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THE WEAK SUBOBJECT CLASSIFIER AXIOM AND MODULES IN SUP

Ulrich Höhle

Résumé. L'axiome du classificateur faible de sous-objet est introduit de telle manière qu'un schéma de compréhension soit disponible. Cependant, le foncteur d'image inverse, lorsqu'il est restreint aux sous-objets classifiables, n'a pas toujours un adjoint à droite. Pour les quantales unitaires arbitraires, la catégorie des modules à droite (ou à gauche) dans Sup satisfait l'axiome du classificateur faible de sous-objet. Les morphismes caractéristiques sont construits en utilisant les catégories enrichies dans les quantales associées aux modules dans Sup. Si le quantale sous-jacent est commutatif, alors les objets puissance faibles existent également.

Abstract. The weak subobject classifier axiom is introduced in such a way that a comprehension scheme is available. However, the inverse image functor restricted to classifiable subobjects need not have a right adjoint. For arbitrary unital quantales, the category of right (left) modules in Sup satisfies the weak subobject classifier axiom. The characteristic morphisms are constructed using the quantale-enriched categories corresponding with modules in Sup. If the underlying quantale is commutative, then also weak power objects exist.

Keywords. Weak Subobject Classifier, Inverse Image Functor, Quantale, Module in Sup, Enriched Category, Weak Power Object.

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1. Introduction

The subobject classifier axiom is one of the prominent axioms of topos theory. In this paper we weaken this axiom and give up the principle that *every* subobject of a pre-described class of subobjects is classifiable by a unique characteristic morphism (cf. [12, Section 14]). This approach leads to the weak subobject classifier axiom. Since the weak subobject classifier is unique up to an isomorphism, the existence of a weak subobject classifier is always an invariant of a finitely complete category.

Moreover, in any complete epi-mono-category the weak subobject classifier axiom gives rise to a comprehension scheme in the sense of Lawvere (cf. [8]). If additionally a symmetric and monoidal closed structure is imposed, then weak power objects are available. In particular, the weak power object of the unit object is isomorphic to the underlying weak subobject classifier. Since in general neither diagonal arrows nor projections of the tensor product exist, we only focus on the construction of the universal quantifier. If the underlying category is an epi-mono-category and the unique arrow from the unit object to the terminal object is an epimorphism, then the existence of the universal quantifier based on objects follows from the weak subobject classifier axiom. On the other hand the weak subobject classifier axiom does not imply that the restriction of the inverse image functor to classifiable subobjects has in general a right adjoint (cf. Example 4.9). In this sense an analogue of the doctrinal diagram of Kock and Wraith is not available (cf. [6]).

Significant examples of categories satisfying the weak subobject classifier axiom, but not being a topos (resp. quasitopos), appear in the study of modules in the category Sup of complete lattices and join preserving maps. Let $\mathfrak{Q}$ be a unital quantale, then the category of right (left) $\mathfrak{Q}$-modules satisfies the weak subobject classifier axiom. We emphasize that the construction of characteristic morphisms is based here on the underlying $\mathfrak{Q}$-enriched categories associated with right $\mathfrak{Q}$-modules. Moreover, if $\mathfrak{Q}$ is commutative, then there exists a well known symmetric, monoidal closed structure on the category of $\mathfrak{Q}$-modules (cf. [4]). In this context weak power objects exist, and there is again a close relationship between universal quantifiers based on $\mathfrak{Q}$-modules and the respectively associated, $\mathfrak{Q}$-enriched categories (cf. Proposition 5.1).

Finally, as first steps toward categorical logic for $\mathfrak{Q}$-modules, we include the conjunction, the implication, the element relation and the universal quantifier based on $\mathfrak{Q}$-modules as truth arrows.

In order to fix some basic facts we begin with preliminaries on quantales and a survey on modules in Sup.

2. Preliminaries

First we point out that Sup is a symmetric, monoidal, closed category. If X, Y and Z are complete lattices, then a map $X \times Y \xrightarrow{b} Z$ is called a *bimorphim* if b is join-preserving in each variable separately. Due to the universal property of the tensor product $\otimes$ in Sup every bimorphism

$$X \times Y \xrightarrow{b} Z$$

can be identified with a unique join-preserving map $X \otimes Y \xrightarrow{\varphi_b} Z$ making the following diagram commutative:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\otimes} & X \otimes Y \\ {\scriptstyle b}\downarrow & \swarrow {\scriptstyle \varphi_b} & \\ Z & & \end{array} \tag{2.1}$$

where $X \times Y \xrightarrow{\otimes} X \otimes Y$ is the *universal bimorphism* from $X \times Y$ to $X \otimes Y$. We also call φ_b the unique join-preserving extension of b.

Further, due to the monoidal closedness of Sup for every object Z of Sup the endofunctor $\_ \otimes Z$ has a right adjoint functor $[Z, \_]$, where $[Z, Y]$ is the complete lattice of all join-preserving maps $Z \to Y$ ordered pointwise. Then for each join-preserving map $X \otimes Z \xrightarrow{\varphi} Y$ there exists a unique join-preserving map $X \xrightarrow{\ulcorner\varphi\urcorner} [Z, Y]$ such that the following diagram is commutative:

$$\begin{array}{ccc} X \otimes Z & \xrightarrow{\ulcorner\varphi\urcorner \otimes 1_Z} & [Z, Y] \otimes Z \\ & {\scriptstyle \varphi} \searrow & \downarrow {\scriptstyle ev_Y} \\ & & Y \end{array} \tag{2.2}$$

where ev_Y is the *evaluation arrow* — i.e. the Y-component of the counit of the adjoint situation $Z \otimes \_ \dashv [Z, \_]$. In this context $\ulcorner\varphi\urcorner$ is called the *monoidal adjoint arrow* of φ.

Since Sup has a self-duality determined by the construction of right adjoint maps, we introduce the following notation. The dual lattice of a complete lattice X is denoted by $X^\dagger$ and the corresponding dual order by $\leq^\dagger$. Then the tensor product $X \otimes Y$ has the form $[X, Y^\dagger]^\dagger$ up to an isomorphism (cf. [4]).

A *quantale* is a semigroup $\mathfrak{Q}$ in Sup — i.e. a complete lattice $\mathfrak{Q}$ provided with an associative, binary operation $\mathfrak{Q} \otimes \mathfrak{Q} \xrightarrow{m} \mathfrak{Q}$ in the sense of Sup (cf. [9, (1) on p. 170]). Then the bimorphism $\mathfrak{Q} \times \mathfrak{Q} \xrightarrow{*} \mathfrak{Q}$ determined by m (cf. (2.1)) is a semigroup operation in Set, which is join-preserving in each variable separately. This bimorphism $*$ is also called the *quantale multiplication* of $\mathfrak{Q}$. The right implication $\searrow$ and left implication $\swarrow$ of $*$ are determined by:

$$\alpha \searrow \beta = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta \,\} \quad \text{and} \quad \beta \swarrow \alpha = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \gamma * \alpha \leq \beta \,\}.$$

Since both types of implications are bimorphisms, they have always unique extensions to join-preserving maps $\mathfrak{Q} \otimes \mathfrak{Q}^\dagger \xrightarrow{\varphi_\searrow} \mathfrak{Q}^\dagger$ and $\mathfrak{Q}^\dagger \otimes \mathfrak{Q} \xrightarrow{\varphi_\swarrow} \mathfrak{Q}^\dagger$, respectively. Finally, the zero element in $\mathfrak{Q}$ coincides with the universal lower bound $\perp$ of $\mathfrak{Q}$.

An element $\alpha \in \mathfrak{Q}$ is *left-sided* (resp. *right-sided*) if $\top * \alpha \leq \alpha$ (resp. $\alpha * \top \leq \alpha$), where $\top$ is the universal upper bound in $\mathfrak{Q}$. An element of $\mathfrak{Q}$ is *two-sided* if it is left- and right-sided.

A *unital quantale* is a monoid in Sup (cf. [9, (1), (2) on p. 170]). The unit $\mathbb{1} \xrightarrow{e} \mathfrak{Q}$ will always be identified with the corresponding element $e \in \mathfrak{Q}$. Typical examples of a unital quantale arise from complete lattices X and are given by the complete lattice $[X, X]$ of all join-preserving self-maps of X provided with the composition as quantale multiplication.

3. Survey on modules in Sup

In this section we begin with a review of some basic properties of modules in Sup. Therefore let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale with unit e.

A complete lattice X provided with a left action $\mathfrak{Q} \otimes X \xrightarrow{\ell_X} X$ of $\mathfrak{Q}$ on X (cf. [9, p. 174]) is called a *left $\mathfrak{Q}$-module* in Sup (cf. [4]). Hence ℓ_X

can be identified with its bimorphism $\mathfrak{Q} \times X \xrightarrow{\odot} X$ (cf. (2.1)) satisfying the following additional axioms:

(L1) $\beta \odot (\alpha \odot x) = (\beta * \alpha) \odot x, \qquad \alpha, \beta \in \mathfrak{Q},\ x \in X,$

(L2) $e \odot x = x, \qquad x \in X.$

Let $X = (X, \ell_X)$ and $Y = (Y, \ell_Y)$ be left $\mathfrak{Q}$-modules. A join-preserving map $X \xrightarrow{h} Y$ is a *left $\mathfrak{Q}$-module morphism* if h also preserves the respective left actions — i.e. the commutativity of the following diagram holds:

$$\begin{array}{ccc} \mathfrak{Q} \otimes X & \xrightarrow{1_{\mathfrak{Q}} \otimes h} & \mathfrak{Q} \otimes Y \\ {\scriptstyle \ell_X}\downarrow & & \downarrow{\scriptstyle \ell_Y} \\ X & \xrightarrow{\quad h \quad} & Y \end{array}$$

The complete lattice $[X, Y]$ of left $\mathfrak{Q}$-module morphisms $X \to Y$ is ordered pointwise in the sense of Y. Hence joins in $[X, Y]$ are computed pointwise, but not meets.

