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NOTES ON LIMITS OF ACCESSIBLE CATEGORIES

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Résumé. Soient κ un cardinal régulier, $\lambda < \kappa$ un cardinal infini plus petit, et K une catégorie κ-accessible qui admet les colimites de chaînes indexées par λ. Nous démontrons que diverses constructions catégoriques appliquées à K, comme les équifiers et inserters produisent de nouvelles catégories κ-accessibles E, et que les objets κ-présentables de E admettent une caractérisation naturelle. En particulier, si C est une catégorie κ-petite, alors la catégorie des foncteurs $\mathsf{C} \longrightarrow \mathsf{K}$ est aussi κ-accessible et ses objets κ-présentables sont exactement les foncteurs à valeurs dans la sous-catégorie des objets κ-présentables de K. Nous discutons aussi la préservation de la κ-accessibilité par les pseudo-limites coniques, les limites lax et oplax et les pseudo-limites à poids. Une partie de ces résultats peuvent se retrouver dans une note non-publiée de Ulmer de 1977. Ce travail est motivé par la théorie des modules plats et des faisceaux quasi-cohérents.

Abstract. Let κ be a regular cardinal, $\lambda < \kappa$ be a smaller infinite cardinal, and K be a κ-accessible category where colimits of λ-indexed chains exist. We show that various category-theoretic constructions applied to K, such as the inserter and the equifier, produce κ-accessible categories E again, and the most obvious expected description of the full subcategory of κ-presentable objects in E in terms of κ-presentable objects in K holds true. In particular, if C is a κ-small category, then the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$ is κ-accessible, and its κ-presentable objects are precisely all the functors from C to the κ-presentable objects of K. We proceed to discuss the preservation of κ-accessibility by conical pseudolimits, lax and oplax limits, and weighted pseudolimits. The results of this paper go back to an unpublished 1977 preprint of Ulmer. Our motivation comes from the theory of flat modules and flat quasi-coherent sheaves.

Keywords. κ-directed colimits, κ-presentable objects, κ-accessible categories, limits of categories, products, inserters, equifiers, lax and oplax limits, weighted pseudolimits, diagram categories, flat modules, diagrams and complexes of modules.

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Introduction

Let κ be a cardinal and K be a category such that all the objects of K are κ-filtered colimits of (suitably defined) "objects of small size relative to κ". Suppose E is the category of objects from K or collections of objects from K with a certain additional structure and/or some equations imposed. Is every object of E a κ-filtered colimit of objects whose underlying objects from K have small size relative to κ?

To specify the context of the discussion, let κ be a regular cardinal and K be a κ-accessible category (in the sense of [19, §2.1] or [1, Chapter 2]). Let C be a κ-small category, and let $\mathsf{E} = \mathsf{Fun}(\mathsf{C}, \mathsf{K})$ be the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$. Ideally, one may wish to claim that the category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is κ-accessible and its κ-presentable objects are precisely all the functors $\mathsf{C} \longrightarrow \mathsf{K}_{<\kappa}$, where $\mathsf{K}_{<\kappa}$ is the full subcategory of κ-presentable objects in K. But is it true?

The "ideal" state of affairs described in the previous paragraph was claimed as a general result in a 1988 paper [18, Lemma 5.1]. A general outline of a proof of the lemma was presented in [18]; the details were declared to be "direct calculations" and omitted. A refutation came in the recent preprint [12, Theorem 1.3]. The ideal state of affairs does not hold in general.

The assertions of [12, Theorem 1.3] provide a complete characterization of all small categories C such that the "ideal" statement holds *for all κ-accessible categories* K. All such categories C are essentially κ-small, but being essentially κ-small is *not* enough. The category C needs to be also *well-founded* in the sense of the definition in [12].

But are there some κ-accessible categories K that are better behaved than some other ones, with respect to the question at hand? Another theorem from [12] tells that there are. According to [12, Theorem 1.2], if the category C is κ-small and the category K is locally κ-presentable (in the sense of [10] or [1, Chapter 1]), then the functor category $\mathsf{Fun}(\mathsf{C},\mathsf{K})$ is locally κ-presentable and its full subcategory of κ-presentable objects is $\mathsf{Fun}(\mathsf{C},\mathsf{K}_{<\kappa}) \subset \mathsf{Fun}(\mathsf{C},\mathsf{K})$.

Are there any better behaved κ-accessible categories beyond the locally κ-presentable ones? The present paper purports to answer this question by generalizing the result of [12, Theorem 1.2].

We show that the following much weaker version of local presentability is sufficient to guarantee the "ideal state of affairs": it is enough to assume existence of an infinite cardinal $\lambda < \kappa$ such that colimits of all λ-indexed chains of objects and morphisms exist in K. If this is the case and K is κ-accessible, then for any κ-small category C the category $\mathsf{Fun}(\mathsf{C},\mathsf{K})$ is also κ-accessible, and the κ-presentable objects of $\mathsf{Fun}(\mathsf{C},\mathsf{K})$ are precisely all the functors $\mathsf{C} \longrightarrow \mathsf{K}_{<\kappa}$. This is the result of our Theorem 6.1.

Let us mention that the idea of our condition on a category K involving a pair of cardinals $\lambda < \kappa$ is certainly not new. It appeared in the discussion of *pseudopullbacks* in [6, Proposition 3.1] and [28, Theorem 2.2] (and our arguments in this paper bear some similarity to the one in [6]). The fact that this condition is sufficient for the "ideal" result on accessibility of diagram categories $\mathsf{Fun}(\mathsf{C},\mathsf{K})$ (our Theorem 6.1) seems to be if not quite new, then a "well-forgotten old" discovery, however.

The discussion in the beginning of this introduction suggests that we are also interested in other category-theoretic constructions beyond the categories of functors or diagrams; and indeed we are.

Limits of accessible categories are mentioned in the title of this paper. There are many relevant concepts of limits of categories, the most general ones being the weighted pseudolimits or weighted bilimits [19, §5.1], [13], [5]. All of them can be built from certain elementary building blocks.

We discuss the *Cartesian product* (easy), the *equifier* (a representative case for our techniques), the *inserter* (difficult), and the *pseudopullback* (for which our result is already known in relatively recent literature [6, 28]), as well as the nonadditive and the additive/k-linear diagram categories. The pseudopullbacks and the diagram categories are built from the products, the inserters, and the equifiers.

In fact, all weighted pseudolimits and weighted bilimits can be built from products, inserters, and equifiers, up to category equivalence [13, 5]. Hence the importance of our detailed discussion of the products, the inserters, and the equifiers in the general context of limits of accessible categories.

In all the settings (with the exception of the trivial case of the Cartesian products), our results are very similar. The main assumptions are that κ is a regular cardinal and $\lambda < \kappa$ is a smaller infinite cardinal (so the case of finitely accessible categories, $\kappa = \aleph_0$, is excluded). The category K is assumed to be κ-accessible with colimits of λ-indexed chains. If this is the case, then the category E of (collections of) objects from K with an additional structure satisfying some equations is also κ-accessible (again with colimits of λ-indexed chains), and the κ-presentable objects of E are precisely those whose underlying objects are κ-presentable in K.

We do not dare to speculate on what the author of the paper [18] might have in mind back in 1988, but the proofs of our results seem to follow the general outline suggested in [18, proof of Lemma 5.1]. They are, indeed, "direct calculations" (which, however, get complicated at times).

In fact, our results go back all the way to late 1970s, to an unpublished 1977 preprint of Ulmer [29]. The very concept and terminology of an *accessible category* was only introduced by Makkai and Paré in their 1989 book [19]. Accordingly, the exposition in [29] was written mainly in the generality of locally presentable categories (which had been known since the 1971 book of Gabriel and Ulmer [10]).

The main results of [29] relevant in our context are [29, Theorem 3.8 and Corollary 3.9]. These are stated for locally presentable categories, followed by a remark [29, Remark 3.11(II)] explaining that the assertions are actually valid for some (what we would now call) accessible categories. This work of Ulmer was subsequently taken up and developed in the 1984 dissertation of Bird [4], which was also written in the generality of locally presentable categories. Ulmer's remark [29, Remark 3.11(II)] was not taken up, and apparently remained almost forgotten.

The topic of limits of accessible categories was studied by Makkai and Paré [19, §5.1] using methods which seem to be quite different from those of Ulmer. The Limit Theorem of Makkai and Paré [19, Theorem 5.1.6] claimed that all weighted bilimits of accessible categories are accessible, but offered no cardinality estimate on the accessibility rank. The fact that a tight estimate can be obtained from Ulmer's results was not realized. See our Corollary 9.2.

The present author learned about the existence of Ulmer's preprint from [28, paragraph after Pseudopullback Theorem 2.2], where the knowledge about Ulmer's work is attributed to Porst. Still, no traces of such knowledge can be found in Porst's own earlier paper [22] (cf. [23, Remark 3.2] and [26]). I only got hold of my copy of Ulmer's preprint after the first version of the present paper, with my own detailed proofs of the main results, was already available on the `arXiv`.

Let us explain our motivation now. In terms of the intended applications, we are primarily interested in the "minimal cardinality" case $\kappa = \aleph_1$ and $\lambda = \aleph_0$. The examples we care about arise from flat modules over rings, flat quasi-coherent sheaves over schemes, flat comodules, and flat contramodules.

It is shown in the preprint [27, Theorem 2.4] that the category $X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$ of flat quasi-coherent sheaves on a quasi-compact quasi-separated scheme X is $\aleph_1$-accessible. More genenerally, the same holds for any countably quasi-compact, countably quasi-separated scheme [27, Theorem 3.5]. The $\aleph_1$-presentable objects of $X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$ are the locally countably presentable flat quasi-coherent sheaves, i. e., the quasi-coherent sheaves $\mathcal{F}$ on X such that the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is flat and countably presented for all affine open subschemes $U \subset X$ (equivalently, for the affine open subschemes U_α appearing in some fixed affine open covering $X = \bigcup_\alpha U_\alpha$ of the scheme X). Obviously, all directed colimits, and in particular directed colimits of $\aleph_0$-indexed chains, exist in $\mathsf{K} = X\text{–}\mathsf{Qcoh}_{\mathsf{fl}}$. So the results of this paper are applicable to this category.

The results of [27] were extended to certain noncommuative stacks and noncommutative ind-affine ind-schemes in the preprint [23]. Specifically, let $\mathcal{C}$ be a (coassociative, counital) coring over a noncommutative ring A. According to [23, Theorem 3.1], the category of A-flat left $\mathcal{C}$-comodules $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$ is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of of $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$ are the A-countably presentable A-flat left $\mathcal{C}$-comodules. Once again, it is obvious that all directed colimits exist in $\mathcal{C}\text{–}\mathsf{Comod}_{A\text{-}\mathsf{fl}}$; so the results of the present paper can be applied. There is also a version for flat contramodules over certain topological rings [23, Theorem 10.1], where the results of the present paper are applicable as well.

Some results about constructing A-pure acyclic complexes of A-flat $\mathcal{C}$-comodules as $\aleph_1$-filtered colimits of A-pure acyclic complexes of A-countably presentable A-flat $\mathcal{C}$-comodules are discussed in [23, Section 4]. A contramodule version can be found in [23, Section 11]. The techniques developed in the present paper are used throughout the current (new) versions of the papers [27] and [23]. The same methods are also used in the preprint [25], where accessibility of categories of modules of finite flat dimension and two-sided/F-totally acyclic flat resolutions is discussed, and in the preprint [26], where we discuss local presentability and accessibility ranks of the categories of corings and coalgebras over rings.

In the present paper, we do not go into any details on sheaves, comodules, or contramodules, restricting ourselves to "toy examples" of diagrams and complexes of modules over a noncommutative ring R. It is easy to see that the category of flat left R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is κ-accessible for any regular cardinal κ; the κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ are those flat R-modules that are κ-presentable in the category of arbiratry R-modules $R\text{–}\mathsf{Mod}$. Applying the results of this paper, we obtain descriptions of diagrams of flat modules and pure acyclic complexes of flat modules as directed colimits (recovering, in particular, a weaker version of a result from the papers [9, 21] with very general category-theoretic methods).

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1. Preliminaries

We use the book [1] as the main background reference source on the foundations of the theory of accessible categories.

Let κ be a regular cardinal. We refer to [1, Definition 1.4, Theorem 1.5, Definition 1.13(1), and Remark 1.21] for the discussion of *κ-directed posets* vs. *κ-filtered categories* and, accordingly, *κ-directed* vs. *κ-filtered diagrams* and their colimits.

Let K be a category in which all κ-directed (equivalently, κ-filtered) colimits exist. An object $S \in \mathsf{K}$ is said to be *κ-presentable* [1, Definitions 1.1 and 1.13(2)] if the functor $\mathrm{Hom}_{\mathsf{K}}(S, -): \mathsf{K} \longrightarrow \mathsf{Sets}$ preserves κ-directed colimits. We will denote by $\mathsf{K}_{<\kappa} \subset \mathsf{K}$ the full subcategory of κ-presentable objects in K.

A category K with κ-directed colimits is called *κ-accessible* [1, Definition 2.1] if there is a set S of κ-presentable objects in K such that every object of K is a κ-directed colimit of objects from S. In any κ-accessible category, there is only a set of isomorphism classes of κ-presentable objects; in fact, the κ-presentable objects of K are precisely the retracts of the objects from S [1, Remarks 1.9 and 2.2(4)].

