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FIBRATIONS OF DOUBLE GROUPOIDS, I: ALGEBRAIC PROPERTIES AND HOMOTOPY SEQUENCES.

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Résumé. Nous introduisons la notion de fibration de double groupoïdes, que nous définissons comme un foncteur double possédant une propriété spécifique de remplissage-relèvement. Nous en étudions les propriétés fondamentales, notamment en établissant des suites exactes d'homotopie, parmi lesquelles figurent une suite de Mayer-Vietoris associée à un changement de base, ainsi que la suite d'homotopie propre à une fibration. Nous construisons également le module croisé fondamental d'une fibration.

Abstract. We introduce the notion of fibrations of double groupoids, defined as double functors with a specific filling-lifting property, and study their main properties. In particular, we establish exact homotopy sequences, including a Mayer-Vietoris sequence arising from a change of base, and the homotopy sequence associated to a fibration. We also construct the fundamental crossed module of a fibration.

Keywords. Double groupoid, Fibration, Homotopy groups, Crossed module, Geometric realization.

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Introduction and summary.

Double groupoids, that is, groupoid objects in the category of groupoids, were introduced by Ehresmann [22, 23, 24] in the late 20th century and have since been studied by several researchers due to their connections with various areas of mathematics. In particular, (small) double groupoids have garnered interest in algebraic topology, largely thanks to the work of Brown, Higgins, Porter, and others, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see, for example, the survey [7] and the references therein).

This is the third paper in a series exploring some purely algebraic properties of double groupoids using methods inspired by the topological context. In [17], we addressed the homotopy types obtained from double groupoids satisfying a quite natural filling condition. Like topological spaces, these double groupoids have associated homotopy groups, which are defined combinatorially using only their algebraic structure. Thus, the notion of weak equivalence between such double groupoids arises, and a corresponding homotopy category is defined. A main result states that the homotopy category of double groupoids with the filling property is equivalent to the homotopy category of all topological spaces with the property that the nth homotopy group at any base point vanishes for $n \geq 3$ (that is, the category of homotopy 2-types). Similar to the theory of Postnikov invariants with homotopy 2-types, the paper [19] provides a precise and purely algebraic classification for weak equivalence classes of double groupoids by three-cohomology classes.

This work and its companion paper [18] deal with *fibrations of double groupoids*, which we introduce as those double functors between double groupoids $F : \mathcal{A} \to \mathcal{B}$ that always solve certain filling-lifting problems on morphisms and boxes (see Definition 2.1 for precision). For instance, a double groupoid $\mathcal{A}$ has the filling property if and only if the double functor $\mathcal{A} \to *$, from $\mathcal{A}$ to the final double groupoid $*$, is a fibration. If $\mathcal{A}$ and $\mathcal{B}$ are 2-groupoids, regarded as double groupoids where one of the side groupoids of morphisms is discrete, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in our sense is the same as a fibration of 2-groupoids in the sense of Moerdijk and Svensson[33, 34], Hermida [28], Buckley [15], or Hardie, Kamps, and Kieboom [27]. By the equivalence between crossed complexes over groupoids and 2-groupoids, our concept of fibration also generalizes the notion of fibration of crossed modules over groupoids by Brown [6]. In particular, if both $\mathcal{A}$ and $\mathcal{B}$ are groupoids, viewed as double groupoids whose vertical morphisms are all identities and whose boxes are all vertical identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in our sense is the same as a fibration of groupoids in the sense of Grothendieck [25] and Brown [2]. However, our concept of fibration is more restrictive than the notion of *double fibration* proposed by Cruttwell, Lambert, Pronk, and Szyld in [20].

After Section 1, where we briefly establish some notational conventions on double groupoids, Sections 2 and 3 present the concept of fibration between double groupoids and study its basic properties, such as the change of base property, the filling property of fibres, and the path-lifting and homotopy-lifting properties. In Section 3, we also review several necessary definitions and results for the fundamental groupoid and the homotopy groups of a double groupoid. In Section 4, we show how a Mayer-Vietoris type exact sequence on homotopy groups arises from a change in the base of a fibration of double groupoids. This is used in Section 5 to derive a 9-term exact sequence on homotopy groups from a fibre sequence of double groupoids. This section also includes additional information about this homotopy sequence that relates to the actions of fundamental groupoids on the homotopy groups of fibres. In particular, we construct the fundamental crossed module over groupoids of a fibration of double groupoids. Our results in Sections 4 and 5 are deeply inspired by those we generalize, stated by Brown in [2] and Brown, Heath, and Kamps in [10] for groupoids, by Brown in [6] and Howie in [29] for crossed modules over groupoids, and by Hardie, Kamps, and Kieboom in [27] and Kamps and Porter in [30] for 2-groupoids.

Concerning the relationship between fibrations of double groupoids and simplicial and topological fibrations, we refer the reader to the companion paper [18].

