CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-2 (2025)

A RESULT ABOUT CONTINUOUS LATTICES OVER THE SIERPIŃSKI LOCALE

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Résumé. Soit $\mathbf{2}$ la catégorie $\{0 \leq 1\}$ et $\mathbb{S}$ la locale de Sierpiński (telle que $Sh(\mathbb{S}) \simeq [\mathbf{2}, \mathbf{Set}]$). Nous démontrons que

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

où $\mathbf{CtsLat}^{\ll}$ est la catégorie de treillis continus avec morphismes les homomorphismes de treillis qui préservent les bornes supérieures et la relation «way below».

Abstract. Let $\mathbf{2}$ be the category $\{0 \leq 1\}$ and $\mathbb{S}$ the Sierpiński locale (so that $Sh(\mathbb{S}) \simeq [\mathbf{2}, \mathbf{Set}]$). We prove

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

where $\mathbf{CtsLat}^{\ll}$ is the category of continuous lattices with way-below relation preserving suplattice homomorphisms as morphisms.

Keywords. Locale, topos, geometric logic.

Mathematics Subject Classification (2010). 06D22.

Dedicated to Harvey Alison on the occasion of his 50^{th} birthday.

1. Introduction

The aim of this short note is to prove the equivalence as stated in the Abstract. The proof technique used is taken from [HT23a] and hinges firstly on viewing continuous lattices as the rounded ideals of a type of information system.

We then rely on a description of rounded ideals in a presheaf category given in that paper, now applied to a broader class of relations. The description is that the poset of rounded ideals, internal to a presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, can be calculated by first applying the rounded ideals functor in $\mathbf{Set}$ and then applying the 'lax-to-natural' construction $\tilde{(\_)} : [\mathcal{C}^{op}, \mathbf{Pos}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Pos}]$. To complete the proof we show that $\tilde{(\_)}$ is full when applied to presheaves of continuous lattices. For this to work we seem to need to reduce to the case $\mathcal{C} = \{0 \leq 1\}^{op}$, so that the presheaf category $\hat{\mathcal{C}}$ is equivalent to sheaves over the Sierpiński locale $\mathbb{S}$.

After the main result (Harvey's Lemma) we finish with a corollary that has implications for the classification of locally compact locales via localic groupoids.

An Appendix has also been included that consists of a result about suplattices in presheaf categories. The result should be of general interest as it provides a new connection between presheaves of suplattices and suplattices internal to a presheaf topos.

2. Continuous lattices via strong proximity lattices

We take the following terms as understood: poset, ideal (of a poset; i.e. a directed lower closed subset), semilattice, continuous lattice, dcpo, suplattice and way-below relation ($\ll$). Consult for example [J82] for background material. The information system approach to continuous posets, exemplified by [V93], possibly covers the material of this section. However here we follow the more recent exposition given in [K21] and in particular are exploiting the notion of 'strength' as defined in that paper. The results of this section (indeed the whole paper) are constructive and so valid in all toposes; in particular they will be exploited in presheaf toposes in later sections, as we build up to the proof of the main result.

Definition 2.1. *1. A strong proximity join semilattice is a join semilattice S together with a relation $\prec \subseteq S \times S$ such that $\forall a, b, c \in S$*

(i) $a \prec b$ if and only if there exist $d \in S$ with $a \prec d$ and $d \prec b$,

(ii) $a \leq b \prec c$ implies $a \prec c$, and $a \prec b \leq c$ implies $a \prec c$,

(iii) $\{d | d \prec a\}$ is an ideal of S,

(iv) $a \prec b$ implies $a \leq b$; and,

(v) (strong) if $c \prec a \vee b$ then there exists $a_0 \prec a$ and $b_0 \prec b$ such that $c \leq a_0 \vee b_0$.

2. A rounded ideal of a strong proximity join semilattice is an ideal I such that $\forall a \in I$ there exists $b \in I$ such that $a \prec b$. The collection of all rounded ideals of S is written $R\text{-}idl(S)$.

Example 2.2. For any continuous lattice A, $(A, \ll)$ is a strong proximity join semilattice. Further $A \cong R\text{-}idl(A)$; in one direction the isomorphism is $a \mapsto \downarrow\!\!\!\downarrow a$ and directed join (i.e. $I \mapsto \bigvee^{\uparrow} I$) is the inverse.

It is clear from the definition that strong proximity join semilattices are the models of a geometric theory; the morphisms of the corresponding category of models are join semilattice homomorphisms that preserve $\prec$. We denote this category $\vee$-$\mathbf{SPSLat}$ and there is a forgetful functor $U : \mathbf{CtsLat}^{\ll} \longrightarrow \vee\text{-}\mathbf{SPSlat}$.