Let K be a category and $\mathsf{S} \subset \mathsf{K}$ be a set of objects. For any object $K \in \mathsf{K}$, the *canonical diagram* [1, Definition 0.4] of morphisms from objects from S into K is indexed by the small indexing category $\Delta = \Delta_{\mathsf{S},K}$ whose objects $v \in \Delta$ are morphisms $v: D_v \longrightarrow K$ into K from objects $D_v \in \mathsf{S}$. A morphism $a: v \longrightarrow w$ in Δ is a morphism $a: D_v \longrightarrow D_w$ in K making the triangular diagram $D_v \longrightarrow D_w \longrightarrow K$ commutative in K. The canonical diagram $D = D_{\mathsf{S},K}: \Delta \longrightarrow K$ takes an object $v \in \Delta$ to the object $D_v \in \mathsf{K}$, and acts on the morphisms in the obvious way.

Lemma 1.1. *Let K be a κ-accessible category and S be a set of representatives of isomorphism classes of κ-presentable objects in K. Then, for every object $K \in \mathsf{K}$, the canonical diagram $D = D_{\mathsf{S},K}$ of morphisms from objects from S into K (or in other words, its indexing category $\Delta = \Delta_{\mathsf{S},K}$) is κ-filtered. The natural morphism $\varinjlim_{v\in\Delta} D_v \longrightarrow K$ is an isomorphism in K.*

Proof. This is [1, Definition 1.23 and Proposition 2.8(i–ii)]. □

Let K be a category with κ-directed colimits and $\mathsf{A} \subset \mathsf{K}$ be a class of objects (full subcategory). Then we denote by $\varinjlim_{(\kappa)} \mathsf{A} \subset \mathsf{K}$ the class of all objects of K that can be obtained as κ-directed colimits of objects from A.

The following proposition is also essentially well-known. In the particular case of finitely accessible ($\kappa = \aleph_0$) additive categories, it was discussed in [17, Proposition 2.1], [8, Section 4.1], and [15, Proposition 5.11]. (The terminology "finitely presented categories" was used in [8, 15] for what are called finitely accessible categories in [1].)

Proposition 1.2. *Let* K *be a* κ*-accessible category and* $\mathsf{S}\subset\mathsf{K}_{<\kappa}$ *be a set of* κ*-presentable objects in* K*. Then the full subcategory* $\varinjlim_{(\kappa)}\mathsf{S}\subset\mathsf{K}$ *is closed under* κ*-directed colimits in* K*. The category* $\varinjlim_{(\kappa)}\mathsf{S}$ *is* κ*-accessible; the full subcategory of all* κ*-presentable objects of* $\varinjlim_{(\kappa)}\mathsf{S}$ *consists of all the retracts of objects from* S *in* K*. An object* $E\in\mathsf{K}$ *belongs to* $\varinjlim_{(\kappa)}\mathsf{S}$ *if and only if, for every* κ*-presentable object* $T\in\mathsf{K}_{<\kappa}$*, every morphism* $T\longrightarrow E$ *in* K *factorizes through an object from* S*.*

Proof. The key assertion is that if an object $E\in\mathsf{K}$ has the property that every morphism $T\longrightarrow K$ into K from an object $T\in\mathsf{K}_{<\kappa}$ factorizes through some object from S, then $E\in\varinjlim_{(\kappa)}\mathsf{S}$. (All the other assertions follow easily from this one.)

Indeed, let T denote a representative set of κ-presentable objects in K. Consider the canonical diagram $C\colon\Delta_{\mathsf{S}}\longrightarrow\mathsf{K}$ of morphisms into E from objects of S and the canonical diagram $D\colon\Delta_{\mathsf{T}}\longrightarrow\mathsf{K}$ of morphisms into E from objects of T. Then we have $E=\varinjlim_{w\in\Delta_{\mathsf{T}}}D_w$ by Lemma 1.1, and we need to show that $E=\varinjlim_{v\in\Delta_{\mathsf{S}}}C_v$. So it remains to check that the natural functor between the index categories $\delta\colon\Delta_{\mathsf{S}}\longrightarrow\Delta_{\mathsf{T}}$ is cofinal in the sense of [1, Section 0.11].

Let $w\colon D_w\longrightarrow E$ be an object of Δ_{T}. Then $D_w\in\mathsf{T}$, and by assumption the morphism v factorizes as $D_w\xrightarrow{a}S\xrightarrow{v}E$ with $S\in\mathsf{S}$. So $v\colon C_v=S\longrightarrow E$ is an object of Δ_{S}, and we have a morphism $a\colon w\longrightarrow\delta(v)$ in Δ_{T}. This proves condition (a) from [1, Section 0.11]. Since the category Δ_{T} is κ-filtered and the functor δ is fully faithful, condition (b) follows automatically. $\square$

Any cardinal λ can be considered as a totally ordered set, which is a particular case of a poset; and any poset I can be viewed as a category (with the elements of I being the objects, and a unique morphism $i\longrightarrow j$ for every pair of objects $i\le j\in I$). A λ*-indexed chain* (of objects and morphisms) in a category K is a functor $\lambda\longrightarrow\mathsf{K}$, where λ is viewed as a category as explained above.

2. Product

The result of this short section is easy and straightforward; it is only included here for the sake of completeness of the exposition. It is essentially a trivial particular case of [12, Theorem 1.3], and also the correct particular case of an erroneous (generally speaking) argument in [1, proof of Proposition 2.67].

Proposition 2.1. *Let κ be a regular cardinal and $(\mathsf{K}_i)_{i\in I}$ be a family of κ-accessible categories. Assume that the cardinality of the indexing set I is smaller than κ. Then the Cartesian product $\mathsf{K} = \prod_{i\in I}\mathsf{K}_i$ is also a κ-accessible category. An object $S \in \mathsf{K}$, $S = (S_i \in \mathsf{K}_i)_{i\in I}$ is κ-presentable in K if and only if all its components S_i are κ-presentable in K_i.*

Proof. The condition that the cardinality of I is smaller than κ (which is missing in [1, proof of Proposition 2.67]) is needed in order to claim that an object $S \in \mathsf{K}$ is κ-presentable whenever its components $S_i \in \mathsf{K}_i$ are κ-presentable for all i. Essentially, this holds because κ-directed colimits commute with κ-small products in the category of sets (cf. [12, Proposition 2.1]). Once this is established, it remains to observe that every object of K is a κ-directed colimit of such objects S, just as [1, proof of Proposition 2.67] tells. Indeed, let $K = (K_i)_{i\in I} \in \mathsf{K}$ be an object and $(\Xi_i)_{i\in I}$ be nonempty κ-filtered categories such that $K_i = \varinjlim_{\xi_i\in\Xi_i} S_{i,\xi_i}$ in K_i for all $i \in I$ with $S_{i,\xi_i} \in (\mathsf{K}_i)_{<\kappa}$. Then $\Xi = \prod_{i\in I}\Xi_i$ is a κ-filtered category and $K = \varinjlim_{\xi\in\Xi} S_\xi$, where $S_\xi = (S_{i,\xi_i})_{i\in I}$ whenever $\xi = (\xi_i)_{i\in I} \in \prod_{i\in I}\Xi_i$. One also needs to use the fact that any retract of an object $S \in \mathsf{K}$ with κ-presentable components S_i is again an object with κ-presentable components. □

3. Equifier

Let κ be a regular cardinal and λ be a smaller infinite cardinal, i. e., $\lambda < \kappa$. Let K and L be κ-accessible categories in which all λ-indexed chains (of objects and morphisms) have colimits. Let $F, G\colon \mathsf{K} \rightrightarrows \mathsf{L}$ be two parallel functors preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functor F takes κ-presentable objects to κ-presentable objects. Let $\phi, \psi\colon F \rightrightarrows G$ be two parallel natural transformations of functors.

Let $\mathsf{E} \subset \mathsf{K}$ be the full subcategory consisting of all objects $E \in \mathsf{K}$ such that $\phi_E = \psi_E$. This construction of the category E is known as the *equifier* [13, Section 4], [5, Section 1], [1, Lemma 2.76].

The aim of this section is to prove the following theorem going back to the unpublished preprint [29, Theorem 3.8, Corollary 3.9, and Remark 3.11(II)].

Theorem 3.1. *In the assumptions above, the equifier category* E *is* κ*-accessible. The* κ*-presentable objects of* E *are precisely all the objects of* E *that are* κ*-presentable as objects of* K.

We start with the obvious observations that κ-directed colimits (as well as colimits of λ-indexed chains) exist in E and are preserved by the inclusion functor $\mathsf{E} \longrightarrow \mathsf{K}$ (because such colimits exist in K and are preserved by the functor F). It follows immediately that any object of E that is κ-presentable in K is also κ-presentable in E. The proof of the theorem is based the following proposition.

Proposition 3.2. *Let* $E \in \mathsf{E}$ *be an object and* $S \in \mathsf{K}_{<\kappa}$ *be a* κ*-presentable object. Then any morphism* $S \longrightarrow E$ *in* K *factorizes through an object* $U \in \mathsf{E} \cap \mathsf{K}_{<\kappa}$.

Proof. Let $E = \varinjlim_{\xi \in \Xi} T_\xi$ be a representation of the object E as a κ-filtered colimit of κ-presentable objects in the category K. Then we have $G(E) = \varinjlim_{\xi \in \Xi} G(T_\xi)$ in L and $F(S)$, $F(T_\xi) \in \mathsf{L}_{<\kappa}$. There exists an index $\xi_0 \in \Xi$ such that the morphism $S \longrightarrow E$ factorizes through the morphism $T_{\xi_0} \longrightarrow E$ in K.

Since $E \in \mathsf{E}$, we have $\phi_E = \psi_E \colon F(E) \longrightarrow G(E)$. Hence the two compositions

$$F(T_{\xi_0}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_0}) \longrightarrow G(E)$$

are equal to each other in L. Since $G(E) = \varinjlim_{\xi \in \Xi} G(T_\xi)$ and $F(T_{\xi_0}) \in \mathsf{L}_{<\kappa}$, it follows that there exists an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the two compositions

$$F(T_{\xi_0}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} G(T_{\xi_0}) \longrightarrow G(T_{\xi_1})$$

are equal to each other in L.

Similarly, there exists an index $\xi_2 \in \Xi$ together with an arrow $\xi_1 \longrightarrow \xi_2$ in Ξ such that the two compositions

$$F(T_{\xi_1}) \underset{\psi}{\overset{\phi}{\rightrightarrows}} G(T_{\xi_1}) \longrightarrow G(T_{\xi_2})$$

are equal to each other, etc.

Proceeding in this way, we construct a λ-indexed chain of indices $\xi_i \in \Xi$ and arrows $\xi_i \longrightarrow \xi_j$ in Ξ for all $0 \le i < j < \lambda$ such that, for all ordinals $0 \le i < \lambda$, the two compositions

$$F(T_{\xi_i}) \underset{\psi}{\overset{\phi}{\rightrightarrows}} G(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$$

are equal to each other in L. Specifically, for a limit ordinal $k < \lambda$, we just pick an index $\xi_k \in \Xi$ and arrows $\xi_i \longrightarrow \xi_k$ in Ξ for all $i < k$ making the triangles $\xi_i \longrightarrow \xi_j \longrightarrow \xi_k$ commutative in Ξ for all $i < j < k$. This can be done, because $k < \kappa$ and the index category Ξ is κ-filtered. For a successor ordinal $k = i + 1 < \lambda$, the same argument as above in this proof provides the desired arrow $\xi_i \longrightarrow \xi_{i+1}$.

After the construction is finished, it remains to put $U = \varinjlim_{i<\lambda} T_{\xi_i}$. We have $U \in \mathsf{K}_{<\kappa}$, since $\lambda < \kappa$ and the class of all κ-presentable objects in a category with κ-directed colimits is closed under those κ-small colimits that exist in the category [1, Proposition 1.16]. We also have $\phi_U = \psi_U$ by construction, since $F(U) = \varinjlim_{i<\lambda} F(T_{\xi_i})$; so $U \in \mathsf{E}$. □

Proof of Theorem 3.1. Combine Propositions 1.2 and 3.2. □

Remark 3.3. In applications of Theorem 3.1, one may be interested in the *joint equifier* of a family of pairs of natural transformations (cf. [1, Remark 2.76]). Let K be a κ-accessible category and $(\mathsf{L}_i)_{i\in I}$ be a family of κ-accessible categories. Let $F_i, G_i\colon \mathsf{K} \rightrightarrows \mathsf{L}_i$ be a family of pairs of parallel functors, all of them preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functors F_i take κ-presentable objects to κ-presentable objects, and that the cardinality of the indexing set I is smaller than κ. Let $\phi_i, \psi_i\colon F_i \rightrightarrows G_i$ be a family of pairs of parallel natural transformations.

Consider the full subcategory $\mathsf{E} \subset \mathsf{K}$ consisting of all objects $E \in \mathsf{K}$ such that $\phi_{i,E} = \psi_{i,E}$ for all $i \in I$. Then the category E is κ-accessible, and the κ-presentable objects of E are precisely all the objects of E that are κ-presentable as objects of K. This assertion can be deduced from Proposition 2.1 and Theorem 3.1 by passing to the Cartesian product category $\mathsf{L} = \prod_{i\in I} \mathsf{L}_i$. The family of functors $F_i\colon \mathsf{K} \longrightarrow \mathsf{L}_i$ defines a functor $F\colon \mathsf{K} \longrightarrow \mathsf{L}$, the family of functors $G_i\colon \mathsf{K} \longrightarrow \mathsf{L}_i$ defines a functor $G\colon \mathsf{K} \longrightarrow \mathsf{L}$, and the family of pairs of natural transformations ϕ_i, $\psi_i\colon F_i \rightrightarrows G_i$ defines a pair of natural transformations ϕ, $\psi\colon F \rightrightarrows G$. It follows from Proposition 2.1 that all the assumptions of Theorem 3.1 are satisfied by the category L and the pair of functors F, G.

4. Inserter

As in Section 3, we consider a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$. Let K and L be κ-accessible categories in which all λ-indexed chains have colimits. Let $F, G\colon \mathsf{K} \rightrightarrows \mathsf{L}$ be two parallel functors preserving κ-directed colimits and colimits of λ-indexed chains; assume further that the functor F takes κ-presentable objects to κ-presentable objects.

