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ON THE BÉNABOU-ROUBAUD THEOREM

Bruno KAHN

Résumé. On donne une preuve détaillée du théorème de Bénabou-Roubaud. Cette preuve fournit un affaiblissement des hypothèses: l'existence de produits fibrés n'est pas nécessaire dans la catégorie de base, et la condition de "Beck-Chevalley", sous la forme d'une transformation naturelle, peut être affaiblie en demandant seulement que cette dernière soit épi.

Abstract. We give a detailed proof of the Bénabou-Roubaud theorem. As a byproduct, it yields a weakening of its hypotheses: the base category does not need fibre products and the Beck-Chevalley condition, in the form of a natural transformation, can be weakened by only requiring the latter to be epi.

Keywords. Descent, monad, Beck-Chevalley condition

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To the memory of Jacques Roubaud.

Introduction

The Bénabou-Roubaud theorem [2] establishes, under certain conditions, an equivalence of categories between a category of descent data and a category of algebras over a monad. This result is widely cited, but [2] is a note "without proofs" and the ones I know in the literature are a bit terse ([7, pp. 50/51], [8, proof of Lemma 4.1], [11, Th. 8.5]), [9, 3.7]; moreover, [8] and [11] are formulated in more general contexts.

The aim of this note is to provide a detailed proof of this theorem in its original context. This exegesis has the advantage of showing that the original hypotheses can be weakened: it is not necessary to suppose that the base category admits fibred products[1], and the Chevalley property of [2], formulated as an exchange condition, can also be weakened by requiring that the base change morphisms be only epi. I hope this will be useful to some readers. I also provided a proof of the equivalence between Chevalley's property and the exchange condition (attributed to Beck, but see remark 1.1): this result is part of the folklore but, here again, I had difficulty finding a published proof. In Corollary 5.2, I give a condition (probably too strong) for the Eilenberg-Moore comparison functor to be essentially surjective. Finally, I give cases in Proposition 6.1 where the exchange isomorphism holds; this is certainly classical, but it recovers conceptually Mackey's formula for the induced representations of a group (Example 6.3).

Notation and conventions

I keep that of [2]: thus $P : \mathbf{M} \to \mathbf{A}$ is a bifibrant functor in the sense of [5, §10]. If $A \in \mathbf{A}$, we denote by $\mathbf{M}(A)$ the fibre of P above A. For an arrow $a : A_1 \to A_0$ of $\mathbf{A}$, we write $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$ and $a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)$ for the associated inverse and direct image functors (a_* is *left* adjoint to a^*) and η^a, ε^a for the associated unit and counit. We also write $T^a = a^*a_*$ for the associated monad, equipped with its unit η^a and its multiplication $\mu^a = a^*\varepsilon^a a_*$. *We do not assume the existence of fibre products in* $\mathbf{A}$.

In order to simplify calculations, we shall assume that the pseudofunctor $a \mapsto a^*$ is a functor. This can be justified by the fact that it can be rectified; more precisely, the morphism of pseudofunctors $i \mapsto F_i$ of [10, §3, p. 141] is clearly faithful, hence any parallel arrows in its source which become equal in its target are already equal. (One could also use [3, I, Th. 2.4.2 or 2.4.4].) Then one can also choose the left adjoints $a \mapsto a_*$ to form a functor [12, IV.8, Th. 1], which we do.

[1] As was pointed out by the referee, the corresponding arguments are related to Street's notion of descent object relative to a truncated (co)simplicial category as in the beginning of [16]; but a "truncated cyclic category" à la Connes is also lurking in Proposition 4.6 b).

1. Adjoint chases

To elucidate certain statements and proofs, I start by doing two things: 1) "deploy" the single object M_1 of [2] into several, which will allow us to remove the quotation marks from "natural" at the bottom of [2, p. 96], 2) not assume the Beck-Chevalley condition to begin with, which will allow us to clarify the functoriality in the first lemma of the note and to weaken hypotheses.

