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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-2 (2025)

ON THE BÉNABOU-ROUBAUD THEOREM

Bruno KAHN

Résumé. On donne une preuve détaillée du théorème de Bénabou-Roubaud. Cette preuve fournit un affaiblissement des hypothèses: l'existence de produits fibrés n'est pas nécessaire dans la catégorie de base, et la condition de "Beck-Chevalley", sous la forme d'une transformation naturelle, peut être affaiblie en demandant seulement que cette dernière soit épi.

Abstract. We give a detailed proof of the Bénabou-Roubaud theorem. As a byproduct, it yields a weakening of its hypotheses: the base category does not need fibre products and the Beck-Chevalley condition, in the form of a natural transformation, can be weakened by only requiring the latter to be epi.

Keywords. Descent, monad, Beck-Chevalley condition

Mathematics Subject Classification (2020). 18D30, 18C15, 18F20

To the memory of Jacques Roubaud.

Introduction

The Bénabou-Roubaud theorem [2] establishes, under certain conditions, an equivalence of categories between a category of descent data and a category of algebras over a monad. This result is widely cited, but [2] is a note "without proofs" and the ones I know in the literature are a bit terse ([7, pp. 50/51], [8, proof of Lemma 4.1], [11, Th. 8.5]), [9, 3.7]; moreover, [8] and [11] are formulated in more general contexts.

The aim of this note is to provide a detailed proof of this theorem in its original context. This exegesis has the advantage of showing that the original hypotheses can be weakened: it is not necessary to suppose that the base category admits fibred products[1], and the Chevalley property of [2], formulated as an exchange condition, can also be weakened by requiring that the base change morphisms be only epi. I hope this will be useful to some readers. I also provided a proof of the equivalence between Chevalley's property and the exchange condition (attributed to Beck, but see remark 1.1): this result is part of the folklore but, here again, I had difficulty finding a published proof. In Corollary 5.2, I give a condition (probably too strong) for the Eilenberg-Moore comparison functor to be essentially surjective. Finally, I give cases in Proposition 6.1 where the exchange isomorphism holds; this is certainly classical, but it recovers conceptually Mackey's formula for the induced representations of a group (Example 6.3).

Notation and conventions

I keep that of [2]: thus $P : \mathbf{M} \to \mathbf{A}$ is a bifibrant functor in the sense of [5, §10]. If $A \in \mathbf{A}$, we denote by $\mathbf{M}(A)$ the fibre of P above A. For an arrow $a : A_1 \to A_0$ of $\mathbf{A}$, we write $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$ and $a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)$ for the associated inverse and direct image functors (a_* is *left* adjoint to a^*) and η^a, ε^a for the associated unit and counit. We also write $T^a = a^*a_*$ for the associated monad, equipped with its unit η^a and its multiplication $\mu^a = a^*\varepsilon^a a_*$. *We do not assume the existence of fibre products in* $\mathbf{A}$.

In order to simplify calculations, we shall assume that the pseudofunctor $a \mapsto a^*$ is a functor. This can be justified by the fact that it can be rectified; more precisely, the morphism of pseudofunctors $i \mapsto F_i$ of [10, §3, p. 141] is clearly faithful, hence any parallel arrows in its source which become equal in its target are already equal. (One could also use [3, I, Th. 2.4.2 or 2.4.4].) Then one can also choose the left adjoints $a \mapsto a_*$ to form a functor [12, IV.8, Th. 1], which we do.

[1] As was pointed out by the referee, the corresponding arguments are related to Street's notion of descent object relative to a truncated (co)simplicial category as in the beginning of [16]; but a "truncated cyclic category" à la Connes is also lurking in Proposition 4.6 b).

1. Adjoint chases

To elucidate certain statements and proofs, I start by doing two things: 1) "deploy" the single object M_1 of [2] into several, which will allow us to remove the quotation marks from "natural" at the bottom of [2, p. 96], 2) not assume the Beck-Chevalley condition to begin with, which will allow us to clarify the functoriality in the first lemma of the note and to weaken hypotheses.

1.1

Let a be as above; still following the notation of [2], we give ourselves a commutative square

$$\begin{array}{ccc} A_2 & \xrightarrow{a_2} & A_1 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a} \\ A_1 & \xrightarrow{a} & A_0. \end{array} \tag{1}$$

except that we don't require it to be Cartesian. The equality $a_1^* a^* = a_2^* a^*$ yields a base change morphism

$$\chi : (a_2)_* a_1^* \Rightarrow T^a \tag{2}$$

equal to the composition $\varepsilon^{a_2} T^a \circ (a_2)_* a_1^* \eta^a$. Hence a map

$$\xi_{M,N} = \xi : \mathbf{M}(A_1)(T^a M, N) \xrightarrow{\chi_M^*} \mathbf{M}(A_1)((a_2)_* a_1^* M, N) \xrightarrow[\sim]{\text{adj}} \mathbf{M}(A_2)(a_1^* M, a_2^* N) \tag{3}$$

for $M, N \in \mathbf{M}(A_1)$. It goes in the *opposite* direction to the map K^a of [2], which we will find back in (15). (See also Remark 4.4 in that section.)

Remark 1.1. The morphism (2) is sometimes called "Beck transformation". However, it already appears in SGA4 (1963/64) to formulate the proper base change and smooth base change theorems [1, §4]. I have adopted the terminology "base change morphism" in reference to this seminar.

Lemma 1.2 (key lemma)**.** *For any $\varphi \in \mathbf{M}(A_1)(T^a M, N)$, one has*

$$\xi(\varphi) = a_2^* \varphi \circ a_1^* \eta_M^a.$$

Proof. For $\psi \in \mathbf{M}(A_1)((a_2)_* a_1^* M, N)$ one has $\operatorname{adj}(\psi) = a_2^*\psi \circ \eta^{a_2}_{a_1^* M}$, hence

$$\begin{aligned}\xi(\varphi) = \operatorname{adj}(\varphi \circ \chi_M) &= a_2^*(\varphi \circ \chi_M) \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^*(\varphi \circ (\varepsilon^{a_2} T^a \circ (a_2)_* a_1^* \eta^a)_M) \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^*\varphi \circ a_2^*\varepsilon^{a_2}_{T^a M} \circ a_2^*(a_2)_* a_1^* \eta^a_M \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^*\varphi \circ a_2^*\varepsilon^{a_2}_{T^a M} \circ \eta^{a_2}_{a_1^* T^a M} \circ a_1^* \eta^a_M\\ &= a_2^*\varphi \circ a_1^* \eta^a_M\end{aligned}$$

where we successively used the naturality of η^{a_2} and an adjunction identity. □

1.2

Let $A_3 \in \mathbf{A}$ be equipped with "projections" $p_1, p_2, p_3 : A_3 \to A_2$. We assume that the "face identities" $a_1 p_2 = a_1 p_3$, $a_1 p_1 = a_2 p_3$, $a_2 p_1 = a_2 p_2$ are satisfied; we call these morphisms respectively b_1, b_2, b_3.

Canonical example 1.3. $A_2 = A_1 \times_{A_0} A_1$, $A_3 = A_1 \times_{A_0} A_1 \times_{A_0} A_1$, all morphisms given by the natural projections.

We then have maps, for $i < j$

$$\alpha_{ij}(M, N) = \alpha_{ij} : \mathbf{M}(A_2)(a_1^* M, a_2^* N) \to \mathbf{M}(A_3)(b_i^* M, b_j^* N) \tag{4}$$

given by

$$\alpha_{12} = p_3^*, \quad \alpha_{13} = p_2^*, \quad \alpha_{23} = p_1^*$$

hence composite maps

$$\theta_{ij} = \alpha_{ij} \circ \xi : \mathbf{M}(A_1)(T^a M, N) \to \mathbf{M}(A)(b_i^* M, b_j^* N). \tag{5}$$

In addition, we have the multiplication of T^a mentioned in the notations:

$$\mu^a = a^* \varepsilon^a a_* : T^a T^a \Rightarrow T^a. \tag{6}$$

The commutative square[2]

$$\begin{array}{ccc} A_3 & \xrightarrow{p_3} & A_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ A_2 & \xrightarrow{a_1} & A_1 \end{array} \tag{7}$$

[2]Note that it is Cartesian in the canonical example.

yields another base change morphism $\lambda : (p_1)_* p_3^* \Rightarrow a_1^*(a_2)_*$, hence a composition

$$(b_3)_* b_1^* = (a_2)_*(p_1)_* p_3^* a_1^* \overset{(a_2)_*\lambda a_1^*}{\Longrightarrow} (a_2)_* a_1^*(a_2)_* a_1^* \overset{\chi * \chi}{\Longrightarrow} T^a T^a \tag{8}$$

which, together with adjunction, induces a map

$$\rho : \mathbf{M}(A_1)(T^a T^a M, N) \to \mathbf{M}(A_2)(b_1^* M, b_3^* N). \tag{9}$$

Lemma 1.4. *a) The diagram of natural transformations*

$$\begin{array}{ccccc}
(a_2)_*(p_1)_* p_3^* a_1^* & = & (b_3)_* b_1^* & = & (a_2)_*(p_2)_* p_2^* a_1^* \\
{\scriptstyle (a_2)_*\lambda a_1^*}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle (a_2)_*\varepsilon^{p_2} a_1^*} \\
(a_2)_* a_1^*(a_2)_* a_1^* & & & & (a_2)_* a_1^* \\
{\scriptstyle \chi*\chi}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \chi} \\
T^a T^a & & \overset{\mu^a}{\Longrightarrow} & & T^a
\end{array}$$

is commutative.
b) One has $\theta_{13} = \rho \circ \mu_a^*$ (see (5), (6) and (9)).

Proof. a) is a matter of developing the base change morphisms as done for χ just below (2) (see proof of Lemma 1.2). This yields a commutative diagram

$$\begin{array}{ccc}
\mathbf{M}(A_1)((b_3)_* b_1^* M, N) & \xleftarrow{((a_2)_*\varepsilon^{p_2} a_1^*)^*} & \mathbf{M}(A_1)((a_2)_* a_1^* M, N) \\
{\scriptstyle (8)^*}\Big\uparrow & & \Big\uparrow{\scriptstyle \chi^*} \\
\mathbf{M}(A_1)(T^a T^a M, N) & \xleftarrow{(\mu^a)^*} & \mathbf{M}(A_1)(T^a M, N)
\end{array}$$

from which we get b) by developing the adjunction isomorphism for $((b_3)_*, b_3^*)$. $\square$

Let now $M_1, M_2, M_3 \in \mathbf{M}(A_1)$ and $\varphi_{ij} \in \mathbf{M}(A_1)(T^a M_i, M_j)$ be three morphisms. We have a not necessarily commutative square:

$$\begin{array}{ccc}
T^a T^a M_1 & \xrightarrow{T^a \varphi_{12}} & T^a M_2 \\
{\scriptstyle (\mu_a)_{M_1}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_{23}} \\
T^a M_1 & \xrightarrow{\varphi_{13}} & M_3.
\end{array} \tag{10}$$

Write $\hat{\varphi}_{ij} = \theta_{ij}(\varphi_{ij}) : b_i^* M_i \to b_j^* M_j$.

Lemma 1.5. *Let ψ (resp. ψ') be the composition of* (10) *passing through $T^a M_2$ (resp. through $T^a M_1$). Then $\rho(\psi) = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$ and $\rho(\psi') = \hat{\varphi}_{13}$.*

Proof. The first point follows from a standard adjunction calculation similar to the previous ones, and the second follows from lemma 1.4. □

Proposition 1.6. *If* (10) *commutes, we have $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$; the converse is true if ρ is injective in* (9).

Proof. This is obvious in view of Lemma 1.5. □

In (3), assume that $M = N$ is of the form $a^* M_0$ and write $p = aa_1 = aa_2 : A_2 \to A_0$. We have a composition

$$\mathbf{M}(A_1)(M, a^* M_0) \xrightarrow{\sim} \mathbf{M}(A_0)(a_* M, M_0) \\ \xrightarrow{a^*} \mathbf{M}(A_1)(T^a M, a^* M_0) \xrightarrow{\xi} \mathbf{M}(A_2)(a_1^* M, p^* M_0) \quad (11)$$

where the first arrow is the adjunction isomorphism. A new adjoint chase gives:

Lemma 1.7. *The composition* (11) *is induced by a_1^*.* □

2. Exchange condition and weak exchange condition

Now we introduce the

Definition 2.1. A commutative square (1) is said to satisfy the *exchange condition* if the base change morphism (2) is an isomorphism; we say that (1) satisfies the *weak exchange condition* if (2) is epi.

Lemma 2.2 (cf. [13, Prop. 11] and [14, II.3])**.** *The exchange condition of Definition 2.1 is equivalent to the Chevalley condition (C) of [2].*

Proof. Recall this condition: given a commutative square

$$\begin{array}{ccc} M_1' & \xrightarrow{k_1} & M_1 \\ {\scriptstyle \chi'}\downarrow & & \downarrow{\scriptstyle \chi} \\ M_0' & \xrightarrow{k_0} & M_0, \end{array} \quad (12)$$

above (1) (where we take $(i,j)=(1,2)$ to fix ideas), if χ and χ' are Cartesian and k_0 is co-Cartesian, then k_1 is co-Cartesian.

I will show that the exchange condition is equivalent to each of the following two conditions: (C) and

(C') if k_0 and k_1 are co-Cartesian and χ' is Cartesian, then χ is Cartesian.

Let us translate the commutativity of (12) in terms of the square

$$\begin{array}{ccc} \mathbf{M}(A_2) & \xrightarrow{(a_2)_*} & \mathbf{M}(A_1) \\ {\scriptstyle a_1^*}\uparrow & & {\scriptstyle a^*}\uparrow \\ \mathbf{M}(A_1) & \xrightarrow{a_*} & \mathbf{M}(A_0). \end{array} \tag{13}$$

The morphisms of (12) correspond to morphisms $\tilde{k}_0 : a_*M_0' \to M_0$, $\tilde{k}_1 : (a_1)_*M_1' \to M_1$, $\tilde{\chi} : M_1 \to a^*M_0$ and $\tilde{\chi}' : M_1' \to a_2^*M_0'$, which fit in a commutative diagram of $\mathbf{M}(A_1)$:

$$\begin{array}{ccccc} (a_2)_*a_1^*M_0' & & \xrightarrow{\quad c \quad} & & T^aM_0' \\ {\scriptstyle (a_2)_*\tilde{\chi}'}\uparrow & & & & \downarrow{\scriptstyle a^*\tilde{k}_0} \\ (a_2)_*M_1' & \xrightarrow{\tilde{k}_1} & M_1 & \xrightarrow{\tilde{\chi}} & a^*M_0 \end{array}$$

where c is the base change morphism of (2). The cartesianity conditions on χ and χ' (resp. co-cartesianity conditions on k_0 and k_1) amount to requesting the corresponding morphisms decorated with a ˜ to be isomorphisms.

Suppose c is an isomorphism. If $\tilde{\chi}'$ and $\tilde{k}_0$ are isomorphisms, $\tilde{\chi}$ is an isomorphism if and only if $\tilde{k}_1$ is. Thus, the exchange condition implies conditions (C) and (C'). Conversely, M_0' being given, let $\tilde{k}_0$, $\tilde{\chi}$ and $\tilde{\chi}'$ be identities, which successively defines M_0, M_1 and M_1'. The arrow c then defines an arrow $\tilde{k}_1$, which is an isomorphism if and only if so is c. This shows that the exchange condition is implied by (C), and we argue symmetrically for (C') by taking $\tilde{\chi}'$, $\tilde{k}_1$ and $\tilde{k}_0$ to be identities. □

Remarks 2.3. a) This proof did not use the hypothesis that (1) be Cartesian.
b) Under conservativity assumptions for a_2^* or a^*, we obtain converses to (C) and (C').

3. Pre-descent data

Here we come back to the set-up of Section 1: namely, we give ourselves a commutative diagram (1) as in §1.1 and a system (A_3, p_1, p_2, p_3) as in the beginning of §1.2 satisfying the identities of *loc. cit.* In other words, we have a set of objects and morphisms of $\mathbf{A}$

$$(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$$

subject to the relations

$$aa_1 = aa_2, \quad a_1p_2 = a_1p_3, \quad a_1p_1 = a_2p_3, \quad a_2p_1 = a_2p_2.$$

Let $M \in \mathbf{M}(A_1)$ and $v \in \mathbf{M}(A_2)(a_1^*M, a_2^*M)$. We associate to v three morphisms

$$\hat{\varphi}_{ij} = \alpha_{ij}(v) : b_i^*M \to b_j^*M \quad (i < j)$$

where α_{ij} are the maps of (4).

Definition 3.1. We say that v is a *pre-descent datum* on M if the $\hat{\varphi}_{ij}$ satisfy the condition $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$ of Proposition 1.6. We write $\mathbf{D}^{\mathrm{pre}}$ for the category whose objects are pairs (M, v), where v is a pre-descent datum on M, and whose morphisms are those of $\mathbf{M}(A_1)$ which commute with pre-descent data.

Let us introduce the

Hypothesis 3.2. The weak exchange condition is verified by the squares (1) and (7).

Proposition 3.3 (cf. [2, lemme]). *In* (10)*, assume $\varphi_{12} = \varphi_{23} = \varphi_{13} =: \varphi$. If φ satisfies the associativity condition of a T^a-algebra, then $\xi(\varphi)$ in* (3) *is a pre-descent datum; the converse is true under Hypothesis 3.2.*

Proof. In view of Proposition 1.6, it suffices to show that Hypothesis 3.2 implies the injectivity of ρ, which is induced by the composition of the two natural transformations of (8). The second is epi, therefore induces an injection on Hom's, and so does the first by adjunction. □

Corollary 3.4. *Let $\mathbf{M}^a_{\rm ass}$ denote the category of associative T^a-algebras which are not necessarily unital. Then Proposition 3.3 defines a faithful functor $\xi : \mathbf{M}^a_{\rm ass} \to \mathbf{D}^{\rm pre}$ commuting with the forgetful functors to $\mathbf{M}(A_1)$; under Hypothesis 3.2, it is an isomorphism of categories.*

Proof. Commutation of ξ with the forgetful functors is obvious. This already shows that it is faithful; under Hypothesis 3.2, it is essentially surjective by Proposition 3.3 and we see immediately that it is also full. □

4. The unit condition

We keep the hypotheses and notation of Section 3, and introduce an additional ingredient: a "diagonal" morphism $\Delta : A_1 \to A_2$ such that $a_1\Delta = a_2\Delta = 1_{A_1}$.

Definition 4.1. A *descent datum* on M is a pre-descent datum v such that $\Delta^* v = 1_M$. We denote by $\mathbf{D}$ the full subcategory of $\mathbf{D}^{\rm pre}$ given by the descent data.

Let $\mathbf{M}^a \subset \mathbf{M}^a_{\rm ass}$ be the category of T^a-algebras.

Theorem 4.2 (cf. [2, théorème]). *For all $\varphi \in \mathbf{M}(A_1)(T^a M, M)$, we have*

$$\Delta^*\xi(\varphi) = \varphi \circ \eta^a_M. \tag{14}$$

In particular, $\xi(\mathbf{M}^a) \subset \mathbf{D}$ and $\xi : \mathbf{M}^a \to \mathbf{D}$ is an isomorphism of categories under Hypothesis 3.2.

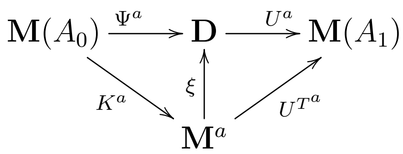
Proof. Suppose that $M = N$ in Lemma 1.2. Applying Δ^* to its identity, we get (14). In particular, if φ is the action of a T^a-algebra then $v = \xi(\varphi)$ verifies $\Delta^* v = 1_M$. We conclude with Corollary 3.4. □

As in [12, VI.3, Th. 1], we have the Eilenberg-Moore comparison functor

$$\begin{aligned} K^a : \mathbf{M}(A_0) &\to \mathbf{M}^a \\ M_0 &\mapsto (a^* M_0, a^*\varepsilon^a_{M_0}). \end{aligned} \tag{15}$$

Lemma 1.7 yields:

Proposition 4.3. *We have* $\xi(a^*\varepsilon^a_{M_0}) = 1_{M_0}$. *In other words, in the diagram*



the left triangle commutes (as well as the right one, trivially). □

Remark 4.4. In [9, 3.7], Janelidze and Tholen construct a functor from $\mathbf{D}$ to $\mathbf{M}$ (same direction as in [2]) by using the *inverses* of the base change morphisms (2).

Remark 4.5. In the canonical example 1.3, a pre-descent datum v satisfies the condition of Definition 4.1 if and only if it is invertible (therefore is a descent datum in the classical sense): this follows from [4, A.1.d pp. 303–304]. In *loc. cit.*, Grothendieck uses an elegant Yoneda argument. It is an issue to see how this result extends to our more general situation: this is done in the next proposition. I am indebted to the referee for prodding me to investigate this.

Note that I merely looked for what is necessary to translate Grothendieck's arguments, and not for the greatest generality.

Proposition 4.6. *Let* $(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$ *be as in Section 3. Let* $M \in \mathbf{M}(A_1)$ *and let* $v \in \mathbf{M}(A_2)(a_1^*M, a_2^*M)$ *be a pre-descent datum as in Definition 3.1. Further, let* Δ *be as in the beginning of the present section. Consider the following conditions:*

(i) $\Delta^*v = 1_M$ *(i.e.* v *is a descent datum).*

(ii) v *is invertible.*

Then:
a) $(ii) \Rightarrow (i)$ *under one of the following conditions: there exists a morphism* s_1 *(resp.* s_2*) from* A_2 *to* A_3 *such that*

$$p_1 s_1 = \Delta a_2, \quad p_2 s_1 = p_3 s_1 = 1$$

(resp.

$$p_1 s_2 = p_2 s_2 = 1, \quad p_3 s_2 = \Delta a_2).$$

b) $(i) \Rightarrow (ii)$ *under the following condition: there exists an involution* σ *of* A_2 *and a morphism* $\Gamma : A_2 \to A_3$ *such that*

$$p_1\Gamma = \sigma, \quad p_2\Gamma = \Delta a_1, \quad p_3\Gamma = 1_{A_2}.$$

(In the case of the canonical example 1.3, we may take for s_1 and s_2 the partial diagonals, for σ the exchange of factors and for Γ the graph of a_1, given in formula by $(\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2, \alpha_1)$.)

Proof. The predescent condition on v is

$$p_2^* v = p_1^* v \circ p_3^* v. \tag{16}$$

a) Applying s_1^* to (16), we get

$$v = a_2^* \Delta^* v \circ v$$

hence $a_2^* \Delta^* v = 1_{A_2}$ and

$$\Delta^* v = \Delta^* a_2^* \Delta^* v = 1_{A_1}.$$

Same reasoning with s_2, *mutatis mutandis*. Note that with s_1 (*resp.* s_2), it suffices to assume that v is right (*resp.* left) cancellable.

b) Applying Γ^* to (16), we get

$$1_{A_2} = a_1^* \Delta^* v = \sigma^* v \circ v.$$

Applying now σ^*, we also get $v \circ \sigma^* v = 1_{A_2}$. □

5. A supplement

Recall [6, Ex. 8.7.8] that a category is called *Karoubian* if any idempotent endomorphism has an image.

Proposition 5.1. *Let* a^* *be fully faithful and* $\mathbf{M}(A_0)$ *Karoubian. Let* $\varphi : T^a M \to M$ *satisfy the identity* $\varphi \circ \eta_M^a = 1_M$. *Then there exists* $M_0 \in \mathbf{M}(A_0)$ *and an isomorphism* $\nu : M \xrightarrow{\sim} a^* M_0$ *such that* $\varphi = \nu^{-1} \circ a^* \varepsilon_{M_0}^a \circ T^a \nu$.

Proof. Let e denote the idempotent $\eta^a_M\varphi \in \operatorname{End}_{\mathbf{M}(A_1)}(T^aM)$. By hypothesis, $e = a^*\tilde{e}$ where $\tilde{e}$ is an idempotent of $\operatorname{End}_{\mathbf{M}(A_0)}(a_*M)$, with image M_0. Then a^*M_0 is isomorphic to the image M of e via a morphism ν as in the statement, such that

$$\nu \circ \varphi = a^*\pi, \quad a^*\iota \circ \nu = \eta^a_M$$

where $\iota\pi$ is the epi-mono factorization of $\tilde{e}$.

To finish, it is enough to see that $a^*\pi = a^*\varepsilon^a_{M_0} \circ T^a\nu$. But we also have

$$\eta^a_{a^*M_0} \circ \nu = T^a\nu \circ \eta^a_M = T^a\nu \circ a^*\iota \circ \nu$$

hence $\eta^a_{a^*M_0} = T^a\nu \circ a^*\iota$. This concludes the proof, since $\eta^a_{a^*M_0} \circ a^*\varepsilon^a_{M_0}$ is the epi-mono factorisation of the idempotent of $\operatorname{End}(T^a a^* M_0)$ with image a^*M_0. □

We thus obtain the following complement:

Corollary 5.2. *Assume Hypothesis 3.2, and also that a^* is fully faithful and $\mathbf{M}(A_0)$ Karoubian. Then*
a) every unital T^a-algebra is associative;
b) K^a is essentially surjective. □

Can one weaken the full faithfulness assumption in this corollary? The following lemma does not seem sufficient:

Lemma 5.3. *Let $M, N \in \mathbf{M}(A_1)$. Then the map*

$$a^* : \mathbf{M}(A_0)(a_*M, a_*N) \to \mathbf{M}(A_1)(T^aM, T^aN)$$

has a retraction r given by $r(f) = \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M$. More generally, we have an identity of the form $r(a^*g \circ f) = g \circ r(f)$.

