be uncountable instead. Then the resulting assertion can be obtained as a particular case of Corollary 5.1. Consider the category of morphisms $\mathsf{A} = \mathsf{K}^{\to}$, the zero category $\mathsf{B} = \{0\}$, and the category $\mathsf{C} = \mathsf{K}$. Let $\Theta_{\mathsf{A}}\colon \mathsf{A} \longrightarrow \mathsf{C}$ be the cokernel functor $f \longmapsto \operatorname{coker}(f)$ and $\Theta_{\mathsf{B}}\colon \mathsf{B} \longrightarrow \mathsf{C}$

be the zero functor. Then the pseudopullback D is the category of epimorphisms $\mathsf{D} = \mathsf{K}^{\mathsf{epi}}$. All the assumptions of Corollary 5.1 (with $\lambda = \aleph_0$) are satisfied; so the corollary tells that $\mathsf{K}^{\mathsf{epi}}$ is κ-accessible and provides the desired description of κ-presentable objects.

Similarly, assuming κ to be uncountable, one can deduce Lemma 10.9 from Lemmas 10.6 and 10.7 using Corollary 5.1. Consider the category of R-module epimorphisms $\mathsf{A} = R\text{–}\mathsf{Mod}^{\mathsf{epi}}$, the category of morphisms of flat R-modules $\mathsf{B} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$, and the category of R-module morphisms $\mathsf{C} = R\text{–}\mathsf{Mod}^{\to}$. Let $\Theta_{\mathsf{A}}\colon \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_{\mathsf{B}}\colon \mathsf{B} \longrightarrow \mathsf{C}$ be the natural inclusions. Then the pseudopullback D is the category of surjective morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$, and Corollary 5.1 is applicable.

10.4 Short exact sequences of flat modules

Now we can deduce the following three corollaries of Lemma 10.9.

Corollary 10.11. *Let R be an associative ring and κ be a regular cardinal. Then any surjective morphism of κ-presentable flat R-modules is a direct summand of a κ-small directed colimit of surjective morphisms of finitely generated projective R-modules (in the category $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$).*

Proof. This follows from Lemma 10.9 in view of [1, proof of Theorem 2.11 (iv) ⇒ (i)] for $\mathsf{K} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$, $\lambda = \aleph_0$, and $\mu = \kappa$. The Govorov–Lazard characterization of flat modules [11, 16] implies that all finitely presentable flat R-modules are projective. By Lemma 10.9, the category of surjective morphisms of flat R-modules is finitely accessible, and its finitely presentable objects are the surjective morphisms of finitely generated projective R-modules. So all surjective morphisms of flat R-modules are directed colimits of surjective morphisms of finitely generated projective R-modules.

Let A denote the set of all κ-small directed colimits of surjective morphisms of finitely generated projective R-modules. Following the argument in [1, proof of Theorem 2.11 (iv) ⇒ (i)] and [1, Example 2.13(1)], all the objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are κ-directed colimits of objects from A. Thus all the κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are direct summands of objects from A. □

The next corollary is a generalization of [23, Lemma 4.1].

Corollary 10.12. *Let R be an associative ring and κ be a regular cardinal. Then the kernel of any surjective morphism of κ-presentable flat R-modules is a κ-presentable flat R-module.*

Proof. Follows from Corollary 10.11, as the kernel of any surjective morphism of finitely generated projective R-modules is a finitely generated projective R-module. For another proof, see [24, Corollary 4.7]. $\square$

Given a ring R and a full subcategory $\mathsf{L} \subset R\text{–}\mathsf{Mod}$, let us denote by $\mathsf{L}^{\mathsf{ses}}$ the category of all short exact sequences in $R\text{–}\mathsf{Mod}$ with the terms belonging to L.

Corollary 10.13. *For any associative ring R and any regular cardinal κ, the category of short exact sequences of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ is κ-accessible. The full subcategory of κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ is the category $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{ses}}$ of all short exact sequences of κ-presentable flat R-modules.*

Proof. By Corollary 10.12, the obvious equivalence of categories $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}} \simeq R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ identifies $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{surj}}$ with $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{ses}}$. This makes the desired assertion a restatement of Lemma 10.9. $\square$

10.5 Pure acyclic complexes of flat modules

Finally, we can present our "toy application" of Corollary 5.1. An acyclic complex of flat R-modules is said to be *pure acyclic* if its modules of cocycles are flat.

The following corollary is essentially a weaker version of the result of [9, Theorem 2.4 (1) $\Leftrightarrow$ (3)] or [21, Theorem 8.6 (ii) $\Leftrightarrow$ (iii)]. Our argument produces it as an application of general category-theoretic principles. See [27, Theorem 4.2] and [23, Corollaries 4.5 and 11.4] for a quasi-coherent sheaf, a comodule, and a contramodule version.

Corollary 10.14. *Let R be an associative ring and κ be an uncountable regular cardinal. Then any pure acyclic complex of flat R-modules is a κ-directed colimit of pure acyclic complexes of κ-presentable flat R-modules.*

Proof. The point is that a pure acyclic complex of flat R-modules $F^\bullet$ is the same thing as a collection of short exact sequences of flat R-modules $0 \longrightarrow G^n \longrightarrow F^n \longrightarrow H^n \longrightarrow 0$ together with a collection of isomorphisms $H^n \simeq G^{n+1}$, $n \in \mathbb{Z}$. This means that the category of pure acyclic complexes of flat R-modules can be constructed from the category of short exact sequences of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ using Cartesian products (as in Section 2) and the isomorpher construction from Remark 5.2.

Specifically, put $\mathsf{H} = \prod_{n\in\mathbb{Z}} R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ and $\mathsf{G} = \prod_{n\in\mathbb{Z}} R\text{–}\mathsf{Mod}_{\mathsf{fl}}$. Let $P\colon \mathsf{H} \longrightarrow \mathsf{G}$ be the functor taking a collection of short exact sequences $(0 \to G^n \to F^n \to H^n \to 0)_{n\in\mathbb{Z}}$ to the collection of modules $(H^n)_{n\in\mathbb{Z}}$, and let $Q\colon \mathsf{H} \longrightarrow \mathsf{G}$ be the functor taking the same collection of short exact sequences to the collection of modules $(G^{n+1})_{n\in\mathbb{Z}}$. Then the resulting isomorpher category D is the category of pure acyclic complexes of flat R-modules. Given the results of Proposition 10.2 and Corollary 10.13, it follows from Proposition 2.1 and Remark 5.2 that the category D is κ-accessible and the pure acyclic complexes of κ-presentable flat R-modules are precisely all the κ-presentable objects of D. $\square$

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THE TOPOLOGY OF CRITICAL PROCESSES, II
(THE FUNDAMENTAL CATEGORY)

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Résumé. La topologie algébrique dirigée étudie des espaces équipés d'une forme de direction, avec l'objectif d'inclure les processus non réversibles. Dans l'extension présente nous voulons couvrir aussi les *processus critiques*, indivisibles et inarrêtables.

La première partie de cette série a introduit les *espaces contrôlés*, en examinant comment ils peuvent modeler les processus critiques issus de divers domaines, du changement d'état dans une cellule de mémoire à l'action d'un thermostat ou un siphon. Ici nous construisons la catégorie fondamentale de ces espaces.

Abstract. Directed Algebraic Topology studies spaces equipped with a form of direction, to include models of non-reversible processes. In the present extension we also want to cover *critical processes*, indecomposable and unstoppable.

The first part of this series introduced *controlled spaces*, examining how they can model critical processes in various domains, from the change of state in a memory cell to the action of a thermostat or a siphon. We now construct the fundamental category of these spaces.

Keywords. Directed algebraic topology, homotopy theory, fundamental category.

Mathematics Subject Classification (2010). 55M, 55P, 55Q.

Introduction

0.1 Critical processes and controlled spaces

Directed Algebraic Topology is an extension of Algebraic Topology, dealing with 'spaces' where the paths need not be reversible; the general aim is including the representation of *irreversible processes*. A typical setting for this study, the category $\mathrm{d}\mathsf{Top}$ of directed spaces, or d-spaces, was introduced and studied in [G1]–[G3]; it is frequently employed in the theory of concurrency: see the book [FGHMR] and many articles cited in a previous paper [G5].

The present series is devoted to a further extension, where the paths can also be non-decomposable in order to include *critical processes*, indivisible and unstoppable – either reversible or not. For instance: quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, the action of a siphon, moving in a no-stop road, etc.

To this effect the category of d-spaces was extended in Part I [G5] to the category $\mathrm{c}\mathsf{Top}$ of *controlled spaces*, or *c-spaces*: an object is a topological space equipped with a set $X^\sharp$ of continuous mappings $a\colon [0,1] \to X$, called *controlled paths*, or *c-paths*, that satisfies three axioms:

(csp.0) (*constant paths*) the trivial loops at the endpoints of a controlled path are controlled,

(csp.1) (*concatenation*) the concatenation of consecutive controlled paths is controlled,

(csp.2) (*global reparametrisation*) the reparametrisation of a controlled path by a surjective increasing map $[0,1] \to [0,1]$ is controlled.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths. Their category $\mathrm{c}\mathsf{Top}$ contains the category $\mathrm{d}\mathsf{Top}$ of d-spaces as a full subcategory, reflective and coreflective: a c-space is a d-space if and only if it is *flexible*, which means that each point is flexible (its trivial loop is controlled) and every controlled path is flexible (all its restrictions are controlled).

Here we deal with the fundamental category of controlled spaces. Part III will study more advanced methods of computations of the latter, with applications to models of critical processes and concurrency. The homotopy theory of c-spaces will be dealt with in Part IV.

0.2 Standard intervals

The difference between these two settings – directed and controlled spaces – shows clearly in two structures of the euclidean interval $\mathbb{I} = [0, 1]$, the starting point of homotopy in each setting, represented as follows (and better described in 1.3)

$$\underset{0}{\vdash}\!\!\longrightarrow\!\!\underset{1}{\dashv} \quad \uparrow\!\mathbb{I} \qquad\qquad \underset{0}{\bullet}\!\!\twoheadrightarrow\!\!\underset{1}{\bullet} \quad \mathrm{c}\mathbb{I} \tag{1}$$

The left figure shows the *standard d-interval* $\uparrow\!\mathbb{I}$ (in $\mathrm{d}\mathsf{Top}$): its directed paths are all the (weakly) increasing maps $\mathbb{I} \to \mathbb{I}$. It may be viewed as the essential model of a non-reversible process, or a one-way route in transport networks. Its fundamental category $\uparrow\!\Pi_1(\uparrow\!\mathbb{I})$, as defined in [G1, G3], is the ordered set $[0, 1]$, with one arrow $t \to t'$ for each pair $t \leqslant t'$ in $[0, 1]$.

The right figure shows the *standard c-interval* $\mathrm{c}\mathbb{I}$, or *one-jump interval* (in $\mathrm{c}\mathsf{Top}$): its controlled paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at 0 or 1. It models a non-reversible unstoppable process, or a one-way no-stop route. Its fundamental category $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I})$, as defined here, is the ordinal $\mathbf{2}$, with one non-trivial arrow $0 \to 1$ and two identities.

0.3 Outline

The basic definitions and the main examples of Part I are recalled in Section 1; in particular, every c-space has two associated d-spaces, the generated d-space $\hat{X}$ and the flexible part $\mathsf{Fl}\,X$, by the reflector and coreflector of the embedding $\mathrm{d}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ (see 1.2). Then we introduce in Section 2 two weak forms of flexibility which will play a role in our study: preflexible and border flexible c-spaces.

Section 3 reviews the basic part of the homotopy theory of d-spaces studied in [G1, G3], including the construction of the fundamental category $\uparrow\!\Pi_1(X)$ of a d-space; some of these results are already extended or adapted to c-spaces.

In the next two sections we introduce the fundamental category $\uparrow\!\Pi_1(X)$ of a c-space, an extension of the previous case. Its vertices are the flexible points of X; its arrows come out of a complex construction, based on the hybrid square $c\mathbb{I}\times{\uparrow}\mathbb{I}$: they are equivalence classes of controlled paths (parametrised on $c\mathbb{I}$), up to flexible deformations (parametrised on ${\uparrow}\mathbb{I}$).

The main results can be found in Section 5: the construction of ${\uparrow}\Pi_1(X)$, its relationship with the fundamental category of the associated d-spaces $\hat{X}$ and $\mathsf{Fl}\,X$ (in 5.2 and 5.3), its homotopy invariance (in 5.4) and its calculation for covering maps of c-spaces (in 5.8). All this is based on the technical analysis of Section 4. We end by calculating the fundamental category of the basic c-spaces, in 5.9.

0.4 Comments

In the present extension we reach models of phenomena that have no place in the previous settings of Directed Algebraic Topology, and peculiar 'shapes', like the *one-stop circle* $c\mathbb{S}^1$, the *n-stop circle* $c_n\mathbb{S}^1$, or the higher controlled spheres and tori described in Part I. The fundamental category of the new spaces is often quite simple.

We also loose some good properties of the theory of d-spaces. For instance, the interval $c\mathbb{I}$ is not exponentiable in $c\mathsf{Top}$ (see 4.7(b)), and the associated cylinder functor $I(X) = X\times c\mathbb{I}$ has no right adjoint: there is no path endofunctor. We manage to extend the fundamental category, by allowing c-paths to be deformed by flexible homotopies, and one can also extend directed singular homology, but new methods of calculation are needed: the van Kampen theorem and the Mayer-Vietoris sequence are based on the subdivision of paths and homological chains, which is no longer permitted. On the other hand, the theory of covering maps can be extended to the present case (Theorem 5.8(b)).

Essentially, the previous setting of d-spaces extends classical topology by breaking the symmetry of reversion: directed paths need no longer be reversible. This further extension to c-spaces breaks a flexibility feature that d-spaces still retain: paths can no longer be subdivided, and this has drastic consequences.

Critical processes and transport networks are often represented by graphs, in an effective way as far as they do not interact with continuous variation. We want to show that they can also be modelled by structured spaces, in a theory that includes classical topology and 'non-reversible spaces'. Controlled spaces can thus unify aspects of continuous and discrete mathematics, interacting with hybrid control systems and others sectors of Control Theory [Br, He].

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0.5 Notation and conventions

A continuous mapping between topological spaces is called a *map*. $\mathbb{R}$ denotes the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0, 1]$. The identity path $\mathrm{id}\,\mathbb{I}$ is written as $\underline{i}$. The open and semiopen intervals of the real line are denoted by square brackets, like $]0, 1[$, $[0, 1[$ etc. A space is *locally compact* if every point has a local basis of compact neighbourhoods; the Hausdorff property is not assumed.

A *preorder* relation, generally written as $x \prec y$, is assumed to be reflexive and transitive; an *order*, often written as $x \leqslant y$, is also assumed to be anti-symmetric. A mapping which preserves (resp. reverses) preorders is said to be *increasing* (resp. *decreasing*), always used in the weak sense.

As usual, a preordered set X is identified with the small category whose objects are the elements of X, with one arrow $x \to x'$ when $x \prec x'$ and none otherwise.

The binary variable α takes values $0, 1$, which are generally written as $-, +$ in superscripts and subscripts. The symbol $\subset$ denotes weak inclusion.

