CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXVI-1 (2025)

A CONSTRUCTIVE ACCOUNT OF THE KAN–QUILLEN MODEL STRUCTURE AND OF KAN'S Ex^∞ FUNCTOR

Simon Henry

Résumé. Nous donnons une preuve constructive de l'existence d'une structure de catégorie de modèles cartésienne fermée et propre sur la catégorie des ensembles simpliciaux, dont les cofibrations génératrices sont les inclusions de bords et les cofibrations triviales génératrices les inclusion de cornets. La différence principale avec l'approche classique est que toutes les inclusions ne sont pas des cofibrations (seulement celles satisfaisant certaines conditions de décidabilités) et tous les objets ne sont pas cofibrants.
La preuve repose sur trois ingrédients principaux: D'abord, l'existence d'une structure de catégorie de modèles faible sur les ensembles simpliciaux, ensuite l'interaction avec la version semi-simpliciale de cette structure et enfin l'utilisation du foncteur Ex^∞ de Kan, et plus précisement de la preuve directe de S.Moss que l'application $X \to \mathrm{Ex}^\infty X$ est une cofibration anodyne, dont nous montrons qu'elle est constructive si on suppose que X est cofibrant.

Abstract. We give a fully constructive proof that there is a proper cartesian ω-combinatorial model structure on the category of simplicial sets, whose generating cofibrations and trivial cofibrations are the usual boundary inclusion and horn inclusion. The main difference with classical mathematics is that constructively not all monomorphisms are cofibrations (only those satisfying some decidability conditions) and not every object is cofibrant.
The proof relies on three main ingredients: First, our construction of a weak model categories on simplicial sets, then the interplay with the semi-simplicial versions of this weak model structure and finally, the use of Kan Ex^∞-

functor, and more precisely of S.Moss' direct proof that the natural map $X \to \mathrm{Ex}^\infty X$ is an anodyne cofibration, which we show is constructive when X is cofibrant.

Keywords. Model categories, constructive mathematics, simplicial sets, Ex^∞-functor.
Mathematics Subject Classification (2020). 03F55, 55U35,55U40,18N50.

Contents

1 Introduction **2**

2 Constructing the model structure **11**
- 2.1 Review of the weak model structures 11
- 2.2 The simplicial model structure 15

3 Kan Ex^∞-functor **25**
- 3.1 Degeneracy quotient and questions of decidability 26
- 3.2 P-structures 34
- 3.3 Kan Ex and Sd functors 41
- 3.4 S.Moss' proof that $X \to \mathrm{Ex}\, X$ is an anodyne cofibration . . 46
- 3.5 Applications 54

1. Introduction

The goal of this paper is to give a fully constructive proof of the existence of the usual Kan–Quillen model structure on simplicial sets, and of some of its classical properties. "Constructive" here can be taken to mean "Without the axiom of choice and the law of excluded middle", or a bit more precisely as "in the internal logic of an elementary topos with a natural number objects".

This work was supported by the Operational Programme Research, Development and Education Project "Postdoc@MUNI" (No. CZ.02.2.69/0.0/0.0/16_027/0008360); and by the Natural Sciences and Engineering Research Council of Canada (NSERC), funding reference number RGPIN-2020-067.

It can also be formalized in Aczel's (CZF) [1] and probably in considerably weaker foundations as well, see Remark 1.6. Our main theorem is:

1.1 Theorem. *There is a proper cartesian Quillen model structure on the category of simplicial sets such that:*

- *The trivial fibrations are the morphisms with the right lifting property against all boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$.*
- *Cofibrations are the monomorphisms $f : A \to B$ which are "level wise complemented" (i.e. for all integers n for each $b \in B([n])$ it is decidable if $b \in A([n])$ or not), and such that for all $b \in B([n]) - A([n])$ it is decidable if b is a degenerate cell or not.*
- *The fibrations are the "Kan fibrations", i.e. they are the morphisms with the right lifting property against the horn inclusion: $\Lambda^k[n] \hookrightarrow \Delta[n]$. Dually trivial cofibrations are the retract of ω-transfinite compositions of pushouts of coproducts of horn inclusions.*

Note that assuming the law of excluded middle the class of cofibrations boils down to the class of all monomorphisms and hence one recovers the usual Kan–Quillen model structure.
After we announced this result, two other proofs, have been found by N. Gambino, C. Sattler and K. Szumilo and appeared in [6].

This theorem is obtained by patching together the following results: Theorem 2.2.9 gives the existence of a model structure with the appropriate cofibrations and trivial fibrations, Proposition 2.2.10 gives left properness, Proposition 3.5.1 shows that the fibrations and trivial cofibrations are indeed as specified here and Proposition 3.5.2 shows that it is also right proper. Cartesianness was already known, but reproved as Proposition 3.2.6.

One can also say a few words about the equivalences of the model structure of Theorem 1.1: they are defined (as Definition 2.2.3) using the forgetful functor to semi-simplicial sets and the weak model structure on semi-simplicial sets constructed in Theorem 5.5.6 of [8]. Concretely, this means that a map between Kan complexes is a weak equivalence if it admit an inverse up to homotopy as a semi-simplicial maps. For general map, we need to first take fibrant replacement and then use the previous definition. Note that Proposition 2.2.2 shows that this notion of equivalence is compatible with the notion we used in [8]. Moreover, Proposition 5.2.6 of [8] shows that weak equivalences admit the usual characterization in terms of homotopy groups, as long as the homotopy groups are defined not as quotient sets but as *setoids*.

As we do not assume the axiom of choice, one needs to make precise some details regarding Theorem 1.1: a "structure of fibration" (resp. trivial fibration) on a map f is the choice of a solution to each lifting problem of a horn inclusion (resp. boundary inclusion) against f. No uniformity condition is required on these lifts. A fibration (resp. trivial fibration) is a morphism which admits at least one structure of fibration (resp. trivial fibration), but the choice of the structure is considered irrelevant.
More generally, we will follow the convention that (unless exceptionally stated otherwise) every statement of the form $\forall a, \exists b$ should be interpreted as the existence of a function that given "a" produces a "b". In particular, when one says that a morphism has the lifting property against some set of arrows it means that one has a function that produces a solution to each lifting problem. We will use the convention constantly in the present paper, i.e. every time we say that "there exists" some x, we mean that one specific x has been chosen for each possible value of the parameters involved in the statement.

As fibrations and trivial fibrations are defined by the right lifting property against a small set of morphisms between finitely presented objects, it is very easy to apply a constructive version of the small object argument to show that one has two weak factorization systems, which will be called as follows:

1.2 Definition.

- The weak factorization system cofibrantly generated by the boundary inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ is called "cofibrations/trivial fibrations".
- The weak factorization system cofibrantly generated by the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ is called "Anodyne cofibrations/Kan fibrations".

We have discussed the constructive validity of the small object argument in appendix C of [8], though there are probably other references doing this.
Note that anodyne cofibrations will in the end be the trivial cofibrations, and Kan fibrations will be what we have called fibrations in the statement of the main Theorem 1.1, but this will be one of the last results we will prove. In the meantime we will distinguish between Kan fibrations and "strong fibrations" and between anodyne cofibrations and "trivial cofibrations" (these two other concepts being defined as Definition 2.2.3). Simplicial sets whose map to the terminal simplicial set is a Kan fibration will be called either Kan complexes, or fibrant simplicial sets.

1.3 Remark. Before going any further, we should pause here to insist on a very important remark: one of the key differences between what we are doing in the present paper and the usual construction of the Kan–Quillen model structure in classical mathematics is that the cofibrations are no longer exactly the monomorphisms. It can be shown, see for example Proposition 5.1.4 in [8], that the class of cofibrations generated by the boundary inclusion, i.e. the class of arrow which have the left lifting property against all trivial fibration is exactly the class of cofibrations described in the statement of Theorem 1.1. In particular one has:

Not every simplicial set is cofibrant !
A simplicial set X is cofibrant if and only if it is decidable whether a cell of X is degenerate or not.

This introduces some changes compared to the classical situation, for example the left properness of the model structure on simplicial sets is no longer automatic, and the assumption that certain objects need to be cofibrant tends to appear in a lot of results. Compare for example Corollary 3.3.4, Proposition 3.3.5 and Proposition 3.4.1 to their classical counterparts.
One can also show the classical Eilenberg-Zilber lemma, asserting that a cell $x \in X([n])$ can be written uniquely as $\sigma^* y$ for σ a degeneracy and y a non-degenerate cells holds if and only if X is cofibrant. A general constructive version of the Eilenberg-Zilber lemma can be found as Lemma 5.1.2 in [8] and does implies that the statement above holds for cofibrant simplicial sets. The converse (that the validity of the Eilenberg-Zilber lemma implies

cofibrancy of X) is immediate from the decidability of equality between morphisms of the category Δ: if a cell is written $\sigma^* y$ with y non-degenerate one can decide if it is degenerate or not depending on if σ is the identity (an isomorphism) or not.

The general structure of the proof of this theorem (and in fact of the paper) is as follows:

- In Section 2.1 we review the existence of a "weak model structure" on simplicial sets and semi-simplicial sets from [8], which is our starting point.
- In Section 2.2, more precisely in Theorem 2.2.9, we will (up to a technical detail, see the Remark 1.4 below) extend this to a model structure on the category of simplicial sets with cofibrations (and trivial fibrations) as specified above, but we will not show that trivial cofibrations are the same as anodyne cofibrations, or equivalently that the fibrations (called "strong fibrations") are the Kan fibrations. This part is based on the use of semi-simplicial sets.
- Left properness of this model structure also follows from semi-simplicial techniques (see Proposition 2.2.10).
- The overall goal[1] of Section 3 is to introduce Kan's Ex^∞-functor. This is done following the work of S. Moss from [15], which can be made constructive at the cost of only minor modification. This will allow us to show that the fibrations of the model structure above are exactly the Kan fibrations (Proposition 3.5.1) and to prove the right properness of this model structure (Proposition 3.5.2), as well as to fix a small gap in constructiveness of Section 2.2 (see the remark below).

1.4 Remark. The gap we are referring too in this last point is that in Section 2.2, the "strong fibrations" (i.e. the fibrations of the model structure on simplicial sets) are defined as the map having the right lifting property against all cofibrations which are equivalences. It is unclear if they can be

[1] We will give a more detailed account of its contents at the beginning of this section.

defined by a lifting property against a small set and hence if trivial cofibration/strong cofibration do form a weak factorization system as a model category structure should require. In Proposition 2.2.7 we give a formal argument that shows it is the case, but it is unlikely that this argument can be made constructive. What definitely solve the problem constructively is the proof in Proposition 3.5.1 that this factorization is actually just the "anodyne cofibrations/Kan fibrations" factorization, but this require all the material of Section 3.
This being said, the reader should note that even before Section 3, it holds constructively that the factorization as an anodyne cofibration followed by a Kan fibration of an arrow with fibrant target is a "trivial cofibration/strong fibration" factorization (because of the third point of Lemma 2.2.6). Hence it holds constructively, even without the results of Section 3, that any arrow with fibrant target admits such a factorization, i.e. one already has something like a right[2]) semi-model category without invoking the properties of Kan Ex^{∞} functor.

1.5 Remark. The fact that we need to invoke the good properties of Kan's Ex^{∞} functor to show that the class of fibration is indeed the class of Kan fibrations of course remind us of D-C.Cisinski's approach to the construction of Kan–Quillen model structure in [3]. We do not really know how deep are the similarities between our proof and D-C.Cisinski's proof. Our initial plan on this problem was actually to try to see if this approach of Cisinski can be made constructive or not. While we definitely do not exclude the possibility that this is the case, it seemed to represent a considerably harder task than what we have achieved here. One of the problems is that Cisinski's theory relies heavily on a set theoretical argument similar to the one we mention in the proof of Proposition 2.2.7, whose constructiveness seems unlikely, and we have not been able to separate his proof that fibrations are the Kan fibrations from this set theoretic argument. The other problem being simply that Cisinski's approach, while very elegant, relies on a considerable amount of machinery whose constructivity would have to be carefully checked, while S. Moss approach, while more technical is considerably more self-contained.

[2]More precisely, we have a right semi-model structure in the sense of Fresse from [4], but not in the sense of Spitzweck from [17].

1.6 Remark. Finally, I only said that "constructive" meant something like internal logic of an elementary topos with a natural number object for simplicity, but everything is actually completely predicative for some, relatively strong, sense of this word. I believe that everything can be formalized within the internal logic of an "Arithmetic universe", i.e. a pretopos with parametrized list objects (see for example [13]). Such a formalization of course requires some modifications: for example it wouldn't make sense to say that a morphism "is a fibration" in the sense that "there exists a structure of fibrations on the morphisms" as the set of all "structure of fibration" on a given morphism cannot be defined, but it would make sense to consider a morphism endowed with a structure of fibration, and to show that given such a pair one can perform some construction.
Although working in such framework in an explicit way forces us to be extremely careful about a huge number of details and makes everything considerably more complicated, and would make the paper considerably longer. For this reason we will not do it explicitly. It seems to me that this is typically the sort of thing that should be done with a proof assistant.
There is one part of this claim that I have not checked carefully: Whether such a weak framework is sufficient to use the case of the small object argument that we need, i.e. construct the cofibration/trivial fibration and the anodyne cofibration/Kan fibration factorization systems (generated respectively by boundary inclusion and horn inclusion) on simplicial set and semi-simplicial sets, though it seems reasonable that a complicated encoding using list object can achieve this. More precisely this should follow from the fact that the initial model theorem for partial horn theories of Vickers and Palmgren in [16] is believed to be provable internally in an arithmetic universe, and the factorization obtained from R.Garner's version of the small object argument (from [7]) are constructed as certain initial structure that can be described using partial horn logic.

1.7 Remark. In a joint paper with Nicola Gambino [5], we will show that this Quillen model structure on simplicial sets admit all the necessary structure to interpret homotopy type theory, with type and context being interpreted as bifibrant objects. This was the main motivation for the present paper and the two papers have been written in close connection. I would also like to thanks Nicola Gambino for the helpful comments he made about earlier versions of the present paper.

1.8 Notation. Δ and Δ_+ denotes the category of finite non-empty ordinal, respectively with non-decreasing map and non-decreasing injection between them. $\widehat{\Delta}$ is the category of simplicial sets and $\widehat{\Delta_+}$ is the category of semi-simplicial sets (see 2.1.2). One denotes by $\Delta[n]$ and $\Delta_+[n]$ the representable simplicial and semi-simplicial sets corresponding to the ordinal $[n] = \{0, \dots, n\}$. Our usual notation for the boundary of the n-simplex and its k-th horn, both for simplicial and semi-simplicial versions are: $\partial\Delta[n] \quad \Lambda^k[n] \quad \partial\Delta_+[n] \quad \Lambda^k_+[n]$
The boundary inclusion map is denotes ∂_n or $\partial[n] : \partial\Delta[n] \to \Delta[n]$, the i-th face map is denoted $\partial^i[n]$ or ∂^i_n or just $\partial^i : \Delta[n-1] \to \Delta[n]$, for the map corresponding to the order preserving injection from $[n-1]$ to $[n]$ which only skip i. The degeneracy $\Delta[n+1] \to \Delta[n]$ that hits i twice is denoted σ^i. Given a simplicial or semi-simplicial sets X, the image of a cell $x \in X_n$ be the i-th face map is denoted $d_i x$.

