CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
VOLUME LXVI-2 (2025)

THE TOPOLOGY OF CRITICAL PROCESSES, IV
(THE HOMOTOPY STRUCTURE)

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Résumé. La Topologie Algébrique Dirigée étudie des espaces équipés d'une forme de direction, avec l'objectif d'inclure les processus non réversibles. Dans l'extension présente nous voulons couvrir aussi les *processus critiques*, indivisibles et inarrêtables, du changement d'état dans une cellule de mémoire à l'action d'un thermostat.

Les parties précédentes de cette série ont introduit les *espaces contrôlés*, en examinant comment il peuvent modeler les processus critiques issus de divers domaines, et étudié leur catégorie fondamentale. Ici on traite leur structure formelle d'homotopie.

Abstract. Directed Algebraic Topology studies spaces equipped with a form of direction, to include models of non-reversible processes. In the present extension we also want to cover 'critical processes', indecomposable and unstoppable – from the change of state in a memory cell to the action of a thermostat.

The previous parts of this series introduced *controlled spaces*, examining how they can model critical processes in various domains, and studied their fundamental category. Here we deal with their formal homotopy theory.

Keywords. Directed algebraic topology, homotopy theory, fundamental category, concurrent process.

Mathematics Subject Classification (2010). 55M, 55P, 55Q, 68Q85.

Introduction

0.1 Directed and controlled spaces

Directed Algebraic Topology is an extension of Algebraic Topology, dealing with 'spaces' where the paths need not be reversible; the general aim is including the representation of *irreversible processes*. The category $\mathrm{d}\mathsf{Top}$ of directed spaces, or d-spaces [G2], is a typical setting for this study; it is frequently employed in the theory of concurrency: see [FGHMR]. Homotopy in $\mathrm{d}\mathsf{Top}$ is based on the standard directed interval $\uparrow\mathbb{I}$, whose paths are the (weakly) increasing continuous mappings $[0,1] \to [0,1]$. It is the basic model of a non-reversible process, or a one-way road.

This article belongs to a series devoted to a further extension, where the paths can also be non-decomposable in order to include *critical processes*, indivisible and unstoppable – either reversible or not. For instance: quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, the action of a siphon, moving in a no-stop road, etc.

Part I [G3] introduced the category $\mathrm{c}\mathsf{Top}$ of controlled spaces, or c-spaces, examining how they can model critical processes; the definition is reviewed in 1.1. The previous setting of d-spaces is embedded in $\mathrm{c}\mathsf{Top}$ as the full, reflective and coreflective subcategory of 'flexible' c-spaces. Homotopy in $\mathrm{c}\mathsf{Top}$ is based on the standard controlled interval $\mathrm{c}\mathbb{I}$, whose paths are the increasing continuous mappings $[0,1] \to [0,1]$ which are either surjective or constant at 0 or 1. It is the basic model of a non-reversible unstoppable process, or a one-way no-stop road.

Parts II and III [G4, G5] introduced and studied the fundamental category of controlled spaces, as a functor

$$\uparrow\Pi_1 \colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \qquad (1)$$

which extends the functor $\uparrow\Pi_1 \colon \mathrm{d}\mathsf{Top} \to \mathsf{Cat}$ studied in [G2].

The extension is not obvious, essentially because the cylinder functor of $\mathrm{c}\mathsf{Top}$ does not preserve pushouts. The problem was overcome with a hybrid use of 'general' and 'flexible' homotopies, and some new methods of computation. Covering maps work well ([G4], Theorem 5.8), but the van Kampen theorem for the fundamental category of d-spaces cannot be extended, as it is based on the subdivision of paths.

Here we study the homotopy theory of c-spaces, including homotopy equivalences, homotopy constructions like cones and suspension, and the relationship with cubical sets.

0.2 Outline

In Section 1 we analyse the formal theory of homotopy in $\mathrm{c}\mathsf{Top}$, following the classification of directed settings in [G2]: it is a symmetric dI2-category with concatenation pushout of the interval (to concatenate paths), no concatenation pushout of the cylinder functor (homotopies cannot be concatenated), and no path functor (the cylinder functor has no right adjoint). For comparison, $\mathrm{d}\mathsf{Top}$ is a dIP4-homotopical category, with far stronger properties. The breaking of symmetries and 'extended symmetries' in $\mathrm{c}\mathsf{Top}$ is examined in 1.6. In 1.7 and 1.8 we review the basic elements of homotopy theory in Cat.

Composite homotopies of c-spaces are introduced in Section 2, together with forms of directed homotopy equivalence, contractibility and connectedness adequate to the present setting.

Sections 3 and 4 deal with homotopy pushouts, cones and suspension. Weak flexibility properties (see 3.1) are used to counteract the fact that pushouts are not preserved by the cylinder functor.

The flexible interval $\uparrow\mathbb{I}$ (the basis of homotopy for d-spaces) produces a second homotopy structure for c-spaces, denoted as $\mathrm{c}\mathsf{Top}_F$ and examined in Section 5. In fact, we have already seen in Part II (Theorems 5.4 and 5.5) that the fundamental category functor is invariant up to flexible homotopy.

Finally, Section 6 is about cubical sets. A cubical set has diverse geometric realisations: as a topological space (the classical realisation, pasting topological cubes $\mathbb{I}^n$), as a directed space (pasting directed cubes $\uparrow\mathbb{I}^n$), and as a controlled space (pasting controlled cubes $\mathrm{c}\mathbb{I}^n$). All this can be combined: a cubical set labelled in $\mathrm{c}\mathsf{Top}$ comprises all these instances and their aggregations (see 6.7).

0.3 An overview of this series

(a) Reviewing our aims, Part I explores how the new controlled spaces can model concrete, critical processes and their interaction with continuous variation.

Controlled spaces can thus unify aspects of continuous and discrete mathematics. Moreover, they can interpret phenomena of diverse domains in a single system of mathematical models, which can be combined together and studied with extensions of the usual tools of Algebraic Topology.

(b) Following this program, Parts II and III introduce and investigate the fundamental category of controlled spaces. This is used, in particular, to classify obstructions in problems related to concurrency (Part III, Section 2).

Now, the fundamental category studies *controlled paths* up to *flexible homotopy*. Homotopy properties are more involved here than in the tamer world of directed spaces, and we have to blend the standard homotopy structure, based on the controlled interval $\mathrm{c}\mathbb{I}$, with the flexible one, based on the directed interval $\uparrow\!\mathbb{I}$.

(c) The present part aims to make clear the role of these homotopy structures on the category of controlled spaces, in the general frame of directed homotopy built in the book [G2]. In our opinion, the interest and beauty of peculiar structures, like controlled cones and spheres (in Section 4), might be a sufficient impulse to study their formal world. Of course, this study is not concluded here.

In a marginal way, we can add that labelled cubical sets, considered in 6.7, might be used to model traffic networks where roads (possibly one-way, or no-stop, or including delays) interact with planar areas, e.g. parking lots and desert lands; or rivers and canals interact with lakes and sees.

0.4 Terminology and notation

A continuous mapping between topological spaces, possibly structured, is called a *map*. $\mathbb{R}$ denotes the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0, 1]$. Similarly $\mathbb{R}^n$ and $\mathbb{I}^n$ are euclidean spaces. $\mathbb{S}^n$ is the n-dimensional sphere. The open and semiopen intervals of the real line are always denoted by square brackets, like $]0, 1[$, $[0, 1[$, etc.

The symbol $\subset$ denotes weak inclusion. The binary variable α takes values $0, 1$, also written, respectively, as $-, +$ in superscripts and subscripts. Marginal remarks are written in small characters.

The previous papers [G3, G4, G5] of this series are cited as Part I, Part II and Part III, respectively. The reference I.2, or II.3, or III.1.4, points to Section 2 of Part I, or Section 3 of Part II, or Subsection 1.4 of Part III.

0.5 Acknowledgements

The author is grateful to the Referee for an exceptionally extensive, accurate report, raising many interesting points about motivation, clarity and possible developments.

1. The standard homotopy structure of controlled spaces

We examine the formal homotopy structure of c-spaces, following the classification of such structures in [G2].

The component $\lambda X\colon F(X) \to G(X)$ of a natural transformation between functors is often written as $\lambda\colon FX \to GX$.

1.1 Controlled spaces

The category $\mathrm{c}\mathsf{Top}$ of *controlled spaces*, or *c-spaces*, was introduced in Part I [G3]; we briefly recall the main definitions.

An object X is a topological space equipped with a set $X^\sharp$ of continuous mappings $a\colon [0,1] \to X$, called *controlled paths*, or *c-paths*, that satisfies three axioms:

(csp.0) (*constant paths*) the trivial loops at the endpoints of a controlled path are controlled,

(csp.1) (*concatenation*) the concatenation of consecutive controlled paths is controlled,

(csp.2) (*global reparametrisation*) the reparametrisation $a\rho$ of a controlled path a by a surjective (weakly) increasing map $\rho\colon [0,1] \to [0,1]$ is controlled.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths.

The reversor functor $R\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ of the category of c-spaces sends X to the opposite object $RX = X^{\mathrm{op}}$, with reversed selected paths.

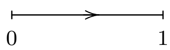
The category $\mathrm{c}\mathsf{Top}$ contains the category $\mathrm{d}\mathsf{Top}$ of d-spaces (studied in [G2]) as a full subcategory, reflective and coreflective: a c-space is a d-space if and only if it is *flexible*, which means that each point is flexible (its trivial loop is controlled) and every controlled path is flexible (all its restrictions are controlled). Both categories are complete and cocomplete, with limits (resp. colimits) calculated as in Top and enriched with initial (resp. final) structures. The terminal object is the singleton $\{*\}$ of $\mathrm{d}\mathsf{Top}$, also called the flexible singleton when viewed in $\mathrm{c}\mathsf{Top}$.

The reflector $\mathrm{c}\mathsf{Top} \to \mathrm{d}\mathsf{Top}$ (cf. I.1.8, that is Section 1.8 of Part I) takes a c-space X to the *generated d-space* $\hat{X}$, with the same underlying topological space and the d-structure generated by the c-paths; the unit of the adjunction is the *reshaping* $X \to \hat{X}$, whose underlying map is the identity. The coreflector takes X to the *flexible part* $\mathsf{Fl}\, X$, namely the subspace of flexible points $|X|_0$ (called the *flexible support*) with the d-structure of the flexible c-paths; the counit is the inclusion $\mathsf{Fl}\, X \to X$.

1.2 Structured intervals and lines

(a) In $\mathrm{d}\mathsf{Top}$ the *standard d-interval* $\uparrow\!\mathbb{I}$ has the d-structure generated by the identity $\mathrm{id}\,\mathbb{I}$: the directed paths are all the increasing maps $\mathbb{I} \to \mathbb{I}$. It plays the role of the standard interval in this category, because the directed paths of any d-space X coincide with the d-maps $\uparrow\!\mathbb{I} \to X$.

It may be viewed as an essential model of a non-reversible process, or a one-way road in transport networks. It is represented as

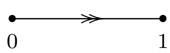


$\uparrow\!\mathbb{I}$ (2)

Similarly, the directed line $\uparrow\!\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$.

(b) In $\mathrm{c}\mathsf{Top}$ the *standard c-interval* $\mathrm{c}\mathbb{I}$, or *one-jump interval*, has the same support, with the c-structure generated by the identity $\mathrm{id}\,\mathbb{I}$: the controlled paths are the surjective increasing maps $\mathbb{I} \to \mathbb{I}$ and the trivial loops at 0 or 1. The controlled paths of any c-space X coincide with the c-maps $\mathrm{c}\mathbb{I} \to X$.

It can model a *non-reversible unstoppable process*, or a *one-way no-stop road*. It is represented as



$\mathrm{c}\mathbb{I}$ (3)

marking by a bullet the isolated flexible points: here the endpoints of the interval. The *controlled line* $\mathrm{c}\mathbb{R}$ has for directed paths all the increasing maps $\mathbb{I} \to \mathbb{R}$ whose image is an interval $[k, k']$ with integral endpoints.

(c) The interval $\uparrow\mathbb{I}$ is also used in cTop, as the *flexible interval*. Flexible paths of c-spaces and flexible homotopies of c-maps are parametrised on it, as c-maps $\uparrow\mathbb{I} \to X$ and $X \times \uparrow\mathbb{I} \to Y$, respectively (see II.4.1).

(d) We also recall that, in the fundamental category $\uparrow\Pi_1(X)$ of a c-space X (introduced in II.5.1)

- the vertices are the flexible points of X,

- the arrows are equivalence classes $[a]\colon x \to y$ of c-paths $a\colon x \to y$; the equivalence relation is generated by flexible homotopies $\varphi\colon a' \to a''$ with fixed endpoints, which are c-maps $\varphi\colon \mathrm{c}\mathbb{I} \times \uparrow\mathbb{I} \to X$ defined on the *hybrid square* $\mathrm{c}\mathbb{I} \times \uparrow\mathbb{I}$.

1.3 The standard dI2-structure

In cTop the standard interval $\mathrm{c}\mathbb{I}$ has a structure formed of the following operations.

The first-order part was already seen in II.4.1: two faces ∂^α ($\alpha = \pm$), a degeneracy e and a reflection r. We also have a second-order part which involves the standard square $\mathrm{c}\mathbb{I}^2 = \mathrm{c}\mathbb{I} \times \mathrm{c}\mathbb{I}$: two *connections*, or main operations g^α (already used in Part II) and a *transposition* s

$$\{*\} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \mathrm{c}\mathbb{I} \overset{g^\alpha}{\longleftarrow} \mathrm{c}\mathbb{I}^2 \qquad r\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\mathrm{op}}, \quad s\colon \mathrm{c}\mathbb{I}^2 \to \mathrm{c}\mathbb{I}^2, \tag{4}$$

$$\partial^\alpha(*) = \alpha, \qquad g^-(t,t') = \max(t,t'), \qquad g^+(t,t') = \min(t,t'),$$
$$r(t) = 1 - t, \qquad s(t,t') = (t',t).$$

As a consequence, the (standard) cylinder endofunctor

$$I_c = - \times \mathrm{c}\mathbb{I}\colon \mathrm{cTop} \to \mathrm{cTop},$$

written as I if it is clear that we are working in cTop, has natural transformations, written as above

$$1 \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} I \overset{g^\alpha}{\longleftarrow} I^2 \qquad r\colon IR \to RI, \qquad s\colon I^2 \to I^2, \tag{5}$$

that satisfy the following equations

$$\begin{array}{ll}
e\partial^\alpha = 1, \qquad eg^\alpha = e(Ie) = e(eI) & (degeneracy),\\
g^\alpha(Ig^\alpha) = g^\alpha(g^\alpha I) & (associativity),\\
g^\alpha(I\partial^\alpha) = 1 = g^\alpha(\partial^\alpha I) & (unitarity),\\
g^\beta(I\partial^\alpha) = \partial^\alpha e = g^\beta(\partial^\alpha I) & (absorbency, \text{ for } \alpha \neq \beta),\\
(RrR)r = 1, \qquad (Re)r = eR, & \\
r(\partial^+ R) = R\partial^-, \qquad r(g^+ R) = (Rg^-)r_2 & (reflection),\\
ss = 1, \qquad (Ie)s = eI, \qquad s(I\partial^\alpha) = \partial^\alpha I, & \\
(Rs)r_2 = r_2(sR), \qquad g^\alpha s = g^\alpha & (transposition),
\end{array} \tag{6}$$

where $r_2 = (rI)(Ir)\colon I^2R(X) \to RI^2(X)$ is the double reflection, namely: $r_2(x, t, t') = (x, -t, -t')$.