The first paper [G5] of this series is cited as Part I; the reference I.2 or I.2.3 points to Section 2 or Subsection 2.3 of Part I, respectively.

1. Controlled and directed spaces

We recall the definition of controlled space, or c-space, introduced in Part I. Their category cTop is an extension of the category dTop of directed spaces, or d-spaces, studied in [G1]–[G3] and commonly used in concurrency; we generally refer to the book [G3]. The term ‘selected path’ is used in both cases.

1.1 Controlled spaces

As defined in Part I, a *controlled space* X, or *c-space*, is a topological space equipped with a set $X^\sharp$ of (continuous) maps $a\colon \mathbb{I} \to X$, called *controlled paths*, or *c-paths*, that satisfies three axioms:

(csp.0) (*constant paths*) the trivial loops at the endpoints of a controlled path are controlled,

(csp.1) (*concatenation*) the controlled paths are closed under path concatenation: if the consecutive paths a, b are controlled, their concatenation $a * b$ is also,

(csp.2) (*global reparametrisation*) the controlled paths are closed under pre-composition with every surjective increasing map $\rho\colon \mathbb{I} \to \mathbb{I}$: if a is a controlled path, $a\rho$ is also.

The controlled paths are also closed under general n-ary concatenations, based on arbitrary partitions $0 = t_0 < t_1 < ... < t_n = 1$ of $\mathbb{I}$ (as shown in I.1.2). The underlying topological space is written as $U(X)$, or $|X|$, and called the *support* of X.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths. Their category is written as $\mathrm{c}\mathsf{Top}$.

Reversing c-paths, by the involution $r(t) = 1 - t$, yields the *opposite* c-space $RX = X^{\mathrm{op}}$, where $a \in (X^{\mathrm{op}})^\sharp$ if and only if ar belongs to $X^\sharp$. We have thus the *reversor* endofunctor

$$R\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad\qquad RX = X^{\mathrm{op}}. \tag{2}$$

The c-space X is *reversible* if $X = X^{\mathrm{op}}$. More generally, it is *reversive* if it is isomorphic to X^{op}.

Controlled spaces have all limits and colimits, which are topological limits and colimits with the initial or final structure determined by the structural maps.

In a c-space X, a point x is *flexible* if its trivial loop e_x is controlled. The *flexible support* $|X|_0$ is the subspace of these points, and can be empty. A c-path is *flexible* if all its restrictions are controlled. The c-space itself is *flexible* if every point and every selected path are flexible.

The singleton space has two c-structures: the discrete one $D_c\{*\}$, with no controlled paths, and the *flexible singleton* $\{*\}$ where the trivial loop is selected; this is the terminal object and the unit of the cartesian product.

The category $\mathrm{c}\mathsf{Top}$ has two forgetful functors to topological spaces

$$U\colon \mathrm{c}\mathsf{Top} \to \mathsf{Top}, \qquad\qquad U_0\colon \mathrm{c}\mathsf{Top} \to \mathsf{Top}, \tag{3}$$

where $U(X) = |X|$ is the topological support and $U_0(X) = |X|_0$ is the flexible support. U has both adjoints, U_0 has only the left one: it preserves limits and sums (I.1.7(e)).

We say that the c-space X_1 is *finer* than X_2 if they have the same topological support X and $X_1^\sharp \subset X_2^\sharp$, which means that the identity map of X is a c-map $X_1 \to X_2$; the latter is called a *reshaping*.

We shall see later, in 4.9, that requiring that *all trivial loops be controlled* would be a serious hindrance.

1.2 Directed spaces

Previously, a *directed space* X, or *d-space*, was also defined as a topological space with a set $X^\sharp$ of selected paths, called *directed paths*, or *d-paths*, under stronger axioms: the selected paths are stable under: (all) constant paths, concatenations and *partial* reparametrisations (by increasing endomaps of the interval, not assumed to be surjective) [G3].

A d-space is the same as a flexible c-space, and can also be defined in this way. The category $\mathrm{d}\mathsf{Top}$ of d-spaces and d-maps is a full subcategory of $\mathrm{c}\mathsf{Top}$, reflective and coreflective

$$\begin{aligned} \hat{}\colon \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top} \qquad & \text{(the } \textit{reflector}\text{)}, \\ \mathsf{Fl}\colon \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top} \qquad & \text{(the } \textit{coreflector}\text{)}. \end{aligned} \tag{4}$$

In the first construction the *generated d-space* $\hat{X}$ of a c-space X has the same support with the d-structure generated by the c-paths, i.e. the finest containing them; it can be obtained by stabilising the latter under constant paths, restriction and general concatenation. The unit of the adjunction is the reshaping $X \to \hat{X}$; the counit is the identity $\hat{Y} = Y$, for a d-space Y.

In the second construction the *flexible part* $\mathsf{Fl}\,X$ is the flexible support $|X|_0$ with the d-structure of the flexible c-paths. The counit is the inclusion $\mathsf{Fl}\,X \to X$, the unit is the identity $Y = \mathsf{Fl}\,Y$, for a d-space Y.

Also $\mathrm{d}\mathsf{Top}$ has all limits and colimits, preserved by the embedding in $\mathrm{c}\mathsf{Top}$.

1.3 Structured intervals and cubes

(a) In dTop the *standard d-interval* $\uparrow\mathbb{I}$ has the d-structure generated by the identity $\underline{i} = \mathrm{id}\,\mathbb{I}$: the directed paths are all the increasing maps $\mathbb{I} \to \mathbb{I}$. It plays the role of the standard interval in this category, because the directed paths of any d-space X coincide with the d-maps $\uparrow\mathbb{I} \to X$.

It may be viewed as an essential model of a non-reversible process, or a one-way road in transport networks. It will be represented as in figure (1) of the Introduction.

Similarly, the directed line $\uparrow\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$.

(b) In cTop the *standard c-interval* $\mathrm{c}\mathbb{I}$, or *one-jump interval*, has the same support, with the c-structure generated by the identity $\underline{i}$: the controlled paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at 0 or 1. The controlled paths of any c-space X coincide with the c-maps $\uparrow\mathbb{I} \to X$.

It can model a *non-reversible unstoppable process*, or a *one-way no-stop road*. It is also represented in figure (1), marking by a bullet the isolated flexible points: in this case, the endpoints of the interval.

The controlled line $\mathrm{c}\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$ whose image is an interval $[k, k']$ with integral endpoints.

We shall also use other c-structures of the compact interval, already examined in I.2.4.

(c) In the *two-jump interval* $\mathrm{c}\mathbb{J}$, the non-trivial c-paths are the increasing maps $\mathbb{I} \to \mathbb{I}$ whose image is either $[0, 1/2]$, or $[1/2, 1]$, or $[0, 1]$

$$\underset{0}{\bullet} \twoheadrightarrow \underset{1/2}{\bullet} \twoheadrightarrow \underset{1}{\bullet} \qquad \mathrm{c}\mathbb{J} \tag{5}$$

This c-space can model a non-reversible two-stage process. Formally, it parametrises the ordinary concatenation of two c-paths, see 4.2.

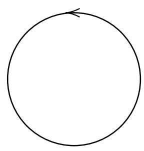
(d) The *reversible c-interval* $\mathrm{c}\mathbb{I}^\sim$ has a c-structure generated by the identity $\underline{i}$ and the reversion $r\colon \mathbb{I} \to \mathbb{I}$. It can model a reversible unstoppable process. The reversible c-paths of a c-space X coincide with the c-maps $\mathrm{c}\mathbb{I}^\sim \to X$.

(e) We shall also use the *delayed intervals* $\mathrm{c}_-\mathbb{I}$ and $\mathrm{c}_+\mathbb{I}$ of I.2.4(b). Each of these c-structure is generated by a single map $\mathbb{I} \to \mathbb{I}$, namely $\rho(t) = 0 \vee (2t - 1)$ or $\sigma(t) = 2t \wedge 1$, respectively. They can model irreversible non-stoppable processes with inertia, or inductance.

(f) The powers $\mathbb{I}^n$ and $\mathbb{R}^n$ inherit various controlled structures. On $\mathbb{I}^2$ we shall mostly use the d-square $\uparrow\!\mathbb{I}^2$, the c-square $\mathrm{c}\mathbb{I}^2$ and the hybrid square $\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I}$ (cf. I.2.7).

1.4 Structured spheres

(a) The *standard d-circle* $\uparrow\!\mathbb{S}^1$ can be obtained as an orbit space



$$\uparrow\!\mathbb{S}^1 = (\uparrow\!\mathbb{R})/\mathbb{Z} \tag{6}$$

with respect to the action of the group of integers on the directed line $\uparrow\!\mathbb{R}$ (by translations): the directed paths of $\uparrow\!\mathbb{S}^1$ are the projections of the increasing paths in the line.

$\uparrow\!\mathbb{S}^1$ can also be obtained as the coequaliser in $\mathrm{d}\mathsf{Top}$ of the following pair of maps

$$\partial^-, \partial^+ \colon \{*\} \rightrightarrows \uparrow\!\mathbb{I}, \qquad\qquad \partial^-(*) = 0, \quad \partial^+(*) = 1, \tag{7}$$

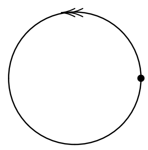
that is the quotient $\uparrow\!\mathbb{I}/\partial\mathbb{I}$ which identifies the points of the boundary $\partial\mathbb{I} = \{0, 1\}$.

(b) More generally, the *directed* n*-dimensional sphere* is defined, for $n > 0$, as the quotient of the directed cube $\uparrow\!\mathbb{I}^n$ modulo the equivalence relation that collapses its boundary $\partial\mathbb{I}^n$ to a point

$$\uparrow\!\mathbb{S}^n = (\uparrow\!\mathbb{I}^n)/(\partial\mathbb{I}^n) \quad (n > 0), \qquad \uparrow\!\mathbb{S}^0 = \mathbb{S}^0 = \{-1, 1\}, \tag{8}$$

while $\uparrow\!\mathbb{S}^0$ has the discrete topology and the natural (discrete) d-structure.

(c) The *standard c-circle* $\mathrm{c}\mathbb{S}^1$, or *one-stop circle*, can also be defined as an orbit space



$$\mathrm{c}\mathbb{S}^1 = (\mathrm{c}\mathbb{R})/\mathbb{Z} \tag{9}$$

for the action of the group of integers on the controlled line $\mathrm{c}\mathbb{R}$, by translations. The controlled paths of $\mathrm{c}\mathbb{S}^1$ are the projections of the controlled paths in the line: here this means an anticlockwise path (in the ordered plane) which is a loop at $* = [0]$, the only flexible point (corresponding to $(1, 0)$ in the plane).

The c-space $\mathrm{c}\mathbb{S}^1$ can also be obtained as the coequaliser in cTop of the following pair of maps

$$\partial^-, \partial^+ \colon \{*\} \rightrightarrows \mathrm{c}\mathbb{I}, \qquad \partial^-(*) = 0, \quad \partial^+(*) = 1. \tag{10}$$

(d) More generally, the *n-stop c-circle* $\mathrm{c}_n\mathbb{S}^1$ $(n > 0)$ is constructed in I.2.6(b) as the orbit space

$$\mathrm{c}_n\mathbb{S}^1 = (\mathrm{c}_n\mathbb{R})/\mathbb{Z} \qquad\qquad (\mathrm{c}_1\mathbb{S}^1 = \mathrm{c}\mathbb{S}^1), \tag{11}$$

where the c-paths of $\mathrm{c}_n\mathbb{R}$ are the increasing paths whose image is an interval $[k/n, k'/n]$, for integers $k \leqslant k'$.

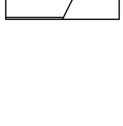
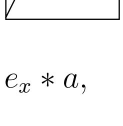
(e) The *standard c-sphere* $\mathrm{c}\mathbb{S}^n$ is defined as a quotient of the cube $\mathrm{c}\mathbb{I}^n$ (for $n > 0$)

$$\mathrm{c}\mathbb{S}^n = (\mathrm{c}\mathbb{I}^n)/(\partial\mathbb{I}^n) \quad (n > 0), \qquad \mathrm{c}\mathbb{S}^0 = \mathbb{S}^0 = \{-1, 1\}. \tag{12}$$

1.5 Identities and associativity

Concatenation of paths and the various forms of reparametrisation have a complex relationship. Here we recall two well-known points.

(a) The constant loops act as identities up to the equivalence relation generated by global reparametrisation. In fact, the following surjective increasing maps $\rho, \sigma \colon \mathbb{I} \to \mathbb{I}$ reparametrise any path a, from x to y, as $e_x * a$ or $a * e_y$, respectively:

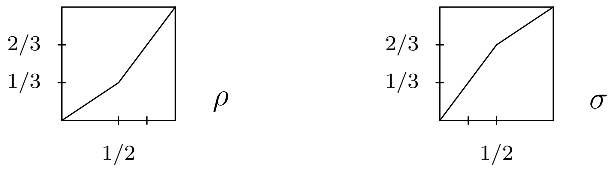
 ρ  σ

$$\rho(t) = 0 \vee (2t - 1), \qquad \sigma(t) = 2t \wedge 1, \tag{13}$$

$$a\rho = e_x * a, \qquad\qquad a\sigma = a * e_y.$$

These maps were used in I.2.4(b), as the *past-* (resp. *future-*) *delayed reparametrisation*. Since $\rho \leqslant \underline{i} \leqslant \sigma$, there are directed homotopies with fixed endpoints $a\rho \to a \to a\sigma$ in $\mathrm{d}\mathsf{Top}$ [G3], which also work in the present setting, as we shall see in Lemma 4.6. (These homotopies are reversible in Top, but are not for our directed structures.)

(b) All n-ary concatenations are equivalent, up to invertible reparametrisation (cf. I.1.2). In particular, let $\rho\colon \mathbb{I} \to \mathbb{I}$ be the obvious piecewise affine invertible reparametrisation that takes the partition $0 < 1/2 < 3/4 < 1$ to the regular partition $0 < 1/3 < 2/3 < 1$, while $\sigma\colon \mathbb{I} \to \mathbb{I}$ does the same on $0 < 1/4 < 1/2 < 1$



(14)

Now, if $d = a * b * c$ is the regular concatenation of three consecutive paths, based on the partition $0 < 1/3 < 2/3 < 1$

$$d\rho = a * (b * c), \qquad d\sigma = (a * b) * c. \tag{15}$$

Again $\rho \leqslant \underline{i} \leqslant \sigma$, and there are homotopies with fixed endpoints $d\rho \to d \to d\sigma$, which work in $\mathrm{d}\mathsf{Top}$ and will also work in the present setting, by Lemma 4.6.

2. Weak flexibility

We now introduce weak forms of flexibility that will be important for the construction of the fundamental category, and still hold in basic c-spaces like $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $\mathrm{c}\mathbb{R}$, $\mathrm{c}\mathbb{S}^1$ and all their products (and limits) – although all of them are rigid c-spaces, in the sense of I.1.6.

X is always a c-space.