1.9 Notation. Finally, we will define many different classes of maps between simplicial and semi-simplicial sets. To help the reader navigate this, we list them all here and recall their definition. This is not meant to be read at this point, but used as a reference latter if the reader needs to remember what a certain class of maps is. In particular, many of the claim we make here will be properly justified latter in the paper.
In the category $\widehat{\Delta}$ of simplicial sets, we consider the following classes of maps:

- *Trivial fibrations* are the map with the right lifting property against the boundary inclusions $\partial\Delta[n] \to \Delta[n]$.
- *Cofibrations* are the map with the left lifting property against trivial fibration. They are also the retracts of ω-transfinite compositions of pushouts of coproducts of boundary inclusions. It is shown as proposition 5.1.4 of [8] that cofibrations can be characterized as inclusion satisfying some decidability conditions as stated in Theorem 1.1.
- *Kan fibrations* are the map with the right lifting property against the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$.
- *Anodyne cofibrations* are the map with the left lifting property against Kan fibrations. Equivalently, they are the retract of ω-transfinite compositions of pushouts of coproducts of horn inclusions.

- *weak equivalences* where defined for arrow between objects that are either fibrant or cofibrant in [8] as maps that are invertible in the homotopy category (the homotopy category being defined using homotopy class of maps between bifibrant objects). In the present paper we extend the definition to general objects by redefining weak equivalences of simplicial sets as the map that are weak equivalences of the underlying semi-simplicial sets. The usual characterization using homotopy groups can also be used as long as homotopy groups are defined as setoids, see proposition 5.2.6 of [8].

- *trivial cofibrations* are the maps that are both cofibrations and weak equivalences. We show that trivial cofibrations and anodyne cofibrations are the same in Proposition 3.5.1, but this is one of the last result of the paper, so almost everywhere in the paper these class are assumed to be potentially different.

- *Strong fibrations* are the map that have the right lifting property agains trivial cofibrations. It also follows from Proposition 3.5.1 that they are the same as Kan fibration.

- *Degeneracy quotient* and *degeneracy detecting maps* is a unique factorization system on $\widehat{\Delta}$ which is studied in Section 3.1. It mostly serves as a technical tool to establish decidability conditions that are central to make the proofs in Section 3.4 constructive.

In the category $\widehat{\Delta_+}$ of semi-ssimplicial sets, we consider the following classes of maps:

- *Trivial fibrations* and *Kan fibrations* are defined as the maps with the right lifting property against respectively the semi-simplicial boundary inclusion $\partial\Delta_+[n] \to \Delta_+[n]$ and the semi-simplicial horn inclusion $\Lambda^k_+[n] \hookrightarrow \Delta_+[n]$.

- *Cofibrations* and *anodyne cofibrations* are defined as the map with the left lifting property against respectively *trivial fibrations* and *Kan fibrations*. They can also be described as the maps that are retracts of ω-transfinite compositions of pushouts of coproducts of respectively the boundary inclusion and the horn inclusion. Semi-simplicial cofibrations have been shown in Section 5 of [8] to be exactly the inclusion that are levelwise complemented (the precise statement is in Theorem 5.5.6 of [8], the proof is the same as for Proposition 5.1.4).

- *Weak equivalences* are the maps that are invertible in the homotopy category of the weak model structure defined by the maps above. Because every semi-simplicial set is cofibrant, the notion makes sense for arbitrary maps (in a weak model category, only objects which are either fibrant or cofibrant have an image in the homotopy category).
- *Trivial cofibrations* are the cofibrations which are weak equivalences. In $\widehat{\Delta}$ trivial cofibration and anodyne cofibrations are not expected to be the same. Trivial cofibrations have the right lifting property against all Kan fibrations between fibrant objects, but not against general Kan fibrations.
- Of course, one could also define the class of *strong fibrations*, as the maps with the right lifting property against all trivial cofibrations, but the notion turn out to serve no purpose in the present paper.

2. Constructing the model structure

2.1 Review of the weak model structures

2.1.1. One of the achievement of [8], which is the starting point of the present paper, is the construction of a "weak model structure" on the category of simplicial sets where fibrations (between fibrant objects) and cofibrations (between cofibrant objects) are as specified above.
More explicitly this means that there is a class of maps called "equivalences[3]" in the category of simplicial sets that are either fibrant or cofibrant (in the sense above) such that:

- Weak equivalences (between objects that are either fibrants or cofibrant) contains isomorphisms, are stable under composition and satisfies 2-out-of-3 (and the stronger 2-out-of-6 property).

[3]In most of the literature this are called weak equivalence, though we can't think of any reasons to keep the adjective "weak" other than history, so we will simply drop it.

- A cofibration between cofibrant objects is a weak equivalence if and only if it has the left lifting properties against all fibrations between fibrant objects (such a map is called a trivial cofibration).
- A fibrations between fibrant objects is a trivial fibration if and only if it is a weak equivalence[4].
- The localization of the category of fibrant or cofibrant objects at the weak equivalences can be described as the category of fibrant and cofibrant objects with homotopy classes of maps between them. Where the homotopy relation is defined as usual, using equivalently a path object or a cylinder object. This localization is called the homotopy category.
- The weak equivalences are exactly the morphisms that are invertible in the homotopy category (which proves the first point immediately).

One can deduce from this various characterization of weak equivalences: for example, a map from a cofibrant object to a fibrant object is a weak equivalence if and only if it can be factored as a trivial cofibration followed by a trivial fibration. Note that at this point it does not makes sense to ask whether a map $X \to Y$ is a weak equivalence if one of X or Y is neither fibrant nor cofibrant.

2.1.2. In [8, theorem 5.5.6] we also showed that a similar "weak model structure" exists on the category of semi-simplicial sets. Semi-simplicial sets are "simplicial sets without degeneracies", i.e. collection of sets $X_0, \dots, X_n, \dots$ with "face maps" satisfying the same relations as the face maps of a simplicial sets. Equivalently they are presheaves on the category Δ_+ of finite non-empty ordinals and injective order preserving maps between them.
The generating cofibrations in the category of semi-simplicial sets are the semi-simplicial boundary inclusion:

$$\partial\Delta_+[n] \hookrightarrow \Delta_+[n],$$

[4]Here we use the fact that that trivial fibrations are characterized by a lifting property against cofibration between cofibrant objects, which might not be the case in a general weak model category.

where $\partial\Delta_+[n]$ and $\Delta_+[n]$ respectively denotes the semi-simplicial subset of non-degenerate cells in $\Delta[n]$ and $\partial\Delta[n]$. Note that the $\Delta_+[n]$ also corresponds to the representable semi-simplicial sets, so that a morphism $\Delta_+[n] \to X$ is the same as an n-cell of X and a morphism $\partial\Delta_+[n] \to X$ is the data of a collection of n cells of dimension $n-1$ with compatible boundary exactly as simplicial morphisms from $\partial\Delta[n]$ to a simplicial sets X. In particular a morphism $f : X \to Y$ of simplicial sets is a trivial fibration if and only if its image by the forgetful functor to semi-simplicial sets is a trivial fibration (in the sense that it has the right lifting property against the generating cofibration).
As there are no degeneracies in $\widehat{\Delta_+}$ the description of cofibrations simplifies to just "levelwise complemented monomorphism" i.e. the class of monomorphism $f : X \to Y$ such that for each n, and for each $y \in Y([n])$ it is decidable whether $y \in X([n])$ or not (this is also discussed in [8, theorem 5.5.6]). In particular, every semi-simplicial set is cofibrant.
Similarly, a morphism of semi-simplicial sets is said to be a Kan fibration when it has the lifting property against the semi-simplicial version of the horn inclusion $\Lambda^k_+[n] \hookrightarrow \Delta_+[n]$, where $\Lambda^k_+[n]$ and $\Delta_+[n]$ respectively denotes respectively the semi-simplicial sets of non-degenerate cells in $\Lambda^k[n]$ and $\Delta[n]$). As above a simplicial morphism between simplicial sets is a Kan fibration if and only if its image by the forgetful functor to simplicial sets is a Kan fibration of semi-simplicial sets.
In this weak model structure on semi-simplicial sets, the cofibrations are as described above, the fibrant objects are the semi-simplicial Kan complexes and the fibrations and trivial fibrations between fibrant objects are the Kan fibrations and trivial fibrations. The big difference with the model structure on simplicial sets is that as every semi-simplicial set is cofibrant, the classes of weak equivalences is defined between arbitrary objects of the category. Note that we do not claim that every trivial cofibration (i.e. cofibration which is an equivalence) is an anodyne cofibration (i.e. a retract of a transfinite composition of pushout of coproducts of semi-simplicial horn inclusion) : the anodyne cofibration have the left lifting property against all Kan fibrations, the trivial cofibration only against Kan fibration between Kan complexes.

2.1.3 Remark. Note that it is well known, even classically, that this model structure cannot be a Quillen model structure. As every object is cofibrant, it can be seen by a combinatorial argument that, at least classically, it is a "right semi-model structure" in the sense of [2]). But for example the codiagonal map $\Delta_+[0] \coprod \Delta_+[0] \to \Delta_+[0]$, where $\Delta_+[0]$ denotes the representable semi-simplicial sets by the ordinal $[0] = \{0\}$ is easily seen to have the lifting property of trivial fibrations (there is no higher cells to lift !) while it is clearly not a weak equivalence.

The forgetful functor from simplicial sets to semi-simplicial sets is very well behaved: we showed in [8, theorem 5.5.6] that it is both a left and right Quillen equivalence, and we will prove as Proposition 2.2.2 that it preserves and detect weak equivalences without any assumption of fibrancy/cofibrancy. As all object in $\widehat{\Delta_+}$ are cofibrant, this will allow to remove some assumption of cofibrancy in various places.

Sketch of proof of 2.1.1. We finish this section by presenting the main steps of the argument given in [8] of the existence of the weak model structure on simplicial sets, i.e. all the claims made in 2.1.1. The details of this can be found in [8], but we hope the following summary will be of help to the reader. The proof for semi-simplicial sets is similar.
The first (and essentially only) important technical step is the proof of the so-called "pushout-product" or "corner-product" conditions for the simplicial generating cofibrations and trivial cofibrations. This follows from a completely constructive results of Joyal (theorem 3.2.2 of [11]), in [8] it corresponds to Lemma 5.2.2 (and how it is used in the proof of Theorem 5.2.1 in Corollary 5.2.3). In the present paper we also reproduce a different proof of this claim as Proposition 3.2.6, which is due to S. Moss (see [15, 2.12]).
From the corner-product condition one deduces formally[5] all the usual properties of stability of cofibrations, anodyne cofibrations, fibrations, and trivial fibrations under product and exponential expected in a cartesian model category (see Proposition 3.2.6 and the comment directly below it).
This allows to construct nicely behaved cylinder objects as $\Delta[1] \times X$ and path objects as $X^{\Delta[1]}$, whose legs are appropriately (trivial) (co)fibrations as soon as X is (co)fibrant. More generally, one can construct relative path objects for any fibration $X \twoheadrightarrow Y$ and relative cylinder objects for any cofibration $A \hookrightarrow Y$. Having such relative cylinders and path objects is the definition

[5] using the so-called "Joyal-Tierney calculus" presented in the appendix of [10], though this types of manipulation were known before, maybe in a less elegant or general way.

of weak model structure that we gave in section 2 of [8]. The precise observation that one gets a weak model structure from such a tensor product satisfying the corner-product condition is essentially the construction done in section 3 of [8], summarized by theorem 3.2 there.
Then all the claims made in 2.1.1 follows from the general theory of weak model structure developed in section 2.1 and 2.2 of [8]. We sketch the general strategy here, though at this point we recommend looking directly at subsection 2.1 and 2.2 of [8] which are mostly self contained.
One uses these cylinders and path objects to define the homotopy relation between maps from a cofibrant object to a fibrant object. Using the lifting property one show that the homotopy relation with respect to any cylinder object is equivalent to the homotopy relation with respect to any path object and that these define an equivalence relation compatible to pre-composition and post-composition. The proof is essentially the same as in a full Quillen model structure: the definition of weak model structure is exactly tailored so that the usual proof of these claims can be applied.
This allows to give a first definition of the homotopy category as the category whose objects are the fibrant-cofibrant objects and the maps are the homotopy class of maps. One then proves formally that this homotopy category is equivalent to various localization (see Theorem 2.2.6 in [8]), the last one being the localization of the category of simplicial sets that are either fibrant or cofibrant at all trivial cofibrations with cofibrant domains and all trivial fibrations with fibrant targets. One can then define weak equivalences as the arrow that are invertible in this localization, and one automatically have 2-out-of-6 and all the other good properties of weak equivalences. The fact that trivial fibrations with fibrant target are exactly the fibration (with fibrant targets) that are equivalence is a little harder and use again the properties of the relative path objects (see proposition 2.2.10 in [8]), and similarly for cofibrations. □

2.2 The simplicial model structure

To obtain that simplicial sets form a full Quillen model structure we first need to extend the meaning of "equivalences" so that it makes sense also for arrows between objects that are neither fibrant nor cofibrant. We will do this by exploiting the forgetful functor from the category of simplicial sets to the

category $\widehat{\Delta_+}$ of semi-simplicial sets. As in the category of semi-simplicial sets every object is cofibrant the notion of weak equivalence there is defined for arbitrary arrows, and we will show it is reasonable to define equivalences of simplicial sets as arrows that are equivalences of the underlying semi-simplicial sets.
We start by the following observation:

2.2.1 Lemma.

1. *If $f : X \to Y$ is an anodyne cofibration in $\widehat{\Delta}$, then its image in $\widehat{\Delta_+}$ is also an anodyne cofibration, and in particular is an equivalence.*

2. *Let $f : X \twoheadrightarrow Y$ be a trivial fibration in $\widehat{\Delta}$. Then the image of f in $\widehat{\Delta_+}$ is an equivalence.*

Note that in the second case, it is obvious that f is a trivial fibration in $\widehat{\Delta_+}$, but this is not enough to deduce that is is an equivalence in general, unless its target is fibrant, as $\widehat{\Delta_+}$ only has a weak model structure.

Proof.

- This is corollary 5.5.15.(ii) of [8].

