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THE TOPOLOGY OF CRITICAL PROCESSES, III
(COMPUTING HOMOTOPY)

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Résumé. La topologie algébrique dirigée étudie des espaces équipés d'une forme de direction, avec l'objectif d'inclure les processus non réversibles. Dans l'extension présente nous voulons couvrir aussi les *processus critiques*, indivisibles et inarrêtables.

Les parties précédentes de cette série ont introduit les *espaces contrôlés* et leur catégorie fondamentale. Ici on étudie comment calculer cette dernière. La structure d'homotopie de ces espaces sera examinée dans la Partie IV.

Abstract. Directed Algebraic Topology studies spaces equipped with a form of direction, to include models of non-reversible processes. In the present extension we also want to cover *critical processes*, indecomposable and unstoppable.

The previous parts of this series introduced *controlled spaces* and their fundamental category. Here we study how to compute the latter. The homotopy structure of these spaces will be examined in Part IV.

Keywords. Directed algebraic topology, homotopy theory, fundamental category, concurrent process.

Mathematics Subject Classification (2010). 55M, 55P, 55Q, 68Q85.

Introduction

0.1 Directed and controlled spaces

Directed Algebraic Topology is an extension of Algebraic Topology, dealing with 'spaces' where the paths need not be reversible; the general aim is including the representation of *irreversible processes*. A typical setting for this study, the category $\mathrm{d}\mathsf{Top}$ of directed spaces, or d-spaces, was introduced and studied in [G1]–[G3]; it is often employed in the theory of concurrency, cf. [FGHMR].

The present series is devoted to a further extension, where the paths can also be non-decomposable in order to include *critical processes*, indivisible and unstoppable – either reversible or not. For instance: quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, the action of a siphon, moving in a no-stop road, etc.

To this effect the category of d-spaces was extended in Part I [G4] to the category $\mathrm{c}\mathsf{Top}$ of *controlled spaces*, or *c-spaces*: an object is a topological space equipped with a set $X^\sharp$ of continuous mappings $a\colon [0,1] \to X$, called *controlled paths*, or *c-paths*, which are closed under concatenation and global reparametrisation (by surjective increasing endomaps of the interval) and include all the constant paths at the endpoints of c-paths.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths. Their category $\mathrm{c}\mathsf{Top}$ contains the category $\mathrm{d}\mathsf{Top}$ of d-spaces as a full subcategory, reflective and coreflective: a c-space is a d-space if and only if it is *flexible*, which means that each point is flexible (its trivial loop is controlled) and every controlled path is flexible (all its restrictions are controlled).

Every c-space X has two associated d-spaces, the generated d-space $\hat{X}$ and the flexible part $\mathsf{Fl}\,X$, by the reflector and coreflector of the embedding $\mathrm{d}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ (Section 1.2 of Part I).

0.2 The fundamental category

Part II [G5] defines and studies the fundamental category of controlled spaces, as a functor

$$\uparrow\!\Pi_1\colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \tag{1}$$

that extends the fundamental category of d-spaces [G1, G3] and the fundamental groupoid of topological spaces.

There are two natural transformations (see Section 5.2 of Part II)

$$\uparrow\!\Pi_1(\mathsf{Fl}\,X) \longrightarrow \uparrow\!\Pi_1(X) \longrightarrow \uparrow\!\Pi_1(\hat{X}) \qquad (2)$$

induced by the embeddings $\mathsf{Fl}\,X \to X \to \hat{X}$ (the counit of the coreflector and the unit of the reflector of d-spaces).

These functors need not be faithful, as we shall see in 1.3, but Theorem 5.3(b) of Part II says that $\uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ is a full embedding when the c-space X is *preflexible*, that is all the c-paths of $\hat{X}$ between flexible points of X are already controlled in the latter.

The present Part III is an immediate continuation of Part II, devoted to computing the fundamental category of c-spaces. The definitions and results of Part II are taken for granted and only referred to.

Part IV will study the homotopy structure of c-spaces, their homotopy equivalences and their links with cubical sets. In particular, we shall analyse the formal theory of homotopy in $\mathrm{c}\mathsf{Top}$, following the classification of directed settings in [G3].

0.3 Outline

In Section 1 we calculate the fundamental category of the c-spaces introduced so far, and others, applying Theorems 5.3 (on preflexible c-spaces) and 5.8 (on covering maps of c-spaces) of Part II, and developing peculiar techniques adequate to the present framework. The relationship between the fundamental category of c-spaces and d-spaces is discussed in 1.6, where we show that the theorem of Seifert-van Kampen fails for c-spaces.