Proposition 2.3. *For any strong proximity join semilattice S, $R\text{-}idl(S)$ is a continuous lattice. Define $\bar{\phi}(I) = \bigcup^{\uparrow}\{\downarrow^{S'} \phi(a) | a \in I\}$, for each strong proximity join semilattice homomorphism $\phi : S \longrightarrow S'$; then by taking $R\text{-}idl(\phi) = \bar{\phi}$ on morphisms we have defined a functor $R\text{-}idl : \vee\text{-}\mathbf{SPSlat} \longrightarrow \mathbf{CtsLat}^{\ll}$.*

In the statement of the Proposition we use the notation $\downarrow^S a = \{b | b \prec a\}$ for any element a of a strong proximity join semilattice S.

Proof. The bottom of $R\text{-}idl(S)$ is $\downarrow^S 0$, and $I \vee J = \downarrow \{a \vee b | a \in I, b \in J\}$ (use the roundedness of I and J to check that this set, which is clearly an ideal, is rounded). The directed union of rounded ideals is a rounded ideal so $R\text{-}idl(S)$ is a complete lattice. Any rounded ideal I is the directed union of $\downarrow^S a$ for each $a \in I$. Therefore $I \ll J$ iff $\exists j \in J$ such that $I \subseteq \downarrow^S j$ from which it is clear that $R\text{-}idl(S)$ is continuous.

Let $\phi : S \longrightarrow S'$ be a morphism of strong proximity join semilattices. Certainly $\bar{\phi}(\downarrow^S 0) \subseteq \downarrow^{S'} 0$ as $i \prec^S 0$ implies $i = 0$ and $\phi(0) = 0$. That $\bar{\phi}$ preserves directed joins follows essentially by definition of union. For preservation of binary joins by $\bar{\phi}$ it is therefore clearly sufficient to verify $\bar{\phi}(\downarrow^S a \vee b) \subseteq \bar{\phi}(\downarrow^S a) \vee \bar{\phi}(\downarrow^S b)$ for any pair $a, b \in S$. This amounts to verifying that for any $d \prec^{S'} \phi(c)$, for some c with $c \prec^S a \vee b$, that

$d \leq c_0 \vee c_1$ for some c_0, c_1 such that there exists $a_0 \prec^S a$ and $b_0 \prec^S b$ with $c_0 \prec^{S'} \phi(a_0)$ and $c_1 \prec^{S'} \phi(b_0)$. Use $c \prec^S a \vee b$ and the strength of the proximity lattice S to find a_0 and b_0 for which then $\phi(c) \leq \phi(a_0 \vee b_0)$. But $\phi(a_0 \vee b_0) = \phi(a_0) \vee \phi(b_0)$ so by the strength of S' there exists the c_0 and c_1 required.

To complete our check that $\bar{\phi}$ is a morphism of $\mathbf{CtsLat}^{\ll}$ we must check that it preserves $\ll$. For this we need to verify that if $I \subseteq \downarrow^S j$ for some $j \in J$ that $\bar{\phi}(I) \subseteq \downarrow^S j'$ for some $j' \in \bar{\phi}(J)$. But by roundedness of J and preservation of $\prec$ by ϕ, $j \in J$ implies $\phi(j) \in \bar{\phi}(J)$ and we can see that $\bar{\phi} \downarrow^S j \subseteq \downarrow^{S'} \phi(j)$ as ϕ preserves $\prec$. These last two observations combine to show that $\bar{\phi}$ preserves $\ll$.

Finally, it is clear that we have defined a functor. Preservation of identity is trivial and $\bar{\psi}\bar{\phi} = \overline{\psi\phi}$ because $\prec$ is preserved. $\square$

Notice that the isomorphism $A \cong R\text{-}idl(A)$ of Example 2.2 is natural in A; more explicitly there is a natural isomorphism $R\text{-}idl \circ U \cong Id_{\mathbf{CtsLat}^{\ll}}$, though we will not notate the forgetful functor $U : \mathbf{CtsLat}^{\ll} \longrightarrow \bigvee\text{-}\mathbf{SPSlat}$ in what follows.