Proof. For $f : T^aM \to T^aN$ and $g : a_*N \to a_*P$, we have

$$r(a^*g \circ f) = \varepsilon^a_{a_*P} \circ a_*a^*g \circ a_*f \circ a_*\eta^a_M = g \circ \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M = g \circ r(f).$$

Taking $f = 1_{T^aM}$, we obtain that r is a retraction. □

6. Appendix: a case where the exchange condition is verified

Let $\mathcal{A}$ be a category. Take for $\mathbf{A}$ the category of presheaves of sets on $\mathcal{A}$. Write $\int A$ for the category associated to $A \in \mathbf{A}$ by the Grothendieck construction [5, §8]. Recall its definition in this simple case: the objects of $\int A$ are pairs (X, a) where $X \in \mathcal{A}$ and $a \in A(X)$, and a morphism from (X, a) to (Y, b) is a morphism $f \in \mathcal{A}(X, Y)$ such that $A(f)(x) = y$.

Let $\mathcal{C}$ be another category. We take for $\mathbf{M}$ the fibred category of representations of $\mathbf{A}$ in $\mathcal{C}$: for $A \in \mathbf{A}$, an object of $\mathbf{M}(A)$ is a functor from $\int A$ to $\mathcal{C}$. For all $a \in \mathbf{A}(A_1, A_0)$ we have an obvious pull-back functor $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$, which has a left adjoint a_* (direct image) given by the usual colimit if $\mathcal{C}$ is cocomplete. We can then ask whether the exchange condition is true for Cartesian squares of $\mathcal{A}$.

Proposition 6.1. *This is the case if $\mathcal{C}$ is the category of sets $\mathbf{Set}$, and more generally if $\mathcal{C}$ admits a forgetful functor $\Omega : \mathcal{C} \to \mathbf{Set}$ with a left adjoint L such that (L, Ω) satisfies the conditions of Beck's theorem [12, VI.7, Th. 1].*

Proof. First suppose $\mathcal{C} = \mathbf{Set}$; to verify that (2) is a natural isomorphism, it is enough to test it on representable functors. Consider Diagram (13) again. For $(c, \gamma) \in \int A_1$ and $(d, \delta) \in \int A_1$ (with $c, d \in \mathcal{A}$ and $\gamma \in A_1(c)$, $\delta \in A_1(d)$), we have

$$\begin{aligned} T^a y(c,\gamma)(d,\delta) = a^* y(c, a(\gamma))(d,\delta) = y(c, a(\gamma))(d, a(\delta)) \\ = \{\varphi \in \mathcal{A}(d,c) \mid \varphi^* a(\gamma) = a(\delta)\} \end{aligned}$$

and

$$\begin{aligned} (a_2)_* a_1^* y(c,\gamma)(d,\delta) &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} a_1^* y(c,\gamma)(e,\eta) \\ &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} y(c,\gamma)(e, a_1(\eta)) \\ &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} \{\psi \in \mathcal{A}(e,c) \mid \psi^*\gamma = a_1(\eta)\}. \end{aligned}$$

We have

$$(d,\delta) \downarrow a_2 = \{(e, \eta, \eta_2, \theta) \in \mathcal{A} \times A_1(e) \times_{A_0(e)} A_1(e) \times \mathcal{A}(d, e) \mid \theta^* \eta_2 = \delta\}.$$

This category has the initial set $\{(d,\eta_1,\delta,1_d)\mid a(\eta_1)=a(\delta)\}$, so

$$(a_2)_*a_1^*y(c,\gamma)(d,\delta)=\coprod_{\{(\eta_1\in A_1(d)|a(\eta_1)=a(\delta)\}}\{\varphi\in\mathcal{A}(d,c)\mid\varphi^*\gamma=\eta_1\}$$
$$=\{\varphi\in\mathcal{A}(d,c)\mid a(\varphi^*\gamma)=a(\delta)\}$$

and the map $(a_2)_*a_1^*y(c,\gamma)(d,\delta)\to(a_2)_*(a^{12})^*y(c,\gamma)(d,\delta)$ is clearly equal to the identity.

General case: let us write more precisely $\mathbf{M}^{\mathcal{C}}(A)=\mathbb{CAT}(\int A,\mathcal{C})$. The functors L and Ω induce pairs of adjoint functors (same notation)

$$L:\mathbf{M}^{\mathbf{Set}}(A)\leftrightarrows\mathbf{M}^{\mathcal{C}}(A):\Omega.$$

These two functors commute with pull-backs; as L is a left adjoint, it also commutes with direct images. Therefore, in the above situation, the base change morphism $\chi_M:(a_2)_*a_1^*M\to T^aM$ is an isomorphism when $M\in\mathbf{M}^{\mathcal{C}}(A_1)$ is of the form LX for $X\in\mathbf{M}^{\mathbf{Set}}(A_1)$. For any M, we have its canonical presentation [12, (5) p. 153]

$$(L\Omega)^2M\rightrightarrows L\Omega M\to M \tag{17}$$

whose image by Ω is a split coequaliser (*loc. cit.*). Given the hypothesis that Ω creates such coequalisers, (17) is a coequaliser. Since pull-backs are cocontinuous, as well as direct images (again, as left adjoints), (17) remains a coequaliser after applying the functors $(a_2)_*(a^{12})^*$ and T^a. Finally, a coequaliser of isomorphisms is an isomorphism. □

Examples 6.2 (for $\mathcal{C}$). Varieties (category of groups, abelian groups, rings. . .): [12, VI.8, Th. 1].

Example 6.3 (for $\mathcal{A}$). The category with one object $\underline{G}$ associated with a group G: then $\mathbf{A}$ is the category of G-sets. Let us take for $\mathcal{C}$ the category of R-modules where R is a commutative ring. If $A\in\mathbf{A}$ is G-transitive, $\int A$ is a connected groupoid, which is equivalent to $\underline{H}$ for the stabilizer H of any element of A; thus, $\mathbf{M}(A)$ is equivalent to $\mathbf{Rep}_R(H)$. If $a:A_1\to A_0$ is the morphism of $\mathbf{A}$ defined by an inclusion $K\subset H\subset G$ ($A_1=G/K$, $A_0=G/H$), then a^* is restriction from H to K and a_* is induction $V\mapsto RH\otimes_{RK}V$. From Proposition 6.1, we thus recover conceptually the Mackey formula of [15, 7.3, Prop. 22], proven "by hand" in *loc. cit.*

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-2 (2025)

A RESULT ABOUT CONTINUOUS LATTICES OVER THE SIERPIŃSKI LOCALE

Christopher TOWNSEND

Résumé. Soit $\mathbf{2}$ la catégorie $\{0 \leq 1\}$ et $\mathbb{S}$ la locale de Sierpiński (telle que $Sh(\mathbb{S}) \simeq [\mathbf{2}, \mathbf{Set}]$). Nous démontrons que

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

où $\mathbf{CtsLat}^{\ll}$ est la catégorie de treillis continus avec morphismes les homomorphismes de treillis qui préservent les bornes supérieures et la relation «way below».

Abstract. Let $\mathbf{2}$ be the category $\{0 \leq 1\}$ and $\mathbb{S}$ the Sierpiński locale (so that $Sh(\mathbb{S}) \simeq [\mathbf{2}, \mathbf{Set}]$). We prove

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

where $\mathbf{CtsLat}^{\ll}$ is the category of continuous lattices with way-below relation preserving suplattice homomorphisms as morphisms.

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Dedicated to Harvey Alison on the occasion of his 50^{th} birthday.

1. Introduction

The aim of this short note is to prove the equivalence as stated in the Abstract. The proof technique used is taken from [HT23a] and hinges firstly on viewing continuous lattices as the rounded ideals of a type of information system.

We then rely on a description of rounded ideals in a presheaf category given in that paper, now applied to a broader class of relations. The description is that the poset of rounded ideals, internal to a presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, can be calculated by first applying the rounded ideals functor in $\mathbf{Set}$ and then applying the 'lax-to-natural' construction $\tilde{(\_)} : [\mathcal{C}^{op}, \mathbf{Pos}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Pos}]$. To complete the proof we show that $\tilde{(\_)}$ is full when applied to presheaves of continuous lattices. For this to work we seem to need to reduce to the case $\mathcal{C} = \{0 \leq 1\}^{op}$, so that the presheaf category $\hat{\mathcal{C}}$ is equivalent to sheaves over the Sierpiński locale $\mathbb{S}$.

After the main result (Harvey's Lemma) we finish with a corollary that has implications for the classification of locally compact locales via localic groupoids.

An Appendix has also been included that consists of a result about suplattices in presheaf categories. The result should be of general interest as it provides a new connection between presheaves of suplattices and suplattices internal to a presheaf topos.

2. Continuous lattices via strong proximity lattices

We take the following terms as understood: poset, ideal (of a poset; i.e. a directed lower closed subset), semilattice, continuous lattice, dcpo, suplattice and way-below relation ($\ll$). Consult for example [J82] for background material. The information system approach to continuous posets, exemplified by [V93], possibly covers the material of this section. However here we follow the more recent exposition given in [K21] and in particular are exploiting the notion of 'strength' as defined in that paper. The results of this section (indeed the whole paper) are constructive and so valid in all toposes; in particular they will be exploited in presheaf toposes in later sections, as we build up to the proof of the main result.

Definition 2.1. *1. A strong proximity join semilattice is a join semilattice S together with a relation $\prec \subseteq S \times S$ such that $\forall a, b, c \in S$*

(i) $a \prec b$ if and only if there exist $d \in S$ with $a \prec d$ and $d \prec b$,

(ii) $a \leq b \prec c$ implies $a \prec c$, and $a \prec b \leq c$ implies $a \prec c$,

(iii) $\{d | d \prec a\}$ is an ideal of S,

(iv) $a \prec b$ implies $a \leq b$; and,

(v) (strong) if $c \prec a \vee b$ then there exists $a_0 \prec a$ and $b_0 \prec b$ such that $c \leq a_0 \vee b_0$.

2. A rounded ideal of a strong proximity join semilattice is an ideal I such that $\forall a \in I$ there exists $b \in I$ such that $a \prec b$. The collection of all rounded ideals of S is written $R\text{-}idl(S)$.

Example 2.2. For any continuous lattice A, $(A, \ll)$ is a strong proximity join semilattice. Further $A \cong R\text{-}idl(A)$; in one direction the isomorphism is $a \mapsto \twoheaddownarrow a$ and directed join (i.e. $I \mapsto \bigvee^{\uparrow} I$) is the inverse.

It is clear from the definition that strong proximity join semilattices are the models of a geometric theory; the morphisms of the corresponding category of models are join semilattice homomorphisms that preserve $\prec$. We denote this category $\vee$-$\mathbf{SPSLat}$ and there is a forgetful functor $U : \mathbf{CtsLat}^{\ll} \longrightarrow \vee\text{-}\mathbf{SPSlat}$.

Proposition 2.3. *For any strong proximity join semilattice S, $R\text{-}idl(S)$ is a continuous lattice. Define $\bar{\phi}(I) = \bigcup^{\uparrow}\{\downarrow^{S'} \phi(a) | a \in I\}$, for each strong proximity join semilattice homomorphism $\phi : S \longrightarrow S'$; then by taking $R\text{-}idl(\phi) = \bar{\phi}$ on morphisms we have defined a functor $R\text{-}idl$: $\vee$-$\mathbf{SPSlat} \longrightarrow \mathbf{CtsLat}^{\ll}$.*

In the statement of the Proposition we use the notation $\downarrow^S a = \{b | b \prec a\}$ for any element a of a strong proximity join semilattice S.

Proof. The bottom of $R\text{-}idl(S)$ is $\downarrow^S 0$, and $I \vee J = \downarrow \{a \vee b | a \in I, b \in J\}$ (use the roundedness of I and J to check that this set, which is clearly an ideal, is rounded). The directed union of rounded ideals is a rounded ideal so $R\text{-}idl(S)$ is a complete lattice. Any rounded ideal I is the directed union of $\downarrow^S a$ for each $a \in I$. Therefore $I \ll J$ iff $\exists j \in J$ such that $I \subseteq \downarrow^S j$ from which it is clear that $R\text{-}idl(S)$ is continuous.

Let $\phi : S \longrightarrow S'$ be a morphism of strong proximity join semilattices. Certainly $\bar{\phi}(\downarrow^S 0) \subseteq \downarrow^{S'} 0$ as $i \prec^S 0$ implies $i = 0$ and $\phi(0) = 0$. That $\bar{\phi}$ preserves directed joins follows essentially by definition of union. For preservation of binary joins by $\bar{\phi}$ it is therefore clearly sufficient to verify $\bar{\phi}(\downarrow^S a \vee b) \subseteq \bar{\phi}(\downarrow^S a) \vee \bar{\phi}(\downarrow^S b)$ for any pair $a, b \in S$. This amounts to verifying that for any $d \prec^{S'} \phi(c)$, for some c with $c \prec^S a \vee b$, that

$d \leq c_0 \vee c_1$ for some c_0, c_1 such that there exists $a_0 \prec^S a$ and $b_0 \prec^S b$ with $c_0 \prec^{S'} \phi(a_0)$ and $c_1 \prec^{S'} \phi(b_0)$. Use $c \prec^S a \vee b$ and the strength of the proximity lattice S to find a_0 and b_0 for which then $\phi(c) \leq \phi(a_0 \vee b_0)$. But $\phi(a_0 \vee b_0) = \phi(a_0) \vee \phi(b_0)$ so by the strength of S' there exists the c_0 and c_1 required.

To complete our check that $\bar{\phi}$ is a morphism of $\mathbf{CtsLat}^{\ll}$ we must check that it preserves $\ll$. For this we need to verify that if $I \subseteq \downarrow^S j$ for some $j \in J$ that $\bar{\phi}(I) \subseteq \downarrow^S j'$ for some $j' \in \bar{\phi}(J)$. But by roundedness of J and preservation of $\prec$ by ϕ, $j \in J$ implies $\phi(j) \in \bar{\phi}(J)$ and we can see that $\bar{\phi} \downarrow^S j \subseteq \downarrow^{S'} \phi(j)$ as ϕ preserves $\prec$. These last two observations combine to show that $\bar{\phi}$ preserves $\ll$.

Finally, it is clear that we have defined a functor. Preservation of identity is trivial and $\bar{\psi}\bar{\phi} = \overline{\psi\phi}$ because $\prec$ is preserved. $\square$

Notice that the isomorphism $A \cong R\text{-}idl(A)$ of Example 2.2 is natural in A; more explicitly there is a natural isomorphism $R\text{-}idl \circ U \cong Id_{\mathbf{CtsLat}^{\ll}}$, though we will not notate the forgetful functor $U : \mathbf{CtsLat}^{\ll} \longrightarrow \vee\text{-}\mathbf{SPSlat}$ in what follows.

3. Background presheaf topos results

This section consists of three subsections where we recall in turn some results about constructions and characterisations of lattice theoretic properties in presheaf toposes. The results are all effectively well known. In the first subsection we recall the $\tilde{(\_)}$ construction which given a presheaf of posets returns another presheaf of posets, but on morphisms sends lax natural transformations to natural transformations. Next we recall how the $\tilde{(\_)}$ construction can be used to give an explicit description of the rounded ideal completion in a presheaf topos. Finally we recall that for any dcpo (suplattice) homomorphism $\alpha : A \longrightarrow B$ in a presheaf topos $\hat{\mathcal{C}}$, that $\alpha_a : A(a) \longrightarrow B(a)$ is a dcpo (suplattice) homomorphism for every object a of $\mathcal{C}$.

3.1 The lax-to-natural functor $\tilde{(\_)}$

We recall the $\tilde{(\_)}$ construction from [HT23a], which is a lax right adjoint to the forgetful functor that embeds $[\mathcal{C}^{op}, \mathbf{Pos}]$ into the category with the same objects (presheaves of posets) but with lax natural transformations as morphisms. We will not exploit this lax universal property here, relying instead on [HT23a] for properties of $\tilde{(\_)}$, but we will need to describe it explicitly.

If $F : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is a presheaf of posets on some category $\mathcal{C}$ then we define $\tilde{F}$ by

$$\tilde{F}(a) = \{(x_f) \in \Pi_{f:b \longrightarrow a} F(b) | F(g)x_f \leq x_{fg}, \forall c \xrightarrow{g} b \xrightarrow{f} a\}.$$

In terms of its action on morphisms we have $[\tilde{F}(f)((x_h))]_g = x_{fg}$ for any $f : b \longrightarrow a$ and $g : c \longrightarrow b$. If $\phi : F \xrightarrow{\leq} G$ is a lax natural transformation (i.e. $G(f)\phi_b \leq \phi_a F(f)$ for all $f : b \longrightarrow a$ of $\mathcal{C}$) then we define a natural transformation $\tilde{\phi} : \tilde{F} \longrightarrow \tilde{G}$ by $\tilde{\phi}_a((x_f)) = (\phi_b(x_f))$. We know from [HT23a] that it is faithful; in fact, for any natural transformation $\alpha : \tilde{F} \longrightarrow \tilde{G}$ there is a lax natural transformation $\psi^\alpha : F \xrightarrow{\leq} G$ such that $\phi = \psi^{\tilde{\phi}}$ for any lax natural transformation $\phi : F \xrightarrow{\leq} G$. Further $\psi^{(\_)}$ has the properties that $\psi^{Id} = Id$ and $\psi^\alpha \psi^\beta \leq \psi^{\alpha\beta}$. The explicit formula for ψ^α is $\psi^\alpha_a(x) = (\alpha_a((F(f)(x))_f))_{Id_a}$.

3.2 R-idl in a presheaf topos

In this subsection we recall the approach taken in [HT23a] to constructing R-idl in a presheaf topos $\hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$. As made clear in Section 4 of that paper, any construction of sets of subsets, each determined by geometric sequents, can be calculated by first applying the construction in $\mathbf{Set}$ at each object (and morphism) to obtain a new presheaf, and then applying the $\tilde{(\_)}$ construction to that presheaf. Put another we are saying that R-$idl_{\hat{\mathcal{C}}}(S)$, as a poset in $\hat{\mathcal{C}}$, is naturally isomorphic to $\widetilde{R\text{-}idl \circ S}$, for any strong proximity lattice S in the topos $\hat{\mathcal{C}}$. (Given that S is the model of a geometric theory in a presheaf topos, it is the same thing as a functor $\mathcal{C}^{op} \longrightarrow \vee\text{-}\mathbf{SPSLat}$; e.g. D1.2.14 (i) of [J02].)

3.3 Dcpo and suplattice homomorphisms in $\hat{\mathcal{C}}$

Lemma 3.1. *Let $\alpha : A \longrightarrow B$ be an internal dcpo (suplattice) homomorphism in a presheaf topos $\hat{\mathcal{C}}$ between two internal dcpos (suplattices) A*

and B. Then $\alpha_a : A(a) \longrightarrow B(a)$ *is a dcpo (suplattice) homomorphism for every object* a *of* $\mathcal{C}$*. Further, if* A *is an internal suplattice then for any morphism* $f : b \longrightarrow a$ *of* $\mathcal{C}$ $A(f)$ *is a suplattice homomorphism.*

Proof. Recall that for any geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ its direct image defines a functor $f_* : \mathbf{dcpo}_{\mathcal{F}} \longrightarrow \mathbf{dcpo}_{\mathcal{E}}$ ([T04]). The techniques of (i) in Lemma C1.6.9 of [J02] can then be applied to complete the proof. To provide more detail, note that $A(a)$ is isomorphic to $\gamma_*(a^*A)$ where γ is the unique geometric morphism $\widehat{\mathcal{C}/a} \xrightarrow{a} \hat{\mathcal{C}} \longrightarrow \mathbf{Set}$. (Here $a : \widehat{\mathcal{C}/a} \longrightarrow \hat{\mathcal{C}}$ is the geometric morphism corresponding to the pullback adjunction $\Sigma_{Y(a)} \dashv Y(a)^*$ under $\widehat{\mathcal{C}/a} \simeq \hat{\mathcal{C}}/Y(a)$; its inverse image is logical.) $\square$

4. Main result

Lemma 4.1. *(Harvey's lemma).*

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

Proof. Consider

$$\begin{array}{rcl} \Psi : [\mathbf{2}, \mathbf{CtsLat}^{\ll}] & \longrightarrow & \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})} \\ F & \mapsto & \tilde{F} \end{array}$$

This is well defined because $F \cong R\text{-}idl \circ F$ (recall our earlier comment that $R\text{-}idl \circ U \cong Id$) and so $\tilde{F} \cong \widetilde{R\text{-}idl \circ F} \cong R\text{-}idl_{Sh(\mathbb{S})}F$, a continuous lattice in $Sh(\mathbb{S})$. It is clearly essentially surjective as any continuous lattice A in $Sh(\mathbb{S})$ has $A \cong R\text{-}idl_{Sh(\mathbb{S})}(A)$. (In fact, all this holds for any $\hat{\mathcal{C}}$.)

We have commented already that $\tilde{(\_)}$ is faithful, so we just need to prove fullness of Ψ to complete the proof. Say we have $\alpha : \tilde{F} \longrightarrow \tilde{G}$, a $\ll$ preserving internal suplattice homomorphism (in $Sh(\mathbb{S})$). Because it is $\ll$ preserving, its right adjoint α_* is directed join preserving; i.e. a dcpo homomorphism. Indeed, this is an equivalent characterisation of $\ll$ preserving for a suplattice homomorphism and we will call on this characterisation below. By Lemma 3.1 we know that $\alpha_i : \tilde{F}(i) \longrightarrow \tilde{G}(i)$ (respectively

$(\alpha_*)_i : \tilde{G}(i) \longrightarrow \tilde{F}(i)$) are suplattice (respectively dcpo) homomorphisms for $i = 0, 1$.

Now, the obvious choice for a natural transformation $\psi : F \longrightarrow G$ such that $\tilde{\psi} = \alpha$ is ψ^α (described in Section 3.1). To show that this works and so to complete the proof we must show: (a) $[\widetilde{\psi^\alpha}]_i = \alpha_i$ for $i = 0, 1$, (b) ψ_i^α is a suplattice homomorphism for $i = 0, 1$, (c) ψ^α is a natural transformation and not just a lax natural transformation; and, (d) ψ_i^α preserves $\ll$ for $i = 0, 1$.

We will take each in turn but first let us note that by construction $\tilde{F}(1) = F(1)$ (there is only the identity morphism away from 1). From this it is clear $\alpha_1 = \psi_1^\alpha = [\widetilde{\psi^\alpha}]_1$ and so (a), (b) and (d) are actually immediate at $i = 1$. The suplattice $\tilde{F}(0)$ is also easy to describe: it consists of pairs (x_0, x_1) with $x_i \in F(i)$, $i = 0, 1$ and $F(\leq)(x_0) \leq x_1$. Further the function $\tilde{F}(\leq) : \tilde{F}(0) \longrightarrow \tilde{F}(1)$ is projection $(x_0, x_1) \mapsto x_1$; this is just a repetition of the definition of $\tilde{(\_)}$. Notice from this that by naturality of α therefore for any (x_0, x_1) in $\tilde{F}(0)$ we have $\pi_2\alpha_0(x_0, F(\leq)x_0) \leq \pi_2\alpha_0(x_0, x_1) = \alpha_1\pi_2(x_0, x_1) = \psi_1^\alpha x_1$; this will be used in our verification of (a) which is the next step.

(a) As α is an internal suplattice homomorphism we know that

$$\begin{array}{ccc} \tilde{F}(0) & \xrightarrow{\alpha_0} & \tilde{G}(0) \\ \Big\uparrow{\scriptstyle \Sigma_{\tilde{F}(\leq)}} & & \Big\uparrow{\scriptstyle \Sigma_{\tilde{G}(\leq)}} \\ \tilde{F}(1) & \xrightarrow{\alpha_1} & \tilde{G}(1) \end{array}$$

commutes where $\Sigma_{\tilde{F}(\leq)} \dashv \tilde{F}(\leq)$ and similarly for $\tilde{G}$. See the proof of Lemma C1.6.9 [J02] for details on how internal suplattices, as presheaves, have left adjoints for their transition functions and for references to see how internal suplattice homomorphisms must commute with these left adjoints (or e.g. Proposition 3.7 of [T04] for effectively the same material). We can give an explicit description of $\Sigma_{\tilde{F}(\leq)}$: it is $x_1 \mapsto (0, x_1)$; this is clear as $\tilde{F}(\leq)$

is projection. Therefore we can calculate:

$$\begin{aligned}
\alpha_0(x_0,x_1) &= \alpha_0((x_0,F(\leq)x_0)\vee(0,x_1)) \\
&= \alpha_0(x_0,F(\leq)x_0)\vee\alpha_0\Sigma_{\tilde{F}(\leq)}(x_1) \\
&= \alpha_0(x_0,F(\leq)x_0)\vee\Sigma_{\tilde{G}(\leq)}\alpha_1(x_1) \\
&= (\psi_0^\alpha x_0,\pi_2\alpha_0(x_0,F(\leq)x_0))\vee(0,\psi_1^\alpha x_1) \text{ by def. of } \psi^\alpha \\
&= (\psi_0^\alpha x_0,\pi_2\alpha_0(x_0,F(\leq)x_0)\vee\psi_1^\alpha x_1) \\
&= (\psi_0^\alpha x_0,\psi_1^\alpha x_1) \text{ by an earlier remark} \\
&= [\widetilde{\psi^\alpha}]_0(x_0,x_1)
\end{aligned}$$

(b) By construction ψ_0^α is the composite

$$F(0)\xrightarrow{(Id,F(\leq))}\tilde{F}(0)\xrightarrow{\alpha_0}\tilde{G}(0)\xrightarrow{\pi_1}G(0).$$

By noting that joins in $\tilde{F}(0)$ and $\tilde{G}(0)$ are calculated pointwise it is clear that each factor in the composite is a suplattice homomorphism, and so ψ_0^α is a suplattice homomorphism.

(c) We know ψ^α is lax so this part of the proof amounts to checking

$$\psi_1^\alpha F(\leq)\leq G(\leq)\psi_0^\alpha.$$

In (a) we established $\alpha_0(x_0,x_1)=(\psi_0^\alpha(x_0),\psi_1^\alpha(x_1))$. By uniqueness of adjoints we therefore know that $(\alpha_*)_0(y_0,y_1)=([\psi_0^\alpha]_*(y_0),[\psi_1^\alpha]_*(y_1))$. From the definition of ψ^{α_*} we therefore know that $\psi_i^{\alpha_*}=[\psi_i^\alpha]_*$. But ψ^{α_*} is a lax natural natural transformation so we know that

$$F(\leq)\psi_0^{\alpha_*}\leq\psi_1^{\alpha_*}G(\leq).$$

Part (c) therefore follows by taking adjoint transpose (twice) of this last inequality.