In the same line, Section 2 briefly considers how the analysis of obstructions, a typical problem in concurrency, can be dealt with replacing the d-spaces used in [G3], Chapter 3 (and elsewhere) with rigid c-spaces. This leads to a far simpler analysis, but a less rich one.

Finally, in Section 3, we prove that the fundamental category of a border flexible c-space can be simply defined by general deformations of controlled paths, instead of using their flexible deformations – as in the general case.

Acknowledgments. The author is indepted to the Referee for many helpful suggestions.

0.4 Notation and conventions

A continuous mapping between topological spaces is called a *map*. $\mathbb{R}$ denotes the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0, 1]$. The identity path $\mathrm{id}\,\mathbb{I}$ is written as $\underline{i}$. The open and semiopen intervals of the real line are denoted by square brackets, like $]0, 1[$, $[0, 1[$ etc.

A *preorder* relation is assumed to be reflexive and transitive; an *order* is also anti-symmetric. A mapping which preserves (resp. reverses) preorders is said to be *increasing* (resp. *decreasing*), always used in the weak sense.

As usual, a preordered set X is identified with the small category whose objects are the elements of X, with one arrow $x \to x'$ when x precedes x' and none otherwise.

The binary variable α takes values $0, 1$, which are generally written as $-, +$ in superscripts and subscripts. The symbol $\subset$ denotes weak inclusion.

The previous papers [G4, G5] of this series are cited as Part I and Part II, respectively; the reference I.2 or II.3.4, for instance, points to Section 2 of Part I or Subsection 3.4 of Part II.

1. Calculating the fundamental category

This section studies how to compute the fundamental category of c-spaces. Using Theorem II.5.3(b) on preflexible c-spaces, many of these results can be deduced from the fundamental category of the generated d-spaces, already computed in [G3]; but a direct calculation can often be simple and more significant.

The new aspects which appear here, with respect to the theory of d-spaces, are highlighted in 1.6.

The symbols $\mathbf{2}, \mathbf{3}, \mathbf{N}, \mathbf{Z}, \mathbf{R}$ denote ordered sets, and the associated categories; the ordered sets 2, 3 and $D|\mathbf{Z}|$ are discrete. $\mathbb{N}$ is the one-object category associated to the additive monoid of the natural numbers.

1.1 Elementary calculations

We begin by examining the basic c-spaces, showing that many of them are 1-simple, in the sense of II.5.1: their fundamental category is a preorder; of course, the controlled circles $\mathrm{c}\mathbb{S}^1$ and $\mathrm{c}_n\mathbb{S}^1$ are not. (Some of these results are already in II.5.9.)

(a) The fundamental categories of $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $\mathrm{c}\mathbb{R}$ are the following ordered sets:

$$\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}) = \mathbf{2}, \qquad \uparrow\!\Pi_1(\mathrm{c}\mathbb{J}) = \mathbf{3}, \qquad \uparrow\!\Pi_1(\mathrm{c}\mathbb{R}) = \mathbf{Z}. \tag{3}$$

As to $\mathrm{c}\mathbb{I}$, the identity $\underline{i}\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is 2-equivalent to any other c-path $\rho\colon 0 \to 1$, by Lemma II.4.6(c): in fact, ρ is a global reparametrisation, and therefore $\rho = \underline{i}\rho \sim_2 \underline{i}$, so that there is precisely one arrow $[\underline{i}]$ from 0 to 1, in the fundamental category. At each flexible point, 0 or 1, there is only one loop $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$, the trivial one.

As to $\mathrm{c}\mathbb{J}$ and $\mathrm{c}\mathbb{R}$, two c-paths $a, b\colon x \to y$ in any of them are always 2-equivalent, since they are in the one-jump c-structure of $[x, y]$, isomorphic to $\mathrm{c}\mathbb{I}$.

For these preflexible spaces the components of the natural transformations $\uparrow\!\Pi_1(\mathsf{Fl}\,X) \to \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ of (2) become inclusions of ordered sets:

$$2 \to \mathbf{2} \to [0,1], \qquad 3 \to \mathbf{3} \to [0,2], \qquad D|\mathbf{Z}| \to \mathbf{Z} \to \mathbf{R}. \tag{4}$$

(b) The argument used above for $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I})$ also applies to the delayed intervals $\mathrm{c}_-\mathbb{I}$ and $\mathrm{c}_+\mathbb{I}$, in II.1.3(e)

$$\uparrow\!\Pi_1(\mathrm{c}_-\mathbb{I}) = \uparrow\!\Pi_1(\mathrm{c}_+\mathbb{I}) = \mathbf{2}, \tag{5}$$

whose c-structure is also generated by a single map $\mathbb{I} \to \mathbb{I}$. These c-spaces are not preflexible, but their fundamental category is still full in $\uparrow\!\Pi_1(\uparrow\!\mathbb{I})$.