- We first assume that X is cofibrant. In this case one can construct a strong cylinder object for X using the cartesian structure of simplicial sets:

$$X \coprod X \hookrightarrow \Delta[1] \times X \overset{\sim}{\to} X$$

  with the two maps $X \hookrightarrow \Delta[1] \times X$ being anodyne cofibrations (this follows from the fact that X is cofibrant and the corner-product conditions). Because of point 2, this produces a strong cylinder object for the underlying semi-simplicial set of X in the category of semi-simplicial sets.

  In $\widehat{\Delta_+}$, every object is cofibrant, and the arrow $f : X \to Y$ is still a trivial fibration, so one can find some dotted lifting for the following two squares in $\widehat{\Delta_+}$:

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & X \\
\big\downarrow & \overset{s}{\nearrow} & \big\downarrow \\
Y & \longrightarrow & Y
\end{array}
\qquad
\begin{array}{ccccc}
X \coprod X & & \xrightarrow{(Id_X, sf)} & & X \\
\big\downarrow & & \overset{h}{\nearrow} & & \big\downarrow \\
\Delta[1] \times X & \longrightarrow & X & \longrightarrow & Y
\end{array}
$$

In particular, s is a section of f, i.e. $fs = Id_Y$, and h a homotopy between Id_X and sf. Hence s is an inverse of f in the homotopy category of $\widehat{\Delta_+}$, which makes f an equivalence in $\widehat{\Delta_+}$.

In the general case (when we do not assume that X is cofibrant), one takes a cofibrant replacement (with a trivial cofibration) $X^c \overset{\sim}{\twoheadrightarrow} X$ and the result above applies to both the trivial fibration $X^c \overset{\sim}{\twoheadrightarrow} X$ and the composite trivial fibration $X^c \overset{\sim}{\twoheadrightarrow} Y$. By 2-out-of-3 for weak equivalences in $\widehat{\Delta_+}$ this implies that the map $X \overset{\sim}{\twoheadrightarrow} Y$ is indeed an equivalence in $\widehat{\Delta_+}$.

□

2.2.2 Proposition. *For a morphism $f : X \to Y$ between simplicial sets that are either fibrant or cofibrant the following are equivalent:*

- *f is an equivalence for the weak model structure in $\widehat{\Delta}$.*
- *The image of f in $\widehat{\Delta_+}$ is an equivalence for the weak model structure on $\widehat{\Delta_+}$*

Proof. If Y is cofibrant, then one can take a fibrant replacement $Y \overset{\sim}{\hookrightarrow} Y^f$. The map $Y \overset{\sim}{\hookrightarrow} Y^f$ is an equivalence both in $\widehat{\Delta}$ and $\widehat{\Delta_+}$, so in both categories f is an equivalence if and only if the composite $X \to Y^f$ is an equivalence, so it is enough to prove the result when Y is fibrant. A similar argument using a cofibrant replacement allows to assume that X is cofibrant.
Assuming both X cofibrant and Y fibrant, one factors f as an anodyne cofibration (with cofibrant domain) followed by a Kan fibration (with fibrant target). The anodyne cofibration is an equivalence in both categories, hence (in both categories) f is an equivalence if and only if the Kan fibration part

is a trivial fibration. But for a map in $\widehat{\Delta}$, being a trivial fibration in $\widehat{\Delta}$ and in $\widehat{\Delta_+}$ are the exact same condition (the lifting property only involves face operations, no degeneracies).

□

This last proposition makes the following definition very reasonable:

2.2.3 Definition.

- An arrow in $\widehat{\Delta}$ is said to be an *equivalence* if its image by the forgetful functor to $\widehat{\Delta_+}$ is an equivalence for the semi-simplicial version of the Kan–Quillen weak model structure mentioned in 2.1.2.
- A *trivial cofibration* is a cofibration which is also an equivalence.
- A *strong fibration* is an arrow that has the right lifting property against all trivial cofibrations.

We remind that the reader, that we will prove in Proposition 3.5.1 that these notions of strong fibrations and trivial cofibrations are equivalent to the usual notion of Kan fibrations and anodyne cofibrations.

2.2.4 Remark. With this definition it is immediate that:

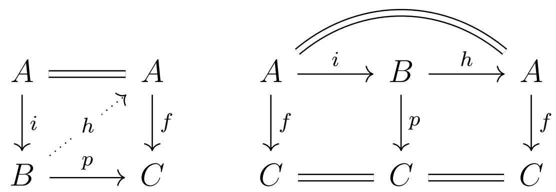
- Isomorphisms are equivalences, and equivalences are stable under composition, satisfies the 2-out-of-3 and even the 2-out-of-6 properties.
- Anodyne cofibrations are trivial cofibrations. Indeed they are cofibrations by definition and they are equivalences in the sense of Definition 2.2.3 by point 1 of Lemma 2.2.1.
- As a consequence, strong fibrations are Kan fibrations.
- Trivial fibrations, defined by the right lifting property against boundary inclusions, are both strong fibrations because they have the right lifting property against all cofibrations, and equivalences because of point 2 of Lemma 2.2.1.

- A Kan fibration (or strong fibrations) with fibrant target is a trivial fibration if and only if it is an equivalence (this follows from proposition Proposition 2.2.2 and the fact that this fact holds in weak model categories).

Maybe it is a good point to recall the following very classical lemma that we will use constantly in this paper:

2.2.5 Lemma. *Assume that a map f is factored as $f = pi$. If i has the left lifting property against f, then f is a retract of p. If p has the right lifting property against f then f is a retract of i.*

Proof. We only prove the first half of the claim, the second is just the dual statement. One form a morphism h as the dotted diagonal filler in first square below (obtained by the lifting property of i against f), which can then be used to form a retract diagram:



□

2.2.6 Lemma.

(i) A cofibration is a trivial cofibration if and only if it has the left lifting property against all Kan fibrations between Kan complexes.

(ii) An arrow whose target is a Kan complex is a trivial cofibration if and only if it is anodyne.

(iii) An arrow whose target is a Kan complex is a strong fibration if and only if it is a Kan fibration.

(iv) A map is a trivial fibration if and only if it is a strong fibration and an equivalence.

Because of the third point it is equivalent for a simplicial set X that $X \to 1$ is a a Kan fibration (i.e. X is a Kan complex) and that $X \to 1$ is a strong fibration. One will simply say that X is fibrant.

Proof.

(i) Let $f : A \hookrightarrow B$ be a cofibration that is also an equivalence, and we consider a lifting problem of f against a Kan fibration between Kan complexes:

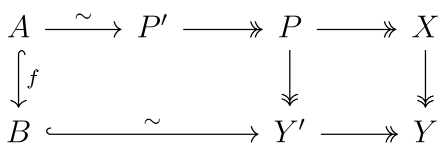
$$\begin{array}{ccc} A & \xrightarrow{v} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{u} & Y \end{array}$$

In the special case where both u and v are equivalences, then by 2-out-of-3, the map p is also an equivalence. As it is a Kan fibration between Kan complexes it is also a trivial fibration, and hence the lifting problem has a solution because f is a cofibration. We will now show that one can bring back the general case to this situation:

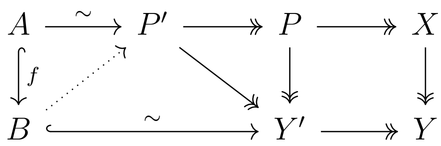
One can factor u as an anodyne cofibration followed by a Kan fibration: $B \overset{\sim}{\hookrightarrow} Y' \twoheadrightarrow Y$ and complete the diagram above by forming the pullback $P = Y' \times_Y X$:

$$\begin{array}{ccccc} A & \xrightarrow{v'} & P & \twoheadrightarrow & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow \\ B & \overset{\sim}{\hookrightarrow} & Y' & \twoheadrightarrow & Y \end{array}$$

The map v' can be factorized as an anodyne cofibration followed by a Kan fibration:



The case treated above, where the two horizontal maps are equivalences, allows to produce a dotted diagonal lifting of the form:



and this concludes the proof in the general case.

Conversely, assume $i : A \hookrightarrow B$ is a cofibration that has the left lifting property against all Kan fibrations between Kan complexes. One needs to show that i is an equivalence. By taking an anodyne cofibration $B \overset{\sim}{\hookrightarrow} B^f$ to a fibrant object the composite $A \hookrightarrow B^f$ still has the announced lifting property so one can freely assume that B is fibrant in order to show that i is an equivalence. Under that assumption one factors i as an anodyne cofibration followed by a Kan fibration, the Kan fibration has a fibrant target so it has the right lifting property against i. Hence by the retract lemma 2.2.5, i is a retract of the anodyne cofibration part of the factorization, hence it is an anodyne cofibration itself, so that we can conclude that i is an equivalence.

(ii) As mentioned in Remark 2.2.4, anodyne cofibrations are trivial cofibrations. So we only need to show the converse. In the proof of point (1), we have shown in the first part that a trivial cofibration have the left lifting property against fibrations between fibrant objects and in the second that any map with this lifting property and whose target is a Kan complex is an anodyne cofibration. Together this indeed shows that a trivial cofibration whose target is a Kan complexe is an anodyne cofibration.

(iii) We have mentioned already that strong fibrations are Kan fibrations, and (i) shows that Kan fibrations between Kan complexes are strong fibrations.

(iv) Trivial fibration have the right lifting property against all cofibrations, in particular against trivial cofibrations hence they are strong fibrations, and point 2 of Lemma 2.2.1 shows they are equivalences. For the other direction, the proof is essentially the dual the proof of (i). Let p be a strong fibration that is also a weak equivalence, and consider a lifting problem of p against a cofibration:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ B & \longrightarrow & Y \end{array}$$

By factoring the map $A \to X$ into a cofibration $A \to A'$ followed by a trivial fibrations and taking the pushout of $A \hookrightarrow B$ along this map $A \to A'$ one reduces the problem to the case where the top map is an equivalence. One can then factor the bottom map as a cofibration followed a trivial fibration:

$$\begin{array}{ccccccc} A & \hookrightarrow & A' & & \xrightarrow{\quad\sim\quad} & & X \\ \big\downarrow & & \big\downarrow & & & \nearrow & \big\downarrow p \\ B & \hookrightarrow & B' & \hookrightarrow & Y' & \xrightarrow{\sim} & Y \end{array}$$

where the dotted arrow exists because the composed cofibration $A \hookrightarrow Y'$ is a weak equivalence by the 2-out-of-3 properties, and hence has the left lifting property against p. This provides a dotted filling for the initial square.

□

In order to conclude that one has a model structure on simplicial sets, one needs one more proposition.

2.2.7 Proposition. *Any morphism can be factored as a trivial cofibration followed by a strong fibration.*

Again, we will show in Proposition 3.5.1 that this factorization system is actually the same as the anodyne cofibrations/Kan fibrations factorization system, i.e. that trivial cofibrations are the same anodyne cofibrations and that strong fibration are the same as Kan fibrations. Note that at this point it is immediate that anodyne cofibrations are trivial cofibrations, and hence that fibrations are Kan fibrations.

Proof. We will give two proofs of this claim. The first one follows from [14], more precisely its Theorem 3.2, which is not known to be constructive but allows to give a simple and direct proof of the present proposition.
In order to fix the issue with constructivity one gives a second, considerably less direct proof: as mentioned above in Proposition 3.5.1 we will prove independently of the present proposition that trivial cofibrations are the same as anodyne cofibrations, hence showing that the weak factorization mentioned in the proposition exists and is simply the anodyne cofibration/Kan fibration weak factorization system (whose existence follows from the small object arguments).
We still give the first proof as we believe it is interesting on its own as it allows to construct the model structure on simplicial sets without needing to invoke Kan Ex^{∞}-functor.
Theorem 3.2 of [14] claims that the 2-category of presentable categories endowed with a class of cellular morphisms generated by a set of morphisms is closed under pseudo-pullback, and that these pullback are constructed explicitly: the underlying category is the pullback of categories, and the class of cellular morphisms are the morphisms whose image in each component are in the specified classes. We apply this to the following square:

$$\begin{array}{ccc} P & \longrightarrow & (\text{Kan-Cplx}, \text{TrivFib}) \\ \downarrow & \lrcorner & \downarrow \\ (\widehat{\Delta}, Cof) & \longrightarrow & (\text{Kan-Cplx}, \text{All arrows}) \end{array}$$

here "Cof" denotes the class of cofibration in $\widehat{\Delta}$ which is generated by a set. Kan-Cplx denotes the category of "algebraic Kan complexes", i.e. simplicial set endowed with chosen lifting against horn inclusion and of morphisms compatible to these choices of lifting. The functor $\widehat{\Delta} \to$ Kan-Cplx sends any simplicial set to the "free algebraic Kan complexes it generates",i.e. the left adjoint to the forgetful functor from algebraic Kan complex to simplicial set, or equivalently the functor sending a simplicial set to its canonical fibrant replacement as produced by R.Garner version of the small object argument. The class TrivFib is the left class of the weak factorization on Kan-Cplx cofibrantly generated by the image of the horn inclusion in $\widehat{\Delta}$. The right class of the weak factorization system is hence exactly the class of morphisms whose image by the forgetful functor to $\widehat{\Delta}$ are Kan fibrations. It follows that the morphism in $\widehat{\Delta}$ which are sent to "trivial cofibrations" in Kan-Cplx are exactly the arrows that have the left lifting property against all Kan fibration between Kan complexes. Hence in this case the pullback is the category of simplicial sets with as set of cellular morphisms the maps that are both cofibrations and have the left lifting property against Kan fibration between Kan complexes, i.e. the "trivial cofibrations" as defined above, hence this class of arrow is generated by a set, and hence by the small object argument it is one half of a weak factorization system. □

2.2.8 Remark. After writting this paper, the non-constructive argument used in the proof of Proposition 2.2.7 have been considerably generalized in section 4 of [9], leading to the general notion of "left and right saturation" of a combinatorial or accessible pre-model category. This is a special case of left saturation of a combinatorial pre-model category.

2.2.9 Theorem. *There is a model structure on the category of simplicial sets such that:*

- *The equivalences are as in Definition 2.2.3.*
- *The cofibrations and trivial fibrations are the same as in Theorem 1.1.*
- *The fibrations are the strong fibration of Definition 2.2.3.*

Proof. We have two weak factorization systems, trivial cofibrations have been defined as the cofibrations that are equivalences, and it was shown in

Lemma 2.2.6 that trivial fibrations are the (strong) fibrations that are equivalences. Equivalences are stable by composition, satisfies 2-out-of-6 and contains isomorphisms by definition, so this concludes the proof. □

2.2.10 Proposition. *The model structure of Theorem 2.2.9 is left proper, i.e. the pushout of a weak equivalence along a cofibration is a weak equivalence.*

Proof. Given a pushout square in the category of simplicial sets:

$$\begin{array}{ccc} A & \xrightarrow{\sim} & C \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & D \end{array}$$

then as the forgetful functor to semi-simplicial sets preserves all colimits, this square is again a pushout in the category of semi-simplicial sets. In this category every object is cofibrant, and pushout along a cofibration between cofibrant objects is a left Quillen functor hence preserves equivalences between cofibrant objects, hence f is an equivalence in the category of semi-simplicial sets, and hence is an equivalence in $\widehat{\Delta}$ by Definition 2.2.3. □

3. Kan Ex$^\infty$-functor

The goal of this section is to introduce Kan's Ex and Ex$^\infty$ functors and to use them in Section 3.5 to prove the remaining claim concerning the simplicial model structure. Most of the results here were (in their classical form) originally proved by Kan in [12] (often with quite different proof than the ones we will provide here), but we will mostly follow the approach of S.Moss in [15] which we will make constructive by only adjusting some details.