1. Some conventions on double groupoids.

The notion of a double groupoid is well-known; in this preliminary section, we specify some basic terminology and notational conventions. We will work only with small double groupoids, so that in a double groupoid $\mathcal{A}$ we have a set of objects (usually denoted by $a, b, c, \ldots$), horizontal morphisms between them $(f, g, h, \ldots)$, vertical morphisms between them $(x, y, z, \ldots)$, both with composition written by juxtaposition, and boxes $(\alpha, \beta, \gamma, \ldots)$, usually depicted as

$$\begin{array}{ccc} d & \xleftarrow{\;f\;} & b \\ {\scriptstyle y}\big\uparrow & \alpha & \big\uparrow{\scriptstyle x} \\ c & \xleftarrow[\;g\;]{} & a \end{array} \tag{1}$$

where the horizontal morphisms f and g are, respectively, its vertical target and source, and the vertical morphisms y and x are its respective horizontal target and source. The horizontal composition of boxes is denoted by the symbol $\circ_{\rm h}$:

$$\begin{array}{ccccc} \cdot & \xleftarrow{\;f'\;} & \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle z}\big\uparrow & \alpha' & \big\uparrow & \alpha & \big\uparrow{\scriptstyle x} \\ \cdot & \xleftarrow[\;g'\;]{} & \cdot & \xleftarrow[\;g\;]{} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xleftarrow{\;f'f\;} & \cdot \\ {\scriptstyle z}\big\uparrow & \alpha' \circ_{\rm h} \alpha & \big\uparrow{\scriptstyle x} \\ \cdot & \xleftarrow[\;g'g\;]{} & \cdot \end{array}$$

and the vertical composition of boxes is denoted by the symbol $\circ_{\rm v}$:

$$\begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle y}\big\uparrow & \alpha & \big\uparrow{\scriptstyle x} \\ \cdot & \longleftarrow & \cdot \\ {\scriptstyle y'}\big\uparrow & \alpha' & \big\uparrow{\scriptstyle x'} \\ \cdot & \xleftarrow[\;h\;]{} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle yy'}\big\uparrow & \alpha \circ_{\rm v} \alpha' & \big\uparrow{\scriptstyle xx'} \\ \cdot & \xleftarrow[\;h\;]{} & \cdot \end{array}$$

Horizontal and vertical identities on objects and morphisms are respectively denoted by $\mathrm{I}^{\rm h}_a$, $\mathrm{I}^{\rm v}_a$, $\mathrm{I}^{\rm v}_f$, $\mathrm{I}^{\rm h}_x$, and $\mathrm{I}_a = \mathrm{I}^{\rm v}_{\mathrm{I}^{\rm h}_a} = \mathrm{I}^{\rm h}_{\mathrm{I}^{\rm v}_a}$, depicted as

$$a = = a \qquad \begin{array}{c} a \\ \| \\ a \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ \| & \mathrm{I}^{\rm v}_f & \| \\ \cdot & \xleftarrow[\;f\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & = = & \cdot \\ {\scriptstyle x}\big\uparrow & \mathrm{I}^{\rm h}_x & \big\uparrow{\scriptstyle x} \\ \cdot & = = & \cdot \end{array} \qquad \begin{array}{ccc} a & = = & a \\ \| & \mathrm{I}_a & \| \\ a & = = & a \end{array}$$

and horizontal and vertical inverses of boxes are respectively denoted by $\alpha^{-\rm h}$, $\alpha^{-\rm v}$, and $\alpha^{-\rm hv} = (\alpha^{-\rm h})^{-\rm v} = (\alpha^{-\rm v})^{-\rm h}$; that is, for α as in (1),

$$\begin{array}{ccc} \cdot & \xleftarrow{\;f^{-1}\;} & \cdot \\ {\scriptstyle x}\big\uparrow & \alpha^{-\rm h} & \big\uparrow{\scriptstyle y} \\ \cdot & \xleftarrow[\;g^{-1}\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;g\;} & \cdot \\ {\scriptstyle y^{-1}}\big\uparrow & \alpha^{-\rm v} & \big\uparrow{\scriptstyle x^{-1}} \\ \cdot & \xleftarrow[\;f\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;g^{-1}\;} & \cdot \\ {\scriptstyle x^{-1}}\big\uparrow & \alpha^{-\rm hv} & \big\uparrow{\scriptstyle y^{-1}} \\ \cdot & \xleftarrow[\;f^{-1}\;]{} & \cdot \end{array}$$

We will frequently use the coherence theorem by Dawson and Paré [21, Theorem 1.2], which ensures that *if a compatible arrangement of boxes in a double groupoid is composable in two different ways, the resulting pasted boxes are equal*. Throughout the paper, when we refer to an equality between pasting diagrams of boxes in a double groupoid, we means that the resulting pasted boxes are the same.

2. Fibrations between double groupoids.

A double functor $F : \mathcal{A} \to \mathcal{B}$ between double groupoids maps objects, horizontal and vertical morphisms, and boxes in $\mathcal{A}$ to objects, horizontal and vertical morphisms, and boxes in $\mathcal{B}$, respectively, in such a way that it preserves compositions and identities.