According to a classification of homotopy structures defined by a cylinder endofunctor, the category $\mathrm{c}\mathsf{Top}$, equipped with the functors I, R and the previous operations, is a *symmetric dI2-category* ([G2], 4.2): the previous equations are the axioms of this structure. Moreover, the existence of the terminal object and pushouts makes $\mathrm{c}\mathsf{Top}$ into a *dI1-homotopical category* ([G2], 1.7.0).

The present structure is made *concrete* fixing the flexible singleton $\{*\}$ as the standard point ([G2], 1.2.4). The c-space $\mathrm{c}\mathbb{I} = I(\{*\})$ is a symmetric dI2-interval for the cartesian product ([G2], 4.2.8).

1.4 Higher properties

(a) The category $\mathrm{d}\mathsf{Top}$ of directed spaces has a far richer structure. In particular the cylinder functor $I_d = -\times\uparrow\mathbb{I}$ preserves all pushouts (which allows the concatenation of homotopies) and has a right adjoint, the cocylinder functor $P_d = (-)^{\uparrow\mathbb{I}}$, or path functor. Adding the previous operations and others, we have a *symmetric dIP4-homotopical category* ([G2], 4.2.6).

The homotopy structure of Cat, reviewed in 1.7, is also of this kind.

(b) The category Top of topological spaces has a *reversible* structure of this kind: the reversor endofunctor is the identity.

The same holds for the category $\mathrm{Ch}_\bullet(\mathsf{D})$ of unbounded chain complexes on an additive category, or $\mathrm{Ch}_+(\mathsf{D})$ of positive chain complexes on an additive category with kernels ([G2], 4.4).

(c) On the other hand, the classification of $\mathrm{c}\mathsf{Top}$ as a symmetric dI2-category and a dI1-homotopical category cannot be improved, essentially because pushouts are not preserved by the cylinder functor.

(i) It is not a *dI3-category* ([G2], 4.2.2): the concatenation pushout $JX = IX +_X IX$, in particular, is not preserved by the functor I, as remarked for $X = \{*\}$ and the two-jump interval $J(\{*\}) = \mathrm{c}\mathbb{J}$, in II.4.7(c).

(ii) It is not a *dIP1-category* ([G2], 1.2.2): the cylinder functor $- \times \mathrm{c}\mathbb{I}$ has no right adjoint, by the same reason.

(iii) We shall see in 3.8(b) that $\mathrm{c}\mathsf{Top}$ is not a *symmetric dI1-homotopical category* (as defined in [G2], 4.1.4): the cylinder I does not preserve the 'cylindrical colimits' that produce the homotopy pushouts.

This makes the use of pushouts in $\mathrm{c}\mathsf{Top}$ complicated, as we have already seen in Part II for the construction of the fundamental category, and will also see in the present study of homotopy equivalences, homotopy pushouts, cones and suspension. Flexibility properties will be used, to link these constructions to the more regular ones of $\mathrm{d}\mathsf{Top}$.

1.5 The splitting property

The *standard concatenation pushout* – pasting two copies of the standard interval, one after the other – is realised in $\mathrm{c}\mathsf{Top}$ as $\mathrm{c}\mathbb{J}$, the two-jump structure on the euclidean interval $[0, 1]$, generated by the paths c^-, c^+ (see II.4.2)

$$\begin{array}{ccc} \{*\} & \xrightarrow{\partial^+} & \mathrm{c}\mathbb{I} \\ {\scriptstyle \partial^-}\downarrow & & \downarrow{\scriptstyle c^-} \\ \mathrm{c}\mathbb{I} & \xrightarrow[c^+]{} & \mathrm{c}\mathbb{J} \end{array} \qquad \begin{array}{l} c^-(t) = t/2, \\ c^+(t) = (t+1)/2. \end{array} \qquad (7)$$

Adding the *concatenation map* $\kappa\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J}$ (a *reshaping*, $\mathrm{c}\mathbb{I}$ being finer than $\mathrm{c}\mathbb{J}$, cf. I.1.7), the regular concatenation of two consecutive c-paths $a', a''\colon \mathrm{c}\mathbb{I} \to X$ is expressed as

$$a' * a'' = a\kappa\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J} \to X \qquad (ac^- = a',\ ac^+ = a''). \qquad (8)$$

This procedure is frequent in homotopy theory. For instance, chain complexes have a similar behaviour: pasting two copies of the interval (or a cylinder) yields a different object, related to the former by a (non-invertible) concatenation map. The same happens in Cat (see 1.7) and for cubical sets (see Section 6).

On the other hand, in dTop (and Top) the standard concatenation pushout can be realised as the interval itself, letting κ be the identity map (see II.3.2), so that every path a is the concatenation of two paths, uniquely determined as ac^- and ac^+. We can introduce the term of a *splittable homotopy structure* to express the property that the concatenation map is invertible.

1.6 Breaking the symmetries of classical topology

A topological space X has 'intrinsic symmetries', which act on its singular cubes $\mathbb{I}^n \to X$.

They are generated by the standard reversion r and the standard transposition s

$$r\colon \mathbb{I} \to \mathbb{I}, \qquad r(t) = 1 - t, \qquad\qquad s\colon \mathbb{I}^2 \to \mathbb{I}^2, \qquad s(t, t') = (t', t). \tag{9}$$

Their n-dimensional versions

$$\begin{aligned} r_i &= \mathbb{I}^{i-1} \times r \times \mathbb{I}^{n-i}\colon \mathbb{I}^n \to \mathbb{I}^n \qquad\qquad (i = 1, ..., n), \\ s_i &= \mathbb{I}^{i-1} \times s \times \mathbb{I}^{n-i-1}\colon \mathbb{I}^n \to \mathbb{I}^n \qquad (i = 1, ..., n-1), \end{aligned} \tag{10}$$

span the group of symmetries of the n-cube, namely the hyperoctahedral group $(\mathbb{Z}/2)^n \rtimes S_n$ (a semidirect product): the reversions r_i commute with each other and generate the first factor, while the transpositions s_i generate the symmetric group S_n. This group acts on the set of n-cubes $\mathbb{I}^n \to X$.

Topological spaces have thus both kinds of symmetries. Directed algebraic topology allows one to break the first kind, and also the second in some settings.

(a) *Reversion.* The prime effect of the reversion $r\colon \mathbb{I} \to \mathbb{I}$ is reversing the paths, in any topological space. This map also gives the reversion of homotopies, by the reversion $\mathrm{id} \times r$ of the cylinder functor $I = - \times \mathbb{I}\colon$ Top $\to$ Top.

Controlled spaces (as well as preordered spaces and directed spaces) lack a reversion, replaced by a *reflection pair* (R, r) consisting of the reversor

$R\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ and a reflection $r\colon IR \to RI$ for the cylinder functor. This behaviour is shared by all the structures for directed homotopy considered in [G2]; the reversible case is a particular instance, with R the identity functor.

(b) *Transposition.* Coming back to topological spaces, the transposition $s(t, t') = (t', t)$ of the standard square $\mathbb{I}^2$ yields the transposition symmetry of the iterated cylinder functor $I^2 = - \times \mathbb{I}^2\colon \mathsf{Top} \to \mathsf{Top}$ ([G2], 1.1.1).

This second-order symmetry, acting on I^2, also exists in $\mathrm{c}\mathsf{Top}$ (as we have seen above), $\mathrm{p}\mathsf{Top}$ and $\mathrm{d}\mathsf{Top}$, but does not exist in other directed structures, e.g. for cubical sets (see 6.3, 6.8). *Its role, within directed algebraic topology, is double-edged.* On the one hand, its presence has an important consequence, the homotopy invariance of the cylinder functor – as proved in (18). On the other hand, it restricts the interest of directed homology, preventing a good relation of the latter with suspension ([G2], Section 2.2).

In fact the (pre)ordered group $\uparrow\! H_1(\uparrow\!\mathbb{S}^1) = \uparrow\!\mathbb{Z}$ of directed homology has the canonical order, while $\uparrow\! H_2(\uparrow\!\mathbb{S}^2)$ only gets the chaotic preorder. Essentially, we cannot reverse the d-path that generates the former, but we can transpose the d-square that generates the latter, so that its homology class and the opposite are both weakly positive. (In a cubical set there is a finer control of cubes, see 6.8.)

(c) *Restriction.* Finally, we already remarked that the present extension to c-spaces breaks a flexibility feature of d-spaces: paths can no longer be subdivided.

Formally, this can be traced back to the action on paths of the monoid of restrictions (in I.1.2), formed of the affine endomaps

$$\rho\colon \mathbb{I} \to \mathbb{I} \qquad \rho(t) = (t_2 - t_1)t + t_1 \qquad (0 \leqslant t_1 < t_2 \leqslant 1). \tag{11}$$

In Top, $\mathrm{p}\mathsf{Top}$ and $\mathrm{d}\mathsf{Top}$ these are endomaps of the standard interval ($\mathbb{I}$ or $\uparrow\!\mathbb{I}$), and act on any path restricting it to the subinterval $[t_1, t_2]$ (reparametrised on $[0, 1]$). In $\mathrm{c}\mathsf{Top}$ they are not endomaps of $\mathrm{c}\mathbb{I}$: path-restriction is prevented in the standard homotopy structure (but allowed in the flexible structure examined in Section 5).

One might say that there is now a breaking of ‘extended symmetries’, forming a monoid instead of a group.

1.7 Categories and directed homotopy

The fundamental category functor $\uparrow\Pi_1\colon \mathrm{cTop} \to \mathsf{Cat}$ takes flexible homotopies of c-spaces (and homotopies of d-spaces) to directed homotopies in Cat, the cartesian closed category of small categories (cf. Theorem II.5.4, and Proposition 5.3 here). We briefly review here the homotopy structure of Cat – a symmetric dIP4-homotopical category (cf. [G2], 4.3.2).

The *reversor* functor R takes a small category to the opposite one

$$R\colon \mathsf{Cat} \to \mathsf{Cat}, \qquad R(X) = X^{\mathrm{op}}, \tag{12}$$

where X^{op} has precisely the same objects as X, with $X^{\mathrm{op}}(x, y) = X(y, x)$ and the opposite composition, so that R is strictly involutive.

(a) The *standard homotopy structure* of Cat is based on the cartesian product and the *directed interval* $\uparrow\mathbf{i} = \mathbf{2} = \{0 \to 1\}$, an ordinal. Its cartesian powers $\mathbf{2}^n$ are ordered sets (viewed as categories); the standard point is the terminal category $\mathbf{2}^0 = 1$.

Faces, degeneracy, reflection, connections and transposition are order preserving mappings (and functors)

$$\mathbf{1} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \mathbf{2} \overset{g^\alpha}{\longleftarrow} \mathbf{2}^2 \qquad r\colon \mathbf{2} \to \mathbf{2}^{\mathrm{op}}, \qquad s\colon \mathbf{2}^2 \to \mathbf{2}^2, \tag{13}$$

defined by the same formulas as in (4).

A *point* $x\colon \mathbf{1} \to X$ of the small category X is an object of the latter. A (directed) *path* $a\colon \mathbf{2} \to X$ from x to x' is an arrow $a\colon x \to x'$ of X, their concatenation is the composition, strictly associative and unitary.

The concatenation pushout gives here the ordinal category $\uparrow\mathbf{j} = \mathbf{3}$. The obvious concatenation map $\kappa\colon \mathbf{2} \to \mathbf{3}$, $\kappa(0 \to 1) = (0 \to 2)$ is not invertible: this homotopy structure is not splittable.

The (directed) *cylinder* functor $IX = X \times \mathbf{2}$ has a right adjoint: $PY = Y^{\mathbf{2}}$, the category of morphisms of Y. A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, represented by a functor $X \times \mathbf{2} \to Y$ or equivalently $X \to Y^{\mathbf{2}}$, is the same as a natural transformation between the functors f, g. Given two parallel arrows $a, b\colon x \to x'$ in X, a homotopy with fixed endpoints $a \to b$ is a commutative square $\mathbf{2} \times \mathbf{2} \to X$ with trivial vertical edges and *equal* horizontal edges $a = b$. The fundamental category of X is the category X itself. (This homotopy structure is 'one-dimensional', akin to that of 1-cubical sets.)

The operations of homotopies (vertical composition and whisker composition) coincide with the 2-categorical structure of Cat, which implies that the homotopy structure is *2-regular*, in the sense of [G2], 4.2.3.

A *future homotopy equivalence* $(f, g; \varphi, \psi)$ between small categories X, Y ([G2], 3.3.1) is a four-tuple of functors and natural transformations (that need not be invertible)

$$f\colon X \rightleftarrows Y : g, \qquad \varphi\colon \mathrm{id} \to gf, \qquad \psi\colon \mathrm{id} \to fg. \tag{14}$$

'Directed homotopy equivalence' in Cat is studied in [G2], Chapter 3, combining the future and past homotopy equivalences of categories in complex forms, aiming to classify the fundamental categories of directed spaces; some examples were recalled in II.7. (Different homotopy structures on Cat are studied or cited in [Mi].)

(b) Ordinary equivalence of categories is a stricter, far simpler notion, based on the *reversible homotopy structure* $\mathsf{Cat}_{\mathbf{i}}$ produced by the *reversible interval* $\mathbf{i}$. The latter is the indiscrete groupoid on two objects, formed by an isomorphism $u\colon 0 \to 1$ and its inverse

$$\mathbf{i} = \{0 \rightleftarrows 1\}, \qquad r\colon \mathbf{i} \to \mathbf{i}, \qquad r(u) = u^{-1}, \tag{15}$$

with the obvious reversion r, defined above. This gives a *reversible cylinder* functor $X \times \mathbf{i}$, with right adjoint $Y^{\mathbf{i}}$ (the full subcategory of $Y^{\mathbf{2}}$ whose objects are the isomorphisms of Y); a *reversible homotopy* $\varphi\colon f \to g\colon X \to Y$ is the same as a natural isomorphism of functors.

(c) Both structures on Cat give the same homotopies for groupoids, represented by the restriction of the cylinder (or cocylinder) of $\mathsf{Cat}_{\mathbf{i}}$ to the full subcategory Gpd of small groupoids.

1.8 Coherent homotopy equivalence of categories

A future homotopy equivalence $(f, g; \varphi, \psi)$ in Cat is said to be *coherent*, or a *future equivalence* ([G2], 3.3.1), if it satisfies the following *coherence* *conditions*:

$$\begin{array}{c} f\colon X \rightleftarrows Y : g, \qquad \varphi\colon \mathrm{id}\,X \to gf, \qquad \psi\colon \mathrm{id}\,Y \to fg, \\ f\varphi = \psi f\colon f \to fgf, \qquad \varphi g = g\psi\colon g \to gfg. \end{array} \tag{16}$$

This structure plays a primary role in [G2]. It can be seen as a symmetric version of an adjunction, with two units (although f and g do not determine each other). Its dual, a *past equivalence*, has two counits.

