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Almost Cofibrations

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Résumé. Dans cet article nous étudions une généralisation de la propriété d'extension de l'homotopie ainsi que la notion associée de *almost-cofibration* pour les espaces topologiques. Après avoir présenté quelques caractéristiques nouvelles et intéressantes de cette notion, nous montrons que tout plongement fermé d'espaces compacts métrisables est une *almost-cofibration*. De plus, il s'avère que la catégorie des espaces compacts métrisables, avec les *almost-cofibrations* et les équivalences de forme forte, possède la structure d'une catégorie de cofibrations dont la catégorie d'homotopie est sa catégorie de forme forte.
Abstract. We study a generalization of the homotopy extension property together with the related notion of almost-cofibration of topological spaces. After giving some new and interesting features of such a notion we show that every closed embedding of compact metrizable spaces is an almost-cofibration. Moreover, it turns out that the category of compact metrizable spaces, together with almost-cofibrations and strong shape equivalences has the structure of a cofibration category whose homotopy category is its strong shape category.

Keywords. HEP, RWHEP, almost-cofibration, strong shape equivalence, cofibration category, compact metrizable space.
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Introduction

We consider a variation of the usual homotopy extension property (HEP), called the rather weak homotopy extension property (RWHEP), that was in-

troduced in [1], see also [8]. The maps having the RWHEP are characterized by the fact that they are exactly those maps inducing levelwise fibrations in the 2-category $[\mathbf{Top}, \mathbf{Gpd}]$, where $\mathbf{Top}$ denotes the category of topological spaces, while $\mathbf{Gpd}$ is the category of groupoids and their homomorphisms (functors). In this paper we are interested in certain features of this property, for instance, in contrast to the HEP, the RWHEP is preserved when passing to the category of inverse systems and also passes to limits of inverse systems. A map having the RWHEP with respect to all spaces is called here an almost-cofibration. In particular, it is proved that every closed embedding of compact metrizable spaces is an almost-cofibration.
Almost-cofibrations and strong shape equivalences [11] give the category $\mathcal{C}$ of compact metrizable spaces the structure of a cofibration category [14].
In [2] it was proved that the strong shape category of $\mathcal{C}$ is obtained by localizing at the class Σ of strong shape equivalences, that is $Ssh(\mathcal{C}) = \mathcal{C}[\Sigma^{-1}]$. It then follows that $Ssh(\mathcal{C})$ actually is the homotopy category of a cofibration category.

1. Preliminaries

A category enriched over $\mathbf{Gpd}$ is just a 2-category whose 2-cells are all invertible. The category $\mathbf{Top}$ of topological spaces and continuous maps will be considered with its enrichment over $\mathbf{Gpd}$. Given two spaces X, Y, the groupoid $\mathbf{Gpd}(X, Y)$ has points the maps $X \to Y$ while a path $\alpha : f \to g$ is a track connecting the two maps, that is $\alpha = [H]$ is the relative homotopy class of a homotopy $H : X \times I \to Y$ connecting f to g. It is often called the *track groupoid of Y under X* [1]. $\mathbf{Anr}$ will denote the full subcategory of $\mathbf{Top}$ whose objects are the spaces having the homotopy type of compact absolute neighborhood retracts for metrizable spaces (Anr-spaces).

$\mathbf{Gpd}$ is enriched over itself, the homotopies being the natural isomorphisms of functors. A homomorphism of groupoids is a homotopy equivalence if and only if it is an equivalence of categories.
Every ordinary category can be considered as a category enriched over $\mathbf{Gpd}$ with only identity homotopies.

Both the categories $\mathbf{Top}$ and $\mathbf{Gpd}$ are closed model categories [12], [7] with the following structure:

(a) $\mathbf{Top}$: the weak equivalences are the homotopy equivalences, the fibrations and the cofibrations are the Hurewicz fibrations and the Hurewicz cofibrations.