Let E be the category of pairs (K, ϕ), where $K \in \mathsf{K}$ is an object and $\phi\colon F(K) \longrightarrow G(K)$ is a morphism in L. This construction of the category E is known as the *inserter* [13, Section 4], [5, Section 1], [19, Section 5.1.1], [1, Section 2.71].

The aim of this section is to prove the following theorem, which also goes back to the unpublished preprint [29, Theorem 3.8, Corollary 3.9, and Remark 3.11(II)].

Theorem 4.1. *In the assumptions above, the inserter category* E *is* κ*-accessible. The* κ*-presentable objects of* E *are precisely all the pairs* (S, ψ) *where* S *is a* κ*-presentable object of* K.

We start with the obvious observations that κ-directed colimits (as well as colimits of λ-indexed chains) exist in E and are preserved by the forgetful functor $\mathsf{E} \longrightarrow \mathsf{K}$ (because such colimits exists in K and are preserved by the functor F).

The proof of the theorem is based on three propositions. It uses the same idea as the proof of Theorem 3.1 above, but the details are much more complicated in the case of Theorem 4.1.

Proposition 4.2. *Let $(S, \psi) \in \mathsf{E}$ be an object such that $S \in \mathsf{K}_{<\kappa}$. Then $(S, \psi) \in \mathsf{E}_{<\kappa}$.*

Proof. The assumptions concerning cardinal λ are not needed for this proposition. Essentially, the assertion holds because κ-directed colimits commute with finite limits in the category of sets (cf. [12, Proposition 2.1]). To be more specific, it helps to observe that, given an object (K, ϕ) in E, the set of morphisms $\mathrm{Hom}_{\mathsf{E}}((S, \psi), (K, \phi))$ is computed as the equalizer of the natural pair of maps

$$\mathrm{Hom}_{\mathsf{K}}(S, K) \underset{f \mapsto G(f) \circ \psi}{\overset{f \mapsto \phi \circ F(f)}{\rightrightarrows}} \mathrm{Hom}_{\mathsf{L}}(F(S), G(K)).$$

Then one needs to use the assumptions that the functor G preserves κ-directed colimits and the functor F takes κ-presentable objects to κ-presentable objects. $\square$

Denote by $\mathsf{E}'_{<\kappa} \subset \mathsf{E}$ the full subcategory formed by all the pairs $(S, \psi) \in \mathsf{E}$ with $S \in \mathsf{K}_{<\kappa}$. By Proposition 4.2, we have $\mathsf{E}'_{<\kappa} \subset \mathsf{E}_{<\kappa}$.

Proposition 4.3. *Let $E = (K, \phi) \in \mathsf{E}$ be an object. Consider the canonical diagram $C = D_E$ of morphisms into E from (representatives of isomorphism classes of) objects $B = (S, \psi) \in \mathsf{E}'_{<\kappa}$, with the indexing category $\Delta = \Delta_E$. Then the indexing category Δ is κ-filtered.*

Proposition 4.4. *In the context of Proposition 4.3, consider also the canonical diagram $D = D_K$ of morphisms into K from (representatives of isomorphism classes of) objects $S \in \mathsf{K}_{<\kappa}$, with the indexing category Δ_K. Then the natural functor between the indexing categories $\Delta_E \longrightarrow \Delta_K$ is cofinal (in the sense of [1, Section 0.11]).*

The proofs of Propositions 4.3 and 4.4 are based on the following lemma.

Lemma 4.5. *Let $E = (K, \phi) \in \mathsf{E}$ be an object, let $S, T \in \mathsf{K}_{<\kappa}$ be κ-presentable objects, and let $\sigma \colon F(S) \longrightarrow G(T)$ be a morphism in L. Let*

$S \longrightarrow T$ and $T \longrightarrow K$ be morphisms in K. Assume that the pentagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T) \arrow[rr] & & G(K) \end{tikzcd}$$

is commutative in L. Then there exists an object $B = (U, \psi) \in \mathsf{E}'_{<\kappa}$ together with a morphism $(U, \psi) \longrightarrow (K, \phi)$ in E and a morphism $T \longrightarrow U$ in K such that the pentagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(U) \arrow[d, "\psi"] \\ G(T) \arrow[rr] & & G(U) \end{tikzcd}$$

is commutative in L and the triangular diagram $T \longrightarrow U \longrightarrow K$ is commutative in K.

Proof. Let $K = \varinjlim_{\xi\in\Xi} T_\xi$ be a representation of the object K as a κ-filtered colimit of κ-presentable objects in the category K. Then we have $G(K) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ in L and $F(S)$, $F(T_\xi) \in \mathsf{L}_{<\kappa}$. There exists an index $\xi_0 \in \Xi$ such that the morphism $T \longrightarrow K$ factorizes through the morphism $T_{\xi_0} \longrightarrow K$ in K. Then the heptagonal diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(T_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T) \arrow[rr] & & G(T_{\xi_0}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L.

Since $G(K) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ and $F(T_{\xi_0}) \in \mathsf{L}_{<\kappa}$, there exists an index $\xi_1 \in \Xi$ such that the composition $F(T_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_1}) \longrightarrow G(K)$ in L:

$$\begin{tikzcd} F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

Moreover, since $G(K) = \varinjlim_{\xi\in\Xi} G(T_\xi)$ and $F(S) \in \mathsf{L}_{<\kappa}$, one can choose the index ξ_1 together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the hexagonal diagram

$$\begin{tikzcd}
F(S) \arrow[r] \arrow[d, "\sigma"'] & F(T) \arrow[r] & F(T_{\xi_0}) \arrow[d, "\psi_0"] \\
G(T) \arrow[r] & G(T_{\xi_0}) \arrow[r] & G(T_{\xi_1})
\end{tikzcd}$$

is commutative in L. Notice that the pentagonal diagram

$$\begin{tikzcd}
F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(T_{\xi_1}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(T_{\xi_1}) \arrow[rr] & & G(K)
\end{tikzcd}$$

is also commutative in L.

Hence one can choose an index $\xi_2 \in \Xi$ together with an arrrow $\xi_1 \longrightarrow \xi_2$ in Ξ such that the composition $F(T_{\xi_1}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_2}) \longrightarrow G(K)$:

$$\begin{tikzcd}
F(T_{\xi_1}) \arrow[r] \arrow[d, "\psi_1"'] & F(K) \arrow[d, "\phi"] \\
G(T_{\xi_2}) \arrow[r] & G(K)
\end{tikzcd}$$

and the square diagram

$$\begin{tikzcd}
F(T_{\xi_0}) \arrow[r] \arrow[d, "\psi_0"'] & F(T_{\xi_1}) \arrow[d, "\psi_1"] \\
G(T_{\xi_1}) \arrow[r] & G(T_{\xi_2})
\end{tikzcd}$$

is commutative in L. Then the pentagonal diagram

$$\begin{tikzcd}
F(T_{\xi_1}) \arrow[r] \arrow[d, "\psi_1"'] & F(T_{\xi_2}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(T_{\xi_2}) \arrow[rr] & & G(K)
\end{tikzcd}$$

is commutative in L.

Proceeding in this way, we construct a λ-indexed chain of indices $\xi_i \in \Xi$ and arrows $\xi_i \longrightarrow \xi_j$ in Ξ for all $0 \le i < j < \lambda$ together with morphisms $\psi_i \colon F(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$ in L such that, for all ordinals $0 \le i < \lambda$, the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L and, for all ordinals $0 \le i < j < \lambda$, the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[d, "\psi_j"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(T_{\xi_{j+1}}) \end{tikzcd}$$

is commutative in L.

Specifically, similarly to the proof of Proposition 3.2, for a limit ordinal $k < \lambda$, we just pick an index $\xi_k \in \Xi$ and arrows $\xi_i \longrightarrow \xi_k$ in Ξ for all $i < k$ making the triangles $\xi_i \longrightarrow \xi_j \longrightarrow \xi_k$ commutative in Ξ for all $i < j < k$. For a successor ordinal $k = j + 1 < \lambda$, we choose an index $\xi_{j+1} \in \Xi$ together with an arrow $\xi_j \longrightarrow \xi_{j+1}$ in Ξ such that the composition $F(T_{\xi_j}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_{j+1}}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(T_{\xi_j}) \arrow[r] \arrow[d, "\psi_j"'] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{j+1}}) \arrow[r] & G(K) \end{tikzcd}$$

and the square diagram

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[d, "\psi_j"] \\ G(T_{\xi_{i+1}}) \arrow[r] & G(T_{\xi_{j+1}}) \end{tikzcd}$$

is commutative in L for all $i < j$. The latter condition can be satisfied because the pentagonal diagrams

$$\begin{tikzcd} F(T_{\xi_i}) \arrow[r] \arrow[d, "\psi_i"'] & F(T_{\xi_j}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(T_{\xi_{i+1}}) \arrow[rr] & & G(K) \end{tikzcd}$$

are commutative in L for all $i < j$ and the index category Ξ is κ-filtered.

After the construction is finished, it remains to put $U = \varinjlim_{i<\lambda} T_{\xi_i}$, and define $\psi\colon F(U) \longrightarrow G(U)$ to be the colimit of the morphisms $\psi_i\colon F(T_{\xi_i}) \longrightarrow G(T_{\xi_{i+1}})$. It is important here that $F(U) = \varinjlim_{i<\lambda} F(T_{\xi_i})$. We have $U \in \mathsf{K}_{<\kappa}$ for the reason explained in the proof of Proposition 3.2. $\square$

Proof of Proposition 4.3. Firstly, let $v_a\colon (S_a, \psi_a) \longrightarrow (K, \phi)$ be a family of morphisms into (K, ϕ) from objects $(S_a, \psi_a) \in \mathsf{E}'_{<\kappa}$, with the set of indices a having cardinality smaller than κ. We need to show that there is a morphism $u\colon (T, \tau) \longrightarrow (K, \phi)$ into (K, ϕ) from an object $(T, \tau) \in \mathsf{E}'_{<\kappa}$ such that all the morphisms v_a factorize through u. For this purpose, choose a representation $K = \varinjlim_{\xi\in\Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$.

Then there exists an index $\xi_0 \in \Xi$ such that all the morphisms $v_a\colon S_a \longrightarrow K$ factorize through the morphism $S_{\xi_0} \longrightarrow K$ in K. The hexagonal diagram

$$\begin{tikzcd} F(S_a) \arrow[r] \arrow[d, "\psi_a"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\ G(S_a) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K) \end{tikzcd}$$

is commutative in L for all indices a. Therefore, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(S_{\xi_1}) \longrightarrow G(K)$:

$$\begin{tikzcd} F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\ G(S_{\xi_1}) \arrow[r] & G(K) \end{tikzcd}$$

and the pentagonal diagrams

$$\begin{tikzcd}
F(S_a) \arrow[rr] \arrow[d, "\psi_a"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] \\
G(S_a) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1})
\end{tikzcd}$$

are commutative in L for all a. Then the pentagonal diagram

$$\begin{tikzcd}
F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(S_{\xi_1}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(S_{\xi_1}) \arrow[rr] & & G(K)
\end{tikzcd}$$

is also commutative in L. It remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and use Lemma 4.5.

Secondly, let $v\colon (P, \pi) \longrightarrow (K, \phi)$ be a morphism into (K, ϕ) from an object $(P, \pi) \in \mathsf{E}'_{<\kappa}$, and let $w_a\colon (R, \rho) \longrightarrow (P, \pi)$ be a family of parallel morphisms into (P, π) from an object $(R, \rho) \in \mathsf{E}'_{<\kappa}$, with the set of indices a having cardinality smaller than κ. Assume that all the morphisms $vw_a\colon (R, \rho) \longrightarrow (K, \phi)$ are equal to each other. We need to show that the morphism $v\colon (P, \pi) \longrightarrow (K, \phi)$ can be factorized as $(P, \pi) \xrightarrow{u} (U, \psi) \longrightarrow (K, \phi)$ in such a way that $(U, \psi) \in \mathsf{E}'_{<\kappa}$ and all the morphisms $uw_a\colon (R, \rho) \longrightarrow (U, \psi)$ are equal to each other.

For this purpose, choose a representation $K = \varinjlim_{\xi\in\Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_0 \in \Xi$ such that the morphism $v\colon P \longrightarrow K$ factorizes through the morphism $S_{\xi_0} \longrightarrow K$ and all the compositions $R \xrightarrow{w_a} P \longrightarrow S_{\xi_0}$ are equal to each other. The hexagonal diagram

$$\begin{tikzcd}
F(P) \arrow[r] \arrow[d, "\pi"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] \\
G(P) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K)
\end{tikzcd}$$

is commutative in L.

Therefore, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(S_{\xi_1}) \longrightarrow G(K)$:

$$\begin{tikzcd}
F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\
G(S_{\xi_1}) \arrow[r] & G(K)
\end{tikzcd}$$

and the pentagonal diagram

$$\begin{tikzcd}
F(P) \arrow[rr] \arrow[d, "\pi"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] \\
G(P) \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1})
\end{tikzcd}$$

is commutative in L. Once again, it remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and refer to Lemma 4.5. $\square$

Proof of Proposition 4.4. Firstly, let $P \longrightarrow K$ be a morphism into K from an object $P \in \mathsf{K}_{<\kappa}$. We need to show that there exists an object $(U, \psi) \in \mathsf{E}'_{<\kappa}$ together with a morphism $(U, \psi) \longrightarrow (K, \phi)$ in E and a morphism $P \longrightarrow U$ in K such that the triangular diagram $P \longrightarrow U \longrightarrow K$ is commutative in K.