Future equivalences compose, in the same way as adjunctions ([G2], 3.3.3), and give an equivalence relation between small categories. Two categories are future equivalent if and only if they can be embedded as full reflective subcategories of a common one ([G2], Theorem 3.3.5).

A property is invariant for future equivalences if and only if it is preserved by full reflective embeddings and by their reflectors.

All this works because the homotopy structure of Cat is 2-regular, as remarked above. On the other hand, these coherence conditions are generally too strong for the homotopy equivalence of topological spaces (or d-spaces), and only one of them is required for strong deformation retracts.

Here we shall make a limited use of coherent future homotopy equivalences of c-spaces: see 2.6.

2. Homotopy equivalence and connectedness of c-spaces

After reviewing the homotopies of c-spaces, from II.4, we introduce formal composite homotopies to cover their lack of a vertical composition. Then we study forms of directed homotopy equivalence and connectedness adequate to the present setting.

2.1 Composing homotopies

Homotopies of controlled spaces, and their structure, are defined by the cylinder functor and the structure examined above.

(a) As we have already said in II.4.1, a (standard, or general) homotopy $\varphi\colon f \to g\colon X \to Y$ of c-spaces is represented by a map $\varphi\colon IX \to Y$, with faces $f = \varphi\partial^-$ and $g = \varphi\partial^+$. The representative map is written as $\hat{\varphi}\colon IX \to Y$ when useful. The degeneracy map $e\colon IX \to X$ gives the trivial homotopy $0_f\colon f \to f\colon X \to Y$, represented by $fe\colon IX \to Y$.

The reflection $r\colon IR \to RI$ transforms a homotopy $\varphi\colon f \to g\colon X \to Y$ into the *reflected* one

$$\varphi^{\mathrm{op}}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}, \quad (\varphi^{\mathrm{op}})\hat{} = R(\hat{\varphi})r\colon IRX \to RY. \tag{17}$$

The transposition $s\colon I^2 \to I^2$ makes the cylinder functor $I\colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ homotopy invariant: the homotopy φ can be transformed into a homotopy $I\varphi\colon If \to Ig\colon IX \to IY$

$$\begin{gathered} (I\varphi)\hat{} = I(\hat{\varphi})s\colon I^2X \to IY, \\ I(\hat{\varphi})s(\partial^- IX) = I(\hat{\varphi})(I\partial^- X) = If, \qquad I(\hat{\varphi})s(\partial^+ IX) = Ig, \end{gathered} \tag{18}$$

modifying the map $I(\hat{\varphi})$, which does not have the correct faces.

(b) There is a *whisker composition* of maps and homotopies

$$X' \xrightarrow{\;h\;} X \underset{g}{\overset{f}{\underset{\downarrow\varphi}{\rightrightarrows}}} Y \xrightarrow{\;k\;} Y' \tag{19}$$

$$k{\circ}\varphi{\circ}h\colon kfh \to kgh\colon X' \to Y', \quad (k{\circ}\varphi{\circ}h)\hat{} = (k\hat{\varphi})(Ih)\colon IX' \to Y',$$

also written as $k\varphi h$. This ternary operation satisfies obvious relations of associativity and identities (cf. [G2], 1.2.3).

(c) On the other hand, homotopies of c-spaces have no vertical composition: two consecutive homotopies $f \to g \to h$ (between parallel c-maps) cannot be concatenated, because of the failure of the concatenation pushout recalled above (see II.4.7(c)).

We introduce thus a *composite homotopy* $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \dot{\to} f''\colon X \to Y$ (denoted by a dot-marked arrow) as a finite sequence of consecutive homotopies between maps $f_i\colon X \to Y$

$$f' = f_0 \xrightarrow{\varphi_1} f_1 \xrightarrow{\varphi_2} f_2 \; ... \xrightarrow{\varphi_n} f_n = f''. \tag{20}$$

Their whisker composition with maps is obvious

$$k\varphi h = (k\varphi_1 h,\; k\varphi_2 h, ...,\; k\varphi_n h). \tag{21}$$

The *vertical composition* $\psi\varphi$ with a second composite homotopy $\psi\colon f'' \dot{\to} f'''\colon X \to Y$ is just word concatenation, and is associative.

Of course these two composition laws do not satisfy the middle-four interchange and do not produce a 2-category.

2.2 Future and past homotopy equivalences

We already recalled that directed homotopy equivalence comes out in two basic forms, which can be combined in various ways studied in [G2]. Here we have to extend this approach, working with composite homotopies.

(a) A *future homotopy equivalence* between the c-spaces X and Y, or *homotopy equivalence in the future*, will be a four-tuple $(f, g; \varphi, \psi)$ of maps and composite homotopies (starting from the identity maps of the spaces)

$$f \colon X \rightleftarrows Y : g, \qquad \varphi \colon \mathrm{id}\, X \twoheadrightarrow gf, \qquad \psi \colon \mathrm{id}\, Y \twoheadrightarrow fg. \tag{22}$$

We speak of a future homotopy equivalence *in one step* when φ and ψ are mere homotopies.

Future homotopy equivalences compose: given a second

$$h \colon Y \rightleftarrows Z : k, \qquad \vartheta \colon \mathrm{id}\, Y \twoheadrightarrow kh, \qquad \zeta \colon \mathrm{id}\, Z \twoheadrightarrow hk, \tag{23}$$

their composite is obtained by whisker composition and vertical composition (as in the horizontal composition of adjunctions)

$$\begin{gathered} hf \colon X \rightleftarrows Z : gk, \\ (g\vartheta f)\varphi \colon \mathrm{id}\, X \twoheadrightarrow gkhf, \qquad (h\psi k)\zeta \colon \mathrm{id}\, Z \twoheadrightarrow hfgk. \end{gathered} \tag{24}$$

Being future homotopy equivalent c-spaces is thus an equivalence relation.

A *past homotopy equivalence* between the c-spaces X, Y is a four-tuple $(f, g; \varphi, \psi)$

$$f \colon X \rightleftarrows Y : g, \qquad \varphi \colon gf \twoheadrightarrow \mathrm{id}\, X, \qquad \psi \colon fg \twoheadrightarrow \mathrm{id}\, Y, \tag{25}$$

where the composite homotopies start from the composed maps. The reversor R turns future into past and conversely (see (17)), and we will mostly deal with the former case.

(b) More particularly, with the following structure

$$i \colon X_0 \rightleftarrows X : p, \qquad \mathrm{id}\, X_0 = pi, \qquad \varphi \colon \mathrm{id}\, X \twoheadrightarrow ip, \tag{26}$$

we say that the c-space X_0 is a *future deformation retract* of X, or embedded in X as a future deformation retract; this can always be realised with a c-subspace $X_0 = \mathrm{Im}\, i \subset X$. Again, we speak of a future deformation retract *in one step* when we can realise the previous structure with a homotopy $\varphi\colon \mathrm{id}\, X \to ip$, as will often be the case in the examples below.

(c) We say that the c-spaces X, Y are *coarsely c-homotopy equivalent* if they are linked by the equivalence relation generated by future and past homotopy equivalence.

2.3 Contractible c-spaces

We say that a c-space is *future contractible* if it is future homotopy equivalent to the terminal singleton $\{*\}$. Here we only need a pair of maps i, p and one composite homotopy φ

$$i\colon \{*\} \rightleftarrows X : p, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow ip \qquad (pi = \mathrm{id}\,\{*\}), \tag{27}$$

and $\{*\}$ is embedded as a future deformation retract of X, at a flexible point $x_0 = i(*)$. We also say that X is *future contractible to* x_0.

Equivalently, we have a composite homotopy φ such that

$$\begin{gathered} \varphi = (\varphi_1, ..., \varphi_n)\colon f_0 \twoheadrightarrow f_n\colon X \to X, \\ f_0 = \mathrm{id}\, X, \qquad f_n \text{ is constant at } x_0 \in |X|_0. \end{gathered} \tag{28}$$

We say that the c-space X is *coarsely c-contractible* if it is coarsely c-homotopy equivalent to $\{*\}$: there exists a finite sequence of c-spaces $X, X_1, ..., X_{n-1}, \{*\}$ such that each of them is future or past homotopy equivalent to the next.

Examples. (a) The standard interval $\mathrm{c}\mathbb{I}$ is past contractible to 0 and future contractible to 1, in one step, with homotopies supplied by the connections $g^\alpha\colon \mathrm{c}\mathbb{I}^2 \to \mathrm{c}\mathbb{I}$ recalled in 1.3

$$\begin{aligned} &\partial^-\colon \{*\} \rightleftarrows \mathrm{c}\mathbb{I} : e, \qquad g^+\colon \partial^- e \to \mathrm{id}\, \mathrm{c}\mathbb{I}, \qquad e\partial^- = \mathrm{id}\,, \\ &\partial^+\colon \{*\} \rightleftarrows \mathrm{c}\mathbb{I} : e, \qquad g^-\colon \mathrm{id}\, \mathrm{c}\mathbb{I} \to \partial^+ e, \qquad e\partial^+ = \mathrm{id}\,. \end{aligned} \tag{29}$$

(b) The two-jump interval $\mathrm{c}\mathbb{J}$ is also past contractible to 0 and future contractible to 1, with homotopies $g^\alpha\colon \mathrm{c}\mathbb{J} \times \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{J}$.

(c) The standard c-line $\mathrm{c}\mathbb{R}$ is not future contractible, because there is no point x_0 such that each flexible point x has a c-path $x \to x_0$. (The necessity of this property is proved in Proposition 2.5(a).) By R-duality it is not past contractible either.

However, it is easy to prove that $\mathrm{c}\mathbb{R}$ is coarsely c-contractible: it has a future deformation retract $\mathrm{c}[0, +\infty[$ (in one step), which is past contractible to 0 (in one step).

2.4 Controlled connection

(a) In a c-space X the existence of a c-path between two points gives a reflexive and transitive relation in $|X|_0$, and we consider the equivalence relation generated by the latter. The equivalence class $[x]_c$ of a flexible point is called a *controlled component*, or *c-component*, of X; it is a topological subspace of $|X|_0$.

X is said to be *c-connected* if it has precisely one c-component. (The empty c-space is not.) If $f\colon X \to Y$ is a c-map whose restriction $|X|_0 \to |Y|_0$ is surjective and X is c-connected, Y is also. A product of c-spaces is c-connected if and only if all its factors are.

(b) We denote as $\uparrow\!\Pi_0(X)$ the quotient of the set $|X|_0$ modulo this equivalence relation, that is the set of controlled components. We have thus a functor $\uparrow\!\Pi_0\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}$, with an obvious action on a c-map $f\colon X \to Y$

$$f_*\colon \uparrow\!\Pi_0(X) \to \uparrow\!\Pi_0(Y), \qquad f_*[x]_c = [f(x)]_c. \tag{30}$$

Equivalently, $\uparrow\!\Pi_0(X)$ is the set of connected components of the fundamental category $\uparrow\!\Pi_1(X)$.

(c) *Examples*. All the ‘basic’ c-spaces are c-connected: the intervals $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $c_-\mathbb{I}$, $c_+\mathbb{I}$, $\uparrow\!\mathbb{I}$, the lines $\mathrm{c}\mathbb{R}$, $c_n\mathbb{R}$, $\uparrow\!\mathbb{R}$, the spheres $\mathrm{c}\mathbb{S}^1$, $c_n\mathbb{S}^1$, $\mathrm{c}\mathbb{S}^n$, $\uparrow\!\mathbb{S}^n$, and all their products. Their non-trivial sums are not, of course.

We recall, from I.2.4(b), that the *past-delayed c-interval* $c_-\mathbb{I}$ is the standard interval $[0, 1]$ with the c-structure generated by the past-delayed reparametrisation $f(t) = 0 \vee (2t-1)$. The *future-delayed c-interval* $c_+\mathbb{I}$ is based on the future-delayed reparametrisation $g(t) = 2t \wedge 1$.

(d) *Remarks*. The interval $\mathrm{c}\mathbb{I}$ is c-connected but its flexible support $\{0, 1\}$ is a disconnected topological space. The same happens for all the c-structures of intervals and lines considered above, except the d-spaces $\uparrow\!\mathbb{I}$ and $\uparrow\!\mathbb{R}$.

This fact becomes clearer using the *path-support* $|X|_1$ of the c-space X, defined in II.2.1(d) as the topological subspace of $|X|$ formed by the union of the images of all c-paths in X, so that $|X|_0 \subset |X|_1 \subset |X|$. If X is c-connected, the space $|X|_1$ is path connected: in fact every point of the latter is linked to a flexible point by the restriction of a c-path. The converse fails, as for the 'diagonal' c-structure of the square $\mathbb{I}^2$, in I.2.7(d) and II.2.2(b).

2.5 Proposition

(a) If the c-space X is future contractible to x_0, every flexible point x has a c-path $x \to x_0$.

(b) A coarsely contractible c-space X is always c-connected.

Proof. It is sufficient to prove (a). According to (28) we have a composite homotopy $\varphi = (\varphi_1, ..., \varphi_n)\colon f_0 \twoheadrightarrow f_n\colon X \to X$, where $f_0 = \mathrm{id}\, X$ and f_n is constant at $x_0 \in |X|_0$. For every flexible point x the c-maps $\varphi_i(x, -)\colon \mathrm{c}\mathbb{I} \to X$ form a sequence of consecutive c-paths $x \to f_1(x) \to ... \to x_0$, whose concatenation is a c-path $x \to x_0$. □

2.6 Coherent homotopy equivalence

(a) As in 1.8, a *coherent future homotopy equivalence* of c-spaces will be a four-tuple $(f, g; \varphi, \psi)$ of maps and homotopies (in one-step) satisfying the *coherence conditions*

$$\begin{array}{lll} f\colon X \rightleftarrows Y : g, & \varphi\colon \mathrm{id}\, X \to gf, & \psi\colon \mathrm{id}\, Y \to fg, \\ f\varphi = \psi f\colon f \to fgf, & & \varphi g = g\psi\colon g \to gfg. \end{array} \tag{31}$$

These structures cannot be composed (even for mere topological spaces). R-duality gives the corresponding case *in the past*.

(b) If the map f (for instance) is a reshaping, its topological support is an identity. Therefore, loosely speaking, φ and ψ are represented by the same mapping $\hat{\varphi} = \hat{\psi}$, which is at the same time a c-map $X \times \mathrm{c}\mathbb{I} \to X$ and $Y \times \mathrm{c}\mathbb{I} \to Y$; the first coherence condition is automatically satisfied, and the second means that $\hat{\varphi}(g \times \mathrm{id}\,) = g\hat{\varphi}$.