(c) The fundamental category of the directed circle $\uparrow\!\mathbb{S}^1$, as described in [G3], 3.2.7(d), is the subcategory of the groupoid $\Pi_1\mathbb{S}^1$ formed of the classes of anticlockwise paths (in $\mathbb{R}^2$). Each monoid $\uparrow\!\pi_1(\uparrow\!\mathbb{S}^1, x)$ is isomorphic to the additive monoid $\mathbb{N}$ of natural numbers.

Applying Theorem II.5.3(b), the fundamental category of the one-stop circle $\mathrm{c}\mathbb{S}^1$ amounts to the fundamental monoid at the unique flexible point x_0 (the point 1 of the complex plane)

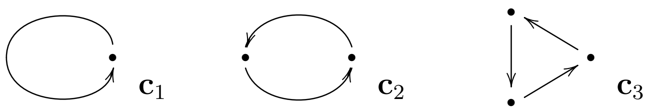
$$\uparrow\!\Pi_1(\mathrm{c}\mathbb{S}^1)(x_0, x_0) = \uparrow\!\pi_1(\uparrow\!\mathbb{S}^1, x_0) = \mathbb{N}. \tag{6}$$

Without using $\uparrow\!\Pi_1(\uparrow\!\mathbb{S}^1)$ this is also proved by Theorem II.5.8(b) applied to the exponential map $\mathrm{c}\mathbb{R} \to \mathrm{c}\mathbb{S}^1$.

Therefore two c-loops a, b in $\mathrm{c}\mathbb{S}^1$ are 2-equivalent if and only if they have the same length $2k\pi$ (in radians), if and only if they both turn k times ($k \geqslant 0$) around the circle, anticlockwise.

(d) More generally, the fundamental category of the preflexible n-stop circle $\mathrm{c}_n\mathbb{S}^1$ (see II.1.4(d) is the full subcategory of the fundamental category of $(\mathrm{c}_n\mathbb{S}^1)\hat{} = \uparrow\mathbb{S}^1 = \uparrow\mathbb{R}/\mathbb{Z}$ on n flexible points, the vertices $[i/n]$ (for $i = 0, ..., n-1$) of an inscribed n-gon.

$\uparrow\Pi_1(\mathrm{c}_n\mathbb{S}^1)$ is thus the category $\mathbf{c}_n$ freely generated by n arrows disposed as follows on the edges of an n-gon



(7)

Again, this result can also be obtained using the covering map of c-spaces $p_n \colon \mathrm{c}_n\mathbb{R} \to \mathrm{c}_n\mathbb{S}$.

(e) For the preflexible c-space X on the euclidean interval $[0, 3]$ described in I.2.3(e) we have a mixed situation; essentially, the paths in $[1, 2]$ behave as in $\mathrm{c}\mathbb{I}$, while those in $[0, 1]$ or $[2, 3]$ behave as in $\uparrow\mathbb{I}$.

1.2 Higher dimensional c-spaces

(a) Applying Theorem II.5.6 on cartesian products, we get the following fundamental categories

$$\begin{gathered}\uparrow\Pi_1(\mathrm{c}\mathbb{I}^n) = \mathbf{2}^n, \qquad \uparrow\Pi_1(\mathrm{c}\mathbb{J}^n) = \mathbf{3}^n,\\ \uparrow\Pi_1(\mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{J}) = \mathbf{2}\times\mathbf{3},\\ \uparrow\Pi_1(\mathrm{c}\mathbb{R}^n) = \mathbf{Z}^n, \qquad \uparrow\Pi_1(\mathrm{c}\mathbb{T}^n) = \mathbb{N}^n,\end{gathered} \tag{8}$$

which are (partially) ordered sets, except the last. The controlled n-torus $\mathrm{c}\mathbb{T}^n$ was defined in I.2.6(d) as the cartesian power $(\mathrm{c}\mathbb{S}^1)^n$, or equivalently as the orbit c-space $(\mathrm{c}\mathbb{R}^n)/\mathbb{Z}^n$; its fundamental category amounts to the monoid $\mathbb{N}^n$ at the only flexible point.

(b) The fundamental category of all the higher c-spheres $\mathrm{c}\mathbb{S}^n$, for $n \geqslant 2$, is trivial: the discrete category $\mathbf{1}$.