2.1 Preflexible and border flexible c-spaces

(a) If S is a subset of the flexible support $|X|_0$ of X, we can form a finer c-space $X_{|S}$ on the same support selecting the c-paths of X whose endpoints belong to S. We say that the c-space $X_{|S}$ is *full* in X, or *obtained from X by restricting the flexible support to S*. We shall see, in Theorem 5.3(a), that $\uparrow\Pi_1(X_{|S})$ is the full subcategory of $\uparrow\Pi_1(X)$ with vertices in S.

For instance the c-spaces $\mathrm{c}\mathbb{I}$ and $\mathrm{c}\mathbb{J}$ are full in $\uparrow\mathbb{I}$, which is just finer than $\mathbb{I}$. Both relationships, being finer or full, are preserved by products.

(b) We say that X is *preflexible* if it is full in the generated d-space $\hat{X}$, which means that every c-path of $\hat{X}$ between flexible points of X is already a c-path of the latter. The interest of this property will be evident in Theorem 5.3(b).

(c) We say that X is *border flexible* if its controlled paths are closed under 'cutting out delays at the endpoints'. More precisely, we are asking that, for every c-path a of X which is constant on $[0, t_1]$ and $[t_2, 1]$, the *border restriction $a\rho$* be still a controlled path, for

$$\rho\colon \mathbb{I} \to \mathbb{I}, \qquad \rho(t) = t_1 + (t_2 - t_1)t \qquad (0 \leqslant t_1 < t_2 \leqslant 1). \tag{16}$$

Plainly, every preflexible c-space is border flexible; the converse is false: see 2.2(b).

(d) We also introduce the *path-support* $|X|_1$ of the c-space X as the topological subspace of $|X|$ formed by the union of the images of all c-paths in X, so that $|X|_0 \subset |X|_1 \subset |X|$. A c-map can be restricted to the path-supports. We say that X has a *total path-support* if $|X|_1 = |X|$.

2.2 Examples and remarks

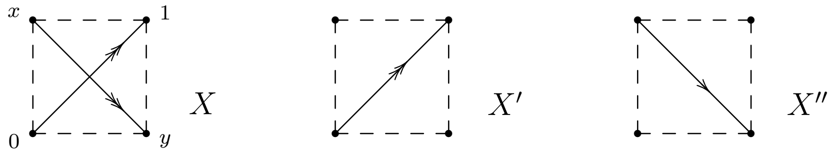
(a) Besides all d-spaces, many c-spaces we have considered in Part I and here are preflexible: for instance $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$ and $\mathrm{c}\mathbb{I}^\sim$ (in $\uparrow\mathbb{I}$), $\mathrm{c}\mathbb{R}$ (in $\uparrow\mathbb{R}$), $\mathrm{c}\mathbb{S}^1$ (in $\uparrow\mathbb{S}^1$), and all their limits (by Proposition 2.3).

The delayed intervals $\mathrm{c}_-\mathbb{I}$ and $\mathrm{c}_+\mathbb{I}$ recalled in 1.3(e) are not even border flexible: the preflexible space generated by any of them (according to Proposition 2.3(a)) is the standard interval $\mathrm{c}\mathbb{I}$.

It is not difficult to prove that the c-spheres $\mathrm{c}\mathbb{S}^n$ are not border flexible, for $n \geqslant 2$.

(b) The 'diagonal' c-structure X of the square $\mathbb{I}^2$ described in I.2.7(d) is border flexible (obviously) and not preflexible (as shown below); it does not have a total path-support.

We recall that the c-paths of X are generated by two diagonal paths, $t \mapsto (t, t)$ and $t \mapsto (t, 1-t)$, represented in the left figure below



(17)

The flexible points are the four vertices of the square. The generated d-space $\hat{X}$ has also c-paths $0 \to y$ and $x \to 1$, proving that X is not preflexible.

We also note that the structure of X is generated by two finer c-structures X', X'' of the square, with the same flexible points and non-trivial c-paths generated by one of the previous diagonals. X is thus the pushout of three preflexible spaces, X' and X'' over X_0: the latter is the square $\mathbb{I}^2$ with the trivial loops at the vertices (the intersection of the structures of X' and X'').

(c) The quotient of the interval $X = \mathrm{c}[0,2] \subset \mathrm{c}\mathbb{R}$ modulo the equivalence relation that collapses its second half to a point

$$X/[1,2] \;\cong\; \mathrm{c}_+\mathbb{I}, \qquad (18)$$

is not border flexible. It is the pushout in $\mathrm{c}\mathsf{Top}$ of two maps of border flexible c-spaces, the inclusion $\mathrm{c}[1,2] \to \mathrm{c}[0,2]$ and the map $\mathrm{c}[1,2] \to \{*\}$.

(d) *Remarks*. In a border flexible c-space, initial or final delays cannot be required; but let us note that they can never be prevented – a global reparametrisation can always produce them.

On the other hand, 'internal' delays can be required, as in the border-flexible *middle-delay* interval $\mathrm{c}_M\mathbb{I}$, with the c-structure generated by the map

$$\mathrm{c}_M(t) = ((3t \wedge 1) \vee (3t-1))/2 \qquad \textit{(middle-delay map)}.$$

(e) Full c-spaces were already considered in I.3.5, in the equivalent perspective of excluding the flexible points of $|X|_0 \setminus S$.

2.3 Proposition and Definition (Reflectors and limits)

(a) Preflexible c-spaces form a full, reflective subcategory $\mathrm{c}_{\mathrm{pf}}\mathsf{Top}$ *of* $\mathrm{c}\mathsf{Top}$. *The reflector*

$$(-)^{\mathrm{pf}} \colon \mathrm{c}\mathsf{Top} \to \mathrm{c}_{\mathrm{pf}}\mathsf{Top}, \qquad X \mapsto X^{\mathrm{pf}}, \qquad (19)$$

gives the generated preflexible space $X^{\rm pf}$*, with the same flexible points and all the c-paths of* $\hat{X}$ *between them; maps stay 'unchanged'. The unit is the reshaping* $X \to X^{\rm pf}$*.*

As in every full reflective subcategory, preflexible c-spaces are closed in $\mathrm{c}\mathsf{Top}$ *under limits. All colimits can be obtained taking the colimit in* $\mathrm{c}\mathsf{Top}$ *and applying the reflector. Sums are preserved by the inclusion, but pushouts are not – in general.*

(b) Similarly, border flexible c-spaces form a full, reflective subcategory $\mathrm{c_{bf}}\mathsf{Top}$ *of* $\mathrm{c}\mathsf{Top}$*. The reflector*

$$(-)^{\rm bf}\colon \mathrm{c}\mathsf{Top} \to \mathrm{c_{bf}}\mathsf{Top}, \qquad\qquad X \mapsto X^{\rm bf}, \tag{20}$$

gives the generated border flexible space $X^{\rm bf}$*, with the least c-structure containing the border restrictions of the original c-paths, as specified in 2.1(c).*

Maps stay 'unchanged'; the unit is the reshaping $X \to X^{\rm bf}$*. Obviously* X *and* $X^{\rm bf}$ *generate the same d-space.*

Again, border flexible c-spaces are closed in $\mathrm{c}\mathsf{Top}$ *under limits. All colimits can be obtained taking the colimit in* $\mathrm{c}\mathsf{Top}$ *and applying the reflector. Sums are preserved by the inclusion and pushouts need not be.*

(c) Controlled spaces with a total path-support form a full coreflective subcategory of $\mathrm{c}\mathsf{Top}$*, closed under colimits and products.*

Proof. (a) The reshaping $X \to X^{\rm pf}$ is a universal arrow from X to the inclusion $\mathrm{c_{pf}}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$, which creates all limits.

Closure under sums is trivial, while this does not work with pushouts, as we have seen in 2.2(b). All colimits can be constructed as specified above; or directly, by final structures.

(b) The c-space X is indeed finer than $X^{\rm bf}$: if $a' = a\rho$ is a border restriction of a c-path of X, we can reconstruct $a = a'\sigma$ by a global reparametrisation of a' that brings back the delays.

Technically, a is constant on two intervals $[1, t_1]$ and $[t_2, 1]$, with $t_1 < t_2$, and we let:

$$\rho(t) = t_1 + (t_2 - t_1)t, \qquad \sigma(t) = ((t - t_1)/(t_2 - t_1) \vee 0) \wedge 1, \tag{21}$$

so that $a\rho\sigma = a$. The reshaping $X \to X^{\rm bf}$ is a universal arrow from X to the inclusion $\mathrm{c_{bf}}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$, and the rest works as in (a), taking into account example 2.2(c) for pushouts.

(c) The coreflector takes a c-space X to its path-support $|X|_1$ equipped with the same c-paths. Moreover the path-support functor $|-|_1\colon \mathrm{c}\mathsf{Top} \to \mathsf{Top}$ preserves products. $\square$

2.4 Lemma

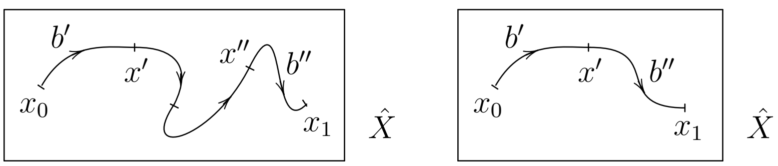
(a) Let a be a c-path of the d-space $\hat{X}$ generated by the c-space X. We suppose that one of these conditions is satisfied:

(i) a is not constant,
(ii) a is constant at a point x of the path-support $|X|_1$.

Then a is a restriction of some c-path $b\colon x_0 \to x_1$ of $\hat{X}$ between flexible points of X. We can choose b so that a is its middle restriction*, on* $[1/3, 2/3]$.

(b) If the c-space X is preflexible, every c-path of $\hat{X}$ satisfying (i) or (ii) is the middle restriction of some c-path of X.

Proof. We prove (a), which trivially implies (b). In case (i) a non-constant c-path $a\colon x' \to x''$ of the d-space $\hat{X}$ is a general concatenation of c-paths $a_1, ..., a_n$ which are restrictions of c-paths $b_1, ..., b_n$ of X. There is thus some c-path $b'\colon x_0 \to x'$ of $\hat{X}$ starting from a flexible point of X: either a restriction of b_1, as in the left figure below



(22)

or a trivial loop at $x_0 = x'$, if x' is a flexible point of X. Symmetrically there is in $\hat{X}$ a c-path $b''\colon x'' \to x_1$ reaching a flexible point of X.

Now, a is the middle restriction of the regular concatenation $b = b' * a * b''\colon x_0 \to x_1$.

Case (ii) is obvious: there is a c-path $b\colon x_0 \to x_1$ *of* X whose image contains x and we can take in $\hat{X}$ suitable restrictions $b'\colon x_0 \to x$ and $b''\colon x \to x_1$ of b, as in the right figure above; or we take two trivial loops, if x is already a flexible point of X. Again, a is the middle restriction of the regular concatenation $b' * a * b''\colon x_0 \to x_1$ in $\hat{X}$. $\square$

2.5 Theorem (Flexibility and cartesian products)

(a) The flexible c-space $(\Pi_i X_i)\hat{}$ *generated by a product is finer than* $\Pi_i \hat{X}_i$*, and can be strictly finer.*

(b) For any family (X_i) *of preflexible c-spaces with a total path-support (see 2.1(d)):*

$$(\Pi X_i)\hat{} \;=\; \Pi \hat{X}_i. \tag{23}$$

Proof. (a) The flexible structure of $\Pi \hat{X}_i$ contains the structure of ΠX_i, and also of $(\Pi X_i)\hat{}$. The inclusion can be strict, as shown in I.2.7(b) for the products $X \times \mathrm{c}\mathbb{I}$ or $X \times {\uparrow}\mathbb{I}$, where $X = D_c\{*\}$ has an empty path-support. One could also use the diagonal c-space of 2.2(b), which is not preflexible.

(b) We prove that, if all X_i are preflexible c-spaces with a total path-support, every c-path $a = \langle a_i \rangle$ of $\Pi \hat{X}_i$ is also controlled in $(\Pi X_i)\hat{}$.

By Lemma 2.4(b), each a_i is the middle restriction $b_i\rho$ of some c-path b_i of X_i, always applying the strictly increasing affine map $\rho\colon \mathbb{I} \to \mathbb{I}$ with image $[1/3, 2/3]$. Therefore $a = \langle b_i \rangle \rho$ is the restriction of a c-path of the product, and belongs to the structure $(\Pi X_i)\hat{}$. □

2.6 Corollary

(a) If X *is a preflexible c-space with a total path-support*

$$(X \times \mathrm{c}\mathbb{I})\hat{} = \hat{X} \times {\uparrow}\mathbb{I}, \qquad (X \times {\uparrow}\mathbb{I})\hat{} = \hat{X} \times {\uparrow}\mathbb{I}. \tag{24}$$

(b) In particular: $(\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I})\hat{} = {\uparrow}\mathbb{I}^2$.

3. Elementary homotopy theory of d-spaces

We recall here the elementary part of homotopy theory in the category $\mathrm{d}\mathsf{Top}$ of d-spaces and the construction of their fundamental category [G3], which will be later extended to c-spaces. Some new results on c-spaces are already inserted in Proposition 3.3 and Theorem 3.9.

3.1 Directed homotopy

Homotopy in $\mathrm{d}\mathsf{Top}$ is based on the directed interval $\uparrow\!\mathbb{I}$ and the reversor endofunctor $R\colon \mathrm{d}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$.

Inside this theory, a map $a\colon {\uparrow}\mathbb{I} \to X$ is simply called a *path* – or a directed path when we want to stress the difference with the paths of the underlying space UX. Homotopies are represented by maps $\varphi\colon X{\times}{\uparrow}\mathbb{I} \to Y$, defined on the (directed) cylinder $X{\times}{\uparrow}\mathbb{I}$. This works by a complex structure on the interval and the cylinder functor (developed in [G3], Chapters 1 and 4), of which we recall here the initial part.

The first-order structure of the interval in $\mathrm{d}\mathsf{Top}$ consists of four maps: two *faces* (∂^-, ∂^+), a *degeneracy* (e) and a *reflection* (r)

$$\begin{array}{lll} \partial^\alpha\colon \{*\} \rightleftarrows {\uparrow}\mathbb{I} : e, & \qquad r\colon {\uparrow}\mathbb{I} \to {\uparrow}\mathbb{I}^{\mathrm{op}} & \qquad (\alpha = 0, 1), \\ \partial^\alpha(*) = \alpha,\; e(t) = *, & \qquad r(t) = 1 - t. & \end{array} \tag{25}$$

(The same structure works in Top, using the euclidean interval $\mathbb{I}$ and a trivial reversor R, the identity of the category.)

The *cylinder* endofunctor I_d (written as I if it is clear that we are working in $\mathrm{d}\mathsf{Top}$)

$$I_d\colon \mathrm{d}\mathsf{Top} \to \mathrm{d}\mathsf{Top}, \qquad\qquad I_d = -\times{\uparrow}\mathbb{I}, \tag{26}$$

inherits from this structure four natural transformations, with the same names and symbols:

$$\begin{array}{lll} \partial^\alpha\colon 1 \rightleftarrows I : e, & \qquad r\colon IR \to RI & \qquad (\alpha = 0, 1), \\ \partial^\alpha(x) = (x, \alpha), & \qquad e(x, t) = x, & \qquad r(x, t) = (x, 1 - t). \end{array} \tag{27}$$

The component $\partial^\alpha X = X{\times}\partial^\alpha$ on the d-space X is simply written as ∂^α, when this is not ambiguous; similarly for the other natural transformations.