Obviously left $\mathfrak{Q}$-modules and left $\mathfrak{Q}$-module morphisms form a category denoted by $\mathsf{Mod}_\ell(\mathfrak{Q})$. Referring to [2] $\mathsf{Mod}_\ell(\mathfrak{Q})$ is complete and cocomplete. Further, it is well known that the forgetful functor $\mathcal{U}\colon \mathsf{Mod}_\ell(\mathfrak{Q}) \to \mathsf{Sup}$ has a left adjoint functor $\mathcal{F}\colon \mathsf{Sup} \to \mathsf{Mod}_\ell(\mathfrak{Q})$ acting on objects and morphisms as follows (cf. [9, p. 174]):

$$\mathcal{F}(X) = \mathfrak{Q} \otimes X \quad \text{and} \quad X \xrightarrow{h} Y, \quad \mathfrak{Q} \otimes X \xrightarrow{\mathcal{F}(h) = 1_{\mathfrak{Q}} \otimes h} \mathfrak{Q} \otimes Y.$$

If $\mathcal{M}$ is the monad induced by the adjoint situation $\mathcal{F} \dashv \mathcal{U}$, then $\mathsf{Mod}_\ell(\mathfrak{Q})$ is isomorphic to the category of $\mathcal{M}$-algebras. In particular, all finite limits in $\mathsf{Mod}_\ell(\mathfrak{Q})$ can be computed at the level of Sup.

Moreover, right actions $X \otimes \mathfrak{Q} \xrightarrow{r_X} X$ in Sup are defined similarly and can again be identified with bimorphisms $X \times \mathfrak{Q} \xrightarrow{\boxdot} X$ satisfying now the properties:

(R1) $(x \boxdot \alpha) \boxdot \beta = x \boxdot (\alpha * \beta), \qquad \alpha, \beta \in \mathfrak{Q},\ x \in X,$

(R2) $x \boxdot e = x, \qquad x \in X.$

The respective results corresponding to the previous ones of left $\mathfrak{Q}$-modules holds also for right $\mathfrak{Q}$-modules. In particular, all finite limits in the category $\mathsf{Mod}_r(\mathfrak{Q})$ of right $\mathfrak{Q}$-modules are again computed at the level of Sup.

As a next step we present a fundamental relationship between left and right $\mathfrak{Q}$-modules in Sup (cf. [2, Fact 1 on p. 207]).

Theorem 3.1 *The self-duality of* Sup *determined by the construction of right adjoint maps can be lifted to a contravariant isomorphism between the categories* $\mathsf{Mod}_\ell(\mathfrak{Q})$ *and* $\mathsf{Mod}_r(\mathfrak{Q})$.

Proof. Let (X, ℓ_X) be a left $\mathfrak{Q}$-module and $X^\dagger \xrightarrow{\ell_X^\vdash} (\mathfrak{Q} \otimes X)^\dagger = [\mathfrak{Q}, X^\dagger]$ be the right adjoint map of its left action ℓ_X. Then we introduce a right action $X^\dagger \otimes \mathfrak{Q} \xrightarrow{r_{X^\dagger}} X^\dagger$ of $\mathfrak{Q}$ on $X^\dagger$ by:

$$\begin{array}{ccc} X^\dagger \otimes \mathfrak{Q} & \xrightarrow{\ell_X^\vdash \otimes 1_{\mathfrak{Q}}} & [\mathfrak{Q}, X^\dagger] \otimes \mathfrak{Q} \\ & {\scriptstyle r_{X^\dagger}} \searrow & \big\downarrow {\scriptstyle ev_{X^\dagger}} \\ & & X^\dagger \end{array}$$

The bimorphism $X^\dagger \times \mathfrak{Q} \xrightarrow{\boxdot^\dagger} X^\dagger$ determined by $r_{X^\dagger}$ in the sense of (2.1) has the form (cf. [1, Def. 5.1.2]):

$$x \boxdot^\dagger \alpha = \bigvee \{\, z \in X \mid \alpha \odot z \le x \,\}, \qquad \alpha \in \mathfrak{Q},\, x \in X. \tag{3.1}$$

It is easily seen that $\odot$ satisfies (L1) and (L2) if and only if $\boxdot^\dagger$ satisfies (R1) and (R2). Further, the formation of right adjoint maps determines a contravariant functor $\Gamma\colon \mathsf{Mod}_\ell(\mathfrak{Q}) \to \mathsf{Mod}_r(\mathfrak{Q})$.

On the other hand, every right action r_X on X induces a left action $\ell_{X^\dagger}$ on the dual lattice $X^\dagger$ of X as follows. First we compute the monoidal adjoint

$$X \xrightarrow{\ulcorner r_X \urcorner} [\mathfrak{Q}, X]$$

of r_x (cf. (2.2)). Then the left action $\ell_{X^\dagger}$ on $X^\dagger$ is given by the right adjoint map $\mathfrak{Q} \otimes X^\dagger \xrightarrow{\ell_{X^\dagger} = (\ulcorner r_X \urcorner)^\vdash} X^\dagger$. The bimorphism $\mathfrak{Q} \times X^\dagger \xrightarrow{\odot^\dagger} X^\dagger$ determined by $\ell_{X^\dagger}$ has the form

$$\alpha \odot^\dagger x = \bigvee \{\, z \in X \mid z \boxdot \alpha \le x \,\}, \qquad \alpha \in \mathfrak{Q},\, x \in X, \tag{3.2}$$

where $\boxdot$ is the bimorphism corresponding to r_X. Because $\Gamma(X^\dagger, \ell_{X^\dagger}) = (X, r_X)$, Γ is a contravariant isomorphism. $\square$

We illustrate the previous theorem by a simple example.

Example 3.2 Let $\mathfrak{Q} = (\mathfrak{Q}, m, e)$ be a monoid in Sup (which can also be viewed as a unital quantale). It follows from the associativity and unit axiom of monoids that m can be read as left action of $\mathfrak{Q}$ on $\mathfrak{Q}$ or as right action of $\mathfrak{Q}$ on $\mathfrak{Q}$. Obviously, $\mathfrak{Q}$ is the free left (resp. right) $\mathfrak{Q}$-module on a singleton. Hence every monomorphism in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is an injective map.
(a) If we consider m as a left action on $\mathfrak{Q}$, then the bimorphism corresponding to the right action $r_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by m in the sense of Theorem 3.1 has the form (cf. (3.1)):

$$\gamma \boxdot^\dagger \alpha = \bigvee\{\, \beta \in \mathfrak{Q} \mid \alpha * \beta \leq \gamma \,\} = \alpha \searrow \gamma, \qquad \alpha, \gamma \in \mathfrak{Q}. \tag{3.3}$$

Hence the right action $r_{\mathfrak{Q}^\dagger}$ is uniquely determined by the right implication of the quantale multiplication $*$. In particular, if $\mathfrak{Q}^\dagger \otimes \mathfrak{Q} \xrightarrow{c_{\mathfrak{Q}^\dagger \mathfrak{Q}}} \mathfrak{Q} \otimes \mathfrak{Q}^\dagger$ is the relevant component of the symmetry in Sup, then $r_{\mathfrak{Q}^\dagger} = \varphi_\searrow \circ c_{\mathfrak{Q}^\dagger \mathfrak{Q}}$. Moreover, if $\gamma \in \mathfrak{Q}$ is right-sided, then $\alpha \searrow \gamma$ is also right-sided for all $\alpha \in \mathfrak{Q}$. Hence, if $\mathbb{R}(\mathfrak{Q})$ is the subquantale of all right-sided elements of $\mathfrak{Q}$, then $\mathbb{R}(\mathfrak{Q})^\dagger$ is a right $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$. By abuse of notation we denote the right action in $\mathbb{R}(\mathfrak{Q})^\dagger$ again by $r_{\mathfrak{Q}^\dagger}$.
(b) If we consider m as a right action on $\mathfrak{Q}$, then the left action $\ell_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by m in the sense of Theorem 3.1 (cf. (3.2)) is uniquely determined by the left implication — i.e.

$$\alpha \odot^\dagger \gamma = \bigvee\{\, \beta \in \mathfrak{Q} \mid \beta * \alpha \leq \gamma \,\} = \gamma \swarrow \alpha, \qquad \alpha, \gamma \in \mathfrak{Q}. \tag{3.4}$$

In particular, $\ell_{\mathfrak{Q}^\dagger} = \varphi_\swarrow \circ c_{\mathfrak{Q}\mathfrak{Q}^\dagger}$. Moreover, if $\mathbb{L}(\mathfrak{Q})$ is the subquantale of all left-sided elements of $\mathfrak{Q}$, then by analogy with (a) the complete lattice $\mathbb{L}(\mathfrak{Q})^\dagger$ is a left $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^\dagger, \ell_{\mathfrak{Q}^\dagger})$. By abuse of notation we denote the left action in $\mathbb{L}(\mathfrak{Q})^\dagger$ again by $\ell_{\mathfrak{Q}^\dagger}$.

It follows immediately from Theorem 3.1 and Example 3.2 that every epimorphism in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is surjective. Moreover, every epimorphism is the coequalizer of its kernel pair and every monomorphism is the equalizer of its cokernel pair. In particular, $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is an epi-mono-category.

With regard to Section 4 we recall that the terminal and initial objects coincide and form consequently the null object in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$). Hence $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is a pointed category and every left (resp. right) $\mathfrak{Q}$-module X has a unique global point $0 \to X$ and is represented by the universal lower bound of X.

The next proposition is a non-commutative version of [2, Lem. 3.1.27].