(d) Apply the same reasoning as (b), but now to $[\psi_0^\alpha]_*$ which we have established is equal to $\psi_0^{\alpha_*}$. Having a right adjoint that preserves directed joins implies preservation of $\ll$. □

Remark 4.2. I have not been able to establish whether we *must* restrict to $\mathcal{C}=\{0\leq 1\}^{op}$ for the result to work. I expect so because in the proof we are exploiting a gluing construction which is, itself, tied to having an open/closed decomposition.

We finish with a corollary that has implications for the classification of locally compact locales via localic groupoids. Define $\mathbf{CtsFrm}^{\ll}$ to be the full subcategory of $\mathbf{CtsLat}^{\ll}$ consisting of continuous frames; i.e. continuous lattices that are also frames. Note that the morphisms are not frame homomorphisms; they are suplattice homomorphisms with directed join preserving right adjoints.

Corollary 4.3.

$$[\mathbf{2}, \mathbf{CtsFrm}^{\ll}] \simeq \mathbf{CtsFrm}^{\ll}_{\mathbf{Sh}(\mathbb{S})}$$

Proof. A continuous lattice is always a preframe (e.g. Lemma VII 4.1 of [J82]; but straightforward lattice theory). Therefore a continuous lattice is a continuous frame if and only if it satisfies the distributive law. The proof of the main lemma gives an explicit description of $\tilde{F}(i)$ in terms of $F(i)$ for $i = 0, 1$ and we noticed $\tilde{F}(1) = F(1)$. So it just needs to be checked that assuming $F(1)$ is distributive, $\tilde{F}(0)$ is distributive if and only if $F(0)$ is. This is immediate from the explicit description as binary meet and join in $\tilde{F}(0)$ are calculated pointwise. □

It is expected that we can construct a classifying localic groupoid for locally compact locales, using for example the approach of [HT23b] (or via an explicit construction of the points of the localic groupoid via the locale $\mathbb{S}^{\mathbb{N}}$; G. Manuell, private communication). That is, we expect that there exists a localic groupoid $\mathbb{G}_{\mathfrak{LK}}$ such that for any locale X the category $\mathfrak{LK}^{\cong}_{Sh(X)}$ of locally compact locales internal to the topos $Sh(X)$ (with isomorphisms as morphisms) is equivalent to the category of principal $\mathbb{G}_{\mathfrak{LK}}$-bundles over X. Now [HT23a] shows that $\mathbb{S}$-homotopies between principal bundles (over the classifying localic groupoid for compact Hausdorff locales) correspond to locale maps between compact Hausdorff locales (and the same correspondence for discrete locales is easy from the definition of presheaf topos). So it might be hoped that the same holds for locally compact locales. The Corollary rules this out: locally compact locales in $Sh(\mathbb{S})$ correspond externally to $\ll$ preserving suplattice homomorphisms and these do not correspond to locale maps.

5. Appendix: Internal suplattices

Below is a result about the relationship between $[\mathcal{C}^{op}, \mathbf{Sup}]$ and $\mathbf{Sup}_{\hat{\mathcal{C}}}$ which should be of general interest.

Proposition 5.1. *Let $\mathcal{C}$ be a small cartesian category (i.e. small and finitely complete). Then the $\tilde{(\_)}$ construction determines a functor:*

$$\tilde{(\_)} : [\mathcal{C}^{op}, \mathbf{Sup}] \longrightarrow \mathbf{Sup}_{\hat{\mathcal{C}}}$$

Proof. We split the proof into two parts:
(a) If $F : \mathcal{C}^{op} \longrightarrow \mathbf{Sup}$ is a functor then $\tilde{F}$ is an internal suplattice in $\hat{\mathcal{C}}$.
(b) If $\phi : F \longrightarrow G$ is a natural transformation then $\tilde{\phi} : \tilde{F} \longrightarrow \tilde{G}$ is an internal suplattice homomorphism in $\hat{\mathcal{C}}$.

(a) We rely on Lemma C1.6.9 of [J02] which shows that a presheaf $L : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is an internal suplattice if and only if (i) $L(a)$ is a suplattice for every object a of $\mathcal{C}$, (ii) $L(f) : L(a) \longrightarrow L(b)$ has a right and left adjoint for every morphism $f : b \longrightarrow a$; and, (iii) Beck-Chevalley holds for left adjoints; that is, for any pullback diagram

$$\begin{array}{ccc} a \times_d b & \xrightarrow{\pi_2} & b \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle k} \\ a & \xrightarrow{l} & d \end{array}$$

in $\mathcal{C}$ the square

$$\begin{array}{ccc} L(a \times_d b) & \xleftarrow{L(\pi_2)} & L(b) \\ \downarrow{\scriptstyle \Sigma_{\pi_1}} & & \downarrow{\scriptstyle \Sigma_k} \\ L(a) & \xleftarrow{L(l)} & L(d) \end{array}$$

commutes where Σ_h is the left adjoint of $L(h)$ for any morphism h of $\mathcal{C}$.

We verify (i), (ii) and (iii) for $\tilde{F}$ where $F : \mathcal{C}^{op} \longrightarrow \mathbf{Sup}$.

For (i) note that if $(x^i_f)_{f:b \longrightarrow a}$ is an indexed ($i \in I$) collection of elements of $\tilde{F}(a)$ then $(\bigvee_{i \in I} x^i_f)_f$ is in $\tilde{F}(a)$ because $F(g)$ preserves arbitrary joins for all $g : c \longrightarrow b$, and can readily be seen to be the join of the $(x^i_f)_f$s. So $\tilde{F}(a)$ is a suplattice for each a. Notice that arbitrary meet is similarly defined pointwise; i.e. $(\bigwedge_{i \in I} x^i_f)_f$ is the meet of the $(x^i_f)_f$s.

Next (ii) is straightforward because arbitrary joins and meets are defined pointwise so it is easy to see that they are preserved by $\tilde{F}(f)$ (and we know that a monotone map between complete lattices has a right(left) adjoint iff it preserves arbitrary joins(meets)). For example, for joins,

$$(\tilde{F}(f)(\bigvee_i x^i_h))_g = \bigvee_i x^i_{fg} = [\bigvee_i \tilde{F}(f)(x^i_h)]_g.$$

For (iii) by uniqueness of adjoints we only need to prove $\tilde{F}(l)\Sigma_k \leq \Sigma_{\pi_1}\tilde{F}(\pi_2)$. Recall that quite generally if $\phi : A \longrightarrow B$ preserves arbitrary meets then its left adjoint is given by $\Sigma_\phi(b) = \bigwedge\{a | b \leq \phi(a)\}$. So checking (iii) amounts to checking for each $n : a' \longrightarrow a$ and each $(x_g)_{g:c \longrightarrow b} \in \tilde{F}(b)$ that

$$(\bigwedge\{(x'_r) | x_g \leq x'_{kg}, \forall g : c \longrightarrow b\})_{ln} \qquad \text{(A)}$$

is less than or equal to

$$(\bigwedge\{(y_m) | x_{\pi_2 t} \leq y_{\pi_1 t}, \forall t : c \longrightarrow a \times_d b\})_n \qquad \text{(B)}$$

Our strategy is to find for each $(y_m) \in \tilde{F}(a)$ in the meet (B) an $(x'_r) \in \tilde{F}(d)$ such that $x_g \leq x'_{kg}$ for all $g : c \longrightarrow b$. From this we know that A $\leq x'_{ln}$ for each $n : a' \longrightarrow a$ and the check of (iii) can be completed by verifying $x'_{ln} \leq y_n$ for each $n : a' \longrightarrow a$.

Define, for $r : d' \longrightarrow d$,

$$x'_r = [F(\pi_2^{d'})]_* y_{\pi_1}$$

where $\pi_2^{d'} : a \times_d d' \longrightarrow d'$ and we are using ϕ_* to denote the right adjoint of any ϕ (and $F(f)$, being a suplattice homomorphism, has a right adjoint for each f). We first check that (x'_r) is in $\tilde{F}(d)$; that is, do we have

$F(t)x'_r \leq x'_{rt}$ for every $t : d'' \longrightarrow d'$? Because $F(Id_a \times t)y_{\pi_1} \leq y_{\pi_1(Id_a \times t)}$ (as $(y_m) \in \tilde{F}(a)$) this can be confirmed by verifying $F(t)[F(\pi_2^{d'})]_* y_{\pi_1} \leq [F(\pi_2^{d''})]_* F(Id_a \times t)y_{\pi_1}$. This last is easy to verify as it is equivalent to $F(\pi_2^{d''})F(t)[F(\pi_2^{d'})]_* y_{\pi_1} \leq F(Id_a \times t)y_{\pi_1}$, $t\pi_2^{d''} = \pi_2^{d'}(Id_a \times t)$ and $F(\pi_2^{d'})[F(\pi_2^{d'})]_* \leq Id_{a \times_d d'}$.

For $x_g \leq x'_{kg}$, given a $g : c \longrightarrow b$ note that $x'_{kg} = [F(\pi_2^c)]_* y_{\pi_1(Id_a \times g)}$. So we must but check $F(\pi_2^c)(x_g) \leq y_{\pi_1(Id_a \times g)}$. This follows because $F(\pi_2^c)(x_g) \leq x_{g\pi_2^c}$ (as (x_g) is in $\tilde{F}(b)$) and $x_{g\pi_2^c} = x_{\pi_2^b(Id_a \times g)} \leq y_{\pi_1(Id_a \times g)}$ where the last inequality follows as (y_m) is in the meet (B).

So to complete our strategy for checking (iii) we must verify that $x'_{ln} \leq y_n$ for any $n : a' \longrightarrow a$. Using that the pullback of the composite ln along l is $\pi_1(Id_a \times n)$, where $\pi_1 : a \times_d a \longrightarrow a$, the calculation is:

$$\begin{aligned} x'_{ln} &= [F(\pi_2^{a'})]_* y_{\pi_1(Id_a \times n)} \\ &\leq [F(\pi_2^{a'})]_* [F(n, Id_{a'})]_* F(n, Id_{a'}) y_{\pi_1(Id_a \times n)} \\ &= [F(Id_{a'})]_* F(n, Id_{a'}) y_{\pi_1(Id_a \times n)} \\ &\leq y_n \end{aligned}$$

where the last is because $(y_m) \in \tilde{F}(a)$ (and, of course, n factors as $\pi_1(Id_a \times n)(n, Id_{a'})$).

(b) We prove that $\tilde{\phi}$ is an internal suplattice homomorphism. This follows provided we can verify that $\tilde{\phi}_a \Sigma_{\tilde{F}(f)} \leq \Sigma_{\tilde{G}(f)} \tilde{\phi}_b$ for all $f : b \longrightarrow a$ (this can be seen from the constructions shown in the proof of C1.6.9 of [J02]; Proposition 3.7 of [T04] also provides a route).

For each $(y_g) \in \tilde{F}(b)$ we must verify

$$\begin{aligned} &\tilde{\phi}_a(\bigwedge\{(x_r) | y_g \leq x_{fg}, \forall g : c \longrightarrow b\}) \leq \\ &\bigwedge\{(z_t) | [\tilde{\phi}_b((y_g))]_g \leq z_{fg}, \forall g : c \longrightarrow b\} \end{aligned}$$

Given $(z_t) \in \tilde{G}(a)$ then it is in the meet of the right hand side iff $\phi_c(y_g) \leq z_{fg}$ for all $g : c \longrightarrow b$. For any such (z_t) define (x_r) by $x_r = [\phi_{a'}]_*(z_r)$ for each $r : a' \longrightarrow a$. We check that $(x_r) \in \tilde{F}(a)$; i.e. that $F(d)x_r \leq x_{rd}$, or equivalently $F(d)[\phi_{a'}]_* z_r \leq [\phi_{a''}]_* z_{rd}$ for each $d : a'' \longrightarrow a'$. But given that $G(d)z_r \leq z_{rd}$ this will follow if $F(d)[\phi_{a'}]_* \leq [\phi_{a''}]_* G(d)$, which is true as ϕ is natural.

To complete we must verify (1) $y_g \leq x_{fg}$ for all g and (2) $(\tilde{\phi}_a((x_r)))_t \leq (z_t)$. For (1), as $\phi_c(y_g) \leq z_{fg}$, $y_g \leq [\phi_c]_* z_{fg} = x_{fg}$. For (2), this amounts to checking $\phi_{a'} x_r \leq z_r$ for each $r : a' \longrightarrow a$ which is again immediate from the definition of (x_r). □

Remark 5.2. I believe that the restriction to cartesian $\mathcal{C}$ can be seen to be unnecessary, using techniques from [T04]; however the proof becomes a bit more involved.

Remark 5.3. Part (b) of the proof does not work if we only assume that ϕ is a lax natural transformation. For example, take $\mathcal{C} = \{0 \leq 1\}^{op}$. If $X, Y : \{0 \leq 1\} \longrightarrow \mathbf{Set}$ then a lax natural transformation from $PX : \{0 \leq 1\} \longrightarrow \mathbf{Sup}$ to $PY : \{0 \leq 1\} \longrightarrow \mathbf{Sup}$ is the same data as a pair of relations $R(0) \subseteq X(0) \times Y(0)$ and $R(1) \subseteq X(1) \times Y(1)$ such that

$$\{Y(\leq)j_0 | \exists i_0 \in I_0, i_0 R(0) j_0\} \subseteq \{j_1 | \exists i'_0 \in I_0, (X(\leq)i'_0) R(1) j_1\}$$

for each $I_0 \subseteq X(0)$. But this is not sufficient for R to be a subfunctor of $X \times Y$, and so cannot correspond to an internal suplattice homomorphism in $\hat{\mathcal{C}}$.

Remark 5.4. The proposition can be used to prove the main lemma (Lemma 4.1) without using information systems. For example, $\twoheaddownarrow : F \longrightarrow idl \circ F$ is a natural transformation if $F : \mathcal{C}^{op} \longrightarrow \mathbf{CtsLat}^{\ll}$ and so $\tilde{\twoheaddownarrow}$ is an internal suplattice homomorphism which is a splitting for $\bigvee^{\uparrow} : \widetilde{idl \circ F} \longrightarrow \tilde{F}$. This shows that $\tilde{F}$ is an internal continuous lattice because $\widetilde{idl \circ F} \cong idl_{\hat{\mathcal{C}}} F$ and so we have exhibited $\tilde{F}$ as a dcpo retract of $idl_{\hat{\mathcal{C}}} A$ for some internal poset A of $\hat{\mathcal{C}}$.

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Almost Cofibrations

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Résumé. Dans cet article nous étudions une généralisation de la propriété d'extension de l'homotopie ainsi que la notion associée de *almost-cofibration* pour les espaces topologiques. Après avoir présenté quelques caractéristiques nouvelles et intéressantes de cette notion, nous montrons que tout plongement fermé d'espaces compacts métrisables est une *almost-cofibration*. De plus, il s'avère que la catégorie des espaces compacts métrisables, avec les *almost-cofibrations* et les équivalences de forme forte, possède la structure d'une catégorie de cofibrations dont la catégorie d'homotopie est sa catégorie de forme forte.
Abstract. We study a generalization of the homotopy extension property together with the related notion of almost-cofibration of topological spaces. After giving some new and interesting features of such a notion we show that every closed embedding of compact metrizable spaces is an almost-cofibration. Moreover, it turns out that the category of compact metrizable spaces, together with almost-cofibrations and strong shape equivalences has the structure of a cofibration category whose homotopy category is its strong shape category.

Keywords. HEP, RWHEP, almost-cofibration, strong shape equivalence, cofibration category, compact metrizable space.
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Introduction

We consider a variation of the usual homotopy extension property (HEP), called the rather weak homotopy extension property (RWHEP), that was in-

troduced in [1], see also [8]. The maps having the RWHEP are characterized by the fact that they are exactly those maps inducing levelwise fibrations in the 2-category $[\mathbf{Top}, \mathbf{Gpd}]$, where $\mathbf{Top}$ denotes the category of topological spaces, while $\mathbf{Gpd}$ is the category of groupoids and their homomorphisms (functors). In this paper we are interested in certain features of this property, for instance, in contrast to the HEP, the RWHEP is preserved when passing to the category of inverse systems and also passes to limits of inverse systems. A map having the RWHEP with respect to all spaces is called here an almost-cofibration. In particular, it is proved that every closed embedding of compact metrizable spaces is an almost-cofibration.
Almost-cofibrations and strong shape equivalences [11] give the category $\mathcal{C}$ of compact metrizable spaces the structure of a cofibration category [14].
In [2] it was proved that the strong shape category of $\mathcal{C}$ is obtained by localizing at the class Σ of strong shape equivalences, that is $Ssh(\mathcal{C}) = \mathcal{C}[\Sigma^{-1}]$. It then follows that $Ssh(\mathcal{C})$ actually is the homotopy category of a cofibration category.

1. Preliminaries

A category enriched over $\mathbf{Gpd}$ is just a 2-category whose 2-cells are all invertible. The category $\mathbf{Top}$ of topological spaces and continuous maps will be considered with its enrichment over $\mathbf{Gpd}$. Given two spaces X, Y, the groupoid $\mathbf{Gpd}(X, Y)$ has points the maps $X \to Y$ while a path $\alpha : f \to g$ is a track connecting the two maps, that is $\alpha = [H]$ is the relative homotopy class of a homotopy $H : X \times I \to Y$ connecting f to g. It is often called the *track groupoid of Y under X* [1]. $\mathbf{Anr}$ will denote the full subcategory of $\mathbf{Top}$ whose objects are the spaces having the homotopy type of compact absolute neighborhood retracts for metrizable spaces (Anr-spaces).

$\mathbf{Gpd}$ is enriched over itself, the homotopies being the natural isomorphisms of functors. A homomorphism of groupoids is a homotopy equivalence if and only if it is an equivalence of categories.
Every ordinary category can be considered as a category enriched over $\mathbf{Gpd}$ with only identity homotopies.

Both the categories $\mathbf{Top}$ and $\mathbf{Gpd}$ are closed model categories [12], [7] with the following structure:

(a) $\mathbf{Top}$: the weak equivalences are the homotopy equivalences, the fibrations and the cofibrations are the Hurewicz fibrations and the Hurewicz cofibrations.

(b) $\mathbf{Gpd}$: the weak equivalences are the homorphisms that are equivalences of categories, the cofibrations are the homomorphisms that are injective on objects. The fibrations are the homomorphisms $\varphi : G \to H$ having the following *source lifting property* as described by Brown in [1]:

(1.1.1) *for every $x \in G$ and every path $\beta : \varphi(x) \to \bullet$ in H, there exists a path $\alpha : x \to \bullet$ in G such that $\varphi(\alpha) = \beta$.*

For $\mathcal{A}$ a (small) category and $\mathcal{K}$ any 2-category, consider the functor 2-category $[\mathcal{A}, \mathcal{K}]$. If $F, G : \mathcal{A} \to \mathcal{K}$ are 2-functors a 2-natural transformation $\tau : F \Rightarrow G$ is a *level equivalence*, respectively a *level fibration, level cofibration*, if $\tau_A : F(A) \to G(A)$ is an equivalence, respectively a fibration, cofibration in $\mathcal{K}$, for all $A \in \mathcal{A}$ (whatever "equivalence, fibration, cofibration" could mean).
Following [5], the functor category $[\mathcal{A}, \mathbf{Gpd}]$ can be equipped with the so called projective model structure. There the weak equivalences are the level equivalences, the fibrations are the level fibrations and the cofibrations are those natural transformations having the left lifting property with respect to level trivial fibrations.

From now on we denote by $\mathcal{C}$ the category of compact metrizable spaces. Moreover, by $\mathcal{K}$ we mean both a class of topological spaces and the full subcategory of $\mathbf{Top}$ it generates.

Let us recall the following theorem ([10], I.5.2, Thm. 7 and Cor. 4) for later use.

Theorem 1.1. *Every space $X \in \mathcal{C}$ can be represented as the inverse limit of an inverse system $\mathbf{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$ in $\mathbf{Anr}$.*

We refer to [10] for all that concerns inverse systems and the construction of the category $\mathrm{Pro}(\mathbf{Top})$. Let us only recall that $\mathbf{X}$ is a contravariant functor

$\mathbf{X} : \Lambda \to \mathbf{Top}$, where $(\Lambda, \leq)$ is a cofinite, strongly directed set [4], $X_\lambda = \mathbf{X}(\lambda)$ and $x_{\lambda\lambda'} = \mathbf{X}(\lambda \leq \lambda')$.
A morphism $\mathbf{p} : X \to \mathbf{X}$ is a natural cone, that is a family $\mathbf{p} = \{p_\lambda : X \to X_\lambda \mid \lambda \in \Lambda\}$ of maps such that $x_{\lambda\lambda'} \circ p_{\lambda'} = p_\lambda$, for $\lambda \leq \lambda'$.

2. Almost-cofibrations

For each topological space X, the representable (covariant) 2-functor

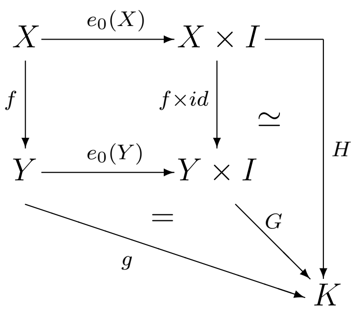
$$\mathbf{Gpd}(X, -) : \mathbf{Top} \to \mathbf{Gpd}$$

sends a space K to the groupoid $\mathbf{Gpd}(X, K)$, a map $f : K \to H$ to the functor $f^*_K = \mathbf{Gpd}(X, f) : \mathbf{Gpd}(X, K) \to \mathbf{Gpd}(X, H)$, $a \mapsto f \circ a$, and a track $\alpha = [H] : f \Rightarrow g : K \to H$ to the natural isomorphism $\mathbf{Gpd}(X, \alpha) : \mathbf{Gpd}(X, f) \Rightarrow \mathbf{Gpd}(X, g)$ induced by α in the evident way.
Let $f : X \to Y$ be a map and let $\mathcal{K} \subset \mathbf{Top}$ be a class of spaces. The natural transformation

$$f^* = \mathbf{Gpd}(f, -) : \mathbf{Gpd}(Y, -) \Rightarrow \mathbf{Gpd}(X, -) : \mathbf{K} \to \mathbf{Gpd}$$

is a *level fibration*, resp. *level equivalence* , if $f^*_K : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$ is a fibration, respectively an equivalence of groupoids, for all $K \in \mathcal{K}$.
The fact that f^*_K is a fibration of groupoids amounts, by (1.1.1), to the following property

(2.1.1) *for every g and H such that $H \circ e_0(X) = g \circ f$, there is a $G : Y \times I \to K$ with $G \circ e_0(Y) = g$ and $G \circ (f \times id) \simeq H$. In diagram*



Such a consideration leads to the following generalization of the classical Homotopy Extension Property (HEP):

Definition 2.1. *A map $f : X \to Y$ has the almost homotopy extension property (RWHEP) with respect to a space K if, for every $g : Y \to K$ and $H : X \times I \to K$ such that $H \circ e_0(X) = g \circ f$, there is a $G : Y \times I \to K$ with $G \circ e_0(Y) = g$ and $G \circ (f \times 1) \simeq H$.*
If $\mathcal{K} \subset \mathbf{Top}$, $f \in RWHEP(\mathcal{K})$ means that f has the RWHEP with respect to all $K \in \mathcal{K}$. If $f \in RWHEP(\mathbf{Top})$, f will be called an almost-cofibration.

Then, it is clear that

Theorem 2.2. *Let $f : X \to Y$ be a map of spaces. The following are equivalent*

(a) $f \in RWHEP(\mathcal{K})$

(b) the homomorphism $f_K^* : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$ is a fibration of groupoids, for all $K \in \mathcal{K}$.

Remark 2.3. We point out that the RWHEP was introduced by R. Brown [1] and also that our almost-cofibrations are called rather weak cofibrations in [8].

Let us recall the following facts:

(a) A map $f : X \to Y$ has the *homotopy extension property* with respect to a class $\mathcal{K}$ of spaces, written $f \in \mathrm{HEP}(\mathcal{K})$, if : given a map $g : Y \to K$, $K \in \mathcal{K}$, and a homotopy $H : X \times I \to K$ starting at $g \circ f$, there is a homotopy $G : Y \times I \to K$ starting at g and such that $G \circ (f \times id) = H$. $f : X \to Y$ is a *(Hurewicz) cofibration* if it has the HEP with respect to all topological spaces. This amounts to the following diagram to be a weak pushout in $\mathbf{Top}$

$$\begin{array}{ccc} X & \xrightarrow{e_0(X)} & X \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times id} \\ Y & \xrightarrow{e_0(Y)} & Y \times I \end{array}$$

Here I denotes the unit interval $[0, 1]$ and $e_0(X) : X \to X \times I$ is the map $e_0(X)(x) = (x, 0)$.

(b) given a map $f : X \to Y$, its *mapping cylinder* $M(f)$ is obtained as the pushout

$$\begin{array}{ccc} X & \xrightarrow{e_0(X)} & X \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_f} \\ Y & \xrightarrow{j_f} & M(f) \end{array}$$

$M(f)$ is then the quotient space of the disjoint union $X \times [0,1] \sqcup Y$ modulo the relation which identifies each point $(x, 0)$ with $f(x)$.