For this purpose, choose a representation $K = \varinjlim_{\xi \in \Xi} T_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $T_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_1 \in \Xi$ such that the morphism $P \longrightarrow K$ factorizes through the morphism $T_{\xi_1} \longrightarrow K$ in K and the composition $F(P) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(T_{\xi_1}) \longrightarrow G(K)$ in L:

$$\begin{tikzcd}
F(P) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\
G(T_{\xi_1}) \arrow[r] & G(K)
\end{tikzcd}$$

It remains to put $S = P$ and $T = T_{\xi_1}$, and refer to Lemma 4.5.

Secondly, let (R', ρ') and (R'', ρ'') be two objects of $\mathsf{E}'_{<\kappa}$, let

$$(R', \rho') \longrightarrow (K, \phi) \longleftarrow (R'', \rho'')$$

be two morphisms in E, and let $R' \longleftarrow P \longrightarrow R''$ be two morphisms in K such that the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & K & R'' \arrow[l] \end{tikzcd}$$

is commutative in K. We need to show that there exists an object $(U,\psi) \in \mathsf{E}_{<\kappa}$ together with two morphisms $(R',\rho') \longrightarrow (U,\psi) \longleftarrow (R'',\rho'')$ and a morphism $(U,\psi) \longrightarrow (K,\phi)$ in E such that the two triangular diagrams

$$\begin{tikzcd} (R',\rho') \arrow[r] \arrow[dr] & (U,\psi) \arrow[d] & (R'',\rho'') \arrow[l] \arrow[dl] \\ & (K,\phi) & \end{tikzcd}$$

are commutative in E and the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & U & R'' \arrow[l] \end{tikzcd}$$

is commutative in K.

For this purpose, choose a representation $K = \varinjlim_{\xi\in\Xi} S_\xi$ of the object $K \in \mathsf{K}$ as a κ-filtered colimit of κ-presentable objects $S_\xi \in \mathsf{K}_{<\kappa}$. Then there exists an index $\xi_0 \in \Xi$ such that both the morphisms $R' \longrightarrow K$ and $R'' \longrightarrow K$ factorize through the morphism $S_{\xi_0} \longrightarrow K$ in K and the square diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ R' \arrow[r] & S_{\xi_0} & R'' \arrow[l] \end{tikzcd}$$

is commutative in K. So the whole diagram

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] & \\
R' \arrow[r] \arrow[dr] & S_{\xi_0} \arrow[d] & R'' \arrow[l] \arrow[dl] \\
& K &
\end{tikzcd}$$

is commutative. Then the two hexagonal diagrams

$$\begin{tikzcd}
F(R') \arrow[r] \arrow[d, "\rho'"'] & F(S_{\xi_0}) \arrow[r] & F(K) \arrow[d, "\phi"] & F(S_{\xi_0}) \arrow[l] & F(R'') \arrow[l] \arrow[d, "\rho''"] \\
G(R') \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(K) & G(S_{\xi_0}) \arrow[l] & G(R'') \arrow[l]
\end{tikzcd}$$

are commutative in L.

Hence one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the composition $F(S_{\xi_0}) \longrightarrow F(K) \longrightarrow G(K)$ factorizes through the morphism $G(\xi_1) \longrightarrow G(K)$:

$$\begin{tikzcd}
F(S_{\xi_0}) \arrow[r] \arrow[d, "\sigma"'] & F(K) \arrow[d, "\phi"] \\
G(S_{\xi_1}) \arrow[r] & G(K)
\end{tikzcd}$$

and the two pentagonal diagrams

$$\begin{tikzcd}
F(R') \arrow[rr] \arrow[d, "\rho'"'] & & F(S_{\xi_0}) \arrow[d, "\sigma"] & & F(R'') \arrow[ll] \arrow[d, "\rho''"] \\
G(R') \arrow[r] & G(S_{\xi_0}) \arrow[r] & G(S_{\xi_1}) & G(S_{\xi_0}) \arrow[l] & G(R'') \arrow[l]
\end{tikzcd}$$

are commutative in L. Then it remains to put $S = S_{\xi_0}$ and $T = S_{\xi_1}$, and refer to Lemma 4.5. $\square$

Finally, we are ready to prove the theorem.

Proof of Theorem 4.1. By Proposition 4.2, all the pairs $(S,\psi)\in\mathsf{E}$ with $S\in\mathsf{K}_{<\kappa}$ are κ-presentable in E. It is also clear that the full subcategory $\mathsf{E}'_{<\kappa}$ of all such pairs (S,ψ) is closed under retracts in E (since the full subcategory $\mathsf{K}_{<\kappa}$ is closed under retracts in K). Let $\mathsf{S}\subset\mathsf{E}$ be a set of representatives of isomorphism classes of objects from $\mathsf{E}'_{<\kappa}$. In view of [1, Remarks 1.9 and 2.2(4)] (see the discussion in Section 1), it suffices to prove that, for every object $E\in\mathsf{E}$, the indexing category $\Delta=\Delta_E$ of the canonical diagram $C=D_E$ of morphisms into E from objects of S is κ-filtered, and that $E=\varinjlim_{v\in\Delta}C_v$.

The former assertion is the result of Proposition 4.3. To prove the latter one, notice that by Lemma 1.1 we have $K=\varinjlim_{w\in\Delta_K}D_w$ in K, where $D\colon\Delta_K\longrightarrow\mathsf{K}$ is the canonical diagram of morphisms into K from representatives of isomorphisms classes of objects from $K_{<\kappa}$. Since the natural functor $\delta\colon\Delta\longrightarrow\Delta_K$ between the indexing categories is cofinal by Proposition 4.4, it follows that $K=\varinjlim_{v\in\Delta_E}D_{\delta(v)}$ in K. As the forgetful functor $\mathsf{E}\longrightarrow\mathsf{K}$ is conservative and preserves κ-filtered colimits, we can conclude that $E=\varinjlim_{v\in\Delta_E}C_v$ in E. $\square$

Remark 4.6. In applications of Theorem 4.1, one may be interested in the *joint inserter* of a family of pairs of functors. Let K be a κ-accessible category and $(\mathsf{L}_i)_{i\in I}$ be a family of κ-accessible categories. Let F_i, $G_i\colon\mathsf{K}\rightrightarrows\mathsf{L}_i$ be a family of pairs of parallel functors, all of them preserving κ-directed colimits and colimits of λ-indexed chains. Assume further that the functors F_i take κ-presentable objects to κ-presentable objects, and that the cardinality of the indexing set I is smaller than κ.

Let E be the category of pairs (K,ϕ), where $K\in\mathsf{K}$ is an object and $\phi=(\phi_i)_{i\in I}$ is a family of morphisms $\phi_i\colon F_i(K)\longrightarrow G_i(K)$ in L_i. Then the category E is κ-accessible, and the κ-presentable objects of E are precisely all the pairs (S,ψ) where S is a κ-presentable object of K. This assertion can be deduced from Proposition 2.1 and Theorem 4.1 by passing to the Cartesian product category $\mathsf{L}=\prod_{i\in I}\mathsf{L}_i$. The family of functors $F_i\colon\mathsf{K}\longrightarrow\mathsf{L}_i$ defines a functor $F\colon\mathsf{K}\longrightarrow\mathsf{L}$, and the family of functors $G_i\colon\mathsf{K}\longrightarrow\mathsf{L}_i$ defines a functor $G\colon\mathsf{K}\longrightarrow\mathsf{L}$. It follows from Proposition 2.1 that all the assumptions of Theorem 4.1 are satisfied by the category L and the pair of functors F, G.

5. Pseudopullback

As in Sections 3 and 4, we consider a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$. Let A, B, and C be κ-accessible categories in which all λ-indexed chains (of objects and morphisms) have colimits. Let $\Theta_{\mathsf{A}} : \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_{\mathsf{B}} \colon \mathsf{B} \longrightarrow \mathsf{C}$ be two functors preserving κ-directed colimits and colimits of λ-indexed chains, and taking κ-presentable objects to κ-presentable objects.

Let D be the category of triples (A, B, θ), where $A \in \mathsf{A}$ and $B \in \mathsf{B}$ are objects and $\theta \colon \Theta_{\mathsf{A}}(A) \simeq \Theta_{\mathsf{B}}(B)$ is an isomorphism in C. This construction of the category D is known as the *pseudopullback* [6, Proposition 3.1], [28, Section 2]. The aim of this section is to deduce the following corollary of Theorems 3.1 and 4.1.

Corollary 5.1. *In the assumptions above, the category* D *is* κ*-accessible. The* κ*-presentable objects of* D *are precisely all the triples* (A, B, θ)*, where* A *is a* κ*-presentable object of* A *and* B *is a* κ*-presentable object of* B.

Proof. This result, going back to [29, Remark 3.2(I), Theorem 3.8, Corollary 3.9, and Remark 3.11(II)], appears in the recent literature as [6, Proposition 3.1], [28, Pseudopullback Theorem 2.2]. So we include this proof for the sake of completeness of the exposition and for illustrative purposes.

The point is that the pseudopullback can be constructed as a combination of products, inserters, and equifiers. Put $\mathsf{K} = \mathsf{A} \times \mathsf{B}$ and $\mathsf{L} = \mathsf{C} \times \mathsf{C}$, and consider the following pair of parallel functors $F, G \colon \mathsf{K} \longrightarrow \mathsf{L}$. The functor F takes a pair of objects $(A, B) \in \mathsf{A} \times \mathsf{B}$ to the pair of objects $(\Theta_{\mathsf{A}}(A), \Theta_{\mathsf{B}}(B)) \in \mathsf{C} \times \mathsf{C}$. The functor G takes a pair of objects $(A, B) \in \mathsf{A} \times \mathsf{B}$ to the pair of objects $(\Theta_{\mathsf{B}}(B), \Theta_{\mathsf{A}}(A)) \in \mathsf{C} \times \mathsf{C}$. Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of quadruples $(A, B, \theta', \theta'')$, where $A \in \mathsf{A}$ and $B \in \mathsf{B}$ are objects, while $\theta' \colon \Theta_{\mathsf{A}}(A) \longrightarrow \Theta_{\mathsf{B}}(B)$ and $\theta'' \colon \Theta_{\mathsf{B}}(B) \longrightarrow \Theta_{\mathsf{A}}(A)$ are arbitrary morphisms.

Theorem 4.1 together with Proposition 2.1 tell that the category E is κ-presentable, and the κ-presentable objects of E are precisely all the quadruples $(A, B, \theta', \theta'')$ such that A is a κ-presentable object of A and B is a κ-presentable object of B.

It remains to apply the joint equifier construction of Section 3 and Remark 3.3 to the family of two pairs of parallel natural transformations $(\mathrm{id}, \theta' \circ$

θ'') and $(\mathrm{id},\ \theta'' \circ \theta')$ of functors $\mathsf{E} \longrightarrow \mathsf{C}$ in order to produce the full subcategory $\mathsf{D} \subset \mathsf{E}$ of all quadruples $(A, B, \theta', \theta'')$ such that θ' and θ'' are mutually inverse isomorphisms $\Theta_{\mathsf{A}}(A) \simeq \Theta_{\mathsf{B}}(B)$. Then Theorem 3.1 tells that the category D is κ-accessible and describes its full subcategory of κ-presentable objects, as desired. □

Remark 5.2. Alternatively, one can consider what we would call the *isomorpher* construction for two parallel functors between two categories $P,\ Q\colon \mathsf{H} \rightrightarrows \mathsf{G}$. (It appears in the literature under the name of the "iso-inserter" [13, Section 4], [5, Section 1].) The isomorpher category D consists of all pairs (H, θ), where $H \in \mathsf{H}$ is an object and $\theta\colon P(H) \simeq Q(H)$ is an isomorphism in G.

One can observe that the pseudopullback and the isomorpher constructions are actually equivalent, in the sense that they can be reduced to one another. Given a pair of functors $\Theta_{\mathsf{A}}\colon \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_{\mathsf{B}}\colon \mathsf{B} \longrightarrow \mathsf{C}$, one can put $\mathsf{H} = \mathsf{A} \times \mathsf{B}$ and $\mathsf{G} = \mathsf{C}$, and denote by $P\colon \mathsf{H} \longrightarrow \mathsf{G}$ and $Q\colon \mathsf{H} \longrightarrow \mathsf{G}$ the compositions $\mathsf{A} \times \mathsf{B} \longrightarrow \mathsf{A} \longrightarrow \mathsf{C}$ and $\mathsf{A} \times \mathsf{B} \longrightarrow \mathsf{B} \longrightarrow \mathsf{C}$. In this context, the two constructions of the category D agree.

Conversely, given a pair of parallel functors $P, Q\colon \mathsf{H} \rightrightarrows \mathsf{G}$, put $\mathsf{A} = \mathsf{B} = \mathsf{H}$ and $\mathsf{C} = \mathsf{H} \times \mathsf{G}$. Let the functor $\Theta_{\mathsf{A}}\colon \mathsf{A} \longrightarrow \mathsf{C}$ take an object $H' \in \mathsf{H}$ to the pair $(H', P(H')) \in \mathsf{H} \times \mathsf{G}$ and the functor $\Theta_{\mathsf{B}}\colon \mathsf{B} \longrightarrow \mathsf{C}$ take an object $H'' \in \mathsf{H}$ to the pair $(H'', Q(H'')) \in \mathsf{H} \times \mathsf{G}$. Then an isomorphism $\Theta_{\mathsf{A}}(H') \simeq \Theta_{\mathsf{B}}(H'')$ in C means a pair of isomorphisms $H' \simeq H''$ in H and $P(H') \simeq Q(H'')$ in G. Up to a category equivalence, the datum of two objects $H', H'' \in \mathsf{H}$ endowed with such two isomorphisms is the same thing as a single object $H \in \mathsf{H}$ together with an isomorphism $P(H) \simeq Q(H)$ in G. Thus, in this context, the two constructions of the category D agree as well.