Proposition 3.3 *Let (X, r_X) be a right $\mathfrak{Q}$-module, (Y, ℓ_Y) be a left $\mathfrak{Q}$-module and $(Y^\dagger, r_{Y^\dagger})$ be the right $\mathfrak{Q}$-module induced by (Y, ℓ_Y) in the sense of Theorem 3.1. Further, let $\boxdot$ and $\odot$ be the bimorphisms determined by r_X and ℓ_Y respectively. Then a join-reversing map $X \xrightarrow{f} Y$ is a right $\mathfrak{Q}$-module morphism $(X, r_X) \xrightarrow{f} (Y^\dagger, r_{Y^\dagger})$ if and only if the following equivalence holds for all $\alpha \in \mathfrak{Q}$, $x \in X$ and $y \in Y$:*

$$y \leq f(x \boxdot \alpha) \iff \alpha \odot y \leq f(x).$$

Proof. Let $(X, r_X) \xrightarrow{f} (Y^\dagger, r_{Y^\dagger})$ be a right $\mathfrak{Q}$-module morphism. Then the definition of $\boxdot^\dagger$ (cf. (3.1)) implies that the following chain of equivalences holds:

$$y \leq f(x \boxdot \alpha) \iff y \leq f(x) \boxdot^\dagger \alpha \iff \alpha \odot y \leq f(x).$$

Conversely, if we assume that $y \leq f(x \boxdot \alpha)$ if and only if $\alpha \odot y \leq f(x)$ for all $\alpha \in \mathfrak{Q}, x \in X, y \in Y$, then we obtain:

$$\begin{aligned} f(x \boxdot \alpha) \leq f(x \boxdot \alpha) &\iff \alpha \odot f(x \boxdot \alpha) \leq f(x) \\ &\iff f(x \boxdot \alpha) \leq f(x) \boxdot^\dagger \alpha. \end{aligned}$$

Further, the definition of $\boxdot^\dagger$ implies that $\alpha \odot (f(x) \boxdot^\dagger \alpha) \leq f(x)$. Referring again to the previous equivalence we obtain $f(x) \boxdot^\dagger \alpha \leq f(x \boxdot \alpha)$. Hence $X \xrightarrow{f} Y^\dagger$ is a right $\mathfrak{Q}$-module morphism and the assertion follows. □

If, in the previous proposition, we interchange right $\mathfrak{Q}$-modules and left $\mathfrak{Q}$-modules, then we can give a respective characterization of left $\mathfrak{Q}$-module morphisms.

Proposition 3.4 *Let (X, ℓ_X) be a left $\mathfrak{Q}$-module, (Y, r_Y) be a right $\mathfrak{Q}$-module and $(Y^\dagger, \ell_{Y^\dagger})$ be the left $\mathfrak{Q}$-module induced by (Y, r_Y) in the sense of Theorem 3.1. Further, let $\odot$ and $\boxdot$ be the bimorphisms determined by ℓ_X and r_Y respectively. Then a join-reversing map $X \xrightarrow{f} Y$ is a left $\mathfrak{Q}$-module morphism $(X, \ell_X) \xrightarrow{f} (Y^\dagger, \ell_{Y^\dagger})$ if and only if the following equivalence holds for all $\alpha \in \mathfrak{Q}$, $x \in X$ and $y \in Y$:*

$$y \leq f(\alpha \odot x) \iff y \boxdot \alpha \leq f(x).$$

As a second step we point out that the self-duality in Sup also permits to associate a $\mathfrak{Q}$-enriched category with every right $\mathfrak{Q}$-module.

For the convenience of the reader we review the details of this construction. By analogy with the situation in $\mathsf{Mod}_\ell(\mathfrak{Q})$ we first compute the right adjoint map

$$X^\dagger \xrightarrow{r_X^\vdash} (X \otimes \mathfrak{Q})^\dagger = [X, \mathfrak{Q}^\dagger]$$

of the right action $X \otimes \mathfrak{Q} \xrightarrow{r_X} X$, and in a second step we construct a join-preserving map $X^\dagger \otimes X \xrightarrow{\varphi} \mathfrak{Q}^\dagger$ by applying the evaluation arrow:

$$\begin{array}{ccc} X^\dagger \otimes X & \xrightarrow{r_X^\vdash \otimes 1_X} & [X, \mathfrak{Q}^\dagger] \otimes X \\ & \overset{\varphi}{\searrow} & \big\downarrow ev_{\mathfrak{Q}^\dagger} \\ & & \mathfrak{Q}^\dagger \end{array}$$

The bimorphism $X^\dagger \times X \xrightarrow{\hom_X} \mathfrak{Q}^\dagger$ determined by φ (cf. (2.1)) has the form:

$$\hom_X(x, y) = \bigvee \{ \alpha \in \mathfrak{Q} \mid y \boxdot \alpha \leq x \}, \qquad x, y \in X. \tag{3.5}$$

If we now consider $\hom_X$ as $\mathfrak{Q}$-valued map defined on the cartesian product $X \times X$ in Set and fix the given order on X, then we can reformulate the lattice-theoretic properties of $\hom_X$ as follows:

> The map $\hom_X$ is meet-preserving in the first variable and join-reversing in the second variable. (3.6)

Further, we conclude from (R1) and (R2) that $\hom_X$ is a $\mathfrak{Q}$-valued hom-object assignment — i.e. the $\mathfrak{Q}$-enriched composition law and the existence of $\mathfrak{Q}$-enriched identities

$$\hom_X(x,y)*\hom_X(z,x) \leq \hom_X(z,y) \text{ and } e \leq \hom_X(x,x), \quad x,y \in X,$$

hold in the framework given by the monoidal biclosed category determined by $\mathfrak{Q}$ (cf. [5]).

As a consequence of this construction we now show that $\hom_X$ gives rise to specific module morphisms in Sup, which will play a significant role in Section 4.

For this purpose, let us consider the given right action r_X on X and the right action $r_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by the left action on $\mathfrak{Q}$ (cf. Example 3.2 (a)). If we fix the first variable in $\hom_X$, then it follows immediately from Proposition 3.3, (3.5) and (R1) that

$$(X, r_X) \xrightarrow{\hom_X(x,\_\_)} (\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$$

is a right $\mathfrak{Q}$-module morphism for all $x \in X$.

On the other hand, if we fix the second variable in $\hom_X$, then we consider the respective left actions $\ell_{X^\dagger}$ and $\ell_{\mathfrak{Q}^\dagger}$ on $X^\dagger$ and $\mathfrak{Q}^\dagger$ induced by the respective right actions on X and on $\mathfrak{Q}$ (cf. Example 3.2 (b)) in the sense of Theorem 3.1. Now we refer to (3.2) and obtain:

$$\begin{aligned} \beta \leq \hom_X(\gamma \odot^\dagger x, y) &\iff y \boxdot \beta \leq \gamma \odot^\dagger x \\ &\iff (y \boxdot \beta) \boxdot \gamma \leq x \\ &\iff \beta * \gamma \leq \hom_X(x,y). \end{aligned}$$

Hence Proposition 3.4 implies that $(X^\dagger, \ell_{X^\dagger}) \xrightarrow{\hom_X(\_\_,y)} (\mathfrak{Q}^\dagger, \ell_{\mathfrak{Q}^\dagger})$ is a left $\mathfrak{Q}$-module morphism for all $y \in X$.

We can summarize the previous observations in the following formulae:

$$\begin{aligned} \hom_X(\alpha \odot^\dagger x, y) &= \hom_X(x,y) \swarrow \alpha \text{ and} \\ \hom_X(x, y \boxdot \alpha) &= \alpha \searrow \hom_X(x,y). \end{aligned} \tag{3.7}$$

Finally, it can be shown that the $\mathfrak{Q}$-enriched category $(X, \hom_X)$ is skeletal and cocomplete. In this context we recall that $\mathsf{Mod}_r(\mathfrak{Q})$ is isomorphic to the category of cocomplete and skeletal $\mathfrak{Q}$-enriched categories — a result, which has been established by I. Stubbe in 2006 in the more general context of quantaloid-enriched categories (cf. [11] and [2, Sect. 3.3.3]).

4. The weak subobject classifier axiom

In this section we present a weakening of the subobject classifier axiom and explore its first categorical consequences. As a motivating example we reveal the special role of categories of modules in Sup.

Definition 4.1 *Let C be a finitely complete category with terminal object T. Further, let Ω be an object of C and $T \overset{t}{\rightarrowtail} \Omega$ be a global point of Ω.*

(a) A monomorphism $U \overset{\psi}{\rightarrowtail} X$ is called (t,Ω)-classifiable *if there exists a morphism $X \xrightarrow{\varphi} \Omega$ such that*

$$\begin{array}{ccc} U & \xrightarrow{!_U} & T \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{\varphi} & \Omega \end{array} \tag{4.1}$$

is a pullback square. In particular, φ is said to be a classifying morphism *of the monomorphism $U \overset{\psi}{\rightarrowtail} X$.*

(b) The pair (t,Ω) is called a weak subobject classifier *if the following conditions are satisfied:*

(WS1) *If X is an object of C, then every global point $T \rightarrowtail X$ of X is (t,Ω)-classifiable.*

(WS2) *If a monomorphism $U \overset{\psi}{\rightarrowtail} X$ is (t,Ω)-classifiable, then it is uniquely (t,Ω)-classifiable — i.e. φ in the pullback (4.1) is uniquely determined by ψ.*

Obviously every (t,Ω)-classifiable monomorphism is an equalizer (cf. [12, Prop. 14.3]).