(c) As a typical example, the reshaping $f\colon c_+\mathbb{I} \to \mathrm{c}\mathbb{I}$ of the future-delayed interval (in 2.4(c)) can be completed to a coherent future homotopy equivalence

$$\begin{gathered} f\colon c_+\mathbb{I} \rightleftarrows \mathrm{c}\mathbb{I} : g, \qquad g(s) = 2s \wedge 1, \\ \hat{\varphi}(s,t) = (1+t)s \wedge 1, \qquad \hat{\varphi}(s,0) = s, \qquad \hat{\varphi}(s,1) = 2s \wedge 1. \end{gathered} \tag{32}$$

In fact, we have to verify three points.

(i) The map $\hat{\varphi}\colon \mathbb{I}^2 \to \mathbb{I}$ is a c-map $\mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$. This is obvious, because $\hat{\varphi}$ is increasing in both variables and preserves the flexible points.

(ii) The same map $\hat{\varphi}$ is a c-map $c_+\mathbb{I}\times\mathrm{c}\mathbb{I} \to c_+\mathbb{I}$. Indeed, given a c-path $a = \langle h, k\rangle\colon \mathrm{c}\mathbb{I} \to c_+\mathbb{I}\times\mathrm{c}\mathbb{I}$, the c-map $h\colon \mathrm{c}\mathbb{I} \to c_+\mathbb{I}$ is increasing between flexible points and either constant at 0, or constant at 1 on a non-degenerate interval $[\tau, 1]$, at least. In both cases the path $\hat{\varphi}a$ is increasing between flexible points and has a final delay: either $\hat{\varphi}(0, k(t)) = 0$, for all t, or

$$\hat{\varphi}(h(t), k(t)) = (1 + h(t)) \wedge 1 = 1, \quad \text{for } t \geqslant \tau.$$

(iii) $\hat{\varphi}(g\times\mathrm{id}\,) = g\hat{\varphi}$, because:

$$\begin{aligned} \hat{\varphi}(g(s), t) &= (1+t)(2s \wedge 1) \wedge 1 = (1+t)2s \wedge (1+t) \wedge 1 \\ &= (1+t)2s \wedge 1 = g\hat{\varphi}(s,t). \end{aligned}$$

(d) The previous homotopy φ is strict, that is $\hat{\varphi}(s, -)$ is constant for $s = 0, 1$ (the flexible points of $\mathrm{c}\mathbb{I}$ and $c_+\mathbb{I}$). Applying Theorem III.3.3 we deduce that f and g induce two functors $\uparrow\!\Pi_1(c_+\mathbb{I}) \rightleftarrows \uparrow\!\Pi_1(\mathrm{c}\mathbb{I})$ which are inverse to each other. Both categories are isomorphic to the ordinal **2**.

3. Homotopy pushouts and flexibility

Taking on the study of the standard homotopy structure cTop, we deal now with homotopy pushouts. Their derived constructions, cones and suspension, are deferred to the next section.

Homotopy pushouts are well-known in Top [Ma], and also in Cat as *cocomma squares*. They are studied in [G1, G2] in general dI1-categories ([G2], Section 1.3) and symmetric dI1-homotopical categories ([G2], Sections 1.7, 4.1), in particular for directed spaces and pointed directed spaces.

The 'defective' homotopy structure of cTop requires a specific study; flexibility properties link these constructions to those of d-spaces, far better behaved.

X will denote a c-space.

3.1 Weak flexibility

We begin by recalling some weak forms of flexibility studied in II.2, which will also have a role in analysing the present constructions.

(a) We say that X is *preflexible* if it is 'full' in the generated d-space $\hat{X}$, which means that every c-path of $\hat{X}$ between flexible points of X is already a c-path of the latter. Then the fundamental category $\uparrow\Pi_1(X)$ is the full subcategory of $\uparrow\Pi_1(\hat{X})$ with vertices in the flexible support $|X|_0$, as proved in Theorem II.5.3(b).

Preflexible c-spaces form a full, reflective subcategory of cTop. They are closed in cTop under limits and sums; they are not closed under pushouts.

(b) We say that X is *border flexible* if one can restrict every c-path, by cutting out delays at the endpoints; more precisely, if $a\colon \mathrm{c}\mathbb{I} \to X$ is a c-path constant on $[0, t_1]$ and $[t_2, 1]$, the restriction of a to the interval $[t_1, t_2]$ (reparametrised on $\mathbb{I}$, see II.2.1(c)) is still a c-path.

Again, border flexible c-spaces form a full, reflective subcategory of cTop. They are closed in cTop under limits and sums, but not under pushouts. All preflexible c-spaces are border flexible; the converse is false.

(c) The *path-support* $|X|_1$ has already been recalled in 2.4(d). We say that X has a *total path-support* if $|X|_1 = |X|$.

(d) *Examples*. Besides all d-spaces, many basic c-spaces we have considered in Parts I and II are preflexible, with a total path-support: for instance $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$ and $\mathrm{c}\mathbb{I}^\sim$ (in $\uparrow\mathbb{I}$), $\mathrm{c}\mathbb{R}$ (in $\uparrow\mathbb{R}$), $\mathrm{c}\mathbb{S}^1$ (in $\uparrow\mathbb{S}^1$), and their products.

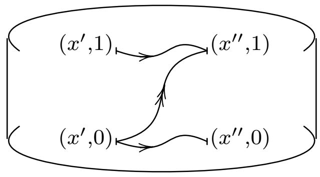
The delayed intervals $c_-\mathbb{I}$ and $c_+\mathbb{I}$ (in 2.4(c)) are not even border flexible, like the higher c-spheres $\mathrm{c}\mathbb{S}^n$, for $n \geqslant 2$ (see 4.6).

The 'diagonal' c-structure X of the square $\mathbb{I}^2$ described in I.2.7(d) and II.2.2(b) is border flexible and not preflexible; its path-support is not total.

3.2 The cylinder

The flexible points of the cylinder $IX = X \times \mathrm{c}\mathbb{I}$ form the set $|X|_0 \times \{0, 1\}$, contained in the bases of the cylinder.

A c-path $\langle a, h\rangle \colon \mathrm{c}\mathbb{I} \to X \times \mathrm{c}\mathbb{I}$ in the cylinder IX consists of a c-path $a \colon x' \to x''$ in the c-space X and a c-path h in the interval $\mathrm{c}\mathbb{I}$. There are three cases, as in the following figure, distinguished by the map $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$:



$X \times \mathrm{c}\mathbb{I}$ (33)

(i) a path $\langle a, e_0\rangle \colon (x', 0) \to (x'', 0)$ in the lower base $\partial^- X = X \times \{0\}$,

(ii) a path $\langle a, e_1\rangle \colon (x', 1) \to (x'', 1)$ in the upper base $\partial^+ X = X \times \{1\}$,

(iii) a path $\langle a, h\rangle \colon (x', 0) \to (x'', 1)$, where $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is a global reparametrisation, that is a surjective increasing map.

If the c-space X is preflexible or border flexible, so is IX. In fact these properties are closed under products and $\mathrm{c}\mathbb{I}$ satisfies them.

3.3 Homotopy pushouts

The structure of cTop as a symmetric dI1-category and the existence of pushouts ensures the existence of homotopy pushouts ([G2], 1.3). These are introduced in their 'standard form', also to get a functor.

(a) Let $f \colon X \to Y$ and $g \colon X \to Z$ be two c-maps with the same domain. The *standard homotopy pushout*, or *h-pushout*, *from f to g* is a four-tuple $(W; u, v; \lambda)$ satisfying the following universal property:

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & \lambda \nearrow & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[u]{} & W \end{array} \qquad \lambda \colon uf \to vg \colon X \to W, \qquad (34)$$

- for every similar four-tuple $(W'; u', v'; \lambda')$ there is precisely one map $h \colon W \to W'$ such that $u' = hu$, $v' = hv$, $\lambda' = h\lambda$.

The object W is determined up to isomorphism; its construction is deferred to the next subsection. It will be denoted as $I_c(f, g)$, or $I(f, g)$, and called a *double mapping cylinder*. Let us note that it is a directed notion, from f to g; the reflection $r\colon IRX \to RIX$ induces an isomorphism $r^I\colon I(Rg, Rf) \to RI(f, g)$, called the *reflection of an h-pushout*. When f or g is $\mathrm{id}\, X$, one has a *mapping cylinder*, $I(X, g)$ or $I(f, X)$.

(b) The cylinder IX itself is the h-pushout of the pair $(\mathrm{id}\, X, \mathrm{id}\, X)$, by means of the structural homotopy ∂ represented by the identity of the cylinder

$$\begin{array}{ccc} X & \xrightarrow{\ \mathrm{id}\ } & X \\ {\scriptstyle \mathrm{id}}\downarrow & \overset{\partial}{\nearrow} & \downarrow{\scriptstyle \partial^+} \\ X & \xrightarrow[\partial^-]{} & IX \end{array} \qquad\qquad \begin{array}{c} \partial\colon \partial^- \to \partial^+\colon X \to IX, \\ \hat{\partial} = \mathrm{id}\, IX. \end{array} \qquad (35)$$

In fact ∂ establishes a bijection between maps $\varphi\colon IX \to Y$ and homotopies $\varphi\partial\colon \varphi\partial^- \to \varphi\partial^+\colon X \to Y$, by the very definition of the latter.

(c) It is easy to see that h-pushouts give a functor

$$I(-, -)\colon \mathrm{c}\mathsf{Top}^{\vee} \to \mathrm{c}\mathsf{Top}, \qquad (36)$$

where $\vee$ is the *formal-span* category: $\bullet \leftarrow \bullet \rightarrow \bullet$. (The verification can be found in [G2], 1.3.7, for all dI1-categories with h-pushouts.)

The corresponding functor for d-spaces carries coherent triples of homotopies to homotopies, as it happens in every symmetric dI1-homotopical category ([G2], Theorem 4.1.6). This does not apply here: see 3.8(b).

3.4 The construction

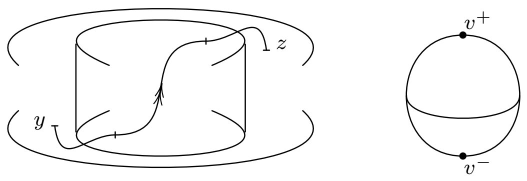
(a) The homotopy pushout $I(f, g)$ can be constructed using the cylinder $IX = X \times \mathrm{c}\mathbb{I}$ and the ordinary colimit of the left solid diagram below, called a *cylindrical colimit*. It amounts to three ordinary pushouts, as shown at the right hand

$$\begin{array}{ccccc} & & X & \xrightarrow{\ g\ } & Z \\ & & {\scriptstyle \partial^+}\downarrow & & \vdots \\ X & \xrightarrow{\partial^-} & IX & & \vdots\, {\scriptstyle v} \\ {\scriptstyle f}\downarrow & & & \searrow{\scriptstyle \lambda} & \downarrow \\ Y & - & - \underset{u}{-} & \dashrightarrow & I(f, g) \end{array} \qquad \begin{array}{ccccc} & & X & \xrightarrow{\ g\ } & Z \\ & & {\scriptstyle \partial^+}\downarrow & & \downarrow \\ X & \xrightarrow{\partial^-} & IX & \dashrightarrow & I(X, g) \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \dashrightarrow & I(f, X) & \dashrightarrow & I(f, g) \end{array} \qquad (37)$$

(b) The space $I(f, g)$ results thus of the pasting of the cylinder IX with the spaces Y, Z, under the following identifications (for $x \in X$)

$$I(f, g) = (Y + IX + Z)/\sim, \qquad [x, 0] = [f(x)], \quad [x, 1] = [g(x)]. \quad (38)$$

The construction is made clear by the following pictures. In the first f and g are injective, and we have drawn a path from a point $y \in Y$ to a point $z \in Z$ (see the following theorem)



(39)

In the second f and g are the terminal map $X \to \{*\}$, and $I(f, g)$ is the suspension ΣX where each base if IX is collapsed to a point (see 4.3).

The structural maps $u\colon Y \to I(f, g)$ and $v\colon Z \to I(f, g)$ are always injective, while the map $\lambda\colon IX \to I(f, g)$ is certainly injective *outside* of the bases $\partial^\alpha X$; it is injective 'everywhere' if and only if both f and g are.

(c) We have a similar construction $I_d(f, g)$ in $\mathrm{d}\mathsf{Top}$, based on the directed cylinder $I_d(X) = X \times \uparrow\mathbb{I}$, and in Top, based on the ordinary cylinder $X \times \mathbb{I}$. The three constructions give the same topological space, since the forgetful functors $\mathrm{c}\mathsf{Top} \to \mathsf{Top}$ and $\mathrm{d}\mathsf{Top} \to \mathsf{Top}$ preserve colimits and commute with the cylinders.

3.5 Theorem (The paths)

(a) In the h-pushout $I(f, g)$ of (37)*, the c-paths are of the following three kinds (and there are none from a point of Z to a point of Y).*

(i) A c-path $y \to y'$ between points of Y is a c-path of Y (embedded in $I(f, g)$ by u),

(ii) A c-path $z \to z'$ between points of Z is a c-path of Z (embedded in $I(f, g)$ by v),

(iii) A c-path from $y \in Y$ to $z \in Z$.

For the last case, we begin to say that a regular *c-path from y to z (as in figure* (39)*) is formed by the regular concatenation $a = ua_1 * \lambda a_2 * va_3$ of the images of three c-paths in Y, IX and Z*

$$\begin{gathered} a_1 \colon y \to f(x_1) \textit{ in } Y, \qquad a_3 \colon g(x_3) \to z \textit{ in } Z, \\ a_2 = \langle b, h \rangle \colon (x', 0) \to (x'', 1), \\ f(x_1) = f(x'), \qquad\qquad g(x_3) = g(x''), \end{gathered} \tag{40}$$

where $b \colon x' \to x'$ in X and $h \colon 0 \to 1$ in $\mathrm{c}\mathbb{I}$.

To get all c-paths from y to z we allow:

- *discarding a_1 when $y \in f(X)$ and discarding a_3 when $z \in g(X)$,*
- *global reparametrisations of the paths previously obtained.*

(b) The structural maps $u \colon Y \to I(f, g)$ and $v \colon Z \to I(f, g)$ are embeddings *of c-spaces, that is they induce an isomorphism onto their images.*

(c) In $\mathrm{d}\mathsf{Top}$ *we have similar results, replacing I with I_d, c-paths with d-paths, and global reparametrisations with the partial ones (for paths of kind (iii)).*

Proof. (a) The paths listed above form a c-structure on the topological colimit $|I(f, g)|$, as they are closed under trivial loops at the endpoints, concatenation and global reparametrisation.

The (injective) mappings u, v are obviously c-maps. To verify that $\lambda \colon IX \to I(f, g)$ is a c-map we note that, in a c-path $a = \langle b, h \rangle \colon \mathrm{c}\mathbb{I} \to IX = X \times \mathrm{c}\mathbb{I}$, the map $h \colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is either constant at 0, or constant at 1, or a path $0 \to 1$. In the first case $a = \partial^- b \colon \mathrm{c}\mathbb{I} \to X \to IX$ and $\lambda a = \lambda \partial^- b = u(fb)$ is a c-path of Y embedded in $I(f, g)$, of kind (i). In the second case $a = \partial^+ b$ is similarly a path of kind (ii). In the last, λa is a path of $I(f, g)$ of kind (iii).