Assume that the categories H and G are κ-accessible with colimits of λ-indexed chains (for a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$). Assume further that the functors F and G preserve κ-directed colimits and colimits of λ-indexed chains, and that they take κ-presentable objects to κ-presentable objects. Then it follows from Proposition 2.1 and Corollary 5.1 that the isomorpher category D is κ-accessible, and the κ-presentable objects of D are precisely all the pairs (H, θ) with $H \in \mathsf{H}_{<\kappa}$.

6. Diagram categories

In this section, we discuss two constructions: the category of functors $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ and the category of k-linear functors $\mathsf{Fun}_k(\mathsf{A}, \mathsf{K})$. The former one is of interest to the general category theory, while the latter one is relevant for additive category theory, module theory, complexes in additive categories, etc.

Let us start with the nonadditive case. Given a small category C and a category K, we denote by $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ the category of functors $\mathsf{C} \longrightarrow \mathsf{K}$.

Recall that a category K is called *locally κ-presentable* [1, Definitions 1.9 and 1.17] if K is κ-accessible and all colimits exist in K. The following theorem is a generalization of [12, Theorem 1.2] from the case of locally κ-presentable categories to the case of κ-accessible categories with colimits of λ-indexed chains (for some fixed infinite cardinal $\lambda < \kappa$). It is also a correct version of [18, Lemma 5.1] (which was shown to be erroneous in full generality in [12, Theorem 1.3]).

A category C is said to be *κ-small* if the cardinality of the set of all objects and morphisms in C is smaller than κ.

Theorem 6.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let C be a κ-small category. Let K be a κ-accessible category in which all λ-indexed chains (of objects and morphisms) have colimits. Then the category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is κ-accessible. The full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{K}_{<\kappa})$ is precisely the full subcategory of all κ-presentable objects in $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$.*

Proof. Similarly to the proof Corollary 5.1, the point is that the diagram category can be constructed as a combination of products, inserters, and equifiers. Let $\mathsf{K}' = \prod_{c \in \mathsf{C}} \mathsf{K}$ be the Cartesian product of copies of the category K indexed by the objects of the category C, and let $\mathsf{L}' = \prod_{(c \to d) \in \mathsf{C}} \mathsf{K}$ be the similar product of copies of K indexed by the morphisms of the category C. Proposition 2.1 tells that the categories K' and L' are κ-accessible, and describes their full subcategories of κ-presentable objects.

Define a pair of parallel functors $F, G \colon \mathsf{K}' \longrightarrow \mathsf{L}'$ as follows. The functor F assigns to a collection of objects $(K_c \in \mathsf{K})_{c \in \mathsf{C}} \in \mathsf{K}'$ the collection of objects $(L_{c \to d})_{(c \to d) \in \mathsf{C}} \in \mathsf{L}'$ given by the rules $L_{c \to d} = K_c$ for any morphism $c \longrightarrow d$ in C. Similarly, the functor G assigns to a collection of objects $(K_c \in \mathsf{K})_{c \in \mathsf{C}} \in \mathsf{K}'$ the collection of objects $(L_{c \to d})_{(c \to d) \in \mathsf{C}} \in \mathsf{L}'$ given by the rules $L_{c \to d} = K_d$ for any morphism $c \longrightarrow d$ in C.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of all "nonmultiplicative functors" $\mathsf{C} \longrightarrow \mathsf{K}$. An object $E \in \mathsf{E}$ is a rule assigning to every object $c \in \mathsf{C}$ an object $E_c \in \mathsf{K}$ and to every morphism $c \longrightarrow d$ in C a morphism $E_c \longrightarrow E_d$ in K. The conditions of compatibility with the compositions of morphisms and with the identity morphisms are *not* imposed. Morphisms of "nonmultiplicative functors" (i. e., the morphisms in E) are similar to the usual morphisms of functors; so the desired functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$ is a full subcategory in E.

Theorem 4.1 tells that the category E is κ-accessible, and describes its full subcategory of κ-presentable objects. Now the desired full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{K}) \subset \mathsf{E}$ can be produced as a joint equifier category, as in Section 3 and Remark 3.3. There are two kinds of pairs of parallel natural transformations to be equified.

Firstly, for every composable pair of morphisms $b \longrightarrow c \longrightarrow d$ in C, we have a pair of parallel functors $F_{b\to c\to d}$, $G_{b\to c\to d}\colon \mathsf{E} \rightrightarrows \mathsf{K}$ and a pair of parallel natural transformations $\phi_{b\to c\to d}$, $\psi_{b\to c\to d}\colon F_{b\to c\to d} \rightrightarrows G_{b\to c\to d}$. The functor $F_{b\to c\to d}$ takes an object $E \in \mathsf{E}$ to the object $E_b \in \mathsf{K}$, and the functor $G_{b\to c\to d}$ takes an object $E \in \mathsf{E}$ to the object $E_d \in \mathsf{K}$. The natural transformation $\phi_{b\to c\to d}$ acts by the composition of the morphisms $E_b \longrightarrow E_c \longrightarrow E_d$ in K assigned to the morphisms $b \longrightarrow c$ and $c \longrightarrow d$ by the datum of the object E. The natural transformation $\psi_{b\to c\to d}$ acts by the morphism $E_b \longrightarrow E_d$ assigned to the composition of the morphisms $b \longrightarrow c \longrightarrow d$ in C by the datum of the object E.

Secondly, for every object $c \in \mathsf{C}$, we have a pair of parallel functors $F_c = G_c\colon \mathsf{E} \longrightarrow \mathsf{K}$ and a pair of parallel natural transformations ϕ_c, $\psi_c\colon F_c \rightrightarrows G_c$. The functor $F_c = G_c$ takes an object $E \in \mathsf{E}$ to the object $E_c \in \mathsf{K}$. The natural transformation ϕ_c acts by the morphism $E_c \longrightarrow E_c$ in K assigned to the identity morphism id_c in C by the datum of the object E; while ψ_c is the identity natural transformation.

The resulting joint equifier is the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{K})$. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. $\square$

Now let k be a commutative ring. A *k-linear category* A is a category enriched in k-modules. This means that, for any two objects a and $b \in \mathsf{A}$, the set of morphisms $\mathrm{Hom}_{\mathsf{A}}(a, b)$ is a k-module, and the composition maps

$\mathrm{Hom}_{\mathsf{A}}(b,c) \times \mathrm{Hom}_{\mathsf{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(a,c)$ are k-bilinear.

Suppose given a set of objects a and, for every pair of objects a, b, a *generating set* of morphisms $\mathrm{Gen}(a,b)$. Then one can construct the k-linear category B on the given set of objects *freely generated* by the given generating sets of morphisms. For every pair of objects a, b, the free k-module $\mathrm{Hom}_{\mathsf{B}}(a,b)$ has a basis consisting of all the formal compositions $g_n \cdots g_1$, $n \geq 0$, where $g_i \in \mathrm{Gen}(c_i, c_{i+1}),\ c_1 = a,\ c_{n+1} = b$.

Furthermore, suppose given a *set of defining relations* $\mathrm{Rel}(a,b) \subset \mathrm{Hom}_{\mathsf{B}}(a,b)$ for every pair of objects a, b. Then one can construct the two-sided ideal of morphisms $\mathsf{J} \subset \mathsf{B}$ generated by all the relations, and pass to the k-linear quotient category $\mathsf{A} = \mathsf{B}/\mathsf{J}$ by the ideal J.

Abusing terminology, we will say that a k-linear category A is *κ-presented* if it has the form $\mathsf{A} = \mathsf{B}/\mathsf{J}$ as per the construction above, where the set of objects $\{a\}$, the set of all generators $\coprod_{a,b} \mathrm{Gen}(a,b)$, and the set of all relations $\coprod_{a,b} \mathrm{Rel}(a,b)$ all have the cardinalities smaller than κ. In another terminology, one could say that A is “the path category of a κ-small quiver with a κ-small set of relations”.

A k-linear category K is said to be *κ-accessible* if it is κ-accessible as an abstract category. Given a small k-linear category A and a k-linear category K, we denote by $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ the (k-linear) category of k-linear functors $\mathsf{A} \longrightarrow \mathsf{K}$. The following theorem is a k-linear version of Theorem 6.1.

Theorem 6.2. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let k be a commutative ring, let A be a κ-presented k-linear category, and let K be a κ-accessible k-linear category in which all λ-indexed chains have colimits. Then the category $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ is κ-accessible. The full subcategory $\mathsf{Fun}_k(\mathsf{A},\mathsf{K}_{<\kappa})$ is precisely the full subcategory of all κ-presentable objects in $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$.*

Proof. The argument is similar to the proof of Theorem 6.1, with the only difference that one works with the generating morphisms and defining relations in A instead of all morphisms and all compositions in C. Let $\mathsf{K}' = \prod_{a\in\mathsf{A}} \mathsf{K}$ be the Cartesian product of copies of the category K indexed by the objects of the category A, and let $\mathsf{L}' = \prod_{a,b\in\mathsf{A}} \prod_{(a\to b)\in\mathrm{Gen}(a,b)} \mathsf{K}$ be the similar product of copies of K indexed by the set of generating morphisms $\coprod_{a,b} \mathrm{Gen}(a,b)$. Proposition 2.1 tells that the categories K' and L' are κ-accessible, and describes their full subcategories of κ-presentable objects.

Define a pair of parallel functors $F, G\colon \mathsf{K}' \longrightarrow \mathsf{L}'$ as follows. The functor F assigns to a collection of objects $(K_a \in \mathsf{K})_{a\in\mathsf{A}} \in \mathsf{K}'$ the collection of objects $(L_{a\to b})_{(a\to b)\in\mathrm{Gen}(a,b),\,a,b\in\mathsf{A}} \in \mathsf{L}'$ given by the rules $L_{a\to b} = K_a$ for any generating morphism $(a \to b) \in \mathrm{Gen}(a,b)$. Similarly, the functor G assigns to a collection of objects $(K_a \in \mathsf{K})_{a\in\mathsf{A}} \in \mathsf{K}'$ the collection of objects $(L_{a\to b})_{(a\to b)\in\mathrm{Gen}(a,b),\,a,b\in\mathsf{A}} \in \mathsf{L}'$ given by the rules $L_{a\to b} = K_b$ for any generating morphism $(a \to b) \in \mathrm{Gen}(a,b)$.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is naturally equivalent to the category $\mathsf{Fun}_k(\mathsf{B},\mathsf{K})$, where B is the "path category of the quiver without relations" constructed in the discussion preceding the formulation of the theorem. Theorem 4.1 tells that the category E is κ-accessible, and defines its full subcategory of κ-presentable objects. The category $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$ we are interested in is a full subcategory in $\mathsf{E} = \mathsf{Fun}_k(\mathsf{B},\mathsf{K})$ consisting of all the "quiver representations in K for which the relations are satisfied". The full subcategory $\mathsf{Fun}_k(\mathsf{A},\mathsf{K}) \subset \mathsf{Fun}_k(\mathsf{B},\mathsf{K})$ can be produced as a joint equifier category, as in Section 3 and Remark 3.3.

The pairs of parallel natural transformations to be equified are indexed by elements of the set of defining relations $\coprod_{a,b}\mathrm{Rel}(a,b)$. Given a defining relation $r \in \mathrm{Rel}(a,b)$, we have a pair of parallel functors F_r, $G_r\colon \mathsf{E} \rightrightarrows \mathsf{K}$ and a pair of natural transformations ϕ_r, $\psi_r\colon F_r \rightrightarrows G_r$. The functor $F_r\colon \mathsf{Fun}_k(\mathsf{B},\mathsf{K}) \longrightarrow \mathsf{K}$ takes a functor $E\colon \mathsf{B} \longrightarrow \mathsf{K}$ to the object $E(a) \in \mathsf{K}$, and the functor G_r takes the functor E to the object $E(b) \in \mathsf{K}$. The natural transformation ϕ_r acts by the morphism $E(r)\colon E(a) \longrightarrow E(b)$. The natural transformation ψ_r acts by the zero morphism $0\colon E(a) \longrightarrow E(b)$ in the k-linear category K.

The resulting joint equifier is the category of k-linear functors $\mathsf{Fun}_k(\mathsf{A},\mathsf{K})$. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. $\square$

7. Brief preliminaries on 2-categories

The aim of this section is to provide a very brief and mostly terminological preliminary discussion for the purposes of the next two Sections 8–9. The reader can find the details by following the references.

Throughout the three sections, for the most part we adopt the policy of benign neglect with respect to set-theoretical issues of size (i. e., the distinction between sets and classes). When specific restrictions on the size matter, we mention them.

In the terminology of higher category theory, the prefix “2-” means strict concepts, while the prefix “bi-” refers to relaxed ones. So 2-categories are strict, while bicategories are relaxed [3].

A 2-*category* is a category enriched in the category of categories Cat (with the monoidal structure on Cat given by the Cartesian product) [14]. In particular, there is the important 2-*category of categories* $\mathbf{Cat}$: categories are the objects, functors are the 1-cells, natural transformations are the 2-cells.

In the terminology of the bicategory theory, one speaks of *morphisms of bicategories* (which are multiplicative and unital on 1-cells up to coherent families of 2-cells) or *homomorphisms of bicategories* (which are multiplicative and unital on 1-cells up to coherent families of *invertible* 2-cells) [3, Section 4]. Even when one is only interested in 2-categories, the notion of a 2-functor may be too strict, and one may want to relax it by considering morphisms of 2-categories (known as *lax functors*), or homomorphisms of 2-categories (known as *pseudofunctors*).