These natural transformations satisfy the identities

$$\begin{array}{ll} e\partial^\alpha = 1\colon \mathrm{id} \to \mathrm{id}\,, & \qquad (RrR)r = 1\colon IR \to IR, \\ (Re)r = eR\colon IR \to R, & \qquad r(\partial^- R) = R\partial^+\colon R \to RI. \end{array} \tag{28}$$

Since $RR = 1$, r is invertible with $r^{-1} = RrR\colon RI \to IR$. Moreover $r(\partial^+ R) = R\partial^-$.

A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$ of d-spaces is defined as a d-map $\varphi\colon IX \to Y$ with faces $\varphi\partial^- = f$ and $\varphi\partial^+ = g$. When we want to distinguish the homotopy from the *map* $IX \to Y$ which represents it, we write the latter as $\hat{\varphi}$. A path in X is the same as a homotopy $a\colon x \to y\colon \{*\} \to X$ between its endpoints, identified to maps $\{*\} \to X$.

Each map $f\colon X \to Y$ has a *trivial* endohomotopy, $e_f\colon f \to f$, represented by $f(eX) = (eY)If\colon IX \to Y$.

Every homotopy $\varphi\colon f \to g\colon X \to Y$ has a *reflected homotopy* between the opposite d-spaces

$$\begin{gathered}\varphi^{\mathrm{op}}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}},\\ (\varphi^{\mathrm{op}})\hat{}\ = R(\hat{\varphi})r\colon IRX \to RIX \to RY,\end{gathered} \tag{29}$$

and $(\varphi^{\mathrm{op}})^{\mathrm{op}} = \varphi,\ (e_f)^{\mathrm{op}} = e_{f^{\mathrm{op}}}$.

There is a *whisker composition* for maps and homotopies

$$\begin{tikzcd} X' \arrow[r, "h"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "\downarrow\varphi"] & Y \arrow[r, "k"] & Y' \end{tikzcd} \tag{30}$$

$$\begin{gathered}k\circ\varphi\circ h\colon kfh \to kgh\colon X' \to Y',\\ (k\circ\varphi\circ h)\hat{} = (k\hat{\varphi})(Ih)\colon IX' \to Y',\end{gathered}$$

which will also be written as $k\varphi h$. This ternary operation satisfies obvious axioms of associativity and identities (cf. [G3], 1.2.3).

3.2 Concatenating paths and homotopies

In dTop the *standard concatenation pushout* – pasting two copies of the d-interval, one after the other – can be realised as $\uparrow\mathbb{I}$ itself, with embeddings c^-, c^+ covering the first or second half of the interval

$$\begin{tikzcd} \{*\} \arrow[r, "\partial^+"] \arrow[d, "\partial^-"'] & \uparrow\mathbb{I} \arrow[d, "c^-"] \\ \uparrow\mathbb{I} \arrow[r, "c^+"'] & \uparrow\mathbb{I} \end{tikzcd} \qquad \begin{aligned} c^-(t) &= t/2,\\ c^+(t) &= (t+1)/2. \end{aligned} \tag{31}$$

Indeed, given two consecutive paths $a, b\colon \uparrow\mathbb{I} \to X$ (with $a\partial^+ = b\partial^-$), their concatenation $a * b$ is a map $\uparrow\mathbb{I} \to X$.

More generally we want to concatenate two consecutive homotopies $\varphi\colon f \to g\colon X \to Y$ and $\psi\colon g \to h\colon X \to Y$, with $\varphi(\partial^+ X) = g = \psi(\partial^- Y)$, defining

$$
\begin{aligned}
&\chi = \varphi * \psi\colon f \to h\colon X \to Y, \\
&\chi(x,t) = \varphi(x, 2t), \qquad \text{for } 0 \leqslant t \leqslant 1/2, \\
&\chi(x,t) = \psi(x, 2t-1), \quad \text{for } 1/2 \leqslant t \leqslant 1.
\end{aligned} \tag{32}
$$

As in Top, we do obtain a map $\chi\colon X \times {\uparrow}\mathbb{I} \to Y$, because of the following proposition.

3.3 Proposition (Concatenating homotopies)

(a) For every d-space X, the functor $X \times -\colon \mathrm{dTop} \to \mathrm{dTop}$ preserves the standard concatenation pushout (31)*, giving the* concatenation pushout of the cylinder functor $I = - \times {\uparrow}\mathbb{I}$

$$
\begin{tikzcd}
X \arrow[r, "\partial^+"] \arrow[d, "\partial^-"'] & X \times {\uparrow}\mathbb{I} \arrow[d, "c^-"] \\
X \times {\uparrow}\mathbb{I} \arrow[r, "c^+"'] & X \times {\uparrow}\mathbb{I}
\end{tikzcd}
\qquad
\begin{aligned}
&c^-(x,t) = (x, t/2), \\
&c^+(x,t) = (x, (t+1)/2).
\end{aligned} \tag{33}
$$

(b) If X is a flexible *c-space, this is also a pushout in* cTop*, so that the functor $X \times -\colon \mathrm{cTop} \to \mathrm{cTop}$ preserves the pushout* (31).

Proof. (a) We copy the proof of [G3], 1.4.9, in a more detailed way that will be used in the next section.

The square (33) becomes a pushout in Top, because $UX \times [0, 1/2]$ and $UX \times [1/2, 1]$ form a finite closed cover of $UX \times \mathbb{I}$, so that each mapping defined on the latter and continuous on these closed parts is continuous.

Consider then the map $\chi\colon UX \times \mathbb{I} \to UY$ obtained by pasting two d-maps $\varphi, \psi\colon X \times {\uparrow}\mathbb{I} \to Y$ on the topological pushout $UX \times \mathbb{I}$, as in (32). We want to prove that χ is a d-map $X \times {\uparrow}\mathbb{I} \to Y$.

Let $\langle a, h\rangle\colon {\uparrow}\mathbb{I} \to IX$ be any d-map, with $a\colon {\uparrow}\mathbb{I} \to X$ and $h\colon {\uparrow}\mathbb{I} \to {\uparrow}\mathbb{I}$.

(i) If the image of h is contained in the first (resp. second half) of $[0, 1]$, then $\chi\langle a, h\rangle$ is directed, because

$$
\begin{aligned}
&\chi(a(t), h(t)) = \varphi(a(t), 2h(t)) \\
&(\text{resp. } \chi(a(t), h(t)) = \varphi(a(t), 2h(t) - 1)),
\end{aligned} \tag{34}
$$

and, in this case, $2h$ (resp. $2h - 1$) is a map $\uparrow\mathbb{I} \to \uparrow\mathbb{I}$ (just an increasing continuous mapping).

(ii) Otherwise, there is some $t_1 \in\ ]0,1[$ such that $h(t_1) = 1/2$, and we can assume that $t_1 = 1/2$ (up to pre-composing with an automorphism of $\uparrow\mathbb{I}$). Now, the path $\chi\langle a,h\rangle\colon \mathbb{I} \to UY$ is directed in Y, because it is the concatenation of the following two directed paths $c_i\colon \uparrow\mathbb{I} \to Y$

$$\begin{aligned} c_1(t) &= \chi(a(t/2), h(t/2)) = \varphi(a(t/2), 2h(t/2)), \\ c_2(t) &= \chi(a((t+1)/2), h((t+1)/2)) \\ &= \psi(a((t+1)/2), 2h((t+1)/2) - 1). \end{aligned} \tag{35}$$

Note that *we are using the splitting property of* a, in the d-space X.

(b) A straightforward consequence: by hypothesis X is a d-space, and the pushout (33) is preserved by the embedding in $\mathrm{c}\mathsf{Top}$. $\square$

3.4 Homotopies of paths

Operating on the standard concatenation pushout (31) with the functors $-\times\uparrow\mathbb{I}$ and $\uparrow\mathbb{I}\times -$, we get the following pushouts

$$\begin{tikzcd} \uparrow\mathbb{I} \arrow[r, "\partial^+\times 1"] \arrow[d, "\partial^-\times 1"'] & \uparrow\mathbb{I}\times\uparrow\mathbb{I} \arrow[d, "\Phi'"] \\ \uparrow\mathbb{I}\times\uparrow\mathbb{I} \arrow[r, "\Psi'"'] & \uparrow\mathbb{I}\times\uparrow\mathbb{I} \end{tikzcd} \qquad \begin{tikzcd} \uparrow\mathbb{I} \arrow[r, "1\times\partial^+"] \arrow[d, "1\times\partial^-"'] & \uparrow\mathbb{I}\times\uparrow\mathbb{I} \arrow[d, "\Phi''"] \\ \uparrow\mathbb{I}\times\uparrow\mathbb{I} \arrow[r, "\Psi''"'] & \uparrow\mathbb{I}\times\uparrow\mathbb{I} \end{tikzcd} \tag{36}$$

that we use now to concatenate double paths, horizontally and vertically.

A *double path* in the d-space X is a d-map $H\colon \uparrow\mathbb{I}^2 \to X$. Its four faces are paths in X, between four points, its vertices

$$\begin{tikzcd} & H\partial_2^+ & \\ H\partial_1^- & \boxed{H} & H\partial_1^+ \\ & H\partial_2^- & \end{tikzcd} \qquad\qquad \begin{tikzcd} t \\ {} \arrow[u] \arrow[r] & s \end{tikzcd} \tag{37}$$

$$\partial_1^\alpha = \partial^\alpha\times\uparrow\mathbb{I}\colon \uparrow\mathbb{I} \to \uparrow\mathbb{I}^2, \qquad \partial_1^\alpha(t) = (\alpha, t),$$
$$\partial_2^\alpha = \uparrow\mathbb{I}\times\partial^\alpha\colon \uparrow\mathbb{I} \to \uparrow\mathbb{I}^2, \qquad \partial_2^\alpha(s) = (s, \alpha).$$

Given two horizontally consecutive double paths $H, K\colon \uparrow\mathbb{I}\times\uparrow\mathbb{I} \to X$ ($H\partial_1^+ = K\partial_1^-$), the left pushout (36) gives a map $H *_1 K\colon \uparrow\mathbb{I}\times\uparrow\mathbb{I} \to X$, the horizontal concatenation of H and K.

Symmetrically, given two double paths $H, L\colon \uparrow\mathbb{I}\times\uparrow\mathbb{I} \to X$ which are vertically consecutive, the right pushout (36) gives the vertical concatenation $H *_2 L\colon \uparrow\mathbb{I}\times\uparrow\mathbb{I} \to X$.

It is evident and well-known that these operations satisfy – even strictly – the middle-four interchange property:

$$(H *_1 K) *_2 (L *_1 M) \;=\; (H *_2 L) *_1 (K *_2 M).$$

In Top this induces the commutativity of π_2 (using double paths with degenerate faces).

A *path-homotopy with fixed endpoints*, denoted as

$$H\colon a \to_2 b\colon x \to y, \tag{38}$$

is a double path $H\colon \uparrow\mathbb{I}^2 \to X$ whose vertical faces $H\partial_1^\alpha$ are trivial loops at two points x, y (constant on the thick edges)

$$\begin{tikzcd} \cdot \arrow[r, dash, "b"] \arrow[dr, phantom, "H"] & \cdot \\ \cdot \arrow[u, dash, "e_x"] \arrow[r, dash, "a"'] & \cdot \arrow[u, dash, "e_y"'] \end{tikzcd} \qquad \begin{array}{l} H(0,t) = x, \quad H(1,t) = y, \\ H(-,0) = a, \quad H(-,1) = b. \end{array} \tag{39}$$

H will also be called a *2-path*, by analogy with the 2-cells of a 2-category. The existence of $H\colon a \to_2 b$ gives a preorder relation $a \preceq_2 b$ consistent with concatenation, because 2-paths are obviously closed under the vertical and horizontal concatenation of double paths.

We now use the *2-equivalence* relation $a' \sim_2 a''$ spanned by the preorder $\preceq_2$: there exists a finite sequence of paths $a' \preceq_2 a_1 \succeq_2 a_2 \ldots \preceq_2 a''$ (between the same endpoints). It is also consistent with concatenation.

A class of paths $[a]$ *up to 2-equivalence* is a class of this equivalence relation. These classes become now the arrows of the fundamental category.

3.5 Definition [G1, G3]

In the *fundamental category* $\uparrow\Pi_1(X)$ of a d-space X:

- the vertices are the points of X,

- the arrows are the equivalence classes $[a]\colon x \to y$ of paths $a\colon x \to y$, up to the 2-equivalence relation $a' \sim_2 a''$ described above (spanned by homotopies with fixed endpoints $a' \to_2 a''$),

- the composition is induced by concatenation of consecutive paths, and the identities are induced by trivial loops

$$[a][b] = [a * b], \qquad 1_x = [e_x]. \tag{40}$$

The operation is well defined, because $\sim_2$ is consistent with path concatenation. Associativity and identities follow from Lemma 3.7, which also implies that a global (increasing) reparametrisation $\rho\colon \mathbb{I} \to \mathbb{I}$ gives $[a] = [a\rho]$.

A d-map $f\colon X \to Y$ produces a functor

$$\begin{gathered} f_* = \uparrow\!\Pi_1(f)\colon \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(Y), \\ f_*(x) = f(x), \qquad f_*[a] = [fa]. \end{gathered} \tag{41}$$

In fact a 2-path $H\colon a' \to_2 a''$ gives a 2-path $fH\colon fa' \to_2 fa''$, and a concatenation $a' * a''$ of paths in X is sent to a concatenation $fa' * fa''$ in Y.

We have thus a functor with values in the category of small categories

$$\uparrow\!\Pi_1\colon \mathrm{d}\mathsf{Top} \to \mathsf{Cat}. \tag{42}$$

The next theorem refers to the directed homotopy structure of Cat, based on the directed interval $\mathbf{2}$ ([G3], 1.2.2, 4.3.2); in this light, a natural transformation is a directed homotopy of functors.

3.6 Theorem (Homotopy invariance, I)

(a) The functor $\uparrow\!\Pi_1$ is homotopy invariant, in the sense that a (directed) homotopy $\varphi\colon f \to g\colon X \to Y$ induces a natural transformation

$$\begin{gathered} \varphi_*\colon f_* \to g_*\colon \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(Y), \\ \varphi_*(x) = [\varphi_x]\colon f(x) \to g(x), \end{gathered} \tag{43}$$

where $\varphi_x = \hat{\varphi}(x \times \underline{i})\colon \{*\} \times \uparrow\!\mathbb{I} \to Y$ is a directed path in Y.

(b) Induction preserves whisker composition: a whisker composition $\psi = k \circ \varphi \circ h\colon kfh \to kgh$ of homotopies and maps (as in (30)*) is taken to the corresponding whisker composition of natural transformations and functors:*

$$\psi_* = k_* \circ \varphi_* \circ h_*\colon k_* f_* h_* \to k_* g_* h_*\colon \uparrow\!\Pi_1(X') \to \uparrow\!\Pi_1(Y'). \tag{44}$$

Proof. (a) Again we write out the proof of [G3], Theorem 3.2.4(c), to extend it later to the present setting.