Further, it follows immediately from the previous definition that every weak subobject classifier is unique up to an isomorphism. Hence the existence of a weak subobject classifier is an invariant of a finitely complete category. In this context we introduce the following terminology:

The morphism t is called the *arrow true*. Further, a morphism φ with codomain Ω is a called a *characteristic morphism*. If φ is uniquely determined by a monomorphism $U \overset{\psi}{\rightarrowtail} X$ in the sense of the pullback in (4.1), then we write χ_ψ instead of φ.

In particular we have the following:

Theorem 4.2 *Let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale with unit e. Then the right $\mathfrak{Q}$-module $(\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$ (cf. Example 3.2 (a)) is the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$, and the arrow true is represented by the universal lower bound of $\mathbb{R}(\mathfrak{Q})^\dagger$.*

Proof. First we notice that the universal lower bound in $\mathbb{R}(\mathfrak{Q})^\dagger$ is the universal upper bound $\top$ in $\mathfrak{Q}$.
(a) Let (X, r_X) be a right $\mathfrak{Q}$-module and $\perp$ be the universal lower bound of X. Further, let $\hom_X$ be the hom-object assignment of the $\mathfrak{Q}$-enriched category $(X, \hom_X)$ associated with (X, r_X) (cf. Section 3). Since $\hom_X(\perp, y)$ is right-sided for all $y \in X$ (cf. (3.5)), the range of $\hom_X(\perp, \_)$ is contained in $\mathbb{R}(\mathfrak{Q})$, and consequently

$$(X, r_X) \xrightarrow{\hom_X(\perp,\_)} (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$$

is a right $\mathfrak{Q}$-module morphism. The right $\mathfrak{Q}$-submodule determined by the pullback of the arrow true $0 \xrightarrow{t} \mathfrak{Q}^\dagger$ along $\hom_X(\perp, \_)$ is given by

$$\{\, y \in X \mid \hom_X(\perp, y) = \top \,\} = \{\, \perp \,\}.$$

Hence (WS1) is satisfied.
(b) Let $(X, r_X) \xrightarrow{\varphi} (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$ be a right $\mathfrak{Q}$-module morphism and U be the right $\mathfrak{Q}$-submodule of (X, r_X) determined by the pullback of the arrow true along φ — i.e.

$$U = \{\, x \in X \mid \varphi(x) = \top \,\}. \tag{4.2}$$

Further, let $\hom_X$ be the hom-object assignment of the $\mathfrak{Q}$-enriched category $(X, \hom_X)$ associated with (X, r_X). Then $y \boxdot \hom_X(x, y) \le x$ holds for all $x, y \in X$. Since φ preserves the respective right actions and is isotone, the previous relation implies $\varphi(y) \boxdot^\dagger \hom_X(x,y) \leq^\dagger \varphi(x)$, which is equivalent to

$$\varphi(x) \leq \hom_X(x,y) \searrow \varphi(y), \quad \text{i.e. } \hom_X(x,y) * \varphi(x) \leq \varphi(y), \quad x, y \in X,$$

where we have also referred to (3.3). Then φ is a cocontinuous contravariant $\mathfrak{Q}$-presheaf on $(X, \hom_X)$ in the terminology of $\mathfrak{Q}$-enriched category theory (cf. [10, Def. 3.1, p. 21 and 23]). Since $\mathfrak{Q}$ is unital, we conclude from (4.2):

$$\bigvee\{\, \hom_X(x,y) \mid x \in U \,\} \leq \varphi(y), \qquad y \in X. \tag{4.3}$$

Now we recall that $\varphi(y)$ is right-sided for all $y \in X$, hence:

$$\varphi(y \boxdot \varphi(y)) = \varphi(y \boxdot \varphi(y)) * \top = \big(\varphi(y) \searrow \varphi(y)\big) * \top = \top,$$

and so $y \boxdot \varphi(y) \in U$. Thus $\varphi(y) \leq \hom_X(y \boxdot \varphi(y), y)$ follows, and the inequality in (4.3) turns into an equality. In particular, the relation:

$$\varphi(y) = \bigvee_{\varphi(x)=\top} \hom_X(x,y) * \top = \bigvee_{\varphi(x)=\top} \hom_X(x \boxdot \top, y) \tag{4.4}$$

holds for all $y \in X$, and so (WS2) is verified. □

By analogy with Theorem 4.2 the left $\mathfrak{Q}$-module $(\mathbb{L}(\mathfrak{Q}^\dagger, \ell_{\mathfrak{Q}^\dagger})$ is the weak subobject classifier in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (cf. Example 3.2 (b)). Indeed, we have only to observe that the hom-object assignment of a left $\mathfrak{Q}$-module (X, ℓ_X) is the hom-object assignment determined by the dual right $\mathfrak{Q}$-module $(X^\dagger, r_{X^\dagger})$ of (X, ℓ_X) — i.e.

$$\hom_X(x,y) = \bigvee\{\, \alpha \in \mathfrak{Q} \mid y \boxdot^\dagger \alpha \leq^\dagger x \,\} = \bigvee\{\, \alpha \in \mathfrak{Q} \mid \alpha \odot x \leq y \,\}$$

for each $x, y \in X$. Consequently the lattice-theoretic properties of $\hom_X$ (cf. (3.6)) are read in $X^\dagger$.

Remark 4.3 Let (X, r_X) be a right $\mathfrak{Q}$-module and $(X, \hom_X)$ be the associated $\mathfrak{Q}$-enriched category. Since (4.4) describes the characteristic morphisms of (X, r_X) in the sense of $\mathsf{Mod}_r(\mathfrak{Q})$, it is easily seen that for every right-sided element $\alpha \in \mathfrak{Q}$ and $x \in X$ the map $X \xrightarrow{\hom_X(x \boxdot \alpha, \_)} \mathbb{R}(\mathfrak{Q}^\dagger)$ is a right $\mathfrak{Q}$-module morphism and the characteristic morphism of the right $\mathfrak{Q}$-submodule $\downarrow(x \boxdot \alpha) = \{\, y \in X \mid y \leq x \boxdot \alpha \,\}$ of X. If $\mathfrak{Q}$ is commutative, then $\mathbb{R}(\mathfrak{Q})$ is the subquantale $\mathbb{I}(\mathfrak{Q})$ of all two-sided elements of $\mathfrak{Q}$ and $\hom_X(x \boxdot \alpha, \_)$ is the elementary tensor $x \otimes_{\mathfrak{Q}} \alpha$ of the tensor product $X \otimes_{\mathfrak{Q}} \mathbb{I}(\mathfrak{Q})$ of $\mathfrak{Q}$-modules, where the action on $\mathbb{I}(\mathfrak{Q})$ is given by the quantale multiplication.

4.1 Classifiable subobjects

Let C be a finitely complete category with terminal object T provided with a weak subobject classifier (t, Ω). We begin with the simple observation that the identity 1_X of an object X is always (t, Ω)-classifiable. The characteristic morphism of 1_X is the composition $X \xrightarrow{!_X} T \overset{t}{\rightarrowtail} \Omega$ and is denoted by true_X.

Further, recall that a subobject of X is an equivalence class of monomorphisms with codomain X. If a representing monomorphism of a subobject S of X is (t, Ω)-classifiable, then it is easily seen that every further representing monomorphism of S is also (t, Ω)-classifiable. Hence, due to the unique classification of (t, Ω)-classifiable monomorphisms with codomain X, every characteristic morphism of X is uniquely determined by its corresponding subobject — i.e. there exists a bijective map between all (t, Ω)-classifiable subobjects of X and all charactertistic morphisms of X.

In a first step we show that the pullback of (t, Ω)-classifiable subobjects is again (t, Ω)-classifiable. In particular, (t, Ω)-classifiable monomorphisms are pullback stable.

Proposition 4.4 *Let $X \xrightarrow{f} Y$ be a morphism and $U \overset{m}{\rightarrowtail} Y$ be a (t, Ω)-classifiable monomorphism. Then the pullback $V \overset{n}{\rightarrowtail} X$ of m along f is again (t, Ω)-classifiable. In particular, if χ_m is the characteristic morphism of m, then $\chi_m \circ f$ is the characteristic morphism of n.*

Proof. We consider the commutative diagram (cf. [12, Proof of Prop. 14.4]):

$$\begin{array}{ccc} V & \overset{n}{\rightarrowtail} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ U & \underset{m}{\rightarrowtail} & Y \\ {\scriptstyle !_U}\big\downarrow & & \big\downarrow{\scriptstyle \chi_m} \\ T & \overset{t}{\rightarrowtail} & \Omega \end{array}$$

Hence the result follows from the Pullback Lemma. □

The formulation of the previous proposition can be seen as an extension of the well known result [12, Prop. 14.4] to the area of weak subobject classifiers.

Now we show that the binary intersection of (t, Ω)-classifiable subobjects is again (t, Ω)-classifiable. For this purpose we will apply (WS1). By analogy with topos theory, the characteristic morphism $\Omega \times \Omega \xrightarrow{\chi_\cap} \Omega$ of the global point $T \xrightarrow{\langle t,t \rangle} \Omega \times \Omega$ will be referred as the *conjunction* in C.

Example 4.5 In $\mathsf{Mod}_r(\mathfrak{Q})$ the conjunction $\chi_\cap$ coincides with the binary meet of right-sided elements. In fact, since the hom-object assignment of the weak subobject classifier $\mathbb{R}(\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$ is given by $\hom_{\mathbb{R}(\mathfrak{Q})^\dagger}(\alpha, \beta) = \beta \swarrow \alpha$, the relation (4.4) implies: $\chi_\cap(\beta_1, \beta_2) = (\beta_1 \swarrow \top) \wedge (\beta_2 \swarrow \top) = \beta_1 \wedge \beta_2$ for all $\beta_1, \beta_2 \in \mathbb{R}(\mathfrak{Q})$.