Definition 2.1. *A double functor $F : \mathcal{A} \to \mathcal{B}$ between double groupoids is a* fibration *if all lifting problems*

(i)

$$\begin{array}{ccc} & & \cdot \\ & & \uparrow{\scriptstyle \exists?} \\ \cdot & \underset{\exists?}{\leftarrow} & a \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} & & \cdot \\ & & \uparrow{\scriptstyle \tilde{x}} \\ \cdot & \underset{\tilde{f}}{\leftarrow} & Fa \end{array}$$

(ii)

$$\begin{array}{ccc} \cdot & \overset{f}{\leftarrow} & \cdot \\ \uparrow & \exists? & \uparrow{\scriptstyle x} \\ \cdot & \leftarrow & \cdot \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} \cdot & \overset{Ff}{\leftarrow} & \cdot \\ \uparrow & \tilde{\alpha} & \uparrow{\scriptstyle Fx} \\ \cdot & \leftarrow & \cdot \end{array}$$

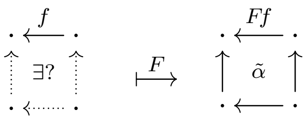
have solution. That is,

(i) If a is an object of $\mathcal{A}$, for any horizontal (resp. vertical) morphism $\tilde{f}$ (resp. $\tilde{x}$) in $\mathcal{B}$ with source Fa, there is a horizontal (resp. vertical) morphism f (resp. x) in $\mathcal{A}$ with source a such that $Ff = \tilde{f}$ (resp. $Fx = \tilde{x}$).

(ii) If f is a horizontal morphism of $\mathcal{A}$ and x is a vertical morphism of $\mathcal{A}$ whose target is the source of f, for any box $\tilde{\alpha}$ of $\mathcal{B}$ with vertical target Ff and horizontal source Fx, there is a box α in $\mathcal{A}$ with vertical target f and horizontal source x such that $F\alpha = \tilde{\alpha}$.

The above fibration condition (i) means that the restrictions of F to the respective groupoids of horizontal and vertical morphisms of $\mathcal{A}$ and $\mathcal{B}$ are both fibrations of groupoids in the sense of Brown [2] or Grothendieck [25]. In fact, if both $\mathcal{A}$ and $\mathcal{B}$ are groupoids, considered as double groupoids with all vertical morphisms as identities and all boxes as vertical identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in the sense of Definition 2.1 is the same as a fibration of groupoids. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are 2-groupoids, regarded as double groupoids whose vertical arrows are all identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ is the same as a fibration of 2-groupoids in the sense of Moerdijk and Svensson [33, 34] (see [33, Lemma 1.7.3]) or, equivalently, a 2-fibration as defined by Hermida [28] or Buckley [15].

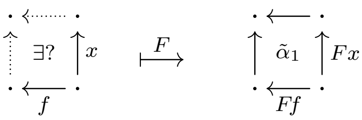
However, our concept of fibration between double groupoids is more restrictive than the notion of *double fibration* proposed by Cruttwell, *et al.* [20]. A double functor between double groupoids $F : \mathcal{A} \to \mathcal{B}$ is a double fibration in the sense of [20, Definition 2.25] whenever its restriction to the groupoids of vertical morphisms is a fibration, and every lifting problem



has a solution. If F is a fibration as in Definition 2.1, we can first select a vertical morphism x with target the source of f, which is carried by F to the vertical target of $\tilde{\alpha}$ and then to find a box α in $\mathcal{A}$ with vertical target f and horizontal source x such that $F\alpha = \tilde{\alpha}$. Thus, every fibration between double groupoids is a double fibration. However, the converse is false because, for example, double fibration does not necessary restrict fibration between the groupoids of horizontal morphisms.

The fibration conditions are more symmetric than they appears:

Lemma 2.2. *If $F : \mathcal{A} \to \mathcal{B}$ is a fibration of double groupoids, then any of the three lifting problems below has a solution.*



$$
\begin{array}{ccc}
\cdot \xleftarrow{\ f\ } \cdot & & \cdot \xleftarrow{\ Ff\ } \cdot \\
x\uparrow \quad \exists? \quad \vdots\uparrow & \overset{F}{\longmapsto} & Fx\uparrow \quad \tilde{\alpha}_2 \quad \uparrow \\
\cdot \leftarrow\cdots \cdot & & \cdot \longleftarrow \cdot
\end{array}
$$

$$
\begin{array}{ccc}
\cdot \leftarrow\cdots \cdot & & \cdot \longleftarrow \cdot \\
x\uparrow \quad \exists? \quad \vdots\uparrow & \overset{F}{\longmapsto} & Fx\uparrow \quad \tilde{\alpha}_3 \quad \uparrow \\
\cdot \xleftarrow[\ f\ ]{} \cdot & & \cdot \xleftarrow[\ Ff\ ]{} \cdot
\end{array}
$$