Let $a\colon x \to x'$ be a path in X, represented by a map $a\colon \uparrow\mathbb{I} \to X$. The homotopy $\varphi\colon X\times\uparrow\mathbb{I} \to Y$ gives a double path φ_a in Y, with the following faces

$$\begin{tikzcd} g(x) \arrow[r, "ga"] & g(x') \\ f(x) \arrow[u, "\varphi_x"] \arrow[r, "fa"'] \arrow[ur, phantom, "\varphi_a"] & f(x') \arrow[u, "\varphi_{x'}"'] \end{tikzcd} \qquad \varphi_a = \hat{\varphi}(a\times\underline{i})\colon \uparrow\mathbb{I}\times\uparrow\mathbb{I} \to Y, \tag{45}$$

where the vertical direction agrees with the present graphic representation of double paths. We have to prove that this double path induces a commutative square in $\uparrow\Pi_1(Y)$: $f_*(a) * \varphi_*(x') = \varphi_*(x) * g_*(a)$.

This comes out nearly for free from the higher structure of the directed interval $\uparrow\mathbb{I}$ (as a *dioid* [G3], 1.1.7), which includes the $\max$ and $\min$ operations g^α (with unit ∂^α)

$$g^\alpha\colon \uparrow\mathbb{I}^2 \to \uparrow\mathbb{I}, \qquad g^-(s,t) = s\vee t, \quad g^+(s,t) = s\wedge t, \tag{46}$$

Working with the double path (45) and the new ones, by horizontal concatenation, we get a double path $(\varphi_x g^+ *_1 \varphi_a) *_1 \varphi_{x'}g^-$ (the parentheses are irrelevant)

$$\begin{tikzcd} f(x) \arrow[r, "\varphi_x"] & g(x) \arrow[r, "ga"] & g(x') \arrow[r, "e_{gx'}"] & g(x') \\ f(x) \arrow[u, "e_{fx}"] \arrow[r, "e_{fx}"'] \arrow[ur, phantom, "\varphi_x g^+"] & f(x) \arrow[u, "\varphi_x"'] \arrow[r, "fa"'] \arrow[ur, phantom, "\varphi_a"] & f(x') \arrow[u, "\varphi_{x'}"] \arrow[r, "\varphi_{x'}"'] \arrow[ur, phantom, "\varphi_{x'}g^-"] & g(x') \arrow[u, "e_{gx'}"'] \end{tikzcd} \tag{47}$$

It is a 2-path from $(e_{fx} * fa) * \varphi_{x'}$ to $(\varphi_x * ga) * e_{gx'}$ in Y, and proves our identity in $\uparrow\Pi_1(Y)$

$$f_*(a) * \varphi_*(x') = [fa][\varphi_{x'}] = [\varphi_x][ga] = \varphi_*(x) * g_*(a).$$

(b) The component $\psi_*(x) = [\psi_x]$ at $x \in X'$ is the 2-equivalence class of the following d-path $kfh(x) \to kgh(x)$ in Y':

$$\psi_x = \hat{\psi}(x \times \underline{i}) = k\hat{\varphi}(Ih)(x \times \underline{i}) = k\hat{\varphi}(h(x) \times \underline{i}) = k\varphi_{hx}.$$

and coincides with the component $(k_* \circ \varphi_* \circ h_*)(x)$. □

3.7 Lemma (Reparametrisation and homotopy)

Let $a\colon x \to y$ be a path in the d-space X.

(a) If $\rho, \sigma\colon \mathbb{I} \to \mathbb{I}$ are global reparametrisations and $\rho \leqslant \sigma$, there exists a 2-path

$$H\colon a\rho \to_2 a\sigma\colon \uparrow\!\mathbb{I} \to X, \qquad H\colon \uparrow\!\mathbb{I}^2 \to X. \tag{48}$$

(b) In particular, using the global reparametrisations (13) *and* (14)*, we get 2-paths of* acceleration *and* associativity*:*

$$e_x * a \to_2 a, \qquad a \to_2 a * e_y, \tag{49}$$

$$a * (b * c) \to_2 a * b * c, \qquad a * b * c \to_2 (a * b) * c. \tag{50}$$

(d) For any global reparametrisation $\rho\colon \mathbb{I} \to \mathbb{I}$ there are global reparametrisations ρ', ρ'' and 2-paths

$$a\rho' \to_2 a \to_2 a\rho'', \qquad a\rho' \to_2 a\rho \to_2 a\rho'', \tag{51}$$

proving that $a \sim_2 a\rho$.

Proof. (a) We build a 2-path $\Phi\colon \rho \to_2 \sigma\colon \uparrow\!\mathbb{I} \to \uparrow\!\mathbb{I}$

$$\Phi\colon \uparrow\!\mathbb{I}^2 \to \uparrow\!\mathbb{I}, \qquad \Phi(s,t) = (1-t)\rho(s) + t\sigma(s), \tag{52}$$

as the affine interpolation from ρ to σ, which is increasing because $\rho \leqslant \sigma$. (This corresponds to a particular case, where a is the identity of $\uparrow\!\mathbb{I}$.)

Then, composing Φ with the path $a\colon \uparrow\!\mathbb{I} \to X$, we get a 2-path

$$a\Phi\colon a\rho \to_2 a\sigma\colon \uparrow\!\mathbb{I} \to X. \tag{53}$$

(b) Follows from (a), using the global reparametrisations $\rho \leqslant \underline{i} \leqslant \sigma$ of (13) and (14).

(c) Follows from (a), using $\rho' = \rho \wedge \underline{i}$ and $\rho'' = \rho \vee \underline{i}$. □

3.8 The cocylinder functor

Coming back to first-order homotopy theory in $\mathrm{d}\mathsf{Top}$, we recall that the cylinder functor $I = - \times {\uparrow}\mathbb{I} \colon \mathrm{d}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$ has a right adjoint, the *cocylinder functor*, or *path functor*

$$P \colon \mathrm{d}\mathsf{Top} \to \mathrm{d}\mathsf{Top}, \qquad P(Y) = Y^{{\uparrow}\mathbb{I}} \qquad (I \dashv P). \tag{54}$$

In fact, the d-interval ${\uparrow}\mathbb{I}$ is exponentiable in $\mathrm{d}\mathsf{Top}$, by Theorem 3.9 (see below). The d-space $P(Y)$ is the set of d-paths $\mathrm{d}\mathsf{Top}({\uparrow}\mathbb{I}, Y)$ equipped with the compact-open topology (induced by the topological path-space $P|Y| = \mathsf{Top}(\mathbb{I}, |Y|)$) and the d-structure described in the statement of 3.9.

A homotopy $\varphi \colon f^- \to f^+ \colon X \to Y$ can thus be equivalently defined

- by a map $\hat{\varphi} \colon X \times {\uparrow}\mathbb{I} \to Y$ with $\hat{\varphi}\partial^\alpha = f^\alpha$, for $\alpha = \pm$,
- by a map $\check{\varphi} \colon X \to Y^{{\uparrow}\mathbb{I}}$ with $\partial^\alpha \check{\varphi} = f^\alpha$ (with respect to the faces $\partial^\alpha \colon P \to 1$ specified below).

The path functor is equipped with a first-order structure dual to that of the cylinder functor, with the same reversor R; it consists of four natural transformations, still called *faces*, *degeneracies* and *reflection*, and denoted by the same symbols (for $a \colon {\uparrow}\mathbb{I} \to X$)

$$\begin{aligned} \partial^\alpha \colon P &\leftrightarrows 1 : e, & \partial^\alpha(a) &= a\partial^\alpha, \qquad e(a) = ae, \\ r \colon RP &\to PR, & r(a) &= (Ra)r \colon {\uparrow}\mathbb{I} \to {\uparrow}\mathbb{I}^{\mathrm{op}} \to X^{\mathrm{op}}. \end{aligned} \tag{55}$$

In $\mathrm{c}\mathsf{Top}$ this part will have a complex reworking: the interval $\mathrm{c}\mathbb{I}$ is not exponentiable, as will be shown in 4.7(b). On the other hand, the interval ${\uparrow}\mathbb{I}$ is still exponentiable in $\mathrm{c}\mathsf{Top}$, as we prove in 3.9(b), and gives rise to a *functor of flexible paths*, that represents flexible homotopies (in (62)).

3.9 Theorem (Exponentiable spaces)

Let K be a d-space on a locally compact support $|K|$.

(a) The d-space K is exponentiable in $\mathrm{d}\mathsf{Top}$. *For every d-space Y*

$$Y^K = \mathrm{d}\mathsf{Top}(K, Y) \subset \mathsf{Top}(|K|, |Y|) \tag{56}$$

is the set of d-maps $K \to Y$, equipped with:

- the compact-open topology (restricted from the topological exponential $|Y|^{|K|}$*)*
- the d-structure where a path $c\colon \mathbb{I} \to |Y^K| \subset |Y|^{|K|}$ *is directed if and only if the corresponding topological map* $\hat{c}\colon \mathbb{I} \times |K| \to |Y|$ *is a d-map* $\uparrow\!\mathbb{I} \times K \to Y$.

(b) The d-space K *is also exponentiable in* $\mathrm{c}\mathsf{Top}$*, extending the definition of the previous internal hom: for every c-space* Y

$$Y^K = \mathrm{c}\mathsf{Top}(K, Y) = \mathrm{d}\mathsf{Top}(K, \mathsf{Fl}\,Y) \subset \mathsf{Top}(|K|, |Y|), \qquad (57)$$

is the set of c-maps $K \to Y$*, with the compact-open topology and the c-structure where a path* $c\colon \mathbb{I} \to |Y^K| \subset |Y|^{|K|}$ *is controlled if and only if the corresponding map* $\hat{c}\colon \mathbb{I} \times |K| \to |Y|$ *is a c-map* $\mathrm{c}\mathbb{I} \times K \to Y$*. Moreover, all the elements of* Y^K *are flexible.*

If Y *is a d-space, we find the same exponential* Y^K*.*

Proof. (a) This is Theorem 1.4.8 of [G3]. We rewrite the proof in a more detailed way, to be used in (b).

In the domain of topological spaces, it is well known that a locally compact space K is exponentiable: the space Y^K is the set of maps $\mathsf{Top}(K, Y)$ with the compact-open topology, and the adjunction consists of the natural bijection

$$\hat{}\colon \mathsf{Top}(X, Y^K) \to \mathsf{Top}(X \times K, Y), \qquad \hat{f}(x, k) = f(x)(k). \qquad (58)$$

Coming back to d-spaces, we verify the axioms (dsp.0)-(dsp.2) for the d-structure of Y^K defined above.

(i) (*Constant paths*) If $c\colon \mathbb{I} \to |Y|^{|K|}$ is constant at the d-map $g\colon K \to Y$, then $\hat{c}$ can be factorised as $gp_2\colon \uparrow\!\mathbb{I} \times K \to K \to Y$, and is a d-map.

(ii) *Concatenation*) Let $c = c_1 * c_2\colon \mathbb{I} \to |Y|^{|K|}$, with $\hat{c}_i\colon \uparrow\!\mathbb{I} \times K \to Y$. By Proposition 3.3(a), the product $- \times K$ preserves the concatenation pushout (31). Therefore $\hat{c}$, as the pasting of $\hat{c}_1, \hat{c}_2$ on this pushout, is a directed map.

(iii) (*Partial reparametrisation*) For $\hat{c}\colon \uparrow\!\mathbb{I} \times K \to Y$ and $h\colon \uparrow\!\mathbb{I} \to \uparrow\!\mathbb{I}$, the map $(ch)\hat{} = \hat{c}(h \times K)$ is directed.

We prove now that (58) restricts to a bijection between $\mathrm{d}\mathsf{Top}(X, Y^K)$ and $\mathrm{d}\mathsf{Top}(X \times K, Y)$.

In fact, we have a chain of equivalent conditions

- $f\colon X \to Y^K$ is a d-map,
- $\forall a \in X^\sharp,\ fa\colon {\uparrow}\mathbb{I} \to Y^K$ is a d-path,
- $\forall a \in X^\sharp,\ (fa)\hat{} = \hat{f}(a\times K)\colon {\uparrow}\mathbb{I}\times K \to X\times K \to Y$ is a d-map,
- $\forall a \in X^\sharp$, $\forall h \in {\uparrow}\mathbb{I}^\sharp$, $\forall b \in K^\sharp$, $\hat{f}\langle ah, b\rangle\colon {\uparrow}\mathbb{I} \to Y$ is a d-map,
- $\hat{f}\colon X\times K \to Y$ is a d-map,

taking into account, for the last equivalence, that $ah\colon {\uparrow}\mathbb{I} \to X$ is an arbitrary d-path.

(b) To verify that the c-structure of Y^K is well formed, point (i) works as above: if $c\colon \mathbb{I} \to Y^K$ is constant at any c-map $g\colon K \to Y$, the associated map $\hat{c}$ can be factorised as $gp_2\colon \mathrm{c}\mathbb{I}\times K \to K \to Y$, and is a c-map. This also proves that all the 'points' of Y^K are flexible.

Point (ii) is also proved as above: we can apply Proposition 3.3(b), since K is a flexible c-space.

Point (iii) is now about *global reparametrisation*. If $h\colon {\uparrow}\mathbb{I} \to {\uparrow}\mathbb{I}$ is a surjective increasing map, $h\times K$ is also surjective and $(ch)\hat{} = \hat{c}(h\times K)$ is controlled.

To prove that (58) restricts to a bijection between $\mathrm{c}\mathsf{Top}(X, Y^K)$ and $\mathrm{c}\mathsf{Top}(X\times K, Y)$ it is sufficient to rewrite the previous chain of equivalences, replacing the interval ${\uparrow}\mathbb{I}$ by $\mathrm{c}\mathbb{I}$ and the prefix 'd' by 'c'.

Finally, if Y is a d-space, the condition that $\hat{c}$ be a c-map $\mathrm{c}\mathbb{I}\times K \to Y$ is equivalent to a d-map ${\uparrow}\mathbb{I}\times K \to Y$, because $(\mathrm{c}\mathbb{I}\times K)\hat{} = {\uparrow}\mathbb{I}\times K$, by Theorem 2.5(b). □

4. Paths and double paths in controlled spaces

We now begin to develop homotopy theory in the category $\mathrm{c}\mathsf{Top}$. We work in a concrete, naive way, to prepare the definition of the fundamental category of a c-space (in the next section). This is achieved by a hybrid theory based on general paths, parametrised on $\mathrm{c}\mathbb{I}$, and their flexible deformations parametrised on ${\uparrow}\mathbb{I}$: in other words, we work with the hybrid square $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I}$.

Loosely speaking, as made precise in 4.5(e), if we work on the standard c-square $\mathrm{c}\mathbb{I}^2$ our homotopies of paths cannot be concatenated; on the other hand, working with the flexible square $\uparrow\mathbb{I}^2$ we would only get the fundamental category of the flexible part of X. But we shall see in Part III that, for a border flexible c-space, one can equivalently use the standard square $\mathrm{c}\mathbb{I}^2$.

X and Y are c-spaces. A controlled path in a c-space is simply called a *path*, or a *general path* if we want to stress that it is not assumed to be flexible; a path in the underlying topological space is called a topological path.