In fact, there is one flexible point, $*$. Every c-path of $\mathrm{c}\mathbb{S}^n$ is a general concatenation of a finite family of c-loops of the form pa, where $a\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^n$ is a c-path of the controlled n-cube, and it is sufficient to prove that each of them is 2-equivalent to the trivial loop (at $*$).

If the path a lies in a face of the cube, pa is already the trivial loop. Otherwise, it is a path $(0, ..., 0) \to (1, ..., 1)$, and it is 2-equivalent to the concatenation $b = b_1 * b_2$ of two c-paths living in some faces, and collapsed to the trivial loop in the quotient c-space. For instance one can take $b_1(t) = (t, 0, ..., 0)$ (on an edge) and $b_2(t) = (1, t, ..., t)$ (in the face $t_1 = 1$).

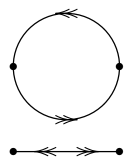
1.3 Other calculations

The following computations of the fundamental category give a better understanding of the natural transformations $\uparrow\!\Pi_1(\mathsf{Fl}\,X) \to \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ of (2). Moreover, they are based on topological arguments which will also be useful in other cases.

(a) The reversible c-interval $\mathrm{c}\mathbb{I}^\sim$ of II.1.3(d) has a c-structure generated by the identity path $\underline{i}$ and the reversion $r\colon \mathbb{I} \to \mathbb{I}$; the flexible points are 0 and 1.

Each c-path $x \to y$ (between flexible points) has an integral length, which is even if $x = y$ and odd if $x \neq y$. We prove below, in Theorem 1.7, that this length is constant up to 2-equivalence, and determines the class of a path in $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^\sim)(x, y)$.

In other words, we shall prove that the obvious c-map $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^\sim$



$$p(x, y) = (x + 1)/2 \qquad (9)$$

induces an isomorphism $p_*\colon \uparrow\!\Pi_1(\mathrm{c}_2\mathbb{S}^1) \to \uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^\sim)$ defined on the category $\mathbf{c}_2$ described in (7). Let us note that p is not a covering map: the flexible points of the basis are not evenly covered; loosely speaking, the selection of c-paths in the domain and codomain 'mends' this failure.

Thus the category $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^\sim)$ is freely generated by two arrows, the classes $[\underline{i}]\colon 0 \to 1$ and $[r]\colon 1 \to 0$; at each vertex it has a fundamental monoid isomorphic to the additive monoid $\mathbb{N}$.

The generated d-space $(c_S\mathbb{I})\hat{} = \mathbb{I}^\sim$ is the reversible d-interval of I.2.4(c), whose fundamental category is the indiscrete groupoid on two objects (with one arrow between any pair of objects).

In this case the functor $\uparrow\Pi_1(X) \to \uparrow\Pi_1(\hat{X})$ is not faithful; moreover $\underline{i}$ and r are reversible c-paths of X whose classes in $\uparrow\Pi_1(X)$ are not invertible.

(b) The fundamental category $\uparrow\Pi_1(c_S\mathbb{I})$ of the growing-siphon interval (in I.3.3(a)) is generated by the following arrows (where r is the reversion path $r(t) = 1 - t$)

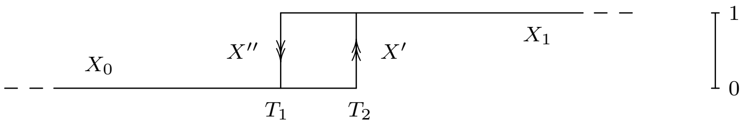
$$(x, x')\colon x \to x', \qquad [r]\colon 1 \to 0 \qquad\qquad (0 \leqslant x < x' \leqslant 1), \qquad (10)$$

under the relation $(x, x')(x', x'') = (x, x'')$, for $0 \leqslant x < x' < x'' \leqslant 1$.

The identity path $\underline{i}$ is flexible and reversible in $c_S\mathbb{I}$, but is not flexibly reversible: the reversed path r is not flexible, and the associated arrow $[\underline{i}] = (0, 1)\colon 0 \to 1$ is not invertible. But it becomes invertible in the fundamental category of $(c_S\mathbb{I})\hat{} = \mathbb{I}^\sim$: also here the functor $\uparrow\Pi_1(X) \to \uparrow\Pi_1(\hat{X})$ is not faithful.

1.4 On-off controller

We now examine the c-space X built in I.3.1(a) to model an on-off controller (e.g. a thermostat) that oversees a variable T (e.g. the temperature), counteracting its rising



(11)

On the left branch X_0 the system is in state 0: the cooling device is off; if the temperature grows to T_2 the device jumps to state 1; then, if the temperature cools to T_1, it goes back to state 0.

