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TOPICS IN THE CATEGORICAL ALGEBRA OF CLOSURE SPACES

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Résumé. Par espace de fermeture nous entendons une paire $(A, \mathcal{C})$, dans laquelle A est un ensemble et $\mathcal{C}$ est un ensemble de sous-ensembles de A fermé sous les intersections arbitraires. Le but de cet article consiste à considérer plusieurs questions qui se posent naturellement dans le cadre de l' algèbre catégorique des espaces de fermeture. Cela inclut l'extensivité (à gauche) de leur catégorie, la description des morphismes de codescente effective, et la description des morphismes de co-revłtement et co-monotones par rapport à une certaine coréflexion des espaces de fermeture dans les ensembles. Cette coréflexion envoie chaque espace de fermeture sur son plus petit sous-ensemble fermé.

Abstract. By a closure space we mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections. The purpose of this paper is to consider several questions that naturally arise in the categorical algebra of closure spaces. This includes (left) extensivity of their category, description of effective codescent morphisms, and description of cocovering and comonotone morphisms with respect to a certain coreflection of closure spaces into sets. That coreflection carries a closure space to its smallest closed subset.

Keywords. closure space, left coextensive category, effective codescent morphism, cocovering, comonotone morphism.

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1. Introduction

There is a number of types of mathematical structures introduced by various authors as 'generalized topological spaces', and several of them were called closure spaces, the paper [8] being one of many useful references; let us also mention the book [7] and the paper [6] for two important links with category theory, omitting many others. Here we briefly repeat from [10]:

- By a *closure space* we will mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections; we will write $A = (A, \mathcal{C}) = (A, \mathcal{C}_A)$ and

  $$\overline{X} = \overline{X}^A = \bigcap_{X \subseteq A' \in \mathcal{C}_A} A'$$

  for a subset X in A. And we say that X is closed in A when $X \in \mathcal{C}_A$, or, equivalently, $\overline{X} = X$.

- A map $f : A' \to A$ of closure spaces is said to be *continuous* if it satisfies (any of) the following three equivalent conditions:

  $$X \in \mathcal{C}_A \Rightarrow f^{-1}(X) \in \mathcal{C}_{A'},$$

  $$X \subseteq A \Rightarrow \overline{f^{-1}(X)} \subseteq f^{-1}(\overline{X}),$$
  $$X' \subseteq A' \Rightarrow f(\overline{X'}) \subseteq \overline{f(X')}.$$

  The category of closure spaces and their continuous maps will be denoted by $\mathbf{CLS}$.

- A continuous map $f : A' \to A$ of closure spaces is said to be *closed* if it satisfies (any of) the following three equivalent conditions:

  $$X' \in \mathcal{C}_{A'} \Rightarrow f(X') \in \mathcal{C}_A,$$

  $$X' \subseteq A' \Rightarrow f(\overline{X'}) \supseteq \overline{f(X')},$$
  $$X' \subseteq A' \Rightarrow f(\overline{X'}) = \overline{f(X')}.$$

- The underlying set functor $U : \mathbf{CLS} \to \mathbf{Sets}$ is *topological* in the sense of categorical topology, and so $\mathbf{CLS}$ is small complete and small cocomplete, and U preserves all existing limits and colimits. In particular, a diagram in $\mathbf{CLS}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ A_1 & \longrightarrow & B \end{array}$$

  is a pullback diagram in $\mathbf{CLS}$ if and only if its U-image is a pullback diagram in $\mathbf{Sets}$ and $\mathcal{C}_A = \{\pi_1^{-1}(X_1) \cap \pi_2^{-1}(X_2) \mid X_1 \in \mathcal{C}_{A_1} \,\&\, X_2 \in \mathcal{C}_{A_2}\}$. We also have $\overline{X} = \pi_1^{-1}(\overline{\pi_1(X)}) \cap \pi_2^{-1}(\overline{\pi_2(X)})$ for every $X \subseteq A$.

The purpose of this paper is to consider several questions that naturally arise in categorical algebra of closure spaces. They could be asked more generally, replacing $\mathbf{CLS}$ with an abstract *topological category* (as defined, e.g., in the survey paper [2]; see also references therein), or even more generally – and doing that systematically could be an interesting future project. Some of it would even be well known, as, for example, a part of Section 3, or Theorem 4.1 (which can be seen as a special case of the dual of Proposition 9.7 of [13]). But here we only consider specifically the case of $\mathbf{CLS}$.