3. Background presheaf topos results

This section consists of three subsections where we recall in turn some results about constructions and characterisations of lattice theoretic properties in presheaf toposes. The results are all effectively well known. In the first subsection we recall the $\tilde{(\_)}$ construction which given a presheaf of posets returns another presheaf of posets, but on morphisms sends lax natural transformations to natural transformations. Next we recall how the $\tilde{(\_)}$ construction can be used to give an explicit description of the rounded ideal completion in a presheaf topos. Finally we recall that for any dcpo (suplattice) homomorphism $\alpha : A \longrightarrow B$ in a presheaf topos $\hat{\mathcal{C}}$, that $\alpha_a : A(a) \longrightarrow B(a)$ is a dcpo (suplattice) homomorphism for every object a of $\mathcal{C}$.

3.1 The lax-to-natural functor $\tilde{(\_)}$

We recall the $\tilde{(\_)}$ construction from [HT23a], which is a lax right adjoint to the forgetful functor that embeds $[\mathcal{C}^{op}, \mathbf{Pos}]$ into the category with the same objects (presheaves of posets) but with lax natural transformations as morphisms. We will not exploit this lax universal property here, relying instead on [HT23a] for properties of $\tilde{(\_)}$, but we will need to describe it explicitly.

If $F : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is a presheaf of posets on some category $\mathcal{C}$ then we define $\tilde{F}$ by

$$\tilde{F}(a) = \{(x_f) \in \Pi_{f:b \longrightarrow a} F(b) | F(g)x_f \leq x_{fg}, \forall c \xrightarrow{g} b \xrightarrow{f} a\}.$$

In terms of its action on morphisms we have $[\tilde{F}(f)((x_h))]_g = x_{fg}$ for any $f : b \longrightarrow a$ and $g : c \longrightarrow b$. If $\phi : F \xrightarrow{\leq} G$ is a lax natural transformation (i.e. $G(f)\phi_b \leq \phi_a F(f)$ for all $f : b \longrightarrow a$ of $\mathcal{C}$) then we define a natural transformation $\tilde{\phi} : \tilde{F} \longrightarrow \tilde{G}$ by $\tilde{\phi}_a((x_f)) = (\phi_b(x_f))$. We know from [HT23a] that it is faithful; in fact, for any natural transformation $\alpha : \tilde{F} \longrightarrow \tilde{G}$ there is a lax natural transformation $\psi^\alpha : F \xrightarrow{\leq} G$ such that $\phi = \psi^{\tilde{\phi}}$ for any lax natural transformation $\phi : F \xrightarrow{\leq} G$. Further $\psi^{(\_)}$ has the properties that $\psi^{Id} = Id$ and $\psi^\alpha \psi^\beta \leq \psi^{\alpha\beta}$. The explicit formula for ψ^α is $\psi^\alpha_a(x) = (\alpha_a((F(f)(x))_f))_{Id_a}$.

3.2 R-idl in a presheaf topos

In this subsection we recall the approach taken in [HT23a] to constructing R-idl in a presheaf topos $\hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$. As made clear in Section 4 of that paper, any construction of sets of subsets, each determined by geometric sequents, can be calculated by first applying the construction in $\mathbf{Set}$ at each object (and morphism) to obtain a new presheaf, and then applying the $\tilde{(\_)}$ construction to that presheaf. Put another we are saying that R-$idl_{\hat{\mathcal{C}}}(S)$, as a poset in $\hat{\mathcal{C}}$, is naturally isomorphic to $\widetilde{R\text{-}idl \circ S}$, for any strong proximity lattice S in the topos $\hat{\mathcal{C}}$. (Given that S is the model of a geometric theory in a presheaf topos, it is the same thing as a functor $\mathcal{C}^{op} \longrightarrow \vee\textbf{-SPSLat}$; e.g. D1.2.14 (i) of [J02].)

3.3 Dcpo and suplattice homomorphisms in $\hat{\mathcal{C}}$

Lemma 3.1. *Let $\alpha : A \longrightarrow B$ be an internal dcpo (suplattice) homomorphism in a presheaf topos $\hat{\mathcal{C}}$ between two internal dcpos (suplattices) A*

and B. Then $\alpha_a : A(a) \longrightarrow B(a)$ is a dcpo (suplattice) homomorphism for every object a of $\mathcal{C}$. Further, if A is an internal suplattice then for any morphism $f : b \longrightarrow a$ of $\mathcal{C}$ $A(f)$ is a suplattice homomorphism.