1.1

Let a be as above; still following the notation of [2], we give ourselves a commutative square

$$\begin{array}{ccc} A_2 & \xrightarrow{a_2} & A_1 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a} \\ A_1 & \xrightarrow{a} & A_0. \end{array} \tag{1}$$

except that we don't require it to be Cartesian. The equality $a_1^* a^* = a_2^* a^*$ yields a base change morphism

$$\chi : (a_2)_* a_1^* \Rightarrow T^a \tag{2}$$

equal to the composition $\varepsilon^{a_2} T^a \circ (a_2)_* a_1^* \eta^a$. Hence a map

$$\xi_{M,N} = \xi : \mathbf{M}(A_1)(T^a M, N) \xrightarrow{\chi_M^*} \mathbf{M}(A_1)((a_2)_* a_1^* M, N) \xrightarrow[\sim]{\text{adj}} \mathbf{M}(A_2)(a_1^* M, a_2^* N) \tag{3}$$

for $M, N \in \mathbf{M}(A_1)$. It goes in the *opposite* direction to the map K^a of [2], which we will find back in (15). (See also Remark 4.4 in that section.)

Remark 1.1. The morphism (2) is sometimes called "Beck transformation". However, it already appears in SGA4 (1963/64) to formulate the proper base change and smooth base change theorems [1, §4]. I have adopted the terminology "base change morphism" in reference to this seminar.

Lemma 1.2 (key lemma)**.** *For any* $\varphi \in \mathbf{M}(A_1)(T^a M, N)$*, one has*

$$\xi(\varphi) = a_2^* \varphi \circ a_1^* \eta_M^a.$$

Proof. For $\psi \in \mathbf{M}(A_1)((a_2)_* a_1^* M, N)$ one has $\operatorname{adj}(\psi) = a_2^* \psi \circ \eta^{a_2}_{a_1^* M}$, hence

$$\begin{aligned}\xi(\varphi) = \operatorname{adj}(\varphi \circ \chi_M) &= a_2^*(\varphi \circ \chi_M) \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^*(\varphi \circ (\varepsilon^{a_2} T^a \circ (a_2)_* a_1^* \eta^a)_M) \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^* \varphi \circ a_2^* \varepsilon^{a_2}_{T^a M} \circ a_2^*(a_2)_* a_1^* \eta^a_M \circ \eta^{a_2}_{a_1^* M}\\ &= a_2^* \varphi \circ a_2^* \varepsilon^{a_2}_{T^a M} \circ \eta^{a_2}_{a_1^* T^a M} \circ a_1^* \eta^a_M\\ &= a_2^* \varphi \circ a_1^* \eta^a_M\end{aligned}$$

where we successively used the naturality of η^{a_2} and an adjunction identity.
□

1.2

Let $A_3 \in \mathbf{A}$ be equipped with "projections" $p_1, p_2, p_3 : A_3 \to A_2$. We assume that the "face identities" $a_1 p_2 = a_1 p_3$, $a_1 p_1 = a_2 p_3$, $a_2 p_1 = a_2 p_2$ are satisfied; we call these morphisms respectively b_1, b_2, b_3.

Canonical example 1.3. $A_2 = A_1 \times_{A_0} A_1$, $A_3 = A_1 \times_{A_0} A_1 \times_{A_0} A_1$, all morphisms given by the natural projections.

We then have maps, for $i < j$

$$\alpha_{ij}(M, N) = \alpha_{ij} : \mathbf{M}(A_2)(a_1^* M, a_2^* N) \to \mathbf{M}(A_3)(b_i^* M, b_j^* N) \tag{4}$$

given by

$$\alpha_{12} = p_3^*, \quad \alpha_{13} = p_2^*, \quad \alpha_{23} = p_1^*$$

hence composite maps

$$\theta_{ij} = \alpha_{ij} \circ \xi : \mathbf{M}(A_1)(T^a M, N) \to \mathbf{M}(A)(b_i^* M, b_j^* N). \tag{5}$$

In addition, we have the multiplication of T^a mentioned in the notations:

$$\mu^a = a^* \varepsilon^a a_* : T^a T^a \Rightarrow T^a. \tag{6}$$

The commutative square[2]

$$\begin{array}{ccc} A_3 & \xrightarrow{p_3} & A_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ A_2 & \xrightarrow{a_1} & A_1 \end{array} \tag{7}$$

[2]Note that it is Cartesian in the canonical example.