How *extensive* (in the sense of [4]) is the category $\mathbf{CLS}$? This question is answered in Section 2.

Three adjunctions,

discrete $\dashv$ *underlying set* $\dashv$ *codiscrete* $\dashv$ *smallest closed subset*,

written as $D \dashv U \dashv C \dashv Z$, between $\mathbf{CLS}$ and the category of sets are considered in Section 3. Unlike in the case of topological spaces, it seems that no reasonable 'locally connected' counterpart of D has a left adjoint admitting a nice Galois theory/theory of covering spaces, by which we mean having a nice special case of constructions of [3]. The adjunctions $D \dashv U$ and $U \dashv C$ are also not interesting from this viewpoint since (D and C are fully faithful while) U has both adjoints. It remains to consider the adjunction $C \dashv Z$, which is done in Sections 5 and 6.

In Section 4 we prove that a morphism in $\mathbf{CLS}$ is an effective codescent morphism (=an effective descent morphism in $\mathbf{CLS}^{\mathrm{op}}$) if and only if it is a subspace inclusion (up to an isomorphism). A possible reference to descent theory convenient for our purposes is any of the surveys [12] and [11], although only very preliminary material from there is needed. More precisely, all we will need to have in mind is the dual form of the following well-known fact: *A morphism in a category with pullbacks and coequalizers is an effective descent morphism whenever it is a pullback stable regular epimorphism with the corresponding pullback functor preserving coequalizers.* Note that here the pullback stability already makes the corresponding pullback functor fully faithful, and, in particular, conservative.

In Section 5 we examine dual forms of some notions considered in [3] in the case of $C \dashv Z$. Specifically, we prove that the coreflection Z has stable counits, and characterize cocoverings (which turn out to be the same as trivial cocoverings) and comonotone morphisms. Then, in Section 6, we make immediate conclusions concerning the resulting factorization systems:

- We have the one first constructed for a general reflection in [5], but not the '(colight, comonotone)'-factorization system.
- On the other hand, there is the obvious (dense, closed subspace inclusions)-factorization system, and as shown in Section 5, comonotone morphisms are the same as closed subspace inclusions.

This paper is dedicated to Bill Lawvere, who was the first to see many unusual adjunctions and their roles.

2. Coproducts and non-distributivity

Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of closure spaces. The coproduct $\sum_{\lambda\in\Lambda} A_\lambda$ is the disjoint union of all A_λ ($\lambda \in \Lambda$), in which a subset X is closed if and only if $X \cap A_\lambda$ is closed in A_λ for each $\lambda \in \Lambda$; writing this we use disjoint union as ordinary union, which we can do here and below without loss of generality. Note that each A_λ is closed in $\sum_{\lambda\in\Lambda} A_\lambda$ if and only if the empty set is closed in each A_λ.

Theorem 2.1. *For every family* $(f_\lambda : A_\lambda \to B_\lambda)_{\lambda\in\Lambda}$ *of morphisms in* **CLS***,* *and every $\mu \in \Lambda$, the diagram*

$$\begin{array}{ccc} A_\mu & \longrightarrow & \sum_{\lambda\in\Lambda} A_\lambda \\ {\scriptstyle f_\mu}\downarrow & & \downarrow{\scriptstyle \sum_{\lambda\in\Lambda} f_\lambda} \\ B_\mu & \longrightarrow & \sum_{\lambda\in\Lambda} B_\lambda \end{array}$$

whose horizontal arrows are coproduct injections, is a pullback diagram.

Proof. Our assertion is true at the level of sets, and all we need to prove is that a subset of A_μ is closed if and only if it is of the form $f_\mu^{-1}(Y) \cap Y'$ for some closed subsets Y of B_μ and Y' of $\sum_{\lambda\in\Lambda} A_\lambda$.

"If": Just note that $f_\mu^{-1}(Y) \cap Y' = f_\mu^{-1}(Y) \cap A_\mu \cap Y'$ and both $f_\mu^{-1}(Y)$ and $A_\mu \cap Y'$ are closed in A_μ, whenever Y is closed in B_μ and Y' is closed in $\sum_{\lambda\in\Lambda} A_\lambda$.