Proof. Since F is a fibration, there are boxes in $\mathcal{A}$

$$
\begin{array}{c}
\cdot \xleftarrow{\ f\ } \cdot \\
\uparrow \quad \alpha_1 \quad \uparrow x^{-1} \\
\cdot \longleftarrow \cdot
\end{array}
,\quad
\begin{array}{c}
\cdot \xleftarrow{\ f^{-1}\ } \cdot \\
\uparrow \quad \alpha_2 \quad \uparrow x \\
\cdot \longleftarrow \cdot
\end{array}
,\quad
\begin{array}{c}
\cdot \xleftarrow{\ f^{-1}\ } \cdot \\
\uparrow \quad \alpha_3 \quad \uparrow x^{-1} \\
\cdot \longleftarrow \cdot
\end{array}
$$

such that $F\alpha_1 = \tilde{\alpha}_1^{-\mathrm{v}}$, $F\alpha_2 = \tilde{\alpha}_2^{-\mathrm{h}}$, and $F\alpha_3 = \tilde{\alpha}_3^{-\mathrm{hv}}$. Then, $\alpha_1^{-\mathrm{v}}$, $\alpha_2^{-\mathrm{h}}$, and $\alpha_3^{-\mathrm{hv}}$ are solutions to the respective lifting problems. $\square$

Let $*$ denote the final double groupoid; that is, the double groupoid with only one object, $*$, one vertical morphism, $\mathrm{I}^{\mathrm{v}}_*$, one horizontal morphism, $\mathrm{I}^{\mathrm{h}}_*$, and one box

$$
\begin{array}{c}
* = = * \\
\| \quad \mathrm{I}_* \quad \| \\
* = = *
\end{array}
$$

If $\mathcal{A}$ is a double groupoid, then the double functor $\mathcal{A} \to *$ is a fibration if and only if $\mathcal{A}$ has the so-called *filling property*: *Any filling problem*

$$
\begin{array}{c}
\cdot \xleftarrow{\ f\ } \cdot \\
\vdots\uparrow \quad \exists? \quad \uparrow x \\
\cdot \leftarrow\cdots \cdot
\end{array}
$$

has a solution. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in weak Hopf algebra theory or in differential geometry (see, for instance, Andruskiewitsch and Natale [1] and Mackenzie [32]). It is also satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.* (see the papers by Brown [3, 4, 7, 8] and the references given there). Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see Brown and Higgins [12], Brown and Spencer [14], Brown, Hardie, Kamps and Porter [9], and Brown, Kamps and Porter [13]), for double groupoid objects in the category of groups (also termed cat^2-groups by Loday [31], see also Porter [35] and Bullejos, Cegarra and Duskin [16]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete (see for instance Moerdijk and Svensson [34] and Hardie, Kamps and Kieboom [26]).

Lemma 2.2 implies the following useful result by Andruskiewitsch and Natale [1, Lemma 1.12].

Corollary 2.3. *In a double groupoid satisfying the filling condition, any of the filling problems below has a solution:*

$$\begin{array}{ccc} \cdot & \dashleftarrow & \cdot \\ \vdots\uparrow & \exists? & \uparrow x \\ \cdot & \underset{f}{\longleftarrow} & \cdot \end{array}\ , \qquad \begin{array}{ccc} \cdot & \dashleftarrow & \cdot \\ x\uparrow & \exists? & \uparrow\vdots \\ \cdot & \underset{f}{\longleftarrow} & \cdot \end{array}\ , \qquad \begin{array}{ccc} \cdot & \overset{f}{\longleftarrow} & \cdot \\ x\uparrow & \exists? & \uparrow\vdots \\ \cdot & \dashleftarrow & \cdot \end{array}\ ,$$

Proposition 2.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids.*

(i) If $\mathcal{B}$ has the filling property, then $\mathcal{A}$ also has the filling property.

(ii) If $\mathcal{A}$ has the filling property and F is onto on objects, then $\mathcal{B}$ has the filling property.

Proof. (i) Suppose $\mathcal{B}$ has the filling property, and let

$$\begin{array}{ccc} \cdot & \overset{f}{\longleftarrow} & \cdot \\ \vdots\uparrow & \exists? & \uparrow x \\ \cdot & \dashleftarrow & \cdot \end{array}$$

be a filling problem in $\mathcal{A}$. Choose a box $\tilde{\alpha}$ in $\mathcal{B}$ of the form

$$\begin{array}{ccc} \cdot & \overset{Ff}{\longleftarrow} & \cdot \\ \uparrow & \tilde{\alpha} & \uparrow Fx \\ \cdot & \longleftarrow & \cdot \end{array}$$

Then, as F is a fibration, we may choose a box α in $\mathcal{A}$ with vertical target f and horizontal source x such that $F\alpha = \tilde{\alpha}$. In particular, α solves the given filling problem in $\mathcal{A}$.

(ii) Suppose $\mathcal{A}$ has the filling property, and let

$$\begin{array}{ccc} \cdot & \xleftarrow{\tilde f} & \cdot \\ \uparrow & \exists? & \uparrow{\scriptstyle \tilde x} \\ \cdot & \leftarrow\cdots & \cdot \end{array}$$

be a filling problem in $\mathcal{B}$. Since F is onto on objects, we can choose an object a of $\mathcal{A}$ such that Fa is the source of $\tilde{x}$. Using that F is a fibration, we can choose a vertical morphism x of $\mathcal{A}$ with source a such that $fx = \tilde{x}$, as well as a horizontal morphism with source the target of x such that $Ff = \tilde{f}$. Since $\mathcal{A}$ has the filling property, we can choose a box α of $\mathcal{A}$

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \alpha & \uparrow{\scriptstyle x} \\ \cdot & \longleftarrow & \cdot \end{array}$$

whose respective horizontal and vertical sources are f and x. Obviously, $F\alpha$ solves the given filling problem in $\mathcal{B}$. □

Proposition 2.5. *In a pullback square of double groupoids*

$$\begin{array}{ccc} \mathcal{B}' \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{G'} & \mathcal{A} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{B}' & \xrightarrow[G]{} & \mathcal{B} \end{array}$$

if F is a fibration, then so also is F'.