Proof. Recall that for any geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ its direct image defines a functor $f_* : \mathbf{dcpo}_{\mathcal{F}} \longrightarrow \mathbf{dcpo}_{\mathcal{E}}$ ([T04]). The techniques of (i) in Lemma C1.6.9 of [J02] can then be applied to complete the proof. To provide more detail, note that $A(a)$ is isomorphic to $\gamma_*(a^*A)$ where γ is the unique geometric morphism $\widehat{\mathcal{C}/a} \xrightarrow{a} \hat{\mathcal{C}} \longrightarrow \mathbf{Set}$. (Here $a : \widehat{\mathcal{C}/a} \longrightarrow \hat{\mathcal{C}}$ is the geometric morphism corresponding to the pullback adjunction $\Sigma_{Y(a)} \dashv Y(a)^*$ under $\widehat{\mathcal{C}/a} \simeq \hat{\mathcal{C}}/Y(a)$; its inverse image is logical.) $\square$

4. Main result

Lemma 4.1. *(Harvey's lemma).*

$$[\mathbf{2}, \mathbf{CtsLat}^{\ll}] \simeq \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})}$$

Proof. Consider

$$\begin{array}{rcl} \Psi : [\mathbf{2}, \mathbf{CtsLat}^{\ll}] & \longrightarrow & \mathbf{CtsLat}^{\ll}_{Sh(\mathbb{S})} \\ F & \mapsto & \tilde{F} \end{array}$$

This is well defined because $F \cong R\text{-}idl \circ F$ (recall our earlier comment that $R\text{-}idl \circ U \cong Id$) and so $\tilde{F} \cong \widetilde{R\text{-}idl \circ F} \cong R\text{-}idl_{Sh(\mathbb{S})}F$, a continuous lattice in $Sh(\mathbb{S})$. It is clearly essentially surjective as any continuous lattice A in $Sh(\mathbb{S})$ has $A \cong R\text{-}idl_{Sh(\mathbb{S})}(A)$. (In fact, all this holds for any $\hat{\mathcal{C}}$.)

We have commented already that $(\tilde{\_})$ is faithful, so we just need to prove fullness of Ψ to complete the proof. Say we have $\alpha : \tilde{F} \longrightarrow \tilde{G}$, a $\ll$ preserving internal suplattice homomorphism (in $Sh(\mathbb{S})$). Because it is $\ll$ preserving, its right adjoint α_* is directed join preserving; i.e. a dcpo homomorphism. Indeed, this is an equivalent characterisation of $\ll$ preserving for a suplattice homomorphism and we will call on this characterisation below. By Lemma 3.1 we know that $\alpha_i : \tilde{F}(i) \longrightarrow \tilde{G}(i)$ (respectively

$(\alpha_*)_i : \tilde{G}(i) \longrightarrow \tilde{F}(i))$ are suplattice (respectively dcpo) homomorphisms for $i = 0, 1$.

Now, the obvious choice for a natural transformation $\psi : F \longrightarrow G$ such that $\tilde{\psi} = \alpha$ is ψ^α (described in Section 3.1). To show that this works and so to complete the proof we must show: (a) $[\widetilde{\psi^\alpha}]_i = \alpha_i$ for $i = 0, 1$, (b) ψ^α_i is a suplattice homomorphism for $i = 0, 1$, (c) ψ^α is a natural transformation and not just a lax natural transformation; and, (d) ψ^α_i preserves $\ll$ for $i = 0, 1$.

We will take each in turn but first let us note that by construction $\tilde{F}(1) = F(1)$ (there is only the identity morphism away from 1). From this it is clear $\alpha_1 = \psi^\alpha_1 = [\widetilde{\psi^\alpha}]_1$ and so (a), (b) and (d) are actually immediate at $i = 1$. The suplattice $\tilde{F}(0)$ is also easy to describe: it consists of pairs (x_0, x_1) with $x_i \in F(i)$, $i = 0, 1$ and $F(\leq)(x_0) \leq x_1$. Further the function $\tilde{F}(\leq) : \tilde{F}(0) \longrightarrow \tilde{F}(1)$ is projection $(x_0, x_1) \mapsto x_1$; this is just a repetition of the definition of $\tilde{(\_)}$. Notice from this that by naturality of α therefore for any (x_0, x_1) in $\tilde{F}(0)$ we have $\pi_2\alpha_0(x_0, F(\leq)x_0) \leq \pi_2\alpha_0(x_0, x_1) = \alpha_1\pi_2(x_0, x_1) = \psi^\alpha_1 x_1$; this will be used in our verification of (a) which is the next step.