Section 3.1 is a preliminary section that is of some independent interest but which will have only a very marginal role in the paper: it will only be used to prove some decidability conditions (more precisely Lemma 3.4.4, which will be an easy consequence of Lemma 3.1.8 and Proposition 3.1.10). As such it can be easily ignored by the reader.

Section 3.2 review the notion of "P-structure" introduced by S.Moss, which is mostly a language to talk more conveniently about "Strongly anodyne cofibrations", i.e. transfinite composition of pushouts of coproducts of horn inclusion. This is a key tool to structure the proof of the main results of Section 3.4.
Section 3.3 introduce Kan's barycentric subdivision functor SD, its right adjoint EX and Kan's Ex^∞ functor and proves some of their basic properties. This is very classical material that we reproduce here mostly for completeness and to discuss some constructive aspect.
Section 3.4 reproduces (with some modifications to make it constructive) S.Moss' proof in [15] that the natural transformation $X \to \text{Ex}^\infty X$ is an anodyne cofibration. Constructively this only works when X is cofibrant. We also noted that S.Moss proof can be used to obtain a result which apparently was not known even classically: for any morphisms $f : X \to Y$ (with X cofibrant) the natural morphism:

$$X \to \text{Ex}^\infty X \times_{\text{Ex}^\infty Y} Y$$

is anodyne. This was known classically when Y is terminal, or when $X \to Y$ is a fibration, and we will actually only use it in these two special cases, but it appears that they can be proved at the same time using S. Moss' argument.
Finally Section 3.5 uses the properties of this functor to conclude that all Kan fibrations are strong fibrations (Proposition 3.5.1) and that the model structure on simplicial sets is indeed right proper (Proposition 3.5.2).

3.1 Degeneracy quotient and questions of decidability

In this section we establish some general results about a notion of "degeneracy quotient" that we will introduce. While the notion might have some interest on its own in other context its only use in the present paper is to prove some decidability results, which will follow from Lemma 3.1.8 below. In fact, the only uses of this section in the present paper is in the proof of the decidability conditions of Lemma 3.4.4. Proposition 3.1.11 is not useful for the present paper, but will serve in some future work, in particular in [5] and it was more natural to include its proof here.

3.1.1 Definition. A morphism $f : X \to Y$ between simplicial sets is said to be *degeneracy detecting* if:

$$\forall x \in X, \quad f(x) \text{ is a degenerate cell} \Rightarrow x \text{ is a degenerate cell}$$

Of course the converse implication is true for any simplicial map, so one has that x is a degenerate cell if and only if $f(x)$ is. One says that a cell $x \in X_n$ is σ-degenerate for some degeneracy $\sigma : [n] \to [m]$ if $x = \sigma^* y$ for some y.

3.1.2 Lemma. *Let $\sigma : [n] \to [m]$ be any degeneracy and $x \in X_n$ any cell. The following are equivalent:*

(i) x is σ-degenerate.

(ii) For all face maps $i : [k] \to [n]$ such that the composite σi is non-injective, the cell $i^* x$ is degenerate.

Proof. If $x = \sigma^* y$ then for any such i, $i^* x = (\sigma i)^* y$ which is degenerate if σi is non-injective, so $(i) \Rightarrow (ii)$.
Conversely, let x satisfy (ii). If σ is the identity the result is trivial. If σ is not injective, then x is in particular a degenerate cell, i.e. there exist a non-trivial degeneracy $s : [n] \to [k]$ such that $x = s^* y$. Note that $y = d^* x$ for $d : [k] \to [n]$ any section of s. If for all section d of s, σd is injective, then Lemma 3.1.3 below shows that s factors as $j\sigma$ for some degeneracy $j : [m] \to [k]$ and $x = s^* y = \sigma^* j^* y$ is indeed σ-degenerate. If now σd is non-injective for some section d of s, then $y = d^* x$ is a degenerate cell by assumptions, hence one can write $x = s'^* y'$ for y' of lower dimension than x and start the argument above again, an induction on the dimension concludes the proof. □

3.1.3 Lemma. *Let $\sigma : [n] \to [m]$ and $s : [n] \to [k]$ be two degeneracy, assume that for all $d : [k] \to [n]$ a section of s, σd is injective, then there exists a (unique) $j : [m] \to [k]$ such that $s = j\sigma$.*

One easily see it is also a necessary condition.

Proof. One needs to show that, under the assumption of the lemma, for any two elements $i, j \in [n]$ if $\sigma i = \sigma j$ then $si = sj$. If $si \neq sj$, then we can find a section d of s such that $dsi = i$ and $dsj = j$, indeed, in order to get a section of s, we just need to chose for each k the value $d(k) \in s^{-1}\{k\}$.

So as long as $si \neq sj$, we can chose $d(si) = i$ and $d(sj) = j$, and for any other $k \in [m]$, we can, for example, take for $d(k)$ the smalest element of the fiber $s^{-1}\{k\}$. Of course, all this is constructively possible because $[m]$ is a finite decidable set. Given such a section d, we have $\sigma j = (\sigma d)(sj)$ and $\sigma i = (\sigma d)(si)$, so the injectivity of σd implies that $\sigma i \neq \sigma j$. As equality in $[n]$ is decidable one can take the contrapositive and concludes the proof. □

3.1.4 Proposition. *Let $f : X \to Y$ be a map between simplicial sets, then the followings conditions are equivalent:*

(i) f is degeneracy detecting.

(ii) If $f(x)$ is σ-degenerate for some degeneracy σ then x is σ-degenerate as well.

(iii) f has the (unique) right lifting property against all the degeneracy map $\Delta[n] \to \Delta[m]$.

Proof. (ii) clearly implies (i) and the converse is immediate from Lemma 3.1.2. The lifting in (iii) is automatically unique as degeneracy are epimorphisms in the presheaf category and this lifting property is a reformulation of (ii). □

Given a simplicial set X, $x \in X([n])$ and $\sigma : [n] \to [m]$ a degeneracy, one defines $X[(x, \sigma)]$ as the pushout:

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{x} & X \\ \downarrow{\scriptstyle \sigma} & & \downarrow \\ \Delta[m] & \longrightarrow & X[(x,\sigma)] \end{array}$$

$X[(x, \sigma)]$ is the universal for map $X \to Y$ making x "σ-degenerate", i.e. given a morphism $f : X \to Y$, it factors as $X \to X[(x, \sigma)]$ if and only if $f(x) = \sigma^* y$ for some $y \in Y([m])$, and such a factorization is unique when it exists.

More generally, given a collection $(x_i \in X([n_i]))_{i \in I}$ and $\sigma_i : [n_i] \to [m_i]$ one can define an object $X[(x_i, \sigma_i)]$ as the pushout of a coproduct of degeneracy maps, which has the following universal property: a morphism $f : X \to Y$ factors (uniquely) through $X \to X[(x_i, \sigma_i)]$ if and only if for all $i \in I$, $f(x_i)$ is a σ_i-degenerate cell.

3.1.5 Definition. A morphisms is said to be a degeneracy quotient if it is obtained as $X \to X[(x_i, \sigma_i)]$ for some collection of $x_i \in X([n_i])$ and $\sigma_i : [n_i] \twoheadrightarrow [m_i]$ as above.

3.1.6 Proposition. *Degeneracy quotient and degeneracy detecting maps form an orthogonal factorization system.*
More precisely, for any morphism $f : X \to Y$ its factorization is obtained as:

$$X \to X[(x_i, \sigma_i)] \to Y$$

where (x_i, σ_i) is the collection of all x_i and σ_i such that $f(x_i)$ is a σ_i-degenerate cell.

Note that this is essentially nothing more than the small object argument, though it is notable that in this case it converges in a single step.

Proof. It is clear from the universal property of $X[(x_i, \sigma_i)]$ that one has a factorization as in the lemma, and the first map is by definition a degeneracy quotient. The map $X[(x_i, \sigma_i)] \to Y$ is degeneracy detecting: given $x \in X[(x_i, \sigma_i)]$, it is the image of a $x_0 \in X$, if the image of x is a degenerate cell in Y, one has $f(x_0) = \sigma^* y$, hence (x_0, σ) appears in the definition of $X[(x_i, \sigma_i)]$, which forces the image of x_0, i.e. x, to be degenerate.
The orthogonality of the two class is relatively immediate as well. Given a lifting problem:

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ X[(x_i, \sigma_i) & \longrightarrow & B \end{array}$$

where the right map is degeneracy detecting, then a diagonal filling exists if and only the image of the x_i in A satisfies the appropriate degeneracy conditions. As their images in B satisfies them because of the existence of the square, and as the map $A \to B$ is degeneracy detecting, this is immediate. $\square$

The following is more or less a reformulation of what is a degeneracy quotient that will be convenient:

3.1.7 Lemma. *An epimorphism of simplicial sets $p : A \to B$ is a degeneracy quotient if and only if for any map $f : A \to X$, the map f factors through p if and only if the following condition holds:*

$$\forall a \in A([n]) \quad p(a) \textit{ is a degenerate cell } \Rightarrow f(a) \textit{ is a degenerate cell.} \qquad \text{(D)}$$

Note that if such a factorization exists then condition (D) holds without any assumption on p, so that if p is a degeneracy quotient then a factorization exists if and only condition (D) holds.

Proof. It follows from Lemma 3.1.2, that condition (D) is equivalent to:

$$\forall a \in A([n]) \quad p(a) \text{ is a } \sigma\text{-degenerate cell} \Rightarrow f(a) \text{ is a } \sigma\text{-degenerate cell.} \qquad \text{(D')}$$

A factorization of f through p is always unique as p is an epimorphism, so saying that f factors through p if and only if condition (D') (or (D)) holds is equivalent to saying that B (endowed with the map $p : A \to B$) has the universal property of $A[(a_i, \sigma_i)]$ where (a_i, σ_i) are all the pairs of $a_i \in A([n])$ such that $p(a_i)$ is a σ_i-degenerate cell. Hence this indeed holds if and only if $A \to B$ is a degeneracy quotient, as because of Proposition 3.1.6, any degeneracy quotient $p : A \to B$ is isomorphic to $A \to A[(a_i, \sigma_i)]$ where (a_i, σ_i) are all the pairs of $a_i \in A([n])$ such that $p(a_i)$ is a σ_i-degenerate cell. □

This observation has a quite interesting consequence that will be extremely useful to us, and in fact is the unique reason why we are interested in degeneracy quotients in the present paper:

3.1.8 Lemma. *Given $p : A \to B$ a degeneracy quotient of finite decidable simplicial sets, and $f : A \to X$ a morphisms to a cofibrant simplicial set, it is decidable if there exists a diagonal lift:*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \overset{?}{\nearrow} & \\ B. & & \end{array}$$

Proof. One can use condition (D) of Lemma 3.1.7 to test whether such a diagonal lift exists. As B is finite and decidable, degeneracy in B is decidable. So for each cell $a \in A$ it is decidable if " $p(a)$ is a degenerate cell $\Rightarrow$ $f(a)$ is a degenerate cell" as both side of the implication are decidable. Moreover this condition is automatically valid for all degenerate cells of A, so it is necessary to test it only on a finite number of cells to know whether f factors through p, which makes the validity of condition (D) decidable and hence the existence of a diagonal lift decidable. □

The following lemma is obvious, but will be a convenient tool to organise the proof that certain maps are degeneracy quotients:

3.1.9 Lemma. *Let $p : A \to B$ be an epimorphism. One considers the equivalence relation $\sim_p$ on A generated by:*

- *If $p(a)$ is a σ-degenerate cell, then $a \sim_p \sigma^* t^* a$ for any section t of σ.*
- *$\sim_p$ is compatible with all the faces and degeneracy maps of A.*

Then p is a degeneracy quotient if and only if any two $a, a' \in A$ such that $pa = pa'$ one has $a \sim_p a'$.

Note that for any morphisms, $a \sim_p a' \Rightarrow pa = pa'$.

Proof. One easily see that $\sim_p$ is exactly the simplicial equivalence relation by which one needs to quotient A to obtain $A[(a_i, \sigma_i)]$ where (a_i, σ_i) is the family of all a_i such that $p(a_i)$ is σ_i degenerate in B. By the second half of Proposition 3.1.6, the map p is a degeneracy quotient if and only if the second maps in the factorization $A \to A[(a_i, \sigma_i)] \to B$ is an isomorphism, which happens if and only if the relation $\sim_p$ is equivalent to $p(a) = p(a')$. □

We continue with a proposition that allows to get many examples of degeneracy quotient (see for example the proof of Lemma 3.4.4).

3.1.10 Proposition. *Let P be a poset with an idempotent order preserving endomorphism π satisfying either $\forall x, \pi x \leqslant x$ or $\forall x, \pi x \geqslant x$. Let $Q = \pi P$. Then the morphism:*

$$N(P) \to N(Q)$$

between the simplicial nerve induced by $\pi : P \to Q$ is a degeneracy quotient.

Proof. We assume that $\pi x \leqslant x$. The other case follows by simply reversing the order relation on P and on all objects of the category Δ. We use Lemma 3.1.9.
Let $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n$ be an element of $N(P)_n$ and assumes that $p_0, \dots, p_{i-1} \in Q$, then one forms

$$p_0 \leqslant p_1 \leqslant \cdots \leqslant p_{i-1} \leqslant \pi p_i \leqslant p_i \leqslant \cdots \leqslant p_n$$

It is an element of $N(P)_{n+1}$ whose image in Q is degenerate, as $\sigma^{i*}(\pi p_0 \leqslant \cdots \leqslant \pi p_n)$. This implies that in $N(P)$:

$$(p_0 \leqslant \cdots \leqslant p_n) \sim (p_0 \leqslant \cdots \leqslant p_{i-1} \leqslant \pi p_i \leqslant p_{i+1} \leqslant \cdots \leqslant p_n)$$

In the sense of the equivalence relation of Lemma 3.1.9. Hence using this for all i from 0 to n, one obtains that for any sequence $p_0 \leqslant \cdots \leqslant p_n$ all the

$$(\pi p_0 \leqslant \cdots \leqslant \pi p_{i-1} \leqslant p_i \leqslant \cdots \leqslant p_n)$$

for $i = 0, \dots, n+1$ are equivalent. In particular any sequence is equivalent to its image by π and finally any two sequences whose image in $N(Q)$ are the same are equivalent.