Proof. (i) By [2, Prop. 2.8], the projection F' restricts giving fibrations both between the groupoids of horizontal and vertical morphisms.

(ii) Suppose given a box α' of $\mathcal{B}'$ and morphisms f and x in $\mathcal{A}$ as in

$$\begin{array}{ccc} \cdot & \xleftarrow{f'} & \cdot \\ \uparrow & \alpha' & \uparrow{\scriptstyle x'} \\ \cdot & \longleftarrow & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ & & \uparrow{\scriptstyle x} \\ & & \cdot \end{array}$$

such that $Gf' = Ff$ and $Gx' = Fx$. Since F is a fibration, there is a solution in $\mathcal{A}$, say α, to the lifting problem

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \exists? & \uparrow{\scriptstyle x} \\ \cdot & \leftarrow & \cdot \end{array} \qquad \overset{F}{\longmapsto} \qquad \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & G\alpha' & \uparrow \\ \cdot & \longleftarrow & \cdot \end{array}$$

Then (α', α) is a box in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ satisfying that $F'(\alpha', \alpha) = \alpha'$. □

Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. If b is an object of $\mathcal{B}$, let $\mathcal{F}_b = F^{-1}(b)$ denote the *double groupoid fibre* of F over b. That is, $\mathcal{F}_b$ is the double subgroupoid of $\mathcal{A}$ with objects those a of $\mathcal{A}$ such that $Fa = b$, vertical morphisms those vertical morphisms x of $\mathcal{A}$ such that $Fx = \mathrm{I}^{\mathrm{v}}_b$, horizontal morphisms those horizontal morphisms f of $\mathcal{A}$ such that $Ff = \mathrm{I}^{\mathrm{h}}_b$, and boxes those α of $\mathcal{A}$ such that $F\alpha = \mathrm{I}_b$. For every object a of $\mathcal{F}_b$, we call the sequence

$$(\mathcal{F}_b, a) \hookrightarrow (\mathcal{A}, a) \xrightarrow{F} (\mathcal{B}, b)$$

a *fibre sequence* of double groupoids.

Proposition 2.6. *In any fibre sequence as above, the double groupoid fibre $\mathcal{F}_b$ has the filling property.*

Proof. This follows from Proposition 2.5, since $\mathcal{F}_b$ occurs in the pullback square of double groupoids

$$\begin{array}{ccc} \mathcal{F}_b & \hookrightarrow & \mathcal{A} \\ \downarrow & & \downarrow{\scriptstyle F} \\ * & \xrightarrow{b} & \mathcal{B} \end{array}$$

□

3. Paths, loops, homotopies, homotopy groups.

In this section, we work under the assumption that the double groupoids satisfy the filling condition.

Let $\mathcal{A}$ be a double groupoid. A *path* in $\mathcal{A}$ from an objet a to an object a' [17, §2], denoted by

$$(f, x) : a \curvearrowright a',$$

is a pair of morphisms (f, x) where x is a vertical morphism from a and f is a horizontal morphism from the target of x to a'; that is, a pair of morphisms of the form

$$\begin{array}{ccc} a' & \xleftarrow{\ f\ } & \cdot \\ & & \big\uparrow{\scriptstyle x} \\ & & a \end{array} \qquad (2)$$

When $a' = a$, we say that $(f, x) : a \curvearrowright a$ is a *loop* with base point a. The *identity loop* at an object a is the loop $(\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a) : a \curvearrowright a$, which is depicted as

$$\begin{array}{ccc} a & = & a \\ & & \| \\ & & a \end{array}$$

Proposition 3.1 (Path-lifting property)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. For every object a of $\mathcal{A}$ and every path $(\tilde{f}, \tilde{x}) : Fa \curvearrowright b$ in $\mathcal{B}$, there exists a path $(f, x) : a \curvearrowright a'$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$.*

Proof. Since F is a fibration, we can choose a vertical morphism in $\mathcal{A}$ with source a, say $x : a \to a''$, such that $Fx = \tilde{x}$. Since Fa'' is the source of $\tilde{f}$, we can also choose a horizontal morphism $f : a'' \to a'$ in $\mathcal{A}$ such that $Ff = \tilde{f}$. Thus $(f, x) : a \curvearrowright a'$ is a path in $\mathcal{A}$ such that $Ff = \tilde{f}$ and $Fx = \tilde{x}$. □

If $(f, x), (g, y) : a \curvearrowright a'$ are two paths in $\mathcal{A}$, then (f, x) is *homotopic* to (g, y), denoted by $(f, x) \simeq (g, y)$, whenever there is a box α in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\ f^{-1}g\ } & \cdot \\ \| & \alpha & \big\uparrow{\scriptstyle yx^{-1}} \\ \cdot & = & \cdot \end{array} \qquad (3)$$

that is, whose horizontal target and vertical sources are identities, its horizontal source is yx^{-1}, and its vertical target is $f^{-1}g$ (see [17, §2]). We call such a box a *homotopy*, and we often write $\alpha : (f, x) \simeq (g, y)$ whenever we wish to display the homotopy.