"Only if": For any subset X of A_μ, we have

$$X = A_\mu \cap (X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda) = f_\mu^{-1}(B_\mu) \cap (X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda),$$

and if X is closed in A_μ, then $X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda$ is closed in $\sum_{\lambda\in\Lambda} A_\lambda$. □

Theorem 2.2. *The category* **CLS** *is* infinitary left extensive*, that is, for every family* $(A_\lambda)_{\lambda\in\Lambda}$ *of closure spaces, the functor*

$$\Sigma : \prod_{\lambda\in\Lambda} (\mathbf{CLS} \downarrow A_\lambda) \to (\mathbf{CLS} \downarrow \sum_{\lambda\in\Lambda} A_\lambda)$$

is fully faithful.

Proof. This is trivial for empty Λ. For non-empty Λ, just note that taking pullbacks of the form

$$\begin{array}{ccc} A_\mu \times_{\sum_{\lambda\in\Lambda} A_\lambda} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A_\mu & \longrightarrow & \sum_{\lambda\in\Lambda} A_\lambda \end{array}$$

determines the right adjoint of Σ, and Theorem 2.1 in fact says that the unit of adjunction is an isomorphism. □

However, Theorem 2.2 can also easily be proved directly. Note also that the term “infinitary left extensive” seems to be used here for the first time, although the term “left (co)extensive” was used in [1].

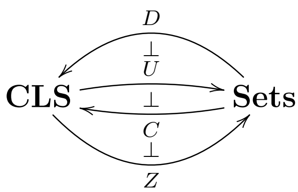
Remark 2.3. Consider Proposition 2.6 and 2.8 in [4]. They say that, in an extensive category, sums (=coproducts) are disjoint and the initial object is strict, respectively. However, the poofs given in [4] show that the same is true in any left extensive category. In particular, these properties hold in $\mathbf{CLS}$. However, in the case of $\mathbf{CLS}$ these properties are obvious anyway.

On the other hand, as the following simple example shows, the category $\mathbf{CLS}$ is not distributive (cf. Proposition 2.2 in [6]), which implies that it is not extensive (and not cartesian closed; and the same applies to the category of finite closure spaces).

Example 2.4. Let $A = \{a, a'\}$, $B = \{b\}$, and $C = \{c\}$ be discrete topological spaces considered as closure spaces (we assume $a \neq a'$ and $b \neq c$ of course). Then we can say that both $A \times (B + C)$ and $(A \times B) + (A \times C)$ have the same underlying set $\{(a, b), (a', b), (a, c), (a', c)\}$, but the set $\{(a, b), (a', c)\}$ is closed in $(A \times B) + (A \times C)$ and not in $A \times (B + C)$.

3. The adjunctions with sets

There are adjunctions



where, for a set S and a closure space A, we have:

3.1. $D(S)$ is the discrete topological space (considered as a closure space) with the underlying set S.

3.2. U is the underlying set functor.

3.3. $C(S)$ is what we will call codiscrete S: it has $UC(S) = S$ and $\mathcal{C}_{C(S)} = \{S\}$.

3.4. $Z(A) = \overline{\emptyset}$ is the smallest element of of $\mathcal{C}_A$. When there is no danger of confusion, we will write $CZ(A) = Z(A) = 0_A$.

Note that D has no left adjoint since it does not preserve, say, binary products, and Z has no right adjoint since it does not preserve, say, the coequalizer of

$$\{a\} \overset{f}{\underset{g}{\rightrightarrows}} \{b, c\} \ ,$$

where $Z(\{a\}) = \emptyset$, $Z(\{b, c\}) = \{b, c\}$, $f(a) = b$, and $g(a) = c$.

4. Equalizers, pushouts, and codescent

The equalizer diagram of two parallel morphisms $f, g : A \to B$ in **CLS** can be described as the inclusion map

$$K = \{a \in A \mid f(a) = g(a)\} \to A \text{ with } \mathcal{C}_K = \{X \cap K \mid X \in \mathcal{C}_A\},$$

and easily obtain

Theorem 4.1. *For $B, E \in$ **CLS** with $B \subseteq E$, the following conditions on the inclusion map $(i : B \to E) \in$ **CLS** are equivalent:*

(a) i is a regular monomorphism;

(b) i is a strong monomorphism;

(c) i is a subspace inclusion, that is, a subset of B is closed in it if and only if it can be presented as the intersection of a closed subset of E with B, or equivalently, if and only if

$$\overline{X}^B = \overline{X}^E \cap B$$

for every $X \subseteq B$.