(b) $\mathbf{Gpd}$: the weak equivalences are the homorphisms that are equivalences of categories, the cofibrations are the homomorphisms that are injective on objects. The fibrations are the homomorphisms $\varphi : G \to H$ having the following *source lifting property* as described by Brown in [1]:

(1.1.1) *for every $x \in G$ and every path $\beta : \varphi(x) \to \bullet$ in H, there exists a path $\alpha : x \to \bullet$ in G such that $\varphi(\alpha) = \beta$.*

For $\mathcal{A}$ a (small) category and $\mathcal{K}$ any 2-category, consider the functor 2-category $[\mathcal{A}, \mathcal{K}]$. If $F, G : \mathcal{A} \to \mathcal{K}$ are 2-functors a 2-natural transformation $\tau : F \Rightarrow G$ is a *level equivalence*, respectively a *level fibration, level cofibration*, if $\tau_A : F(A) \to G(A)$ is an equivalence, respectively a fibration, cofibration in $\mathcal{K}$, for all $A \in \mathcal{A}$ (whatever "equivalence, fibration, cofibration" could mean).
Following [5], the functor category $[\mathcal{A}, \mathbf{Gpd}]$ can be equipped with the so called projective model structure. There the weak equivalences are the level equivalences, the fibrations are the level fibrations and the cofibrations are those natural transformations having the left lifting property with respect to level trivial fibrations.

From now on we denote by $\mathcal{C}$ the category of compact metrizable spaces. Moreover, by $\mathcal{K}$ we mean both a class of topological spaces and the full subcategory of $\mathbf{Top}$ it generates.

Let us recall the following theorem ([10], I.5.2, Thm. 7 and Cor. 4) for later use.

Theorem 1.1. *Every space $X \in \mathcal{C}$ can be represented as the inverse limit of an inverse system $\mathbf{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$ in $\mathbf{Anr}$.*

We refer to [10] for all that concerns inverse systems and the construction of the category $\mathrm{Pro}(\mathbf{Top})$. Let us only recall that $\mathbf{X}$ is a contravariant functor

$\mathbf{X} : \Lambda \to \mathbf{Top}$, where $(\Lambda, \leq)$ is a cofinite, strongly directed set [4], $X_\lambda = \mathbf{X}(\lambda)$ and $x_{\lambda\lambda'} = \mathbf{X}(\lambda \leq \lambda')$.
A morphism $\mathbf{p} : X \to \mathbf{X}$ is a natural cone, that is a family $\mathbf{p} = \{p_\lambda : X \to X_\lambda \,|\, \lambda \in \Lambda\}$ of maps such that $x_{\lambda\lambda'} \circ p_{\lambda'} = p_\lambda$, for $\lambda \leq \lambda'$.

2. Almost-cofibrations

For each topological space X, the representable (covariant) 2-functor

$$\mathbf{Gpd}(X, -) : \mathbf{Top} \to \mathbf{Gpd}$$

sends a space K to the groupoid $\mathbf{Gpd}(X, K)$, a map $f : K \to H$ to the functor $f^*_K = \mathbf{Gpd}(X, f) : \mathbf{Gpd}(X, K) \to \mathbf{Gpd}(X, H)$, $a \mapsto f \circ a$, and a track $\alpha = [H] : f \Rightarrow g : K \to H$ to the natural isomorphism $\mathbf{Gpd}(X, \alpha) : \mathbf{Gpd}(X, f) \Rightarrow \mathbf{Gpd}(X, g)$ induced by α in the evident way.
Let $f : X \to Y$ be a map and let $\mathcal{K} \subset \mathbf{Top}$ be a class of spaces. The natural transformation

$$f^* = \mathbf{Gpd}(f, -) : \mathbf{Gpd}(Y, -) \Rightarrow \mathbf{Gpd}(X, -) : \mathbf{K} \to \mathbf{Gpd}$$

is a *level fibration*, resp. *level equivalence* , if $f^*_K : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$ is a fibration, respectively an equivalence of groupoids, for all $K \in \mathcal{K}$.
The fact that f^*_K is a fibration of groupoids amounts, by (1.1.1), to the following property

(2.1.1) *for every g and H such that $H \circ e_0(X) = g \circ f$, there is a $G : Y \times I \to K$ with $G \circ e_0(Y) = g$ and $G \circ (f \times id) \simeq H$. In diagram*

$$\begin{tikzcd}
X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "f \times id"] \arrow[ddr, bend left, "H"] \arrow[dr, phantom, "\simeq"] & \\
Y \arrow[r, "e_0(Y)"] \arrow[drr, "g"'] & Y \times I \arrow[dr, "G"] \arrow[d, phantom, "="] & \\
& & K
\end{tikzcd}$$

Such a consideration leads to the following generalization of the classical Homotopy Extension Property (HEP):