Let Γ and Δ be two 2-categories. Then 2-functors $\Gamma \longrightarrow \Delta$ form a 2-category $[\Gamma, \Delta]$. The objects of $[\Gamma, \Delta]$ are the 2-functors $\Gamma \longrightarrow \Delta$, the 1-cells of $[\Gamma, \Delta]$ are the 2-natural transformations, and the 2-cells of $[\Gamma, \Delta]$ are called *modifications* [14, Section 1.4]. A 2-functor $\Gamma \longrightarrow \Delta$ is a rule assigning to every object of Γ an object of Δ, to every 1-cell of Γ a 1-cell of Δ, and to every 2-cell of Γ a 2-cell of Δ. A 2-natural transformation is a rule assiging to every object of Γ a 1-cell in Δ. A modification is a rule assigning to every object of Γ a 2-cell in Δ. 2-categories and 2-functors form the 3-*category of* 2-*categories*: 2-categories are the objects, 2-functors are the 1-cells, 2-natural transformations are the 2-cells, and modifications are the 3-cells.

Even when one is only interested in 2-functors rather than the more relaxed concepts of lax functors or pseudofunctors, the notion of a 2-natural transformation may be too strict, and one may want to relax it. Then one can consider *lax natural transformations* (compatible with the action of the 2-functors on 1-cells in Γ up to a coherent family of 2-cells in Δ) or *pseudonatural transformations* (compatible with the action of the 2-functors on the 1-cells in Γ up to a coherent family of *invertible* 2-cells in Δ). In the terminology of [19, §4.1], lax natural transformations are called "transformations", pseudonatural transformations are called "strong transformations", and 2-natural transformations are called "strict transformations". The 2-category of 2-functors $\Gamma \longrightarrow \Delta$, pseudonatural transformations, and modifications is denoted by $\mathsf{Psd}[\Gamma, \Delta]$ in [4], [13, Section 5], [5, Section 2].

In connection with the "lax" notions, the choice of the direction of the (possibly noninvertible) 2-cells providing the relaxed compatibility becomes important. When the direction is reversed, the correspoding notions are called "oplax". For "pseudo" notions, the compatibility 2-cells are assumed to be invertible, and so the choice of the direction in which they act no longer matters.

8. Conical pseudolimits, lax limits, and oplax limits

We denote by $\mathbf{Cat}$ the 2-category of small categories and by $\mathbf{CAT}$ the 2-category of locally small categories (i. e., large categories in which morphisms between any fixed pair of objects form a set). So the categories of morphisms in $\mathbf{CAT}$ need not be even locally small; this will present no problem for our constructions.

Let Γ be a small 2-category and $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be a 2-functor. The (*conical*) *lax limit* of H is a category L whose objects are the following sets of data:

i. for every object $\gamma \in \Gamma$, an object $L_\gamma \in H(\gamma)$ of the category $H(\gamma)$ is given;

ii. for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, a morphism $l_a\colon H(a)(L_\gamma) \longrightarrow L_\delta$ in the category $H(\delta)$ is given.

Here $H(a)\colon H(\gamma) \longrightarrow H(\delta)$ is the functor assigned to the 1-cell $a\colon \gamma \longrightarrow \delta$ by the 2-functor H.

The set of data (i–ii) must satisfy the following conditions:

iii. for every identity 1-cell $a = \mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ, one has $l_{\mathrm{id}_\gamma} = \mathrm{id}_{L_\gamma}\colon L_\gamma \longrightarrow L_\gamma$;

iv. for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, one has $l_{ba} = l_b \circ H(b)(l_a)$ in the category $H(\epsilon)$;

v. for every 2-cell $t\colon a \longrightarrow b$, where a, $b\colon \gamma \rightrightarrows \delta$ is a pair of parallel 1-cells in Γ, the triangular diagram

$$\begin{tikzcd} H(a)(L_\gamma) \arrow[dd, "H(t)_{L_\gamma}"'] \arrow[dr, "l_a"] & \\ & L_\delta \\ H(b)(L_\gamma) \arrow[ur, "l_b"'] & \end{tikzcd}$$

is commutative in the category $H(\delta)$.

Here $H(t)\colon H(a) \longrightarrow H(b)$ is the morphism of functors from the category $H(\gamma)$ to the category $H(\delta)$ assigned to the 2-cell $t\colon a \longrightarrow b$ by the 2-functor H.

A morphism $L \longrightarrow M$ in the category L is the datum of a morphism $L_\gamma \longrightarrow M_\gamma$ in the category $H(\gamma)$ for every object $\gamma \in \Gamma$, satisfying the obvious compatibility condition with the data (ii) for the objects L and M.

The (*conical*) *oplax limit* of the 2-functor H is the category M whose objects are the following sets of data:

i*. for every object $\gamma \in \Gamma$, an object $M_\gamma \in H(\gamma)$ of the category $H(\gamma)$ is given;

ii*. for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, a morphism $m_a\colon M_\delta \longrightarrow H(a)(M_\gamma)$ in the category $H(\delta)$ is given.

The set of data (i*–ii*) must satisfy the following conditions:

iii*. for every identity 1-cell $a = \mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ, one has $m_{\mathrm{id}_\gamma} = \mathrm{id}_{M_\gamma}\colon M_\gamma \longrightarrow M_\gamma$;

iv*. for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, one has $m_{ba} = H(b)(m_a) \circ m_b$ in the category $H(\epsilon)$;

v*. for every 2-cell $t\colon a \longrightarrow b$, where a, $b\colon \gamma \rightrightarrows \delta$ is a pair of parallel

1-cells in Γ, the triangular diagram

$$\begin{tikzcd} & H(a)(M_\gamma) \arrow[dd, "H(t)_{M_\gamma}"] \\ M_\delta \arrow[ur, "m_a"] \arrow[dr, "m_b"'] & \\ & H(b)(M_\gamma) \end{tikzcd}$$

is commutative in the category $H(\delta)$.

A morphism $L \longrightarrow M$ in the category M is the datum of a morphism $L_\gamma \longrightarrow M_\gamma$ in the category $H(\gamma)$ for every object $\gamma \in \Gamma$, satisfying the obvious compatibility condition with the data (ii*) for the objects L and M.

The *pseudolimit* of the 2-functor H is the full subcategory $\mathsf{E} \subset \mathsf{L}$ consisting of all the objects $E \in \mathsf{L}$ such that the morphism $e_a\colon H(a)(E_\gamma) \longrightarrow E_\delta$ in (ii) is an isomorphism in $H(\delta)$ for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ. Equivalently, the pseudolimit E can be defined as the full subcategory $\mathsf{E} \subset \mathsf{M}$ consisting of all the objects $E \in \mathsf{M}$ such that the morphism $e_a\colon E_\delta \longrightarrow H(a)(E_\gamma)$ in (ii*) is an isomorphism in $H(\delta)$ for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ.

Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Denote by $\mathbf{ACC}_{\lambda,\kappa} \subset \mathbf{CAT}$ the following 2-subcategory in $\mathbf{CAT}$. The objects of $\mathbf{ACC}_{\lambda,\kappa}$ are all the κ-accessible categories with colimits of λ-indexed chains. The 1-cells of $\mathbf{ACC}_{\lambda,\kappa}$ are the functors preserving κ-directed colimits and colimits of λ-indexed chains. The 2-cells of $\mathbf{ACC}_{\lambda,\kappa}$ are the (arbitrary) natural transformations.

As usual, we will say that a 2-category is *κ-small* if it has less than κ objects, less than κ 1-cells, and less than κ 2-cells.

Theorem 8.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}_{\lambda,\kappa}$ be a 2-functor. Then the oplax limit M of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{M} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}_{\lambda,\kappa}$. An object $S \in \mathsf{M}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similarly to the proofs of Corollary 5.1 and Theorems 6.1–6.2, one constructs the oplax limit M as a combination of products, inserters, and equifiers.

Let $\mathsf{K} = \prod_{\gamma\in\Gamma} H(\gamma)$ be the Cartesian product of the categories $H(\gamma)$ taken over all objects $\gamma \in \Gamma$, and let $\mathsf{L} = \prod_{(a:\gamma\to\delta)\in\Gamma} H(\delta)$ be the Cartesian product of the categories $H(\delta)$ taken over all the 1-cells $a\colon \gamma \longrightarrow \delta$ in Γ. Consider the following pair of parallel functors $F, G\colon \mathsf{K} \longrightarrow \mathsf{L}$. The functor F takes a collection of objects $(M_\gamma \in H(\gamma))_{\gamma\in\Gamma} \in \mathsf{K}$ to the collection of objects $(M_\delta \in H(\delta))_{(a:\gamma\to\delta)} \in \mathsf{L}$. The functor G takes the same collection of objects $(M_\gamma \in H(\gamma))_{\gamma\in\Gamma} \in \mathsf{K}$ to the collection of objects $(F(a)(M_\gamma) \in H(\delta))_{(a:\gamma\to\delta)} \in \mathsf{L}$.

Then the related inserter category E from Section 4 (cf. Remark 4.6) is the category of all sets of data (i*–ii*) from the definition of the oplax limit above. The conditions (iii*–v*) have not been imposed yet.

Theorem 4.1 tells that E is a κ-accessible category and describes its full subcategory of κ-presentable objects. The desired oplax limit M is a full subcategory $\mathsf{M} \subset \mathsf{E}$ which can be produced as a joint equifier category, as in Section 3 and Remark 3.3. There are three kinds of pairs of parallel natural transformations to be equified, corresponding to the three conditions (iii*–v*).

Firstly, for every object $\gamma \in \Gamma$, we have a pair of parallel functors $F_\gamma = G_\gamma\colon \mathsf{E} \longrightarrow H(\gamma)$ and a pair of parallel natural transformations ϕ_γ, $\psi_\gamma\colon F_\gamma \longrightarrow G_\gamma$. The functor $F_\gamma = G_\gamma$ takes an object $E \in \mathsf{E}$ to the object $E_\gamma \in H(\gamma)$. The natural transformation ϕ_γ acts by the morphism $e_{\mathrm{id}_\gamma}\colon E_\gamma \longrightarrow E_\gamma$ assigned to the identity 1-cell $\mathrm{id}_\gamma\colon \gamma \longrightarrow \gamma$ in Γ by the datum (ii*) for the object $E \in \mathsf{E}$; while ψ_γ is the identity natural transformation.

Secondly, for every composable pair of 1-cells $a\colon \gamma \longrightarrow \delta$ and $b\colon \delta \longrightarrow \epsilon$ in Γ, we have a pair of parallel functors $F_{a,b}$, $G_{a,b}\colon \mathsf{E} \rightrightarrows H(\epsilon)$ and a pair of parallel natural transformations $\phi_{a,b}$, $\psi_{a,b}\colon F_{a,b} \rightrightarrows G_{a,b}$. The functor $F_{a,b}$ takes an object $E \in \mathsf{E}$ to the object $E_\epsilon \in H(\epsilon)$. The functor $G_{a,b}$ takes an object $E \in \mathsf{E}$ to the object $H(ba)(E_\gamma) \in H(\epsilon)$. The natural transformation $\phi_{a,b}$ acts by the morphism $e_{ba}\colon E_\epsilon \longrightarrow H(ba)(E_\gamma)$. The natural transformation $\psi_{a,b}$ acts by the composition of morphisms $H(b)(e_a) \circ e_b\colon E_\epsilon \longrightarrow H(b)(E_\delta) \longrightarrow H(ba)(E_\gamma)$.

Thirdly, for every 2-cell $t\colon a \longrightarrow b$, where a, $b\colon \gamma \rightrightarrows \delta$ is a pair of parallel 1-cells in Γ, we have a pair of parallel functors F_t, $G_t\colon \mathsf{E} \rightrightarrows H(\delta)$ and a pair of parallel natural transformations ϕ_t, $\psi_t\colon F_t \rightrightarrows G_t$. The functor F_t takes an object $E \in \mathsf{E}$ to the object $E_\delta \in H(\delta)$. The functor G_t takes an object $E \in \mathsf{E}$ to the object $H(b)(E_\gamma) \in H(\delta)$. The natural transformation ϕ_t acts by the composition of morphisms $H(t)_{E_\gamma} \circ e_a\colon E_\delta \longrightarrow H(a)(E_\gamma) \longrightarrow H(b)(E_\gamma)$. The natural transformation ψ_t acts by the morphism $e_b\colon E_\delta \longrightarrow H(b)(E_\gamma)$.

The resulting joint equifier is the oplax limit M. Theorem 3.1 tells that this category is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. This proves the first and the third assertions of the theorem, while the second assertion is easy. $\square$

Denote by $\mathbf{ACC}^\kappa_{\lambda,\kappa} \subset \mathbf{ACC}_{\lambda,\kappa}$ the following 2-subcategory in $\mathbf{CAT}$. The objects of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the same as the objects of $\mathbf{ACC}_{\lambda,\kappa}$, i. e., all the κ-accessible categories with colimits of λ-indexed chains. The 1-cells of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the functors preserving κ-directed colimits and colimits of λ-indexed chains, and taking κ-presentable objects to κ-presentable objects. The 2-cells of $\mathbf{ACC}^\kappa_{\lambda,\kappa}$ are the (arbitrary) natural transformations.