Consider a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & E \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ A & \xrightarrow[\iota_1]{} & A +_B E \end{array}$$

in **CLS**, in which i is as in Theorem 4.1. Assuming $A \cap E = \emptyset$ we can present it as

$$A +_B E = A \cup (E \setminus B) \text{ as sets,}$$

$$\mathcal{C}_{A+_BE} = \{A' \cup E' \mid A' \in \mathcal{C}_A,\ E' \subseteq (E \setminus B),\ \alpha^{-1}(A') \cup E' \in \mathcal{C}_E\},$$

with ι_1 being the inclusion map and ι_2 defined by

$$\iota_2(e) = \begin{cases} \alpha(e), \text{ if } e \in B; \\ e, \text{ if } e \in (E \setminus B). \end{cases}$$

We will informally call this diagram the *standard pushout* of α and i.

Lemma 4.2. *Regular monomorphisms in* **CLS** *are pushout stable.*

Proof. Consider the standard pushout above. We have to show that the inclusion map $\iota_1 : A \to A +_B E$ is a subspace inclusion. For an arbitrary $X \in \mathcal{C}_A$, we have $\alpha^{-1}(X) \in \mathcal{C}_B$, and so

$$\alpha^{-1}(X) = B \cap \overline{\alpha^{-1}(X)}^E$$

by 4.1(c). We need to find $A' \in \mathcal{C}_A$ and $E' \subseteq (E \setminus B)$ such that inside the disjoint union $A \cup (E \setminus B)$ we have

$$X = (A' \cup E') \cap A \text{ and } \alpha^{-1}(A') \cup E' \in \mathcal{C}_E,$$

and we claim that we can take $A' = X$ and $E' = \overline{\alpha^{-1}(X)}^E \cap (E \setminus B)$. Indeed, the first equality will hold simply because $A' \subseteq A$ and $E' \subseteq (E \setminus B)$, while

$$\alpha^{-1}(A') \cup E' = \alpha^{-1}(X) \cup (\overline{\alpha^{-1}(X)}^E \cap (E \setminus B))$$

$$= (B \cap \overline{\alpha^{-1}(X)}^E) \cup (\overline{\alpha^{-1}(X)}^E \cap (E \setminus B)) = \overline{\alpha^{-1}(X)}^E \in \mathcal{C}_E.$$

□

Theorem 4.3. *A morphism in* $\mathbf{CLS}$ *is an effective codescent morphism if and only if it is a regular monomorphism, or, equivalently, a strong monomorphism.*

Proof. Thanks to (Theorem 4.1 and) Lemma 4.2, it suffices to prove that if $i : B \to E$ satisfies the equivalent conditions of Theorem 4.1, then the associated pushout functor

$$E +_B (-) : (B \downarrow \mathbf{CLS}) \to (E \downarrow \mathbf{CLS})$$

preserves equalizers. But this follows from the fact that the equalizers are preserved at the level of sets, and that any such pushout functor preserves regular monomorphisms, by Lemma 4.2. □

Remark 4.4. The arguments used in the proof above can be copied for the category of topological spaces and the category of topological spaces whose sets of open sets are closed under arbitrary intersections (= Alexandrov-discrete topological spaces, which are in fact nothing but preordered sets). Therefore, in both of these categories, effective codescent morphisms are the same as regular monomorphisms.

5. The coreflection $Z : \mathbf{CLS} \to \mathbf{Sets}$

We will use dual forms of several categorical notions from [5] and [3], such as the one of coreflection with stable counits, dual to reflection with stable units (introduced in [5]):

Theorem 5.1. *The coreflection* $Z : \mathbf{CLS} \to \mathbf{Sets}$ *has stable counits, that is, it preserves colimits of all diagrams* $A \leftarrow B \to A'$ *in which* $B = CZ(B)$*, or, equivalently,* $\mathcal{C}_B = \{B\}$.

Proof. Since Z (obviously) preserves coproducts, it suffices to prove that it preserves coequalizers of all pairs $f, g : B \to A$ in which $B = CZ(B)$. The coequalizer of such a pair can be presented as the canonical map $p : A \to A/R$, where R is the smallest equivalence relation on A containing the set $S = \{(f(b), g(b)) \mid b \in B\}$ and $\mathcal{C}_{A/R}$ is the set of all subsets of A/R whose inverse images under p belong to $\mathcal{C}_A$. We observe:

(i) Since $B = CZ(B) = 0_B$, the images of f and g are subsets of 0_A.