Finally, the c-structure we have described is generated by the images of the c-paths of Y (by u), of IX (by λ) and Z (by v), and is thus the structure of the colimit c-space.

(b) To prove that the mapping u is an embedding of c-spaces, we only have to consider the topological part, since the c-paths of $u(Y)$ have already been considered.

Let V be open in Y. The subset $W = f^{-1}(V) \times [0, 1/2[$ is open in the cylinder IX. Now $V \cup W$ is open in the topological sum $T = Y + IX + Z$, and saturated for the projection $p\colon T \to I(f,g)$. Thus $p(T)$ is open in $I(f,g)$, and $u(V) = u(Y) \cap p(T)$ is open in $u(Y)$.

(c) The argument is the same. □

3.6 Theorem (Preflexibility, I)

If, in the homotopy pushout (34)*, the c-space X is preflexible with a total path-support (see 3.1), then:*

$$(I_c(f,g))\hat{} = I_d(\hat{f}, \hat{g}). \tag{41}$$

More explicitly, we are considering the following two structures on the same topological space, the h-pushout in Top *of the underlying maps, and saying that they coincide:*

- the d-structure generated by the h-pushout $I_c(f,g)$ in cTop*,*

- the h-pushout in dTop *of the d-maps $\hat{f}\colon \hat{X} \to \hat{Y}$ and $\hat{g}\colon \hat{X} \to \hat{Z}$.*

Proof. The colimit (37) in cTop is preserved by the reflector $\hat{}\colon$ cTop $\to$ dTop. Moreover $(I_cX)\hat{} = I_d(\hat{X})$, by Corollary II.2.6. □

3.7 Theorem (Preflexibility, II)

If, in the homotopy pushout (34)*, the c-space X is flexible, while Y and Z are preflexible, the h-pushout $I(f,g)$ is preflexible.*

Proof. It is a consequence of Theorems 3.5 and 3.6.

Let $a\colon w' \to w''$ be a d-path in $(I_c(f,g))\hat{} = I_d(\hat{f}, \hat{g})$ between flexible points of $I(f,g)$; we have to prove that a is a c-path of $I(f,g)$. By 3.5(c) the d-path a can be of three kinds.

In case (i) (or (ii)) a is a d-path of $\hat{Y}$ (or $\hat{Z}$) between flexible points of Y (or Z), and therefore a c-path of the latter.

In case (iii) it is sufficient to consider a regular d-path from $y \in |Y|_0$ to $z \in |Z|_0$, formed by the regular concatenation $a = ua_1 * \lambda a_2 * va_3$ of the images of three d-paths in $\hat{Y}$, $I_d\hat{X}$ and $\hat{Z}$, as in (40).

By hypothesis, $b\colon x' \to x'$ is a c-path in X. Then $f(x_1) = f(x')$ is a flexible point of Y, and a_1 is a c-path of Y. Similarly a_3 is a c-path of Z. Finally, $h\colon 0 \to 1$ in $\uparrow\mathbb{I}$ is a c-path of $c\mathbb{I}$, and we are done: $a = ua_1 * \lambda a_2 * va_3$ is a c-path of $I(f,g)$. □

3.8 Remarks

(a) In the previous statement we cannot let X be just preflexible. In fact, the cylinder of a preflexible c-space X *is* preflexible, but we shall see in 4.5(b) that its suspension need not be – although in this case the c-spaces $Y = Z = \{*\}$ are even flexible.

(b) We recalled in 1.3 that cTop is a symmetric dI2-category. Moreover, it has all limits and colimits, and in particular all cylindrical colimits. To make it into a symmetric dI1-homotopical category ([G2], 4.1.4) the cylinder I should preserve the cylindrical colimits (as colimits), which would ensure good properties for h-pushouts ([G2], 4.1, 4.2). This is not the case.

In fact, the cylindrical colimit $I(\{*\}, \partial^-)$ amounts to the standard concatenation pushout (7) and we know that this pushout is not preserved by $I = - \times \mathrm{c}\mathbb{I}$ (cf. II.4.7(c)).

4. Cones, suspension and flexibility

Cones and suspension are derived from homotopy pushouts. We go on studying their weak flexibility properties.

X is always a c-space and we write as $p\colon X \to \{*\}$, or p_X, its map to the terminal singleton.

4.1 Mapping cones and cones

(a) A c-map $f\colon X \to Y$ has an *upper mapping cone* $C_c^+ f = I_c(f, p_X)$, also written $C^+ f$, defined as the h-pushout below, at the left

$$\begin{array}{ccc} X & \xrightarrow{\ p\ } & \{*\} \\ {\scriptstyle f}\downarrow & \gamma \nearrow & \downarrow{\scriptstyle v^+} \\ Y & \xrightarrow[u]{\ } & C^+ f \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle p}\downarrow & \gamma \nearrow & \downarrow{\scriptstyle u} \\ \{*\} & \xrightarrow[v^-]{\ } & C^- f \end{array} \qquad\qquad (42)$$

Its structural maps are the *lower base* $u\colon Y \to C^+ f$ and the *upper vertex* $v^+\colon \top \to C^+ f$; then, we have a *structural homotopy* $\gamma\colon uf \to v^+ p\colon X \to C^+ f$, which links uf to a constant map, in a universal way. The upper mapping cone is a functor $C^+\colon \mathrm{cTop}^{\mathbf{2}} \to \mathrm{cTop}$, defined on the category of morphisms of cTop.

Symmetrically, f has a *lower mapping cone* $C_c^- f = I(p_X, f)$ defined as the right h-pushout above (also written as $C^- f$), with a *lower vertex* v^- and an *upper base* u.

Let us note that the terms *upper* and *lower* agree with the vertex: this is consistent with *future* and *past* contractibility, in Lemma 4.2.

(b) In particular, the *upper cone* $C_c^+X = C_c^+(\mathrm{id}\, X) = I_c(X, p_X)$ of a c-space X is given by the h-pushout of the left diagram below, or equivalently by the pushout at the right

$$\begin{array}{ccc} X & \xrightarrow{\;p\;} & \{*\} \\ {\scriptstyle \mathrm{id}}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle v^+} \\ X & \xrightarrow[u]{} & C^+X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\;p\;} & \{*\} \\ {\scriptstyle \partial^+}\downarrow & & \downarrow{\scriptstyle v^+} \\ IX & \xrightarrow[\gamma]{} & C^+X \end{array} \tag{43}$$

$$C^+X = (IX + \{*\})/(\partial^+X + \{*\}), \quad \gamma(x,t) = [x,t], \quad v^+(*) = v^+.$$

It can be calculated as the quotient c-space displayed above: the upper base of the cylinder is collapsed to an upper vertex $v^+ = v^+(*)$, while the lower base is still an embedding:

$$u = \gamma\partial^-\colon X \to IX \to C^+X, \qquad u(x) = [x, 0]. \tag{44}$$

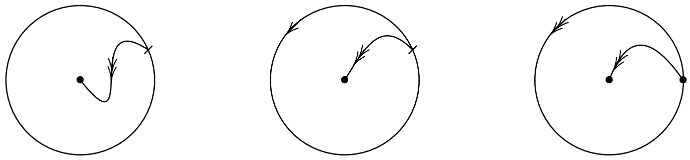
The homotopy $\gamma\colon IX \to C^+X$ allows one to deform – in the cone – the lower base $u\colon X \to C^+X$ to the map $\gamma\partial^+ = v^+p\colon X \to C^+X$, constant at the upper vertex.

In particular, $C^+\varnothing = \{v^+\}$ is a flexible singleton. If X is not empty:

$$C^+X = IX/\partial^+X, \qquad \gamma(x,t) = [x,t], \qquad v^+(*) = [x,1]. \tag{45}$$

The previous pushout defines a functor $C^+\colon \mathrm{cTop} \to \mathrm{cTop}$. As in 3.3(c), it is not homotopy invariant.

(c) *Examples*. The upper cones of $\mathbb{S}^1$, $\uparrow\!\mathbb{S}^1$ and $\mathrm{c}\mathbb{S}^1$, in cTop, can be drawn in the plane, as discs



$C_c^+(\mathbb{S}^1)$ $\quad$ $C_c^+(\uparrow\mathbb{S}^1)$ $\quad$ $C_c^+(\mathrm{c}\mathbb{S}^1)$ $\quad$ (46)

The upper cones of $\mathbb{S}^1$ and $\uparrow\mathbb{S}^1$ in dTop will be recalled in (56).

(d) Dually, by reversor duality, the *lower cone* $C^-X = I(p, X)$ is obtained by collapsing the lower base of IX to a lower vertex $v^- = v^-(*)$.

4.2 Lemma (Cones and contractibility)

A c-space X is future contractible in one step if and only if the base of its upper cone $u\colon X \to C^+X$ has a retraction $h\colon C^+X \to X$.

Proof. (In [G2], Lemma 1.7.3, this result is stated for dI1-homotopical categories.) We use the notation of (43). If $hu = \mathrm{id}\,X$, the map $h\gamma\colon IX \to X$ is a homotopy from $h\gamma\partial^- = hu = \mathrm{id}\,X$ to $h\gamma\partial^+ = hv^+p_X\colon X \to X$, and the latter is a constant endomap.

Conversely, if there is a homotopy $\varphi\colon IX \to X$ with $\varphi\partial^- = \mathrm{id}\,X$ and $\varphi\partial^+ = ip_X\colon X \to X$, we define $h\colon C^+X \to X$ as the unique map such that $h\gamma = \varphi\colon IX \to X$ and $hv^+ = i\colon \top \to X$. Now $hu = h\gamma\partial^- = \varphi\partial^- = \mathrm{id}\,X$. □

4.3 Suspension

The *suspension* $\Sigma_c X = I_c(p_X, p_X)$, or ΣX, of the c-space X is the following colimit and quotient space

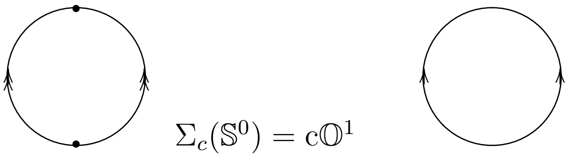
$$\begin{array}{ccccc} & & X & \longrightarrow & \{*\} \\ & & {\scriptstyle\partial^+}\downarrow & & \downarrow{\scriptstyle v^+} \\ X & \xrightarrow{\partial^-} & IX & \xrightarrow{\sigma} & \Sigma X \\ \downarrow & & & & \\ \{*\} & & \xrightarrow{\quad v^- \quad} & & \Sigma X \end{array} \qquad \Sigma X = (\{*\} + IX + \{*\})/R, \qquad (47)$$

where R collapses each base of IX to a point: the lower vertex v^- and the upper vertex v^+. (See the right-hand figure in (39).)

In particular $\Sigma\varnothing = \mathbb{S}^0 = \mathrm{c}\mathbb{S}^0$ (with the natural c-structure of a discrete topological space, the flexible one). If X is not empty, ΣX is the quotient of IX that collapses each bases to a point

$$\Sigma X = IX/R', \qquad \sigma(x,t) = [x,t], \qquad v^\alpha(*) = [x,\alpha]. \tag{48}$$

The suspension of $\mathbb{S}^0$ gives a rigid structure of the circle, represented in the left figure below

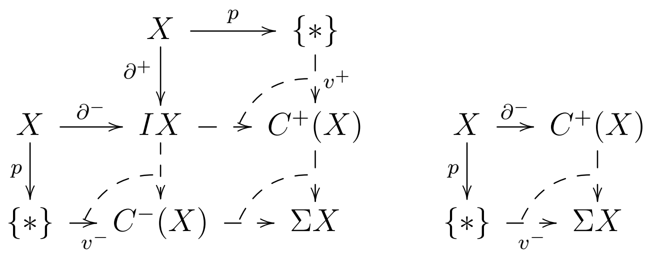


$\Sigma_c(\mathbb{S}^0) = \mathrm{c}\mathbb{O}^1 \qquad \Sigma_d(\uparrow\mathbb{S}^0) = \uparrow\mathbb{O}^1$ (49)

The generated d-space is the suspension $\Sigma_d(\mathbb{S}^0) = \uparrow\mathbb{O}^1$ in dTop ([G3], 1.4.3, 1.7.4), whose non-trivial selected paths are the restrictions of the previous ones. It is called the *ordered circle*, as its d-structure is produced by an obvious (partial) order relation on the topological circle.

4.4 Suspension and cones

The suspension ΣX is linked to the cones $C^\alpha(X)$ by the following diagrams of pushouts



(50)

$$\Sigma X = C^+(X)/\partial^- X = C^-(X)/\partial^+ X, \tag{51}$$

and is the quotient of each cone that collapses its base to a point.

In a splittable homotopy structure like Top and dTop (see 1.5), the suspension can also be obtained as a pasting of two cones on their bases. This cannot be done here: see 4.5(a).

4.5 Examples and flexibility

(a) The cones and suspension of a flexible space X are preflexible, by Theorem 3.7. Of course they are not flexible, generally; for instance this is not the case when X is a non-empty discrete d-space, as in the following examples

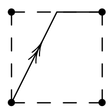
$$C^+\{*\} = C^-\{*\} = \Sigma\{*\} = \mathrm{c}\mathbb{I}, \qquad (52)$$

(53)

including $\Sigma(\mathbb{S}^0)$, represented in (49). (We note that this c-space has two flexible points, while the pasting of $C^-(\mathbb{S}^0)$ and $C^+(\mathbb{S}^0)$ over $\mathbb{S}^0$ has four of them.)

(b) The cones and suspension of a preflexible c-space need not be preflexible, nor even border flexible.

In fact, this is not the case for $C^+(\mathrm{c}\mathbb{I}) = \mathrm{c}\mathbb{I}^2/\partial^+\mathbb{I}$: the projection in $C^+(\mathrm{c}\mathbb{I})$ of the following c-path a in $\mathrm{c}\mathbb{I}^2$ has an unavoidable final delay



$$a(t) = (t, 2t \wedge 1). \qquad (54)$$

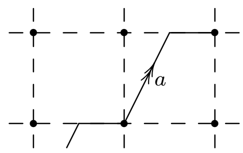
4.6 Higher spheres

The spheres $\mathrm{c}\mathbb{S}^n$ are not border flexible, for $n \geqslant 2$.

We consider the two-dimensional case $\mathrm{c}\mathbb{S}^2 = \mathrm{c}\mathbb{I}^2/\partial\mathbb{I}^2$, the higher ones being similar. It is convenient to view $\mathrm{c}\mathbb{S}^2$ as a quotient of the controlled torus $\mathrm{c}\mathbb{T}^2 = \mathrm{c}\mathbb{R}^2/\mathbb{Z}^2$ (an orbit space, cf. I.2.6) modulo the equivalence relation that collapses the image A of $\mathbb{R}\times\mathbb{Z} \cup \mathbb{Z}\times\mathbb{R}$ (in $\mathrm{c}\mathbb{T}^2$) to the flexible point of the sphere. (Also in cTop 'a quotient of a quotient is a quotient', up to isomorphism.)