yields another base change morphism $\lambda : (p_1)_* p_3^* \Rightarrow a_1^*(a_2)_*$, hence a composition

$$(b_3)_* b_1^* = (a_2)_*(p_1)_* p_3^* a_1^* \overset{(a_2)_*\lambda a_1^*}{\Longrightarrow} (a_2)_* a_1^* (a_2)_* a_1^* \overset{\chi * \chi}{\Longrightarrow} T^a T^a \tag{8}$$

which, together with adjunction, induces a map

$$\rho : \mathbf{M}(A_1)(T^a T^a M, N) \to \mathbf{M}(A_2)(b_1^* M, b_3^* N). \tag{9}$$

Lemma 1.4. *a) The diagram of natural transformations*

$$\begin{array}{ccccc}
(a_2)_*(p_1)_* p_3^* a_1^* & = & (b_3)_* b_1^* & = & (a_2)_*(p_2)_* p_2^* a_1^* \\
{\scriptstyle (a_2)_*\lambda a_1^*}\Downarrow & & & & \Downarrow{\scriptstyle (a_2)_*\varepsilon^{p_2} a_1^*} \\
(a_2)_* a_1^* (a_2)_* a_1^* & & & & (a_2)_* a_1^* \\
{\scriptstyle \chi * \chi}\Downarrow & & & & \Downarrow{\scriptstyle \chi} \\
T^a T^a & & \overset{\mu^a}{\Longrightarrow} & & T^a
\end{array}$$

is commutative.
b) One has $\theta_{13} = \rho \circ \mu_a^*$ (see (5), (6) and (9)).

Proof. a) is a matter of developing the base change morphisms as done for χ just below (2) (see proof of Lemma 1.2). This yields a commutative diagram

$$\begin{array}{ccc}
\mathbf{M}(A_1)((b_3)_* b_1^* M, N) & \xleftarrow{((a_2)_*\varepsilon^{p_2} a_1^*)^*} & \mathbf{M}(A_1)((a_2)_* a_1^* M, N) \\
{\scriptstyle (8)^*}\uparrow & & \uparrow{\scriptstyle \chi^*} \\
\mathbf{M}(A_1)(T^a T^a M, N) & \xleftarrow{(\mu^a)^*} & \mathbf{M}(A_1)(T^a M, N)
\end{array}$$

from which we get b) by developing the adjunction isomorphism for $((b_3)_*, b_3^*)$. □

Let now $M_1, M_2, M_3 \in \mathbf{M}(A_1)$ and $\varphi_{ij} \in \mathbf{M}(A_1)(T^a M_i, M_j)$ be three morphisms. We have a not necessarily commutative square:

$$\begin{array}{ccc}
T^a T^a M_1 & \xrightarrow{T^a \varphi_{12}} & T^a M_2 \\
{\scriptstyle (\mu_a)_{M_1}}\downarrow & & \downarrow{\scriptstyle \varphi_{23}} \\
T^a M_1 & \xrightarrow{\varphi_{13}} & M_3.
\end{array} \tag{10}$$

Write $\hat{\varphi}_{ij} = \theta_{ij}(\varphi_{ij}) : b_i^* M_i \to b_j^* M_j$.

Lemma 1.5. *Let ψ (resp. ψ') be the composition of* (10) *passing through $T^a M_2$ (resp. through $T^a M_1$). Then $\rho(\psi) = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$ and $\rho(\psi') = \hat{\varphi}_{13}$.*

Proof. The first point follows from a standard adjunction calculation similar to the previous ones, and the second follows from lemma 1.4. □

Proposition 1.6. *If* (10) *commutes, we have $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$; the converse is true if ρ is injective in* (9).

Proof. This is obvious in view of Lemma 1.5. □

In (3), assume that $M = N$ is of the form $a^* M_0$ and write $p = aa_1 = aa_2 : A_2 \to A_0$. We have a composition

$$\mathbf{M}(A_1)(M, a^* M_0) \xrightarrow{\sim} \mathbf{M}(A_0)(a_* M, M_0) \xrightarrow{a^*} \mathbf{M}(A_1)(T^a M, a^* M_0) \xrightarrow{\xi} \mathbf{M}(A_2)(a_1^* M, p^* M_0) \quad (11)$$

where the first arrow is the adjunction isomorphism. A new adjoint chase gives:

Lemma 1.7. *The composition* (11) *is induced by a_1^*.* □

2. Exchange condition and weak exchange condition

Now we introduce the

Definition 2.1. A commutative square (1) is said to satisfy the *exchange condition* if the base change morphism (2) is an isomorphism; we say that (1) satisfies the *weak exchange condition* if (2) is epi.