Theorem 4.6 *Let $U_1 \xrightarrow{m_1} X$ and $U_2 \xrightarrow{m_2} X$ be (t, Ω)-classifiable monomorphisms. Then the monomorphism $V \xrightarrow{n} X$ determined by the following pullback square*:

$$\begin{array}{ccc} V & \xrightarrow{m_1'} & U_2 \\ {\scriptstyle m_2'}\downarrow & \searrow^{n} & \downarrow{\scriptstyle m_2} \\ U_1 & \xrightarrow{m_1} & X \end{array} \tag{4.5}$$

is again (t, Ω)-classifiable. Moreover, if χ_{m_i} is the characteristic morphism of $U_i \xrightarrow{m_i} X$ ($i = 1, 2$) and χ_n is the characteristic morphism of $V \xrightarrow{n} X$, then the relation $\chi_n = \chi_\cap \circ \langle \chi_{m_1}, \chi_{m_2} \rangle$ holds.

Proof. In the case of a weak subobject classifier axiom (cf. Definition 4.1) we can also follow the same strategy as in topos theory. We consider the characteristic morphisms $X \xrightarrow{\chi_{m_1}} \Omega$ and $X \xrightarrow{\chi_{m_2}} \Omega$ corresponding to m_1

and m_2 and prove that the outer rectangle of the diagram:

$$\begin{array}{ccc} V & \overset{n}{\rightarrowtail} & X \\ \downarrow & & \downarrow{\scriptstyle \langle \chi_{m_1}, \chi_{m_2} \rangle} \\ T & \overset{\langle t,t \rangle}{\rightarrowtail} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \chi_\cap} \\ T & \overset{t}{\rightarrowtail} & \Omega \end{array}$$

is a pullback square. For this purpose it is sufficient to show that the upper square is a pullback. The commutativity of the upper square is evident. Now let us consider a morphism $Z \xrightarrow{\ell} X$ with $\langle t,t \rangle \circ !_Z = \langle \chi_{m_1}, \chi_{m_2} \rangle \circ \ell$. Then the weak subobject classifier axiom implies that there exist morphisms $Z \xrightarrow{\varphi_i} U_i$ $(i = 1,2)$ such that $m_1 \circ \varphi_1 = \ell = m_2 \circ \varphi_2$. Finally, the pullback square (4.5) guarantees the existence of $Z \xrightarrow{\psi} V$ satisfying $\varphi_1 = m_2' \circ \psi$ and $\varphi_2 = m_1' \circ \psi$. Now we observe $n \circ \psi = m_1 \circ m_2' \circ \psi = m_1 \circ \varphi_1 = \ell$. Hence the assertion is verified.
Finally, the relation $\chi_n = \chi_\cap \circ \langle \chi_{m_1}, \chi_{m_2} \rangle$ follows from the uniqueness of the classification. □

Since the identity 1_Ω of Ω is the characteristic morphism of the arrow true, Theorem 4.6 implies that $\chi_\cap$ is idempotent — i.e. $1_\Omega = \chi_\cap \circ \langle 1_\Omega, 1_\Omega \rangle$. Moreover, the unique classification shows that $(\Omega, \chi_\cap)$ is a commutative monoid in C w.r.t. the monoidal structure determined by the product in C. In particular, the arrow true is the unit of $(\Omega, \chi_\cap)$. Hence $(\Omega, \chi_\cap)$ induces a partial order on the set $\mathrm{HOM}_{\mathsf{C}}(X, \Omega)$ of all characteristic morphisms of (X, r_X) by

$$\chi_1 \le \chi_2 \iff \chi_1 = \chi_\cap \circ \langle \chi_1, \chi_2 \rangle, \qquad \chi_1, \chi_2 \in \mathrm{HOM}_{\mathsf{C}}(X, \Omega).$$

Obviously $(\mathrm{HOM}_{\mathsf{C}}(X, \Omega), \le)$ is a semilattice. Due to the weak subobject classifier axiom, $(\mathrm{HOM}_{\mathsf{C}}(X, \Omega), \le)$ is order-isomorphic to the partially ordered set $\mathrm{sub}_{cl}(X)$ of all (t, Ω)-classifiable subobjects of X.

If C is a complete category, then Theorem 4.6 holds also for any family of (t, Ω)-classifiable subobjects of X. Hence in this case $\mathrm{sub}_{cl}(X)$ is a complete lattice, and for any subobject with representing monomorphism $U \xrightarrow{m} X$ its (t, Ω)-*classifiable hull* exists — i.e. there exists a (t, Ω)-classifiable monomorphism $\widetilde{U} \xrightarrow{\widetilde{m}} X$ determined by the following properties:

(CL1) There exists a (mono)morphism $U \xrightarrow{j_U} \widetilde{U}$ such that $m = \widetilde{m} \circ j_U$.

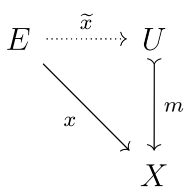
(CL2) For every further (t, Ω)-classifiable monomorphism $V \xrightarrow{n} X$ satisfying the condition $m = n \circ j_{VU}$ with $U \xrightarrow{j_{VU}} V$ there exists a morphism $\widetilde{U} \xrightarrow{\widetilde{j_{VU}}} V$ such that $\widetilde{m} = n \circ \widetilde{j_{VU}}$ holds.

Hence $\widetilde{m}$ is uniquely determined by m up to an isomorphism.

Finally, if C is complete, then for every morphism $X \xrightarrow{f} Y$ the inverse image functor $\mathrm{sub}_{cl}(Y) \xrightarrow{f^{-1}} \mathrm{sub}_{cl}(X)$ has a left adjoint. It is an open question whether f^{-1} has a right adjoint.

4.2 Comprehension scheme

In this subsection we do not only assume that the finitely complete category C satisfies the weak subobject classifier axiom, but also that for every subobject its (t, Ω)-*classifiable hull* exists. Referring to [7, 8] the question arises to which extent the weak subobject classifier axiom is a weakening of the comprehension principle. Following Lawvere, we understand a morphism $E \xrightarrow{x} X$ as an element of X "defined over E" and for every monomorphism $U \xrightarrow{m} X$ we say $x \in m$ if there exists $\widetilde{x}$ such that

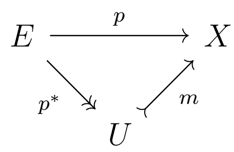


i.e. $x = m \circ \widetilde{x}$. Further, recall that true_E is the composition $E \xrightarrow{!_E} T \xrightarrow{t} \Omega$, where T is the terminal object and t is the arrow true (cf. [7]). Then the weak subobject classifier axiom says the following. Given any "propositional function" (i.e. characteristic morphism of X) $X \xrightarrow{\chi} \Omega$ there exists a monomorphism $\{X|\chi\}$ with codomain X such that for any $E \xrightarrow{x} X$

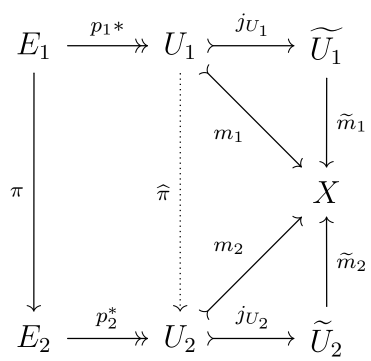
$$x \in \{X|\chi\} \iff \chi \circ x = \text{true}_E,$$

and, conversely, for *every* monomorphism with codomain X there exists the *smallest* (t,Ω)-classifiable monomorphism — i.e. its (t,Ω)-classifiable hull, with codomain X which has a *unique* "characteristic function" χ. If C is an epi-mono-category, then this relationship can be expressed by an adjoint situation (cf. [8]) — i.e. there exists a functor $\mathcal{F}\colon \mathsf{C}/X \to \text{Hom}(X,\Omega)$, which has a right adjoint.

In fact, $\mathcal{F}$ acts on objects as follows. For $E \xrightarrow{p} X$ we first construct the epi-mono-factorization



and subsequently we consider the (t,Ω)-classifiable hull $\widetilde{U} \overset{\widetilde{m}}{\rightarrowtail} X$ of $U \overset{m}{\rightarrowtail} X$. Then $\mathcal{F}(p)$ is given by the characteristic morphism of $\widetilde{U} \overset{\widetilde{m}}{\rightarrowtail} X$. Further, for every morphism $p_1 \xrightarrow{\pi} p_2$ the epi-mono-factorization leads to the following commutative diagram:



Since $\widetilde{m}_2 \circ j_{U_2} \circ \widehat{\pi} = m_1$, we conclude from the universal property (CL2) that there exists a morphism $\widetilde{U}_1 \xrightarrow{\widetilde{j}_{\widetilde{U}_2\widetilde{U}_1}} \widetilde{U}_2$ such that $\widetilde{m}_1 = \widetilde{m}_2 \circ \widetilde{j}_{\widetilde{U}_2\widetilde{U}_1}$. If χ_i is the characteristic morphism of $\widetilde{m}_i$ ($i = 1, 2$), then $\chi_1 = \chi_\cap(\chi_1, \chi_2)$ follows — i.e. $\mathcal{F}(p_1) \leq \mathcal{F}(p_2)$.

On the other hand there exists a functor $\mathcal{G}\colon \mathrm{Hom}(X, \Omega) \to \mathsf{C}/X$ determined by the pullback diagram:

$$\begin{array}{ccc} X & \overset{\widetilde{m}}{\longleftarrow\!\prec} & \widetilde{U} \\ {\scriptstyle\chi}\big\downarrow & & \big\downarrow \\ \Omega & \overset{t}{\longleftarrow\!\prec} & T \end{array}$$

Since there is a natural transformation $\eta\colon \mathrm{id}_{\mathsf{C}} \to \mathcal{G}\mathcal{F}$ with the p-component $E \xrightarrow{\eta_p} \mathcal{G}(\mathcal{F}(p))$ defined by $\eta_p = j_U \circ p^*$ with $\widetilde{m} \circ j_U = m$ and $p = m \circ p^*$, it is not difficult to show that $\mathcal{G}$ is right adjoint to $\mathcal{F}$. In this sense a "comprehension scheme" holds in C.