Proposition 2.4. *Let $f : X \to Y$ be any map and $\mathcal{K}$ a class of spaces. If $f \in \mathrm{HEP}(\mathcal{K})$, then the functor $f^*_K = \mathbf{Gpd}(f, K) : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$ is a fibration in $\mathbf{Gpd}$, for all $K \in \mathbf{K}$. In particular, every map having the* HEP*($\mathcal{K}$) has also the* RWHEP*($\mathcal{K}$).*

Proof. See ([1], 7.2.2) and Thm. 3.2. □

The converse implication does not hold in general. In fact: let $A = \{0\} \cup \bigcup_{n=1}^{\infty}\{\frac{1}{n}\}$ and $Z = [0,1] \times \{0\} \cup A \times [0,1]$. Let $f : A \to [0,1]$ be the inclusion and let maps $G : A \times [0,1] \to Z$, $g : [0,1] \to Z$ be defined by the formulas $G(x,t) = (x,t)$ and $g(y) = (y, 0)$ for $(x,t) \in A \times [0,1]$ and $y \in [0,1]$. Then $f \in \mathrm{HEP}(\mathbf{Anr})$ by the classical homotopy extension theorem, but it is not true that $f \in \mathrm{HEP}(\mathcal{C})$. Indeed, a homotopy $F : [0,1] \times [0,1] \to Z$ such that $F(x,t) = G(x,t)$ and $F(y,0) = g(y)$ for $(x,t) \in A \times [0,1]$ and $y \in [0,1]$ would be a retraction of the locally connected continuum $[0,1]$ to a non-locally connected continuum Z.

Proposition 2.5. *Let $f : X \to Y$ be a map of compact metrizable spaces, then:*

(1) the mapping cylinder $M(f)$ is also a compact metrizable space.

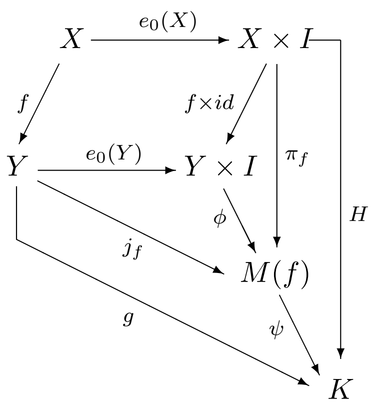
(2) f is a cofibration if and only if $f \in \mathrm{HEP}(\mathcal{C})$.

Proof. (1) $M(f)$ is compact since the category of compact spaces is closed under finite coproducts and quotients. Since every continuous image in a Hausdorff space of a compact metrizable space is metrizable, it suffices to prove that $M(f)$ is Hausdorff. Let $q : X \times [0,1] \sqcup Y \to M(f)$ be the

quotient map and $u, v \in M(f)$. If the two points are both in Y there are disjoint open sets $U \ni u$ and $V \ni v$ in Y. Then $\tilde{U} = q(f^{-1}(U) \times [0,1] \sqcup U)$ and $\tilde{V} = q(f^{-1}(V) \times [0,1] \sqcup V)$ are disjoint open sets in $M(f)$ contaning u and v, respectively. Assume now that $u \in X \times [0,1)$ and $v \in Y$: they have disjoint open neighborhoods given by $X \times [0,t)$, $t < 1$, and $X \times [0,s) \sqcup Y$ for some $0 < s < t$. The case $u, v \in X \times [0,1)$ is obviuos.
(2) Recall that a map is a cofibration if and only if it has the HEP with respect to its mapping cylinder ([9], 2.10). □

Proposition 2.6. *Let $f : X \to Y$ be such that $M(f) \in \mathcal{K}$. Then $f \in RWHEP(\mathcal{K})$ if and only if $f \in RWHEP(M(f))$.*

Proof. Let us consider the following diagram



From the fact that $M(f) \in \mathcal{K}$, there is a map ϕ such that:
- $\phi \circ e_0(Y) = j_f$,
- $\phi \circ (f \times id) \simeq \pi_f$.

Since the middle square is a pushout, there is a map ψ such that:
- $\psi \circ j_f = g$,
- $\psi \circ \pi_f = H$.

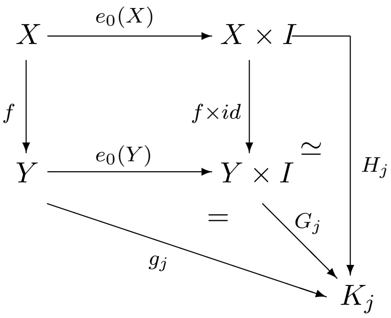
Finally: $(\psi \circ \phi) \circ e_0(Y) = g$ and $(\psi \circ \phi) \circ (f \times id) \simeq H$. □

The next two propositions mark the difference between the HEP and the RWHEP.

Proposition 2.7. *Let $f : X \to Y$. The following are equivalent* *(a)* $f \in RWHEP(\mathcal{K})$

(b) $f \in RWHEP(\text{Pro}(\mathcal{K}))$.

Proof. $(a) \Rightarrow (b)$: let $\mathbf{K} = \{K_j\}_J \in \text{Pro}(\mathcal{K})$ and let $\mathbf{g} : Y \to \mathbf{K}$, $\mathbf{H} : X \times I \to \mathbf{K}$ be such that $\mathbf{H} \circ e_0^Y = \mathbf{g} \circ f$. Since $f \in RWHEP(\mathcal{K})$, there is a diagram

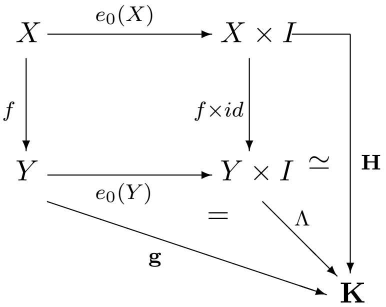


for every $j \in J$. The family of maps $\mathbf{G} = \{G_j : Y \times I \to K_j : j \in J\}$ is a pseudo cone, that is $k_{jj'} \circ G_{j'} \simeq G_j$, for all $j \leq j'$, being $\mathbf{G} \circ e_0(Y) = \mathbf{g}$ and $e_0(Y)$ a homotopy equivalence. Moreover $\mathbf{G} \circ (f \times id) \simeq \mathbf{H}$.
Finally, since $\mathbf{G} \circ e_0(Y) = \mathbf{g}$ is a cone, let us consider the cone

$$\Lambda : Y \times I \xrightarrow{\sigma(Y)} Y \xrightarrow{e_0(Y)} Y \times I \xrightarrow{\mathbf{G}} \mathbf{K}$$

which has the properties:

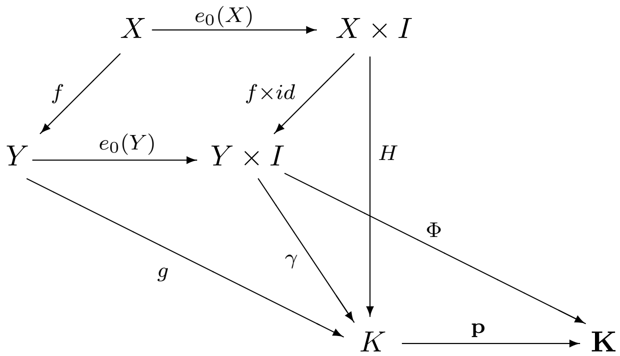
- $\Lambda \circ e_0(Y) = \mathbf{g}$,
- $\Lambda \circ (f \times id) \simeq \mathbf{H}$.



The other direction is clear. □

Proposition 2.8. *Let $f \in RWHEP(\mathcal{K})$ and let $\mathbf{K} = \{K_j\}_{j \in J}$ be an inverse system in $\mathcal{K}$ with inverse limit $\mathbf{p} : K \to \mathbf{K}$. Then $f \in RWHEP(K)$.*

Proof. Let $g : Y \to K,\ H : X \times I \to K$ be such that $H \circ e_0(X) = g \circ f$ and consider the diagram



Since $f \in RWHEP(\text{Pro}(\mathcal{K}))$ (Prop. 2.6), there is a map (cone) $\Phi : Y \times I \to \mathbf{K}$ such that $\Phi \circ e_0(Y) = \mathbf{p} \circ g$ and $\Phi \circ (f \times id) \simeq \mathbf{p} \circ H$. By the universal property of the limit, there is a unique map $\gamma : Y \times I \to K$ with $\Phi = \mathbf{p} \circ \gamma$. Then $\mathbf{p} \circ \gamma \circ e_0(Y) = \mathbf{p} \circ g$, hence $\gamma \circ e_0(Y) = g$.
Note that:

$$\mathbf{p} \circ H \circ e_0(X) = \mathbf{p} \circ g \circ f = \Phi \circ e_0(Y) \circ f =$$

$$= \mathbf{p} \circ \gamma \circ e_0(Y) \circ f = \mathbf{p} \circ \gamma \circ (f \times id) \circ e_0(X),$$

from which it follows

$$H \circ e_0(X) = \gamma \circ (f \times id) \circ e_0(X).$$

Finally: $H \simeq \gamma \circ (f \times id)$, being $e_0(X)$ a homotopy equivalence. □

Theorem 2.9. *Let $f : X \to Y$ be a map of compact metrizable spaces. The following are equivalent*

(a) $f \in RWHEP(\mathbf{Anr})$,

(b) $f \in RWHEP(\mathcal{C})$,

(c) f *is an almost-cofibration.*

Proof. $(a) \Rightarrow (b)$: this follows from Thm. 1.1 and Prop. 2.7. $(b) \Rightarrow (c)$: follows from Prop. 2.5. □

Our main result here is the following

Theorem 2.10. *Every inclusion $i : B \to X$ of a closed set in a compact metrizable space is an almost-cofibration.*

Proof. By the Borsuk's homotopy extension theorem $i : B \to X$ has the HEP($\mathbf{Anr}$), hence also the $\mathrm{RWHEP}(\mathbf{Anr})$. From Theorem 2.8 the assertion follows. □

3. The Homotopy Structure

Definition 3.1. *[11] A map $f : X \to Y$ is a strong shape equivalence if it fulfills the following requirements:*

(ss1) for each map $g : X \to K$, $K \in \mathbf{Anr}$, there is a map $h : Y \to K$ such that $h \circ f \simeq g$,

(ss2) if $h_1, h_2 : Y \to K$ are given maps and $G : X \times I \to K$ is a homotopy $G : h_1 \circ f \simeq h_2 \circ f$, then there is a homotopy $H : Y \times I \to K$, $H : h_1 \simeq h_2$, such that G and $H \circ (f \times 1)$ are homotopic rel end maps.

Since the homotopy H in (ss2) is uniquely determined up to homotopies rel end maps ([3], Prop.1.2), it follows at once that $f : X \to Y$ is a strong shape equivalence whenever the natural transformation

$$\mathbf{Gpd}(f, -) : \mathbf{Gpd}(Y, -) \Rightarrow \mathbf{Gpd}(X, -) : \mathcal{A} \to \mathbf{Gpd}$$

is a level equivalence, that is the functors of groupoids

$$\mathbf{Gpd}(f, K) : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$$

are all equivalences of categories, for all $K \in \mathbf{Anr}$.
Every homotopy equivalence is a strong shape equivalence.

Definition 3.2. *[14] A cofibration category is a category $\mathcal{E}$ equipped with two classes of morphisms Σ and Γ called weak equivalences and cofibrations, respectively, such that the following axioms are satisfied.*

(1) Weak equivalences satisfy the 2-out-of-6 property, i.e., if f, g, h are composable morphisms of $\mathcal{E}$ such that both gf and hg are weak equivalences, then so are f, g and h.

(2) Every isomorphism of $\mathcal{E}$ is an acyclic cofibration.

(3) $\mathcal{E}$ has an initial object, denoted 0.

(4) Every object $X \in \mathcal{E}$ is cofibrant, that is the unique morphism $0 \to X$ is a cofibration.

(5) (Trivial) Cofibrations are stable under pushouts along arbitrary morphisms of $\mathcal{E}$. A trivial cofibrations is morphisms in $\Sigma \cap \Gamma$.

(6) Every morphism of $\mathcal{E}$ factors as a composite of a cofibration followed by a weak equivalence.

In the category $\mathcal{C}$ let us denote Σ = the class of strong shape equivalences and Γ = the class of almost-cofibrations.

Theorem 3.3. $(\mathcal{C}, \Sigma, \Gamma)$ *is a cofibration category.*

Proof. (1) Let

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

be morphisms of $\mathcal{C}$ such that both $g \circ f$ and $h \circ g$ are strong shape equivalences. Then, for every $K \in \mathbf{Anr}$, we have that in

$$\mathbf{Gpd}(Z, K) \xrightarrow{h_K^*} \mathbf{Gpd}(Y, K) \xrightarrow{g_K^*} \mathbf{Gpd}(X, K) \xrightarrow{f_K^*} \mathbf{Gpd}(W, K)$$

both $f_K^* \circ g_K^*$ and $g_K^* \circ h_K^*$ are equivalences in $\mathbf{Gpd}$. Since $\mathbf{Gpd}$ is a model category and every model category has the 2-out-of-6 property, the assertion follows.
(2), (3), (4) and (6) are obvious.
(5) Let

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ A & \xrightarrow[\bar{f}]{} & A \bigsqcup_B Y \end{array}$$

be a pushout in $\mathcal{C}$ with i a (trivial) almost-cofibration. For all $K \in \mathcal{C}$, we get a pullback in $\mathbf{Gpd}$

$$\begin{array}{ccc} \mathbf{Gpd}(A \bigsqcup_B Y, K) & \xrightarrow{f'_*} & \mathbf{Gpd}(A, K) \\ {\scriptstyle \bar{i}_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ \mathbf{Gpd}(Y, K) & \xrightarrow[f_*]{} & \mathbf{Gpd}(B, K) \end{array}$$

with i_* a (trivial) fibration in $\mathbf{Gpd}$. Since (trivial) fibration are stable under pullbacks in the model structure of $\mathbf{Gpd}$, it follows that $\bar{i}_*$ is a (trivial) fibration in $\mathbf{Gpd}$, from which it follows that $\bar{i}$ is a (trivial) almost-cofibration.

□

The strong shape category of compact metrizable spaces is obtained formally inverting the class of strong shape equivalences $SSh(\mathcal{C}) = \mathcal{C}[\Sigma^{-1}]$ [2]. The previous theorem says that it can be represented as the homotopy category of the cofibration category $(\mathcal{C}, \Sigma, \Gamma)$.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
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THE TOPOLOGY OF CRITICAL PROCESSES, IV (THE HOMOTOPY STRUCTURE)

Marco GRANDIS

Résumé. La Topologie Algébrique Dirigée étudie des espaces équipés d'une forme de direction, avec l'objectif d'inclure les processus non réversibles. Dans l'extension présente nous voulons couvrir aussi les *processus critiques*, indivisibles et inarrêtables, du changement d'état dans une cellule de mémoire à l'action d'un thermostat.

Les parties précédentes de cette série ont introduit les *espaces contrôlés*, en examinant comment il peuvent modeler les processus critiques issus de divers domaines, et étudié leur catégorie fondamentale. Ici on traite leur structure formelle d'homotopie.

Abstract. Directed Algebraic Topology studies spaces equipped with a form of direction, to include models of non-reversible processes. In the present extension we also want to cover 'critical processes', indecomposable and unstoppable – from the change of state in a memory cell to the action of a thermostat.

The previous parts of this series introduced *controlled spaces*, examining how they can model critical processes in various domains, and studied their fundamental category. Here we deal with their formal homotopy theory.

Keywords. Directed algebraic topology, homotopy theory, fundamental category, concurrent process.

Mathematics Subject Classification (2010). 55M, 55P, 55Q, 68Q85.

Introduction

0.1 Directed and controlled spaces

Directed Algebraic Topology is an extension of Algebraic Topology, dealing with 'spaces' where the paths need not be reversible; the general aim is including the representation of *irreversible processes*. The category $\mathrm{d}\mathsf{Top}$ of directed spaces, or d-spaces [G2], is a typical setting for this study; it is frequently employed in the theory of concurrency: see [FGHMR]. Homotopy in $\mathrm{d}\mathsf{Top}$ is based on the standard directed interval $\uparrow\mathbb{I}$, whose paths are the (weakly) increasing continuous mappings $[0, 1] \to [0, 1]$. It is the basic model of a non-reversible process, or a one-way road.

This article belongs to a series devoted to a further extension, where the paths can also be non-decomposable in order to include *critical processes*, indivisible and unstoppable – either reversible or not. For instance: quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, the action of a siphon, moving in a no-stop road, etc.

Part I [G3] introduced the category $\mathrm{c}\mathsf{Top}$ of controlled spaces, or c-spaces, examining how they can model critical processes; the definition is reviewed in 1.1. The previous setting of d-spaces is embedded in $\mathrm{c}\mathsf{Top}$ as the full, reflective and coreflective subcategory of 'flexible' c-spaces. Homotopy in $\mathrm{c}\mathsf{Top}$ is based on the standard controlled interval $\mathrm{c}\mathbb{I}$, whose paths are the increasing continuous mappings $[0, 1] \to [0, 1]$ which are either surjective or constant at 0 or 1. It is the basic model of a non-reversible unstoppable process, or a one-way no-stop road.

Parts II and III [G4, G5] introduced and studied the fundamental category of controlled spaces, as a functor

$$\uparrow\Pi_1 \colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \qquad (1)$$

which extends the functor $\uparrow\Pi_1 \colon \mathrm{d}\mathsf{Top} \to \mathsf{Cat}$ studied in [G2].

The extension is not obvious, essentially because the cylinder functor of $\mathrm{c}\mathsf{Top}$ does not preserve pushouts. The problem was overcome with a hybrid use of 'general' and 'flexible' homotopies, and some new methods of computation. Covering maps work well ([G4], Theorem 5.8), but the van Kampen theorem for the fundamental category of d-spaces cannot be extended, as it is based on the subdivision of paths.

Here we study the homotopy theory of c-spaces, including homotopy equivalences, homotopy constructions like cones and suspension, and the relationship with cubical sets.

0.2 Outline

In Section 1 we analyse the formal theory of homotopy in cTop, following the classification of directed settings in [G2]: it is a symmetric dI2-category with concatenation pushout of the interval (to concatenate paths), no concatenation pushout of the cylinder functor (homotopies cannot be concatenated), and no path functor (the cylinder functor has no right adjoint). For comparison, dTop is a dIP4-homotopical category, with far stronger properties. The breaking of symmetries and 'extended symmetries' in cTop is examined in 1.6. In 1.7 and 1.8 we review the basic elements of homotopy theory in Cat.

Composite homotopies of c-spaces are introduced in Section 2, together with forms of directed homotopy equivalence, contractibility and connectedness adequate to the present setting.

Sections 3 and 4 deal with homotopy pushouts, cones and suspension. Weak flexibility properties (see 3.1) are used to counteract the fact that pushouts are not preserved by the cylinder functor.

The flexible interval $\uparrow\mathbb{I}$ (the basis of homotopy for d-spaces) produces a second homotopy structure for c-spaces, denoted as cTop_F and examined in Section 5. In fact, we have already seen in Part II (Theorems 5.4 and 5.5) that the fundamental category functor is invariant up to flexible homotopy.

Finally, Section 6 is about cubical sets. A cubical set has diverse geometric realisations: as a topological space (the classical realisation, pasting topological cubes $\mathbb{I}^n$), as a directed space (pasting directed cubes $\uparrow\mathbb{I}^n$), and as a controlled space (pasting controlled cubes $\mathrm{c}\mathbb{I}^n$). All this can be combined: a cubical set labelled in cTop comprises all these instances and their aggregations (see 6.7).

0.3 An overview of this series

(a) Reviewing our aims, Part I explores how the new controlled spaces can model concrete, critical processes and their interaction with continuous variation.

Controlled spaces can thus unify aspects of continuous and discrete mathematics. Moreover, they can interpret phenomena of diverse domains in a single system of mathematical models, which can be combined together and studied with extensions of the usual tools of Algebraic Topology.

(b) Following this program, Parts II and III introduce and investigate the fundamental category of controlled spaces. This is used, in particular, to classify obstructions in problems related to concurrency (Part III, Section 2).

Now, the fundamental category studies *controlled paths* up to *flexible homotopy*. Homotopy properties are more involved here than in the tamer world of directed spaces, and we have to blend the standard homotopy structure, based on the controlled interval $\mathrm{c}\mathbb{I}$, with the flexible one, based on the directed interval $\uparrow\!\mathbb{I}$.

(c) The present part aims to make clear the role of these homotopy structures on the category of controlled spaces, in the general frame of directed homotopy built in the book [G2]. In our opinion, the interest and beauty of peculiar structures, like controlled cones and spheres (in Section 4), might be a sufficient impulse to study their formal world. Of course, this study is not concluded here.

In a marginal way, we can add that labelled cubical sets, considered in 6.7, might be used to model traffic networks where roads (possibly one-way, or no-stop, or including delays) interact with planar areas, e.g. parking lots and desert lands; or rivers and canals interact with lakes and sees.

0.4 Terminology and notation

A continuous mapping between topological spaces, possibly structured, is called a *map*. $\mathbb{R}$ denotes the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0,1]$. Similarly $\mathbb{R}^n$ and $\mathbb{I}^n$ are euclidean spaces. $\mathbb{S}^n$ is the n-dimensional sphere. The open and semiopen intervals of the real line are always denoted by square brackets, like $]0,1[$, $[0,1[$, etc.

The symbol $\subset$ denotes weak inclusion. The binary variable α takes values $0,1$, also written, respectively, as $-,+$ in superscripts and subscripts. Marginal remarks are written in small characters.

The previous papers [G3, G4, G5] of this series are cited as Part I, Part II and Part III, respectively. The reference I.2, or II.3, or III.1.4, points to Section 2 of Part I, or Section 3 of Part II, or Subsection 1.4 of Part III.

0.5 Acknowledgements

The author is grateful to the Referee for an exceptionally extensive, accurate report, raising many interesting points about motivation, clarity and possible developments.

1. The standard homotopy structure of controlled spaces

We examine the formal homotopy structure of c-spaces, following the classification of such structures in [G2].

The component $\lambda X\colon F(X) \to G(X)$ of a natural transformation between functors is often written as $\lambda\colon FX \to GX$.

1.1 Controlled spaces

The category $\mathrm{c}\mathsf{Top}$ of *controlled spaces*, or *c-spaces*, was introduced in Part I [G3]; we briefly recall the main definitions.

An object X is a topological space equipped with a set $X^\sharp$ of continuous mappings $a\colon [0,1] \to X$, called *controlled paths*, or *c-paths*, that satisfies three axioms:

(csp.0) (*constant paths*) the trivial loops at the endpoints of a controlled path are controlled,

(csp.1) (*concatenation*) the concatenation of consecutive controlled paths is controlled,

(csp.2) (*global reparametrisation*) the reparametrisation $a\rho$ of a controlled path a by a surjective (weakly) increasing map $\rho\colon [0,1] \to [0,1]$ is controlled.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths.

The reversor functor $R\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ of the category of c-spaces sends X to the opposite object $RX = X^{\mathrm{op}}$, with reversed selected paths.

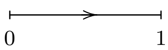
The category $\mathrm{c}\mathsf{Top}$ contains the category $\mathrm{d}\mathsf{Top}$ of d-spaces (studied in [G2]) as a full subcategory, reflective and coreflective: a c-space is a d-space if and only if it is *flexible*, which means that each point is flexible (its trivial loop is controlled) and every controlled path is flexible (all its restrictions are controlled). Both categories are complete and cocomplete, with limits (resp. colimits) calculated as in Top and enriched with initial (resp. final) structures. The terminal object is the singleton $\{*\}$ of $\mathrm{d}\mathsf{Top}$, also called the flexible singleton when viewed in $\mathrm{c}\mathsf{Top}$.

The reflector $\mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$ (cf. I.1.8, that is Section 1.8 of Part I) takes a c-space X to the *generated d-space* $\hat{X}$, with the same underlying topological space and the d-structure generated by the c-paths; the unit of the adjunction is the *reshaping* $X \to \hat{X}$, whose underlying map is the identity. The coreflector takes X to the *flexible part* $\mathsf{Fl}\,X$, namely the subspace of flexible points $|X|_0$ (called the *flexible support*) with the d-structure of the flexible c-paths; the counit is the inclusion $\mathsf{Fl}\,X \to X$.

1.2 Structured intervals and lines

(a) In $\mathrm{d}\mathsf{Top}$ the *standard d-interval* $\uparrow\!\mathbb{I}$ has the d-structure generated by the identity $\mathrm{id}\,\mathbb{I}$: the directed paths are all the increasing maps $\mathbb{I} \to \mathbb{I}$. It plays the role of the standard interval in this category, because the directed paths of any d-space X coincide with the d-maps $\uparrow\!\mathbb{I} \to X$.

It may be viewed as an essential model of a non-reversible process, or a one-way road in transport networks. It is represented as

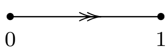


$\uparrow\!\mathbb{I}$ (2)

Similarly, the directed line $\uparrow\!\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$.

(b) In $\mathrm{c}\mathsf{Top}$ the *standard c-interval* $\mathrm{c}\mathbb{I}$, or *one-jump interval*, has the same support, with the c-structure generated by the identity $\mathrm{id}\,\mathbb{I}$: the controlled paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at 0 or 1. The controlled paths of any c-space X coincide with the c-maps $\mathrm{c}\mathbb{I} \to X$.

It can model a *non-reversible unstoppable process*, or a *one-way no-stop road*. It is represented as



$\mathrm{c}\mathbb{I}$ (3)

marking by a bullet the isolated flexible points: here the endpoints of the interval. The *controlled line* $\mathrm{c}\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$ whose image is an interval $[k, k']$ with integral endpoints.

(c) The interval $\uparrow\mathbb{I}$ is also used in cTop, as the *flexible interval*. Flexible paths of c-spaces and flexible homotopies of c-maps are parametrised on it, as c-maps $\uparrow\mathbb{I} \to X$ and $X \times \uparrow\mathbb{I} \to Y$, respectively (see II.4.1).

(d) We also recall that, in the fundamental category $\uparrow\Pi_1(X)$ of a c-space X (introduced in II.5.1)

- the vertices are the flexible points of X,

- the arrows are equivalence classes $[a]\colon x \to y$ of c-paths $a\colon x \to y$; the equivalence relation is generated by flexible homotopies $\varphi\colon a' \to a''$ with fixed endpoints, which are c-maps $\varphi\colon \mathrm{c}\mathbb{I} \times \uparrow\mathbb{I} \to X$ defined on the *hybrid square* $\mathrm{c}\mathbb{I} \times \uparrow\mathbb{I}$.