(ii) As follows from (i), $0_A = p^{-1}(p(0_A))$.

(iii) As follows from (ii), $p(0_A)$ belongs to $\mathcal{C}_{A/R}$.

(iv) Since p is surjective and continuous, (ii) and (iii) imply $p(0_A) = 0_{A/R}$.

This makes

$$B \rightrightarrows 0_A \longrightarrow 0_{A/R}$$

a coequalizer diagram in the category of sets. But this diagram is the same as

$$Z(B) \underset{Z(g)}{\overset{Z(f)}{\rightrightarrows}} Z(A) \xrightarrow{Z(p)} Z(A/R),$$

which completes the proof. □

As follows from Theorem 5.1, a simplified version of Galois theory [9], recalled in [3], applies to the reflection $Z^{\mathrm{op}} : \mathbf{CLS}^{\mathrm{op}} \to \mathbf{Sets}^{\mathrm{op}}$. And the resulting dualization makes a morphism $\alpha : B \to A$ in $\mathbf{CLS}$:

- a *trivial cocovering*, if the diagram

$$\begin{array}{ccc} CZ(B) & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle CZ(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ CZ(A) & \xrightarrow[\varepsilon_A]{} & A \end{array}$$

  in which ε is the counit of the adjunction $C \dashv Z$, is a pushout;

- a *cocovering*, or a *colight morphism*, if there exists a subspace inclusion $i : B \to E$ such that the morphism $\iota_2 : E \to A +_B E$ is a trivial cocovering.

- *covertical* (according to fibration-theoretic terminology), if $Z(\alpha)$ is an isomorphism.

- *comonotone*, if it is a pushout stable covertical morphism.

We are going now to characterize these types of morphisms, except covertical ones, whose definition already characterizes them.

For a closure space A, we will write $\mathrm{int}(A)$ for the set $A \setminus 0_A$ equipped with its induced structure, and call it the *interior* of A, since it is the largest subset of A whose complement is closed. Note that a morphism $\alpha : B \to A$ in **CLS** in general induces only a *partial map* $\mathrm{int}(\alpha) : \mathrm{int}(B) \to \mathrm{int}(A)$; it is a morphism in **CLS** if and only if it is a map, that is, if and only if

$$\alpha^{-1}(0_A) = 0_B.$$

Lemma 5.2. *For any closure space A and a subset A' of $\mathrm{int}(A)$, we have*

$$A' \in \mathcal{C}_{\mathrm{int}(A)} \Leftrightarrow 0_A \cup A' \in \mathcal{C}_A.$$

Proof. "$\Rightarrow$": $A' \in \mathcal{C}_{\mathrm{int}(A)}$ means that $A' = (A \setminus 0_A) \cap A''$ for some $A'' \in \mathcal{C}_A$. Then, since both 0_A and A' are subsets of A'', we have $0_A \cup A' \subseteq A''$. On the other hand, each $a \in A''$ must satisfy one on the following two conditions:

(i) $a \in 0_A$;

(ii) $a \in A \setminus 0_A$, but then a belongs to $(A \setminus 0_A) \cap A'' = A'$.

That is, $0_A \cup A' = A'' \in \mathcal{C}_A$.

The implication "$\Leftarrow$" follows from the equality $A' = (0_A \cup A') \cap (A \setminus 0_A)$. □

Corollary 5.3. *For any closure space A, the assignments*

$$X \mapsto 0_A \cup X \text{ and } Y \mapsto Y \cap (A \setminus 0_A)$$

determine bijections $\mathcal{C}_{\mathrm{int}(A)} \to \mathcal{C}_A$ and $\mathcal{C}_A \to \mathcal{C}_{\mathrm{int}(A)}$ inverse to each other.