In the following picture of the c-plane $\mathrm{c}\mathbb{R}^2$, the points of the flexible support $\mathbb{Z}^2$ are marked with bullets, and the dashed lines form the subspace

$\mathbb{R}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{R}$, collapsed to a point in the sphere



$\mathrm{c}\mathbb{R}^2$ (55)

As in (54), the projection b (in $\mathrm{c}\mathbb{S}^2$) of the c-path a of $\mathrm{c}\mathbb{R}^2$ has an unavoidable final delay, so that $\mathrm{c}\mathbb{S}^2$ is not border flexible.

Remarks. (a) This example is even more 'defective' than the cone $C^+(\mathrm{c}\mathbb{I})$ of 4.5(b): our path b also has *internal* prescribed delays. Thus the generated border flexible c-space $(\mathrm{c}\mathbb{S}^2)^{\mathrm{bf}}$ (defined in II.2.3(b)) is not preflexible.

(b) The controlled circle, and even more the *n-stop c-circle* $\mathrm{c}_n\mathbb{S}^1$ of I.2.6(b), model various concrete processes, like a hand of a watch, a washing machine dial, or a vertical panoramic wheel with n cabins. A railway turntable can be modelled by the reversible c-space generated by $\mathrm{c}_n\mathbb{S}^1$.

A controlled 2-sphere with an assigned distribution of 'stops' (the flexible points) might model a net of stationary satellites around the Earth, at a fixed altitude, providing the stops for a servicing shuttle.

4.7 The cofibre sequences of a map

The cofibre sequences of a map, or Puppe sequences, are defined and studied for a general dI1-homotopical category, in [G2], 1.7.6 – 1.7.9. (A *symmetric* dI1-homotopical structure is not required.) Here we only mention the basic facts obtained by applying these results to $\mathrm{c}\mathsf{Top}$.

Every c-map $f\colon X\to Y$ has a natural *upper cofibre sequence*

$$X \xrightarrow{f} Y \xrightarrow{u} C^+f \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma u} \Sigma(C^+f) \xrightarrow{\Sigma d} \Sigma^2 X \ldots \qquad (56)$$

formed by the base $u\colon Y\to C^+f$ and the *upper differential* $d = d^+(f)\colon C^+f\to\Sigma X$. The latter is defined by the universal property of the upper cone C^+f as an h-pushout, in (42)

$$\begin{gathered} du = v^-p_Y\colon Y\to\Sigma X, \qquad dv^+ = v^+\colon \top\to\Sigma X, \\ d\gamma = \sigma\colon v^-p_X\to v^+p_X\colon X\to\Sigma X. \end{gathered} \qquad (57)$$

Reversor duality gives the *lower cofibre sequence* of f, with the *lower differential* $d = d^-(f) \colon C^- f \to \Sigma X$

$$X \xrightarrow{f} Y \xrightarrow{u} C^- f \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma u} \Sigma(C^- f) \xrightarrow{\Sigma d} \Sigma^2 X \; ... \qquad (58)$$

These sequences give exact sequences in directed homology, as proved in [G2], 2.6.3, for a general dI1-homotopical category. But we already remarked in 1.6(b) the moderate interest of directed homology for d-spaces and c-spaces, beyond degree 1; on the other hand, pointed cubical sets have a 'perfect theory' ([G2], 2.6.4).

5. The flexible homotopy structure of c-spaces

The flexible interval $\uparrow\mathbb{I}$ produces a second homotopy structure on the category of c-spaces, that will be denoted as $\mathrm{c}\mathsf{Top}_F$. It has a secondary role here, although it is better related to the fundamental category functor, by Proposition 5.3 (and Theorem II.5.4).

5.1 The flexible structure

The c-space $\uparrow\mathbb{I}$ is also a symmetric dI2-interval, with the same operations listed above, in (4). Moreover it is exponentiable in $\mathrm{c}\mathsf{Top}$, as proved in Theorem II.3.9(b).

The *flexible cylinder functor*

$$I_F \colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad I_F(X) = X \times \uparrow\mathbb{I}, \qquad (59)$$

extends the cylinder functor I_d of d-spaces, and satisfies the axioms listed in (6): $\mathrm{c}\mathsf{Top}_F$ is also a symmetric dI2-category.

The functor P_F *of flexible paths*, right adjoint to I_F

$$P_F \colon \mathrm{c}\mathsf{Top} \to \mathrm{c}\mathsf{Top}, \qquad P_F(Y) = Y^{\uparrow\mathbb{I}}, \qquad (60)$$

extends the path functor P_d of d-spaces. The adjunction automatically makes $\mathrm{c}\mathsf{Top}_F$ into a symmetric dIP2-homotopical category: the faces, degeneracy, reflection and connections of P_F are mates to those of I_F ([G2], 4.2.1). For instance, the 'new' faces $\partial^\alpha \colon P_F \to 1$ are obtained as $P_F Y \to I_F P_F Y \to Y$

by composing the cylinder faces of $P_F Y$ with the counit of the adjunction (the path evaluation).

Moreover, taking into account that the category cTop has all colimits, preserved by I_F, and all limits, preserved by P_F, the structure cTop_F is a symmetric dIP2-homotopical category ([G2], 4.2.1).

Remarks. (a) The classification of dIP-structures cannot go higher, as shown below.

(b) The fact that every flexible homotopy is a standard one can be formally expressed saying that the identity functor of the category cTop is a lax dI2-functor $H\colon \mathrm{cTop}_F \to \mathrm{cTop}$ of dI2-categories (cf. [G2], 1.2.6 and 4.2.7).

Essentially, we have a strict comparison $RH = HR$ (the reversors of these structures are the same) and a non-invertible comparison $h\colon I_c H \to HI_F$ (given by the reshaping $I_c X \to I_F X$), which agree with the dI2-structure.

5.2 Flexible homotopies

In cTop_F the flexible interval produces flexible homotopies, already introduced in II.4.1: a flexible homotopy $\varphi\colon f \to g\colon X \to Y$ of c-maps is represented by a map $I_F X \to Y$ (or equivalently $X \to P_F Y$).

Again, flexible homotopies 'cannot' be concatenated): more precisely, as proved in II.4.7(b), they are not closed under concatenation within topological homotopies, and cTop_F cannot have a dI3-structure consistent with the topological one (cf. [G2], 4.2.2).

Formally, one can define a concatenation pushout $J_F X = I_F X +_X I_F X$, which is preserved by I_F since $\uparrow\!\mathbb{I}$ is exponentiable. But this is useless without a 'good' concatenation map $I_F X \to J_F X$ that would transform a 'pre-concatenation' $J_F X \to Y$ into a flexible homotopy $I_F X \to Y$: to agree with the underlying topology, this map should be a reshaping, and $I_F X$ need not be finer than $J_F X$, as shown in II.4.7(b) for $X = \mathrm{c}\mathbb{I}$.

Flexible homotopies are dealt with in the same way as the general ones in 2.1: trivial case, reflection, whisker composition, composite flexible homotopies and their operations.

In particular, a *composite flexible homotopy* $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \twoheadrightarrow f''\colon X \to Y$ is a finite sequence of consecutive flexible homotopies between c-maps $f_i\colon X \to Y$

$$f' = f_0 \xrightarrow{\varphi_1} f_1 \xrightarrow{\varphi_2} f_2 \; ... \xrightarrow{\varphi_n} f_n = f''. \qquad (61)$$

Computations are the same as in the standard structure, but involve different spaces: for instance, the trivial flexible homotopy of a c-map $f\colon X \to Y$, represented by $fe\colon X\times{\uparrow}\mathbb{I} \to Y$, is still defined as $fe(x,t) = f(x)$, on a different domain.

Future F-homotopy equivalences and *future F-deformation retracts* are defined as in (22) and (26), using composite flexible homotopies. The reversed notions, *in the past*, are produced by the reflector R. The coherent case is defined as in 2.6.

5.3 Proposition (Homotopy invariance)

(a) A composite flexible homotopy $\varphi = (\varphi_1, ..., \varphi_n)\colon f' \twoheadrightarrow f''\colon X \to Y$ *(as in* (61)*) induces a natural transformation*

$$\varphi_*\colon f'_* \to f''_*\colon {\uparrow}\Pi_1(X) \to {\uparrow}\Pi_1(Y), \qquad (62)$$

the composite of the natural transformations $\varphi_{i*}\colon f_{i-1*} \to f_{i*}$ *of Theorem II.5.4.*

Induction agrees with the operations of composite flexible homotopies: their vertical composition and whisker composition (with c-maps) are sent to the corresponding operations of natural transformations (with functors).

(b) A future F-homotopy equivalence $(f, g; \varphi, \psi)$ *of c-spaces* X, Y *induces a future homotopy equivalence of their fundamental categories (see* (14)*)*

$$f_*\colon {\uparrow}\Pi_1(X) \rightleftarrows {\uparrow}\Pi_1(Y) : g_*, \quad \varphi_*\colon \mathrm{id} \to g_*f_*, \quad \psi_*\colon \mathrm{id} \to f_*g_*. \qquad (63)$$

Proof. A consequence of Theorem II.5.4(b). □

5.4 Flexible contractibility

A *future F-contractible* c-space X is defined as in (27), using a composite flexible homotopy φ

$$i\colon \{*\} \rightleftarrows X : p, \qquad \varphi\colon \mathrm{id}\, X \twoheadrightarrow ip \qquad (pi = \mathrm{id}\, \{*\}). \qquad (64)$$

In this case the singleton $\{*\}$ is a future F-deformation retract of X, at the flexible point $x_0 = i(*)$. We also say that X is *future F-contractible* to x_0.

Here the fundamental category of X is *future contractible* to the singleton category $\mathbf{1}$

$$i_*\colon \mathbf{1} \rightleftarrows \uparrow\Pi_1(X) : p_*, \quad \varphi_*\colon \mathrm{id}\uparrow\Pi_1(X) \to p_*i_* \ \ (p_*i_* = \mathrm{id}\,\mathbf{1}). \qquad (65)$$

Equivalently, $\uparrow\Pi_1(X)$ has a *natural weak terminal object* $x_0 = i(*)$. This means a family of arrows $\varphi(x)\colon x \to x_0$ (indexed on the flexible points $x \in |X|_0$) which is natural, in the sense that every arrow $u\colon x \to x'$ gives a commutative triangle $\varphi(x) = \varphi(x')u\colon x \to x_0$.

We say that X is *coarsely F-contractible* if there exists a finite sequence $X, X_1, ..., X_{n-1}, \{*\}$ where each c-space is future or past F-homotopy equivalent to the next.

The interval $\uparrow\mathbb{I}$ is past F-contractible to 0 and future F-contractible to 1, with flexible homotopies given by the (flexible) connections $g^\alpha\colon \uparrow\mathbb{I}^2 \to \uparrow\mathbb{I}$, as in (29).

5.5 Flexible connectedness

In a c-space X, a flexible path $a\colon x \to x'$ is the same as a c-path in the flexible part $\mathsf{Fl}\,X$ (in 1.1).

Therefore, flexible connectedness in X can be defined by the property of c-connectedness in $\mathsf{Fl}\,X$, characterised by the composed functor

$$\uparrow\Pi_0\mathsf{Fl}\colon \mathrm{c}\mathsf{Top} \to \mathsf{Set}. \qquad (66)$$

The *flexible component* $[x]_F$ of a flexible point of the c-space X will be its c-component in $\mathsf{Fl}\,X$, contained in the controlled component $[x]_c$ and equal to the latter in any d-space. X is *flexibly connected*, or *F-connected*, if it has precisely one flexible component, if and only if $\uparrow\Pi_0\mathsf{Fl}\,(X)$ is a singleton.

Let $n \geqslant 1$. The c-spaces $\mathrm{c}\mathbb{S}^1$, $c_n\mathbb{S}^1$, $\mathrm{c}\mathbb{S}^n$ are trivially flexibly connected, as they have a single flexible point. The d-spaces $\uparrow\mathbb{I}$, $\mathbb{I}^\sim$, $\uparrow\mathbb{R}$, $\uparrow\mathbb{S}^n$ are flexibly connected, as well as the cartesian products of the examples considered so far. The other examples in 2.4(c) are not, which implies that they are not coarsely F-contractible, by the following proposition.

5.6 Proposition

(a) If the c-space X is future F-contractible to x_0, every flexible point x has a flexible path $x \to x_0$.

(b) A coarsely F-contractible c-space X is always flexibly connected.

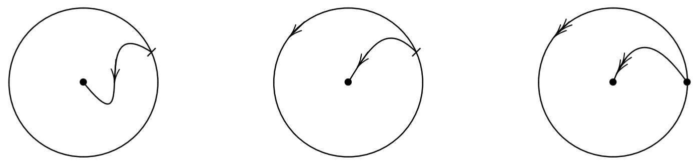
Proof. Point (b) follows from (a), which can be proved as in Proposition 2.5. Here we have flexible homotopies $\varphi_i\colon f_{i-1} \to f_i$, which are c-maps $X \times {\uparrow}\mathbb{I} \to X$ and, at any flexible point x, give a sequence of consecutive flexible paths $x \to f_1(x) \to ... \to x_0$. □

5.7 Flexible homotopy pushouts

The homotopy pushouts of the standard structure cTop_c, studied in Section 3, have parallel constructions in cTop_F, based on the flexible cylinder $I_F = - \times {\uparrow}\mathbb{I}$:

- the *flexible homotopy pushout* $I_F(f, g)$,
- the *flexible mapping cones* $C_F^+ f = I_F(f, p_X)$ and $C_F^- f = I_F(p_X, f)$,
- the *flexible cones* $C_F^+ X = I_F(X, p_X)$ and $C_F^- X = I_F(p_X, X)$,
- the *flexible suspension* $\Sigma_F X = I_F(p_X, p_X)$.

These constructions extend the corresponding ones of dTop, based on its endofunctor $I_d = - \times {\uparrow}\mathbb{I}$. Thus, in the following examples of flexible upper cones, the d-spaces $\mathbb{S}^1$ and ${\uparrow}\mathbb{S}^1$ give the upper cones of dTop, namely $C_F^+(\mathbb{S}^1) = C_d^+(\mathbb{S}^1)$ and $C_F^+({\uparrow}\mathbb{S}^1) = C_d^+({\uparrow}\mathbb{S}^1)$



$C_F^+(\mathbb{S}^1)$ $\quad C_F^+({\uparrow}\mathbb{S}^1)$ $\quad C_F^+(\mathrm{c}\mathbb{S}^1)$ (67)

while $C_F^+(\mathrm{c}\mathbb{S}^1)$ is not a d-space, of course; but note that there are also flexible paths from the flexible point of the circle to the upper vertex.

5.8 Proposition (Flexible cones and contractibility)

(a) The flexible upper cone $C_F^+ X$ is future F-contractible, in one step.

(b) A c-space X is future F-contractible in one step if and only if the base $u\colon X \to C^+_F X$ has a retraction $h\colon C^+_F X \to X$.