□

We finish with a proposition that is useful in a related work [5]):

3.1.11 Proposition. *The class of degeneracy quotients is stable under pullback.*

Proof. First we show that given a pullback of the form:

$$\begin{array}{ccc} P & \longrightarrow & \Delta[n] \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \Delta[k] & \xrightarrow{f} & \Delta[m] \end{array}$$

where σ is a degeneracy map, the map ϕ is a degeneracy quotient. This is proved using Proposition 3.1.10. Indeed in such a pullback P is nerve of the corresponding pullback of posets, that we will also denote P (because the

nerve functor commutes with pullback). We will show that the map $P \to [k]$ is of the form of Proposition 3.1.10. The map $\sigma : [n] \twoheadrightarrow [m]$ is of this form, with the section $[n] \to [m]$ sending each $i \in [m]$ to the smallest element of the fiber, this gives an order preserving idempotent $\pi : [n] \to [n]$ such that $\pi x \leqslant x$. This induce an idempotent on P sending a pair (i,j) (with $i \in [k]$, $j \in [n]$) to $\pi'(i,j) = (i, \pi j)$. This is still an element of P, $\pi'(i,j) \leqslant (i,j)$ it is idempotent, and its image identifies naturally with $[k]$.
Hence $\phi : P \to \Delta[k]$ is indeed a degeneracy quotient by Proposition 3.1.10.
We now show that given any pullback of the form:

$$\begin{array}{ccc} P & \longrightarrow & \Delta[n] \\ {\scriptstyle \phi}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{f} & \Delta[m] \end{array}$$

for a degeneracy σ, the map ϕ is a degeneracy quotient.
Indeed, one write:

$$X = \operatorname*{Colim}_{\Delta[k] \to X} \Delta[k]$$

Given a $x : \Delta[k] \to X$ one writes P_x the pullback:

$$\begin{array}{ccccc} P_x & \longrightarrow & P & \longrightarrow & \Delta[n] \\ {\scriptstyle \phi_x}\downarrow & \lrcorner & {\scriptstyle \phi}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma} \\ \Delta[k] & \longrightarrow & X & \xrightarrow{f} & \Delta[m] \end{array}$$

All map ϕ_x are degeneracy quotient by the first part of the proof. As the category of simplicial sets is a topos, colimits are universal, hence the morphism ϕ is the colimit of the arrows ϕ_x (in the category of arrows). As the class of degeneracy quotient is the left class of an orthogonal factorization system, the colimit ϕ is also a degeneracy quotient. To give an explicit argument: given a lifting problem of ϕ against a degeneracy detecting map one can construct for each x a lifting:

$$\begin{array}{ccccc} P_x & \longrightarrow & P & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_x} & & \downarrow & {\scriptstyle \phi} \nearrow & \downarrow \\ \Delta[k] & \longrightarrow & X & \longrightarrow & B \end{array}$$

By uniqueness of the lifts, they will all be compatible and produces a morphisms from the colimits to A making the square commutes.
Finally we can prove the claim in the proposition. Given a morphism $f : X \to Y$ any degeneracy map $\Delta[n] \to \Delta[m]$ over Y (i.e with $\delta[m] \to Y$) is sent by the pullback functor $\widehat{\Delta}_{/Y} \to \widehat{\Delta}_{/X}$ to a degeneracy quotient. But a general degeneracy quotient is a pushout of coproduct of degeneracy maps, and this coproduct and pushout are preserved by the pullback functor (because the category of simplicial sets is cartesian closed), and coproduct of pushout of degeneracy quotient are degeneracy quotient so this concludes the proof. □

3.2 P-structures

This section recalls the notion of P-structure introduced in [15] with some minor modification to make it more suitable to the constructive context. A "P-structure" on a morphism $f : A \to B$ is essentially a recipe for constructing it as an iterated pushout of coproduct of horn inclusion $\Lambda^i[n] \hookrightarrow \Delta[n]$. The general idea of this definition is that in such an iterated pushout cells are added by pairs: each pushout by a horn inclusion $\Lambda^i[n] \to \Delta[n]$ adds exactly two non-degenerate cells:

(I) The cell P corresponding to the identity of $\Delta[n]$.

(II) The cell F corresponding to the i-th face $\partial^i[n] : \Delta[n-1] \to \Delta[n]$.

These two cells are connected by $F = d_i P$. So if $A \hookrightarrow B$ is constructed by iterating such pushouts, then one can partition the non-degenerate cells of B that are not in A into "type I" and "type II" and there should be a bijection which associates to any type II cell the type I cell that is added by the same pushout. The formal definition looks like this:

3.2.1 Definition. Let $f : A \to B$ be a cofibration of simplicial sets. A P-structure on f is the data of:

- A (decidable) partition of the set of non-degenerate cells of B which are not in A into:

$$B_{\mathrm{I}} \coprod B_{\mathrm{II}}$$

called respectively type I cells and type II cells.

- A bijection $P : B_{\mathrm{II}} \overset{\sim}{\to} B_{\mathrm{I}}$.

Such that:

1. For all $x \in B_{\mathrm{II}}$, $dim(Px) = dim(x) + 1$
2. For all $x \in B_{\mathrm{II}}$, there is a unique i such that $d_i(Px) = x$.
3. Every cell of B_{II} has finite P-height (see Definition 3.2.2 and Lemma 3.2.3 below).

Recall that, if $f : A \to B$ is a cofibration, it is decidable whether a cell is in A or not, and for cells not in A it is decidable whether they are degenerate or non-degenerate. So a P-structure gives a partition of the cells of B into for parts: the cells of A, the degenerate cells of B not in A, the type I cells and the type II cells.

In [15], the last condition of this definition was formulated as a well-foundness condition. Well-foundness is a tricky notion constructively so we prefer to avoid it. It should be clear to the reader that the condition we will now explain is equivalent to well-foundness if one assumes classical logic, or if one has a nice enough notion of well-foundness constructively. Intuitively this last condition just asserts that the "recipe" given by the P-structure to construct B from A as an iterated pushout of horn inclusion is indeed well-founded, i.e. can be executed. We will formulate it by introducing for each cell $b \in B$ a set:

$$Ant(b)$$

which corresponds to the set of cells that needs to be constructed before b in the process described by P. In [15] the well-foundness condition is essentially that the order relation generated by $b' \in Ant(b)$ is well-founded. As each $Ant(b)$ is a finite set this is equivalent to the fact that for each b there is an integer k such that when iterating $Ant(b)$ more than k times one has only cells in A. It is this second definition that we will use in our constructive context.

More precisely: Given a cell $b \in B_{\text{II}}$, let i be the unique integer such that $d_i Px = x$, one defines:

$$Ant_0(b) = \{d_j P(b) | j \neq i\}$$

And one defines the set $Ant(b)$ as the union of $Ant_0(b)$ and all (iterated) faces of cells appearing in $Ant_0(b)$.
Similarly, if $b = Pb'$ is type I, one defines:

$$Ant(b) = Ant(b')$$

Finally, if $b \in A$:

$$Ant(b) = \emptyset$$

and if b is not in A but degenerate, then

$$Ant(b) = Ant(b')$$

where b' is the unique non-degenerate cell such that $b = \sigma^* b'$.
One also defines $Ant_{\text{II}}(b)$ to be the set of non-degenerate type II cell in $Ant_0(b)$. Note that in all cases $Ant(b)$ and $Ant_0(b)$ are Kurawtowski-finite[6] sets, and as the subset of type II cell is decidable, $Ant_{\text{II}}(b)$ is also Kurawtowski-finite. One defines $Ant^k(b)$ and $Ant^k_{\text{II}}(b)$ by:

$$Ant^1(b) = Ant(b) \qquad Ant^k(b) = \bigcup_{c \in Ant b} Ant^{k-1} c$$

$$Ant^1_{\text{II}}(b) = Ant_{\text{II}}(b) \qquad Ant^k_{\text{II}}(b) = \bigcup_{c \in Ant_{\text{II}} b} Ant^{k-1}_{\text{II}} c$$

[6] A set X is said to be Kuratowski-finite if $\exists n, \exists x_1, \dots, x_n \in X, \forall x \in X, x = x_1$ or $\dots$ or $x = x_n$.

Note that when applied to a non-degenerate type II cell $b \in B$, all elements of $Ant_{\text{II}}(b)$ (and hence of $Ant^k_{\text{II}}(b)$ as well) are non-degenerate type II cells of the same dimension as b.

3.2.2 Definition.

- One says that b has finite P-height if there exists an integer k such that:

$$Ant^k(b) = \emptyset$$

- One says that b has finite weak P-height if there is an integer k such that:

$$Ant^k_{\text{II}}(b) = \emptyset$$

Note that for each given k and $b \in B$, as the sets $Ant^k(b)$ and $Ant^k_{\text{II}}(b)$ are Kuratowski-finite it is decidable whether or not $Ant^k(b)$ and $Ant^k_{\text{II}}(b)$ are empty. In particular, assuming b has finite (weak) P-height there is smallest integer k, called the (weak) P-height of b, such that $Ant^k_{(\text{II})}(b) = \emptyset$. But in general it might not be decidable whether b has finite (weak) P-height or not.

3.2.3 Lemma. *Let $f : A \hookrightarrow B$ be a cofibration with a P-structure satisfying all the conditions of Definition 3.2.1 but the last. Then the following are equivalent:*

- *Every $b \in B$ has finite P-height.*
- *Every non-degenerate type II cell $b \in B_{II}$ has finite weak P-height.*

Proof. It is clear that $Ant^k_{\text{II}}(b) \subset Ant^k(b)$ hence the first condition implies the second. Conversely, assume that every non-degenerate $b \in B_{\text{II}}$ has finite weak P-height. We will prove by double induction on both the dimension and the weak P-height that all cells of B have finite P-height.
First we assume that all cells of dimension $< n$ have finite P-height. Cells of A have P-height zero. All cells of B of dimension n that are either degenerate or of type I satisfies $Ant(b) = Ant(b')$ for some b' of dimension strictly

less than n, hence for b' of finite P-height by the induction assumption. As $Ant^k(b) = Ant^k(b')$ this implies that b has finite P-height as well.
It remains to show that all non-degenerate n-cells of type II in B have finite P-height. We do that by induction on their weak P-height.
Indeed for a general type II cell b, $Ant(b)$ is constituted of:

- Degenerate or type I cell, that are already known to have finite P-height.
- Faces of cell in $Ant_0(b)$ which are hence of dimension $< n$ and hence are known to be of finite P-height.
- Non-degenerate type II cells that are hence elements of $Ant_{\mathrm{II}}(b)$, but

  $$\emptyset = Ant^k_{\mathrm{II}}(b) = \bigcup_{c \in Ant_{\mathrm{II}} b} Ant^{k-1}_{\mathrm{II}} c$$

  hence all $c \in Ant_{\mathrm{II}} b$ have weak P-height at most $k-1$, and hence they all have finite P-height by induction.

So all elements of $Ant(b)$ have finite P-height, let m be the maximum of all these P-height, one has that:

$$Ant^{m+1}(b) = \bigcup_{c \in Ant(b)} Ant^m(b) = \emptyset$$

□

3.2.4 Lemma. *A cofibration with a P-structure is anodyne. More precisely it is a ω-transfinite composition of pushouts of coproducts of horn inclusions.*

A map will be called "strongly anodyne" if it admits a P-structure.

Proof. Let $A \hookrightarrow B$ be a cofibration with a P-structure.
Let $B_k \subset B$ be the subset of B of cells of P-height at most k. One has $B_0 = A$, and B_k is a sub-simplicial set. Indeed, for every cell $b \in B$ all faces of b appear in $Ant(b)$ or are such that $Ant(d_i b) = Ant(b)$ and all degeneracies of b satisfies $Ant(\sigma^* b) = Ant(b)$, hence they all have P-height at most k.

Let U be the set of non-degenerate type II cell of B of P-height exactly k. For each $u \in U$, let i_u be the unique integer such that $d_{i_u} P(u) = u$.
Then the corresponding map $\Delta[n] \overset{Pu}{\to} B_k$ sends $\Lambda^{i_u}[n]$ to B_{k-1} and both u and Pu are in $B_k - B_{k-1}$.
Hence taking the pushout:

$$\begin{array}{ccc} \Lambda^{i_u}[n] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ B_{k-1} & \longrightarrow & R \end{array}$$

produces the simplicial set $R \subset B_k$ whose cells are all those of B_{k-1}, u and Pu and all their degeneracy. Taking the pushout by the coproduct of all these horn inclusions for all $u \in U$ gives $B_{k-1} \to B_k$.
Hence $B = \bigcup B_k$ is a ω-transfinite composition of the maps $B_k \to B_{k+1}$ which are all pushouts of coproducts horn inclusions.

□

Classically one also has the converse: any transfinite composition of pushouts of coproducts horn inclusions has a canonical P-structure. Constructively this sort of statement is somehow problematic, mostly because the general notion of "transfinite composition" requires a notion of ordinal to be formulated appropriately, but it works perfectly fine if one restricts to ω-composition:

3.2.5 Proposition. *The class of strongly anodyne cofibration contains all horn inclusion and is stable under pushout and ω-transfinite*[7] *composition. Any morphism can be factored as a strongly anodyne cofibration followed by a Kan fibration, and any anodyne cofibration is a retract of a strongly anodyne cofibration.*

Proof. Horn inclusion have a trivial P-structure with one cell of type I and one cell of type II. It is easy to see that coproduct, pushout and transfinite composition of strongly anodyne cofibration have P-structure induced by the P-structure we start from, for example if $A \hookrightarrow B$ has a P-structure, then

[7] Here the restriction to "ω" is only to avoid the discussion of what is an ordinal constructively.

$C \to B \coprod_A C$ has a P-structure where a cell in $B \coprod_A C$ is type I or II if and only if it is type I or II for the P-structure on $A \hookrightarrow B$ and the map P is the same as the one on B, and similarly for coproducts and transfinite compositions.
It follows that the factorization of the map as an anodyne cofibration followed by a Kan fibration obtained by the small object argument is a strongly anodyne cofibration as it is constructed as a ω-transfinite composition of pushout of coproduct of horn inclusion. Finally any anodyne cofibration j can be factored as a strongly anodyne cofibration followed by a Kan fibration, and the usual retract lemma 2.2.5 shows that j is a retract of the strongly anodyne cofibration part of the factorization.