4.1 Paths and homotopies

(a) In $\mathrm{c}\mathsf{Top}$ we want to use both the standard c-interval $\mathrm{c}\mathbb{I}$ and the flexible interval $\uparrow\mathbb{I}$. They have a similar first-order structure (two faces, a degeneracy and a reflection), already examined for $\uparrow\mathbb{I}$ in (25)

$$\begin{aligned} &\partial^\alpha\colon \{*\} \ \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}}\ \mathrm{c}\mathbb{I} : e, \qquad && r\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\mathrm{op}}, \\ &\partial^\alpha\colon \{*\} \ \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}}\ \uparrow\mathbb{I} : e, \qquad && r\colon \uparrow\mathbb{I} \to \uparrow\mathbb{I}^{\mathrm{op}}. \end{aligned} \tag{59}$$

A map $x\colon \{*\} \to X$ is a flexible point in the c-space X; a map $a\colon \mathrm{c}\mathbb{I} \to X$ is a c-path in X, also called a path, or a general path.

A map $a\colon \uparrow\mathbb{I} \to X$ is a flexible path. The latter is *flexibly reversible* if also the reversed topological path $a^\sharp = ar$ is a flexible path of the c-space X; this is equivalent to a c-map $\mathbb{I}^\sim \to X$, defined on the reversible d-interval (whose d-structure is generated by $\underline{i}$ and r, see I.2.4(c)).

(b) A (general) *homotopy* $\varphi\colon f \to g\colon X \to Y$ is a map $\varphi\colon X \times \mathrm{c}\mathbb{I} \to Y$, with faces $f = \varphi(1\times\partial^-)$ and $g = \varphi(1\times\partial^+)$. In particular, a homotopy $a\colon x \to y\colon \{*\} \to X$ is a general path (between flexible points), identifying $\{*\}\times\mathrm{c}\mathbb{I}$ with $\mathrm{c}\mathbb{I}$.

A *flexible homotopy* $\varphi\colon f \to g$ is a map $\varphi\colon X\times\uparrow\mathbb{I} \to Y$ with faces as above. A flexible homotopy $a\colon x \to y\colon \{*\} \to X$ is the same as a flexible path in X. A flexible homotopy $\varphi\colon X\times\uparrow\mathbb{I} \to Y$ is *flexibly reversible* if also the reflected topological homotopy $\varphi^\sharp = \varphi(X\times r)$ is a flexible homotopy.

Given a general (resp. flexible, flexibly reversible) homotopy $\varphi\colon f \to g$, every flexible point $x \in X$ gives a general (resp. flexible, flexibly reversible) path $\varphi(x, -)\colon f(x) \to g(x)$ in Y. A mere point gives a topological path. (According to the structure of X, it can give something more: see 4.4(b).)

(c) The reversor R transforms a general homotopy $\varphi\colon f \to g\colon X \to Y$ into the *reflected* one

$$\begin{gathered} \varphi^{\mathrm{op}}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}, \\ (\varphi^{\mathrm{op}})\hat{} = R(\hat{\varphi})(X^{\mathrm{op}}\times r)\colon X^{\mathrm{op}}\times \mathrm{c}\mathbb{I} \to (X\times \mathrm{c}\mathbb{I})^{\mathrm{op}} \to Y^{\mathrm{op}}. \end{gathered} \tag{60}$$

(As in 3.1, the representative map $IX \to Y$ is written as $\hat{\varphi}$ when it should be distinguished from the homotopy φ.) The flexible case works similarly

$$(\varphi^{\mathrm{op}})\hat{} = R(\hat{\varphi})(X^{\mathrm{op}}\times r)\colon X^{\mathrm{op}}\times {\uparrow}\mathbb{I} \to (X\times {\uparrow}\mathbb{I})^{\mathrm{op}} \to Y^{\mathrm{op}}. \tag{61}$$

(d) General and flexible homotopies have a whisker composition with maps, as in (30). (We shall see that they cannot be concatenated: this is verified in 4.7(b), (c), for the special case $X = \mathrm{c}\mathbb{I}$.)

More formally, we are working with the *standard cylinder functor* I_c and the *flexible cylinder functor* I_F

$$\begin{aligned} &I_c\colon \mathsf{cTop} \to \mathsf{cTop}, &&I_c = -\times \mathrm{c}\mathbb{I}, \\ &I_F\colon \mathsf{cTop} \to \mathsf{cTop}, &&I_F = -\times {\uparrow}\mathbb{I}, \end{aligned} \tag{62}$$

supplying the category cTop with structures that will be investigated in a sequel.

(e) Within flexible c-spaces, both general and flexible homotopies coincide with the homotopies of d-spaces. The second point is obvious, the first comes from Corollary 2.6: if X and Y are flexible, a general homotopy $\varphi\colon X\times \mathrm{c}\mathbb{I} \to Y$ amounts to a map defined on $(X\times \mathrm{c}\mathbb{I})\hat{} = X\times {\uparrow}\mathbb{I}$.

We now want to form the fundamental category of a c-space.

4.2 Concatenation of c-paths

(a) The concatenation of general paths is formalised in a slightly more complex way than for d-spaces (in 3.2), because of the failure of the path-splitting property.

The standard concatenation pushout of c-spaces is realised as $\mathrm{c}\mathbb{J}$, the two-jump structure on the euclidean interval $[0, 1]$ (recalled in 1.3(c))

$$\begin{tikzcd} \{*\} \arrow[r, "\partial^+"] \arrow[d, "\partial^-"'] & \mathrm{c}\mathbb{I} \arrow[d, "c^-"] \\ \mathrm{c}\mathbb{I} \arrow[r, "c^+"'] & \mathrm{c}\mathbb{J} \end{tikzcd} \qquad \begin{aligned} &c^-(t) = t/2, \\ &c^+(t) = (t+1)/2. \end{aligned} \tag{63}$$

Now, given two maps $a', a''\colon \mathrm{c}\mathbb{I} \to X$ such that $a'\partial^+ = a''\partial^-$, we get a map $a\colon \mathrm{c}\mathbb{J} \to X$ such that $ac^- = a'$, $ac^+ = a''$, which 'is' also a path $\mathrm{c}\mathbb{I} \to X$, because $\mathrm{c}\mathbb{I}$ has a finer structure.

More formally, we can introduce the *concatenation map* $\kappa\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J}$, a reshaping, and define $a' * a'' = a\kappa\colon \mathrm{c}\mathbb{I} \to X$.

This procedure is not infrequent in homotopy theory. For instance, chain complexes of abelian groups have a similar behaviour: pasting two copies of the interval (or a cylinder) yields a different object, related to the former by a concatenation map (cf. [G3], Section 4.4, where we mostly work with the cocylinder). The same happens in Cat, whose two-jump interval is the ordinal **3**.

(b) The problem here is that the pushout (63) need *not* be preserved by a functor $X \times -\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$.

Trying to adapt the proof of Proposition 3.3 to the category $\mathrm{c}\mathsf{Top}$ *and* this pushout, we note that:

- case (i) still holds: here it can only occur for a c-map $h\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ constant at 0 (resp. 1), and then $2h$ (resp. $2h - 1$) is the same map,

- case (ii) fails: the argument only works if the path $a\colon \mathrm{c}\mathbb{I} \to X$ can be splitted as $a = a' * a''$.

(c) Finally, we have proved that: *if X is a flexible c-space*, the functor $X\times-$ (or equivalently the functor $-\times X$) preserves the pushout (63), in $\mathrm{c}\mathsf{Top}$.

We shall use the fact that the product by $\uparrow\mathbb{I}$ works, using flexible homotopies of general paths. (In this case the preservation of all colimits already follows from $\uparrow\mathbb{I}$ being exponentiable in $\mathrm{c}\mathsf{Top}$, by Theorem 3.9(b).)

4.3 Double paths and 2-paths

(a) A map $H\colon \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to X$ represents a flexible homotopy between two general paths $H(1\times\partial^\alpha)\colon \mathrm{c}\mathbb{I}\times\{*\} \to X$

$$H\colon a \to b\colon \mathrm{c}\mathbb{I} \to X, \qquad H(-,0) = a, \quad H(-,1) = b. \tag{64}$$

H will be called a *hybrid double path*, as it is parametrised on the hybrid square $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I}$; the latter is less fine than $\mathrm{c}\mathbb{I}^2$, and gives a stronger condition on H.

(b) In particular, a *hybrid 2-path* is a map $H\colon \mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I} \to X$ which is constant on the vertical faces of the square; it will be written as $H\colon a \to_2 b\colon x \to y$, where the c-paths a and b are its horizontal faces: see figure (39).

(c) Marginally, we also consider *general double paths* $H\colon \mathrm{c}\mathbb{I}^2 \to X$; such a map gives a *general 2-path* $H\colon a \to_2 b\colon x \to y$ if the vertical faces $H(\alpha, -)$ are constant.

(d) All these notions extend the corresponding ones for dTop: if X is a d-space

- a general path $\mathrm{c}\mathbb{I} \to X$ is the same as a d-path $\uparrow\!\mathbb{I} \to X$ (since $(\mathrm{c}\mathbb{I})\hat{} = \uparrow\!\mathbb{I}$),

- a hybrid 2-path $\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I} \to X$ is the same as a 2-path $\uparrow\!\mathbb{I}^2 \to X$ (because $(\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I})\hat{} = \uparrow\!\mathbb{I}^2$, by 2.6).

(e) A homotopy $\varphi\colon f \to g\colon X \to Y$ will be said to be ***strict*** if it is constant at each flexible point $x \in X$: the path $\varphi(x, -)\colon f(x) \to g(x)$ is a trivial loop in Y. Then f and g have the same restriction $f_0 = g_0\colon |X|_0 \to |Y|_0$ to the flexible supports.

This notion is of interest if the c-space X has few flexible points, while it gives a trivial homotopy if X is a d-space. In particular, a strict (resp. strict flexible) homotopy between c-paths is the same as a 2-path (resp. hybrid 2-path).

Strict homotopies will be important for border flexible c-spaces, in Part III.

4.4 Complements

Let $H\colon \mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I} \to X$ be a hybrid 2-path between c-paths $a, b\colon x \to y$.

(a) At any $t \in \uparrow\!\mathbb{I}$ (always a flexible point) we get an *intermediate* c-path

$$H_t = H(-, t)\colon \mathrm{c}\mathbb{I} \to X, \qquad\qquad H_t\colon x \to y, \tag{65}$$

that varies continuously (in the path space $X^{\mathbb{I}}$) from $H_0 = a$ to $H_1 = b$. These paths cover $\mathrm{Im}\, H$, in the sense that $\mathrm{Im}\, H = \cup_t \mathrm{Im}\, H_t$. This proves that a hybrid 2-path H between c-paths $x \to y$ satisfies

$$\mathrm{Im}\, H \subset |X(x, y)|, \tag{66}$$

where $X(x, y)$ denotes the c-subspace of X formed by the union of the images of all c-paths $x \to y$ in X.

(b) On the other hand, the only flexible points of $\mathrm{c}\mathbb{I}$ are the endpoints, where we get e_x and e_y. At any other $s \in \mathrm{c}\mathbb{I}$ we get a topological path $H(s, -)\colon a(s) \to b(s)$, which is actually increasing for the *extended path preorder* $x \preceq x'$ of X (determined by the existence of a generated path $x \to x'$ in $\hat{X}$, see I.1.7(c)).

In fact the hybrid double path H 'is' also a d-map $(\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I})\hat{} \to \hat{X}$, and $(\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I})\hat{} = {\uparrow}\mathbb{I}^2$ (by 2.6), so that $H(s, -)$ 'is' a d-map ${\uparrow}\mathbb{I} \to \hat{X}$.

(c) Let x, y be two flexible points of the c-space X. The subspace $X(x, y)$ covered by the c-paths $x \to y$ of X gives a functor ${\uparrow}\Pi_1(X(x, y)) \to {\uparrow}\Pi_1(X)$ which is bijective for the arrows $x \to y$.

Surjectivity is obvious. Injectivity is a trivial consequence of (66): every hybrid 2-path between c-paths $x \to y$ has image in $X(x, y)$.

(d) An easy construction shows that a general 2-path $H\colon \mathrm{c}\mathbb{I}^2 \to X$ between c-paths $x \to y$ also has a continuous family of intermediate paths $Hw_t\colon \mathrm{c}\mathbb{I} \to X$, which vary from $Hw_0 = a * e_y$ to $Hw_1 = e_x * b$, and cover $\mathrm{Im}\, H$. Therefore, property (66) is still satisfied.

4.5 The concatenation of hybrid double paths

Hybrid double paths have a horizontal concatenation $H *_1 K$, and no vertical concatenation.

(a) *Horizontal concatenation*. The concatenation pushout (63) is preserved by the product by ${\uparrow}\mathbb{I}$, a flexible c-space, as we have seen in 4.2(c)

$$\begin{tikzcd}
\mathrm{c}\mathbb{I} \arrow[r, "\partial^+ \times 1"] \arrow[d, "\partial^- \times 1"'] & \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \arrow[d, "c^- \times 1"] \\
\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \arrow[r, "c^+ \times 1"'] & \mathrm{c}\mathbb{J} \times {\uparrow}\mathbb{I}
\end{tikzcd} \qquad (67)$$

Given two horizontally consecutive hybrid double paths $H, K\colon \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to X$, this pushout gives a map $L\colon \mathrm{c}\mathbb{J} \times {\uparrow}\mathbb{I} \to X$. Using the reshaping $\kappa' = \kappa \times \underline{i}\colon \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to \mathrm{c}\mathbb{J} \times {\uparrow}\mathbb{I}$, we define the horizontal pasting of H and K

as the composed map

$$H *_1 K = L\kappa' \colon \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to X. \tag{68}$$

(b) There is no vertical concatenation of hybrid double paths: this is proved in 4.7(b).

(c) The existence of a hybrid 2-path $a \to_2 b$ in X is a reflexive relation (between paths in X, with the same endpoints), consistent with path concatenation by the horizontal concatenation (a).

Transitivity fails, as stated in (b), but this can be overcome: we write as $a' \preceq_2 a''$ the preordering spanned by the previous relation: there exists a finite sequence $a' \to_2 a_1 \to_2 a_2 \ldots \to_2 a''$ of hybrid 2-paths. We write as $a' \sim_2 a''$ the equivalence relation, called *flexible 2-equivalence*, spanned by the latter (or by $\to_2$): there exists a finite sequence

$$a' \quad \longrightarrow_2 \quad a_1 \;\; {}_2\!\longleftarrow \quad a_2 \;\; \ldots \quad \longrightarrow_2 \quad a''$$

of hybrid 2-paths, forward or backward. Both these relations are consistent with concatenation.

A class of paths $[a]$ *up to 2-equivalence* will be a class of this equivalence relation, and an arrow of the fundamental category of X, in 5.1.

(d) The failure of transitivity is not a real problem here: in any case we must use the equivalence relation generated by the relation $a' \to_2 a''$, and it makes little difference whether the latter is transitive or not. (It will make a difference in the general theory of homotopies, where we do not want to miss direction: see Part III.)

On the other hand, the failure of point (a) would have been crucial: the congruence of categories generated by a relation between parallel arrows which is not consistent with composition involves all the objects, and is too 'uncontrolled' to allow non-trivial calculations.