4.3 (t, Ω)-classifiable subobjects in $\mathsf{Mod}_r(\mathfrak{Q})$

Let $(t, \Omega) = (t, (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ be the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$ (cf. Theorem 4.2) and (X, r_X) be a right $\mathfrak{Q}$-module. Since in $\mathsf{Mod}_r(\mathfrak{Q})$ the characteristic morphism $\chi_\cap$ is the binary *meet* in $\mathbb{R}(\mathfrak{Q})$ (cf. Example 4.5), the complete lattice $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ ($\cong \mathrm{sub}_{cl}(X, r_X)$) is the dual lattice of the complete lattice $[X, \mathbb{R}(\mathfrak{Q})^\dagger]$ of all characteristic morphisms of (X, r_X) ordered pointwise in $\mathbb{R}(\mathfrak{Q})^\dagger$.

As a first step we give a characterization of characteristic morphisms.

Proposition 4.7 *Let (X, r_X) be a right $\mathfrak{Q}$-module and $\hom_X$ be the associated hom-object assignment. Then for every characteristic morphism $X \xrightarrow{\chi} \mathbb{R}(\mathfrak{Q})^\dagger$ there exists a unique element $x \in X$ satisfying the following conditions*

$$x \boxdot \top = x \quad \textit{and} \quad \chi(y) = \textit{hom}_X(x, y), \quad y \in X. \tag{4.6}$$

Proof. The anti-symmetry of the partial order on X implies the uniqueness of the element x in (4.6). In order to confirm the existence of x we proceed as follows:

$$x = \bigvee \{\, z \in X \mid e \leq \chi(z) \,\}.$$

Since $\chi(z)$ is right-sided for all $z \in X$ and χ itself is join-reversing, we obtain $\chi(x) = \top$. Now we observe $\chi(x \boxdot \top) = \top \searrow \top = \top$ — i.e. $x \boxdot \top \leq x$ and consequently $x \boxdot \top = x$. Since every right $\mathfrak{Q}$-module morphism is also a $\mathfrak{Q}$-functor in the sense of $\mathfrak{Q}$-enriched category theory, the relation $\hom_X(x, \_) \leq \chi$ holds. On the other hand we observe $e \leq \chi(y) \searrow \chi(y) = \chi(y \boxdot \chi(y))$. Hence $y \boxdot \chi(y) \leq x$ — i.e. $\chi(y) \leq \hom_X(x, y)$, and the relation (4.6) is verified. $\square$

As an immediate corollary from Remark 4.3 and Proposition 4.7 we obtain that every (t, Ω)-classifiable subobject of (X, r_X) is a right $\mathfrak{Q}$-submodule U having the following form:

$$\exists x \in X \text{ with } x \boxdot \top = x \quad \text{such that} \quad U = \downarrow\! x = \{\, y \in X \mid y \leq x \,\} \tag{4.7}$$

Hence, for an arbitrary right $\mathfrak{Q}$-submodule U of (X, r_X), its (t, Ω)-classifiable hull $\widetilde{U}$ is given by $\widetilde{U} = \downarrow\!(\bigvee U)$.

Remark 4.8 Let $\mathfrak{Q}$ be an integral quantale (i.e. the unit is the universal upper bound of $\mathfrak{Q}$), then $\mathfrak{Q}^\dagger$ is the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$. Since $\mathfrak{Q}^\dagger$ is a $\mathfrak{Q}$-bimodule, for every right $\mathfrak{Q}$-module (X, r_X) the complete lattice $[X, \mathfrak{Q}^\dagger]$ of all characteristic morphisms on X is a *left* $\mathfrak{Q}$-module with the left action $\ell_{[X,\mathfrak{Q}^\dagger]}$ determined by:

$$(\alpha \odot \chi)(x) = \chi(x) \swarrow \alpha, \qquad \alpha \in \mathfrak{Q},\ x \in X.$$

Since $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})) = [X, \mathbb{R}(\mathfrak{Q})^\dagger]^\dagger$, we may conclude from Theorem 3.1 that $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ is a right $\mathfrak{Q}$-module and its dual right action of $\ell_{[X,\mathbb{R}(\mathfrak{Q})^\dagger]}$ has the form:

$$f \boxdot^\dagger \alpha = \bigwedge \{\, g \in [X, \mathbb{R}(\mathfrak{Q})^\dagger] \mid f(x) * \alpha \leq g(x) \text{ for all } x \in X \,\}.$$

Further, let us invoke again Theorem 3.1 and consider the dual left $\mathfrak{Q}$-module $(X^\dagger, \ell_{X^\dagger})$ of (X, r_X).Then we conclude from Proposition 4.7 and (3.7) that there exists a left $\mathfrak{Q}$-module isomorphism $X^\dagger \xrightarrow{\eta_X} [X, \mathfrak{Q}^\dagger]$ defined by:

$$\eta_X(x) = \hom_X(x, \_), \qquad x \in X.$$

If $(X, r_X) \xrightarrow{f} (Y, r_Y)$ is a right $\mathfrak{Q}$-module morphism, then the right adjoint *left* $\mathfrak{Q}$-module morphism $(Y^\dagger, \ell_{Y^\dagger}) \xrightarrow{f^\vdash} (X^\dagger, \ell_{X^\dagger})$ satisfies the following chain of equivalences for all $x \in X$ and $y \in Y$:

$$\begin{aligned} \alpha \leq \hom_X(y, f(x)) &\iff f(x) \boxdot \alpha \leq y \\ &\iff x \boxdot \alpha \leq f^\vdash(y) \iff \alpha \leq \hom_X(f^\vdash(y), x). \end{aligned}$$

Hence the diagram

$$\begin{array}{ccc} (Y^\dagger, \ell_{Y^\dagger}) & \xrightarrow{\eta_Y} & [Y, \mathfrak{Q}^\dagger] \\ {\scriptstyle f^\vdash}\downarrow & & \downarrow{\scriptstyle \Theta_f} \\ (X^\dagger, r_{X^\dagger}) & \xrightarrow{\eta_X} & [X, \mathfrak{Q}^\dagger] \end{array}$$

is commutative, where Θ_f is given by $\Theta_f(\chi) = \chi \circ f$ for all $\chi \in [Y, \mathfrak{Q}^\dagger]$ (cf. Proposition 4.4). So we obtain that in the case of integral quantales the restriction of the inverse image functor f^{-1} to (t, Ω)-classifiable subobjects of (Y, r_Y) is equivalent to the right adjoint left $\mathfrak{Q}$-module morphism $f^\vdash$ of f.

Finally, let us consider the case of arbitrary unital quantales. Then we need some more terminology. An element x of a right $\mathfrak{Q}$-module (X, r_X) is *well-sided* if $x \boxdot \top = x$. The set $\mathbb{W}(X)$ of all well-sided elements of X is a complete sublattice of X in the sense of Sup, but not necessarily a right $\mathfrak{Q}$-submodule of X. The inclusion map $\mathbb{W}(X) \hookrightarrow X$ is meet-preserving.

If $(X, r_X) \xrightarrow{f} (Y, r_Y)$ is a right $\mathfrak{Q}$-module morphism, then the right adjoint $f^\vdash$ of f viewed as morphism in Sup factors through $\mathbb{W}(Y)^\dagger$ in the following way:

$$\begin{array}{ccc} \mathbb{W}(Y)^\dagger & \xhookrightarrow{\iota_{Y^\dagger}} & Y^\dagger \\ {\scriptstyle f^*}\vdots & & \downarrow{\scriptstyle f^\vdash} \\ \mathbb{W}(X)^\dagger & \xrightarrow{\iota_{X^\dagger}} & X^\dagger \end{array}$$

Since in this situation $\mathbb{W}(Y)^\dagger \xrightarrow{\eta_Y} [Y, \mathbb{R}(\mathfrak{Q})^\dagger]$ is only an order isomorphism (cf. Proposition 4.7), we refer to Remark 4.8 and conclude that the restriction of f^{-1} to $\mathrm{sub}_{cl}(Y, r_Y)$ has a right adjoint functor $\mathrm{sub}_{cl}(X, r_X) \xrightarrow{\forall_f}$ $\mathrm{sub}_{cl}(Y, r_Y)$ if and only if f^* is meet-preserving in the respective orders of $\mathbb{W}(Y)^\dagger$ and $\mathbb{W}(X)^\dagger$.

As an illustration of this situation we include the following simple example.

Example 4.9 Let $\mathfrak{Q}$ be a unital quantale without zero divisors. Further, let us view $\mathfrak{Q}$ as right $\mathfrak{Q}$-module w.r.t. the right quantale multiplication. Then the complete sublattice $\mathbb{W}(\mathfrak{Q})$ coincides with the subquantale $\mathbb{R}(\mathfrak{Q})$ of all right-sided elements of $\mathfrak{Q}$. In general $\mathbb{R}(\mathfrak{Q})$ is not a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}$.