Lemma 2.2 (cf. [13, Prop. 11] and [14, II.3])**.** *The exchange condition of Definition 2.1 is equivalent to the Chevalley condition (C) of [2].*

Proof. Recall this condition: given a commutative square

$$\begin{array}{ccc} M_1' & \xrightarrow{k_1} & M_1 \\ {\scriptstyle \chi'}\downarrow & & \downarrow{\scriptstyle \chi} \\ M_0' & \xrightarrow{k_0} & M_0, \end{array} \quad (12)$$

above (1) (where we take $(i, j) = (1, 2)$ to fix ideas), if χ and χ' are Cartesian and k_0 is co-Cartesian, then k_1 is co-Cartesian.

I will show that the exchange condition is equivalent to each of the following two conditions: (C) and

(C') if k_0 and k_1 are co-Cartesian and χ' is Cartesian, then χ is Cartesian.

Let us translate the commutativity of (12) in terms of the square

$$\begin{array}{ccc} \mathbf{M}(A_2) & \xrightarrow{(a_2)_*} & \mathbf{M}(A_1) \\ {\scriptstyle a_1^*}\uparrow & & {\scriptstyle a^*}\uparrow \\ \mathbf{M}(A_1) & \xrightarrow{a_*} & \mathbf{M}(A_0). \end{array} \tag{13}$$

The morphisms of (12) correspond to morphisms $\tilde{k}_0 : a_* M'_0 \to M_0$, $\tilde{k}_1 : (a_1)_* M'_1 \to M_1$, $\tilde{\chi} : M_1 \to a^* M_0$ and $\tilde{\chi}' : M'_1 \to a_2^* M'_0$, which fit in a commutative diagram of $\mathbf{M}(A_1)$:

$$\begin{array}{ccccc} (a_2)_* a_1^* M'_0 & & \xrightarrow{\quad c \quad} & & T^a M'_0 \\ {\scriptstyle (a_2)_*\tilde{\chi}'}\uparrow & & & & \downarrow{\scriptstyle a^*\tilde{k}_0} \\ (a_2)_* M'_1 & \xrightarrow{\tilde{k}_1} & M_1 & \xrightarrow{\tilde{\chi}} & a^* M_0 \end{array}$$

where c is the base change morphism of (2). The cartesianity conditions on χ and χ' (resp. co-cartesianity conditions on k_0 and k_1) amount to requesting the corresponding morphisms decorated with a ˜ to be isomorphisms.

Suppose c is an isomorphism. If $\tilde{\chi}'$ and $\tilde{k}_0$ are isomorphisms, $\tilde{\chi}$ is an isomorphism if and only if $\tilde{k}_1$ is. Thus, the exchange condition implies conditions (C) and (C'). Conversely, M'_0 being given, let $\tilde{k}_0$, $\tilde{\chi}$ and $\tilde{\chi}'$ be identities, which successively defines M_0, M_1 and M'_1. The arrow c then defines an arrow $\tilde{k}_1$, which is an isomorphism if and only if so is c. This shows that the exchange condition is implied by (C), and we argue symmetrically for (C') by taking $\tilde{\chi}'$, $\tilde{k}_1$ and $\tilde{k}_0$ to be identities. □

Remarks 2.3. a) This proof did not use the hypothesis that (1) be Cartesian. b) Under conservativity assumptions for a_2^* or a^*, we obtain converses to (C) and (C').

3. Pre-descent data

Here we come back to the set-up of Section 1: namely, we give ourselves a commutative diagram (1) as in §1.1 and a system (A_3, p_1, p_2, p_3) as in the beginning of §1.2 satisfying the identities of *loc. cit.* In other words, we have a set of objects and morphisms of $\mathbf{A}$

$$(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$$

subject to the relations

$$aa_1 = aa_2, \quad a_1p_2 = a_1p_3, \quad a_1p_1 = a_2p_3, \quad a_2p_1 = a_2p_2.$$

Let $M \in \mathbf{M}(A_1)$ and $v \in \mathbf{M}(A_2)(a_1^*M, a_2^*M)$. We associate to v three morphisms

$$\hat{\varphi}_{ij} = \alpha_{ij}(v) : b_i^*M \to b_j^*M \quad (i < j)$$

where α_{ij} are the maps of (4).