1.3 The standard dI2-structure

In cTop the standard interval $\mathrm{c}\mathbb{I}$ has a structure formed of the following operations.

The first-order part was already seen in II.4.1: two faces ∂^α $(\alpha = \pm)$, a degeneracy e and a reflection r. We also have a second-order part which involves the standard square $\mathrm{c}\mathbb{I}^2 = \mathrm{c}\mathbb{I} \times \mathrm{c}\mathbb{I}$: two *connections*, or main operations g^α (already used in Part II) and a *transposition* s

$$\{*\} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \mathrm{c}\mathbb{I} \overset{g^\alpha}{\leftleftarrows} \mathrm{c}\mathbb{I}^2 \qquad r\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\mathrm{op}}, \quad s\colon \mathrm{c}\mathbb{I}^2 \to \mathrm{c}\mathbb{I}^2, \tag{4}$$

$$\partial^\alpha(*) = \alpha, \qquad g^-(t,t') = \max(t,t'), \qquad g^+(t,t') = \min(t,t'),$$
$$r(t) = 1 - t, \qquad\qquad s(t,t') = (t',t).$$

As a consequence, the (standard) cylinder endofunctor

$$I_c = - \times \mathrm{c}\mathbb{I}\colon \mathrm{cTop} \to \mathrm{cTop},$$

written as I if it is clear that we are working in cTop, has natural transformations, written as above

$$1 \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} I \overset{g^\alpha}{\leftleftarrows} I^2 \qquad r\colon IR \to RI, \qquad s\colon I^2 \to I^2, \tag{5}$$

that satisfy the following equations

$$
\begin{array}{ll}
e\partial^\alpha = 1, \qquad eg^\alpha = e(Ie) = e(eI) & (degeneracy), \\
g^\alpha(Ig^\alpha) = g^\alpha(g^\alpha I) & (associativity), \\
g^\alpha(I\partial^\alpha) = 1 = g^\alpha(\partial^\alpha I) & (unitarity), \\
g^\beta(I\partial^\alpha) = \partial^\alpha e = g^\beta(\partial^\alpha I) & (absorbency, \text{ for } \alpha \neq \beta), \\
(RrR)r = 1, \qquad (Re)r = eR, & \\
r(\partial^+ R) = R\partial^-, \qquad r(g^+ R) = (Rg^-)r_2 & (reflection), \\
ss = 1, \qquad (Ie)s = eI, \qquad s(I\partial^\alpha) = \partial^\alpha I, & \\
(Rs)r_2 = r_2(sR), \qquad g^\alpha s = g^\alpha & (transposition),
\end{array}
\tag{6}
$$

where $r_2 = (rI)(Ir)\colon I^2R(X) \to RI^2(X)$ is the double reflection, namely: $r_2(x, t, t') = (x, -t, -t')$.

According to a classification of homotopy structures defined by a cylinder endofunctor, the category $\mathrm{c}\mathsf{Top}$, equipped with the functors I, R and the previous operations, is a *symmetric dI2-category* ([G2], 4.2): the previous equations are the axioms of this structure. Moreover, the existence of the terminal object and pushouts makes $\mathrm{c}\mathsf{Top}$ into a *dI1-homotopical category* ([G2], 1.7.0).

The present structure is made *concrete* fixing the flexible singleton $\{*\}$ as the standard point ([G2], 1.2.4). The c-space $\mathrm{c}\mathbb{I} = I(\{*\})$ is a symmetric dI2-interval for the cartesian product ([G2], 4.2.8).

1.4 Higher properties

(a) The category $\mathrm{d}\mathsf{Top}$ of directed spaces has a far richer structure. In particular the cylinder functor $I_d = -\times{\uparrow}\mathbb{I}$ preserves all pushouts (which allows the concatenation of homotopies) and has a right adjoint, the cocylinder functor $P_d = (-)^{{\uparrow}\mathbb{I}}$, or path functor. Adding the previous operations and others, we have a *symmetric dIP4-homotopical category* ([G2], 4.2.6).

The homotopy structure of Cat, reviewed in 1.7, is also of this kind.

(b) The category Top of topological spaces has a *reversible* structure of this kind: the reversor endofunctor is the identity.

The same holds for the category $\mathrm{Ch}_\bullet(\mathsf{D})$ of unbounded chain complexes on an additive category, or $\mathrm{Ch}_+(\mathsf{D})$ of positive chain complexes on an additive category with kernels ([G2], 4.4).

(c) On the other hand, the classification of $\mathrm{c}\mathsf{Top}$ as a symmetric dI2-category and a dI1-homotopical category cannot be improved, essentially because pushouts are not preserved by the cylinder functor.

(i) It is not a *dI3-category* ([G2], 4.2.2): the concatenation pushout $JX = IX +_X IX$, in particular, is not preserved by the functor I, as remarked for $X = \{*\}$ and the two-jump interval $J(\{*\}) = \mathrm{c}\mathbb{J}$, in II.4.7(c).

(ii) It is not a *dIP1-category* ([G2], 1.2.2): the cylinder functor $- \times \mathrm{c}\mathbb{I}$ has no right adjoint, by the same reason.

(iii) We shall see in 3.8(b) that $\mathrm{c}\mathsf{Top}$ is not a *symmetric dI1-homotopical category* (as defined in [G2], 4.1.4): the cylinder I does not preserve the 'cylindrical colimits' that produce the homotopy pushouts.

This makes the use of pushouts in $\mathrm{c}\mathsf{Top}$ complicated, as we have already seen in Part II for the construction of the fundamental category, and will also see in the present study of homotopy equivalences, homotopy pushouts, cones and suspension. Flexibility properties will be used, to link these constructions to the more regular ones of $\mathrm{d}\mathsf{Top}$.

1.5 The splitting property

The *standard concatenation pushout* – pasting two copies of the standard interval, one after the other – is realised in $\mathrm{c}\mathsf{Top}$ as $\mathrm{c}\mathbb{J}$, the two-jump structure on the euclidean interval $[0, 1]$, generated by the paths c^-, c^+ (see II.4.2)

$$\begin{array}{ccc} \{*\} & \xrightarrow{\partial^+} & \mathrm{c}\mathbb{I} \\ {\scriptstyle \partial^-}\downarrow & & \downarrow{\scriptstyle c^-} \\ \mathrm{c}\mathbb{I} & \xrightarrow[c^+]{} & \mathrm{c}\mathbb{J} \end{array} \qquad\qquad \begin{array}{l} c^-(t) = t/2, \\ c^+(t) = (t+1)/2. \end{array} \qquad (7)$$

Adding the *concatenation map* $\kappa\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J}$ (a *reshaping*, $\mathrm{c}\mathbb{I}$ being finer than $\mathrm{c}\mathbb{J}$, cf. I.1.7), the regular concatenation of two consecutive c-paths $a', a''\colon \mathrm{c}\mathbb{I} \to X$ is expressed as

$$a' * a'' = a\kappa\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J} \to X \qquad\qquad (ac^- = a', \;\; ac^+ = a''). \qquad (8)$$

This procedure is frequent in homotopy theory. For instance, chain complexes have a similar behaviour: pasting two copies of the interval (or a cylinder) yields a different object, related to the former by a (non-invertible) concatenation map. The same happens in Cat (see 1.7) and for cubical sets (see Section 6).

On the other hand, in dTop (and Top) the standard concatenation pushout can be realised as the interval itself, letting κ be the identity map (see II.3.2), so that every path a is the concatenation of two paths, uniquely determined as ac^- and ac^+. We can introduce the term of a *splittable homotopy structure* to express the property that the concatenation map is invertible.

1.6 Breaking the symmetries of classical topology

A topological space X has 'intrinsic symmetries', which act on its singular cubes $\mathbb{I}^n \to X$.

They are generated by the standard reversion r and the standard transposition s

$$r\colon \mathbb{I} \to \mathbb{I}, \qquad r(t) = 1 - t, \qquad\qquad s\colon \mathbb{I}^2 \to \mathbb{I}^2, \qquad s(t,t') = (t',t). \tag{9}$$

Their n-dimensional versions

$$\begin{aligned} r_i &= \mathbb{I}^{i-1} \times r \times \mathbb{I}^{n-i}\colon \mathbb{I}^n \to \mathbb{I}^n \qquad\qquad (i = 1, ..., n), \\ s_i &= \mathbb{I}^{i-1} \times s \times \mathbb{I}^{n-i-1}\colon \mathbb{I}^n \to \mathbb{I}^n \qquad (i = 1, ..., n-1), \end{aligned} \tag{10}$$

span the group of symmetries of the n-cube, namely the hyperoctahedral group $(\mathbb{Z}/2)^n \rtimes S_n$ (a semidirect product): the reversions r_i commute with each other and generate the first factor, while the transpositions s_i generate the symmetric group S_n. This group acts on the set of n-cubes $\mathbb{I}^n \to X$.

Topological spaces have thus both kinds of symmetries. Directed algebraic topology allows one to break the first kind, and also the second in some settings.

(a) *Reversion*. The prime effect of the reversion $r\colon \mathbb{I} \to \mathbb{I}$ is reversing the paths, in any topological space. This map also gives the reversion of homotopies, by the reversion $\mathrm{id} \times r$ of the cylinder functor $I = - \times \mathbb{I}\colon$ Top $\to$ Top.

Controlled spaces (as well as preordered spaces and directed spaces) lack a reversion, replaced by a *reflection pair* (R, r) consisting of the reversor

$R\colon \mathrm{cTop} \to \mathrm{cTop}$ and a reflection $r\colon IR \to RI$ for the cylinder functor. This behaviour is shared by all the structures for directed homotopy considered in [G2]; the reversible case is a particular instance, with R the identity functor.

(b) *Transposition.* Coming back to topological spaces, the transposition $s(t, t') = (t', t)$ of the standard square $\mathbb{I}^2$ yields the transposition symmetry of the iterated cylinder functor $I^2 = - \times \mathbb{I}^2\colon \mathsf{Top} \to \mathsf{Top}$ ([G2], 1.1.1).

This second-order symmetry, acting on I^2, also exists in cTop (as we have seen above), pTop and dTop, but does not exist in other directed structures, e.g. for cubical sets (see 6.3, 6.8). *Its role, within directed algebraic topology, is double-edged.* On the one hand, its presence has an important consequence, the homotopy invariance of the cylinder functor – as proved in (18). On the other hand, it restricts the interest of directed homology, preventing a good relation of the latter with suspension ([G2], Section 2.2).

In fact the (pre)ordered group $\uparrow\! H_1(\uparrow\! \mathbb{S}^1) = \uparrow\! \mathbb{Z}$ of directed homology has the canonical order, while $\uparrow\! H_2(\uparrow\! \mathbb{S}^2)$ only gets the chaotic preorder. Essentially, we cannot reverse the d-path that generates the former, but we can transpose the d-square that generates the latter, so that its homology class and the opposite are both weakly positive. (In a cubical set there is a finer control of cubes, see 6.8.)

(c) *Restriction.* Finally, we already remarked that the present extension to c-spaces breaks a flexibility feature of d-spaces: paths can no longer be subdivided.

Formally, this can be traced back to the action on paths of the monoid of restrictions (in I.1.2), formed of the affine endomaps

$$\rho\colon \mathbb{I} \to \mathbb{I} \qquad \rho(t) = (t_2 - t_1)t + t_1 \qquad (0 \leqslant t_1 < t_2 \leqslant 1). \tag{11}$$

In Top, pTop and dTop these are endomaps of the standard interval ($\mathbb{I}$ or $\uparrow\!\mathbb{I}$), and act on any path restricting it to the subinterval $[t_1, t_2]$ (reparametrised on $[0, 1]$). In cTop they are not endomaps of $\mathrm{c}\mathbb{I}$: path-restriction is prevented in the standard homotopy structure (but allowed in the flexible structure examined in Section 5).

One might say that there is now a breaking of ‘extended symmetries’, forming a monoid instead of a group.

1.7 Categories and directed homotopy

The fundamental category functor $\uparrow\Pi_1\colon \mathrm{cTop} \to \mathsf{Cat}$ takes flexible homotopies of c-spaces (and homotopies of d-spaces) to directed homotopies in Cat, the cartesian closed category of small categories (cf. Theorem II.5.4, and Proposition 5.3 here). We briefly review here the homotopy structure of Cat – a symmetric dIP4-homotopical category (cf. [G2], 4.3.2).

The *reversor* functor R takes a small category to the opposite one

$$R\colon \mathsf{Cat} \to \mathsf{Cat}, \qquad R(X) = X^{\mathrm{op}}, \tag{12}$$

where X^{op} has precisely the same objects as X, with $X^{\mathrm{op}}(x, y) = X(y, x)$ and the opposite composition, so that R is strictly involutive.

(a) The *standard homotopy structure* of Cat is based on the cartesian product and the *directed interval* $\uparrow\mathbf{i} = \mathbf{2} = \{0 \to 1\}$, an ordinal. Its cartesian powers $\mathbf{2}^n$ are ordered sets (viewed as categories); the standard point is the terminal category $\mathbf{2}^0 = 1$.

Faces, degeneracy, reflection, connections and transposition are order preserving mappings (and functors)

$$\mathbf{1} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \mathbf{2} \overset{g^\alpha}{\longleftarrow} \mathbf{2}^2 \qquad r\colon \mathbf{2} \to \mathbf{2}^{\mathrm{op}}, \qquad s\colon \mathbf{2}^2 \to \mathbf{2}^2, \tag{13}$$

defined by the same formulas as in (4).

A *point* $x\colon \mathbf{1} \to X$ of the small category X is an object of the latter. A (directed) *path* $a\colon \mathbf{2} \to X$ from x to x' is an arrow $a\colon x \to x'$ of X, their concatenation is the composition, strictly associative and unitary.

The concatenation pushout gives here the ordinal category $\uparrow\mathbf{j} = \mathbf{3}$. The obvious concatenation map $\kappa\colon \mathbf{2} \to \mathbf{3}$, $\kappa(0 \to 1) = (0 \to 2)$ is not invertible: this homotopy structure is not splittable.

The (directed) *cylinder* functor $IX = X \times \mathbf{2}$ has a right adjoint: $PY = Y^{\mathbf{2}}$, the category of morphisms of Y. A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, represented by a functor $X \times \mathbf{2} \to Y$ or equivalently $X \to Y^{\mathbf{2}}$, is the same as a natural transformation between the functors f, g. Given two parallel arrows $a, b\colon x \to x'$ in X, a homotopy with fixed endpoints $a \to b$ is a commutative square $\mathbf{2} \times \mathbf{2} \to X$ with trivial vertical edges and *equal* horizontal edges $a = b$. The fundamental category of X is the category X itself. (This homotopy structure is 'one-dimensional', akin to that of 1-cubical sets.)

The operations of homotopies (vertical composition and whisker composition) coincide with the 2-categorical structure of Cat, which implies that the homotopy structure is *2-regular*, in the sense of [G2], 4.2.3.

A *future homotopy equivalence* $(f, g; \varphi, \psi)$ between small categories X, Y ([G2], 3.3.1) is a four-tuple of functors and natural transformations (that need not be invertible)

$$f\colon X \rightleftarrows Y : g, \qquad \varphi\colon \mathrm{id} \to gf, \qquad \psi\colon \mathrm{id} \to fg. \tag{14}$$

'Directed homotopy equivalence' in Cat is studied in [G2], Chapter 3, combining the future and past homotopy equivalences of categories in complex forms, aiming to classify the fundamental categories of directed spaces; some examples were recalled in II.7. (Different homotopy structures on Cat are studied or cited in [Mi].)

(b) Ordinary equivalence of categories is a stricter, far simpler notion, based on the *reversible homotopy structure* $\mathsf{Cat}_{\mathbf{i}}$ produced by the *reversible interval* $\mathbf{i}$. The latter is the indiscrete groupoid on two objects, formed by an isomorphism $u\colon 0 \to 1$ and its inverse

$$\mathbf{i} = \{0 \rightleftarrows 1\}, \qquad r\colon \mathbf{i} \to \mathbf{i}, \qquad r(u) = u^{-1}, \tag{15}$$

with the obvious reversion r, defined above. This gives a *reversible cylinder* functor $X \times \mathbf{i}$, with right adjoint $Y^{\mathbf{i}}$ (the full subcategory of $Y^{\mathbf{2}}$ whose objects are the isomorphisms of Y); a *reversible homotopy* $\varphi\colon f \to g\colon X \to Y$ is the same as a natural isomorphism of functors.

(c) Both structures on Cat give the same homotopies for groupoids, represented by the restriction of the cylinder (or cocylinder) of $\mathsf{Cat}_{\mathbf{i}}$ to the full subcategory Gpd of small groupoids.

1.8 Coherent homotopy equivalence of categories

A future homotopy equivalence $(f, g; \varphi, \psi)$ in Cat is said to be *coherent*, or a *future equivalence* ([G2], 3.3.1), if it satisfies the following *coherence* *conditions*:

$$\begin{gathered} f\colon X \rightleftarrows Y : g, \qquad \varphi\colon \mathrm{id}\,X \to gf, \qquad \psi\colon \mathrm{id}\,Y \to fg, \\ f\varphi = \psi f\colon f \to fgf, \qquad \varphi g = g\psi\colon g \to gfg. \end{gathered} \tag{16}$$

This structure plays a primary role in [G2]. It can be seen as a symmetric version of an adjunction, with two units (although f and g do not determine each other). Its dual, a *past equivalence*, has two counits.

Future equivalences compose, in the same way as adjunctions ([G2], 3.3.3), and give an equivalence relation between small categories. Two categories are future equivalent if and only if they can be embedded as full reflective subcategories of a common one ([G2], Theorem 3.3.5).

A property is invariant for future equivalences if and only if it is preserved by full reflective embeddings and by their reflectors.

All this works because the homotopy structure of Cat is 2-regular, as remarked above. On the other hand, these coherence conditions are generally too strong for the homotopy equivalence of topological spaces (or d-spaces), and only one of them is required for strong deformation retracts.

Here we shall make a limited use of coherent future homotopy equivalences of c-spaces: see 2.6.

2. Homotopy equivalence and connectedness of c-spaces

After reviewing the homotopies of c-spaces, from II.4, we introduce formal composite homotopies to cover their lack of a vertical composition. Then we study forms of directed homotopy equivalence and connectedness adequate to the present setting.

2.1 Composing homotopies

Homotopies of controlled spaces, and their structure, are defined by the cylinder functor and the structure examined above.

(a) As we have already said in II.4.1, a (standard, or general) homotopy $\varphi\colon f \to g\colon X \to Y$ of c-spaces is represented by a map $\varphi\colon IX \to Y$, with faces $f = \varphi\partial^-$ and $g = \varphi\partial^+$. The representative map is written as $\hat{\varphi}\colon IX \to Y$ when useful. The degeneracy map $e\colon IX \to X$ gives the trivial homotopy $0_f\colon f \to f\colon X \to Y$, represented by $fe\colon IX \to Y$.

The reflection $r\colon IR \to RI$ transforms a homotopy $\varphi\colon f \to g\colon X \to Y$ into the *reflected* one

$$\varphi^{\rm op}\colon g^{\rm op} \to f^{\rm op}\colon X^{\rm op} \to Y^{\rm op}, \quad (\varphi^{\rm op})\hat{} = R(\hat{\varphi})r\colon IRX \to RY. \qquad (17)$$

The transposition $s\colon I^2 \to I^2$ makes the cylinder functor $I\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ homotopy invariant: the homotopy φ can be transformed into a homotopy $I\varphi\colon If \to Ig\colon IX \to IY$

$$\begin{gathered}(I\varphi)\hat{} = I(\hat{\varphi})s\colon I^2X \to IY,\\ I(\hat{\varphi})s(\partial^-IX) = I(\hat{\varphi})(I\partial^-X) = If, \qquad I(\hat{\varphi})s(\partial^+IX) = Ig,\end{gathered} \qquad (18)$$

modifying the map $I(\hat{\varphi})$, which does not have the correct faces.

(b) There is a *whisker composition* of maps and homotopies

$$X' \xrightarrow{\ h\ } X \underset{g}{\overset{f}{\underset{\downarrow\varphi}{\rightrightarrows}}} Y \xrightarrow{\ k\ } Y' \qquad (19)$$

$$k{\circ}\varphi{\circ}h\colon kfh \to kgh\colon X' \to Y', \quad (k{\circ}\varphi{\circ}h)\hat{} = (k\hat{\varphi})(Ih)\colon IX' \to Y',$$

also written as $k\varphi h$. This ternary operation satisfies obvious relations of associativity and identities (cf. [G2], 1.2.3).

(c) On the other hand, homotopies of c-spaces have no vertical composition: two consecutive homotopies $f \to g \to h$ (between parallel c-maps) cannot be concatenated, because of the failure of the concatenation pushout recalled above (see II.4.7(c)).

We introduce thus a *composite homotopy* $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \dot{\to} f''\colon X \to Y$ (denoted by a dot-marked arrow) as a finite sequence of consecutive homotopies between maps $f_i\colon X \to Y$

$$f' = f_0 \xrightarrow{\varphi_1} f_1 \xrightarrow{\varphi_2} f_2 \ ... \xrightarrow{\varphi_n} f_n = f''. \qquad (20)$$

Their whisker composition with maps is obvious

$$k\varphi h = (k\varphi_1 h,\ k\varphi_2 h, ...,\ k\varphi_n h). \qquad (21)$$

The *vertical composition* $\psi\varphi$ with a second composite homotopy $\psi\colon f'' \dot{\to} f'''\colon X \to Y$ is just word concatenation, and is associative.

Of course these two composition laws do not satisfy the middle-four interchange and do not produce a 2-category.

2.2 Future and past homotopy equivalences

We already recalled that directed homotopy equivalence comes out in two basic forms, which can be combined in various ways studied in [G2]. Here we have to extend this approach, working with composite homotopies.

(a) A *future homotopy equivalence* between the c-spaces X and Y, or *homotopy equivalence in the future*, will be a four-tuple $(f, g; \varphi, \psi)$ of maps and composite homotopies (starting from the identity maps of the spaces)

$$f\colon X \rightleftarrows Y : g, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow gf, \qquad \psi\colon \mathrm{id}\, Y \twoheadrightarrow fg. \tag{22}$$

We speak of a future homotopy equivalence *in one step* when φ and ψ are mere homotopies.

Future homotopy equivalences compose: given a second

$$h\colon Y \rightleftarrows Z : k, \qquad \vartheta\colon \mathrm{id}\, Y \twoheadrightarrow kh, \qquad \zeta\colon \mathrm{id}\, Z \twoheadrightarrow hk, \tag{23}$$

their composite is obtained by whisker composition and vertical composition (as in the horizontal composition of adjunctions)

$$\begin{gathered} hf\colon X \rightleftarrows Z : gk, \\ (g\vartheta f)\varphi\colon \mathrm{id}\, X \twoheadrightarrow gkhf, \qquad (h\psi k)\zeta\colon \mathrm{id}\, Z \twoheadrightarrow hfgk. \end{gathered} \tag{24}$$

Being future homotopy equivalent c-spaces is thus an equivalence relation.

A *past homotopy equivalence* between the c-spaces X, Y is a four-tuple $(f, g; \varphi, \psi)$

$$f\colon X \rightleftarrows Y : g, \qquad \varphi\colon gf \twoheadrightarrow \mathrm{id}\, X, \qquad \psi\colon fg \twoheadrightarrow \mathrm{id}\, Y, \tag{25}$$

where the composite homotopies start from the composed maps. The reversor R turns future into past and conversely (see (17)), and we will mostly deal with the former case.

(b) More particularly, with the following structure

$$i\colon X_0 \rightleftarrows X : p, \qquad \mathrm{id}\, X_0 = pi, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow ip, \tag{26}$$

we say that the c-space X_0 is a *future deformation retract* of X, or embedded in X as a future deformation retract; this can always be realised with a c-subspace $X_0 = \mathrm{Im}\, i \subset X$. Again, we speak of a future deformation retract *in one step* when we can realise the previous structure with a homotopy $\varphi\colon \mathrm{id}\, X \to ip$, as will often be the case in the examples below.

(c) We say that the c-spaces X, Y are *coarsely c-homotopy equivalent* if they are linked by the equivalence relation generated by future and past homotopy equivalence.

2.3 Contractible c-spaces

We say that a c-space is *future contractible* if it is future homotopy equivalent to the terminal singleton $\{*\}$. Here we only need a pair of maps i, p and one composite homotopy φ

$$i\colon \{*\} \rightleftarrows X : p, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow ip \qquad (pi = \mathrm{id}\,\{*\}), \tag{27}$$

and $\{*\}$ is embedded as a future deformation retract of X, at a flexible point $x_0 = i(*)$. We also say that X is *future contractible to* x_0.

Equivalently, we have a composite homotopy φ such that

$$\begin{gathered} \varphi = (\varphi_1, ..., \varphi_n)\colon f_0 \twoheadrightarrow f_n\colon X \to X, \\ f_0 = \mathrm{id}\, X, \qquad f_n \text{ is constant at } x_0 \in |X|_0. \end{gathered} \tag{28}$$

We say that the c-space X is *coarsely c-contractible* if it is coarsely c-homotopy equivalent to $\{*\}$: there exists a finite sequence of c-spaces $X, X_1, ..., X_{n-1}, \{*\}$ such that each of them is future or past homotopy equivalent to the next.

Examples. (a) The standard interval $\mathrm{c}\mathbb{I}$ is past contractible to 0 and future contractible to 1, in one step, with homotopies supplied by the connections $g^\alpha\colon \mathrm{c}\mathbb{I}^2 \to \mathrm{c}\mathbb{I}$ recalled in 1.3

$$\begin{aligned} &\partial^-\colon \{*\} \rightleftarrows \mathrm{c}\mathbb{I} : e, \qquad g^+\colon \partial^- e \to \mathrm{id}\,\mathrm{c}\mathbb{I}, \qquad e\partial^- = \mathrm{id}\,, \\ &\partial^+\colon \{*\} \rightleftarrows \mathrm{c}\mathbb{I} : e, \qquad g^-\colon \mathrm{id}\,\mathrm{c}\mathbb{I} \to \partial^+ e, \qquad e\partial^+ = \mathrm{id}\,. \end{aligned} \tag{29}$$

(b) The two-jump interval $\mathrm{c}\mathbb{J}$ is also past contractible to 0 and future contractible to 1, with homotopies $g^\alpha\colon \mathrm{c}\mathbb{J} \times \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J}$.