Further we fix an element $\alpha \in \mathfrak{Q} \setminus \{\bot\}$. The left translation in $\mathfrak{Q}$ by α — i.e.

$$f_\alpha(\gamma) = \alpha * \gamma, \qquad \gamma \in \mathfrak{Q},$$

is a right $\mathfrak{Q}$-module morphism $\mathfrak{Q} \xrightarrow{f_\alpha} \mathfrak{Q}$. Then the right adjoint left $\mathfrak{Q}$-module morphism $\mathfrak{Q}^\dagger \xrightarrow{(f_\alpha)^\vdash} \mathfrak{Q}^\dagger$ of f_α has the form $(f_\alpha)^\vdash(\gamma) = \alpha \searrow \gamma$ with $\gamma \in \mathfrak{Q}$. Since $\mathfrak{Q}$ does not have zero divisors, the relation $(f_\alpha)^\vdash(\bot) = \bot$ follows. Hence the restriction of $(f_\alpha)^\vdash$ to $\mathbb{R}(\mathfrak{Q})^\dagger$ is meet-preserving in $\mathbb{R}(\mathfrak{Q})^\dagger$ (i.e. the restriction of the inverse image functor $(f_\alpha)^{-1}$ to $\mathrm{sub}_{cl}(\mathfrak{Q}, *)$ has a right adjoint) if and only if for all nonempty subsets A of $\mathbb{R}(\mathfrak{Q})$ the following relation holds:

$$\alpha \searrow (\bigvee A) = \bigvee_{\gamma \in A} (\alpha \searrow \gamma). \tag{4.8}$$

There exist unital quantales without zero divisors, in which (4.8) is violated. For example let us consider the idemptotent, non-commutative and unital quantale C_4^r on the 4-chain with $\bot < a < e < \top$, where e is the unit and a satisfies the properties $\top * a = \top$ and $a * \top = a$. Then the tensor product $C_4^r \otimes C_4^r$ in the sense of quantales (cf. [2, p. 92]) is a unital quantale without zero divisors, in which (4.8) is violated for certain non-zero elements of $C_4^r \otimes C_4^r$. The details are as follows. The subquantale $\mathbb{R}(C_4^r \otimes C_4^r)$ of all right-sided elements consists of six elements:

$$\bot,\ \delta = a \otimes a,\ \alpha = \top \otimes a,\ \beta = a \otimes \top,\ \gamma = (\top \otimes a) \vee (a \otimes \top),\ \top = \top \otimes \top$$

with $\delta = \alpha \wedge \beta$ and $\gamma = \alpha \vee \beta$. Now we consider the left translation f_γ on

$C_4^r \otimes C_4^r$ by γ. Then $\gamma \searrow (\alpha \vee \beta) = \gamma \searrow \gamma = \top$ and

$\gamma \searrow \alpha = (a \otimes \top) \searrow (\top \otimes a) = \bot$ and $\gamma \searrow \beta = (\top \otimes a) \searrow (a \otimes \top) = \bot$,

hence the relation (4.8) is violated, and consequently the restriction of the inverse image functor $(f_\gamma)^{-1}$ to $\mathsf{sub}_{cl}(C_4^r \otimes C_4^r)$ does not have a right adjoint. But on the other hand, if we consider the left translation f_α on $C_4^r \otimes C_4^r$ by α, then the relation (4.8) is satisfied and the restriction of the inverse image functor $(f_\alpha)^{-1}$ to $\mathsf{sub}_{cl}(C_4^r \otimes C_4^r)$ has a right adjoint.

4.4 The implication as truth arrow in $\mathsf{Mod}_r(\mathfrak{Q})$

Let $\mathfrak{Q}^\dagger \times \mathfrak{Q}^\dagger \xrightarrow{\pi_1} \mathfrak{Q}^\dagger$ be the projection onto the first coordinate. By analogy to topos theory we consider the equalizer in $\mathsf{Mod}_r(\mathfrak{Q})$

$$U \rightarrowtail \mathbb{R}(\mathfrak{Q})^\dagger \times \mathbb{R}(\mathfrak{Q})^\dagger \underset{\chi_\cap}{\overset{\pi_1}{\rightrightarrows}} (\mathbb{R}(\mathfrak{Q})^\dagger$$

and observe that the (t, Ω)-classifiable hull of U coincides with $\mathbb{R}(\mathfrak{Q})^\dagger \times \mathbb{R}(\mathfrak{Q})^\dagger$. This is the motivation to avoid the direct product in $\mathsf{Mod}_r(\mathfrak{Q})$ and to lift the tensor product of Sup to $\mathsf{Mod}_r(\mathfrak{Q})$ as follows. Let (X, r_X) be a right $\mathfrak{Q}$-module and Y be a complete lattice. Then on the tensor product $X \otimes Y$ we consider the right action $(X \otimes Y) \otimes \mathfrak{Q} \xrightarrow{r} X \otimes Y$ determined on elementary tensors by:

$$(x \otimes y) \boxdot \alpha = (x \boxdot \alpha) \otimes y, \qquad \alpha \in \mathfrak{Q},\ x \in X,\ y \in Y.$$

Since every tensor is a join of elementary tensors, the corresponding hom-object assignment has the following form:

$$\hom_{X \otimes Y}(f, g) = \bigwedge_{x \otimes y \leq g} \{\, \alpha \in \mathfrak{Q} \mid y \leq f(x \boxdot \alpha) \,\}, \qquad f, g \in X \otimes Y. \quad (4.9)$$

We apply this situation to the right $\mathfrak{Q}$-module $\mathfrak{Q}$ provided with the right multiplication as right action and the complete lattice $\mathbb{R}(\mathfrak{Q})^\dagger$. After these preparations we now consider the following tensor:

$$f = \bigvee \{\, \mu \otimes \nu \mid \mu \in \mathfrak{Q},\ \nu \in \mathbb{R}(\mathfrak{Q})^\dagger,\ \mu \leq \nu \,\},$$

where $\leq$ is the order in $\mathfrak{Q}$. We view f as the «tensorial» analogue of the equalizer $\underset{\chi_\cap}{\overset{\pi_1}{\rightrightarrows}}$.

If $\mu \in \mathfrak{Q}$, then $\mu \leq \mu * \top$, and for all $g \in \mathfrak{Q} \otimes \mathbb{R}(\mathfrak{Q})^\dagger$ the following chain of equivalences hold:

$$\mu \otimes (\mu * \top) \leq^\dagger g \iff \mu * \top \leq^\dagger g(\mu) \iff g(\mu) \leq \mu * \top.$$

Thus the explicit form of f is given by $f(\alpha) = \alpha * \top$ for all $\alpha \in \mathfrak{Q}$, and the characteristic morphism of the (t, Ω)-classifiable subobject $\downarrow f$ has the following form:

$$\begin{aligned}\chi(\alpha \otimes \beta) &= \mathrm{hom}_{\mathfrak{Q}\otimes\mathbb{R}(\mathfrak{Q})^\dagger}(f, \alpha \otimes \beta) = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \beta \leq^\dagger f(\alpha * \gamma)\,\} \\ &= \bigvee\{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma * \top \leq \beta\,\} = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta\,\} = \alpha \searrow \beta.\end{aligned}$$

Hence $\chi = \mathrm{hom}_{\mathfrak{Q}\otimes\mathbb{R}(\mathfrak{Q})^\dagger}(f, \_)$ coincides with the join-preserving extension $\varphi_\searrow$ of the right implication $\mathfrak{Q} \times \mathbb{R}(\mathfrak{Q})^\dagger \xrightarrow{\searrow} \mathbb{R}(\mathfrak{Q})^\dagger$ viewed as bimorphism. In this sense we consider the characteristic morphism χ of $\downarrow f$ as the *implication* in $\mathsf{Mod}_r(\mathfrak{Q})$.

Let c be the symmetry in Sup, and let us consider the restriction of the quantale multiplication in its second factor to $\mathbb{R}(\mathfrak{Q})$. Then the right adjoint of the implication $\mathbb{R}(\mathfrak{Q}) \xrightarrow{\chi^\vdash} [\mathfrak{Q}, \mathbb{R}(\mathfrak{Q})]$ coincides with the monoidal adjoint of $\mathbb{R}(\mathfrak{Q}) \otimes \mathfrak{Q} \xrightarrow{c_{\mathbb{R}(\mathfrak{Q})\mathfrak{Q}}} \mathfrak{Q} \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{m} \mathbb{R}(\mathfrak{Q})$. This observation underlines the close relationship between the implication in $\mathsf{Mod}_r(\mathfrak{Q})$ and the given quantale multiplication in $\mathfrak{Q}$.

5. Weak power object

If a symmetric and monoidal closed structure is imposed on a finitely complete category C with a weak subobject classifier, then we can always have weak power objects in the following sense. Let $\otimes$ be the tensor product in C, and $[X, \_]$ be the right adjoint functor of $\_ \otimes X$ for every object X. Further, let (t, Ω) be the weak subobject classifier. Now we can pull back the evaluation arrow $[X, \Omega] \otimes X \xrightarrow{ev_\Omega} \Omega$ along the arrow true t and obtain a (t, Ω)-classifiable monomorphism $\epsilon_X \overset{\varepsilon}{\rightarrowtail} [X, \Omega] \otimes X$. Then for every (t, Ω)-classifiable monomorphism $R \overset{r}{\rightarrowtail} Y \otimes X$ there exists a unique morphism $Y \xrightarrow{f_r} [X, \Omega]$

such that the following diagram is a pullback square:

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & Y \otimes X \\ \downarrow & & \vdots\, {\scriptstyle f_r \otimes 1_X} \\ \epsilon_X & \overset{\varepsilon}{\rightarrowtail} & [X, \Omega] \otimes X \end{array} \tag{5.1}$$

Since $[X, \Omega]$ is uniquely determined up to an isomorphism by the pullback (5.1), we also call $[X, \Omega]$ the *weak power object* of X and the subobject $(\varepsilon_X, \varepsilon)$ the *element relation* in $[X, \Omega] \otimes X$.

Let $\mathbb{1}$ be the unit object of the tensor product $\otimes$. Then it is not difficult to show that the weak power object $[\mathbb{1}, \Omega]$ is isomorphic to the weak subobject classifier Ω.