□

We finish this section by mentioning a very important example where this machinery applies, mostly to serve as an example of how it can be used.
Given two morphisms $f : A \to B$ and $g : X \to Y$ between simplicial sets one defines as usual $f \ulcorner\!\times g$ the cartesian "corner-product" or "pushout-product" of f and g as the morphism:

$$f \ulcorner\!\times g : (A \times Y) \coprod_{A \times X} (B \times X) \to B \times Y,$$

one then has the following well known proposition, which we have referred to in the introduction as the corner-product conditions, and which is a key point in establishing the existence of the weak model structure on simplicial sets. It also corresponds to the fact the model structure on simplicial sets that we are constructing is cartesian.

3.2.6 Proposition. *If i and j are cofibrations, then $i \ulcorner\!\times j$ is a cofibration as well. If one of them is anodyne then $i \ulcorner\!\times j$ is also anodyne.*

As usual (following for example the appendix of [10]) this implies the dual condition, that if $i : A \to B$ is a cofibration and $p : Y \to X$ is a fibration, then the map $[B, Y] \to [B, X] \times_{[A,X]} [A, Y]$ is a fibration (the brackets denotes the cartesian exponential in simplicial sets), and it is a trivial fibration as soon as either i is anodyne or p is a trivial fibration.

Proof. By usual abstract manipulation (see for example the appendix of [10]) it is sufficient to show it when i and j are generating cofibrations/generating

anodyne cofibration. If i and j are generating cofibrations it is very easy to check that $i \ulcorner\!\!\times j$ is a cofibration as defined in the statement of our main theorem 1.1. It remains to check that if i is one of the generating cofibrations, i.e. $\partial\Delta[n] \hookrightarrow \Delta[n]$ for some n, and j is one of the generating anodyne cofibrations, i.e. $\Lambda^k[m] \hookrightarrow \Delta[m]$ for some k, m, then $i \ulcorner\!\!\times j$ is an anodyne cofibration. This is done by constructing an explicit P-structure on $i \ulcorner\!\!\times j$.
The first direct proof of this claim that we know of is in [11] (theorem 3.2.2), here we follow the proof of S.Moss' in 2.12 of [15] to show how P-structures work. We only treat the case $k < m$ for simplicity. The case $k > 0$ can be treated in a completely similar way, by simply reversing the order relation on the $[n]$, which allows to deduce the missing case $k = m$.
A p-cell of $\Delta[n] \times \Delta[m]$ is an order preserving function $[p] \to [n] \times [m]$. It is non-degenerate if and only if it is an injective function. The domain D of $i \ulcorner\!\!\times j$ is:

$$\left(\Delta[n] \times \Lambda^k[m]\right) \coprod_{\partial\Delta[n] \times \Lambda^k[m]} \left(\partial\Delta[n] \times \Delta[m]\right) = \left(\Delta[n] \times \Lambda^k[m]\right) \bigcup \left(\partial\Delta[n] \times \Delta[m]\right)$$

It corresponds to the morphisms $[p] \to [n] \times [m]$ such that either they skip a column or they skip a row other than k, where we consider that $[n] = \{0, \dots, n\}$ numbers the column of $[n] \times [m]$ and $[m] = \{0, \dots, k, \dots, m\}$ numbers the row. So the only non-degenerate cell of $\Delta[n] \times \Delta[m]$ that are not in D are injection $[k] \to [n] \times [m]$ whose first projection takes all possible value, and whose second projection takes all possible values except maybe k.
One says that a cell is type II if either it skip the k^{th} row by going directly from $(a, k-1)$ to $(a+1, k+1)$, in which case one defines Px by adding the intermediate step $(a, k-1), (a+1, k), (a+1, k+1)$, or if the last point where the k^{th} row is reached, is (a, k) followed by $(a+1, k+1)$ in which case Px is defined by inserting the intermediate step: $(a, k), (a, k+1), (a+1, k+1)$. It is an easy exercise to check that this defines a P-structure.

□

3.3 Kan Ex and Sd functors

Consider the barycentric subdivision functor $\Delta \to \widehat{\Delta}$:

$$\Delta[n] \mapsto \mathrm{SD}\,\Delta[n] := N\mathcal{K}([n])$$

Where $\mathcal{K}([n])$ denotes the set of *finite non-empty decidable* subsets of $[n]$. Functoriality in $[n]$ is given by direct image of subsets on $\mathcal{K}[n])$.
This extends to an adjunction:

$$\mathrm{SD} : \widehat{\Delta} \leftrightarrows \widehat{\Delta} : \mathrm{EX}$$

with:

$$(\mathrm{EX}\,X)_n = \mathrm{Hom}(\mathrm{SD}\,\Delta[n], X) \qquad \mathrm{SD}\,X = \operatorname*{Colim}_{\Delta[n]\to X} \mathrm{SD}\,\Delta[n]$$

The barycentric subdivision construction has a nice expression not just for the $\Delta[n]$, but also for all objects which are in the image of the functor $\widehat{\Delta_+} \to \widehat{\Delta}$, indeed:

3.3.1 Proposition. *The composite:*

$$\widehat{\Delta_+} \to \widehat{\Delta} \overset{\mathrm{SD}}{\to} \widehat{\Delta}$$

is the functor sending a semi-simplicial set X to $N(\Delta_+/X)$.

One can note that as the category Δ_+/X is directed, the nerve $N(\Delta_+/X)$ is itself the image of the semi-simplicial set of its non-degenerate cells. We won't make any use of this remark though.

Proof. This functors $X \mapsto N(\Delta_+/X)$ preserves colimits, because it can be rewritten as:

$$N(\Delta_+/X)_k = \coprod_{F:[k]\to\Delta_+} X(F(k))$$

which is levelwise a coproduct of colimits-preserving functor.
Hence we are comparing to colimits preserving functor, so it is enough to show they are isomorphic when restricted to representable. But $\Delta_+/[n] \simeq \mathcal{K}[n]$ functorially on map of Δ_+ so this concludes the proof.

□

3.3.2 Proposition. SD *preserves cofibrations and anodyne cofibrations,* EX *preserves fibrations and trivial fibrations.*

Proof. It is enough to check that the image of the generating cofibrations and generating anodyne cofibrations by SD are cofibrations and anodyne cofibrations respectively.
In both case one can use Proposition 3.3.1 to compute SD on the generators as they are image of semi-simplicial maps. This makes the results immediate for cofibrations:

$$\mathrm{SD}\,\partial\Delta[n] \to \mathrm{SD}\,\Delta[n]$$

is the morphism $N(\mathcal{K}[n] - \{[n]\}) \to N(\mathcal{K}[n])$ which is clearly a levelwise complemented monomorphisms between finite decidable, hence cofibrant, simplicial sets.
For the generating anodyne cofibrations,

$$\mathrm{SD}\,\Lambda^i[n] \to \mathrm{SD}\,\Delta[n]$$

is the morphisms $N(\mathcal{K}[n] - \{[n], [n] - \{i\}\}) \to N(\mathcal{K}[n])$. It can then be checked completely explicitly that this is a (strongly) anodyne cofibrations, see Proposition 2.14 of [15] for an explicit description of a P-structure.

□

There is a natural transformation:

$$\mathrm{SD}\,\Delta[n] \to \Delta[n]$$

Which is induced by the order preserving function:

$$\max : \mathcal{K}[n] \to [n]$$

sending each (decidable) subset of $[n]$ to its maximal element. By Kan extension, this gives us natural transformations:

$$\mathrm{SD} \xrightarrow{m} Id \qquad Id \xrightarrow{n} \mathrm{EX}$$

One can hence define a sequences of functors:

$$X \xrightarrow{n_x} \operatorname{Ex} X \xrightarrow{n_{\operatorname{Ex} X}} \operatorname{Ex}^2 X \xrightarrow{n_{\operatorname{Ex}^2 X}} \dots \xrightarrow{n_{\operatorname{Ex}^{k-1} X}} \operatorname{Ex}^k X \xrightarrow{n_{\operatorname{Ex}^k X}} \dots \longrightarrow \operatorname{Ex}^\infty X$$

with Ex^∞ the colimit.

3.3.3 Lemma. *For each k, n, there is a (dotted) arrow Ψ_n^k making the following triangle commute.*

$$\begin{array}{ccc} \operatorname{Sd}^2 \Lambda^k[n] & \xrightarrow{\operatorname{Sd}(m_{\Lambda^k[n]})} & \operatorname{Sd} \Lambda^k[n] \\ & \searrow \quad \nearrow_{\Psi_n^k} & \\ & \operatorname{Sd}^2 \Delta[n] & \end{array}$$

Proof. The proof given in [3] as proposition 2.1.39 is purely combinatorial and constructive. □

3.3.4 Corollary. *For every cofibrant simplicial set X, $\operatorname{Ex}^\infty X$ is a Kan complex.*

The proof that follows essentially comes from [3]. If one does not assume that X is cofibrant it still applies to prove that X has the "existential" right lifting property against horn inclusion, but it does not seem possible to give a uniform choice of solution to all lifting problems without this assumption. Without such a uniform choice of lifting against horn inclusion one cannot construct solution to lifting problems against more complicated anodyne cofibrations that involves an infinite number of pushout of horn inclusion, unless we assume the axiom of choice.

Proof. Lemma 3.3.3 allows to show that given any solid diagram as below, there is a dotted filling:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \operatorname{Ex} X \\ \downarrow & & \downarrow{\scriptstyle n_{\operatorname{Ex} X}} \\ \Delta[n] & \dashrightarrow & \operatorname{Ex}^2 X \end{array}$$

Indeed, through the adjunction the map $\Lambda^k[n] \to \operatorname{Ex} X$ corresponds to an arrow $\operatorname{Sd} \Lambda^k[n] \to X$, which due to Lemma 3.3.3 can be extended in:

$$\begin{array}{ccccc} \operatorname{Sd}^2 \Lambda^k[n] & \xrightarrow{\operatorname{Sd} m_{\Lambda^k[n]}} & \operatorname{Sd} \Lambda^k[n] & \longrightarrow & X \\ {\scriptstyle \operatorname{Sd}^2 \_}\downarrow & \nearrow_{\psi^k_n} & & & \\ \operatorname{Sd}^2 \Delta[n] & & & & \end{array}$$

The resulting map $\operatorname{Sd}^2 \Delta[n] \to X$ corresponds to a map $\Delta[n] \to \operatorname{Ex}^2 X$ which has exactly the right property to make the square above commute. Now by smallness of $\Lambda^k[n]$, any map $\Lambda^k[n] \to \operatorname{Ex}^\infty X$ factors in $\operatorname{Ex}^k X$, the observation above produces a canonical filling in $\Delta[n] \to \operatorname{Ex}^{k+1} X$. The choice of the filling, seen as taking values in $\operatorname{Ex}^\infty X$, in general depends on k, but if one further assume that X is cofibrant, than by Lemma 3.4.4, the maps $\operatorname{Ex}^k X \to \operatorname{Ex}^{k+1} X$ are all levelwise decidable inclusion, so there is a smallest k such that the map $\Lambda^k[n] \to \operatorname{Ex}^\infty X$ factors into $\operatorname{Ex}^k X$ and this produces a canonical solution to the lifting problem. □

3.3.5 Proposition. *If $f : X \to Y$ is a fibration (resp. a trivial fibration) with X and Y cofibrant then $\operatorname{Ex}^\infty f : \operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ is also a fibration (resp. a trivial fibration).*

Similarly to what happen with Corollary 3.3.4, without the assumption that X and Y are cofibrant it is only possible to obtain the "existential" form of the lifting property and no canonical choice of lifting.

Proof. Given a lifting problem:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \operatorname{Ex}^\infty X \\ \downarrow{\scriptstyle \sim} & & \downarrow \\ \Delta[n] & \longrightarrow & \operatorname{Ex}^\infty Y \end{array}$$

There is an i such that it factors into:

$$\begin{array}{ccccc} \Lambda^k[n] & \longrightarrow & \mathrm{Ex}^i X & \longrightarrow & \mathrm{Ex}^\infty X \\ \downarrow{\sim} & & \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & \mathrm{Ex}^i Y & \longrightarrow & \mathrm{Ex}^\infty Y \end{array}$$

Moreover, assuming X and Y are cofibrant, Lemma 3.4.4 shows that $\mathrm{Ex}^i X \subset \mathrm{Ex}^{i+1} X$ are levelwise decidable inclusion, so (by finiteness of $\Lambda^k[n]$ and $\Delta[n]$) the set of i such that a factorization as above exists is decidable, and hence there is a smallest such i. Proposition 3.3.2 shows that $\mathrm{Ex}^i f$ is a fibration, so the first square has a diagonal lifting and this concludes the proof. □

3.4 S.Moss' proof that $X \to \mathrm{Ex}\, X$ is an anodyne cofibration

Let $f : X \to Y$ be a simplicial morphism. One has a square:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Ex}^\infty X & \longrightarrow & \mathrm{Ex}^\infty Y \end{array}$$

Our goal in this section is to show that when X is cofibrant the induced map:

$$X \to \mathrm{Ex}^\infty X \underset{Ex^\infty Y}{\times} Y$$

is a strong anodyne cofibration. Note that if $Y = \Delta[0]$ is the terminal object, then $Ex^\infty(Y) = Y$ hence the statement above boils down to the fact that $X \to Ex^\infty X$ is a strong anodyne cofibration. The idea to consider this morphism comes form D.C Cisinski's book [3, Cor 2.1.32], but the proof below follows closely the proof given by S.Moss in [15] that $X \to \mathrm{Ex}^\infty X$ is a strong anodyne cofibration.

Following the argument given in [3, Cor 2.1.32] (reproduced in the proof of Corollary 3.4.7 below), it will be enough to show:

3.4.1 Proposition. *Given $f : X \to Y$ a simplicial morphism, with X cofibrant, then the morphism:*

$$X \to \operatorname{Ex} X \underset{\operatorname{Ex} Y}{\times} Y$$

is strongly anodyne.

The proof will be concluded in 3.4.6, essentially, we will construct an explicit P-structure on this map. This construction is mostly due to S.Moss in [15]. In addition to the dependency in Y, the main new contributions of this paper in this section is to show that assuming X is cofibrant one can show that sufficiently many decidability conditions can be proved to make S.Moss' argument constructive. In order to do that properly one needs to completely reproduce his argument.
Following, [15] one introduces two functions between the $\operatorname{Sd} \Delta[n]$.
Let $j_n^k : \operatorname{Sd} \Delta[n] \to \operatorname{Sd} \Delta[n]$ and $r_n^k : \operatorname{Sd} \Delta[n+1] \to \operatorname{Sd} \Delta[n]$ be the maps defined at the level of posets by:

$$j_n^k\{i\} = \begin{cases} \{i\} & \text{if } i \leqslant k \\ \{0, \dots, i\} & \text{if } i > k \end{cases} \qquad r_n^k\{i\} = \begin{cases} \{i\} & \text{if } i \leqslant k \\ \{0, \dots, i-1\} & \text{if } i = k+1 \\ \{i-1\} & \text{if } i > k+1 \end{cases}$$

Both extended to non-singleton elements as binary join preserving maps. These functions satisfies a certain number of equations, we list here those that we will need, they are all due to S.Moss.