Definition 2.1. *A map $f : X \to Y$ has the almost homotopy extension property (RWHEP) with respect to a space K if, for every $g : Y \to K$ and $H : X \times I \to K$ such that $H \circ e_0(X) = g \circ f$, there is a $G : Y \times I \to K$ with $G \circ e_0(Y) = g$ and $G \circ (f \times 1) \simeq H$.*
If $\mathcal{K} \subset \mathbf{Top}$, $f \in RWHEP(\mathcal{K})$ means that f has the RWHEP with respect to all $K \in \mathcal{K}$. If $f \in RWHEP(\mathbf{Top})$, f will be called an almost-cofibration.

Then, it is clear that

Theorem 2.2. *Let $f : X \to Y$ be a map of spaces. The following are equivalent*

(a) $f \in RWHEP(\mathcal{K})$

(b) the homomorphism $f_K^* : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$ is a fibration of groupoids, for all $K \in \mathcal{K}$.

Remark 2.3. We point out that the RWHEP was introduced by R. Brown [1] and also that our almost-cofibrations are called rather weak cofibrations in [8].

Let us recall the following facts:

(a) A map $f : X \to Y$ has the *homotopy extension property* with respect to a class $\mathcal{K}$ of spaces, written $f \in \mathrm{HEP}(\mathcal{K})$, if : given a map $g : Y \to K$, $K \in \mathcal{K}$, and a homotopy $H : X \times I \to K$ starting at $g \circ f$, there is a homotopy $G : Y \times I \to K$ starting at g and such that $G \circ (f \times id) = H$. $f : X \to Y$ is a *(Hurewicz) cofibration* if it has the HEP with respect to all topological spaces. This amounts to the following diagram to be a weak pushout in $\mathbf{Top}$

$$\begin{tikzcd} X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "f \times id"] \\ Y \arrow[r, "e_0(Y)"] & Y \times I \end{tikzcd}$$

Here I denotes the unit interval $[0, 1]$ and $e_0(X) : X \to X \times I$ is the map $e_0(X)(x) = (x, 0)$.

(b) given a map $f : X \to Y$, its *mapping cylinder* $M(f)$ is obtained as the pushout

$$\begin{tikzcd} X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "\pi_f"] \\ Y \arrow[r, "j_f"] & M(f) \end{tikzcd}$$

$M(f)$ is then the quotient space of the disjoint union $X \times [0,1] \sqcup Y$ modulo the relation which identifies each point $(x,0)$ with $f(x)$.

Proposition 2.4. *Let $f : X \to Y$ be any map and $\mathcal{K}$ a class of spaces. If $f \in$ $\mathrm{HEP}(\mathcal{K})$, then the functor $f^*_K = \mathbf{Gpd}(f,K) : \mathbf{Gpd}(Y,K) \to \mathbf{Gpd}(X,K)$ is a fibration in $\mathbf{Gpd}$, for all $K \in \mathbf{K}$. In particular, every map having the* HEP*($\mathcal{K}$) has also the* RWHEP*($\mathcal{K}$).*

Proof. See ([1], 7.2.2) and Thm. 3.2. □

The converse implication does not hold in general. In fact: let $A = \{0\} \cup \bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$ and $Z = [0,1] \times \{0\} \cup A \times [0,1]$. Let $f : A \to [0,1]$ be the inclusion and let maps $G : A \times [0,1] \to Z$, $g : [0,1] \to Z$ be defined by the formulas $G(x,t) = (x,t)$ and $g(y) = (y,0)$ for $(x,t) \in A \times [0,1]$ and $y \in [0,1]$. Then $f \in \mathrm{HEP}(\mathbf{Anr})$ by the classical homotopy extension theorem, but it is not true that $f \in \mathrm{HEP}(\mathcal{C})$. Indeed, a homotopy $F : [0,1] \times [0,1] \to Z$ such that $F(x,t) = G(x,t)$ and $F(y,0) = g(y)$ for $(x,t) \in A \times [0,1]$ and $y \in [0,1]$ would be a retraction of the locally connected continuum $[0,1]$ to a non-locally connected continuum Z.

Proposition 2.5. *Let $f : X \to Y$ be a map of compact metrizable spaces, then:*

(1) the mapping cylinder $M(f)$ is also a compact metrizable space.

(2) f is a cofibration if and only if $f \in \mathrm{HEP}(\mathcal{C})$.