Definition 3.1. We say that v is a *pre-descent datum* on M if the $\hat{\varphi}_{ij}$ satisfy the condition $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$ of Proposition 1.6. We write $\mathbf{D}^{\mathrm{pre}}$ for the category whose objects are pairs (M, v), where v is a pre-descent datum on M, and whose morphisms are those of $\mathbf{M}(A_1)$ which commute with pre-descent data.

Let us introduce the

Hypothesis 3.2. The weak exchange condition is verified by the squares (1) and (7).

Proposition 3.3 (cf. [2, lemme]). *In* (10)*, assume $\varphi_{12} = \varphi_{23} = \varphi_{13} =: \varphi$. If φ satisfies the associativity condition of a T^a-algebra, then $\xi(\varphi)$ in* (3) *is a pre-descent datum; the converse is true under Hypothesis 3.2.*

Proof. In view of Proposition 1.6, it suffices to show that Hypothesis 3.2 implies the injectivity of ρ, which is induced by the composition of the two natural transformations of (8). The second is epi, therefore induces an injection on Hom's, and so does the first by adjunction. □

Corollary 3.4. *Let $\mathbf{M}^a_{\mathrm{ass}}$ denote the category of associative T^a-algebras which are not necessarily unital. Then Proposition 3.3 defines a faithful functor $\xi : \mathbf{M}^a_{\mathrm{ass}} \to \mathbf{D}^{\mathrm{pre}}$ commuting with the forgetful functors to $\mathbf{M}(A_1)$; under Hypothesis 3.2, it is an isomorphism of categories.*

Proof. Commutation of ξ with the forgetful functors is obvious. This already shows that it is faithful; under Hypothesis 3.2, it is essentially surjective by Proposition 3.3 and we see immediately that it is also full. □

4. The unit condition

We keep the hypotheses and notation of Section 3, and introduce an additional ingredient: a "diagonal" morphism $\Delta : A_1 \to A_2$ such that $a_1\Delta = a_2\Delta = 1_{A_1}$.

Definition 4.1. A *descent datum* on M is a pre-descent datum v such that $\Delta^* v = 1_M$. We denote by $\mathbf{D}$ the full subcategory of $\mathbf{D}^{\mathrm{pre}}$ given by the descent data.

Let $\mathbf{M}^a \subset \mathbf{M}^a_{\mathrm{ass}}$ be the category of T^a-algebras.

Theorem 4.2 (cf. [2, théorème]). *For all $\varphi \in \mathbf{M}(A_1)(T^a M, M)$, we have*

$$\Delta^*\xi(\varphi) = \varphi \circ \eta^a_M. \tag{14}$$

In particular, $\xi(\mathbf{M}^a) \subset \mathbf{D}$ and $\xi : \mathbf{M}^a \to \mathbf{D}$ is an isomorphism of categories under Hypothesis 3.2.

Proof. Suppose that $M = N$ in Lemma 1.2. Applying Δ^* to its identity, we get (14). In particular, if φ is the action of a T^a-algebra then $v = \xi(\varphi)$ verifies $\Delta^* v = 1_M$. We conclude with Corollary 3.4. □

As in [12, VI.3, Th. 1], we have the Eilenberg-Moore comparison functor

$$\begin{aligned} K^a : \mathbf{M}(A_0) &\to \mathbf{M}^a \\ M_0 &\mapsto (a^* M_0, a^*\varepsilon^a_{M_0}). \end{aligned} \tag{15}$$

Lemma 1.7 yields:

Proposition 4.3. *We have $\xi(a^*\varepsilon^a_{M_0}) = 1_{M_0}$. In other words, in the diagram*

$$\begin{array}{ccccc} \mathbf{M}(A_0) & \xrightarrow{\Psi^a} & \mathbf{D} & \xrightarrow{U^a} & \mathbf{M}(A_1) \\ & {\scriptstyle K^a}\searrow & \uparrow{\scriptstyle \xi} & \nearrow{\scriptstyle U^{T^a}} & \\ & & \mathbf{M}^a & & \end{array}$$

the left triangle commutes (as well as the right one, trivially). □

Remark 4.4. In [9, 3.7], Janelidze and Tholen construct a functor from $\mathbf{D}$ to $\mathbf{M}$ (same direction as in [2]) by using the *inverses* of the base change morphisms (2).