Theorem 5.4. *The following conditions on a morphism $\alpha : B \to A$ in* **CLS** *are equivalent:*

(a) α is a trivial cocovering;

(b) $\mathrm{int}(\alpha)$ is an isomorphism (in **CLS***);*

(c) α induces bijections

$$B \setminus 0_B \to A \setminus 0_A \text{ and } \mathcal{C}_A \to \mathcal{C}_B \text{ (where } A' \mapsto \alpha^{-1}(A')).$$

Proof. The morphism $\varepsilon_B : CZ(B) \to B$ is obviously a subspace inclusion. Therefore, and according to the standard pushout construction in Section 4, assuming for simplicity that A and B are disjoint, we can reformulate condition (a) as:

(a′) The map $\tilde{\alpha} : 0_A \cup (B \setminus 0_B) \to A$ defined by

$$\tilde{\alpha}(x) = \begin{cases} x, \text{ if } x \in 0_A; \\ \alpha(x), \text{ if } x \in (B \setminus 0_B). \end{cases}$$

is bijective, and a subset A' of A belongs to $\mathcal{C}_A$ if and only if it is of the form

$$A' = \tilde{\alpha}(0_A \cup B') = 0_A \cup \alpha(B')$$

with $B' \subseteq B \setminus 0_B$ and $0_B \cup B' \in \mathcal{C}_B$, or, equivalently, $B' \in \mathcal{C}_{\mathrm{int}(B)}$.

It is easy to see that the map $\tilde{\alpha}$ defined in (a′) is a bijection if and only if

$$\mathrm{int}(\alpha) : \mathrm{int}(B) \to \mathrm{int}(A)$$

is a morphism in **CLS** that is a bijective map. This allows us to argue as follows.

(a′)⇒(b): Suppose (a′) holds. To prove (b) is to prove that if B' belongs to $\mathcal{C}_{\mathrm{int}(B)}$, then $\mathrm{int}(\alpha)(B')$ belongs to $\mathcal{C}_{\mathrm{int}(A)}$. We have

$$\mathrm{int}(\alpha)(B') = \alpha(B') = (0_A \cup \alpha(B')) \cap (A \setminus 0_A),$$

which belongs to $\mathcal{C}_{\mathrm{int}(A)}$ since, by (a′), $0_A \cup \alpha(B')$ belongs to $\mathcal{C}_A$.

(b)⇒(a′): Applying Corollary 5.3 and then (b), we obtain

$$A' \in \mathcal{C}_A \Leftrightarrow \exists_{X \in \mathcal{C}_{\mathrm{int}(A)}} A' = 0_A \cup X \Leftrightarrow \exists_{B' \in \mathcal{C}_{\mathrm{int}(B)}} A' = 0_A \cup \alpha(B'),$$

which gives (a′).

(b)⇔(c) easily follows from Corollary 5.3. □

Remark 5.5. As also easily follows from Corollary 5.3, the inverse of the bijection $\mathcal{C}_A \to \mathcal{C}_B$ in 5.4(c) is given by $B' \mapsto 0_A \cup \alpha(B')$.

Theorem 5.6. *Every cocovering is a trivial cocovering.*

Proof. Suppose $\alpha : B \to A$ is a cocovering, and let $i : B \to E$ be a subspace inclusion such that $\iota_2 : E \to A +_B E$ is a trivial cocovering. Then ι_2 induces a bijection $E \setminus 0_E \to A +_B E \setminus 0_{A+_BE}$. Using our standard pushout of α and i, we see that since the map $\iota_2 : E \to A \cup (E \setminus B)$ maps B to A via α and maps $E \setminus B$ identically to itself, it induces maps

$$B \setminus 0_E \to A \setminus 0_{A+_BE} \text{ and } (E \setminus B) \setminus 0_E \to (E \setminus B) \setminus 0_{A+_BE},$$

which then both must be bijections. Since

$$A \setminus 0_{A+_BE} = A \setminus A \cap (0_{A+_BE}) = A \setminus 0_A,$$

the first of these bijections is in fact the bijection $B \setminus 0_B \to A \setminus 0_A$ induced by α.

Now, thanks to Theorem 5.4(c)⇒(a), it only remains to prove that the canonical map $\mathcal{C}_A \to \mathcal{C}_B$ is a bijection. For, using the same $i : B \to E$, consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_{A+_BE} & \longrightarrow & \mathcal{C}_A & \xrightarrow{\subseteq} & \{X \subseteq A \mid 0_A \subseteq X\} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle X \mapsto \alpha^{-1}(X)} \\ \mathcal{C}_E & \longrightarrow & \mathcal{C}_B & \xrightarrow[\subseteq]{} & \{Y \subseteq B \mid 0_B \subseteq Y\} \end{array}$$

of canonical maps. The left-hand square shows that $\mathcal{C}_A \to \mathcal{C}_B$ is surjective, while the right-hand square shows that it is injective. □

Theorem 5.7. *The following conditions on a morphism $\alpha : B \to A$ in* **CLS** *are equivalent:*

(a) α is comonotone;

(b) α is injective and closed.