It is also convenient to recall the concept of *naming arrows* in the context of symmetric monoidal closed categories (cf. [3, page 78]). Let $X \xrightarrow{f} Y$ be an arrow and $\mathbb{1} \otimes X \xrightarrow{\ell_X} X$ be the X-component of the natural isomorphism $\mathbb{1} \otimes \_ \xrightarrow{\ell} \mathrm{id}_{\mathsf{C}}$. Then the monoidal adjoint $\ulcorner f \circ \ell_X \urcorner$ of $f \circ \ell_X$ is called the *name* of f and is denoted by $\ulcorner f \urcorner$.

Further, let T be the terminal object in C. If C is an epi-mono-category and the unique arrow $\mathbb{1} \xrightarrow{!_{\mathbb{1}}} T$ is an epimorphism, then the universal quantifier exists in the following sense. Let $\ulcorner \mathrm{true}_X \urcorner$ be the name of true_X. Then the commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{1} \otimes X & & & & \\ {\scriptstyle !_{\mathbb{1}} \otimes 1_X} \downarrow & & & \searrow {\scriptstyle \mathrm{true}_X \circ \ell_X} & \\ T \otimes X & \xrightarrow{!_{T \otimes X}} & T & \overset{t}{\rightarrowtail} & \Omega \end{array}$$

implies the decomposition $\ulcorner \mathrm{true}_X \urcorner = \ulcorner (t \circ !_{T \otimes X}) \urcorner \circ !_{\mathbb{1}}$. Hence the image of $\ulcorner \mathrm{true}_X \urcorner$ is the global point $T \overset{\ulcorner t \circ !_{T \otimes X} \urcorner}{\rightarrowtail} [X, \Omega]$ and is thus (t, Ω)-classifiable according to (WS1). The characteristic morphism of the image of $\ulcorner \mathrm{true}_X \urcorner$ is the *universal quantifier* of X, which we denote by $\forall_X$. Since in general the tensor product does not have projections, we leave the construction of the existential quantifier as an open question.

In the following we briefly sketch the situation in the category of modules on a unital quantale $\mathfrak{Q}$. In order to have a symmetric monoidal closed structure we have to assume that $\mathfrak{Q}$ is *commutative* (cf. [4]). The complete lattice $[X, Y]$ of all $\mathfrak{Q}$-module morphisms $(X, \odot) \xrightarrow{f} (Y, \odot)$ is a $\mathfrak{Q}$-module provided with the pointwise defined action. Then for $(x, y) \in X \times Y$ the elementary tensor $x \otimes_{\mathfrak{Q}} y$ is determined by

$$x \otimes_{\mathfrak{Q}} y = \bigwedge\{ f \in [X, Y^{\dagger}] \mid x \otimes y \leq f \}$$

where $\otimes$ is the tensor product in Sup. Hence the action on $x \otimes_{\mathfrak{Q}} y$ has the form

$$\alpha \odot (x \otimes_{\mathfrak{Q}} y) = \bigwedge\{ f \in [X, Y^{\dagger}] \mid x \otimes_{\mathfrak{Q}} y \leq \alpha \odot^{\dagger} f \}, \qquad \alpha \in \mathfrak{Q},$$

and the well known relation $(\alpha \odot x) \otimes_{\mathfrak{Q}} y = x \otimes_{\mathfrak{Q}} (\alpha \odot y) = \alpha \odot (x \otimes_{\mathfrak{Q}} y)$ follows from Proposition 3.3. In this context, we recall that the category $\mathsf{Mod}(\mathfrak{Q})$ of $\mathfrak{Q}$-modules is symmetric and monoidal closed w.r.t. $\otimes_{\mathfrak{Q}}$ (cf. [4, 2]).

Since $\mathfrak{Q}$ is commutative, the weak subobject classifier in $\mathsf{Mod}(\mathfrak{Q})$ is given by the dual $\mathfrak{Q}$-module $\mathbb{I}(\mathfrak{Q})^{\dagger}$ of all two-sided elements of $\mathfrak{Q}$. Then the weak power object of a $\mathfrak{Q}$-module X is the $\mathfrak{Q}$-module $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ of all characteristic morphisms of X. In this situation we point out that the $\mathfrak{Q}$-module HOM$(X, \mathbb{I}(\mathfrak{Q}^{\dagger}))$ coincides with the tensor product $X \otimes_{\mathfrak{Q}} \mathbb{I}(\mathfrak{Q})$.

Since, for commutative quantales, the complete sublattice $\mathbb{W}(X)$ of all well-sided elements of X is a $\mathfrak{Q}$-submodule of X, we can express Proposition 4.7 in this context as follows. Let $\hom_X$ be the hom-object assignment associated with X and $\mathbb{W}(X)^{\dagger}$ be the dual $\mathfrak{Q}$-module of $\mathbb{W}(X)$. Then there exists a $\mathfrak{Q}$-module isomorphism $\mathbb{W}(X)^{\dagger} \xrightarrow{\eta_X} [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ determined by

$$\eta_X(x) = \hom_X(x, \_), \qquad x \in \mathbb{W}(X).$$

Hence we can also identify $\mathbb{W}(X)^{\dagger}$ with the weak power object of X. In particular $\mathbb{W}(X) \cong X \otimes_{\mathfrak{Q}} \mathbb{I}(X)$.

Moreover, by abuse of notation let us denote the restriction of $\hom_X$ to $\mathbb{W}(X)^{\dagger} \times X$ again by $\hom_X$. Since $\hom_X$ is a bimorphism (cf. (3.6) and

(3.7)), the evaluation arrow $[X, \mathbb{I}^{\dagger}(\mathfrak{Q})] \otimes X \xrightarrow{ev_{\mathbb{I}(\mathfrak{Q})^{\dagger}}} \mathbb{I}(\mathfrak{Q})^{\dagger}$ is the unique extension of $\hom_X$ to a $\mathfrak{Q}$-module morphism $\mathbb{W}(X)^{\dagger} \otimes_{\mathfrak{Q}} X \xrightarrow{\varphi} \mathbb{I}(\mathfrak{Q})^{\dagger}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{W}(X)^{\dagger} \times X & \xrightarrow{\otimes_{\mathfrak{Q}}} & \mathbb{W}(X)^{\dagger} \otimes_{\mathfrak{Q}} X \\ & \searrow^{\hom_X} & \Big\downarrow{\scriptstyle \varphi \cong ev_{\mathbb{I}(\mathfrak{Q})^{\dagger}}} \\ & & \mathbb{I}(\mathfrak{Q})^{\dagger} \end{array}$$

Hence the (t, Ω)-classifiable subobject of the element relation ϵ_X is given by

$$\varepsilon_X = \downarrow f \;\text{ where }\; f = \bigvee \{ x \otimes_{\mathfrak{Q}} y \mid (x, y) \in \mathbb{W}(X)^{\dagger} \times X,\ \hom_X(x, y) = \top \}.$$

Finally, we recall that the underlying, commutative and unital quantale $\mathfrak{Q}$ viewed as $\mathfrak{Q}$-module is the unit object of $\mathsf{Mod}(\mathfrak{Q})$ (cf. [4, 2]). It is easily seen that the unique arrow $\mathfrak{Q} \xrightarrow{!_{\mathfrak{Q}}} T$ is an epimorphism. Hence for every $\mathfrak{Q}$-module the universal quantifier $\forall_X$ exists.

Proposition 5.1 *Let X be a $\mathfrak{Q}$-module and $\mathbb{W}(X)^{\dagger} \xrightarrow{\eta_X} [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ be the isomorphism identifying characteristic morphisms with well-sided elements. If $\hom_X$ is the hom-object assignment associated with X, then the universal quantifier of X has the form*:

$$\forall_X(\eta_X(x)) = \hom_X(x, \top), \qquad x \in \mathbb{W}(X)^{\dagger}.$$

Proof. Let us recall that the universal lower bound in $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ is the constant characteristic morphism of X attaining $\top$ for all $x \in X$ — i.e. true_X. Then the image of the name $\ulcorner \mathrm{true}_X \urcorner$ coincides with the unique global point $T \rightarrowtail [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$. In order to compute the corresponding characteristic morphism of the global point $T \rightarrowtail [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ we have to associate the hom-object assignment with the $\mathfrak{Q}$-module $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$. Referring to Section 3 it is important to understand that in this context we always have to read a $\mathfrak{Q}$-module as a *right* $\mathfrak{Q}$-module. Therefore, if $\varphi, \psi \in [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$, the hom-object assignment of $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ is determined by:

$$\hom_{[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]}(\varphi, \psi) = \bigvee \{ \alpha \in \mathfrak{Q} \mid \alpha \searrow \psi \leq^{\dagger} \varphi \} = \bigwedge_{z \in X} \big(\varphi(x) \searrow \psi(x) \big).$$

Now we choose $x \in \mathbb{W}(X)^\dagger$. Referring to Remark 4.3 we obtain for $\varphi = \eta_X(x)$:

$$\begin{aligned}\forall_X(\eta_X(x)) &= \bigwedge_{z\in X}(\top \searrow \eta_X(x)(z)) = \bigwedge_{z\in X}\eta_X(x)(z)\\ &= \bigwedge_{z\in X}\hom_X(x,z) = \hom_X(x,\top).\end{aligned}$$

Hence the assertion is verified. □

If we understand the universal upper bound in X as *true*, then the universal quantifier applied to the (t,Ω)-classifiable subobject $\downarrow x$ corresponding to $\eta_X(x)$ can be interpreted as the extent to which x is true.

In this sense there exists a close relationship between hom-object assignments of $\mathfrak{Q}$-modules and truth arrows in $\mathsf{Mod}(\mathfrak{Q})$.

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Ulrich Höhle
Fakultät für Mathematik und Naturwissenschaften, Bergische Universität
D-42097 Wuppertal, Germany
uhoehle@uni-wuppertal.de