Remark 4.5. In the canonical example 1.3, a pre-descent datum v satisfies the condition of Definition 4.1 if and only if it is invertible (therefore is a descent datum in the classical sense): this follows from [4, A.1.d pp. 303–304]. In *loc. cit.*, Grothendieck uses an elegant Yoneda argument. It is an issue to see how this result extends to our more general situation: this is done in the next proposition. I am indebted to the referee for prodding me to investigate this.

Note that I merely looked for what is necessary to translate Grothendieck's arguments, and not for the greatest generality.

Proposition 4.6. *Let $(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$ be as in Section 3. Let $M \in \mathbf{M}(A_1)$ and let $v \in \mathbf{M}(A_2)(a_1^*M, a_2^*M)$ be a pre-descent datum as in Definition 3.1. Further, let Δ be as in the beginning of the present section. Consider the following conditions:*

(i) $\Delta^*v = 1_M$ *(i.e. v is a descent datum).*

(ii) *v is invertible.*

Then:
a) $(ii) \Rightarrow (i)$ under one of the following conditions: there exists a morphism s_1 (resp. s_2) from A_2 to A_3 such that

$$p_1 s_1 = \Delta a_2, \quad p_2 s_1 = p_3 s_1 = 1$$

(resp.

$$p_1 s_2 = p_2 s_2 = 1, \quad p_3 s_2 = \Delta a_2).$$

b) $(i) \Rightarrow (ii)$ *under the following condition: there exists an involution* σ *of* A_2 *and a morphism* $\Gamma : A_2 \to A_3$ *such that*

$$p_1\Gamma = \sigma, \quad p_2\Gamma = \Delta a_1, \quad p_3\Gamma = 1_{A_2}.$$

(In the case of the canonical example 1.3, we may take for s_1 and s_2 the partial diagonals, for σ the exchange of factors and for Γ the graph of a_1, given in formula by $(\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2, \alpha_1)$.)

Proof. The predescent condition on v is

$$p_2^*v = p_1^*v \circ p_3^*v. \tag{16}$$

a) Applying s_1^* to (16), we get

$$v = a_2^*\Delta^*v \circ v$$

hence $a_2^*\Delta^*v = 1_{A_2}$ and

$$\Delta^*v = \Delta^*a_2^*\Delta^*v = 1_{A_1}.$$

Same reasoning with s_2, *mutatis mutandis*. Note that with s_1 (*resp.* s_2), it suffices to assume that v is right (*resp.* left) cancellable.

b) Applying Γ^* to (16), we get

$$1_{A_2} = a_1^*\Delta^*v = \sigma^*v \circ v.$$

Applying now σ^*, we also get $v \circ \sigma^*v = 1_{A_2}$. □

5. A supplement

Recall [6, Ex. 8.7.8] that a category is called *Karoubian* if any idempotent endomorphism has an image.

Proposition 5.1. *Let* a^* *be fully faithful and* $\mathbf{M}(A_0)$ *Karoubian. Let* $\varphi : T^aM \to M$ *satisfy the identity* $\varphi \circ \eta_M^a = 1_M$*. Then there exists* $M_0 \in \mathbf{M}(A_0)$ *and an isomorphism* $\nu : M \xrightarrow{\sim} a^*M_0$ *such that* $\varphi = \nu^{-1} \circ a^*\varepsilon_{M_0}^a \circ T^a\nu$*.*

Proof. Let e denote the idempotent $\eta^a_M\varphi \in \mathrm{End}_{\mathbf{M}(A_1)}(T^aM)$. By hypothesis, $e = a^*\tilde{e}$ where $\tilde{e}$ is an idempotent of $\mathrm{End}_{\mathbf{M}(A_0)}(a_*M)$, with image M_0. Then a^*M_0 is isomorphic to the image M of e via a morphism ν as in the statement, such that

$$\nu \circ \varphi = a^*\pi, \quad a^*\iota \circ \nu = \eta^a_M$$

where $\iota\pi$ is the epi-mono factorization of $\tilde{e}$.