Proposition 3.2 (Path homotopy-lifting property)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. Suppose $(g, y) : a \curvearrowright a'$ is a path in $\mathcal{A}$, $(\tilde{f}, \tilde{x}) : Fa \curvearrowright Fa'$ is a path in $\mathcal{B}$, and $\tilde{\alpha} : (\tilde{f}, \tilde{x}) \simeq (Fg, Fy)$ is a homotopy in $\mathcal{B}$. Then, there is a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$ and there is a homotopy $\alpha : (f, x) \simeq (g, y)$ such that $F\alpha = \tilde{\alpha}$.*

Proof. Using the filling property, we can select a box $\tilde\beta$ in $\mathcal{B}$ of the form

$$\begin{array}{ccc} Fa' & \overset{\tilde f}{\longleftarrow} & \cdot \\ {\scriptstyle \tilde z}\uparrow & \tilde\beta & \uparrow{\scriptstyle \tilde x} \\ \cdot & \underset{\tilde h}{\longleftarrow} & Fa \end{array}$$

and construct the box $\tilde\gamma$ of $\mathcal{B}$ by

$$\begin{array}{ccc} Fa' & \overset{Fg}{\longleftarrow} & \cdot \\ {\scriptstyle \tilde z}\uparrow & \tilde\gamma & \uparrow{\scriptstyle Fy} \\ \cdot & \underset{\tilde h}{\longleftarrow} & Fa \end{array} \quad = \quad \begin{array}{ccccc} Fa' & \overset{\tilde f}{\longleftarrow} & \cdot & \overset{\tilde f^{-1}\,Fg}{\longleftarrow} & \cdot \\ \uparrow & & \| & \tilde\alpha & \uparrow{\scriptstyle Fy\,\tilde x^{-1}} \\ {\scriptstyle \tilde z}\,| & \tilde\beta & \cdot & = = = & \cdot \\ | & & {\scriptstyle \tilde x}\uparrow & \mathrm{I}^{\mathrm{h}} & \uparrow{\scriptstyle \tilde x} \\ \cdot & \underset{\tilde h}{\longleftarrow} & Fa & = = = & Fa \end{array}$$

Since F is a fibration, we can choose a box γ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} a' & \overset{g}{\longleftarrow} & \cdot \\ {\scriptstyle z}\uparrow & \gamma & \uparrow{\scriptstyle y} \\ \cdot & \underset{h}{\longleftarrow} & a \end{array}$$

such that $F\gamma = \tilde\gamma$, and then (since $Fz = \tilde z$ and $Fh = \tilde h$) we can also choose a box β of the form

$$\begin{array}{ccc} a' & \overset{f}{\longleftarrow} & \cdot \\ {\scriptstyle z}\uparrow & \beta & \uparrow{\scriptstyle x} \\ \cdot & \underset{h}{\longleftarrow} & a \end{array}$$

such that $F\beta = \tilde\beta$. Then $(f,x) : a \curvearrowright a'$ is a path in $\mathcal{A}$ satisfying that $(Ff, Fx) = (\tilde f, \tilde x)$. Furthermore, if $\alpha : (f,x) \simeq (g,y)$ is the homotopy defined by

$$
\begin{array}{ccc}
\cdot & \xleftarrow{f^{-1}g} & \cdot \\
\| & \alpha & \uparrow{\scriptstyle yx^{-1}} \\
\cdot & = & \cdot
\end{array}
\;=\;
\begin{array}{ccccc}
\cdot & \xleftarrow{f^{-1}} & \cdot & \xleftarrow{g} & \cdot \\
{\scriptstyle x}\uparrow & \beta^{\text{-h}} & \uparrow{\scriptstyle z} & \gamma & \uparrow{\scriptstyle y} \\
\cdot & \xleftarrow[h^{-1}]{} & \cdot & \xleftarrow[h]{} & \cdot \\
{\scriptstyle x^{-1}}\uparrow & & \mathrm{I}^{\text{h}} & & \uparrow{\scriptstyle x^{-1}} \\
\cdot & & = & & \cdot
\end{array}
$$