Proof. (a)⇒(b): Choose any $B_0 \in \mathcal{C}_B$ and consider the pushout

$$\begin{array}{ccc} B & \xrightarrow{b \mapsto b} & B' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ A & \xrightarrow[\iota_1]{} & A' \end{array}$$

in which $B' = B$ as sets and $\mathcal{C}'_B = \{Y \in \mathcal{C}_B \mid B_0 \subseteq Y\}$. This allows us to put:

$$A' = A \text{ as sets, } \iota_1(a) = a,\ \iota_2(b) = \alpha(b),$$

for all $a \in A$ and $b \in B$, and

$$\mathcal{C}'_A = \{X \in \mathcal{C}_A \mid B_0 \subseteq \alpha^{-1}(X)\}.$$

According to this presentation of our pushout, $0_{A'}$ is the smallest closed subset of A containing $\alpha(0_{B'}) = \alpha(B_0)$. That is,

$$0_{A'} = \overline{\alpha(B_0)}^A.$$

Since, by (a), ι_2 is covertical, we conclude that the restriction of α on B_0 must be injective and $\alpha(B_0)$ is closed in A; in particular, taking $B_0 = B$ gives injectivity of α.

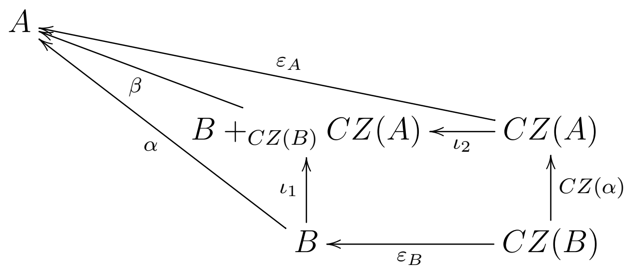
(b)⇒(a): Let us change our notation. We can assume, without loss of generality, that α is a closed subspace inclusion and we will rename it as $i : B \to E$. On the other hand, by $\alpha : B \to A$ we will denote now an arbitrary morphism in **CLS** (with the same B). We have to show that $\iota_1 : A \to A +_B E$ is covertical, and, in terms of the standard pushout of α and i, this simply means that 0_A is the smallest closed subset in $A +_B E$. For, we recall that a subset of $A +_B E$ is closed if and only if it is of the form $A' \cup E'$ with $A' \in \mathcal{C}_A$, $E' \subseteq (E \setminus B)$, and $\alpha^{-1}(A') \cup E' \in \mathcal{C}_E$. Since every closed subset of B is closed in E, the smallest closed subset of $A +_B E$ is $0_A \cup \emptyset = 0_A$. □

6. Two factorization systems on CLS

As follows from the results of [5], recalled in [3], and Theorem 5.1, the category **CLS** admits the *(trivial cocoverings, covertical morphisms)-factorization system* $(\mathbf{E}, \mathbf{M})$, for which:

- $\mathbf{E}$ is the class of all trivial cocoverings, defined as in Section 5;
- $\mathbf{M}$ is the class of all covertical morphisms, defined as in Section 5;

- the $(\mathbf{E}, \mathbf{M})$-factorization of a morphism $\alpha : B \to A$ in $\mathbf{CLS}$ is constructed as $\alpha = \beta\iota_1$ in the diagram



  in which the square part is a pushout and β is induced by α and ε_A.

Note that the diagram above is shaped as diagram (4.2) in [5] and diagram (3.5) in [3], except that the directions of all arrows are opposite since we consider the dual situation.

Now, trying to follow [3], could we (co)localize $\mathbf{E}$ and (co)stabilize $\mathbf{M}$ to obtain '(colight, comonotone)'-factorization system? Obviously not, since every cocovering is trivial (Theorem 5.6) while not every covertical morphism is comonotone (as immediately follows from Theorem 5.7). Nevertheless, we do have a factorization system $(\mathbf{E}', \mathbf{M}')$ on $\mathbf{CLS}$, in which $\mathbf{M}'$ is the class of all comonotone morphisms: we just need to take $\mathbf{E}'$ to be the class of (obviously defined) *dense* morphisms. This can also be seen as a consequence of Theorem 2.4 in [7].

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