Theorem 8.2. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}^\kappa_{\lambda,\kappa}$ be a 2-functor. Then the lax limit L of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{L} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. An object $S \in \mathsf{L}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similar to the proof of Theorem 8.1, with the directions of some arrows suitably reversed as needed. $\square$

Theorem 8.3. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $H\colon \Gamma \longrightarrow \mathbf{ACC}^\kappa_{\lambda,\kappa}$ be a 2-functor. Then the pseudolimit E of the 2-functor H (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. For every object $\gamma \in \Gamma$, the natural forgetful/projection functor $\mathsf{E} \longrightarrow H(\gamma)$ belongs to $\mathbf{ACC}^\kappa_{\lambda,\kappa}$. An object $S \in \mathsf{E}$ is κ-presentable if and only if, for every object $\gamma \in \Gamma$, the image S_γ of S in $H(\gamma)$ is κ-presentable.*

Proof. Similar to the proofs of Theorems 8.1 and 8.2, with the only difference that it is convenient to use the isomorpher construction of Remark 5.2 instead of the inserter construction of Theorem 4.1. The equifier construction of Theorem 3.1 still needs to be used. (Cf. [13, Propositions 4.4 and 5.1] and [5, Proposition 2.1].) $\square$

Remark 8.4. The notions of (op)lax limit and pseudolimit are somewhat relaxed. The related strict notion is the 2-*limit* of categories. 2-limits of categories are *not* well-behaved in connection with accessible categories, generally speaking [19, paragraph after Proposition 5.1.1], [1, Example 2.68]. The well-behaved ones among the (weighted) 2-limits are called *flexible limits* in [5]. Still, the (op)lax limits and pseudolimits are strict enough to be defined *up to isomorphism of categories* (as per the constructions above) rather than just up to category equivalence.

The case of the pseudopullback is instructive. Let Γ be the following small 2-category. The 2-category Γ has three objects A, B, and C, and two nonidentity 1-cells $a\colon A \longrightarrow C$ and $b\colon B \longrightarrow C$. There are no nonidentity 2-cells in Γ. Hence a 2-functor $H\colon \Gamma \longrightarrow \mathbf{CAT}$ is the same thing as a triple of categories A, B, and C together with a pair of functors $\Theta_\mathsf{A}\colon \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_\mathsf{B}\colon \mathsf{B} \longrightarrow \mathsf{C}$, as in Section 5. Then [19, paragraph after Proposition 5.1.1] explains that the 2-pullbacks, i. e., the 2-limits of 2-functors $H\colon \Gamma \longrightarrow \mathbf{CAT}$, do *not* preserve accessibility of categories.

The (op)lax limits and pseudolimits are better behaved and preserve accessibility, as per the theorems above in this section; but one has to be careful. Looking into these constructions, one can observe that the definition of the pseudopullback in Section 5 was, strictly speaking, an abuse of terminology. The pseudolimit E of a 2-functor $H\colon \Gamma \longrightarrow \mathbf{CAT}$ is the category of all quintuples $(A, B, C, \theta_a, \theta_b)$, where $A \in \mathsf{A}$, $B \in \mathsf{B}$, and $C \in \mathsf{C}$ are three objects and $\theta_a\colon \Theta_\mathsf{A}(A) \simeq C$, $\theta_b\colon \Theta_\mathsf{B}(B) \simeq C$ are two isomorphisms (cf. [6, Proposition 3.1], [28, Pseudopullback Theorem 2.2]). The pseudopullback D as defined in Section 5 is *naturally equivalent* to the pseudolimit E of the 2-functor H, but *not* isomorphic to it.

The even more relaxed notion of a limit of categories defined up to a category equivalence is called the *bilimit* [19, Section 5.1.1], [13, Section 6]. In the terminology of [19, Section 5.1.1], the pseudolimits are called *strong bilimits*.

9. Weighted pseudolimits

Let Γ be a small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor (so the category $W(\gamma)$ is small for every $\gamma \in \Gamma$). The 2-functor W is called a *weight*.

Let $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be another 2-functor. The *weighted pseudolimit* $\{W, H\}_p$ [5, Sections 1–2] (called "indexed pseudolimit" in the terminology of [13, Sections 2 and 5] or "strong weighted bilimit" in the terminology of [19, Section 5.1.1]) can be simply constructed as the category of 1-cells $W \longrightarrow H$ in the 2-category of pseudonatural transformations $\mathsf{Psd}[\Gamma, \mathbf{CAT}]$ (mentioned in Section 7). So $\{W, H\}_p = \mathsf{Psd}[\Gamma, \mathbf{CAT}](W, H)$ [13, formula (5.5)].

The strict version of the same construction is the *weighted 2-limit* $\{W, H\}$, which can be defined as the category of 1-cells $W \longrightarrow H$ in the 2-category of 2-natural transformations $[\Gamma, \mathbf{CAT}]$; so $\{W, H\} = [\Gamma, \mathbf{CAT}](W, H)$ [13, formula (2.5)]. It is explained in [13, Section 4] or [5, Section 1] how to obtain the inserters, equifiers, and isomorphers (iso-inserters) as particular cases of weighted 2-limits. Up to category equivalence, they are also particular cases of weighted pseudolimits.

Taking Γ to be the 2-category with a single object, a single 1-cell, and a single 2-cell, one obtains the construction of the diagram category (as in Theorem 6.1), called the "cotensor product" in [13, Section 3], [5, Section 1], as the particular case of the weighted 2-limit or weighted pseudolimit.

Taking W to be the 2-functor assigning to every object $\gamma \in \Gamma$ the category with a single object and a single morphism, one obtains the construction of the pseudolimit from Section 8 as a particular case of weighted pseudolimit. To distinguish them from the more general weighted pseudolimits, the pseudolimits from Section 8 are called *conical pseudolimits* [13, Sections 3 and 5], [5, Sections 1–2].

The notation $\mathbf{ACC}^{\kappa}_{\lambda,\kappa} \subset \mathbf{ACC}_{\lambda,\kappa} \subset \mathbf{CAT}$ was introduced in Section 8.

Theorem 9.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let Γ be a κ-small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor such that the category $W(\gamma)$ is κ-small for every object $\gamma \in \Gamma$. Let $H\colon \Gamma \longrightarrow \mathbf{ACC}^{\kappa}_{\lambda,\kappa}$ be a 2-functor. Then the weighted pseudolimit $\{W, H\}_p$ (computed in $\mathbf{CAT}$, as per the construction above) belongs to $\mathbf{ACC}^{\kappa}_{\lambda,\kappa}$.*

Proof. The point is that all weighted pseudolimits can be constructed in terms of products, inserters, and equifiers [13, Proposition 5.2], [5, Proposition 2.1]; so the assertion follows from Proposition 2.1, Theorem 3.1, and Theorem 4.1. The same argument applies also to all weighted bilimits [13, Section 6] and all flexible weighted 2-limits [5, Theorem 4.9 and Remark 7.6]. ☐

Corollary 9.2. *Let λ and κ be infinite regular cardinals such that $\lambda \triangleleft \kappa$ in the sense of [19, §2.3] or [1, Definition 2.12]. Let Γ be a κ-small 2-category and $W\colon \Gamma \longrightarrow \mathbf{Cat}$ be a 2-functor such that the category $W(\gamma)$ is κ-small for every object $\gamma \in \Gamma$. Let $H\colon \Gamma \longrightarrow \mathbf{CAT}$ be a 2-functor such that, for every object $\gamma \in \Gamma$, the category $H(\gamma)$ is λ-accessible, and for every 1-cell $a\colon \gamma \longrightarrow \delta$ in Γ, the functor $H(a)\colon H(\gamma) \longrightarrow H(\delta)$ preserves λ-directed colimits and takes κ-presentable objects to κ-presentable objects. Then the weighted pseudolimit $\{W, H\}_p$ (computed in $\mathbf{CAT}$, as per the construction above) is a κ-accessible category.*

Proof. Follows immediately from Theorem 9.1. ☐

Remark 9.3. The assertion of Theorem 9.1 captures many, but not all the aspects of the preceding results in this paper. In particular, Theorems 3.1 and 4.1 are *not* particular cases of Theorem 9.1, if only because the assumptions of Theorems 3.1–4.1 are more general. Indeed, in the assumptions of Theorems 3.1–4.1 the functor F is required to belong to $\mathbf{ACC}^{\kappa}_{\lambda,\kappa}$, while the functor G may belong to the wider 2-category $\mathbf{ACC}_{\lambda,\kappa}$. In other words, the functor G *need not* take κ-presentable objects to κ-presentable objects. This subtlety, which was emphasized already in [29, Section 3], manifests itself in the related difference between the formulations of Theorem 8.1, on the one hand, and Theorems 8.2–8.3, on the other hand. It plays an important role in the application to comodules over corings worked out in [23, Theorem 3.1 and Remark 3.2] and in the application to corings in [26, Theorem 4.2].

10. Toy examples

The examples in this section aim to illustrate the main results of this paper in the context of additive categories, modules categories, and flat modules, which served as the main motivation for the present research. We refer to the papers [27, 23, 25, 26] for more substantial applications to flat quasi-coherent sheaves, flat comodules and contramodules, arbitrary and flat coalgebras and corings, and flat/injective (co)resolutions. This section also serves as a reference source for [27, 23, 25, 26], as it contains some results that are useful as building blocks for the more complicated constructions.

10.1 Modules and flat modules

Let R be an associative ring. We denote by $R\text{–}\mathsf{Mod}$ the abelian category of left R-modules and by $R\text{–}\mathsf{Mod}_{\mathsf{fl}} \subset R\text{–}\mathsf{Mod}$ the full subcategory of flat left R-modules.

The following two propositions are fairly standard.

Proposition 10.1. *For any ring R and any regular cardinal κ, the category of R-modules $R\text{–}\mathsf{Mod}$ is locally κ-presentable. The κ-presentable objects of $R\text{–}\mathsf{Mod}$ are precisely all the left R-modules that can be constructed as the cokernel of a morphism of free left R-modules with less than κ generators.* □

Proposition 10.2. *For any ring R and any regular cardinal κ, the category of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is κ-accessible. All directed colimits exist in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ and agree with the ones in $R\text{–}\mathsf{Mod}$. The κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ are precisely all those flat left R-modules that are κ-presentable as objects of $R\text{–}\mathsf{Mod}$.*

Proof. The connection between the present proposition and the previous one fits into the setting described in Proposition 1.2. The assertions for $\kappa = \aleph_0$ are corollaries of the classical Govorov–Lazard theorem [11, 16] characterizing the flat R-modules as the directed colimits of finitely generated projective (or free) R-modules. The general case of an arbitrary regular cardinal κ can be deduced by applying [1, Theorem 2.11 and Example 2.13(1)]. □

For a version of Proposition 10.2 for modules of bounded flat dimension, see [25, Corollary 5.2].

10.2 Diagrams of flat modules

The following two corollaries are our "toy applications" of Theorem 6.2.

Corollary 10.3. *Let k be a commutative ring and R be an associative, unital k-algebra. Let κ be an uncountable regular cardinal and A be a κ-presented k-linear category (in the sense of Section 6). Then any k-linear functor $\mathsf{A} \longrightarrow R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ is a κ-directed colimit of k-linear functors $\mathsf{A} \longrightarrow R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}$ into the category of κ-presentable flat left R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}$.*

Proof. By Proposition 10.2 and Theorem 6.2 (with $\lambda = \aleph_0$), the k-linear functor/diagram category $\mathsf{Fun}_k(\mathsf{A}, R\text{–}\mathsf{Mod}_{\mathsf{fl}})$ is κ-accessible, and $\mathsf{Fun}_k(\mathsf{A}, R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})$ is its full subcategory of κ-presentable objects. $\square$

Corollary 10.4. *Let R be an associative ring and κ be an uncountable regular cardinal. Then any cochain complex of flat R-modules is a κ-directed colimit of complexes of κ-presentable flat R-modules.*

Proof. This is the particular case of Corollary 10.3 for the ring $k = \mathbb{Z}$ and the suitable choice of additive category A describing cochain complexes. The objects of A are the integers $n \in \mathbb{Z}$, the set of generating morphisms is the singleton $\mathrm{Gen}(n, m) = \{d_n\}$ for $m = n+1$ and the empty set otherwise, and the set of defining relations is the singleton $\mathrm{Rel}(n, m) = \{d_{n+1}d_n\}$ for $m = n + 2$ and the empty set otherwise. $\square$

For a quasi-coherent sheaf, a comodule, and a contramodule version of Corollary 10.4, see [27, Theorem 4.1] and [23, Propositions 3.3 and 10.2].

Remark 10.5. For an uncountable regular cardinal κ, the complexes of κ-presentable R-modules are precisely all the κ-presentable objects of the locally finitely presentable (hence locally κ-presentable) abelian category of complexes of R-modules. For $\kappa = \aleph_0$, the finitely presentable objects of the category of complexes of R-modules are the *bounded* complexes of finitely presentable R-modules.

Notice that *not* every complex of flat R-modules is a directed colimit of bounded complexes of finitely presentable flat (i. e., finitely generated projective) R-modules. In fact, the directed colimits of bounded complexes of finitely generated projective R-modules are the *homotopy flat* complexes of flat R-modules [7, Theorem 1.1].

Using the argument from [1, proof of Theorem 2.11 (iv) $\Rightarrow$ (i)] (for $\lambda = \aleph_0$ and $\mu = \kappa$), one can deduce the assertion that any homotopy flat complex of flat R-modules is a κ-directed colimit of homotopy flat complexes of κ-presentable flat R-modules, for any uncountable regular cardinal κ. A quasi-coherent sheaf version of this observation can be found in [27, Theorem 4.5].

10.3 Categories of epimorphisms

For any category K, we denote by $\mathsf{K}^{\to}$ the category of morphisms in K (with commutative squares in K as morphisms in $\mathsf{K}^{\to}$). The following lemma is not difficult.

Lemma 10.6. *For any regular cardinal κ and κ-accessible category K, the category of morphisms $\mathsf{K}^{\to}$ is κ-accessible. The full subcategory of κ-presentable objects in $\mathsf{K}^{\to}$ is the category $(\mathsf{K}_{<\kappa})^{\to}$ of morphisms of κ-presentable objects in K.*

Proof. One has $\mathsf{K}^{\to} = \mathsf{Fun}(\mathsf{C}, \mathsf{K})$ for the obvious finite category C with no nonidentity endomorphisms; so the result of [2, Exposé I, Proposition 8.8.5], [20, page 55], or [12, Theorem 1.3] is applicable. $\square$

For any category K, let us denote by $\mathsf{K}^{\mathsf{epi}} \subset \mathsf{K}^{\to}$ the full subcategory whose objects are all the epimorphisms in K.