3.4.2 Lemma.

$$j^k_n j^h_n = j^h_n j^k_n = j^h_n \qquad 0 \leqslant h \leqslant k \leqslant n \tag{1}$$

$$Id_{\Delta[n]} = r^k_n \circ \mathrm{S\small{D}}\, \partial^{k+1}_{n+1} \qquad 0 \leqslant k \leqslant n \tag{2}$$

$$j^k_n r^k_n = (\mathrm{S\small{D}}\, \sigma^k_n) j^k_{n+1} \qquad 0 \leqslant k \leqslant n \tag{3}$$

$$j^h_n r^k_n = j^h_n(\mathrm{S\small{D}}\, \sigma^k_n) \qquad 0 \leqslant h < k \leqslant n \tag{4}$$

$$r^k_n j^h_{n+1} = j^h_n r^k_n \qquad 0 \leqslant h \leqslant k \leqslant n \tag{5}$$

$$r^k_n(\mathrm{S\small{D}}\, \partial^{i+1}_{n+1}) = (\mathrm{S\small{D}}\, \partial^i_n) r^k_{n-1} \qquad 0 \leqslant k < i \leqslant n \tag{6}$$

$$j^k_n r^k_n r^k_{n+1} = j^k_n r^k_n (\mathrm{S\small{D}}\, \sigma^{k+1}_{n+1}) \qquad 0 \leqslant k \leqslant n \tag{7}$$

$$j^k_{n+1}(\mathrm{S\small{D}}\, \partial^h_{n+1}) j^k_n = j^k_{n+1}(\mathrm{S\small{D}}\, \partial^h_{n+1}) \qquad 0 \leqslant k \leqslant n \textit{ and } 0 \leqslant h \leqslant n+1 \tag{8}$$

$$j^k_n r^k_n (\mathrm{S\small{D}}\, \partial^i_{n+1}) j^{k-1}_n = j^k_n r^k_n (\mathrm{S\small{D}}\, \partial^i_{n+1}) \qquad 0 \leqslant i \leqslant k \leqslant n \tag{9}$$

$$(\mathrm{S\small{D}}\, \sigma^h_n) j^k_{n+1} r^k_{n+1} = j^{k-1}_n r^{k-1}_n (\mathrm{S\small{D}}\, \sigma^h_{n+1}) \qquad 0 \leqslant h < k \leqslant n+1 \tag{10}$$

$$(\mathrm{S\small{D}}\, \sigma^h_n) j^k_{n+1} r^k_{n+1} = j^k_n r^k_n (\mathrm{S\small{D}}\, \sigma^{h+1}_{n+1}) \qquad 0 \leqslant k \leqslant h \leqslant n \tag{11}$$

Proof. All the functions involved are nerve of join preserving maps between the $\mathcal{K}[n]$, so it is enough to check the relations at the level of posets and when functions are evaluated at $\{i\}$, where one has explicit formulas for all of them. □

As functions between the $\mathrm{S\small{D}}\,\Delta[n]$, j^k_n and r^k_n automatically acts on the cells of $\mathrm{E\small{X}}\, X$. One denotes this action by $x \mapsto x j^k_n$ and $x \mapsto x r^k_n$ which is compatible to the identification of cells of $\mathrm{E\small{X}}\, X$ with functions $\mathrm{S\small{D}}\,\Delta[n] \to x$.

By equation (1), the j^k_n are an increasing family of commuting projection whose image defines a series of subsets:

$$X_n = J^0_n \subset J^1_n \subset \ldots J^n_n = (\mathrm{E\small{X}}\, X)_n$$

where the identifications with $(\mathrm{E\small{X}}\, X)_n$ and X_n comes from the fact that j^n_n is the identity, and $j^0_n : \mathcal{K}[n] \to \mathcal{K}[n]$ has image isomorphic to $[n]$, with $j^0_n : \mathcal{K}[n] \to [n]$ being the “Max” function used in the definition of the natural transformation $\mathrm{S\small{D}}\,\Delta[n] \to \Delta[n]$.

3.4.3 Notation. For $X \to Y$ any morphism, we define:

$$\mathrm{E\small{X}}_Y(X) = \mathrm{E\small{X}}\, X \underset{\mathrm{E\small{X}}\, Y}{\times} Y.$$

An n-cell in Ex_Y is a morphism $\text{Sd}\,\Delta[n] \to X$ whose image in Y factors through the map $\text{Sd}\,\Delta[n] \to \Delta[n]$. I.e. it is an n-cell of $x \in (\text{Ex}\,X)_n$ which satisfies:

$$fxj_n^0 = fx$$

Note that because of relation (1) and (5), $\text{Ex}_Y\,X$, as a subsimplicial object of $\text{Ex}\,X$, is stable under the action of j_n^k and r_n^k on $\text{Ex}\,X$. We also denote by J_n^k the image of j_n^k in $(\text{Ex}_Y\,X)_n$.
Before going any further, one needs to state some decidability conditions:

3.4.4 Lemma. *If X is a cofibrant simplicial set, then:*

1. *The inclusion $X \subset \text{Ex}_Y\,X$ is levelwise decidable.*
2. *$\text{Ex}_Y\,X$ is cofibrant and $X \to \text{Ex}_Y\,X$ is a cofibration.*
3. *The sets $J_k^n \subset (\text{Ex}_Y\,X)_n$ are decidable.*

Proof. All these decidability problems correspond to the decidability of a factorization of a map $\text{Sd}\,\Delta[n] \to X$ through some epimorphism $\text{Sd}\,\Delta[n] \to K$. In all these cases we will show that the corresponding epimorphism is a degeneracy quotient using Proposition 3.1.10 and conclude about the decidability using Lemma 3.1.8.

1. It corresponds to the map $\text{Sd}\,\Delta[n] \to \Delta[n]$ which is the nerve of the max function $\mathcal{K}[n] \to [n]$, whose section $i \mapsto \{0, \dots, i\}$ satisfies the condition of Proposition 3.1.10.
2. One just needs to check degeneracy are decidable in $\text{Ex}\,X$, so it is about the epimorphism $\text{Sd}(\sigma) : \text{Sd}\,\Delta[n] \to \text{Sd}\,\Delta[m]$ for any degeneracy σ. It is the nerve of $\sigma : \mathcal{K}[n] \to \mathcal{K}[m]$ which has a section satisfying the condition of Proposition 3.1.10 which sends every $P \in \mathcal{K}[m]$ to $\sigma^{-1}P$
3. It corresponds to the map $j_n^k : \text{Sd}\,\Delta[n] \to j_n^k(\text{Sd}\,\Delta[n])$, which is just is the nerve of the projection $j_n^k : \mathcal{K}[n] \to j_n^k\mathcal{K}[n]$ which is already of the form of Proposition 3.1.10.

□

We can now give the definition of the P-structure on $X \hookrightarrow \text{Ex}_Y X$.

- Type I cells are the non-degenerate cells $v \in \text{Ex}_Y(X)$ which are not[8] in X and can be written as yr_n^k with $y \in J_n^k \subset \text{Ex}_Y X$.
- Point 8 of Lemma 3.4.5 will prove that being type I is decidable. Type II cells are just the cells that are not of type I (and which are non-degenerate and not in X).
- For any cell x one defines Px as xr_n^k where k is the smallest integer such that $x \in J_n^k$, i.e. $x \in J_n^k - J_n^{k-1}$. Lemma 3.4.4 shows that the J_n^k are decidable so there is indeed a unique such integer k.

In order to show that being type I is decidable and that P defined this way defines a bijection from type II cells to type I cells, one needs a few technical lemma that we have regrouped in:

3.4.5 Lemma.

1. *If $x \in J_n^k - J_n^{k-1}$, then $d_{k+1}Px = x$.*
2. *$x \in J_n^k$ if and only if $Px \in J_{n+1}^k$*
3. *If $x \in J_n^{k-1}$ then xr_n^k is degenerate.*
4. *P^2x is always a degenerate cell.*
5. *If x is a degenerate or type I cell or in X, then Px is a degenerate cell.*
6. *If $x \in J_n^k - J_n^{k-1}$ then for all $i \leqslant k$, $d_i(Px) \in J_n^{k-1}$.*
7. *If $x \in J_n^k - J_n^{k-1}$ then for all i, with $k+1 < i \leqslant n+1$, $d_i(Px)$ is either of type I or degenerate.*
8. *A non-degenerate cell x in $(\text{Ex}_Y X)_n - X_n$ is type I if and only Px is a degenerate cell.*

[8]It appears that, because of point 2 of Lemma 3.4.5 and the fact that r_n^0 is the same as $\text{SD}\,\sigma_0$, it is actually a consequence from the rest of the definition that type I cells are not in X.

Proof. 1. $d_{k+1}Px$ is $xr^k_n(\text{SD}\,\partial^{k+1})$ which is equal to x by equation (2).

2. Let k is the smallest value such that $xj^k_n = x$, i.e.$Px = xr^k_n$. Equation (5) gives $xr^k_n j^k_{n+1} = xj^k_n r^k_n = xr^k_n$. Hence $Px \in J^k_{n+1}$, in particular $x \in J^h_n \Rightarrow k \leqslant h \Rightarrow Px \in J^h_{n+1}$. Conversely, if $Px \in J^k_{n+1}$ then:

$$\begin{aligned} xj^k_n &= (Px)(\text{SD}\,\partial^{h+1})j^k_n && (\text{as } x = d_{h+1}Px) \\ &= (Px)j^k_{n+1}(\text{SD}\,\partial^{h+1})j^k_n && (\text{ as } Px \in J^k_{n+1}) \\ &= (Px)j^k_{n+1}(\text{SD}\,\partial^{h+1}) && (\text{by equation (8)}) \\ &= x && (\ Px \in J^k_{n+1} \text{ and } x = d_{h+1}Px) \end{aligned}$$

Hence $x \in J^k_n$.

3. $xr^k_n = xj^{k-1}_n r^k_n$ is a degenerate cell because of equation (4)

4. Let k such that $x \in J^k_n - J^{k-1}_n$, then $Px = xr^k_n = xj^k_n r^k_n$ and $Px \in J^k_{n+1} - J^{k-1}_{n+1}$ because of point 2, hence $P^2x = xr^k_n r^k_{n+1} = xj^k_n r^k_n r^k_{n+1}$ which is a degenerate cell because of equation (7).

5. Equations (10) and (11) show that if x is a degenerate cell then Px is a degenerate cell. If $x \in X$, i.e. $x \in J^0_n$ then $Px = xr^0_n$ but $r^0_n = \text{SD}\,\sigma_0$ so Px is a degenerate cell.

   It follows that if x is of type I, then $x = yr^k_n$ with $y \in J^k_n$ if $y \in J^{k-1}_n$ then x is a degenerate cell because of point 3, hence Px is a degenerate cell because of the first part of the present point, if $y \notin J^{k-1}_n$ then $x = Py$ and hence Px is a degenerate cell because of point 4.

6. This follows immediately from equation (9) as $d_i(Px) = xj^k_n r^k_n(\text{SD}\,\partial^i)$.

7. For $k+1 < i \leqslant n+1$ we have:

$$\begin{aligned} j^k_n r^k_n(\text{SD}\,\partial^i_{n+1}) &= j^k_n(\text{SD}\,\partial^{i-1}_n)r^k_{n-1} && \text{by equation (6)} \\ &= j^k_n(\text{SD}\,\partial^{i-1}_n)j^k_{n-1}r^k_{n-1} && \text{by equation (8)} \end{aligned}$$

   This equation shows that for $x \in J^k_n$, d_iPx is of the form yr^k_{n-1} for $y \in J^k_{n-1}$, namely $y = x(\text{SD}\,\partial^{i-1})j^k_{n-1}$, hence, if d_iPx is a non-degenerate cell, it is of type I.

8. We have shown in Item 5 that if x is type I then Px is a degenerate cell. Conversely let x be a non-degenerate cell such that Px is a degenerate cell. Let k be such that $x \in J_n^k - J_n^{k-1}$. One has $x = d_{k+1}Px$ by point 1 of the lemma, hence $d_{k+1}Px$ is a non-degenerate cell, which means that Px can only be σ_k-degenerate or σ_{k+1}-degenerate (otherwise $d_{k+1}PX$ would also be a degenerate cell). If Px is σ_k-degenerate then $d_kPx = d_{k+1}Px = x$, but by point 6 of the present lemma $d_kPx \in J_n^{k-1}$ so this is impossible. If Px is σ_{k+1}-degenerate then $d_{k+2}Px = d_{k+1}Px = x$ hence point 7 shows that x is of type I.

□

3.4.6. We are now ready to prove Proposition 3.4.1:

Proof. The goal is to show that the type I cell and the operation P we have defined satisfy the condition of Definition 3.2.1, so that the map is a strongly anodyne cofibration because of Lemma 3.2.4.
Point 8 of Lemma 3.4.5 (combined with Lemma 3.4.4) shows that being a type I cell is decidable. So one can indeed defines type II cells as the cells that are not of type I (and non-degenerate nor in the domain) and get a partition of the non-degenerate cells. It also follows from point 8 that if x is a type II cell then Px is a non-degenerate cell, and it is type I (either by definition or because of point 4). Finally, point 2 shows that P preserve the k such that $x \in J_n^k$, as $X \subset \mathrm{Ex}_Y X$ corresponds to J_n^0 it shows that P never sends cell not in X to cell in X. So P restricts into a function from type II cells to type I cells.
We now show that it is a bijection:
If x is a type I cell than it can be written as yr_n^k with $y \in J_n^k$. By point 3 of Lemma 3.4.5, if $y \in J_n^{k-1}$, then $x = yr_n^k$ is a degenerate cell, hence $y \notin J_n^{k-1}$ and hence $x = Py$. By point 5 of Lemma 3.4.5, if y is a degenerate or type I cell then $x = Py$ is a degenerate cell, hence y is a type II cell. This proves the surjectivity of P.
If x is a type II cell and $y = Px$, then $x = d_{k+1}Px$ (because of point 1 of Lemma 3.4.5) where k can be characterized as the unique integer such that $y \in J_{n+1}^k - J_{n+1}^{k-1}$ (because of point 2 of Lemma 3.4.5). Hence P is injective on type II cell and this concludes the proof that P is a bijection between non-degenerate type II cells and non-degenerate type I cells.