To finish, it is enough to see that $a^*\pi = a^*\varepsilon^a_{M_0} \circ T^a\nu$. But we also have

$$\eta^a_{a^*M_0} \circ \nu = T^a\nu \circ \eta^a_M = T^a\nu \circ a^*\iota \circ \nu$$

hence $\eta^a_{a^*M_0} = T^a\nu \circ a^*\iota$. This concludes the proof, since $\eta^a_{a^*M_0} \circ a^*\varepsilon^a_{M_0}$ is the epi-mono factorisation of the idempotent of $\mathrm{End}(T^aa^*M_0)$ with image a^*M_0. □

We thus obtain the following complement:

Corollary 5.2. *Assume Hypothesis 3.2, and also that a^* is fully faithful and $\mathbf{M}(A_0)$ Karoubian. Then*
a) every unital T^a-algebra is associative;
b) K^a is essentially surjective. □

Can one weaken the full faithfulness assumption in this corollary? The following lemma does not seem sufficient:

Lemma 5.3. *Let $M, N \in \mathbf{M}(A_1)$. Then the map*

$$a^* : \mathbf{M}(A_0)(a_*M, a_*N) \to \mathbf{M}(A_1)(T^aM, T^aN)$$

has a retraction r given by $r(f) = \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M$. More generally, we have an identity of the form $r(a^*g \circ f) = g \circ r(f)$.

Proof. For $f : T^aM \to T^aN$ and $g : a_*N \to a_*P$, we have

$$r(a^*g \circ f) = \varepsilon^a_{a_*P} \circ a_*a^*g \circ a_*f \circ a_*\eta^a_M = g \circ \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M = g \circ r(f).$$

Taking $f = 1_{T^aM}$, we obtain that r is a retraction. □

6. Appendix: a case where the exchange condition is verified

Let $\mathcal{A}$ be a category. Take for $\mathbf{A}$ the category of presheaves of sets on $\mathcal{A}$. Write $\int A$ for the category associated to $A \in \mathbf{A}$ by the Grothendieck construction [5, §8]. Recall its definition in this simple case: the objects of $\int A$ are pairs (X, a) where $X \in \mathcal{A}$ and $a \in A(X)$, and a morphism from (X, a) to (Y, b) is a morphism $f \in \mathcal{A}(X, Y)$ such that $A(f)(x) = y$.

Let $\mathcal{C}$ be another category. We take for $\mathbf{M}$ the fibred category of representations of $\mathbf{A}$ in $\mathcal{C}$: for $A \in \mathbf{A}$, an object of $\mathbf{M}(A)$ is a functor from $\int A$ to $\mathcal{C}$. For all $a \in \mathbf{A}(A_1, A_0)$ we have an obvious pull-back functor $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$, which has a left adjoint a_* (direct image) given by the usual colimit if $\mathcal{C}$ is cocomplete. We can then ask whether the exchange condition is true for Cartesian squares of $\mathcal{A}$.

Proposition 6.1. *This is the case if $\mathcal{C}$ is the category of sets $\mathbf{Set}$, and more generally if $\mathcal{C}$ admits a forgetful functor $\Omega : \mathcal{C} \to \mathbf{Set}$ with a left adjoint L such that (L, Ω) satisfies the conditions of Beck's theorem [12, VI.7, Th. 1].*

Proof. First suppose $\mathcal{C} = \mathbf{Set}$; to verify that (2) is a natural isomorphism, it is enough to test it on representable functors. Consider Diagram (13) again. For $(c, \gamma) \in \int A_1$ and $(d, \delta) \in \int A_1$ (with $c, d \in \mathcal{A}$ and $\gamma \in A_1(c)$, $\delta \in A_1(d)$), we have

$$\begin{aligned} T^a y(c,\gamma)(d,\delta) = a^* y(c, a(\gamma))(d,\delta) = y(c, a(\gamma))(d, a(\delta)) \\ = \{\varphi \in \mathcal{A}(d,c) \mid \varphi^* a(\gamma) = a(\delta)\} \end{aligned}$$

and

$$\begin{aligned} (a_2)_* a_1^* y(c,\gamma)(d,\delta) &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} a_1^* y(c,\gamma)(e,\eta) \\ &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} y(c,\gamma)(e, a_1(\eta)) \\ &= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2} \{\psi \in \mathcal{A}(e,c) \mid \psi^*\gamma = a_1(\eta)\}. \end{aligned}$$