Proof. (1) $M(f)$ is compact since the category of compact spaces is closed under finite coproducts and quotients. Since every continuous image in a Hausdorff space of a compact metrizable space is metrizable, it suffices to prove that $M(f)$ is Hausdorff. Let $q : X \times [0,1] \sqcup Y \to M(f)$ be the

quotient map and $u, v \in M(f)$. If the two points are both in Y there are disjoint open sets $U \ni u$ and $V \ni v$ in Y. Then $\tilde{U} = q(f^{-1}(U) \times [0,1] \sqcup U)$ and $\tilde{V} = q(f^{-1}(V) \times [0,1] \sqcup V)$ are disjoint open sets in $M(f)$ contaning u and v, respectively. Assume now that $u \in X \times [0,1)$ and $v \in Y$: they have disjoint open neighborhoods given by $X \times [0,t),\ t < 1$, and $X \times [0,s) \sqcup Y$ for some $0 < s < t$. The case $u, v \in X \times [0,1)$ is obviuos.
(2) Recall that a map is a cofibration if and only if it has the HEP with respect to its mapping cylinder ([9], 2.10). ☐

Proposition 2.6. *Let $f : X \to Y$ be such that $M(f) \in \mathcal{K}$. Then $f \in RWHEP(\mathcal{K})$ if and only if $f \in RWHEP(M(f))$.*

Proof. Let us consider the following diagram

$$
\begin{tikzcd}
 & X \arrow[rr, "e_0(X)"] \arrow[dl, "f"'] & & X \times I \arrow[dl, "f \times id"'] \arrow[dd, "\pi_f"] \arrow[dddr, bend left, "H"] & \\
Y \arrow[rr, "e_0(Y)"] \arrow[drrr, "j_f"'] \arrow[ddrrrr, "g"'] & & Y \times I \arrow[dr, "\phi"'] & & \\
 & & & M(f) \arrow[dr, "\psi"'] & \\
 & & & & K
\end{tikzcd}
$$

From the fact that $M(f) \in \mathcal{K}$, there is a map ϕ such that:
- $\phi \circ e_0(Y) = j_f$,
- $\phi \circ (f \times id) \simeq \pi_f$.

Since the middle square is a pushout, there is a map ψ such that:
- $\psi \circ j_f = g$,
- $\psi \circ \pi_f = H$.

Finally: $(\psi \circ \phi) \circ e_0(Y) = g$ and $(\psi \circ \phi) \circ (f \times id) \simeq H$. ☐

The next two propositions mark the difference between the HEP and the RWHEP.

Proposition 2.7. *Let $f : X \to Y$. The following are equivalent* *(a)* $f \in RWHEP(\mathcal{K})$

(b) $f \in RWHEP(\mathrm{Pro}(\mathcal{K}))$.

Proof. $(a) \Rightarrow (b)$: let $\mathbf{K} = \{K_j\}_J \in \mathrm{Pro}(\mathcal{K})$ and let $\mathbf{g} : Y \to \mathbf{K}$, $\mathbf{H} : X \times I \to \mathbf{K}$ be such that $\mathbf{H} \circ e_0^Y = \mathbf{g} \circ f$. Since $f \in RWHEP(\mathcal{K})$, there is a diagram

$$\begin{tikzcd}
X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "f\times id"'] \arrow[dd, bend left, "H_j"] \\
Y \arrow[r, "e_0(Y)"] \arrow[dr, "g_j"'] & Y \times I \arrow[d, "G_j"] \arrow[r, phantom, "\simeq"] & {} \\
{} \arrow[r, phantom, "="] & K_j
\end{tikzcd}$$

for every $j \in J$. The family of maps $\mathbf{G} = \{G_j : Y \times I \to K_j : j \in J\}$ is a pseudo cone, that is $k_{jj'} \circ G_{j'} \simeq G_j$, for all $j \leq j'$, being $\mathbf{G} \circ e_0(Y) = \mathbf{g}$ and $e_0(Y)$ a homotopy equivalence. Moreover $\mathbf{G} \circ (f \times id) \simeq \mathbf{H}$.
Finally, since $\mathbf{G} \circ e_0(Y) = \mathbf{g}$ is a cone, let us consider the cone

$$\Lambda : Y \times I \xrightarrow{\sigma(Y)} Y \xrightarrow{e_0(Y)} Y \times I \xrightarrow{\mathbf{G}} \mathbf{K}$$

which has the properties:

- $\Lambda \circ e_0(Y) = \mathbf{g}$,
- $\Lambda \circ (f \times id) \simeq \mathbf{H}$.

$$\begin{tikzcd}
X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "f\times id"'] \arrow[dd, bend left, "\mathbf{H}"] \\
Y \arrow[r, "e_0(Y)"'] \arrow[dr, "\mathbf{g}"'] & Y \times I \arrow[d, "\Lambda"] \arrow[r, phantom, "\simeq"] & {} \\
{} \arrow[r, phantom, "="] & \mathbf{K}
\end{tikzcd}$$

The other direction is clear. $\square$

Proposition 2.8. *Let $f \in RWHEP(\mathcal{K})$ and let $\mathbf{K} = \{K_j\}_{j\in J}$ be an inverse system in $\mathcal{K}$ with inverse limit $\mathbf{p} : K \to \mathbf{K}$. Then $f \in RWHEP(K)$.*

Proof. Let $g : Y \to K,\ H : X \times I \to K$ be such that $H \circ e_0(X) = g \circ f$ and consider the diagram

$$\begin{tikzcd}
& X \arrow[rr, "e_0(X)"] \arrow[dl, "f"'] & & X \times I \arrow[dl, "f\times id"'] \arrow[dd, "H"] & \\
Y \arrow[rr, "e_0(Y)"] \arrow[drrr, "g"'] & & Y \times I \arrow[dr, "\gamma"] \arrow[drr, "\Phi"] & & \\
& & & K \arrow[r, "\mathbf{p}"] & \mathbf{K}
\end{tikzcd}$$

Since $f \in RWHEP(\mathrm{Pro}(\mathcal{K}))$ (Prop. 2.6), there is a map (cone) $\Phi : Y \times I \to \mathbf{K}$ such that $\Phi \circ e_0(Y) = \mathbf{p} \circ g$ and $\Phi \circ (f \times id) \simeq \mathbf{p} \circ H$. By the universal property of the limit, there is a unique map $\gamma : Y \times I \to K$ with $\Phi = \mathbf{p} \circ \gamma$. Then $\mathbf{p} \circ \gamma \circ e_0(Y) = \mathbf{p} \circ g$, hence $\gamma \circ e_0(Y) = g$.
Note that:

$$\mathbf{p} \circ H \circ e_0(X) = \mathbf{p} \circ g \circ f = \Phi \circ e_0(Y) \circ f =$$

$$= \mathbf{p} \circ \gamma \circ e_0(Y) \circ f = \mathbf{p} \circ \gamma \circ (f \times id) \circ e_0(X),$$

from which it follows

$$H \circ e_0(X) = \gamma \circ (f \times id) \circ e_0(X).$$

Finally: $H \simeq \gamma \circ (f \times id)$, being $e_0(X)$ a homotopy equivalence. □

Theorem 2.9. *Let $f : X \to Y$ be a map of compact metrizable spaces. The following are equivalent*

(a) $f \in RWHEP(\mathbf{Anr})$,

(b) $f \in RWHEP(\mathcal{C})$,

(c) f *is an almost-cofibration.*

Proof. $(a) \Rightarrow (b)$: this follows from Thm. 1.1 and Prop. 2.7. $(b) \Rightarrow (c)$: follows from Prop. 2.5. □

Our main result here is the following

Theorem 2.10. *Every inclusion $i : B \to X$ of a closed set in a compact metrizable space is an almost-cofibration.*

Proof. By the Borsuk's homotopy extension theorem $i : B \to X$ has the HEP($\mathbf{Anr}$), hence also the $\mathrm{RWHEP}(\mathbf{Anr})$. From Theorem 2.8 the assertion follows. □

3. The Homotopy Structure

Definition 3.1. *[11] A map $f : X \to Y$ is a strong shape equivalence if it fulfills the following requirements:*

(ss1) for each map $g : X \to K$, $K \in \mathbf{Anr}$, there is a map $h : Y \to K$ such that $h \circ f \simeq g$,

(ss2) if $h_1, h_2 : Y \to K$ are given maps and $G : X \times I \to K$ is a homotopy $G : h_1 \circ f \simeq h_2 \circ f$, then there is a homotopy $H : Y \times I \to K$, $H : h_1 \simeq h_2$, such that G and $H \circ (f \times 1)$ are homotopic rel end maps.