Finally if x is a non-degenerate type II cell, and k is such that $x \in J_n^k - J_n^{k-1}$. Point 1 of Lemma 3.4.5 shows that $d_{k+1}(Px) = x$, while point 6 and 7 show that for all $i \neq k+1$, $d_i P x$ is either in J_n^{k-1} or a type I or degenerate cell, hence always distinct from x. So there is indeed a unique i such that $d_i P x = x$, and it is $k+1$.
It remains to prove the "well-foundedness" or "finite height" condition. It follows from point 6 and 7 of Lemma 3.4.5 that given $x \in J_n^k - J_n^{k-1}$ a non-degenerate type II cell, $Ant_{\text{II}}(x) \subset J_n^{k-1}$. In particular, any cell $x \in J_n^k$ has weak P-height at most k, hence by Lemma 3.2.3 this shows that every cell has finite P-height and hence concludes the proof.

□

3.4.7 Corollary. *For any $f : X \to Y$ with X cofibrant, the morphism:*

$$X \to \text{Ex}^\infty X \underset{\text{Ex}^\infty Y}{\times} Y$$

is a strongly anodyne cofibration.

Proof. Consider $\text{Ex}^k X \times_{\text{Ex}^k Y} Y \to Y$ and apply the functor Ex_Y to it. One obtains:

$$\begin{aligned} \text{Ex}_Y \left(\text{Ex}^k X \times_{\text{Ex}^k Y} Y \right) &= \text{Ex} \left(\text{Ex}^k X \times_{\text{Ex}^k Y} Y \right) \times_{\text{Ex}\, Y} Y \\ &= \left(\text{Ex}^{k+1} X \times_{\text{Ex}^{k+1} Y} \text{Ex}\, Y \right) \times_{\text{Ex}\, Y} Y \end{aligned}$$

in the last terms the map from $\left(\text{Ex}^{k+1} X \times_{\text{Ex}^{k+1} Y} \text{Ex}\, Y\right)$ to $\text{Ex}\, Y$ used in the fiber product is just the second projection, so the fiber product simplifies to:

$$\text{Ex}_Y \left(\text{Ex}^k X \underset{\text{Ex}^k Y}{\times} Y \right) = \text{Ex}^{k+1} X \underset{\text{Ex}^{k+1} Y}{\times} Y$$

And the natural map $\text{Ex}^k X \times_{\text{Ex}^k Y} Y \to \text{Ex}_Y \left(\text{Ex}^k X \times_{\text{Ex}^k Y} Y \right)$ corresponds through this identification to:

$$n_{\text{Ex}^k X} \underset{n_{\text{Ex}^k Y}}{\times} Id_Y : \text{Ex}^k X \underset{\text{Ex}^k Y}{\times} Y \to \text{Ex}^{k+1} X \underset{\text{Ex}^{k+1} Y}{\times} Y$$

It follows by induction that the sequence of maps:

$$X \to \operatorname{Ex} X \underset{\operatorname{Ex} Y}{\times} Y \to \cdots \to \operatorname{Ex}^k X \underset{\operatorname{Ex}^k Y}{\times} Y \to \operatorname{Ex}^{k+1} X \underset{\operatorname{Ex}^{k+1} Y}{\times} Y \to \dots$$

are all strong anodyne cofibrations (and all these objects are cofibrant), and the map $X \to \operatorname{Ex}^\infty X \times_{\operatorname{Ex}^\infty Y} Y$ is their transfinite composite (this last claim can either be observed very explicitly, or formally by commutation of directed colimits with finite limits).

□

3.5 Applications

3.5.1 Proposition. *Kan fibrations are the same as the strong fibrations of Definition 2.2.3. Dually, the trivial cofibrations of Definition 2.2.3 are the same as anodyne cofibrations.*

The proof given here, at least the case of a Kan fibration between cofibrant objects, is essentially the proof proposition 2.1.41 of [3].

Proof. We start with the first half: we observed in Remark 2.2.4 that strong fibrations are Kan fibrations. So we only need to show that any Kan fibration is a strong fibration. We first show this claim for $p : A \twoheadrightarrow B$ a Kan fibration between cofibrant object. One has that $\operatorname{Ex}^\infty(f)$ is a Kan fibration (by Proposition 3.3.2) between fibrant objects (because of Corollary 3.3.4), hence it is a strong fibration (by Lemma 2.2.6.(iii)), in particular any pullback of $\operatorname{Ex}^\infty(f)$ is also a strong fibration. This gives a factorization of p:

$$\begin{array}{ccccc} A & \overset{\sim}{\hookrightarrow} & \operatorname{Ex}^\infty(A) \times_{\operatorname{Ex}^\infty(B)} B & \longrightarrow & \operatorname{Ex}^\infty A \\ & {\scriptstyle p} \searrow & \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow {\scriptstyle \operatorname{Ex}^\infty p} \\ & & B & \longrightarrow & \operatorname{Ex}^\infty B \end{array}$$

in an anodyne cofibration (by Corollary 3.4.7) followed by strong fibration as a pullback of the strong fibration $\operatorname{Ex}^\infty(p)$. So p is a retract of the strong fibration part by the retract lemma (2.2.5) and hence is itself a strong fibration.

We now move to the case of a general Kan fibration. We first show that a Kan fibration that is also an equivalence is a trivial fibration. Let $p : X \to Y$

be such a Kan fibration and weak equivalence, one needs to show that it has the right lifting property against all boundary inclusion: $\partial\Delta[n] \hookrightarrow \Delta[n]$, consider such a lifting problem:

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$

One first factors the map $\Delta[n] \to Y$ as a cofibration followed by a trivial fibration and we form a pullback of f along the fibration part to get a diagram:

$$\begin{array}{ccccc} \partial\Delta[n] & \xrightarrow{u} & P & \xrightarrow{\sim} & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ \Delta[n] & \hookrightarrow & Z & \xrightarrow{\sim} & Y \end{array}$$

By 2-out-of-3 the new fibration f' is again a weak equivalence, but note that now the object Z is cofibrant. One can further factor u in a cofibration followed by a trivial fibration:

$$\begin{array}{ccccccc} \partial\Delta[n] & \hookrightarrow & K & \xrightarrow{\sim} & P & \xrightarrow{\sim} & X \\ \downarrow & & & \searrow^{f''} & \downarrow f' & & \downarrow f \\ \Delta[n] & & \hookrightarrow & & Z & \xrightarrow{\sim} & Y \end{array}$$

f'' is a Kan fibration between cofibrant objects, hence is a strong fibration by the first part of the proof, moreover it is an equivalence hence it is a trivial fibration by the last point of Lemma 2.2.6, and hence it has the right lifting property against the boundary inclusion which shows that the morphism f is a trivial fibration as well.

One can then conclude the proof by the same argument as used in the proof of the first part of Lemma 2.2.6: Given a lifting problem of a trivial cofibration against a Kan fibration one can, using appropriate factorization, reduce to the case where the top and bottom map of the lifting square are weak equivalences, in which case the Kan fibration is a weak equivalence by 2-out-of-3 and hence is a trivial fibration by the claim we just proved, and hence has the right lifting property against all cofibrations which concludes the proof.
For the second half of the proposition, given a trivial cofibration j one factors it as an anodyne cofibration followed by a Kan fibration. By the first half of the proof, the Kan fibration is a strong fibration and hence has the right lifting property against j. It immediately follows from the retract lemma 2.2.5 that j is a retract of the anodyne cofibration part of the factorization and hence is an anodyne cofibration itself. □

3.5.2 Proposition. *The model structure of Theorem 2.2.9 is right proper, i.e. the pullback of a weak equivalence along a fibration is again a weak equivalence.*

Proof. We consider a square in $\widehat{\Delta}$:

$$\begin{array}{ccc} P & \xrightarrow{g} & B \\ \downarrow & \lrcorner & \downarrow p \\ C & \xrightarrow[f]{\sim} & A \end{array}$$

where p is a fibration and f is a weak equivalence, and we need to show that g is a weak equivalence. Using Lemma 3.5.3 below, we can freely assume that A, B and C are cofibrants. This implies that the pullback P is also cofibrant because it is a subobject of the product $B \times C$ which is cofibrant because of the cartesianess of the model structure (Proposition 3.2.6), and the explicit description of cofibrant objects in terms of decidability of degenerateness of cell, immediately shows that a subobject of a cofibrant simplicial set is cofibrant.
In this case when all objects are cofibrant, the result follows immediately from an application of Kan's Ex^{∞} functor: It preserves the pullback square (because it is a right adjoint), it sends each object to a fibrant object, when all the object are fibrant the result is true in any (weak) model category (a constructive argument, valid in weak model category is given as corollary 2.4.4

in [8]), and it detects equivalences between cofibrant objects because the morphism $X \to \mathrm{Ex}^\infty X$ is an anodyne cofibration (hence an equivalence) for X cofibrant.

□

3.5.3 Lemma. *Let $\mathcal{C}$ be a Quillen*[9] *model category, if for every pullback diagram*

$$\begin{array}{ccc} P & \xrightarrow{g} & B \\ \downarrow & \lrcorner & \downarrow p \\ C & \xrightarrow[f]{\sim} & A \end{array}$$

in which A, B and C are cofibrant, p is a fibration, if f is a weak equivalence then so too is g. Then $\mathcal{C}$ is right proper: that is the condition also holds without assuming the A, B and C are cofibrant.

Proof. We consider a pullback as in the lemma, and we need to show that the projection map $P \to B$ is a weak equivalence, but without assuming A, B and C are cofibrant. By assumption, we already know this is the case case when A, B and C are cofibrants. The proof will proceed in three steps, where at each step we relax the cofibrancy assumption on one of the three objects:

First step: We assume that C and A are cofibrant (but not neccessarily B). In this case, we consider a cofibrant replacement $B^c \overset{\sim}{\twoheadrightarrow} B$, and we form the pullback:

$$\begin{array}{ccc} Q & \xrightarrow{h} & B^c \\ \sim\downarrow & \lrcorner & \downarrow\sim \\ P & \xrightarrow{g} & B \\ \downarrow & \lrcorner & \downarrow p \\ C & \xrightarrow[f]{\sim} & A \end{array}$$

Then $Q \to P$ is a trivial fibration because it is a pullback of a trivial fibration, the outer rectangle is a pullback as the composite of two pullback squares,

[9] It is actually enough to assume that $\mathcal{C}$ is a left semi-model category, as the proof below shows. We will only use it for the Kan-Quillen model structure.

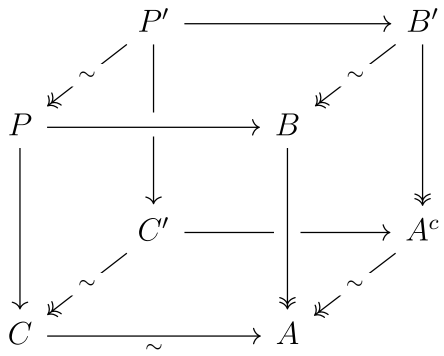
so h is a weak equivalence as the pullback of f along a fibration (with all the object involved cofibrants), hence g is a weak equivalence by 2-out-of-3.

Second step: We only assume that A is cofibrant. We then take a cofibrant replacement $C^c \overset{\sim}{\twoheadrightarrow} C$ of C. and we form the pullback:

$$\begin{array}{ccccc} R & \overset{\sim}{\twoheadrightarrow} & P & \overset{g}{\longrightarrow} & B \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C^c & \overset{\sim}{\twoheadrightarrow} & C & \underset{f}{\overset{\sim}{\longrightarrow}} & A \end{array}$$

as in the previous case, the map $R \to P$ is a trivial fibration because it is a pullback of a trivial fibration. The composite map $R \to B$ is a pullback along the fibration p of the composite weak equivalence $C^c \to A$, so as C^c and A are cofibrant, we deduce from the first step that the composite $R \to B$ is a weak equivalence. By 2-out-of-3, this shows that g is a weak equivalence and concludes the proof for this case.

Third step: We make no cofibrancy assumption. Then we take a cofibrant replacement $A^c \overset{\sim}{\twoheadrightarrow} A$. We then form a cube



where each face is a pullback square. All the diagonal maps are pullback of the trivial fibration $A^c \to A$, and so are trivial fibrations, the map $C' \to A^c$ is a weak equivalence by 2-out-of-3, hence the map $P' \to B'$ is also a weak equivalence as a pullback of a weak equivalence along a fibration (using the fact that A^c is cofibrant and the second step). Hence the map $P \to B$ is a weak equivalence by 2-out-of-3. $\square$

References

[1] Peter Aczel and Michael Rathjen. Notes on constructive set theory. Available from http://www1.maths.leeds.ac.uk/~rathjen/book.pdf, 2010.

[2] Clark Barwick. On left and right model categories and left and right Bousfield localizations. *Homology, Homotopy and Applications*, 12(2):245–320, 2010.

[3] Denis-Charles Cisinski. *Les préfaisceaux comme modèles des types d'homotopie*. Société mathématique de France, 2006.

[4] Benoit Fresse. *Modules over operads and functors*, volume 169 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[5] Nicola Gambino and Simon Henry. Towards a constructive simplicial model of Univalent Foundations. *Journal of the London Mathematical Society*, 105(2):1073–1109, 2022.

[6] Nicola Gambino, Christian Sattler, and Karol Szumiło. The constructive Kan–Quillen model structure: two new proofs. *The Quarterly Journal of Mathematics*, 73(4):1307–1373, 2022.

[7] Richard Garner. Understanding the small object argument. *Applied categorical structures*, 17(3):247–285, 2009.

[8] Simon Henry. Weak model categories in constructive and classical mathematics. *Theory and Applications of Categories*, 35(24):875–958, 2020.

[9] Simon Henry. Combinatorial and accessible weak model categories. *Journal of Pure and Applied Algebra*, 227(2):107191, 2023.

[10] Andre Joyal and Myles Tierney. Quasi-categories vs Segal spaces. *Contemporary Mathematics*, 431, 2007.

[11] André Joyal and Myles Tierney. Notes on simplicial homotopy theory. *Quaderns*, Vol. 47, CRM, Barcelona, 2008.

Available at http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf.

[12] Daniel M Kan. On css complexes. *American Journal of Mathematics*, 79(3):449–476, 1957.

[13] Maria Emilia Maietti and Steven Vickers. An induction principle for consequence in arithmetic universes. *Journal of Pure and Applied Algebra*, 216(8-9):2049–2067, 2012.

[14] Michael Makkai and Jiří Rosický. Cellular categories. *Journal of Pure and Applied Algebra*, 218(9):1652–1664, 2014.

[15] Sean Moss. Another approach to the Kan–Quillen model structure. *Journal of Homotopy and Related Structures*, 15(1):143–165, 2020.

[16] Erik Palmgren and Steven J Vickers. Partial horn logic and cartesian categories. *Annals of Pure and Applied Logic*, 145(3):314–353, 2007.

[17] Markus Spitzweck. *Operads, algebras and modules in model categories and motives*. PhD thesis, Ph. D. thesis (Universität Bonn), 2001.

Simon Henry
Departement of Mathematics and Statistics
University of Ottawa
150 Louis-Pasteur Private,
Ottawa, ON K1N 9A7 (Canada)
Shenry2@Uottawa.ca