The support $|X|$ of our model is a one-dimensional subspace of $\mathbb{R}^2$. The c-structure of X is generated by the c-structures of:

- X_0, X_1, natural intervals where T can vary,

- X', X'', one-jump c-intervals, where T is constant and the state of the system varies.

The flexible part $X_0 + X_1$ of the c-space X is the sum of two natural intervals; its fundamental groupoid $\Pi_1(\mathsf{Fl}\, X)$ is the sum of the indiscrete groupoids on the same sets, categorically equivalent to the discrete groupoid $2 = \{0, 1\}$.

The fundamental category $\uparrow\!\Pi_1(X)$ is equivalent to its skeleton, the full subcategory on two points $x_0 \in X_0$ and $x_1 \in X_1$; the latter is isomorphic to the category $\mathbf{c}_2$ (see (7)).

1.5 Transport networks and labelled graphs

Transport networks are usually modelled in graph theory, in an effective way as far as they do not interact with continuous variation. They can also be modelled by c-spaces, which allows us to combine them with planar or three-dimensional regions, as we have discussed in I.3.4.

The fundamental category can be readily used to study such models. Controlled spaces can thus unify aspects of continuous and discrete mathematics, interacting with hybrid control systems and others sectors of Control Theory [Br, He].

1.6 Comments

(a) The main method of calculation of the fundamental category for complex spaces, the theorem of Seifert-van Kampen, holds true in $\mathrm{d}\mathsf{Top}$, in the fundamental-category version of [G3], 3.2.6, but fails here.

For instance, we have seen that the category $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}) = \mathbf{2}$ has one arrow $0 \to 1$. Now we can cover $\mathrm{c}\mathbb{I}$ with the open subspaces $U = [0, 1[$ and $V =]0, 1]$, which only inherit the trivial loops at 0 and 1, respectively. Their fundamental category has only these trivial arrows, and the pushout over $\uparrow\!\Pi_1(U \cap V)$ (the empty category) gives the discrete category 2.

(b) Nevertheless, we have seen that the fundamental category $\uparrow\!\Pi_1(X)$ of a rigid or 'partially rigid' c-space can be rather easy to compute without this tool – or using it on $\uparrow\!\Pi_1(\hat{X})$ when the original c-space is preflexible.

(c) In many cases $\uparrow\!\Pi_1(X)$ is very small and easy to analyse, while $\uparrow\!\Pi_1(\hat{X})$ gives a finer description, at the price of a complex analysis where the equivalence of categories is totally ineffective. This will show even more clearly in the next section.

1.7 Theorem

The projection $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^{\sim}$ *defined in* (9) *induces an isomorphism of categories* $p_*\colon \uparrow\!\Pi_1(\mathrm{c}_2\mathbb{S}^1) \to \uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^{\sim})$.

Proof. (a) The functor p_* is bijective on the objects, the flexible points. It is also full, because $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^{\sim}$ obviously satisfies the path-lifting property II.5.7(i) within c-paths: every c-path $b\colon y \to y'$ in $\mathrm{c}\mathbb{I}^{\sim}$ has a lifting $a\colon x \to x'$ in $\mathrm{c}\mathbb{I}^{\sim}$, determined by the starting point $x \in F_y$ (unique in the present case).

The length of b is an integer, equal to the length of a measured in half-circles.

To prove that p_* is faithful we shall show that two c-paths $b, b'\colon y_0 \to y_1$ in $\mathrm{c}\mathbb{I}^{\sim}$ which are 2-equivalent *have the same length*, so that any pair of their liftings in $\mathrm{c}_2\mathbb{S}^1$ starting at the same point are also 2-equivalent; in other words one can lift along p the 2-equivalence relation – if not the actual 2-paths.

For the sake of simplicity we suppose that $y_0 = 0$, the case $y_0 = 1$ being similar. We use the path spaces $P(\mathbb{I}) = \mathbb{I}^{\mathbb{I}}$ and $P(\mathbb{I}^2)$ with the compact-open topology, determined by the metric $d(c, c') = \max_t d(c(t), c'(t))$ (and the euclidean metric on $\mathbb{I}$ and $\mathbb{I}^2$).

(b) Let P_n be the subspace of $P(\mathbb{I})$ formed of the c-paths $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\sim}$ starting at 0, of length n; let P be their (disjoint) union. We prove now that each P_n is open in P. (This amounts to saying that the length function $P \to \mathbb{N}$ is continuous, which is not obvious as it fails on the whole path space $\mathbb{I}^{\mathbb{I}}$.)