Since the homotopy H in (ss2) is uniquely determined up to homotopies rel end maps ([3], Prop.1.2), it follows at once that $f : X \to Y$ is a strong shape equivalence whenever the natural transformation

$$\mathbf{Gpd}(f, -) : \mathbf{Gpd}(Y, -) \Rightarrow \mathbf{Gpd}(X, -) : \mathcal{A} \to \mathbf{Gpd}$$

is a level equivalence, that is the functors of groupoids

$$\mathbf{Gpd}(f, K) : \mathbf{Gpd}(Y, K) \to \mathbf{Gpd}(X, K)$$

are all equivalences of categories, for all $K \in \mathbf{Anr}$.
Every homotopy equivalence is a strong shape equivalence.

Definition 3.2. *[14] A cofibration category is a category $\mathcal{E}$ equipped with two classes of morphisms Σ and Γ called weak equivalences and cofibrations, respectively, such that the following axioms are satisfied.*

(1) Weak equivalences satisfy the 2*-out-of-*6 *property, i.e., if f, g, h are composable morphisms of $\mathcal{E}$ such that both gf and hg are weak equivalences, then so are f, g and h.*

(2) Every isomorphism of $\mathcal{E}$ is an acyclic cofibration.

(3) $\mathcal{E}$ has an initial object, denoted 0.

(4) Every object $X \in \mathcal{E}$ is cofibrant, that is the unique morphism $0 \to X$ is a cofibration.

(5) (Trivial) Cofibrations are stable under pushouts along arbitrary morphisms of $\mathcal{E}$. A trivial cofibrations is morphisms in $\Sigma \cap \Gamma$.

(6) Every morphism of $\mathcal{E}$ factors as a composite of a cofibration followed by a weak equivalence.

In the category $\mathcal{C}$ let us denote Σ = the class of strong shape equivalences and Γ = the class of almost-cofibrations.

Theorem 3.3. $(\mathcal{C}, \Sigma, \Gamma)$ *is a cofibration category.*

Proof. (1) Let

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

be morphisms of $\mathcal{C}$ such that both $g \circ f$ and $h \circ g$ are strong shape equivalences. Then, for every $K \in \mathbf{Anr}$, we have that in

$$\mathbf{Gpd}(Z, K) \xrightarrow{h_K^*} \mathbf{Gpd}(Y, K) \xrightarrow{g_K^*} \mathbf{Gpd}(X, K) \xrightarrow{f_K^*} \mathbf{Gpd}(W, K)$$

both $f_K^* \circ g_K^*$ and $g_K^* \circ h_K^*$ are equivalences in $\mathbf{Gpd}$. Since $\mathbf{Gpd}$ is a model category and every model category has the 2-out-of-6 property, the assertion follows.
(2), (3), (4) and (6) are obvious.
(5) Let

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "\bar{i}"] \\ A \arrow[r, "\bar{f}"'] & A \bigsqcup_B Y \end{tikzcd}$$

be a pushout in $\mathcal{C}$ with i a (trivial) almost-cofibration. For all $K \in \mathcal{C}$, we get a pullback in $\mathbf{Gpd}$

$$\begin{tikzcd} \mathbf{Gpd}(A \bigsqcup_B Y, K) \arrow[r, "f'_*"] \arrow[d, "\bar{i}_*"'] & \mathbf{Gpd}(A, K) \arrow[d, "i_*"] \\ \mathbf{Gpd}(Y, K) \arrow[r, "f_*"'] & \mathbf{Gpd}(B, K) \end{tikzcd}$$

with i_* a (trivial) fibration in $\mathbf{Gpd}$. Since (trivial) fibration are stable under pullbacks in the model structure of $\mathbf{Gpd}$, it follows that $\bar{i}_*$ is a (trivial) fibration in $\mathbf{Gpd}$, from which it follows that $\bar{i}$ is a (trivial) almost-cofibration.

□

The strong shape category of compact metrizable spaces is obtained formally inverting the class of strong shape equivalences $SSh(\mathcal{C}) = \mathcal{C}[\Sigma^{-1}]$ [2]. The previous theorem says that it can be represented as the homotopy category of the cofibration category $(\mathcal{C}, \Sigma, \Gamma)$.

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