then,

$$
F\alpha \;=\;
\begin{array}{ccccc}
\cdot & \xleftarrow{\tilde{f}^{-1}} & \cdot & \xleftarrow{\tilde{g}} & \cdot \\
{\scriptstyle \tilde{x}}\uparrow & \tilde{\beta}^{\text{-h}} & \uparrow{\scriptstyle \tilde{z}} & \tilde{\gamma} & \uparrow{\scriptstyle \tilde{y}} \\
\cdot & \xleftarrow[\tilde{h}^{-1}]{} & \cdot & \xleftarrow[\tilde{h}]{} & \cdot \\
{\scriptstyle \tilde{x}^{-1}}\uparrow & & \mathrm{I}^{\text{h}} & & \uparrow{\scriptstyle \tilde{x}^{-1}} \\
\cdot & & = & & \cdot
\end{array}
\;=\;
\begin{array}{ccccccc}
\cdot & \xleftarrow{\tilde{f}^{-1}} & \cdot & \xleftarrow{\tilde{f}} & \cdot & \xleftarrow{\tilde{f}^{-1}Fy} & \cdot \\
 & & & & \| & \tilde{\alpha} & \uparrow{\scriptstyle Fy\,\tilde{x}^{-1}} \\
{\scriptstyle \tilde{x}}\uparrow & \tilde{\beta}^{\text{-h}} & \uparrow{\scriptstyle \tilde{z}} & \tilde{\beta} & \cdot & = & \cdot \\
 & & & & {\scriptstyle \tilde{x}}\uparrow & \mathrm{I}^{\text{h}} & \uparrow{\scriptstyle \tilde{x}} \\
\cdot & \xleftarrow[\tilde{h}^{-1}]{} & \cdot & \xleftarrow[\tilde{h}]{} & \cdot & = & \cdot \\
{\scriptstyle \tilde{x}^{-1}}\uparrow & & & \mathrm{I}^{\text{h}} & & & \uparrow{\scriptstyle \tilde{x}^{-1}} \\
\cdot & & & = & & & \cdot
\end{array}
\;=\tilde{\alpha}.
$$

□

For every pair of objects a and a' of a double groupoid $\mathcal{A}$, by [17, Lemma 2.1], homotopy is an equivalence relation on the set of paths in $\mathcal{A}$ from a to a', and we write $[f,x]$ to denote the homotopy class of a path $(f,x) : a \curvearrowright a'$ in $\mathcal{A}$. These homotopy classes of paths are the morphisms

$$[f,x] : a \to a'$$

of the *fundamental groupoid* of the double groupoid, which is denoted by

$$\Pi\mathcal{A}.$$

The composition of two morphisms $[f,x] : a \to a'$ and $[g,y] : a' \to a''$ in $\Pi\mathcal{A}$ is defined as follows: By the filling condition, we can select a box θ in $\mathcal{A}$ whose horizontal target is y and whose vertical source is f. Thus, we have a diagram in $\mathcal{A}$ of the form

$$
\begin{array}{ccccc}
a'' & \xleftarrow{g} & \cdot & \xleftarrow{f'} & \cdot \\
 & & {\scriptstyle y}\uparrow & \theta & \uparrow{\scriptstyle y'} \\
 & & a' & \xleftarrow[f]{} & \cdot \\
 & & & & \uparrow{\scriptstyle x} \\
 & & & & a
\end{array}
\tag{4}
$$

and we define the composite

$$[g, y] \cdot [f, x] = [gf', y'x] : a \curvearrowright a''. \tag{5}$$

By [17, Lemma 2.2], this composition is well-defined, that is, it *does not depend on the representative paths or on the selection made of the box θ in* (4). By [17, Theorem 2.3], with this composition, $\Pi\mathcal{A}$ is actually a groupoid. The *identity* of an object a is the morphism represented by the identity loop at a,

$$[\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a] : a \to a.$$

The *inverse* in $\Pi\mathcal{A}$ of a morphism $[f, x] : a \to a'$ represented by a path $(f, x) : a \curvearrowright a'$ is the morphism

$$[f, x]^{-1} = [f', x'] : a' \to a$$

represented by the path $(f', x') : a' \curvearrowright a$ defined by the vertical target and the horizontal source of a (any) box γ in $\mathcal{A}$ whose horizontal target is x^{-1} and whose vertical source is f^{-1}, that is, of the form

$$\begin{array}{ccc} a & \xleftarrow{f'} & \cdot \\ {\scriptstyle x^{-1}}\uparrow & \gamma & \uparrow{\scriptstyle x'} \\ \cdot & \xleftarrow[f^{-1}]{} & a' \end{array}$$

The set $\pi_0\mathcal{A}$ [19, §3,1], *of path-connected classes of objects* of a double groupoid $\mathcal{A}$, is

$$\pi_0\mathcal{A} = \pi_0(\Pi\mathcal{A}),$$

the set of iso-classes of objects of its fundamental groupoid.

The group $\pi_1(\mathcal{A}, a)$ [19, §3,2], *of homotopy classes of loops in $\mathcal{A}$ based at a*, is

$$\pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a),$$

the group of automorphisms of a in the fundamental groupoid $\Pi\mathcal{A}$.

The abelian group

$$\pi_2(\mathcal{A}, a)$$

[19, §3,3] consists of all boxes of $\mathcal{A}$ whose horizontal source and target are both $\mathrm{I}^{\mathrm{v}}_a$, the vertical identity of a, and whose vertical source and target are both $\mathrm{I}^{\mathrm{h}}_a$, the horizontal identity of a; that is, of the form

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array}$$

By the Eckman-Hilton argument, the interchange law on $\mathcal{A}$ implies that the operations $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ on $\pi_2(\mathcal{A}, a)$ coincide and are commutative. Thus, $\pi_2(\mathcal{A}, a)$ is an abelian group with addition

$$\sigma + \tau = \sigma \circ_{\mathrm{h}} \tau = \sigma \circ_{\mathrm{v}} \tau,$$

zero $0 = \mathrm{I}_a$, and opposites $-\sigma = \sigma^{-\mathrm{v}} = \sigma^{-\mathrm{h}}$.