(a) As α is an internal suplattice homomorphism we know that

$$\begin{tikzcd}
\tilde{F}(0) \arrow[r, "\alpha_0"] & \tilde{G}(0) \\
\tilde{F}(1) \arrow[r, "\alpha_1"] \arrow[u, "\Sigma_{\tilde{F}(\leq)}"] & \tilde{G}(1) \arrow[u, "\Sigma_{\tilde{G}(\leq)}"']
\end{tikzcd}$$

commutes where $\Sigma_{\tilde{F}(\leq)} \dashv \tilde{F}(\leq)$ and similarly for $\tilde{G}$. See the proof of Lemma C1.6.9 [J02] for details on how internal suplattices, as presheaves, have left adjoints for their transition functions and for references to see how internal suplattice homomorphisms must commute with these left adjoints (or e.g. Proposition 3.7 of [T04] for effectively the same material). We can give an explicit description of $\Sigma_{\tilde{F}(\leq)}$: it is $x_1 \mapsto (0, x_1)$; this is clear as $\tilde{F}(\leq)$

is projection. Therefore we can calculate:

$$\begin{aligned}
\alpha_0(x_0,x_1) &= \alpha_0((x_0,F(\leq)x_0)\vee(0,x_1)) \\
&= \alpha_0(x_0,F(\leq)x_0)\vee\alpha_0\Sigma_{\tilde{F}(\leq)}(x_1) \\
&= \alpha_0(x_0,F(\leq)x_0)\vee\Sigma_{\tilde{G}(\leq)}\alpha_1(x_1) \\
&= (\psi_0^\alpha x_0,\pi_2\alpha_0(x_0,F(\leq)x_0))\vee(0,\psi_1^\alpha x_1) \text{ by def. of } \psi^\alpha \\
&= (\psi_0^\alpha x_0,\pi_2\alpha_0(x_0,F(\leq)x_0)\vee\psi_1^\alpha x_1) \\
&= (\psi_0^\alpha x_0,\psi_1^\alpha x_1) \text{ by an earlier remark} \\
&= [\widetilde{\psi^\alpha}]_0(x_0,x_1)
\end{aligned}$$

(b) By construction ψ_0^α is the composite

$$F(0)\xrightarrow{(Id,F(\leq))}\tilde{F}(0)\xrightarrow{\alpha_0}\tilde{G}(0)\xrightarrow{\pi_1}G(0).$$

By noting that joins in $\tilde{F}(0)$ and $\tilde{G}(0)$ are calculated pointwise it is clear that each factor in the composite is a suplattice homomorphism, and so ψ_0^α is a suplattice homomorphism.

(c) We know ψ^α is lax so this part of the proof amounts to checking

$$\psi_1^\alpha F(\leq)\leq G(\leq)\psi_0^\alpha.$$

In (a) we established $\alpha_0(x_0,x_1)=(\psi_0^\alpha(x_0),\psi_1^\alpha(x_1))$. By uniqueness of adjoints we therefore know that $(\alpha_*)_0(y_0,y_1)=([\psi_0^\alpha]_*(y_0),[\psi_1^\alpha]_*(y_1))$. From the definition of ψ^{α_*} we therefore know that $\psi_i^{\alpha_*}=[\psi_i^\alpha]_*$. But ψ^{α_*} is a lax natural natural transformation so we know that

$$F(\leq)\psi_0^{\alpha_*}\leq\psi_1^{\alpha_*}G(\leq).$$

Part (c) therefore follows by taking adjoint transpose (twice) of this last inequality.

(d) Apply the same reasoning as (b), but now to $[\psi_0^\alpha]_*$ which we have established is equal to $\psi_0^{\alpha_*}$. Having a right adjoint that preserves directed joins implies preservation of $\ll$. □

Remark 4.2. I have not been able to establish whether we *must* restrict to $\mathcal{C}=\{0\leq 1\}^{op}$ for the result to work. I expect so because in the proof we are exploiting a gluing construction which is, itself, tied to having an open/closed decomposition.

We finish with a corollary that has implications for the classification of locally compact locales via localic groupoids. Define $\mathbf{CtsFrm}^{\ll}$ to be the full subcategory of $\mathbf{CtsLat}^{\ll}$ consisting of continuous frames; i.e. continuous lattices that are also frames. Note that the morphisms are not frame homomorphisms; they are suplattice homomorphisms with directed join preserving right adjoints.