(c) The standard c-line $\mathrm{c}\mathbb{R}$ is not future contractible, because there is no point x_0 such that each flexible point x has a c-path $x \to x_0$. (The necessity of this property is proved in Proposition 2.5(a).) By R-duality it is not past contractible either.

However, it is easy to prove that $\mathrm{c}\mathbb{R}$ is coarsely c-contractible: it has a future deformation retract $\mathrm{c}[0, +\infty[$ (in one step), which is past contractible to 0 (in one step).

2.4 Controlled connection

(a) In a c-space X the existence of a c-path between two points gives a reflexive and transitive relation in $|X|_0$, and we consider the equivalence relation generated by the latter. The equivalence class $[x]_c$ of a flexible point is called a *controlled component*, or *c-component*, of X; it is a topological subspace of $|X|_0$.

X is said to be *c-connected* if it has precisely one c-component. (The empty c-space is not.) If $f\colon X \to Y$ is a c-map whose restriction $|X|_0 \to |Y|_0$ is surjective and X is c-connected, Y is also. A product of c-spaces is c-connected if and only if all its factors are.

(b) We denote as $\uparrow\!\Pi_0(X)$ the quotient of the set $|X|_0$ modulo this equivalence relation, that is the set of controlled components. We have thus a functor $\uparrow\!\Pi_0\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}$, with an obvious action on a c-map $f\colon X \to Y$

$$f_*\colon \uparrow\!\Pi_0(X) \to \uparrow\!\Pi_0(Y), \qquad f_*[x]_c = [f(x)]_c. \qquad (30)$$

Equivalently, $\uparrow\!\Pi_0(X)$ is the set of connected components of the fundamental category $\uparrow\!\Pi_1(X)$.

(c) *Examples*. All the ‘basic’ c-spaces are c-connected: the intervals $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $c_-\mathbb{I}$, $c_+\mathbb{I}$, $\uparrow\!\mathbb{I}$, the lines $\mathrm{c}\mathbb{R}$, $c_n\mathbb{R}$, $\uparrow\!\mathbb{R}$, the spheres $\mathrm{c}\mathbb{S}^1$, $c_n\mathbb{S}^1$, $\mathrm{c}\mathbb{S}^n$, $\uparrow\!\mathbb{S}^n$, and all their products. Their non-trivial sums are not, of course.

We recall, from I.2.4(b), that the *past-delayed c-interval* $c_-\mathbb{I}$ is the standard interval $[0, 1]$ with the c-structure generated by the past-delayed reparametrisation $f(t) = 0 \vee (2t - 1)$. The *future-delayed c-interval* $c_+\mathbb{I}$ is based on the future-delayed reparametrisation $g(t) = 2t \wedge 1$.

(d) *Remarks*. The interval $\mathrm{c}\mathbb{I}$ is c-connected but its flexible support $\{0, 1\}$ is a disconnected topological space. The same happens for all the c-structures of intervals and lines considered above, except the d-spaces $\uparrow\!\mathbb{I}$ and $\uparrow\!\mathbb{R}$.

This fact becomes clearer using the *path-support* $|X|_1$ of the c-space X, defined in II.2.1(d) as the topological subspace of $|X|$ formed by the union of the images of all c-paths in X, so that $|X|_0 \subset |X|_1 \subset |X|$. If X is c-connected, the space $|X|_1$ is path connected: in fact every point of the latter is linked to a flexible point by the restriction of a c-path. The converse fails, as for the 'diagonal' c-structure of the square $\mathbb{I}^2$, in I.2.7(d) and II.2.2(b).

2.5 Proposition

(a) If the c-space X is future contractible to x_0, every flexible point x has a c-path $x \to x_0$.

(b) A coarsely contractible c-space X is always c-connected.

Proof. It is sufficient to prove (a). According to (28) we have a composite homotopy $\varphi = (\varphi_1, ..., \varphi_n)\colon f_0 \twoheadrightarrow f_n\colon X \to X$, where $f_0 = \mathrm{id}\, X$ and f_n is constant at $x_0 \in |X|_0$. For every flexible point x the c-maps $\varphi_i(x, -)\colon \mathrm{c}\mathbb{I} \to X$ form a sequence of consecutive c-paths $x \to f_1(x) \to ... \to x_0$, whose concatenation is a c-path $x \to x_0$. □

2.6 Coherent homotopy equivalence

(a) As in 1.8, a *coherent future homotopy equivalence* of c-spaces will be a four-tuple $(f, g; \varphi, \psi)$ of maps and homotopies (in one-step) satisfying the *coherence conditions*

$$\begin{array}{lll} f\colon X \rightleftarrows Y : g, & \varphi\colon \mathrm{id}\, X \to gf, & \psi\colon \mathrm{id}\, Y \to fg, \\ \quad f\varphi = \psi f\colon f \to fgf, & & \varphi g = g\psi\colon g \to gfg. \end{array} \tag{31}$$

These structures cannot be composed (even for mere topological spaces). R-duality gives the corresponding case *in the past*.

(b) If the map f (for instance) is a reshaping, its topological support is an identity. Therefore, loosely speaking, φ and ψ are represented by the same mapping $\hat{\varphi} = \hat{\psi}$, which is at the same time a c-map $X \times \mathrm{c}\mathbb{I} \to X$ and $Y \times \mathrm{c}\mathbb{I} \to Y$; the first coherence condition is automatically satisfied, and the second means that $\hat{\varphi}(g \times \mathrm{id}\,) = g\hat{\varphi}$.

(c) As a typical example, the reshaping $f\colon c_+\mathbb{I} \to \mathrm{c}\mathbb{I}$ of the future-delayed interval (in 2.4(c)) can be completed to a coherent future homotopy equivalence

$$\begin{gathered} f\colon c_+\mathbb{I} \rightleftarrows \mathrm{c}\mathbb{I} : g, \qquad g(s) = 2s \wedge 1, \\ \hat{\varphi}(s,t) = (1+t)s \wedge 1, \qquad \hat{\varphi}(s,0) = s, \qquad \hat{\varphi}(s,1) = 2s \wedge 1. \end{gathered} \tag{32}$$

In fact, we have to verify three points.

(i) The map $\hat{\varphi}\colon \mathbb{I}^2 \to \mathbb{I}$ is a c-map $\mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$. This is obvious, because $\hat{\varphi}$ is increasing in both variables and preserves the flexible points.

(ii) The same map $\hat{\varphi}$ is a c-map $c_+\mathbb{I}\times\mathrm{c}\mathbb{I} \to c_+\mathbb{I}$. Indeed, given a c-path $a = \langle h, k\rangle\colon \mathrm{c}\mathbb{I} \to c_+\mathbb{I}\times\mathrm{c}\mathbb{I}$, the c-map $h\colon \mathrm{c}\mathbb{I} \to c_+\mathbb{I}$ is increasing between flexible points and either constant at 0, or constant at 1 on a non-degenerate interval $[\tau, 1]$, at least. In both cases the path $\hat{\varphi}a$ is increasing between flexible points and has a final delay: either $\hat{\varphi}(0, k(t)) = 0$, for all t, or

$$\hat{\varphi}(h(t), k(t)) = (1 + h(t)) \wedge 1 = 1, \quad \text{for } t \geqslant \tau.$$

(iii) $\hat{\varphi}(g\times\mathrm{id}\,) = g\hat{\varphi}$, because:

$$\begin{aligned} \hat{\varphi}(g(s), t) &= (1+t)(2s \wedge 1) \wedge 1 = (1+t)2s \wedge (1+t) \wedge 1 \\ &= (1+t)2s \wedge 1 = g\hat{\varphi}(s,t). \end{aligned}$$

(d) The previous homotopy φ is strict, that is $\hat{\varphi}(s, -)$ is constant for $s = 0, 1$ (the flexible points of $\mathrm{c}\mathbb{I}$ and $c_+\mathbb{I}$). Applying Theorem III.3.3 we deduce that f and g induce two functors $\uparrow\!\Pi_1(c_+\mathbb{I}) \rightleftarrows \uparrow\!\Pi_1(\mathrm{c}\mathbb{I})$ which are inverse to each other. Both categories are isomorphic to the ordinal **2**.

3. Homotopy pushouts and flexibility

Taking on the study of the standard homotopy structure cTop, we deal now with homotopy pushouts. Their derived constructions, cones and suspension, are deferred to the next section.

Homotopy pushouts are well-known in Top [Ma], and also in Cat as *cocomma squares*. They are studied in [G1, G2] in general dI1-categories ([G2], Section 1.3) and symmetric dI1-homotopical categories ([G2], Sections 1.7, 4.1), in particular for directed spaces and pointed directed spaces.

The 'defective' homotopy structure of cTop requires a specific study; flexibility properties link these constructions to those of d-spaces, far better behaved.

X will denote a c-space.

3.1 Weak flexibility

We begin by recalling some weak forms of flexibility studied in II.2, which will also have a role in analysing the present constructions.

(a) We say that X is *preflexible* if it is 'full' in the generated d-space $\hat{X}$, which means that every c-path of $\hat{X}$ between flexible points of X is already a c-path of the latter. Then the fundamental category $\uparrow\Pi_1(X)$ is the full subcategory of $\uparrow\Pi_1(\hat{X})$ with vertices in the flexible support $|X|_0$, as proved in Theorem II.5.3(b).

Preflexible c-spaces form a full, reflective subcategory of cTop. They are closed in cTop under limits and sums; they are not closed under pushouts.

(b) We say that X is *border flexible* if one can restrict every c-path, by cutting out delays at the endpoints; more precisely, if $a\colon \mathrm{c}\mathbb{I} \to X$ is a c-path constant on $[0, t_1]$ and $[t_2, 1]$, the restriction of a to the interval $[t_1, t_2]$ (reparametrised on $\mathbb{I}$, see II.2.1(c)) is still a c-path.

Again, border flexible c-spaces form a full, reflective subcategory of cTop. They are closed in cTop under limits and sums, but not under pushouts. All preflexible c-spaces are border flexible; the converse is false.

(c) The *path-support* $|X|_1$ has already been recalled in 2.4(d). We say that X has a *total path-support* if $|X|_1 = |X|$.

(d) *Examples*. Besides all d-spaces, many basic c-spaces we have considered in Parts I and II are preflexible, with a total path-support: for instance $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$ and $\mathrm{c}\mathbb{I}^{\sim}$ (in $\uparrow\mathbb{I}$), $\mathrm{c}\mathbb{R}$ (in $\uparrow\mathbb{R}$), $\mathrm{c}\mathbb{S}^1$ (in $\uparrow\mathbb{S}^1$), and their products.

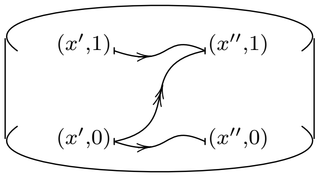
The delayed intervals $c_{-}\mathbb{I}$ and $c_{+}\mathbb{I}$ (in 2.4(c)) are not even border flexible, like the higher c-spheres $\mathrm{c}\mathbb{S}^n$, for $n \geqslant 2$ (see 4.6).

The 'diagonal' c-structure X of the square $\mathbb{I}^2$ described in I.2.7(d) and II.2.2(b) is border flexible and not preflexible; its path-support is not total.

3.2 The cylinder

The flexible points of the cylinder $IX = X \times \mathrm{c}\mathbb{I}$ form the set $|X|_0 \times \{0, 1\}$, contained in the bases of the cylinder.

A c-path $\langle a, h\rangle \colon \mathrm{c}\mathbb{I} \to X \times \mathrm{c}\mathbb{I}$ in the cylinder IX consists of a c-path $a \colon x' \to x''$ in the c-space X and a c-path h in the interval $\mathrm{c}\mathbb{I}$. There are three cases, as in the following figure, distinguished by the map $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$:



$X \times \mathrm{c}\mathbb{I}$ (33)

(i) a path $\langle a, e_0\rangle \colon (x', 0) \to (x'', 0)$ in the lower base $\partial^- X = X \times \{0\}$,

(ii) a path $\langle a, e_1\rangle \colon (x', 1) \to (x'', 1)$ in the upper base $\partial^+ X = X \times \{1\}$,

(iii) a path $\langle a, h\rangle \colon (x', 0) \to (x'', 1)$, where $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is a global reparametrisation, that is a surjective increasing map.

If the c-space X is preflexible or border flexible, so is IX. In fact these properties are closed under products and $\mathrm{c}\mathbb{I}$ satisfies them.

3.3 Homotopy pushouts

The structure of cTop as a symmetric dI1-category and the existence of pushouts ensures the existence of homotopy pushouts ([G2], 1.3). These are introduced in their 'standard form', also to get a functor.

(a) Let $f \colon X \to Y$ and $g \colon X \to Z$ be two c-maps with the same domain. The *standard homotopy pushout*, or *h-pushout*, *from f to g* is a four-tuple $(W; u, v; \lambda)$ satisfying the following universal property:

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[u]{} & W \end{array} \qquad \lambda \colon uf \to vg \colon X \to W, \tag{34}$$

- for every similar four-tuple $(W'; u', v'; \lambda')$ there is precisely one map $h \colon W \to W'$ such that $u' = hu$, $v' = hv$, $\lambda' = h\lambda$.

The object W is determined up to isomorphism; its construction is deferred to the next subsection. It will be denoted as $I_c(f,g)$, or $I(f,g)$, and called a *double mapping cylinder*. Let us note that it is a directed notion, from f to g; the reflection $r\colon IRX \to RIX$ induces an isomorphism $r^I\colon I(Rg,Rf) \to RI(f,g)$, called the *reflection of an h-pushout*. When f or g is $\mathrm{id}\,X$, one has a *mapping cylinder*, $I(X,g)$ or $I(f,X)$.

(b) The cylinder IX itself is the h-pushout of the pair $(\mathrm{id}\,X, \mathrm{id}\,X)$, by means of the structural homotopy ∂ represented by the identity of the cylinder

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & {\scriptstyle \partial}\nearrow & \downarrow{\scriptstyle \partial^+} \\ X & \xrightarrow[\partial^-]{} & IX \end{array} \qquad \begin{array}{c} \partial\colon \partial^- \to \partial^+\colon X \to IX, \\ \hat{\partial} = \mathrm{id}\, IX. \end{array} \qquad (35)$$

In fact ∂ establishes a bijection between maps $\varphi\colon IX \to Y$ and homotopies $\varphi\partial\colon \varphi\partial^- \to \varphi\partial^+\colon X \to Y$, by the very definition of the latter.

(c) It is easy to see that h-pushouts give a functor

$$I(-,-)\colon \mathrm{c}\mathsf{Top}^{\vee} \to \mathrm{c}\mathsf{Top}, \qquad (36)$$

where $\vee$ is the *formal-span* category: $\bullet \leftarrow \bullet \rightarrow \bullet$. (The verification can be found in [G2], 1.3.7, for all dI1-categories with h-pushouts.)

The corresponding functor for d-spaces carries coherent triples of homotopies to homotopies, as it happens in every symmetric dI1-homotopical category ([G2], Theorem 4.1.6). This does not apply here: see 3.8(b).

3.4 The construction

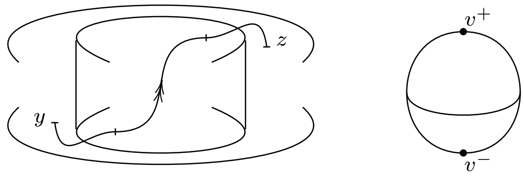
(a) The homotopy pushout $I(f,g)$ can be constructed using the cylinder $IX = X \times \mathrm{c}\mathbb{I}$ and the ordinary colimit of the left solid diagram below, called a *cylindrical colimit*. It amounts to three ordinary pushouts, as shown at the right hand

$$\begin{array}{ccccc} & & X & \xrightarrow{g} & Z \\ & & {\scriptstyle \partial^+}\downarrow & & \vdots \\ X & \xrightarrow{\partial^-} & IX & & \vdots{\scriptstyle v} \\ {\scriptstyle f}\downarrow & & & \searrow{\scriptstyle \lambda} & \downarrow \\ Y & & \dashrightarrow_{u} & & I(f,g) \end{array} \qquad \begin{array}{ccccc} & & X & \xrightarrow{g} & Z \\ & & {\scriptstyle \partial^+}\downarrow & & \downarrow \\ X & \xrightarrow{\partial^-} & IX & \dashrightarrow & I(X,g) \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \dashrightarrow & I(f,X) & \dashrightarrow & I(f,g) \end{array} \qquad (37)$$

(b) The space $I(f,g)$ results thus of the pasting of the cylinder IX with the spaces Y, Z, under the following identifications (for $x \in X$)

$$I(f,g) = (Y + IX + Z)/\sim, \qquad [x,0] = [f(x)], \quad [x,1] = [g(x)]. \quad (38)$$

The construction is made clear by the following pictures. In the first f and g are injective, and we have drawn a path from a point $y \in Y$ to a point $z \in Z$ (see the following theorem)



(39)

In the second f and g are the terminal map $X \to \{*\}$, and $I(f,g)$ is the suspension ΣX where each base if IX is collapsed to a point (see 4.3).

The structural maps $u\colon Y \to I(f,g)$ and $v\colon Z \to I(f,g)$ are always injective, while the map $\lambda\colon IX \to I(f,g)$ is certainly injective *outside* of the bases $\partial^\alpha X$; it is injective 'everywhere' if and only if both f and g are.

(c) We have a similar construction $I_d(f,g)$ in $\mathrm{d}\mathsf{Top}$, based on the directed cylinder $I_d(X) = X \times \uparrow\!\mathbb{I}$, and in Top, based on the ordinary cylinder $X \times \mathbb{I}$. The three constructions give the same topological space, since the forgetful functors $\mathrm{c}\mathsf{Top} \to \mathsf{Top}$ and $\mathrm{d}\mathsf{Top} \to \mathsf{Top}$ preserve colimits and commute with the cylinders.

3.5 Theorem (The paths)

(a) In the h-pushout $I(f,g)$ of (37)*, the c-paths are of the following three kinds (and there are none from a point of Z to a point of Y).*

(i) A c-path $y \to y'$ between points of Y is a c-path of Y (embedded in $I(f,g)$ by u),

(ii) A c-path $z \to z'$ between points of Z is a c-path of Z (embedded in $I(f,g)$ by v),

(iii) A c-path from $y \in Y$ to $z \in Z$.

For the last case, we begin to say that a regular *c-path from y to z (as in figure* (39)*) is formed by the regular concatenation $a = ua_1 * \lambda a_2 * va_3$ of the images of three c-paths in Y, IX and Z*

$$\begin{gathered} a_1 \colon y \to f(x_1) \text{ in } Y, \qquad a_3 \colon g(x_3) \to z \text{ in } Z, \\ a_2 = \langle b, h \rangle \colon (x', 0) \to (x'', 1), \\ f(x_1) = f(x'), \qquad g(x_3) = g(x''), \end{gathered} \tag{40}$$

where $b \colon x' \to x'$ in X and $h \colon 0 \to 1$ in $\mathrm{c}\mathbb{I}$.

To get all c-paths from y to z we allow:

- *discarding a_1 when $y \in f(X)$ and discarding a_3 when $z \in g(X)$,*
- *global reparametrisations of the paths previously obtained.*

(b) The structural maps $u \colon Y \to I(f, g)$ and $v \colon Z \to I(f, g)$ are embeddings *of c-spaces, that is they induce an isomorphism onto their images.*

(c) In $\mathrm{d}\mathsf{Top}$ *we have similar results, replacing I with I_d, c-paths with d-paths, and global reparametrisations with the partial ones (for paths of kind (iii)).*

Proof. (a) The paths listed above form a c-structure on the topological colimit $|I(f, g)|$, as they are closed under trivial loops at the endpoints, concatenation and global reparametrisation.

The (injective) mappings u, v are obviously c-maps. To verify that $\lambda \colon IX \to I(f, g)$ is a c-map we note that, in a c-path $a = \langle b, h \rangle \colon \mathrm{c}\mathbb{I} \to IX = X \times \mathrm{c}\mathbb{I}$, the map $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is either constant at 0, or constant at 1, or a path $0 \to 1$. In the first case $a = \partial^- b \colon \mathrm{c}\mathbb{I} \to X \to IX$ and $\lambda a = \lambda \partial^- b = u(fb)$ is a c-path of Y embedded in $I(f, g)$, of kind (i). In the second case $a = \partial^+ b$ is similarly a path of kind (ii). In the last, λa is a path of $I(f, g)$ of kind (iii).

Finally, the c-structure we have described is generated by the images of the c-paths of Y (by u), of IX (by λ) and Z (by v), and is thus the structure of the colimit c-space.

(b) To prove that the mapping u is an embedding of c-spaces, we only have to consider the topological part, since the c-paths of $u(Y)$ have already been considered.

Let V be open in Y. The subset $W = f^{-1}(V) \times [0, 1/2[$ is open in the cylinder IX. Now $V \cup W$ is open in the topological sum $T = Y + IX + Z$, and saturated for the projection $p\colon T \to I(f,g)$. Thus $p(T)$ is open in $I(f,g)$, and $u(V) = u(Y) \cap p(T)$ is open in $u(Y)$.

(c) The argument is the same. □

3.6 Theorem (Preflexibility, I)

If, in the homotopy pushout (34)*, the c-space X is preflexible with a total path-support (see 3.1), then:*

$$(I_c(f,g))\hat{} = I_d(\hat{f},\hat{g}). \tag{41}$$

More explicitly, we are considering the following two structures on the same topological space, the h-pushout in Top *of the underlying maps, and saying that they coincide:*

- *the d-structure generated by the h-pushout $I_c(f,g)$ in* cTop,
- *the h-pushout in* dTop *of the d-maps $\hat{f}\colon \hat{X} \to \hat{Y}$ and $\hat{g}\colon \hat{X} \to \hat{Z}$.*

Proof. The colimit (37) in cTop is preserved by the reflector $\hat{}\colon$ cTop $\to$ dTop. Moreover $(I_cX)\hat{} = I_d(\hat{X})$, by Corollary II.2.6. □

3.7 Theorem (Preflexibility, II)

If, in the homotopy pushout (34)*, the c-space X is flexible, while Y and Z are preflexible, the h-pushout $I(f,g)$ is preflexible.*

Proof. It is a consequence of Theorems 3.5 and 3.6.

Let $a\colon w' \to w''$ be a d-path in $(I_c(f,g))\hat{} = I_d(\hat{f},\hat{g})$ between flexible points of $I(f,g)$; we have to prove that a is a c-path of $I(f,g)$. By 3.5(c) the d-path a can be of three kinds.

In case (i) (or (ii)) a is a d-path of $\hat{Y}$ (or $\hat{Z}$) between flexible points of Y (or Z), and therefore a c-path of the latter.

In case (iii) it is sufficient to consider a regular d-path from $y \in |Y|_0$ to $z \in |Z|_0$, formed by the regular concatenation $a = ua_1 * \lambda a_2 * va_3$ of the images of three d-paths in $\hat{Y}$, $I_d\hat{X}$ and $\hat{Z}$, as in (40).

By hypothesis, $b\colon x' \to x'$ is a c-path in X. Then $f(x_1) = f(x')$ is a flexible point of Y, and a_1 is a c-path of Y. Similarly a_3 is a c-path of Z. Finally, $h\colon 0 \to 1$ in $\uparrow\mathbb{I}$ is a c-path of $\mathrm{c}\mathbb{I}$, and we are done: $a = ua_1 * \lambda a_2 * va_3$ is a c-path of $I(f,g)$. □

3.8 Remarks

(a) In the previous statement we cannot let X be just preflexible. In fact, the cylinder of a preflexible c-space X *is* preflexible, but we shall see in 4.5(b) that its suspension need not be – although in this case the c-spaces $Y = Z = \{*\}$ are even flexible.

(b) We recalled in 1.3 that cTop is a symmetric dI2-category. Moreover, it has all limits and colimits, and in particular all cylindrical colimits. To make it into a symmetric dI1-homotopical category ([G2], 4.1.4) the cylinder I should preserve the cylindrical colimits (as colimits), which would ensure good properties for h-pushouts ([G2], 4.1, 4.2). This is not the case.

In fact, the cylindrical colimit $I(\{*\}, \partial^-)$ amounts to the standard concatenation pushout (7) and we know that this pushout is not preserved by $I = - \times \mathrm{c}\mathbb{I}$ (cf. II.4.7(c)).

4. Cones, suspension and flexibility

Cones and suspension are derived from homotopy pushouts. We go on studying their weak flexibility properties.

X is always a c-space and we write as $p\colon X \to \{*\}$, or p_X, its map to the terminal singleton.

4.1 Mapping cones and cones

(a) A c-map $f\colon X \to Y$ has an *upper mapping cone* $C_c^+ f = I_c(f, p_X)$, also written $C^+ f$, defined as the h-pushout below, at the left

$$\begin{array}{ccc} X & \xrightarrow{\;p\;} & \{*\} \\ {\scriptstyle f}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle v^+} \\ Y & \xrightarrow[u]{} & C^+ f \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle p}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle u} \\ \{*\} & \xrightarrow[v^-]{} & C^- f \end{array} \qquad (42)$$

Its structural maps are the *lower base* $u\colon Y \to C^+ f$ and the *upper vertex* $v^+\colon \top \to C^+ f$; then, we have a *structural homotopy* $\gamma\colon uf \to v^+ p\colon X \to C^+ f$, which links uf to a constant map, in a universal way. The upper mapping cone is a functor $C^+ \colon \mathrm{cTop}^{\mathbf{2}} \to \mathrm{cTop}$, defined on the category of morphisms of cTop.

Symmetrically, f has a *lower mapping cone* $C_c^- f = I(p_X, f)$ defined as the right h-pushout above (also written as $C^- f$), with a *lower vertex* v^- and an *upper base* u.

Let us note that the terms *upper* and *lower* agree with the vertex: this is consistent with *future* and *past* contractibility, in Lemma 4.2.