Proof. (a) The pushout in (43) (at the right hand) is preserved by the product $-\times{\uparrow}\mathbb{I}$ (by an exponentiable object of cTop). There is thus precisely one map $\varphi\colon C^+_F X\times{\uparrow}\mathbb{I} \to C^+_F X$ such that

$$\begin{aligned}\varphi(\gamma\times{\uparrow}\mathbb{I}) = \gamma(X\times g^-)\colon X\times{\uparrow}\mathbb{I}\times{\uparrow}\mathbb{I} \to X\times{\uparrow}\mathbb{I} \to C^+_F X,\\ \varphi(v^+\times{\uparrow}\mathbb{I}) = v^+q\colon {\uparrow}\mathbb{I} \to \{*\} \to C^+_F X,\end{aligned} \tag{68}$$

taking into account that $\gamma(X\times g^-)(\partial^+\times{\uparrow}\mathbb{I})(x,t) = [x, 1\vee t] = v^+$.

This c-map is a flexible homotopy, from $\mathrm{id}\, C^+_F X$ to the constant map $C^+_F X \to C^+_F X$ at v^+

$$\begin{aligned}(\varphi\partial^-)[x,s] &= \varphi(\gamma(x,s),0) = \varphi(\gamma\times{\uparrow}\mathbb{I})(x,s,0)\\ &= \gamma(x, g^-(s,0)) = \gamma(x,s) = [x,s],\\ (\varphi\partial^+)[x,s] &= ... = \gamma(x,g^-(s,1)) = \gamma(x,1) = v^+.\end{aligned}$$

(b) As in Lemma 4.2. □

6. Cubical sets and their realisations

Cubical sets have a well-known non-symmetric monoidal structure. As a consequence, the obvious directed interval ${\uparrow}\mathbf{i}$, freely generated by a 1-cube (see 6.2) gives rise to a *left cylinder* ${\uparrow}\mathbf{i}\otimes K$ and a *right cylinder* $K\otimes{\uparrow}\mathbf{i}$, and two notions of directed homotopy interchanged by an endofunctor, the ‘transposer’ S.

Classically, cubical sets are viewed as combinatorial structures modelling relatively simple topological spaces, by their geometric realisation (see 6.4). But they can also model d-spaces, by a *directed* geometric realisation (in 6.5), and c-spaces, by a *controlled* geometric realisation (in 6.6).

Part of this material comes from [G2], Section 1.6.

6.1 Cubical sets

Every topological space X has an associated *cubical set* $\square X$, with components $\square_n X = \mathsf{Top}(\mathbb{I}^n, X)$, the set of *singular n-cubes* of X. Its faces and degeneracies, for $\alpha = 0, 1$ and $i = 1, ..., n$

$$\partial_i^\alpha \colon \square_n X \to \square_{n-1} X, \qquad e_i \colon \square_{n-1} X \to \square_n X, \tag{69}$$

come out (contravariantly) from the faces and degeneracies of the standard cubes $\mathbb{I}^n$, written in the same way

$$\begin{aligned} &\partial_i^\alpha = \mathbb{I}^{i-1} \times \partial^\alpha \times \mathbb{I}^{n-i} \colon \mathbb{I}^{n-1} \to \mathbb{I}^n, \\ &\qquad \partial_i^\alpha(t_1, ..., t_{n-1}) = (t_1, ..., \alpha, ..., t_{n-i}), \\ &e_i = \mathbb{I}^{i-1} \times e \times \mathbb{I}^{n-i} \colon \mathbb{I}^n \to \mathbb{I}^{n-1}, \\ &\qquad e_i(t_1, ..., t_n) = (t_1, ..., \hat{t}_i, ..., t_n). \end{aligned} \tag{70}$$

As usual, $\hat{t}_i$ means that the coordinate t_i is omitted.

Generally, a *cubical set* K is a sequence of sets K_n $(n \geqslant 0)$, together with mappings, called *faces* (∂_i^α) and *degeneracies* (e_i)

$$\begin{gathered} K = ((K_n), (\partial_i^\alpha), (e_i)), \\ \partial_i^\alpha = \partial_{ni}^\alpha \colon K_n \to K_{n-1}, \qquad e_i = e_{ni} \colon K_{n-1} \to K_n, \end{gathered} \tag{71}$$

(for $\alpha = \pm$ and $i = 1, ..., n$) that satisfy the *cubical* relations

$$\begin{gathered} \partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_{i+1}^\alpha \ \ (j \leqslant i), \qquad e_j e_i = e_{i+1} e_j \ \ (j \leqslant i), \\ \partial_i^\alpha e_j = e_j \partial_{i-1}^\alpha \ (j < i), \ \text{ or id } (j = i), \ \text{ or } \ e_{j-1}.\partial_i^\alpha \ (j > i). \end{gathered} \tag{72}$$

Elements of K_n are called *n-cubes*, and *vertices* or *edges* for $n = 0$ or 1, respectively. Each n-cube $x \in K_n$ has 2^n vertices: $\partial_1^\alpha \partial_2^\beta \partial_3^\gamma(x)$ for $n = 3$. Given a vertex $x \in K_0$, the *totally degenerate n-cube at x* is obtained by applying n degeneracy operators to the given vertex, in any (legitimate) way:

$$e^n(x) = e_{i_n} ... \, e_{i_2} e_{i_1}(x) \in K_n \qquad\qquad (1 \leqslant i_j \leqslant j). \tag{73}$$

A *morphism* $f = (f_n) \colon K \to L$ is a sequence of mappings $f_n \colon K_n \to L_n$ commuting with faces and degeneracies.

All this forms a category Cub which has all limits and colimits and is cartesian closed. It is the presheaf category of functors $K \colon \underline{\mathrm{I}}^{\mathrm{op}} \to$ Set, where $\underline{\mathrm{I}}$ is the subcategory of Set consisting of the *elementary cubes* 2^n, together with the maps $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates. One can see [G2], Sections 1.6.7 and A1.8; or [GM] for cubical sets with a richer structure, including connections and symmetries.

The terminal object $\top$ is freely generated by one vertex $*$ and will also be written as $\{*\}$ (although each of its components is a singleton). The initial object is empty, i.e. all its components are; all the other cubical sets have non-empty components in every degree.

We shall make use of two covariant involutive endofunctors, the *reversor* R and the *transposer* S

$$\begin{aligned} &R\colon \mathsf{Cub} \to \mathsf{Cub}, \qquad RK = K^{\mathrm{op}} = ((K_n), (\partial_i^{-\alpha}), (e_i)),\\ &S\colon \mathsf{Cub} \to \mathsf{Cub}, \qquad SK = ((K_n), (\partial_{n+1-i}^{\alpha}), (e_{n+1-i})),\\ &RR = \mathrm{id}\,, \qquad SS = \mathrm{id}\,, \qquad RS = SR. \end{aligned} \tag{74}$$

(The meaning of $-\alpha$, for $\alpha = \pm$, is obvious.) The functor R reverses the 1-dimensional direction, while S reverses the 2-dimensional one; plainly, they commute. If $x \in K_n$, the same element viewed in K^{op} will often be written as x^{op}, so that $\partial_i^-(x^{\mathrm{op}}) = (\partial_i^+ x)^{\mathrm{op}}$.

We say that a cubical set K is *reversive* if $RK \cong K$, and *permutative* if $SK \cong K$.

The category Cub has a *non*-symmetric monoidal structure [Ka, BH]

$$(K \otimes L)_n = (\Sigma_{p+q=n}\, K_p \times L_q)/\sim_n, \tag{75}$$

where $\sim_n$ is the equivalence relation generated by identifying $(e_{r+1}x, y)$ with $(x, e_1 y)$, for all $(x, y) \in K_r \times L_s$ (where $r + s = n - 1$). The equivalence class of (x, y) is written as $x \otimes y$.

We refer to [G2], 1.6.3, for a more detailed description.

6.2 Standard models

The *elementary directed interval* $\uparrow\!\mathbf{i} = \mathbf{2}$ is freely generated by a 1-cube, written as u

$$0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \qquad \partial_1^+(u) = 1. \tag{76}$$

This cubical set is reversive and permutative.

The *elementary directed n-cube* $(n \geqslant 0)$ is its n-th tensor power $\uparrow\mathbf{i}^{\otimes n} = \uparrow\mathbf{i} \otimes ... \otimes \uparrow\mathbf{i}$ ([G2], 1.6.3). It is freely generated by its n-cube $u^{\otimes n}$, still reversive and permutative, and can also be defined as the representable presheaf $\underline{\mathrm{I}}(-, 2^n) \colon \underline{\mathrm{I}}^{\mathrm{op}} \to \mathsf{Set}$. The *elementary directed square* $\uparrow\mathbf{i}^{\otimes 2} = \uparrow\mathbf{i} \otimes \uparrow\mathbf{i}$ can be represented as follows, with the generator $u \otimes u$, its faces and vertices

$$\begin{array}{ccc} 00 & \xrightarrow{0\otimes u} & 01 \\ {\scriptstyle u\otimes 0}\downarrow & {\scriptstyle u\otimes u} & \downarrow{\scriptstyle u\otimes 1} \\ 10 & \xrightarrow[1\otimes u]{} & 11 \end{array} \qquad\qquad \begin{array}{cc} \bullet & \xrightarrow[2]{} \\ \downarrow{\scriptstyle 1} & \end{array} \tag{77}$$

The face $\partial_1^-(u \otimes u) = 0 \otimes u$ is drawn as orthogonal to direction 1 (and directions are chosen so that the labelling of vertices agrees with matrix indexing). For each cubical object K, $\mathsf{Cub}(\uparrow\mathbf{i}^{\otimes n}, K) = K_n$, by Yoneda Lemma.

The *directed (integral) line* $\uparrow\mathbf{z}$ is generated by (countably many) vertices $n \in \mathbb{Z}$ and edges u_n, from $\partial_1^-(u_n) = n$ to $\partial_1^+(u_n) = n + 1$. The *directed integral interval* $\uparrow[i, j]_{\mathbf{z}}$ is the obvious cubical subset with vertices in the integral interval $[i, j]_{\mathbf{z}}$ (and all cubes whose vertices lie there); in particular, $\uparrow\mathbf{i} = \uparrow[0, 1]_{\mathbf{z}}$.

The *elementary directed circle* $\uparrow\mathbf{s}^1$ is generated by one 1-cube u with equal faces

$$* \xrightarrow{u} * \qquad\qquad \partial_1^-(u) = \partial_1^+(u). \tag{78}$$

Similarly, the *elementary directed n-sphere* $\uparrow\mathbf{s}^n$ (for $n > 1$) is generated by one n-cube u all whose faces are totally degenerate (see (73)), hence equal

$$\partial_i^\alpha(u) = e^{n-1}(\partial_1^-)^n(u) \qquad\qquad (\alpha = \pm;\ i = 1, ..., n). \tag{79}$$

Moreover, $\uparrow\mathbf{s}^0$ is the discrete cubical set on two vertices. The *elementary directed n-torus* is a tensor power of $\uparrow\mathbf{s}^1$

$$\uparrow\mathbf{t}^n = (\uparrow\mathbf{s}^1)^{\otimes n}. \tag{80}$$

We also consider the *ordered circle* $\uparrow\mathbf{o}^1$, generated by two edges with the same faces

$$0 \underset{u''}{\overset{u'}{\rightrightarrows}} 1 \qquad\qquad \partial_1^\alpha(u') = \partial_1^\alpha(u''), \tag{81}$$

which is a 'cubical model' of the ordered circle $\uparrow\mathbb{O}^1$, a d-space recalled in 4.3. (The latter is the directed geometric realisation of the former, in the sense of 6.5.)

Starting from $\uparrow\mathbf{s}^0$, the *unpointed suspension* provides all $\uparrow\mathbf{o}^n$ ([G2], 1.7.6) while the *pointed suspension* provides all $\uparrow\mathbf{s}^n$ ([G2], 2.3.2). Of course, these models have the same geometric realisation $\mathbb{S}^n$ (as a topological space) and the same homology; but their *directed* homology is different ([G2], 2.1.4): the models $\uparrow\mathbf{s}^n$ are more interesting: they have a non-trivial order in directed homology.

All these cubical sets are reversive and permutative.

6.3 The homotopy structure

As shown in [G2], 1.6.5, the category of cubical sets has a left dIP1-structure Cub_L defined by the left cylinder functor $I(K) = \uparrow\mathbf{i} \otimes K$, and a right dIP1-structure Cub_R defined by the right cylinder functor $SIS(K) = K \otimes \uparrow\mathbf{i}$. These structures are made isomorphic by the symmetriser $S\colon \mathsf{Cub}_L \to \mathsf{Cub}_R$.

The left cylinder P has a simple description: it shifts down all the components, discarding the faces and degeneracies of index 1, which are then used to build three natural transformations, the faces and degeneracy of P

$$\begin{aligned} &P\colon \mathsf{Cub} \to \mathsf{Cub}, \qquad && PK = ((K_{n+1}), (\partial^\alpha_{i+1}), (e_{i+1})), \\ &\partial^\alpha = \partial^\alpha_1\colon PK \to K, && e = e_1\colon K \to PK. \end{aligned} \tag{82}$$

Symmetrically, the right cylinder SPS shifts down all components, discarding the faces and degeneracies of highest index.

The transposition of spaces, d-spaces and c-spaces is partially surrogated here by an 'external transposition', $s\colon PSPS \to SPSP$, whose components are identities ([G2], 1.6.5).

Coming back to the discussion of symmetries in 1.6, we note that Cub breaks both symmetries of topological spaces, reversion and transposition. This has heavy consequences for homotopy theory, as remarked in 1.6, and strong advantages for homology, recalled in 6.8.

6.4 The classical geometric realisation

We have already recalled the functor

$$\square\colon \mathsf{Top} \to \mathsf{Cub}, \qquad \square X = \mathsf{Top}(\mathbb{I}^\bullet, X), \tag{83}$$

which assigns to a topological space X the singular cubical set of n-cubes $\mathbb{I}^n \to X$, produced by the cocubical space of standard cubes

$$\mathbb{I}^\bullet = ((\mathbb{I}^n), (\partial_i^\alpha), (e_i)),$$

a covariant functor $\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathsf{Top}$. The *geometric realisation* $\mathcal{R}K$ of a cubical set is given by its left adjoint

$$\mathcal{R}\colon \mathsf{Cub} \rightleftarrows \mathrm{d}\mathsf{Top} : \square, \qquad \mathcal{R} \dashv \square. \tag{84}$$

The topological space $\mathcal{R}K$ is constructed by pasting a copy of the standard cube $\mathbb{I}^n$ for each n-cube $x \in K_n$, along faces and degeneracies. This colimit comes with a cocone of structural mappings $\hat{x}$ (for $x \in K_n$ and $n \in \mathbb{N}$), coherently with faces and degeneracies of $\mathbb{I}^\bullet$ and K

$$\hat{x}\colon \mathbb{I}^n \to \mathcal{R}K, \qquad \hat{x}\partial_i^\alpha = (\partial_i^\alpha x)\hat{}, \qquad \hat{x}e_i = (e_i x)\hat{}, \tag{85}$$

and $\mathcal{R}K$ has the finest topology making all the structural mappings continuous. (Formally, $\mathcal{R}$ is the coend of the functor $K\mathbb{I}^\bullet\colon \underline{\mathrm{I}}^{\mathrm{op}}\times\underline{\mathrm{I}} \to \mathsf{Set}\times\mathsf{Top} \to \mathsf{Top}$, see [M].)