It will be sufficient to show that any two c-paths $b, b'\colon 0 \to y$ with $d(b, b') < 1/2$ have the same length. If b has length n, it determines a partition of the interval $\mathbb{I}$ in n subintervals

$$\begin{gathered} 0 = t_0 < t_1 < ... < t_n = 1, \\ b(t_0) = 0, \qquad b(t_1) = 1, \ ... \qquad b(t_n) = (1 - (-1)^n)/2, \end{gathered} \tag{12}$$

and is *properly* increasing on $[0, t_1]$, properly decreasing on $[t_1, t_2]$, and so on (by ‘properly’ we mean that it is not constant). There are $n-1$ ‘inversions of monotony’ (each of them occurring on a maximal closed subinterval where b is constant at 1 or 0, alternatively).

The other path b', of length n', has $b'(t_0) = 0$ and $b'(t_1) > 1/2$; because of the form of c-paths in $\mathrm{c}\mathbb{I}^{\sim}$, it must be properly increasing on some (at least one) subinterval of $[0, t_1]$. It also has $b'(t_2) < 1/2$, and must be properly decreasing on some subinterval of $[t_1, t_2]$; and so on. Finally, it has at least as many inversions of monotony as b, and $n' \geqslant n$. By symmetry, $n = n'$.

(c) Let $K\colon c\mathbb{I} \times \uparrow\mathbb{I} \to c\mathbb{I}^{\sim}$ be a hybrid 2-path between the c-paths $b, b'\colon 0 \to y$. Proving that they have the same length will achieve the argument.

The family of c-paths

$$u_t\colon c\mathbb{I} \to c\mathbb{I}^{\sim} \times \uparrow\mathbb{I}, \qquad u_t(s) = (s, t) \qquad\qquad (t \in \mathbb{I}), \tag{13}$$

gives an isometry $u\colon \mathbb{I} \to P(\mathbb{I}^2)$

$$d(u_t, u_{t'}) = \max_s d((s, t), (s, t')) = |t - t'|.$$

Composing u with the map $K_*\colon P(\mathbb{I}^2) \to P(\mathbb{I})$ we get a continuous mapping

$$Ku\colon \mathbb{I} \to P(\mathbb{I}), \qquad t \mapsto K_t = K(-, t)\colon \mathbb{I} \to \mathbb{I}, \tag{14}$$

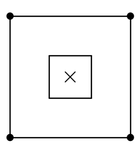
whose values K_t are the intermediate c-paths of K (see II.4.4(a)). They belong to P. Since Ku is defined on a connected space, all of them belong to the same subset P_n, including b and b'. □

2. Analysing obstructions

The analysis of obstructions inside a cubical directed space is a typical problem in concurrency, dealt with in [FGHMR] and many papers (see Part I). It is also studied in [G3], Chapter 3, working with d-spaces. The corresponding problem in rigid c-spaces seems to be far simpler, although it can give a less fine analysis, as shown in 2.3.

2.1 An elementary case

We begin with the 'square annulus' $X \subset c\mathbb{I}^2$ represented below, namely the compact subspace of the standard c-square which is the complement of the open square $]1/3, 2/3[^2$ (marked with a cross); the latter should be viewed as a single obstruction in an unstoppable process



X

$$\begin{array}{ccc} x & \longrightarrow & 1 \\ \uparrow & \times & \uparrow \\ 0 & \longrightarrow & y \end{array} \qquad \uparrow\Pi_1(X) \tag{15}$$

Typically, in the analysis of concurrent processes, the obstruction represents a resource (e.g. a memory storage, an application, a printer) that two (or more) concurrent automata cannot engage at the same time. A path below *or* above the obstruction corresponds to priority of one of them. Modifying the picture, one can represent in a similar way an island in a stream or a one-dimensional obstacle in space-time, as in the Introduction to [G3].

The fundamental category $\uparrow\Pi_1(X)$ is represented in the right diagram above: it is generated by four arrows forming a non-commutative square, and has two arrows $0 \to 1$ (not drawn in the figure).

Applying Theorem II.5.3(b) one can deduce this fact from the fundamental category of the generated d-space $\hat{X} \subset \uparrow\mathbb{I}^2$, determined in [G3], 3.1.1. But a direct proof is rather simple.