(e) If we only work on $\mathrm{c}\mathbb{I}$, by general homotopies of general paths, horizontal and vertical concatenation both fail (they are equivalent, by symmetry): general double paths, parametrised on $\mathrm{c}\mathbb{I}^2$, are not closed under concatenation. This is proved in 4.7(c).

On the other hand, if we only work on ${\uparrow}\mathbb{I}$, by flexible homotopies of flexible paths, we just get the fundamental category ${\uparrow}\Pi_1(\mathsf{Fl}\,X)$ of the flexible part of X: in fact a map ${\uparrow}\mathbb{I} \to X$ is a d-path in $\mathsf{Fl}\,X$, and a map ${\uparrow}\mathbb{I}^2 \to X$ is a double d-path in $\mathsf{Fl}\,X$.

4.6 Lemma

Let $a\colon x \to y$ be a path in the c-space X.

(a) If $\rho, \sigma\colon \mathbb{I} \to \mathbb{I}$ are global reparametrisations and $\rho \leqslant \sigma$, there exists a hybrid 2-path:

$$a\rho \ \to_2 \ a\sigma. \tag{69}$$

(b) In particular, using the global reparametrisations $\rho, \sigma\colon \mathbb{I} \to \mathbb{I}$ of (13) *and* (14)*, we get hybrid 2-paths of* acceleration *and* associativity*:*

$$e_x * a \ \to_2 \ a, \qquad\qquad a \ \to_2 \ a * e_y, \tag{70}$$

$$a * (b * c) \ \to_2 \ a * b * c, \qquad a * b * c \ \to_2 \ (a * b) * c. \tag{71}$$

(c) For any global reparametrisation $\rho\colon \mathbb{I} \to \mathbb{I}$ there exist hybrid 2-paths

$$a\rho' \ \to_2 \ a \ \to_2 \ a\rho'', \qquad a\rho' \ \to_2 \ a\rho \ \to_2 \ a\rho'' \tag{72}$$

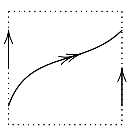
proving that $a \sim_2 a\rho$.

Proof. It is a consequence of Lemma 3.7, once we verify that the increasing interpolation $\Phi\colon \rho \to_2 \sigma$ constructed there

$$\Phi\colon \mathbb{I}^2 \to \mathbb{I}, \qquad \Phi(s,t) = (1-t)\rho(s) + t\sigma(s), \tag{73}$$

is a map $\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to \mathrm{c}\mathbb{I}$.

In fact, the hybrid square $\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I}$ has three kind of controlled paths $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I}$ (see I.2.7(c)), of the form $\langle 0, v\rangle$, or $\langle 1, v\rangle$, or $\langle u, v\rangle$, with increasing functions u, v and u surjective



$\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I}$ (74)

Now the functions $\Phi(0, v(t))$ and $\Phi(1, v(t))$ are constant at 0 and 1, respectively, while $\Phi(u(t), v(t))$ is increasing from 0 to 1. □

4.7 Concatenating double paths

The concatenation of double paths in $\mathrm{d}\mathsf{Top}$ was recalled in 3.4. We end this section by analysing what happens in $\mathrm{c}\mathsf{Top}$ for hybrid, general and flexible double paths. The particular case of 2-paths is dealt with in 4.8.

(a) (*Horizontal concatenation of hybrid double paths*) This case has already been discussed in 4.5(a): we have a concatenation pushout (67), which allows us to define the horizontal concatenation $H *_1 K$ of horizontally consecutive *hybrid double paths*.

(b) (*Vertical concatenation of hybrid double paths*) In the following pushout, P is a c-structure on $\mathbb{I}^2$ generated by the structural maps $1 \times c^{\pm}$

$$\begin{tikzcd} \mathrm{c}\mathbb{I} \arrow[r, "1\times\partial^+"] \arrow[d, "1\times\partial^-"'] & \mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I} \arrow[d, "1\times c^-"] \\ \mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I} \arrow[r, "1\times c^+"'] & P \end{tikzcd} \tag{75}$$

This structure is not $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I}$ (in fact, it is strictly finer): we prove below, in point (e), that the diagonal of the square is not controlled in P. Therefore:

- the interval $\mathrm{c}\mathbb{I}$ is not exponentiable in $\mathrm{c}\mathsf{Top}$, since the functor $\mathrm{c}\mathbb{I}\times -$ does not preserve the concatenation pushout (31) (of the d-interval),

- hybrid double paths are not closed under vertical concatenation (within topological double paths), and flexible homotopies cannot be concatenated in $\mathrm{c}\mathsf{Top}$.

Indeed, the structural maps $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I} \to P$ are vertically consecutive hybrid double paths, and flexible homotopies (of general paths). Their 'topological' vertical concatenation (in Top) is $\mathrm{id}\, P$, which is not a c-map $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I} \to P$.

(c) (*Concatenation of general double paths*) By symmetry it is sufficient to consider the vertical case. The following pushout gives a c-structure Q on the square, different from $\mathrm{c}\mathbb{I}^2$

$$\begin{tikzcd} \mathrm{c}\mathbb{I} \arrow[r, "1\times\partial^+"] \arrow[d, "1\times\partial^-"'] & \mathrm{c}\mathbb{I}^2 \arrow[d, "1\times c^-"] \\ \mathrm{c}\mathbb{I}^2 \arrow[r, "1\times c^+"'] & Q \end{tikzcd} \tag{76}$$

Again, the diagonal of the square is not controlled in Q, which is finer than P in pushout (75), as $\mathrm{c}\mathbb{I}^2$, in the present pushout, is finer than $\mathrm{c}\mathbb{I}\times{\uparrow}\mathbb{I}$.

We also note that the concatenation pushout $\mathrm{c}\mathbb{J}$, in (63), is not preserved by the cylinder functor $-\times\mathrm{c}\mathbb{I}$: in fact, $\mathrm{c}\mathbb{J}\times\mathrm{c}\mathbb{I}$ is distinct from Q, by the same reason.

As a consequence:

- general double paths are not closed under vertical concatenation (within topological double paths),
- general homotopies cannot be concatenated in $\mathrm{c}\mathsf{Top}$.

(d) (*Concatenation of flexible double paths*) Flexible squares have a horizontal and vertical concatenation in $\mathrm{c}\mathsf{Top}$. Indeed a c-map ${\uparrow}\mathbb{I}^2 \to X$ is the same as a d-map ${\uparrow}\mathbb{I}^2 \to \mathsf{Fl}\, X$, and we are just considering d-squares in the flexible parts.

(e) Finally we verify that, in the c-space P of pushout (75), the diagonal map d of $\mathbb{I}^2$ is not a c-path

(77)

In fact any c-path $(0,0) \to (1,1)$ in P, being generated by the c-paths of the lower and upper half, must go through the point $p' = (0, 1/2)$ (as a in the right figure above), or $p'' = (1, 1/2)$ (as b), or both (as c).

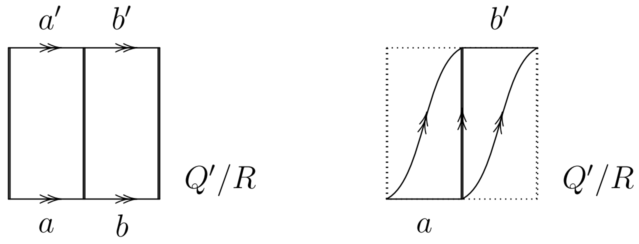
4.8 General 2-paths

We have seen that general double paths are not closed under horizontal or vertical concatenation (within topological double paths), but the argument is not really conclusive for our goal: we have to show that general *2-paths* are not closed under horizontal concatenation.

We start from the following pushout (symmetric with respect to (76))

$$\begin{tikzcd}
\mathrm{c}\mathbb{I} \arrow[r, "\partial^+\times 1"] \arrow[d, "\partial^-\times 1"'] & \mathrm{c}\mathbb{I}^2 \arrow[d, "c^-\times 1"] \\
\mathrm{c}\mathbb{I}^2 \arrow[r, "c^+\times 1"'] & Q'
\end{tikzcd} \qquad (78)$$

We take the quotient Q'/R, modulo the equivalence relation that collapses each of the thick segments to a point, in the left figure below



(79)

Now $c^- \times 1$ and $c^+ \times 1$ induce general 2-paths $a \to_2 a'$ and $b \to_2 b'$, respectively. Topologically, their horizontal concatenation is the canonical projection $p\colon Q' \to Q'/R$, and this is not a c-map $\mathrm{c}\mathbb{I}^2 \to Q'/R$, by an argument similar to that of (77).

The diagonal $d\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^2$ is projected to a path $(0,0) \to (1,1)$ in the support of Q'/R, which is not a c-path: to qualify as such it should admit as a restriction either a or b' or both, as shown in the right figure above.

4.9 Trivial loops

Finally we want to make clear a point concerning the definition of c-spaces in 1.1: if we replace axiom (csp.0) about constant paths with (dsp.0) (all trivial loops are selected), we get a structure – let us say a *$\underline{c}$-space* – where our construction of the fundamental category fails, by the failure of the horizontal concatenation of hybrid double paths, in 4.7(a).

In fact, the standard $\underline{\mathrm{c}}$-interval $\underline{\mathrm{c}}\mathbb{I}$ is the euclidean interval $[0,1]$ with the new structure generated by the identity mapping $\underline{i}$: the selected paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and all the constant ones.

The new hybrid square has two kind of c-paths: the increasing vertical paths $t \mapsto (s_0, v(t))$ and all increasing paths with surjective first projections.

Now the double path $\Phi\colon \mathbb{I}^2 \to \mathbb{I}$ defined in (73) is not a map $\underline{\mathrm{c}}\mathbb{I} \times \uparrow\mathbb{I} \to \underline{\mathrm{c}}\mathbb{I}$: on a vertical path $t \mapsto (s_0, t)$ in $\underline{\mathrm{c}}\mathbb{I} \times \uparrow\mathbb{I}$ we get a path $\Phi(s_0, -)\colon \mathbb{I} \to \mathbb{I}$ from $\rho(s_0)$ to $\sigma(s_0)$, which – generally – is not selected in $\underline{\mathrm{c}}\mathbb{I}$: it is neither surjective nor constant.

5. The fundamental category of controlled spaces

We introduce the fundamental category $\uparrow\!\Pi_1(X)$ of a c-space, extending the fundamental category of a d-space – and therefore the fundamental groupoid $\Pi_1(-)$ of a topological space.

There are also fundamental categories $\uparrow\!\Pi_1(\mathsf{Fl}\,X)$ and $\uparrow\!\Pi_1(\hat{X})$ induced by the coreflector and the reflector of d-spaces, and linked to the previous one by obvious natural transformations. But these 'induced' functors miss the critical aspects of c-spaces.

The invariance of $\uparrow\!\Pi_1$ up to flexible homotopies, proved in Theorem 5.4, will be linked to directed homotopy equivalence of categories [G3], in Part III.

After Theorem 5.3 we already have some of the main ingredients to compute the fundamental category of a c-space; one can read at that point the elementary computations of $\uparrow\!\Pi_1(X)$ in 5.9 (which only marginally rely on Theorem 5.8).

5.1 Definition (The fundamental category of a c-space)

The *fundamental category* $\uparrow\!\Pi_1(X)$ of a controlled space consists of the following items:

- the vertices are the flexible points of X,

- the arrows are the equivalence classes $[a]\colon x \to y$ of general paths $a\colon x \to y$, up to the 2-equivalence relation $a' \sim_2 a''$ spanned by the hybrid 2-paths (see 4.5(c)),

- the composition is induced by concatenation of general paths, and the identities are induced by trivial loops

$$[a][b] = [a * b], \qquad 1_x = [e_x]. \tag{80}$$

The operation is well defined because 2-equivalence is consistent with path concatenation. Identities and associativity follow from hybrid 2-paths of acceleration and associativity, in Lemma 4.6.

A d-map $f\colon X \to Y$ produces a functor between small categories

$$\begin{gathered} f_* = \uparrow\!\Pi_1(f)\colon \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(Y), \\ f_*(x) = f(x), \qquad f_*[a] = [fa] \qquad\qquad (x \in |X|_0), \end{gathered} \tag{81}$$

and we have a functor

$$\uparrow\Pi_1 \colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \tag{82}$$

which extends $\uparrow\Pi_1 \colon \mathrm{d}\mathsf{Top} \to \mathsf{Cat}$, by 4.3(d).

At a flexible point x_0 of the c-space X, the endoarrows of $\uparrow\Pi_1(X)$ form the *fundamental monoid*

$$\uparrow\pi_1(X, x_0) = \uparrow\Pi_1(X)(x_0, x_0). \tag{83}$$

Extending a definition of d-spaces ([G3], 3.2.7), we say that a c-space X is *1-simple* if its fundamental category is a preorder (i.e. the category associated to a preordered set). This means that every hom-set $\uparrow\Pi_1(X)(x, x')$ has at most one arrow: there is one if there is a c-path $x \to x'$ in X, and none otherwise. We shall see that many basic c-spaces are of this kind.

5.2 Induced fundamental categories

We can also use in $\mathrm{c}\mathsf{Top}$ *the homotopy theory of* $\mathrm{d}\mathsf{Top}$, through the reflector $(\hat{\ })\colon \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$ and the coreflector $\mathsf{Fl} : \mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$.

We obtain thus two 'induced' functors, the *fundamental category of generated paths* $\uparrow\Pi_1(\hat{X})$

$$\uparrow\Pi_1(\hat{\ })\colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \qquad X \mapsto \uparrow\Pi_1(\hat{X}), \tag{84}$$

and the *fundamental category of flexible paths* $\uparrow\Pi_1(\mathsf{Fl}\,X)$

$$\uparrow\Pi_1\mathsf{Fl} : \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \qquad X \mapsto \uparrow\Pi_1(\mathsf{Fl}\,X). \tag{85}$$

We recall that the support of $\mathsf{Fl}\,X$ is the subspace $|X|_0$ of flexible points, which are the vertices of the categories $\uparrow\Pi_1(\mathsf{Fl}\,X)$ and $\uparrow\Pi_1(X)$.

The canonical embeddings $\mathsf{Fl}\,X \to X \to \hat{X}$ (the counit and unit of the adjunctions) give two natural transformations, with components

$$\uparrow\Pi_1(\mathsf{Fl}\,X) \longrightarrow \uparrow\Pi_1(X) \longrightarrow \uparrow\Pi_1(\hat{X}). \tag{86}$$

If X is flexible we get two identities. These functors need not be faithful (see Part III), but Theorem 5.3(b) shows an important case where the second is a full embedding.

As we shall see in 5.9, these three categories are strongly different on rigid objects. Both the 'induced' functors $\uparrow\Pi_1(\mathsf{Fl}\,X)$ and $\uparrow\Pi_1(\hat{X})$ are unable to analyse the critical features of c-spaces: the former leaves out the non-flexible paths and the latter makes all paths flexible.