Corollary 4.3.

$$[\mathbf{2}, \mathbf{CtsFrm}^{\ll}] \simeq \mathbf{CtsFrm}^{\ll}_{\mathbf{Sh}(\mathbb{S})}$$

Proof. A continuous lattice is always a preframe (e.g. Lemma VII 4.1 of [J82]; but straightforward lattice theory). Therefore a continuous lattice is a continuous frame if and only if it satisfies the distributive law. The proof of the main lemma gives an explicit description of $\tilde{F}(i)$ in terms of $F(i)$ for $i = 0, 1$ and we noticed $\tilde{F}(1) = F(1)$. So it just needs to be checked that assuming $F(1)$ is distributive, $\tilde{F}(0)$ is distributive if and only if $F(0)$ is. This is immediate from the explicit description as binary meet and join in $\tilde{F}(0)$ are calculated pointwise. ☐

It is expected that we can construct a classifying localic groupoid for locally compact locales, using for example the approach of [HT23b] (or via an explicit construction of the points of the localic groupoid via the locale $\mathbb{S}^{\mathbb{N}}$; G. Manuell, private communication). That is, we expect that there exists a localic groupoid $\mathbb{G}_{\mathfrak{LK}}$ such that for any locale X the category $\mathfrak{LK}^{\cong}_{Sh(X)}$ of locally compact locales internal to the topos $Sh(X)$ (with isomorphisms as morphisms) is equivalent to the category of principal $\mathbb{G}_{\mathfrak{LK}}$-bundles over X. Now [HT23a] shows that $\mathbb{S}$-homotopies between principal bundles (over the classifying localic groupoid for compact Hausdorff locales) correspond to locale maps between compact Hausdorff locales (and the same correspondence for discrete locales is easy from the definition of presheaf topos). So it might be hoped that the same holds for locally compact locales. The Corollary rules this out: locally compact locales in $Sh(\mathbb{S})$ correspond externally to $\ll$ preserving suplattice homomorphisms and these do not correspond to locale maps.

5. Appendix: Internal suplattices

Below is a result about the relationship between $[\mathcal{C}^{op}, \mathbf{Sup}]$ and $\mathbf{Sup}_{\hat{\mathcal{C}}}$ which should be of general interest.

Proposition 5.1. *Let $\mathcal{C}$ be a small cartesian category (i.e. small and finitely complete). Then the $\tilde{(\_)}$ construction determines a functor:*

$$\tilde{(\_)} : [\mathcal{C}^{op}, \mathbf{Sup}] \longrightarrow \mathbf{Sup}_{\hat{\mathcal{C}}}$$

Proof. We split the proof into two parts:
(a) If $F : \mathcal{C}^{op} \longrightarrow \mathbf{Sup}$ is a functor then $\tilde{F}$ is an internal suplattice in $\hat{\mathcal{C}}$.
(b) If $\phi : F \longrightarrow G$ is a natural transformation then $\tilde{\phi} : \tilde{F} \longrightarrow \tilde{G}$ is an internal suplattice homomorphism in $\hat{\mathcal{C}}$.

(a) We rely on Lemma C1.6.9 of [J02] which shows that a presheaf $L : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is an internal suplattice if and only if (i) $L(a)$ is a suplattice for every object a of $\mathcal{C}$, (ii) $L(f) : L(a) \longrightarrow L(b)$ has a right and left adjoint for every morphism $f : b \longrightarrow a$; and, (iii) Beck-Chevalley holds for left adjoints; that is, for any pullback diagram

$$\begin{tikzcd} a \times_d b \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & b \arrow[d, "k"] \\ a \arrow[r, "l"] & d \end{tikzcd}$$

in $\mathcal{C}$ the square

$$\begin{tikzcd} L(a \times_d b) \arrow[d, "\Sigma_{\pi_1}"'] & L(b) \arrow[l, "L(\pi_2)"'] \arrow[d, "\Sigma_k"] \\ L(a) & L(d) \arrow[l, "L(l)"'] \end{tikzcd}$$

commutes where Σ_h is the left adjoint of $L(h)$ for any morphism h of $\mathcal{C}$.

We verify (i), (ii) and (iii) for $\tilde{F}$ where $F : \mathcal{C}^{op} \longrightarrow \mathbf{Sup}$.

For (i) note that if $(x^i_f)_{f:b \longrightarrow a}$ is an indexed ($i \in I$) collection of elements of $\tilde{F}(a)$ then $(\bigvee_{i\in I} x^i_f)_f$ is in $\tilde{F}(a)$ because $F(g)$ preserves arbitrary joins for all $g : c \longrightarrow b$, and can readily be seen to be the join of the $(x^i_f)_f$s. So $\tilde{F}(a)$ is a suplattice for each a. Notice that arbitrary meet is similarly defined pointwise; i.e. $(\bigwedge_{i\in I} x^i_f)_f$ is the meet of the $(x^i_f)_f$s.