We have

$$(d,\delta) \downarrow a_2 = \{(e, \eta, \eta_2, \theta) \in \mathcal{A} \times A_1(e) \times_{A_0(e)} A_1(e) \times \mathcal{A}(d,e) \mid \theta^* \eta_2 = \delta\}.$$

This category has the initial set $\{(d,\eta_1,\delta,1_d)\mid a(\eta_1)=a(\delta)\}$, so

$$(a_2)_*a_1^*y(c,\gamma)(d,\delta)=\coprod_{\{(\eta_1\in A_1(d)\mid a(\eta_1)=a(\delta)\}}\{\varphi\in\mathcal{A}(d,c)\mid \varphi^*\gamma=\eta_1\}$$
$$=\{\varphi\in\mathcal{A}(d,c)\mid a(\varphi^*\gamma)=a(\delta)\}$$

and the map $(a_2)_*a_1^*y(c,\gamma)(d,\delta)\to(a_2)_*(a^{12})^*y(c,\gamma)(d,\delta)$ is clearly equal to the identity.

General case: let us write more precisely $\mathbf{M}^{\mathcal{C}}(A)=\mathbb{CAT}(\int A,\mathcal{C})$. The functors L and Ω induce pairs of adjoint functors (same notation)

$$L:\mathbf{M}^{\mathbf{Set}}(A)\leftrightarrows\mathbf{M}^{\mathcal{C}}(A):\Omega.$$

These two functors commute with pull-backs; as L is a left adjoint, it also commutes with direct images. Therefore, in the above situation, the base change morphism $\chi_M:(a_2)_*a_1^*M\to T^aM$ is an isomorphism when $M\in\mathbf{M}^{\mathcal{C}}(A_1)$ is of the form LX for $X\in\mathbf{M}^{\mathbf{Set}}(A_1)$. For any M, we have its canonical presentation [12, (5) p. 153]

$$(L\Omega)^2M\rightrightarrows L\Omega M\to M \tag{17}$$

whose image by Ω is a split coequaliser (*loc. cit.*). Given the hypothesis that Ω creates such coequalisers, (17) is a coequaliser. Since pull-backs are cocontinuous, as well as direct images (again, as left adjoints), (17) remains a coequaliser after applying the functors $(a_2)_*(a^{12})^*$ and T^a. Finally, a coequaliser of isomorphisms is an isomorphism. □

Examples 6.2 (for $\mathcal{C}$). Varieties (category of groups, abelian groups, rings...): [12, VI.8, Th. 1].

Example 6.3 (for $\mathcal{A}$). The category with one object $\underline{G}$ associated with a group G: then $\mathbf{A}$ is the category of G-sets. Let us take for $\mathcal{C}$ the category of R-modules where R is a commutative ring. If $A\in\mathbf{A}$ is G-transitive, $\int A$ is a connected groupoid, which is equivalent to $\underline{H}$ for the stabilizer H of any element of A; thus, $\mathbf{M}(A)$ is equivalent to $\mathbf{Rep}_R(H)$. If $a:A_1\to A_0$ is the morphism of $\mathbf{A}$ defined by an inclusion $K\subset H\subset G$ ($A_1=G/K$, $A_0=G/H$), then a^* is restriction from H to K and a_* is induction $V\mapsto RH\otimes_{RK}V$. From Proposition 6.1, we thus recover conceptually the Mackey formula of [15, 7.3, Prop. 22], proven "by hand" in *loc. cit.*

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Bruno Kahn
CNRS, Sorbonne Université and Université Paris Cité, IMJ-PRG
Case 247
4 place Jussieu
75252 Paris Cedex 05
France
bruno.kahn@imj-prg.fr