(b) In particular, the *upper cone* $C_c^+ X = C_c^+(\mathrm{id}\, X) = I_c(X, p_X)$ of a c-space X is given by the h-pushout of the left diagram below, or equivalently by the pushout at the right

$$\begin{array}{ccc} X & \xrightarrow{p} & \{*\} \\ {\scriptstyle \mathrm{id}}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle v^+} \\ X & \xrightarrow[u]{} & C^+X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{p} & \{*\} \\ {\scriptstyle \partial^+}\downarrow & & \downarrow{\scriptstyle v^+} \\ IX & \xrightarrow[\gamma]{} & C^+X \end{array} \tag{43}$$

$$C^+X = (IX + \{*\})/(\partial^+X + \{*\}), \quad \gamma(x,t) = [x,t], \quad v^+(*) = v^+.$$

It can be calculated as the quotient c-space displayed above: the upper base of the cylinder is collapsed to an upper vertex $v^+ = v^+(*)$, while the lower base is still an embedding:

$$u = \gamma\partial^- \colon X \to IX \to C^+X, \qquad u(x) = [x, 0]. \tag{44}$$

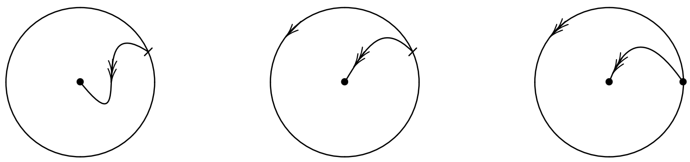
The homotopy $\gamma \colon IX \to C^+X$ allows one to deform – in the cone – the lower base $u \colon X \to C^+X$ to the map $\gamma\partial^+ = v^+p \colon X \to C^+X$, constant at the upper vertex.

In particular, $C^+\varnothing = \{v^+\}$ is a flexible singleton. If X is not empty:

$$C^+X = IX/\partial^+X, \qquad \gamma(x,t) = [x,t], \qquad v^+(*) = [x, 1]. \tag{45}$$

The previous pushout defines a functor $C^+ \colon \mathrm{cTop} \to \mathrm{cTop}$. As in 3.3(c), it is not homotopy invariant.

(c) *Examples*. The upper cones of $\mathbb{S}^1$, $\uparrow\!\mathbb{S}^1$ and $\mathrm{c}\mathbb{S}^1$, in cTop, can be drawn in the plane, as discs



$C_c^+(\mathbb{S}^1)$ $C_c^+(\uparrow\mathbb{S}^1)$ $C_c^+(\mathrm{c}\mathbb{S}^1)$ (46)

The upper cones of $\mathbb{S}^1$ and $\uparrow\mathbb{S}^1$ in dTop will be recalled in (56).

(d) Dually, by reversor duality, the *lower cone* $C^-X = I(p, X)$ is obtained by collapsing the lower base of IX to a lower vertex $v^- = v^-(*)$.

4.2 Lemma (Cones and contractibility)

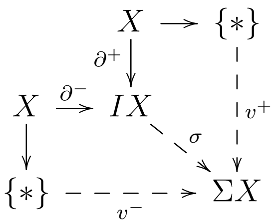
A c-space X is future contractible in one step if and only if the base of its upper cone $u\colon X \to C^+X$ has a retraction $h\colon C^+X \to X$.

Proof. (In [G2], Lemma 1.7.3, this result is stated for dI1-homotopical categories.) We use the notation of (43). If $hu = \mathrm{id}\, X$, the map $h\gamma\colon IX \to X$ is a homotopy from $h\gamma\partial^- = hu = \mathrm{id}\, X$ to $h\gamma\partial^+ = hv^+p_X\colon X \to X$, and the latter is a constant endomap.

Conversely, if there is a homotopy $\varphi\colon IX \to X$ with $\varphi\partial^- = \mathrm{id}\, X$ and $\varphi\partial^+ = ip_X\colon X \to X$, we define $h\colon C^+X \to X$ as the unique map such that $h\gamma = \varphi\colon IX \to X$ and $hv^+ = i\colon \top \to X$. Now $hu = h\gamma\partial^- = \varphi\partial^- = \mathrm{id}\, X$. □

4.3 Suspension

The *suspension* $\Sigma_c X = I_c(p_X, p_X)$, or ΣX, of the c-space X is the following colimit and quotient space



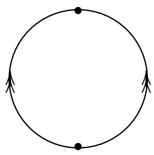
$$\Sigma X = (\{*\} + IX + \{*\})/R, \qquad (47)$$

where R collapses each base of IX to a point: the lower vertex v^- and the upper vertex v^+. (See the right-hand figure in (39).)

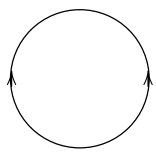
In particular $\Sigma\varnothing = \mathbb{S}^0 = \mathrm{c}\mathbb{S}^0$ (with the natural c-structure of a discrete topological space, the flexible one). If X is not empty, ΣX is the quotient of IX that collapses each bases to a point

$$\Sigma X = IX/R', \qquad \sigma(x,t) = [x,t], \qquad v^\alpha(*) = [x,\alpha]. \tag{48}$$

The suspension of $\mathbb{S}^0$ gives a rigid structure of the circle, represented in the left figure below



$\Sigma_c(\mathbb{S}^0) = \mathrm{c}\mathbb{O}^1$



$\Sigma_d(\uparrow\mathbb{S}^0) = \uparrow\mathbb{O}^1$ (49)

The generated d-space is the suspension $\Sigma_d(\mathbb{S}^0) = \uparrow\mathbb{O}^1$ in $\mathrm{d}\mathsf{Top}$ ([G3], 1.4.3, 1.7.4), whose non-trivial selected paths are the restrictions of the previous ones. It is called the *ordered circle*, as its d-structure is produced by an obvious (partial) order relation on the topological circle.

4.4 Suspension and cones

The suspension ΣX is linked to the cones $C^\alpha(X)$ by the following diagrams of pushouts

$$\begin{array}{ccccc}
 & & X & \xrightarrow{p} & \{*\} \\
 & & \downarrow{\scriptstyle \partial^+} & & \downarrow{\scriptstyle v^+} \\
X & \xrightarrow{\partial^-} & IX & \dashrightarrow & C^+(X) \\
\downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\
\{*\} & \xrightarrow[v^-]{} & C^-(X) & \dashrightarrow & \Sigma X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\partial^-} & C^+(X) \\
\downarrow{\scriptstyle p} & & \downarrow \\
\{*\} & \xrightarrow[v^-]{} & \Sigma X
\end{array} \tag{50}$$

$$\Sigma X = C^+(X)/\partial^- X = C^-(X)/\partial^+ X, \tag{51}$$

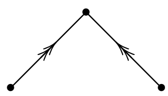
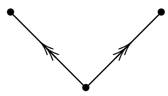
and is the quotient of each cone that collapses its base to a point.

In a splittable homotopy structure like Top and $\mathrm{d}\mathsf{Top}$ (see 1.5), the suspension can also be obtained as a pasting of two cones on their bases. This cannot be done here: see 4.5(a).

4.5 Examples and flexibility

(a) The cones and suspension of a flexible space X are preflexible, by Theorem 3.7. Of course they are not flexible, generally; for instance this is not the case when X is a non-empty discrete d-space, as in the following examples

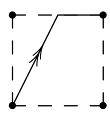
$$C^+\{*\} = C^-\{*\} = \Sigma\{*\} = \mathrm{c}\mathbb{I}, \tag{52}$$


$C^+(\mathbb{S}^0)$

$C^-(\mathbb{S}^0)$ (53)

including $\Sigma(\mathbb{S}^0)$, represented in (49). (We note that this c-space has two flexible points, while the pasting of $C^-(\mathbb{S}^0)$ and $C^+(\mathbb{S}^0)$ over $\mathbb{S}^0$ has four of them.)

(b) The cones and suspension of a preflexible c-space need not be preflexible, nor even border flexible.

In fact, this is not the case for $C^+(\mathrm{c}\mathbb{I}) = \mathrm{c}\mathbb{I}^2/\partial^+\mathbb{I}$: the projection in $C^+(\mathrm{c}\mathbb{I})$ of the following c-path a in $\mathrm{c}\mathbb{I}^2$ has an unavoidable final delay



$$a(t) = (t, 2t \wedge 1). \tag{54}$$

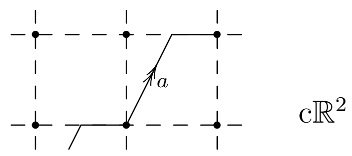
4.6 Higher spheres

The spheres $\mathrm{c}\mathbb{S}^n$ are not border flexible, for $n \geqslant 2$.

We consider the two-dimensional case $\mathrm{c}\mathbb{S}^2 = \mathrm{c}\mathbb{I}^2/\partial\mathbb{I}^2$, the higher ones being similar. It is convenient to view $\mathrm{c}\mathbb{S}^2$ as a quotient of the controlled torus $\mathrm{c}\mathbb{T}^2 = \mathrm{c}\mathbb{R}^2/\mathbb{Z}^2$ (an orbit space, cf. I.2.6) modulo the equivalence relation that collapses the image A of $\mathbb{R}\times\mathbb{Z} \cup \mathbb{Z}\times\mathbb{R}$ (in $\mathrm{c}\mathbb{T}^2$) to the flexible point of the sphere. (Also in $\mathrm{c}\mathsf{Top}$ 'a quotient of a quotient is a quotient', up to isomorphism.)

In the following picture of the c-plane $\mathrm{c}\mathbb{R}^2$, the points of the flexible support $\mathbb{Z}^2$ are marked with bullets, and the dashed lines form the subspace

$\mathbb{R}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{R}$, collapsed to a point in the sphere



(55)

As in (54), the projection b (in $\mathrm{c}\mathbb{S}^2$) of the c-path a of $\mathrm{c}\mathbb{R}^2$ has an unavoidable final delay, so that $\mathrm{c}\mathbb{S}^2$ is not border flexible.

Remarks. (a) This example is even more 'defective' than the cone $C^+(\mathrm{c}\mathbb{I})$ of 4.5(b): our path b also has *internal* prescribed delays. Thus the generated border flexible c-space $(\mathrm{c}\mathbb{S}^2)^{\mathrm{bf}}$ (defined in II.2.3(b)) is not preflexible.

(b) The controlled circle, and even more the *n-stop c-circle* $\mathrm{c}_n\mathbb{S}^1$ of I.2.6(b), model various concrete processes, like a hand of a watch, a washing machine dial, or a vertical panoramic wheel with n cabins. A railway turntable can be modelled by the reversible c-space generated by $\mathrm{c}_n\mathbb{S}^1$.

A controlled 2-sphere with an assigned distribution of 'stops' (the flexible points) might model a net of stationary satellites around the Earth, at a fixed altitude, providing the stops for a servicing shuttle.

4.7 The cofibre sequences of a map

The cofibre sequences of a map, or Puppe sequences, are defined and studied for a general dI1-homotopical category, in [G2], 1.7.6 – 1.7.9. (A *symmetric* dI1-homotopical structure is not required.) Here we only mention the basic facts obtained by applying these results to $\mathrm{c}\mathsf{Top}$.

Every c-map $f\colon X\to Y$ has a natural *upper cofibre sequence*

$$X \xrightarrow{f} Y \xrightarrow{u} C^+f \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma u} \Sigma(C^+f) \xrightarrow{\Sigma d} \Sigma^2 X \ \ldots \qquad (56)$$

formed by the base $u\colon Y\to C^+f$ and the *upper differential* $d = d^+(f)\colon C^+f\to\Sigma X$. The latter is defined by the universal property of the upper cone C^+f as an h-pushout, in (42)

$$\begin{gathered} du = v^-p_Y\colon Y\to\Sigma X, \qquad dv^+ = v^+\colon \top\to\Sigma X, \\ d\gamma = \sigma\colon v^-p_X\to v^+p_X\colon X\to\Sigma X. \end{gathered} \qquad (57)$$

Reversor duality gives the *lower cofibre sequence* of f, with the *lower differential* $d = d^-(f)\colon C^-f \to \Sigma X$

$$X \xrightarrow{f} Y \xrightarrow{u} C^-f \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma u} \Sigma(C^-f) \xrightarrow{\Sigma d} \Sigma^2 X \,... \qquad (58)$$

These sequences give exact sequences in directed homology, as proved in [G2], 2.6.3, for a general dI1-homotopical category. But we already remarked in 1.6(b) the moderate interest of directed homology for d-spaces and c-spaces, beyond degree 1; on the other hand, pointed cubical sets have a 'perfect theory' ([G2], 2.6.4).

5. The flexible homotopy structure of c-spaces

The flexible interval $\uparrow\mathbb{I}$ produces a second homotopy structure on the category of c-spaces, that will be denoted as $\mathrm{c}\mathsf{Top}_F$. It has a secondary role here, although it is better related to the fundamental category functor, by Proposition 5.3 (and Theorem II.5.4).

5.1 The flexible structure

The c-space $\uparrow\mathbb{I}$ is also a symmetric dI2-interval, with the same operations listed above, in (4). Moreover it is exponentiable in $\mathrm{c}\mathsf{Top}$, as proved in Theorem II.3.9(b).

The *flexible cylinder functor*

$$I_F\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad I_F(X) = X \times \uparrow\mathbb{I}, \qquad (59)$$

extends the cylinder functor I_d of d-spaces, and satisfies the axioms listed in (6): $\mathrm{c}\mathsf{Top}_F$ is also a symmetric dI2-category.

The functor P_F *of flexible paths*, right adjoint to I_F

$$P_F\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad P_F(Y) = Y^{\uparrow\mathbb{I}}, \qquad (60)$$

extends the path functor P_d of d-spaces. The adjunction automatically makes $\mathrm{c}\mathsf{Top}_F$ into a symmetric dIP2-homotopical category: the faces, degeneracy, reflection and connections of P_F are mates to those of I_F ([G2], 4.2.1). For instance, the 'new' faces $\partial^\alpha\colon P_F \to 1$ are obtained as $P_F Y \to I_F P_F Y \to Y$

by composing the cylinder faces of P_FY with the counit of the adjunction (the path evaluation).

Moreover, taking into account that the category cTop has all colimits, preserved by I_F, and all limits, preserved by P_F, the structure cTop_F is a symmetric dIP2-homotopical category ([G2], 4.2.1).

Remarks. (a) The classification of dIP-structures cannot go higher, as shown below.

(b) The fact that every flexible homotopy is a standard one can be formally expressed saying that the identity functor of the category cTop is a lax dI2-functor $H\colon \mathrm{cTop}_F \to \mathrm{cTop}$ of dI2-categories (cf. [G2], 1.2.6 and 4.2.7).

Essentially, we have a strict comparison $RH = HR$ (the reversors of these structures are the same) and a non-invertible comparison $h\colon I_cH \to HI_F$ (given by the reshaping $I_cX \to I_FX$), which agree with the dI2-structure.

5.2 Flexible homotopies

In cTop_F the flexible interval produces flexible homotopies, already introduced in II.4.1: a flexible homotopy $\varphi\colon f \to g\colon X \to Y$ of c-maps is represented by a map $I_FX \to Y$ (or equivalently $X \to P_FY$).

Again, flexible homotopies 'cannot' be concatenated): more precisely, as proved in II.4.7(b), they are not closed under concatenation within topological homotopies, and cTop_F cannot have a dI3-structure consistent with the topological one (cf. [G2], 4.2.2).

Formally, one can define a concatenation pushout $J_FX = I_FX +_X I_FX$, which is preserved by I_F since $\uparrow\mathbb{I}$ is exponentiable. But this is useless without a 'good' concatenation map $I_FX \to J_FX$ that would transform a 'pre-concatenation' $J_FX \to Y$ into a flexible homotopy $I_FX \to Y$: to agree with the underlying topology, this map should be a reshaping, and I_FX need not be finer than J_FX, as shown in II.4.7(b) for $X = \mathrm{c}\mathbb{I}$.

Flexible homotopies are dealt with in the same way as the general ones in 2.1: trivial case, reflection, whisker composition, composite flexible homotopies and their operations.

In particular, a *composite flexible homotopy* $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \to f''\colon X \to Y$ is a finite sequence of consecutive flexible homotopies between c-maps $f_i\colon X \to Y$

$$f' = f_0 \xrightarrow{\varphi_1} f_1 \xrightarrow{\varphi_2} f_2 \; ... \xrightarrow{\varphi_n} f_n = f''. \qquad (61)$$

Computations are the same as in the standard structure, but involve different spaces: for instance, the trivial flexible homotopy of a c-map $f\colon X \to Y$, represented by $fe\colon X \times \uparrow\mathbb{I} \to Y$, is still defined as $fe(x,t) = f(x)$, on a different domain.

Future F-homotopy equivalences and *future F-deformation retracts* are defined as in (22) and (26), using composite flexible homotopies. The reversed notions, *in the past*, are produced by the reflector R. The coherent case is defined as in 2.6.

5.3 Proposition (Homotopy invariance)

(a) A composite flexible homotopy $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \twoheadrightarrow f''\colon X \to Y$ (as in (61)*) induces a natural transformation*

$$\varphi_*\colon f'_* \to f''_*\colon \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y), \tag{62}$$

the composite of the natural transformations $\varphi_{i*}\colon f_{i-1*} \to f_{i*}$ of Theorem II.5.4.

Induction agrees with the operations of composite flexible homotopies: their vertical composition and whisker composition (with c-maps) are sent to the corresponding operations of natural transformations (with functors).

(b) A future F-homotopy equivalence $(f, g; \varphi, \psi)$ of c-spaces X, Y induces a future homotopy equivalence of their fundamental categories (see (14)*)*

$$f_*\colon \uparrow\Pi_1(X) \rightleftarrows \uparrow\Pi_1(Y) : g_*, \quad \varphi_*\colon \mathrm{id} \to g_*f_*, \quad \psi_*\colon \mathrm{id} \to f_*g_*. \tag{63}$$

Proof. A consequence of Theorem II.5.4(b). □

5.4 Flexible contractibility

A *future F-contractible* c-space X is defined as in (27), using a composite flexible homotopy φ

$$i\colon \{*\} \rightleftarrows X : p, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow ip \qquad (pi = \mathrm{id}\,\{*\}). \tag{64}$$

In this case the singleton $\{*\}$ is a future F-deformation retract of X, at the flexible point $x_0 = i(*)$. We also say that X is *future F-contractible* to x_0.

Here the fundamental category of X is *future contractible* to the singleton category $\mathbf{1}$

$$i_*\colon \mathbf{1} \rightleftarrows {\uparrow}\Pi_1(X) : p_*, \quad \varphi_*\colon \mathrm{id}\,{\uparrow}\Pi_1(X) \to p_*i_* \;\; (p_*i_* = \mathrm{id}\,\mathbf{1}). \qquad (65)$$

Equivalently, ${\uparrow}\Pi_1(X)$ has a *natural weak terminal object* $x_0 = i(*)$. This means a family of arrows $\varphi(x)\colon x \to x_0$ (indexed on the flexible points $x \in |X|_0$) which is natural, in the sense that every arrow $u\colon x \to x'$ gives a commutative triangle $\varphi(x) = \varphi(x')u\colon x \to x_0$.

We say that X is *coarsely F-contractible* if there exists a finite sequence $X, X_1, ..., X_{n-1}, \{*\}$ where each c-space is future or past F-homotopy equivalent to the next.

The interval ${\uparrow}\mathbb{I}$ is past F-contractible to 0 and future F-contractible to 1, with flexible homotopies given by the (flexible) connections $g^\alpha\colon {\uparrow}\mathbb{I}^2 \to {\uparrow}\mathbb{I}$, as in (29).

5.5 Flexible connectedness

In a c-space X, a flexible path $a\colon x \to x'$ is the same as a c-path in the flexible part $\mathsf{Fl}\,X$ (in 1.1).

Therefore, flexible connectedness in X can be defined by the property of c-connectedness in $\mathsf{Fl}\,X$, characterised by the composed functor

$${\uparrow}\Pi_0\mathsf{Fl}\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}. \qquad (66)$$

The *flexible component* $[x]_F$ of a flexible point of the c-space X will be its c-component in $\mathsf{Fl}\,X$, contained in the controlled component $[x]_c$ and equal to the latter in any d-space. X is *flexibly connected*, or *F-connected*, if it has precisely one flexible component, if and only if ${\uparrow}\Pi_0\mathsf{Fl}\,(X)$ is a singleton.

Let $n \geqslant 1$. The c-spaces $\mathrm{c}\mathbb{S}^1$, $c_n\mathbb{S}^1$, $\mathrm{c}\mathbb{S}^n$ are trivially flexibly connected, as they have a single flexible point. The d-spaces ${\uparrow}\mathbb{I}$, $\mathbb{I}^\sim$, ${\uparrow}\mathbb{R}$, ${\uparrow}\mathbb{S}^n$ are flexibly connected, as well as the cartesian products of the examples considered so far. The other examples in 2.4(c) are not, which implies that they are not coarsely F-contractible, by the following proposition.

5.6 Proposition

(a) If the c-space X is future F-contractible to x_0, every flexible point x has a flexible path $x \to x_0$.

(b) A coarsely F-contractible c-space X is always flexibly connected.

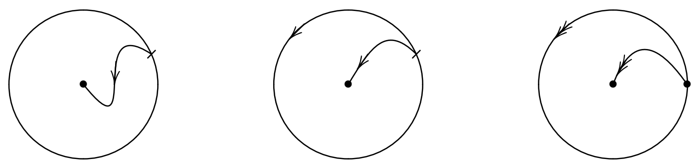
Proof. Point (b) follows from (a), which can be proved as in Proposition 2.5. Here we have flexible homotopies $\varphi_i \colon f_{i-1} \to f_i$, which are c-maps $X \times \uparrow\mathbb{I} \to X$ and, at any flexible point x, give a sequence of consecutive flexible paths $x \to f_1(x) \to ... \to x_0$. □

5.7 Flexible homotopy pushouts

The homotopy pushouts of the standard structure $\mathrm{c}\mathsf{Top}_c$, studied in Section 3, have parallel constructions in $\mathrm{c}\mathsf{Top}_F$, based on the flexible cylinder $I_F = - \times \uparrow\mathbb{I}$:

- the *flexible homotopy pushout* $I_F(f, g)$,
- the *flexible mapping cones* $C_F^+ f = I_F(f, p_X)$ and $C_F^- f = I_F(p_X, f)$,
- the *flexible cones* $C_F^+ X = I_F(X, p_X)$ and $C_F^- X = I_F(p_X, X)$,
- the *flexible suspension* $\Sigma_F X = I_F(p_X, p_X)$.

These constructions extend the corresponding ones of $\mathrm{d}\mathsf{Top}$, based on its endofunctor $I_d = - \times \uparrow\mathbb{I}$. Thus, in the following examples of flexible upper cones, the d-spaces $\mathbb{S}^1$ and $\uparrow\mathbb{S}^1$ give the upper cones of $\mathrm{d}\mathsf{Top}$, namely $C_F^+(\mathbb{S}^1) = C_d^+(\mathbb{S}^1)$ and $C_F^+(\uparrow\mathbb{S}^1) = C_d^+(\uparrow\mathbb{S}^1)$



(67)

$C_F^+(\mathbb{S}^1)$ $\quad$ $C_F^+(\uparrow\mathbb{S}^1)$ $\quad$ $C_F^+(\mathrm{c}\mathbb{S}^1)$

while $C_F^+(\mathrm{c}\mathbb{S}^1)$ is not a d-space, of course; but note that there are also flexible paths from the flexible point of the circle to the upper vertex.

5.8 Proposition (Flexible cones and contractibility)

(a) The flexible upper cone $C_F^+ X$ is future F-contractible, in one step.

(b) A c-space X is future F-contractible in one step if and only if the base $u\colon X \to C^+_F X$ has a retraction $h\colon C^+_F X \to X$.

Proof. (a) The pushout in (43) (at the right hand) is preserved by the product $-\times{\uparrow}\mathbb{I}$ (by an exponentiable object of cTop). There is thus precisely one map $\varphi\colon C^+_F X\times{\uparrow}\mathbb{I} \to C^+_F X$ such that

$$\begin{aligned}\varphi(\gamma\times{\uparrow}\mathbb{I}) = \gamma(X\times g^-)\colon X\times{\uparrow}\mathbb{I}\times{\uparrow}\mathbb{I} \to X\times{\uparrow}\mathbb{I} \to C^+_F X,\\ \varphi(v^+\times{\uparrow}\mathbb{I}) = v^+q\colon {\uparrow}\mathbb{I} \to \{*\} \to C^+_F X,\end{aligned} \tag{68}$$

taking into account that $\gamma(X\times g^-)(\partial^+\times{\uparrow}\mathbb{I})(x,t) = [x, 1\vee t] = v^+$.

This c-map is a flexible homotopy, from $\mathrm{id}\, C^+_F X$ to the constant map $C^+_F X \to C^+_F X$ at v^+

$$\begin{aligned}(\varphi\partial^-)[x,s] &= \varphi(\gamma(x,s),0) = \varphi(\gamma\times{\uparrow}\mathbb{I})(x,s,0)\\ &= \gamma(x, g^-(s,0)) = \gamma(x,s) = [x,s],\\ (\varphi\partial^+)[x,s] &= ... = \gamma(x, g^-(s,1)) = \gamma(x,1) = v^+.\end{aligned}$$

(b) As in Lemma 4.2. □

6. Cubical sets and their realisations

Cubical sets have a well-known non-symmetric monoidal structure. As a consequence, the obvious directed interval ${\uparrow}\mathbf{i}$, freely generated by a 1-cube (see 6.2) gives rise to a *left cylinder* ${\uparrow}\mathbf{i}\otimes K$ and a *right cylinder* $K\otimes{\uparrow}\mathbf{i}$, and two notions of directed homotopy interchanged by an endofunctor, the 'transposer' S.

Classically, cubical sets are viewed as combinatorial structures modelling relatively simple topological spaces, by their geometric realisation (see 6.4). But they can also model d-spaces, by a *directed* geometric realisation (in 6.5), and c-spaces, by a *controlled* geometric realisation (in 6.6).