Next (ii) is straightforward because arbitrary joins and meets are defined pointwise so it is easy to see that they are preserved by $\tilde{F}(f)$ (and we know that a monotone map between complete lattices has a right(left) adjoint iff it preserves arbitrary joins(meets)). For example, for joins,

$$(\tilde{F}(f)(\bigvee_i x^i_h))_g = \bigvee_i x^i_{fg} = [\bigvee_i \tilde{F}(f)(x^i_h)]_g.$$

For (iii) by uniqueness of adjoints we only need to prove $\tilde{F}(l)\Sigma_k \leq \Sigma_{\pi_1}\tilde{F}(\pi_2)$. Recall that quite generally if $\phi : A \longrightarrow B$ preserves arbitrary meets then its left adjoint is given by $\Sigma_\phi(b) = \bigwedge\{a | b \leq \phi(a)\}$. So checking (iii) amounts to checking for each $n : a' \longrightarrow a$ and each $(x_g)_{g:c \longrightarrow b} \in \tilde{F}(b)$ that

$$(\bigwedge\{(x'_r) | x_g \leq x'_{kg}, \forall g : c \longrightarrow b\})_{ln} \qquad \text{(A)}$$

is less than or equal to

$$(\bigwedge\{(y_m) | x_{\pi_2 t} \leq y_{\pi_1 t}, \forall t : c \longrightarrow a \times_d b\})_n \qquad \text{(B)}$$

Our strategy is to find for each $(y_m) \in \tilde{F}(a)$ in the meet (B) an $(x'_r) \in \tilde{F}(d)$ such that $x_g \leq x'_{kg}$ for all $g : c \longrightarrow b$. From this we know that A $\leq x'_{ln}$ for each $n : a' \longrightarrow a$ and the check of (iii) can be completed by verifying $x'_{ln} \leq y_n$ for each $n : a' \longrightarrow a$.

Define, for $r : d' \longrightarrow d$,

$$x'_r = [F(\pi_2^{d'})]_* y_{\pi_1}$$

where $\pi_2^{d'} : a \times_d d' \longrightarrow d'$ and we are using ϕ_* to denote the right adjoint of any ϕ (and $F(f)$, being a suplattice homomorphism, has a right adjoint for each f). We first check that (x'_r) is in $\tilde{F}(d)$; that is, do we have

$F(t)x'_r \leq x'_{rt}$ for every $t : d'' \longrightarrow d'$? Because $F(Id_a \times t)y_{\pi_1} \leq y_{\pi_1(Id_a \times t)}$ (as $(y_m) \in \tilde{F}(a)$) this can be confirmed by verifying $F(t)[F(\pi_2^{d'})]_* y_{\pi_1} \leq [F(\pi_2^{d''})]_* F(Id_a \times t)y_{\pi_1}$. This last is easy to verify as it is equivalent to $F(\pi_2^{d''})F(t)[F(\pi_2^{d'})]_* y_{\pi_1} \leq F(Id_a \times t)y_{\pi_1}$, $t\pi_2^{d''} = \pi_2^{d'}(Id_a \times t)$ and $F(\pi_2^{d'})[F(\pi_2^{d'})]_* \leq Id_{a \times_d d'}$.

For $x_g \leq x'_{kg}$, given a $g : c \longrightarrow b$ note that $x'_{kg} = [F(\pi_2^c)]_* y_{\pi_1(Id_a \times g)}$. So we must but check $F(\pi_2^c)(x_g) \leq y_{\pi_1(Id_a \times g)}$. This follows because $F(\pi_2^c)(x_g) \leq x_{g\pi_2^c}$ (as (x_g) is in $\tilde{F}(b)$) and $x_{g\pi_2^c} = x_{\pi_2^b(Id_a \times g)} \leq y_{\pi_1(Id_a \times g)}$ where the last inequality follows as (y_m) is in the meet (B).

So to complete our strategy for checking (iii) we must verify that $x'_{ln} \leq y_n$ for any $n : a' \longrightarrow a$. Using that the pullback of the composite ln along l is $\pi_1(Id_a \times n)$, where $\pi_1 : a \times_d a \longrightarrow a$, the calculation is:

$$\begin{aligned} x'_{ln} &= [F(\pi_2^{a'})]_* y_{\pi_1(Id_a \times n)} \\ &\leq [F(\pi_2^{a'})]_* [F(n, Id_{a'})]_* F(n, Id_{a'}) y_{\pi_1(Id_a \times n)} \\ &= [F(Id_{a'})]_* F(n, Id_{a'}) y_{\pi_1(Id_a \times n)} \\ &\leq y_n \end{aligned}$$

where the last is because $(y_m) \in \tilde{F}(a)$ (and, of course, n factors as $\pi_1(Id_a \times n)(n, Id_{a'})$).