This realisation is important, since it is well known that the combinatorial homology of a cubical set K coincides with the homology of the CW-space $\mathcal{R}K$ (cf. [Mu] 4.39, for the simplicial case). But we also want finer realisations, retaining more information on the cubes of K: we shall use a d-space (in 6.5), or a c-space (in 6.6, 6.7).

6.5 Directed geometric realisation

Cubical sets also have a realisation as d-spaces, where the n-cube $\uparrow\mathbf{i}^{\otimes n}$ is realised as $\uparrow\mathbb{I}^n$.

We replace the cocubical space $\mathbb{I}^\bullet$ of standard topological cubes $\mathbb{I}^n$ by a directed version, the cocubical d-space $\uparrow\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathrm{d}\mathsf{Top}$ of standard d-cubes

$\uparrow\mathbb{I}^n$, with the corresponding faces and degeneracies

$$\begin{aligned} \partial_i^\alpha &= \uparrow\mathbb{I}^{i-1} \times \partial^\alpha \times \uparrow\mathbb{I}^{n-i} \colon \uparrow\mathbb{I}^{n-1} \to \uparrow\mathbb{I}^n, \\ e_i &= \uparrow\mathbb{I}^{i-1} \times e \times \uparrow\mathbb{I}^{n-i} \colon \uparrow\mathbb{I}^{n+1} \to \uparrow\mathbb{I}^n. \end{aligned} \tag{86}$$

This produces the functor

$$\uparrow\square\colon \mathrm{dTop} \to \mathsf{Cub}, \qquad \uparrow\square_n(X) = \mathrm{dTop}(\uparrow\mathbb{I}^n, X), \tag{87}$$

which assigns to a d-space X the singular cubical set $\uparrow\square(X)$ of its directed n-cubes $\uparrow\mathbb{I}^n \to X$, extending the functor $\square\colon \mathsf{Top} \to \mathsf{Cub}$. Its left adjoint yields the *directed geometric realisation* $\uparrow\mathcal{R}(K)$ of a cubical set K

$$\uparrow\mathcal{R}\colon \mathsf{Cub} \rightleftarrows \mathrm{dTop} : \uparrow\square, \qquad \uparrow\mathcal{R} \dashv \uparrow\square. \tag{88}$$

The d-space $\uparrow\mathcal{R}(K)$ is thus the pasting in dTop of K_n copies of $\uparrow\mathbb{I}^\bullet(2^n) = \uparrow\mathbb{I}^n$ ($n \geqslant 0$), along faces and degeneracies. (Again, the coend of the functor $K\uparrow\mathbb{I}^\bullet\colon \underline{\mathrm{I}}^{\mathrm{op}} \times \underline{\mathrm{I}} \to \mathrm{dTop}$.)

In other words (since a colimit in dTop is the colimit of the underlying topological spaces, equipped with the final d-structure of the structural maps), one starts from the ordinary geometric realisation $\mathcal{R}K$, as a topological space, and equips it with the following d-structure $\uparrow\mathcal{R}(K)$: the d-paths are generated, under concatenation and partial reparametrisation, by the mappings $\hat{x}a\colon \mathbb{I} \to \mathbb{I}^n \to \mathcal{R}K$, where $a\colon \mathbb{I} \to \mathbb{I}^n$ is an order-preserving map and $\hat{x}$ corresponds to some cube $x \in K_n$, in the colimit-construction of $\mathcal{R}K$.

Composing (88) with the adjunction $U \dashv D'$ (in I.1.7) between d-spaces and spaces

$$\mathsf{Cub} \underset{\uparrow\square}{\overset{\uparrow\mathcal{R}}{\rightleftarrows}} \mathrm{dTop} \underset{D'}{\overset{U}{\rightleftarrows}} \mathsf{Top} \tag{89}$$

we get back the ordinary realisation $\mathcal{R} = U(\uparrow\mathcal{R})\colon \mathsf{Cub} \to \mathsf{Top}$. (The d-space $D'X$ associated to a space admits all paths as d-paths.)

Various models of dTop are directed realisations of simple cubical sets already considered in 6.2. For instance, the directed interval $\uparrow\mathbb{I}$ realises $\uparrow\mathbf{i}$; the directed line $\uparrow\mathbb{R}$ realises $\uparrow\mathbf{z}$; the directed sphere $\uparrow\mathbb{S}^n$ realises $\uparrow\mathbf{s}^n$; the ordered circle $\uparrow\mathbb{O}^1$ (cf. 4.3) realises $\uparrow\mathbf{o}^1 = \{0 \rightrightarrows 1\}$.

The directed realisation functor $\uparrow\mathcal{R}\colon \mathsf{Cub} \to \mathrm{dTop}$ is a strong dI1-functor, which means that it commutes with the cylinder functor, up to functorial isomorphism: see [G2], 1.6.7.

6.6 Controlled geometric realisation

Cubical sets can also be realised as c-spaces, turning the n-cube $\uparrow\mathbf{i}^{\otimes n}$ into the standard c-cube $\mathrm{c}\mathbb{I}^n$.

Now we start from the cocubical c-space $\mathrm{c}\mathbb{I}^\bullet\colon \underline{\mathrm{I}} \to \mathrm{cTop}$ of standard c-cubes $\mathrm{c}\mathbb{I}^n$, with the corresponding faces and degeneracies

$$\partial_i^\alpha = \mathrm{c}\mathbb{I}^{i-1}\times\partial^\alpha\times\mathrm{c}\mathbb{I}^{n-i}, \qquad e_i = \mathrm{c}\mathbb{I}^{i-1}\times e\times\mathrm{c}\mathbb{I}^{n-i}. \tag{90}$$

This produces the functor

$$\mathrm{c}\square\colon \mathrm{cTop}\to\mathsf{Cub}, \qquad \mathrm{c}\square_n(X) = \mathrm{cTop}(\mathrm{c}\mathbb{I}^n, X), \tag{91}$$

which assigns to a c-space X the singular cubical set $\mathrm{c}\square(X)$ of its controlled n-cubes $\mathrm{c}\mathbb{I}^n \to X$; it is an extension of the functor $\uparrow\square\colon \mathrm{dTop}\to\mathsf{Cub}$, because the reflector of d-spaces in cTop gives $(\mathrm{c}\mathbb{I}^n)\hat{} = \uparrow\mathbb{I}^n$ (see I.2.7). The *controlled geometric realisation* $\mathrm{c}\mathcal{R}(K)$ of a cubical set is given by the left adjoint functor

$$\mathrm{c}\mathcal{R}\colon \mathsf{Cub} \rightleftarrows \mathrm{cTop} : \mathrm{c}\square, \qquad \mathrm{c}\mathcal{R} \dashv \mathrm{c}\square. \tag{92}$$

The c-space $\mathrm{c}\mathcal{R}(K)$ is thus the pasting in cTop of K_n copies of $\mathrm{c}\mathbb{I}^\bullet(2^n) = \mathrm{c}\mathbb{I}^n$ ($n \geqslant 0$), along faces and degeneracies – the coend of the functor $K\mathrm{c}\mathbb{I}^\bullet\colon \underline{\mathrm{I}}^{\mathrm{op}}\times\underline{\mathrm{I}}\to\mathrm{cTop}$.

Again, $\mathrm{c}\mathcal{R}(K)$ is the geometric realisation $\mathcal{R}K$ with controlled paths generated, under concatenation and global reparametrisation, by the mappings $\hat{x}a\colon \mathbb{I}\to\mathbb{I}^n\to\mathcal{R}K$, where $a\colon \mathrm{c}\mathbb{I}\to\mathrm{c}\mathbb{I}^n$ is a c-path and $\hat{x}$ corresponds to a cube $x\in K_n$ in the colimit-construction of $\mathcal{R}K$.

Composing this adjunction with the canonical adjunctions between c-spaces, d-spaces and spaces (in I.1)

$$\mathsf{Cub} \underset{\mathrm{c}\square}{\overset{\mathrm{c}\mathcal{R}}{\rightleftarrows}} \mathrm{cTop} \underset{\supset}{\overset{\hat{}}{\rightleftarrows}} \mathrm{dTop} \underset{D'}{\overset{U}{\rightleftarrows}} \mathsf{Top} \tag{93}$$

we prove that the controlled realisation is consistent with the previous ones

$$(\mathrm{c}\mathcal{R}(K))\hat{} = \uparrow\mathcal{R}(K), \qquad U((\mathrm{c}\mathcal{R}(K)))\hat{}) = \mathcal{R}K. \tag{94}$$

Also here, various models of cTop are controlled realisations of simple cubical sets: the controlled interval $\mathrm{c}\mathbb{I}$ realises $\uparrow\mathbf{i}$; the controlled line $\mathrm{c}\mathbb{R}$

realises $\uparrow\mathbf{z}$; the controlled sphere $\mathrm{c}\mathbb{S}^n$ realises $\uparrow\mathbf{s}^n$; the controlled ordered circle $\mathrm{c}\mathbb{O}^1$ (in (49)) realises $\uparrow\mathbf{o}^1$.

The controlled realisation functor $\mathrm{c}\mathcal{R}\colon \mathsf{Cub} \to \mathrm{c}\mathsf{Top}$ is also a strong dI1-functor, with the same proof as in [G2], 1.6.7.

6.7 Labelled cubical sets

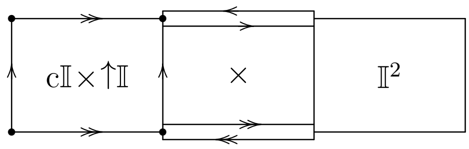
More generally, one can define the geometric realisation $\mathcal{R}K$ of a *c-labelled cubical set*: a cubical set $K = ((K_n), (\partial_i^\alpha), (e_i))$ where each n-cube $x \in K_n$ is labelled with a c-structure $K^n(x)$ on the euclidean n-cube $\mathbb{I}^n$, under a *coherence condition*: all the faces and degeneracies of the euclidean cubes give c-maps

$$\partial_i^\alpha\colon K^{n-1}(\partial_i^\alpha x) \to K^n(x), \qquad e_i\colon K^{n+1}(e_i x) \to K^n(x). \tag{95}$$

The first condition, for instance, means that each face $K^{n-1}(\partial_i^\alpha x)$ has a c-structure (weakly) finer than that induced by $K^n(x)$.

The *geometric realisation* of K is defined as the colimit in $\mathrm{c}\mathsf{Top}$ of the diagram formed by all the c-spaces $K^n(x)$, with the c-maps specified above.

The following 2-dimensional example is about a simpler case, a c-labelled *face-cubical set* (without degeneracies)



(96)

The labels $\mathrm{c}\mathbb{I}$, $\uparrow\mathbb{I}$, $\mathbb{I}$ of the edges are replaced by the symbols we have been using: $\twoheadrightarrow, \to$, or unmarked. The cross in the central rectangle means that there are no 2-cubes inside.

Finally, $\mathcal{R}K$ is a quotient of a sum of c-spaces

$$(\mathrm{c}\mathbb{I} \times \uparrow\mathbb{I}) + \mathbb{I}^2 + \uparrow\mathbb{I} + \uparrow\mathbb{I}^{\mathrm{op}} + \mathrm{c}\mathbb{I} + \mathrm{c}\mathbb{I}^{\mathrm{op}}, \tag{97}$$

modulo the equivalence relation that identifies vertices as shown in the picture.

$\uparrow\mathbb{I}^{\mathrm{op}}$ and $\mathrm{c}\mathbb{I}^{\mathrm{op}}$ can be replaced by their opposites, $\uparrow\mathbb{I}$ and $\mathrm{c}\mathbb{I}$, which are isomorphic to the former; yet, formula (97) makes identifications easier.

6.8 Directed homology

Cubical sets have a directed homology (introduced in [G1]), taking values in preordered abelian groups. For instance, $\uparrow H_n(\uparrow \mathrm{s}^n) = \uparrow \mathbb{Z}$, the ordered abelian group of the integers.

We refer to [G2], Chapter 2, for this theory and its strong relationship with noncommutative geometry.

Obviously, one can define the directed homology of a c-space X, letting

$$\uparrow H_n(X) = \uparrow H_n(\mathrm{c}\Box(X)), \tag{98}$$

but the interest of this issue is not clear.

Already in [G2], the directed homology of cubical sets is far more interesting than the derived directed homology of d-spaces. This comes out of the fact that the directed character of d-spaces (and c-spaces) does not go beyond the one-dimensional level: after selecting some paths and forbidding others, no higher choice is needed.

On the other hand, a cubical set K has privileged selections in any dimension: an element of K_n need not have any counterpart with faces reversed in some direction (for $n \geqslant 1$), nor permuted (for $n \geqslant 2$). This richer choice is paid with many drawbacks, starting with the lack of path-concatenation, which is not needed for homology but obviously needed for homotopy. However, the weak homotopy structure Cub_L can be enriched to a *relative dI-homotopical structure*, by means of the functor $\uparrow\mathcal{R}\colon \mathsf{Cub} \to \mathrm{d}\mathsf{Top}$ ([G2], 5.8.6) which takes values in a good dIP4-homotopical structure.

References

[BH] R. Brown and P.J. Higgins, Tensor products and homotopies for w-groupoids and crossed complexes, J. Pure Appl. Algebra 47 (1987), 1–33.

[FGHMR] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram and M. Raussen, Directed algebraic topology and concurrency, Springer, 2016.

[G1] M. Grandis, Directed combinatorial homology and noncommutative tori (The breaking of symmetries in algebraic topology), Math. Proc. Cambridge Philos. Soc. 138 (2005), 233–262.

[G2] M. Grandis, Directed Algebraic Topology, Models of non-reversible worlds, Cambridge University Press, 2009. Downloadable at: https://www.researchgate.net/publication/267089582

[G3] M. Grandis, The topology of critical processes, I (Processes and models), Cah. Topol. Géom. Différ. Catég. 65 (2024), 3–34.

[G4] M. Grandis, The topology of critical processes, II (The fundamental category), Cah. Topol. Géom. Différ. Catég. 65 (2024), 438–483.

[G5] M. Grandis, The topology of critical processes, III (Computing homotopy), Cah. Topol. Géom. Différ. Catég., to appear.

[GM] M. Grandis and L. Mauri, Cubical sets and their site, Theory Appl. Categ. 11 (2003), No. 8, 185–211.

[Ka] D.M. Kan, Abstract homotopy I, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 1092–1096.

[M] S. Mac Lane, Categories for the working mathematician, Springer, 1971.

[Ma] M. Mather, Pull-backs in homotopy theory, Can. J. Math. 28 (1976), 225–263.

[Mi] E.G. Minian, $\mathcal{C}at$ as a Λ-cofibration category, J. Pure Appl. Algebra 167 (2002), 301–314.

[Mu] J.R. Munkres, Elements of algebraic topology, Perseus Publ., 1984.

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