Part of this material comes from [G2], Section 1.6.

6.1 Cubical sets

Every topological space X has an associated *cubical set* $\square X$, with components $\square_n X = \mathsf{Top}(\mathbb{I}^n, X)$, the set of *singular* n*-cubes* of X. Its faces and degeneracies, for $\alpha = 0, 1$ and $i = 1, ..., n$

$$\partial_i^\alpha \colon \square_n X \to \square_{n-1} X, \qquad e_i \colon \square_{n-1} X \to \square_n X, \tag{69}$$

come out (contravariantly) from the faces and degeneracies of the standard cubes $\mathbb{I}^n$, written in the same way

$$\begin{aligned} &\partial_i^\alpha = \mathbb{I}^{i-1} \times \partial^\alpha \times \mathbb{I}^{n-i} \colon \mathbb{I}^{n-1} \to \mathbb{I}^n, \\ &\qquad \partial_i^\alpha(t_1, ..., t_{n-1}) = (t_1, ..., \alpha, ..., t_{n-i}), \\ &e_i = \mathbb{I}^{i-1} \times e \times \mathbb{I}^{n-i} \colon \mathbb{I}^n \to \mathbb{I}^{n-1}, \\ &\qquad e_i(t_1, ..., t_n) = (t_1, ..., \hat{t}_i, ..., t_n). \end{aligned} \tag{70}$$

As usual, $\hat{t}_i$ means that the coordinate t_i is omitted.

Generally, a *cubical set* K is a sequence of sets K_n $(n \geqslant 0)$, together with mappings, called *faces* (∂_i^α) and *degeneracies* (e_i)

$$\begin{gathered} K = ((K_n), (\partial_i^\alpha), (e_i)), \\ \partial_i^\alpha = \partial_{ni}^\alpha \colon K_n \to K_{n-1}, \qquad e_i = e_{ni} \colon K_{n-1} \to K_n, \end{gathered} \tag{71}$$

(for $\alpha = \pm$ and $i = 1, ..., n$) that satisfy the *cubical* relations

$$\begin{gathered} \partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_{i+1}^\alpha \ \ (j \leqslant i), \qquad e_j e_i = e_{i+1} e_j \ \ (j \leqslant i), \\ \partial_i^\alpha e_j = e_j \partial_{i-1}^\alpha \ (j < i), \ \text{ or } \ \mathrm{id} \ (j = i), \ \text{ or } \ e_{j-1}.\partial_i^\alpha \ (j > i). \end{gathered} \tag{72}$$

Elements of K_n are called *n-cubes*, and *vertices* or *edges* for $n = 0$ or 1, respectively. Each n-cube $x \in K_n$ has 2^n vertices: $\partial_1^\alpha \partial_2^\beta \partial_3^\gamma(x)$ for $n = 3$. Given a vertex $x \in K_0$, the *totally degenerate n-cube at x* is obtained by applying n degeneracy operators to the given vertex, in any (legitimate) way:

$$e^n(x) = e_{i_n} ... \, e_{i_2} e_{i_1}(x) \in K_n \qquad\qquad (1 \leqslant i_j \leqslant j). \tag{73}$$

A *morphism* $f = (f_n) \colon K \to L$ is a sequence of mappings $f_n \colon K_n \to L_n$ commuting with faces and degeneracies.

All this forms a category Cub which has all limits and colimits and is cartesian closed. It is the presheaf category of functors $K \colon \underline{\mathrm{I}}^{\mathrm{op}} \to \mathsf{Set}$, where $\underline{\mathrm{I}}$ is the subcategory of Set consisting of the *elementary cubes* 2^n, together with the maps $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates. One can see [G2], Sections 1.6.7 and A1.8; or [GM] for cubical sets with a richer structure, including connections and symmetries.

The terminal object $\top$ is freely generated by one vertex $*$ and will also be written as $\{*\}$ (although each of its components is a singleton). The initial object is empty, i.e. all its components are; all the other cubical sets have non-empty components in every degree.

We shall make use of two covariant involutive endofunctors, the *reversor* R and the *transposer* S

$$\begin{aligned} &R\colon \mathsf{Cub} \to \mathsf{Cub}, \qquad RK = K^{\mathrm{op}} = ((K_n), (\partial_i^{-\alpha}), (e_i)), \\ &S\colon \mathsf{Cub} \to \mathsf{Cub}, \qquad SK = ((K_n), (\partial_{n+1-i}^{\alpha}), (e_{n+1-i})), \\ &RR = \mathrm{id}\,, \qquad SS = \mathrm{id}\,, \qquad RS = SR. \end{aligned} \tag{74}$$

(The meaning of $-\alpha$, for $\alpha = \pm$, is obvious.) The functor R reverses the 1-dimensional direction, while S reverses the 2-dimensional one; plainly, they commute. If $x \in K_n$, the same element viewed in K^{op} will often be written as x^{op}, so that $\partial_i^-(x^{\mathrm{op}}) = (\partial_i^+ x)^{\mathrm{op}}$.

We say that a cubical set K is *reversive* if $RK \cong K$, and *permutative* if $SK \cong K$.

The category Cub has a *non*-symmetric monoidal structure [Ka, BH]

$$(K \otimes L)_n = (\Sigma_{p+q=n} K_p \times L_q)/ \sim_n, \tag{75}$$

where $\sim_n$ is the equivalence relation generated by identifying $(e_{r+1}x, y)$ with $(x, e_1 y)$, for all $(x, y) \in K_r \times L_s$ (where $r + s = n - 1$). The equivalence class of (x, y) is written as $x \otimes y$.

We refer to [G2], 1.6.3, for a more detailed description.

6.2 Standard models

The *elementary directed interval* $\uparrow\mathbf{i} = \mathbf{2}$ is freely generated by a 1-cube, written as u

$$0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \qquad \partial_1^+(u) = 1. \tag{76}$$

This cubical set is reversive and permutative.

The *elementary directed* n*-cube* $(n \geqslant 0)$ is its n-th tensor power $\uparrow\!\mathbf{i}^{\otimes n} = \uparrow\!\mathbf{i} \otimes ... \otimes \uparrow\!\mathbf{i}$ ([G2], 1.6.3). It is freely generated by its n-cube $u^{\otimes n}$, still reversive and permutative, and can also be defined as the representable presheaf $\underline{\mathrm{I}}(-, 2^n)\colon \underline{\mathrm{I}}^{\mathrm{op}} \to \mathsf{Set}$. The *elementary directed square* $\uparrow\!\mathbf{i}^{\otimes 2} = \uparrow\!\mathbf{i} \otimes \uparrow\!\mathbf{i}$ can be represented as follows, with the generator $u \otimes u$, its faces and vertices

$$\begin{array}{ccc} 00 & \xrightarrow{0\otimes u} & 01 \\ {\scriptstyle u\otimes 0}\downarrow & {\scriptstyle u\otimes u} & \downarrow{\scriptstyle u\otimes 1} \\ 10 & \xrightarrow[1\otimes u]{} & 11 \end{array} \qquad\qquad \begin{array}{cc} \bullet & \xrightarrow[2]{} \\ \downarrow{\scriptstyle 1} & \end{array} \tag{77}$$

The face $\partial_1^-(u \otimes u) = 0 \otimes u$ is drawn as orthogonal to direction 1 (and directions are chosen so that the labelling of vertices agrees with matrix indexing). For each cubical object K, $\mathsf{Cub}(\uparrow\!\mathbf{i}^{\otimes n}, K) = K_n$, by Yoneda Lemma.

The *directed (integral) line* $\uparrow\!\mathbf{z}$ is generated by (countably many) vertices $n \in \mathbb{Z}$ and edges u_n, from $\partial_1^-(u_n) = n$ to $\partial_1^+(u_n) = n + 1$. The *directed integral interval* $\uparrow\![i, j]_{\mathbf{z}}$ is the obvious cubical subset with vertices in the integral interval $[i, j]_{\mathbf{z}}$ (and all cubes whose vertices lie there); in particular, $\uparrow\!\mathbf{i} = \uparrow\![0, 1]_{\mathbf{z}}$.

The *elementary directed circle* $\uparrow\!\mathbf{s}^1$ is generated by one 1-cube u with equal faces

$$* \xrightarrow{u} * \qquad\qquad \partial_1^-(u) = \partial_1^+(u). \tag{78}$$

Similarly, the *elementary directed* n*-sphere* $\uparrow\!\mathbf{s}^n$ (for $n > 1$) is generated by one n-cube u all whose faces are totally degenerate (see (73)), hence equal

$$\partial_i^\alpha(u) = e^{n-1}(\partial_1^-)^n(u) \qquad\qquad (\alpha = \pm;\ i = 1, ..., n). \tag{79}$$

Moreover, $\uparrow\!\mathbf{s}^0$ is the discrete cubical set on two vertices. The *elementary directed* n*-torus* is a tensor power of $\uparrow\!\mathbf{s}^1$

$$\uparrow\!\mathbf{t}^n = (\uparrow\!\mathbf{s}^1)^{\otimes n}. \tag{80}$$

We also consider the *ordered circle* $\uparrow\!\mathbf{o}^1$, generated by two edges with the same faces

$$0 \overset{u'}{\underset{u''}{\rightrightarrows}} 1 \qquad\qquad \partial_1^\alpha(u') = \partial_1^\alpha(u''), \tag{81}$$

which is a 'cubical model' of the ordered circle $\uparrow\mathbb{O}^1$, a d-space recalled in 4.3. (The latter is the directed geometric realisation of the former, in the sense of 6.5.)

Starting from $\uparrow\mathbf{s}^0$, the *unpointed suspension* provides all $\uparrow\mathbf{o}^n$ ([G2], 1.7.6) while the *pointed suspension* provides all $\uparrow\mathbf{s}^n$ ([G2], 2.3.2). Of course, these models have the same geometric realisation $\mathbb{S}^n$ (as a topological space) and the same homology; but their *directed* homology is different ([G2], 2.1.4): the models $\uparrow\mathbf{s}^n$ are more interesting: they have a non-trivial order in directed homology.

All these cubical sets are reversive and permutative.

6.3 The homotopy structure

As shown in [G2], 1.6.5, the category of cubical sets has a left dIP1-structure Cub_L defined by the left cylinder functor $I(K) = \uparrow\mathbf{i} \otimes K$, and a right dIP1-structure Cub_R defined by the right cylinder functor $SIS(K) = K \otimes \uparrow\mathbf{i}$. These structures are made isomorphic by the symmetriser $S\colon \mathsf{Cub}_L \to \mathsf{Cub}_R$.

The left cylinder P has a simple description: it shifts down all the components, discarding the faces and degeneracies of index 1, which are then used to build three natural transformations, the faces and degeneracy of P

$$\begin{array}{ll} P\colon \mathsf{Cub} \to \mathsf{Cub}, & PK = ((K_{n+1}), (\partial^\alpha_{i+1}), (e_{i+1})), \\ \partial^\alpha = \partial^\alpha_1\colon PK \to K, & e = e_1\colon K \to PK. \end{array} \qquad (82)$$

Symmetrically, the right cylinder SPS shifts down all components, discarding the faces and degeneracies of highest index.

The transposition of spaces, d-spaces and c-spaces is partially surrogated here by an 'external transposition', $s\colon PSPS \to SPSP$, whose components are identities ([G2], 1.6.5).

Coming back to the discussion of symmetries in 1.6, we note that Cub breaks both symmetries of topological spaces, reversion and transposition. This has heavy consequences for homotopy theory, as remarked in 1.6, and strong advantages for homology, recalled in 6.8.

6.4 The classical geometric realisation

We have already recalled the functor

$$\square\colon \mathsf{Top} \to \mathsf{Cub}, \qquad \square X = \mathsf{Top}(\mathbb{I}^\bullet, X), \tag{83}$$

which assigns to a topological space X the singular cubical set of n-cubes $\mathbb{I}^n \to X$, produced by the cocubical space of standard cubes

$$\mathbb{I}^\bullet = ((\mathbb{I}^n), (\partial_i^\alpha), (e_i)),$$

a covariant functor $\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathsf{Top}$. The *geometric realisation* $\mathcal{R}K$ of a cubical set is given by its left adjoint

$$\mathcal{R}\colon \mathsf{Cub} \rightleftarrows \mathrm{d}\mathsf{Top} : \square, \qquad \mathcal{R} \dashv \square. \tag{84}$$

The topological space $\mathcal{R}K$ is constructed by pasting a copy of the standard cube $\mathbb{I}^n$ for each n-cube $x \in K_n$, along faces and degeneracies. This colimit comes with a cocone of structural mappings $\hat{x}$ (for $x \in K_n$ and $n \in \mathbb{N}$), coherently with faces and degeneracies of $\mathbb{I}^\bullet$ and K

$$\hat{x}\colon \mathbb{I}^n \to \mathcal{R}K, \qquad \hat{x}\partial_i^\alpha = (\partial_i^\alpha x)\hat{}, \qquad \hat{x}e_i = (e_i x)\hat{}, \tag{85}$$

and $\mathcal{R}K$ has the finest topology making all the structural mappings continuous. (Formally, $\mathcal{R}$ is the coend of the functor $K\mathbb{I}^\bullet\colon \underline{\mathrm{I}}^{\mathrm{op}}\times\underline{\mathrm{I}} \to \mathsf{Set}\times\mathsf{Top} \to \mathsf{Top}$, see [M].)

This realisation is important, since it is well known that the combinatorial homology of a cubical set K coincides with the homology of the CW-space $\mathcal{R}K$ (cf. [Mu] 4.39, for the simplicial case). But we also want finer realisations, retaining more information on the cubes of K: we shall use a d-space (in 6.5), or a c-space (in 6.6, 6.7).

6.5 Directed geometric realisation

Cubical sets also have a realisation as d-spaces, where the n-cube $\uparrow\mathbf{i}^{\otimes n}$ is realised as $\uparrow\mathbb{I}^n$.

We replace the cocubical space $\mathbb{I}^\bullet$ of standard topological cubes $\mathbb{I}^n$ by a directed version, the cocubical d-space $\uparrow\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathrm{d}\mathsf{Top}$ of standard d-cubes

$\uparrow\mathbb{I}^n$, with the corresponding faces and degeneracies

$$\begin{aligned} \partial_i^\alpha &= \uparrow\mathbb{I}^{i-1} \times \partial^\alpha \times \uparrow\mathbb{I}^{n-i} \colon \uparrow\mathbb{I}^{n-1} \to \uparrow\mathbb{I}^n, \\ e_i &= \uparrow\mathbb{I}^{i-1} \times e \times \uparrow\mathbb{I}^{n-i} \colon \uparrow\mathbb{I}^{n+1} \to \uparrow\mathbb{I}^n. \end{aligned} \tag{86}$$

This produces the functor

$$\uparrow\Box \colon \mathrm{dTop} \to \mathsf{Cub}, \qquad \uparrow\Box_n(X) = \mathrm{dTop}(\uparrow\mathbb{I}^n, X), \tag{87}$$

which assigns to a d-space X the singular cubical set $\uparrow\Box(X)$ of its directed n-cubes $\uparrow\mathbb{I}^n \to X$, extending the functor $\Box\colon \mathsf{Top} \to \mathsf{Cub}$. Its left adjoint yields the *directed geometric realisation* $\uparrow\mathcal{R}(K)$ of a cubical set K

$$\uparrow\mathcal{R} \colon \mathsf{Cub} \rightleftarrows \mathrm{dTop} : \uparrow\Box, \qquad \uparrow\mathcal{R} \dashv \uparrow\Box. \tag{88}$$

The d-space $\uparrow\mathcal{R}(K)$ is thus the pasting in dTop of K_n copies of $\uparrow\mathbb{I}^\bullet(2^n) = \uparrow\mathbb{I}^n$ ($n \geqslant 0$), along faces and degeneracies. (Again, the coend of the functor $K\uparrow\mathbb{I}^\bullet \colon \underline{\mathrm{I}}^{\mathrm{op}} \times \underline{\mathrm{I}} \to \mathrm{dTop}$.)

In other words (since a colimit in dTop is the colimit of the underlying topological spaces, equipped with the final d-structure of the structural maps), one starts from the ordinary geometric realisation $\mathcal{R}K$, as a topological space, and equips it with the following d-structure $\uparrow\mathcal{R}(K)$: the d-paths are generated, under concatenation and partial reparametrisation, by the mappings $\hat{x}a \colon \mathbb{I} \to \mathbb{I}^n \to \mathcal{R}K$, where $a\colon \mathbb{I} \to \mathbb{I}^n$ is an order-preserving map and $\hat{x}$ corresponds to some cube $x \in K_n$, in the colimit-construction of $\mathcal{R}K$.

Composing (88) with the adjunction $U \dashv D'$ (in I.1.7) between d-spaces and spaces

$$\mathsf{Cub} \underset{\uparrow\Box}{\overset{\uparrow\mathcal{R}}{\rightleftarrows}} \mathrm{dTop} \underset{D'}{\overset{U}{\rightleftarrows}} \mathsf{Top} \tag{89}$$

we get back the ordinary realisation $\mathcal{R} = U(\uparrow\mathcal{R}) \colon \mathsf{Cub} \to \mathsf{Top}$. (The d-space $D'X$ associated to a space admits all paths as d-paths.)

Various models of dTop are directed realisations of simple cubical sets already considered in 6.2. For instance, the directed interval $\uparrow\mathbb{I}$ realises $\uparrow\mathbf{i}$; the directed line $\uparrow\mathbb{R}$ realises $\uparrow\mathbf{z}$; the directed sphere $\uparrow\mathbb{S}^n$ realises $\uparrow\mathbf{s}^n$; the ordered circle $\uparrow\mathbb{O}^1$ (cf. 4.3) realises $\uparrow\mathbf{o}^1 = \{0 \rightrightarrows 1\}$.

The directed realisation functor $\uparrow\mathcal{R} \colon \mathsf{Cub} \to \mathrm{dTop}$ is a strong dI1-functor, which means that it commutes with the cylinder functor, up to functorial isomorphism: see [G2], 1.6.7.

6.6 Controlled geometric realisation

Cubical sets can also be realised as c-spaces, turning the n-cube $\uparrow\mathbf{i}^{\otimes n}$ into the standard c-cube $\mathrm{c}\mathbb{I}^n$.

Now we start from the cocubical c-space $\mathrm{c}\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathrm{cTop}$ of standard c-cubes $\mathrm{c}\mathbb{I}^n$, with the corresponding faces and degeneracies

$$\partial_i^\alpha = \mathrm{c}\mathbb{I}^{i-1} \times \partial^\alpha \times \mathrm{c}\mathbb{I}^{n-i}, \qquad e_i = \mathrm{c}\mathbb{I}^{i-1} \times e \times \mathrm{c}\mathbb{I}^{n-i}. \tag{90}$$

This produces the functor

$$\mathrm{c}\Box\colon \mathrm{cTop} \to \mathsf{Cub}, \qquad \mathrm{c}\Box_n(X) = \mathrm{cTop}(\mathrm{c}\mathbb{I}^n, X), \tag{91}$$

which assigns to a c-space X the singular cubical set $\mathrm{c}\Box(X)$ of its controlled n-cubes $\mathrm{c}\mathbb{I}^n \to X$; it is an extension of the functor $\uparrow\Box\colon \mathrm{dTop} \to \mathsf{Cub}$, because the reflector of d-spaces in cTop gives $(\mathrm{c}\mathbb{I}^n)\hat{} = \uparrow\mathbb{I}^n$ (see I.2.7). The *controlled geometric realisation* $\mathrm{c}\mathcal{R}(K)$ of a cubical set is given by the left adjoint functor

$$\mathrm{c}\mathcal{R}\colon \mathsf{Cub} \rightleftarrows \mathrm{cTop} : \mathrm{c}\Box, \qquad \mathrm{c}\mathcal{R} \dashv \mathrm{c}\Box. \tag{92}$$

The c-space $\mathrm{c}\mathcal{R}(K)$ is thus the pasting in cTop of K_n copies of $\mathrm{c}\mathbb{I}^\bullet(2^n) = \mathrm{c}\mathbb{I}^n$ ($n \geqslant 0$), along faces and degeneracies – the coend of the functor $K\mathrm{c}\mathbb{I}^\bullet\colon \underline{\mathrm{I}}^{\mathrm{op}} \times \underline{\mathrm{I}} \to \mathrm{cTop}$.

Again, $\mathrm{c}\mathcal{R}(K)$ is the geometric realisation $\mathcal{R}K$ with controlled paths generated, under concatenation and global reparametrisation, by the mappings $\hat{x}a\colon \mathbb{I} \to \mathbb{I}^n \to \mathcal{R}K$, where $a\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^n$ is a c-path and $\hat{x}$ corresponds to a cube $x \in K_n$ in the colimit-construction of $\mathcal{R}K$.

Composing this adjunction with the canonical adjunctions between c-spaces, d-spaces and spaces (in I.1)

$$\mathsf{Cub} \underset{\mathrm{c}\Box}{\overset{\mathrm{c}\mathcal{R}}{\rightleftarrows}} \mathrm{cTop} \underset{\supset}{\overset{\hat{}}{\rightleftarrows}} \mathrm{dTop} \underset{D'}{\overset{U}{\rightleftarrows}} \mathsf{Top} \tag{93}$$

we prove that the controlled realisation is consistent with the previous ones

$$(\mathrm{c}\mathcal{R}(K))\hat{} = \uparrow\mathcal{R}(K), \qquad U((\mathrm{c}\mathcal{R}(K))\hat{}) = \mathcal{R}K. \tag{94}$$

Also here, various models of cTop are controlled realisations of simple cubical sets: the controlled interval $\mathrm{c}\mathbb{I}$ realises $\uparrow\mathbf{i}$; the controlled line $\mathrm{c}\mathbb{R}$

realises $\uparrow\mathbf{z}$; the controlled sphere $\mathrm{c}\mathbb{S}^n$ realises $\uparrow\mathbf{s}^n$; the controlled ordered circle $\mathrm{c}\mathbb{O}^1$ (in (49)) realises $\uparrow\mathbf{o}^1$.

The controlled realisation functor $\mathrm{c}\mathcal{R}\colon \mathsf{Cub} \to \mathrm{c}\mathsf{Top}$ is also a strong dI1-functor, with the same proof as in [G2], 1.6.7.

6.7 Labelled cubical sets

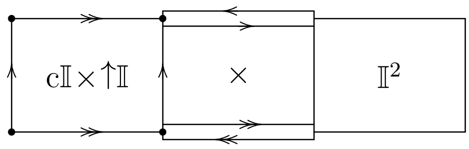
More generally, one can define the geometric realisation $\mathcal{R}K$ of a *c-labelled cubical set*: a cubical set $K = ((K_n), (\partial_i^\alpha), (e_i))$ where each n-cube $x \in K_n$ is labelled with a c-structure $K^n(x)$ on the euclidean n-cube $\mathbb{I}^n$, under a *coherence condition*: all the faces and degeneracies of the euclidean cubes give c-maps

$$\partial_i^\alpha\colon K^{n-1}(\partial_i^\alpha x) \to K^n(x), \qquad e_i\colon K^{n+1}(e_i x) \to K^n(x). \tag{95}$$

The first condition, for instance, means that each face $K^{n-1}(\partial_i^\alpha x)$ has a c-structure (weakly) finer than that induced by $K^n(x)$.

The *geometric realisation* of K is defined as the colimit in $\mathrm{c}\mathsf{Top}$ of the diagram formed by all the c-spaces $K^n(x)$, with the c-maps specified above.

The following 2-dimensional example is about a simpler case, a c-labelled *face-cubical set* (without degeneracies)



(96)

The labels $\mathrm{c}\mathbb{I}$, $\uparrow\mathbb{I}$, $\mathbb{I}$ of the edges are replaced by the symbols we have been using: $\twoheadrightarrow, \to$, or unmarked. The cross in the central rectangle means that there are no 2-cubes inside.

Finally, $\mathcal{R}K$ is a quotient of a sum of c-spaces

$$(\mathrm{c}\mathbb{I}\times\uparrow\mathbb{I}) + \mathbb{I}^2 + \uparrow\mathbb{I} + \uparrow\mathbb{I}^{\mathrm{op}} + \mathrm{c}\mathbb{I} + \mathrm{c}\mathbb{I}^{\mathrm{op}}, \tag{97}$$

modulo the equivalence relation that identifies vertices as shown in the picture.

$\uparrow\mathbb{I}^{\mathrm{op}}$ and $\mathrm{c}\mathbb{I}^{\mathrm{op}}$ can be replaced by their opposites, $\uparrow\mathbb{I}$ and $\mathrm{c}\mathbb{I}$, which are isomorphic to the former; yet, formula (97) makes identifications easier.

6.8 Directed homology

Cubical sets have a directed homology (introduced in [G1]), taking values in preordered abelian groups. For instance, $\uparrow\!H_n(\uparrow\!\mathrm{s}^n) = \uparrow\!\mathbb{Z}$, the ordered abelian group of the integers.

We refer to [G2], Chapter 2, for this theory and its strong relationship with noncommutative geometry.

Obviously, one can define the directed homology of a c-space X, letting

$$\uparrow\!H_n(X) = \uparrow\!H_n(\mathrm{c}\Box(X)), \tag{98}$$

but the interest of this issue is not clear.

Already in [G2], the directed homology of cubical sets is far more interesting than the derived directed homology of d-spaces. This comes out of the fact that the directed character of d-spaces (and c-spaces) does not go beyond the one-dimensional level: after selecting some paths and forbidding others, no higher choice is needed.

On the other hand, a cubical set K has privileged selections in any dimension: an element of K_n need not have any counterpart with faces reversed in some direction (for $n \geqslant 1$), nor permuted (for $n \geqslant 2$). This richer choice is paid with many drawbacks, starting with the lack of path-concatenation, which is not needed for homology but obviously needed for homotopy. However, the weak homotopy structure Cub_L can be enriched to a *relative dI-homotopical structure*, by means of the functor $\uparrow\!\mathcal{R}\colon \mathsf{Cub} \to \mathrm{d}\mathsf{Top}$ ([G2], 5.8.6) which takes values in a good dIP4-homotopical structure.

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