In fact, every c-path $a\colon 0 \to 1$ in X meets the vertical strip

$$S =]1/3, 2/3[\times \mathbb{I}$$

in one connected component of $S \cap X$, either below or above the obstruction. Suppose that a meets the lower component $U =]1/3, 2/3[\times[0, 1/3[$ (open in X). The preimage $a^{-1}(U)$ is an open subinterval of $]0, 1[$ (by continuity and monotony), and we can suppose it is precisely $]1/3, 2/3[$, up to invertible reparametrisation and 2-equivalence. For a second path a' of the same kind and similarly reparametrised, we can suppose that $a(t) \leqslant a'(t)$ for $t \in \mathbb{I}$ (replacing a with $a \wedge a'$).

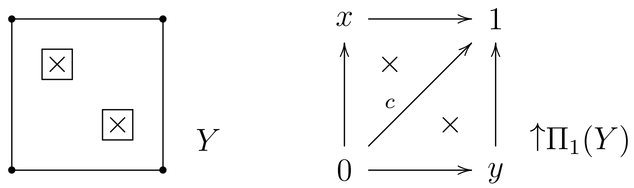
Now the affine interpolation H from a to a' is a hybrid 2-path in $\mathrm{c}\mathbb{I}^2$ and takes the interval $]1/3, 2/3[$ to the rectangle U (by monotony), proving that $a \sim_2 a'$ in X. Similarly, two paths above the obstruction are 2-equivalent in X. Finally, a c-path below the obstruction and another above are not even 2-equivalent in the underlying topological space.

2.2 Two obstructions

We examine now two subspaces $Y, Z \subset \mathrm{c}\mathbb{I}^2$ which arise from two obstructions, either appearing together (with respect to the generated path order, see I.1.8(c)) or one after the other.

In both cases a direct computation is easy, if more complex than in the previous case; alternatively, one can deduce our results from the fundamental category of the generated d-spaces, described in [G3], 3.9.2 and 3.9.4(b).

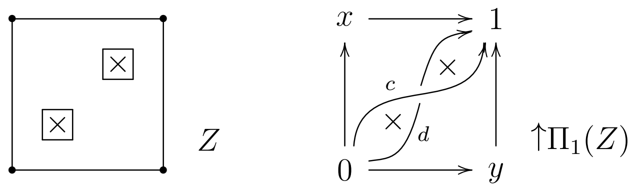
(a) *Simultaneous obstructions*. The first case can be modelled with the subspace Y of $\mathrm{c}\mathbb{I}^2$ represented below



(16)

The fundamental category $\uparrow\Pi_1(Y)$ has again four vertices; from 0 to 1 there are three arrows: $[a]$ (through x), $[b]$ (through y) and $[c]$.

(b) *Consecutive obstructions*. The second case is modelled by $Z \subset \mathrm{c}\mathbb{I}^2$



(17)

In $\uparrow\Pi_1(Z)$ there are now four arrows from 0 to 1: $[a]$ (through x), $[b]$ (through y) and $[c], [d]$.

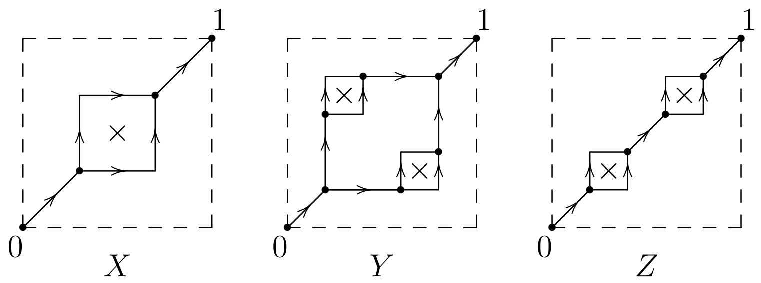
(c) *Comments*. The fundamental category distinguishes these situations, which topology cannot separate: the underlying topological spaces $|Y|$ and $|Z|$ are homeomorphic.

2.3 Obstructions in d-spaces

The d-spaces $\hat{X}, \hat{Y}, \hat{Z}$ generated by the previous c-spaces have the same topological support and the structure induced by the ordered square $\uparrow\mathbb{I}^2$.

Their fundamental category, much more complex than in the previous cases, was studied in [G3], 3.1.1, 3.9.2, 3.9.4(b).

In each case the fundamental category, whose objects are the infinite points of the support, is skeletal and *cannot* be reduced up to equivalence of categories. As analysed in [G3], Section 3.9, it is essentially represented by a 'minimal injective model', future and past equivalent to the given category. Here we get the finite, full subcategories represented below (on 4, 8, 6 objects, respectively), determining the 'branching points' of the process



(18)

A cell marked with a cross is not commutative, while the central cell in $\uparrow\Pi_1(\hat{Y})$ commutes. In $\uparrow\Pi_1(\hat{X})$ there are two arrows $0 \to 1$, in $\uparrow\Pi_1(\hat{Y})$ there are three of them, in $\uparrow\Pi_1(\hat{Z})$ four.