Lemma 10.7. *For any regular cardinal κ and any locally κ-presentable abelian category K, the category of epimorphisms $\mathsf{K}^{\mathsf{epi}}$ is locally κ-presentable. The full subcategory of κ-presentable objects in $\mathsf{K}^{\mathsf{epi}}$ is the category $(\mathsf{K}_{<\kappa})^{\mathsf{epi}}$ of epimorphisms between κ-presentable objects in K.*

Proof. Notice first of all that a morphism in $\mathsf{K}_{<\kappa}$ is an epimorphism in $\mathsf{K}_{<\kappa}$ if and only if it is an epimorphism in K (because the full subcategory $\mathsf{K}_{<\kappa}$ is closed under cokernels in K [1, Proposition 1.16]). Furthermore, the full subcategory $\mathsf{K}^{\mathsf{epi}}$ is closed under colimits in the locally presentable abelian category $\mathsf{K}^{\to}$; so all colimits exist in $\mathsf{K}^{\mathsf{epi}}$. In view of Lemma 10.6 and according to Proposition 1.2, in order to prove the lemma it suffices to check that any morphism from an object of $(\mathsf{K}_{<\kappa})^{\to}$ to an object of $\mathsf{K}^{\mathsf{epi}}$ factorizes through an object of $(\mathsf{K}_{<\kappa})^{\mathsf{epi}}$ in $\mathsf{K}^{\to}$.

Indeed, consider a commutative square diagram in K

$$\begin{tikzcd} S \arrow[r] \arrow[d] & K \arrow[d, two heads] \\ T \arrow[r] & L \end{tikzcd}$$

with an epimorphism $K \twoheadrightarrow L$ and objects $S, T \in \mathsf{K}_{<\kappa}$. Let M be the pullback of the pair of morphisms $K \longrightarrow L$ and $T \longrightarrow L$ in K; then $M \longrightarrow T$ is also an epimorphism (since the category K is assumed to be abelian).

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, two heads] & K \arrow[d, two heads] \\ T \arrow[r, equal] & T \arrow[r] & L \end{tikzcd}$$

Let $M = \varinjlim_{\xi\in\Xi} U_\xi$ be a representation of M as a κ-filtered colimit of κ-presentable objects U_ξ in K, and let V_ξ denote the images of the compositions $U_\xi \longrightarrow M \longrightarrow T$. The κ-filtered colimits are exact functors in K [1, Proposition 1.59]; hence we have $T = \varinjlim_{\xi\in\Xi} V_\xi$. Since $T \in \mathsf{K}_{<\kappa}$, it follows that there exists $\xi_0 \in \Xi$ such that the morphism $V_{\xi_0} \longrightarrow T$ is a retraction (as $V_{\xi_0} \longrightarrow T$ is a monomorphism by construction, this means that $V_{\xi_0} \longrightarrow T$ is actually an isomorphism). Hence the composition $U_{\xi_0} \longrightarrow M \longrightarrow T$ is an epimorphism. Since $S \in \mathsf{K}_{<\kappa}$, one can choose an index $\xi_1 \in \Xi$ together with an arrow $\xi_0 \longrightarrow \xi_1$ in Ξ such that the morphism $S \longrightarrow M$ factorizes through the morphism $U_{\xi_1} \longrightarrow M$. Hence we arrive to the desired factorization

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U_{\xi_1} \arrow[r] \arrow[d, two heads] & K \arrow[d, two heads] \\ T \arrow[r, equal] & T \arrow[r] & L \end{tikzcd}$$

through an object $(U_{\xi_1} \to T) \in (\mathsf{K}_{<\kappa})^{\mathsf{epi}}$. $\square$

Remark 10.8. It follows immediately from the first assertion of Lemma 10.7 that the category $\mathsf{K}^{\mathsf{mono}}$ of monomorphisms in K is also locally κ-presentable. In fact, the categories $\mathsf{K}^{\mathsf{epi}}$ and $\mathsf{K}^{\mathsf{mono}}$ are naturally equivalent; the functors of the kernel of an epimorphism and the cokernel of a monomorphism provide the equivalence. However, the direct analogue of the second assertion of Lemma 10.7 *fails* for monomorphisms (even though the full subcategory $\mathsf{K}^{\mathsf{mono}} \subset \mathsf{K}^{\to}$ is closed under κ-directed colimits by [1, Proposition 1.59]). In fact, a monomorphism i in K is a κ-directed colimit of monomorphisms between κ-presentable objects if and only if i is an admissible monomorphism in the *maximal locally κ-coherent exact structure on* K [24, Corollary 3.3]. In particular, if R is an associative ring that is not left coherent, then any monomorphism $i\colon N \longrightarrow M$ from a finitely generated but not finitely presentable left R-module N to a finitely presentable left R-module M is *not* a directed colimit of monomorphisms of finitely presentable modules in $R\text{–}\mathsf{Mod}$.

Given a ring R and a full subcategory $\mathsf{L} \subset R\text{–}\mathsf{Mod}$, we denote by $\mathsf{L}^{\mathsf{surj}} \subset \mathsf{L}^{\to}$ the full subcategory whose objects are all the surjective morphisms between objects of L.

Lemma 10.9. *For any associative ring R and any regular cardinal κ, the category of surjective morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ is κ-accessible. The κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are the surjective morphisms of κ-presentable flat R-modules.*

Proof. The argument is similar to the proof of Lemma 10.7. In view of Proposition 10.2, Lemma 10.6 is applicable to $\mathsf{K} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}$; so the category of morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$ is κ-accessible and the category of morphisms of κ-presentable flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}^{\to}$ is the full subcategory of κ-presentable objects in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\to}$. According to Proposition 1.2, in order to prove the lemma it suffices to check that any morphism from an object of $R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa}^{\to}$ to an object of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ factorizes through an object of $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{surj}}$.

Following the proof of Lemma 10.7, one needs to observe that if $K \twoheadrightarrow L$ is a surjective morphism of flat R-modules and $T \longrightarrow L$ is a morphism of flat R-modules, then the pullback M (computed in the category $R\text{–}\mathsf{Mod}$) is a flat R-module. Indeed, the kernel F of the morphism $K \longrightarrow L$ is a flat R-module, so the short exact sequence $0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0$ shows that M is a flat R-module, too. The images V_ξ of the morphisms $U_\xi \longrightarrow T$ can be taken in the ambient abelian category $R\text{–}\mathsf{Mod}$. Otherwise, the argument is the same, except that one considers surjective morphisms in $R\text{–}\mathsf{Mod}_{\mathsf{fl}}$ rather than epimorphisms in K. $\square$

Remark 10.10. Alternatively, one can drop the assumption that the category K is abelian in Lemma 10.7, requiring it only to be additive; but assume the cardinal κ to be uncountable instead. Then the resulting assertion can be obtained as a particular case of Corollary 5.1. Consider the category of morphisms $\mathsf{A} = \mathsf{K}^{\to}$, the zero category $\mathsf{B} = \{0\}$, and the category $\mathsf{C} = \mathsf{K}$. Let $\Theta_{\mathsf{A}}\colon \mathsf{A} \longrightarrow \mathsf{C}$ be the cokernel functor $f \longmapsto \operatorname{coker}(f)$ and $\Theta_{\mathsf{B}}\colon \mathsf{B} \longrightarrow \mathsf{C}$

be the zero functor. Then the pseudopullback D is the category of epimorphisms $\mathsf{D} = \mathsf{K}^{\mathsf{epi}}$. All the assumptions of Corollary 5.1 (with $\lambda = \aleph_0$) are satisfied; so the corollary tells that $\mathsf{K}^{\mathsf{epi}}$ is κ-accessible and provides the desired description of κ-presentable objects.

Similarly, assuming κ to be uncountable, one can deduce Lemma 10.9 from Lemmas 10.6 and 10.7 using Corollary 5.1. Consider the category of R-module epimorphisms $\mathsf{A} = R\text{–}\mathsf{Mod}^{\mathsf{epi}}$, the category of morphisms of flat R-modules $\mathsf{B} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\rightarrow}$, and the category of R-module morphisms $\mathsf{C} = R\text{–}\mathsf{Mod}^{\rightarrow}$. Let $\Theta_{\mathsf{A}} \colon \mathsf{A} \longrightarrow \mathsf{C}$ and $\Theta_{\mathsf{B}} \colon \mathsf{B} \longrightarrow \mathsf{C}$ be the natural inclusions. Then the pseudopullback D is the category of surjective morphisms of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$, and Corollary 5.1 is applicable.

10.4 Short exact sequences of flat modules

Now we can deduce the following three corollaries of Lemma 10.9.

Corollary 10.11. *Let R be an associative ring and κ be a regular cardinal. Then any surjective morphism of κ-presentable flat R-modules is a direct summand of a κ-small directed colimit of surjective morphisms of finitely generated projective R-modules (in the category $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\rightarrow}$).*

Proof. This follows from Lemma 10.9 in view of [1, proof of Theorem 2.11 (iv) ⇒ (i)] for $\mathsf{K} = R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$, $\lambda = \aleph_0$, and $\mu = \kappa$. The Govorov–Lazard characterization of flat modules [11, 16] implies that all finitely presentable flat R-modules are projective. By Lemma 10.9, the category of surjective morphisms of flat R-modules is finitely accessible, and its finitely presentable objects are the surjective morphisms of finitely generated projective R-modules. So all surjective morphisms of flat R-modules are directed colimits of surjective morphisms of finitely generated projective R-modules.

Let A denote the set of all κ-small directed colimits of surjective morphisms of finitely generated projective R-modules. Following the argument in [1, proof of Theorem 2.11 (iv) ⇒ (i)] and [1, Example 2.13(1)], all the objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are κ-directed colimits of objects from A. Thus all the κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}}$ are direct summands of objects from A. □

The next corollary is a generalization of [23, Lemma 4.1].

Corollary 10.12. *Let R be an associative ring and κ be a regular cardinal. Then the kernel of any surjective morphism of κ-presentable flat R-modules is a κ-presentable flat R-module.*

Proof. Follows from Corollary 10.11, as the kernel of any surjective morphism of finitely generated projective R-modules is a finitely generated projective R-module. For another proof, see [24, Corollary 4.7]. $\square$

Given a ring R and a full subcategory $\mathsf{L} \subset R\text{–}\mathsf{Mod}$, let us denote by $\mathsf{L}^{\mathsf{ses}}$ the category of all short exact sequences in $R\text{–}\mathsf{Mod}$ with the terms belonging to L.

Corollary 10.13. *For any associative ring R and any regular cardinal κ, the category of short exact sequences of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ is κ-accessible. The full subcategory of κ-presentable objects of $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ is the category $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{ses}}$ of all short exact sequences of κ-presentable flat R-modules.*

Proof. By Corollary 10.12, the obvious equivalence of categories $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{surj}} \simeq R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ identifies $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{surj}}$ with $(R\text{–}\mathsf{Mod}_{\mathsf{fl},<\kappa})^{\mathsf{ses}}$. This makes the desired assertion a restatement of Lemma 10.9. $\square$

10.5 Pure acyclic complexes of flat modules

Finally, we can present our "toy application" of Corollary 5.1. An acyclic complex of flat R-modules is said to be *pure acyclic* if its modules of cocycles are flat.

The following corollary is essentially a weaker version of the result of [9, Theorem 2.4 (1) $\Leftrightarrow$ (3)] or [21, Theorem 8.6 (ii) $\Leftrightarrow$ (iii)]. Our argument produces it as an application of general category-theoretic principles. See [27, Theorem 4.2] and [23, Corollaries 4.5 and 11.4] for a quasi-coherent sheaf, a comodule, and a contramodule version.

Corollary 10.14. *Let R be an associative ring and κ be an uncountable regular cardinal. Then any pure acyclic complex of flat R-modules is a κ-directed colimit of pure acyclic complexes of κ-presentable flat R-modules.*

Proof. The point is that a pure acyclic complex of flat R-modules $F^\bullet$ is the same thing as a collection of short exact sequences of flat R-modules $0 \longrightarrow G^n \longrightarrow F^n \longrightarrow H^n \longrightarrow 0$ together with a collection of isomorphisms $H^n \simeq G^{n+1}$, $n \in \mathbb{Z}$. This means that the category of pure acyclic complexes of flat R-modules can be constructed from the category of short exact sequences of flat R-modules $R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ using Cartesian products (as in Section 2) and the isomorpher construction from Remark 5.2.

Specifically, put $\mathsf{H} = \prod_{n\in\mathbb{Z}} R\text{–}\mathsf{Mod}_{\mathsf{fl}}^{\mathsf{ses}}$ and $\mathsf{G} = \prod_{n\in\mathbb{Z}} R\text{–}\mathsf{Mod}_{\mathsf{fl}}$. Let $P\colon \mathsf{H} \longrightarrow \mathsf{G}$ be the functor taking a collection of short exact sequences $(0 \to G^n \to F^n \to H^n \to 0)_{n\in\mathbb{Z}}$ to the collection of modules $(H^n)_{n\in\mathbb{Z}}$, and let $Q\colon \mathsf{H} \longrightarrow \mathsf{G}$ be the functor taking the same collection of short exact sequences to the collection of modules $(G^{n+1})_{n\in\mathbb{Z}}$. Then the resulting isomorpher category D is the category of pure acyclic complexes of flat R-modules. Given the results of Proposition 10.2 and Corollary 10.13, it follows from Proposition 2.1 and Remark 5.2 that the category D is κ-accessible and the pure acyclic complexes of κ-presentable flat R-modules are precisely all the κ-presentable objects of D. □

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