For every c-space X, the underlying topological space $|X|$ has a natural c-structure, and the reshaping $X \to |X|$ gives another natural transformation $\uparrow\Pi_1(X) \to \Pi_1(|X|)$ with values in the ordinary fundamental category of the support. There is thus a commutative diagram

$$\begin{tikzcd} \uparrow\Pi_1(\mathsf{Fl}\,X) \arrow[r] \arrow[d] & \uparrow\Pi_1(X) \arrow[r] \arrow[d] & \uparrow\Pi_1(\hat{X}) \arrow[d] \\ \Pi_1(|X|_0) \arrow[r] & \Pi_1(|X|) \arrow[r, equal] & \Pi_1(|X|) \end{tikzcd} \qquad (87)$$

5.3 Theorem (Weak flexibility and fundamental category)

(a) If X is full in the c-space X' (see 2.1(a)), $\uparrow\Pi_1(X)$ is the full subcategory of $\uparrow\Pi_1(X')$ with vertices in $|X|_0$.

(b) If the c-space X is preflexible (i.e. full in the generated d-space $\hat{X}$, see 2.1(b)) the component $\uparrow\Pi_1(X) \to \uparrow\Pi_1(\hat{X})$ is a full embedding. In other words $\uparrow\Pi_1(X)$ is the full subcategory of $\uparrow\Pi_1(\hat{X})$ with vertices in $|X|_0$.

Proof. It is sufficient to prove (a). Let x, y be two flexible points of the c-space X. The subspace $X(x, y)$ covered by the c-paths $x \to y$ of X coincides with $X'(x, y)$, and the component $\uparrow\Pi_1(X'(x, y)) \to \uparrow\Pi_1(X')$ is bijective for the arrows $x \to y$, by 4.4(c). $\square$

5.4 Theorem (Homotopy invariance, II)

(a) The fundamental category of c-spaces is invariant in the following sense: a flexible *homotopy $\varphi\colon f \to g\colon X \to Y$ of c-spaces induces a natural transformation*

$$\begin{gathered} \varphi_*\colon f_* \to g_*\colon \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y), \\ \varphi_*(x) = [\varphi_x]\colon f(x) \to g(x), \qquad \varphi_x = \hat{\varphi}(x \times \underline{i})\colon \{*\} \times \uparrow\mathbb{I} \to Y, \end{gathered} \qquad (88)$$

where, for every $x \in |X|_0$, φ_x is a flexible path in Y, from $f(x)$ to $g(x)$.

(b) If $\varphi\colon f \to g$ is a strict flexible homotopy (see 4.3(e)), φ_* is the identity of $f_* = g_*$.

(c) A whisker composition $\psi = k{\circ}\varphi{\circ}h\colon kfh \to kgh$ of flexible homotopies and maps is taken to the corresponding whisker composition of natural transformations and functors.

Proof. (a) Naturality works as in Theorem 3.6(a), for d-spaces. We have only to check that, in the present setting, the three double paths used in (47) are maps $\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I} \to Y$, that is *flexible* double paths. (Of course this would not work for a general homotopy.)

First, the double path $\varphi_a = \varphi(a{\times}\underline{i})$ is a composite $\mathrm{c}\mathbb{I}{\times}\uparrow\!\mathbb{I} \to X{\times}\uparrow\!\mathbb{I} \to Y$. Secondly, the double path $\varphi_x g^+$ is a composite $\uparrow\!\mathbb{I}^2 \to \uparrow\!\mathbb{I} \to Y$, because φ_x is a flexible path; but $\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I}$ is finer than $\uparrow\!\mathbb{I}^2$ and we are done. The third double path $\varphi_{x'}g^-$ works in the same way.

(b) An obvious consequence. (c) By the same argument as in 3.6(b). □

5.5 Theorem

(a) If a is a flexibly reversible c-path in the c-space X (see 4.1), the arrows $[a]$ and $[ar]$ are inverse to each other in $\uparrow\!\Pi_1(X)$.

This need not be true if a is merely flexible and reversible: see Part III.

(b) If $\varphi\colon f \to g\colon X \to Y$ is a flexibly reversible homotopy of c-spaces (see 4.1), the natural transformation $\varphi_*\colon f_* \to g_*$ is invertible.

More generally, the same holds if φ is a flexible homotopy of c-spaces and, for every $x \in X$, the c-path $\varphi_x = \hat{\varphi}(x\times\underline{i})\colon \{*\}\times\uparrow\!\mathbb{I} \to Y$ is flexibly reversible in Y.

Proof. The path $a\colon x \to x'$ is supposed to be a map $a\colon \mathbb{I}^\sim \to X$. The double paths $H(s,t) = a(s \wedge t)$ and $K(s,t) = a((1-s) \wedge t)$

$$\begin{array}{c|c|c|c} & a & ar & \\ \hline e_x & H & a\;\; K & e_x \\ \hline & e_x & e_x & \end{array} \qquad \begin{aligned} &H = ag^+, \\ &K = ag^+(r\times 1), \end{aligned} \tag{89}$$

are maps $\mathbb{I}^{\sim 2} \to \mathbb{I}^\sim \to X$, and therefore $\mathrm{c}\mathbb{I}\times\uparrow\!\mathbb{I} \to X$, that is hybrid double paths.

The latter have a horizontal concatenation $H *_1 K$, which is a hybrid 2-path and proves that $(a * ar) \sim_2 e_x$. Applying the same result to the path $ar\colon x' \to x$, we get $(ar * a) \sim_2 e_{x'}$.

(b) An obvious consequence of Theorem 5.4 and the previous point: in the given hypotheses each component $\varphi_*(x) = [\varphi_x]\colon f(x) \to g(x)$ of the natural transformation φ_* is invertible. □

5.6 Theorem (Products and sums)

(a) The functor $\uparrow\Pi_1\colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}$ *preserves arbitrary products and sums.*

(b) A product or sum of a family (X_i) *of 1-simple c-spaces is 1-simple. Conversely, if their product or sum is 1-simple, all* X_i *are also – provided there is no empty factor in the case of the product.*

Proof. (a) One can apply the same argument that works in Top (and dTop). In a cartesian product of c-spaces, paths and hybrid double paths are detected by the cartesian projections. In a sum, they live in one 'component'.

(b) A consequence of (a). In fact, property (b) holds in Cat, interpreting '1-simple' as being a preorder. Moreover, preorders form a reflective and coreflective subcategory of Cat, so that limits and colimits agree.

(In the elementary homotopy theory of categories described in [G3], 1.1.6, the fundamental category of a category is the category itself.)

□

5.7 Covering maps

(a) The theory of covering maps for topological spaces, a classical topic of Homotopy Theory, can be found in many books on Algebraic Topology, like [Ha, HiW, Hu, Sp].

We recall that a map $p\colon X \to Y$ of topological spaces is called a *covering map*, or a covering projection, if every point $y \in Y$ has an open neighbourhood which is 'evenly covered' by p: this means that the preimage $p^{-1}(U)$ is the disjoint union of open subsets of X, each of which is mapped homeomorphically onto U by p. X is the *total space*, or covering space, of p; Y is the *basis*; $F_y = p^{-1}\{y\}$ is the *fibre* at $y \in Y$ – a discrete subspace of X. The map p is a surjective local homeomorphism, and an open map.

A classical example is the *exponential map*, forming the 'universal covering' of the circle (in the complex plane)

$$p\colon \mathbb{R} \to \mathbb{S}^1, \qquad p(t) = e^{2\pi i t}. \tag{90}$$

(b) Here we only need the basic results on the homotopy lifting property of a covering map $p\colon X \to Y$. (For a proof without prerequisites one can see [G4], Theorem 6.2.9.)

(i) For every path $b\colon \mathbb{I} \to Y$ in the basis and every point x_0 in the fibre at $b(0)$, there is a unique lifting $a\colon \mathbb{I} \to X$ (i.e. $pa = b$) starting at x_0.

(ii) For every 2-path $K\colon b \to_2 b'$ (a map $K\colon \mathbb{I}^2 \to Y$) and every point x_0 in the fibre at $b(0) = b'(0)$, there is a unique lifting $H\colon \mathbb{I}^2 \to X$ (i.e. $pH = K$) to a 2-path $a \to_2 a'$ of paths starting at x_0. (The latter are liftings of the paths b, b'.)

Formally, property (i) is a consequence of (ii); nevertheless, stating it is useful.

(c) Now, we say that a map $p\colon X \to Y$ of c-spaces is a *covering map of c-spaces* if it is a covering map of topological spaces and the path-lifting property (i) holds within c-paths: every lifting of a c-path of the basis is a c-path of the total space.

Then the homotopy-lifting property (ii) automatically holds for any hybrid 2-path $K\colon \mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{J} \to Y$, because its topological lifting $H\colon \mathbb{I}^2 \to |X|$ is a c-map $\mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{J} \to X$ if and only if every c-path $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{J} \to Y$ is lifted to a c-path of X.

We also note that the flexible support of X is the union of the fibres of the flexible points of Y.

Our prime examples are the exponential c-maps

$$\begin{aligned} p\colon \mathrm{c}\mathbb{R} \to \mathrm{c}\mathbb{S}^1, &\qquad p(t) = e^{2\pi i t}, \\ p_n\colon \mathrm{c}_n\mathbb{R} \to \mathrm{c}_n\mathbb{S}^1, &\qquad p_n(t) = e^{2\pi i t}. \end{aligned} \tag{91}$$

This will be applied in 5.9 to determine $\uparrow\!\Pi_1(\mathrm{c}\mathbb{S}^1)$ and $\uparrow\!\Pi_1(\mathrm{c}_n\mathbb{S}^1)$.

The second map can be replaced with $p'_n\colon \mathrm{c}\mathbb{R} \to \mathrm{c}_n\mathbb{S}^1,\ p'_n(t) = e^{2n\pi i t}$.

A *covering map of d-spaces* is similarly defined, using d-maps and d-paths.

5.8 Theorem

Let $p\colon X \to Y$ *be a covering map of topological spaces.*

(a) Let $x_0 \in X$, $y_0 = p(x_0)$ *and* $y \in Y$. *The functor* $p_*\colon \Pi_1(X) \to \Pi_1(Y)$ *induces a bijection of sets*

$$p_*\colon \textstyle\sum_{x \in F_y} \Pi_1(X)(x_0, x) \;\to\; \Pi_1(Y)(y_0, y), \qquad (92)$$

defined on the disjoint union of the sets $\Pi_1(X)(x_0, x)$, *for* $x \in F_y$.

In other words, the functor p_* *is surjective on objects and arrows, and faithful. The composition in* $\Pi_1(Y)$ *is determined by that of* $\Pi_1(X)$, *as any pair of composable arrows in* $\Pi_1(Y)$ *can be lifted to a pair of composable arrows in* $\Pi_1(X)$.

(b) For a covering map $p\colon X \to Y$ *of c-spaces (as defined above) we have the same result for the functor* $p_*\colon {\uparrow}\Pi_1(X) \to {\uparrow}\Pi_1(Y)$.

Proof. The lifting properties (i), (ii) of 5.8 imply that the mapping (92) is surjective and injective.

In case (b), where 2-equivalence is *generated* by hybrid 2-paths, one should use again the fact that all c-paths can be lifted. □

5.9 Elementary calculations

Computing the fundamental category will be studied in Part III, but several results can be simply obtained using Theorem 5.3(b) on preflexible c-spaces (and the fundamental category of the generated d-spaces, already computed in [G3]), or Theorem 5.8 on covering maps of c-spaces. In particular, many basic c-spaces are 1-simple, in the sense of 5.1: their fundamental category is a preorder; of course, the controlled circle is not.

The symbols $\mathbf{2}, \mathbf{3}, \mathbf{N}, \mathbf{Z}, \mathbf{R}$ denote ordered sets and the associated categories; the ordered sets 2, 3 and $D|\mathbf{Z}|$ are discrete. $\mathbb{N}$ is the one-object category associated to the additive monoid of the natural numbers.

(a) By Theorem 5.3(b), the fundamental categories of the preflexible c-spaces $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $\mathrm{c}\mathbb{R}$ are the following ordered sets:

$${\uparrow}\Pi_1(\mathrm{c}\mathbb{I}) = \mathbf{2}, \qquad {\uparrow}\Pi_1(\mathrm{c}\mathbb{J}) = \mathbf{3}, \qquad {\uparrow}\Pi_1(\mathrm{c}\mathbb{R}) = \mathbf{Z}. \qquad (93)$$

For these preflexible spaces the functors (86) become inclusions of ordered sets:

$$2 \to \mathbf{2} \to [0,1], \qquad 3 \to \mathbf{3} \to [0,2], \qquad D|\mathbf{Z}| \to \mathbf{Z} \to \mathbf{R}. \tag{94}$$

(b) The fundamental category of the directed circle $\uparrow\mathbb{S}^1$, as described in [G3], 3.2.7(d), is the subcategory of the groupoid $\Pi_1\mathbb{S}^1$ formed of the classes of anticlockwise paths (in $\mathbb{R}^2$). Each monoid $\uparrow\pi_1(\uparrow\mathbb{S}^1, x)$ is isomorphic to the additive monoid $\mathbb{N}$ of natural numbers.

Applying Theorem 5.3(b), the fundamental category of the one-stop circle $\mathrm{c}\mathbb{S}^1$ amounts to the fundamental monoid at the unique flexible point x_0 (the point 1 of the complex plane)

$$\uparrow\Pi_1(\mathrm{c}\mathbb{S}^1)(x_0, x_0) = \uparrow\pi_1(\uparrow\mathbb{S}^1, x_0) = \mathbb{N}. \tag{95}$$

Without using $\uparrow\Pi_1(\uparrow\mathbb{S}^1))$, this is also proved by Theorem 5.8(b) applied to the exponential map $\mathrm{c}\mathbb{R} \to \mathrm{c}\mathbb{S}^1$. Two c-loops a, b in $\mathrm{c}\mathbb{S}^1$ are 2-equivalent if and only if they have the same length $2k\pi$ (in radians), if and only if they both turn k times ($k \geqslant 0$) around the circle, anticlockwise.

(c) More generally, the fundamental category of the preflexible n-stop circle $\mathrm{c}_n\mathbb{S}^1$ (see (11)) is the full subcategory $\mathbf{c}_n$ of the fundamental category of $(\mathrm{c}_n\mathbb{S}^1)\hat{} = \uparrow\mathbb{S}^1 = \uparrow\mathbb{R}/\mathbb{Z}$ on n flexible points, the vertices $[i/n]$ (for $i = 0, ..., n-1$) of an inscribed n-gon.

Again, this result can also be obtained using the covering map of c-spaces $p_n \colon \mathrm{c}_n\mathbb{R} \to \mathrm{c}_n\mathbb{S}$.

(d) Applying Theorem 5.6 on cartesian products, we get the following fundamental categories, which are (partially) ordered sets

$$\begin{aligned} &\uparrow\Pi_1(\mathrm{c}\mathbb{I}^n) = \mathbf{2}^n, && \uparrow\Pi_1(\mathrm{c}\mathbb{J}^n) = \mathbf{3}^n, \\ &\uparrow\Pi_1(\mathrm{c}\mathbb{I} \times \mathrm{c}\mathbb{J}) = \mathbf{2} \times \mathbf{3}, && \uparrow\Pi_1(\mathrm{c}\mathbb{R}^n) = \mathbf{Z}^n. \end{aligned} \tag{96}$$

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