3. Border flexible c-spaces and strict homotopies

We end by examining the relationship of border flexible c-spaces (defined in II.2.1(c)) with strict homotopies (see II.4.3(e)), expressed in Theorem 3.1.

As a consequence, the fundamental category of a border flexible c-space can be simply defined using c-paths up to homotopy with fixed endpoints (see 3.2). Its invariance up to strict homotopies is stated in Theorem 3.3.

The importance of a simple construction, instead of the hybrid construction of Sections II.4 and II.5, is evident – although it does not apply to essential c-spaces like the delayed intervals and the higher c-spheres, which are not border flexible (see II.2.2).

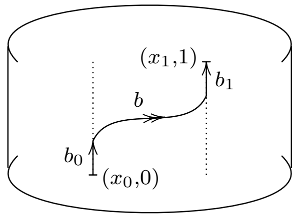
3.1 Theorem (Border flexible c-spaces and homotopies)

Let Y be a border flexible c-space. Every strict homotopy $\varphi\colon X \times \mathrm{c}\mathbb{I} \to Y$ is flexible.

Proof. We are given a c-map $\varphi\colon X \times \mathrm{c}\mathbb{I} \to Y$ which is constant on each fibre $\{x\} \times \mathrm{c}\mathbb{I}$ at a flexible point of X, and we have to prove that φ is also a c-map $X \times \uparrow\mathbb{I} \to Y$.

We take a c-path $b = \langle a, h\rangle\colon \mathrm{c}\mathbb{I} \to X \times \uparrow\mathbb{I}$, where $a\colon x_0 \to x_1$ is a c-path of X (between flexible points) and $h\colon t_0 \to t_1$ is increasing in $\uparrow\mathbb{I}$; we have to prove that φb is controlled in Y.

We insert a path $b_\alpha\colon c\mathbb{I} \to X \times {\uparrow}\mathbb{I}$ in each fibre of the cylinder at the endpoints x_α (for $\alpha = 0, 1$)



$X \times {\uparrow}\mathbb{I}$ (19)

$$b_0 = \langle e_{x_0}, h_0\rangle\colon c\mathbb{I} \to X \times {\uparrow}\mathbb{I}, \qquad h_0\colon 0 \to t_0,$$
$$b_1 = \langle e_{x_1}, h_1\rangle\colon c\mathbb{I} \to X \times {\uparrow}\mathbb{I}, \qquad h_1\colon t_1 \to 1,$$

and we get a c-path $b' = \langle a', h'\rangle = b_0 * b * b_1$ in $X \times {\uparrow}\mathbb{I}$ which is controlled in $X \times c\mathbb{I}$, because h' is an increasing path $0 \to 1$.

Now $\varphi b'$ is controlled in the border flexible c-space Y and each path φb_α is constant (because φ is a strict homotopy). It follows that the middle restriction φb is also controlled in Y. □

3.2 The border flexible case

As a particular case of the previous theorem, if the c-space X is border flexible, a general 2-path $c\mathbb{I}^2 \to X$ is always a hybrid 2-path $c\mathbb{I} \times {\uparrow}\mathbb{I} \to X$ (because H is constant on the vertical edges of $c\mathbb{I}^2$).

Therefore the restricted functor

$$\uparrow\Pi_1\colon c_{\rm bf}\mathsf{Top} \to \mathsf{Cat}, \tag{20}$$

can be equivalently defined using general 2-paths, based on the standard square $c\mathbb{I}^2$, instead of hybrid 2-paths based on $c\mathbb{I} \times {\uparrow}\mathbb{I}$.

The restricted functor is still invariant up to flexible homotopies. But strict homotopies in $c_{\rm bf}\mathsf{Top}$ are always flexible, giving the following result.

3.3 Theorem (Homotopy invariance, III)

A strict homotopy $\varphi\colon f \to g\colon X \to Y$ of border flexible c-spaces induces the identity of the associated functors

$$f_* = g_*\colon {\uparrow}\Pi_1(X) \to {\uparrow}\Pi_1(Y). \tag{21}$$

Proof. By Theorem 3.1, φ is a strict flexible homotopy. Applying Theorem II.5.4(b), φ_* is the identity of $f_* = g_*$. □

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