(b) We prove that $\tilde{\phi}$ is an internal suplattice homomorphism. This follows provided we can verify that $\tilde{\phi}_a \Sigma_{\tilde{F}(f)} \leq \Sigma_{\tilde{G}(f)} \tilde{\phi}_b$ for all $f : b \longrightarrow a$ (this can be seen from the constructions shown in the proof of C1.6.9 of [J02]; Proposition 3.7 of [T04] also provides a route).

For each $(y_g) \in \tilde{F}(b)$ we must verify

$$\begin{aligned} &\tilde{\phi}_a(\bigwedge\{(x_r) | y_g \leq x_{fg}, \forall g : c \longrightarrow b\}) \leq \\ &\bigwedge\{(z_t) | [\tilde{\phi}_b((y_g))]_g \leq z_{fg}, \forall g : c \longrightarrow b\} \end{aligned}$$

Given $(z_t) \in \tilde{G}(a)$ then it is in the meet of the right hand side iff $\phi_c(y_g) \leq z_{fg}$ for all $g : c \longrightarrow b$. For any such (z_t) define (x_r) by $x_r = [\phi_{a'}]_*(z_r)$ for each $r : a' \longrightarrow a$. We check that $(x_r) \in \tilde{F}(a)$; i.e. that $F(d)x_r \leq x_{rd}$, or equivalently $F(d)[\phi_{a'}]_* z_r \leq [\phi_{a''}]_* z_{rd}$ for each $d : a'' \longrightarrow a'$. But given that $G(d)z_r \leq z_{rd}$ this will follow if $F(d)[\phi_{a'}]_* \leq [\phi_{a''}]_* G(d)$, which is true as ϕ is natural.

To complete we must verify (1) $y_g \leq x_{fg}$ for all g and (2) $(\tilde{\phi}_a((x_r)))_t \leq (z_t)$. For (1), as $\phi_c(y_g) \leq z_{fg}$, $y_g \leq [\phi_c]_* z_{fg} = x_{fg}$. For (2), this amounts to checking $\phi_{a'} x_r \leq z_r$ for each $r : a' \longrightarrow a$ which is again immediate from the definition of (x_r). □

Remark 5.2. I believe that the restriction to cartesian $\mathcal{C}$ can be seen to be unnecessary, using techniques from [T04]; however the proof becomes a bit more involved.

Remark 5.3. Part (b) of the proof does not work if we only assume that ϕ is a lax natural transformation. For example, take $\mathcal{C} = \{0 \leq 1\}^{op}$. If $X, Y : \{0 \leq 1\} \longrightarrow \mathbf{Set}$ then a lax natural transformation from $PX : \{0 \leq 1\} \longrightarrow \mathbf{Sup}$ to $PY : \{0 \leq 1\} \longrightarrow \mathbf{Sup}$ is the same data as a pair of relations $R(0) \subseteq X(0) \times Y(0)$ and $R(1) \subseteq X(1) \times Y(1)$ such that

$$\{Y(\leq)j_0 | \exists i_0 \in I_0, i_0 R(0) j_0\} \subseteq \{j_1 | \exists i_0' \in I_0, (X(\leq)i_0') R(1) j_1\}$$

for each $I_0 \subseteq X(0)$. But this is not sufficient for R to be a subfunctor of $X \times Y$, and so cannot correspond to an internal suplattice homomorphism in $\hat{\mathcal{C}}$.

Remark 5.4. The proposition can be used to prove the main lemma (Lemma 4.1) without using information systems. For example, $\twoheaddownarrow : F \longrightarrow idl \circ F$ is a natural transformation if $F : \mathcal{C}^{op} \longrightarrow \mathbf{CtsLat}^{\ll}$ and so $\tilde{\twoheaddownarrow}$ is an internal suplattice homomorphism which is a splitting for $\bigvee^{\uparrow} : \widetilde{idl \circ F} \longrightarrow \tilde{F}$. This shows that $\tilde{F}$ is an internal continuous lattice because $\widetilde{idl \circ F} \cong idl_{\hat{\mathcal{C}}} F$ and so we have exhibited $\tilde{F}$ as a dcpo retract of $idl_{\hat{\mathcal{C}}} A$ for some internal poset A of $\hat{\mathcal{C}}$.

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