Every double functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor between the fundamental groupoids

$$F_* : \Pi\mathcal{A} \to \Pi\mathcal{B},$$

which carries a morphism $[f, x] : a \to a'$ to the morphism

$$F_*[f, x] = [Ff, Fx] : Fa \to Fa'.$$

Hence, for every object a of $\mathcal{A}$, F induces a pointed map

$$F_* : \pi_0(\mathcal{A}, [a]) \to \pi_0(\mathcal{B}, [Fa])$$

and a homomorphism of groups

$$F_* : \pi_1(\mathcal{A}, a) \to \pi_1(\mathcal{B}, Fa).$$

Clearly, there is also an induced homomorphism

$$F_* : \pi_2(\mathcal{A}, a) \to \pi_2(\mathcal{B}, Fa)$$

given by

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array} \quad \mapsto \quad \begin{array}{ccc} Fa & = & Fa \\ \| & F\sigma & \| \\ Fa & = & Fa \end{array}$$

4. The Mayer-Vietoris sequence.

Throughout this section, we consider a pullback square of double groupoids

$$\begin{array}{ccc} \mathcal{B}'\times_{\mathcal{B}}\mathcal{A} & \xrightarrow{G'} & \mathcal{A} \\ {\scriptstyle F'}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathcal{B}' & \xrightarrow{G} & \mathcal{B} \end{array} \tag{6}$$

where F is a fibration and both $\mathcal{B}$ and $\mathcal{B}'$ have the filling property. By Proposition 2.5 F' is a fibration and, by Proposition 2.4, both $\mathcal{A}$ and $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ have the filling property.

Moreover, we fix an object (b',a) of $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$, so that b' is an object of $\mathcal{B}'$ a is an object of $\mathcal{A}$ such that $Gb' = Fa$, and let $b = Fa$.

Theorem 4.1 (Mayer-Vietoris sequence)**.** *There is an exact sequence of homomorphisms of groups and pointed maps*

$$0 \to \pi_2\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big) \xrightarrow{(F'_*,G'_*)} \pi_2(\mathcal{B}',b')\times\pi_2(\mathcal{A},a) \xrightarrow{-G_*+F_*} \pi_2(\mathcal{B},b)$$

$$\xrightarrow{\partial_2} \pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big) \xrightarrow{(F'_*,G'_*)} \pi_1(\mathcal{B}',b')\times\pi_1(\mathcal{A},a) \xrightarrow{G_*^{-1}\cdot F_*} \pi_1(\mathcal{B},b)$$

$$\xrightarrow{\partial_1} \pi_0\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},[b',a]\big) \xrightarrow{(F'_*,G'_*)} (\pi_0\mathcal{B}'\times_{\pi_0\mathcal{B}}\pi_0\mathcal{A},([b'],[a])) \to 1$$

Furthermore, $[\tilde{f}_1,\tilde{x}_1],[\tilde{f}_2,\tilde{x}_2]\in\pi_1(\mathcal{B},b)$ *satisfy* $\partial_1[\tilde{f}_1,\tilde{x}_2]=\partial_1[\tilde{f}_2,\tilde{f}_2]$ *if and only if there are* $[f',x']\in\pi_1(\mathcal{B}',b')$ *and* $[f,x]\in\pi_1(\mathcal{A},a)$ *such that*

$$[\tilde{f}_2,\tilde{x}_2] = G_*[f',x']^{-1}\cdot[\tilde{f}_1,\tilde{x}_1]\cdot F_*[f,x].$$

The meaning of the maps in the sequence is clarified in the proof provided in the following subsections 4.1, 4.2, and 4.3.

If the pullback square (6) is a pullback of groupoids, regarded as double groupoids where the vertical morphisms are all identities and the boxes are all vertical identities, then the Mayer-Vietoris sequence in Theorem 4.1 specializes to the Mayer-Vietoris sequence of Brown, Heath, and Kamps [10, Theorem 2.2] (see also [5, 10.7.6]).

4.1 The connecting homomorphism $\partial_2 : \pi_2(\mathcal{B}, b) \to \pi_1\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big)$.

Let $\beta \in \pi_2(\mathcal{B}, b)$. Since F is a fibration, the lifting problem

$$\begin{array}{ccc} a & \overset{}{\dashleftarrow} & \cdot \\ \| & \exists? & \vdots\!\uparrow \\ a & = & a \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} b & = & b \\ \| & \beta & \| \\ b & = & b \end{array}$$

has solution. Thus, we can choose a box α_β in $\mathcal{A}$ of the form

$$\begin{array}{ccc} a & \overset{f_\beta}{\longleftarrow} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array} \tag{7}$$

such that $F\alpha_\beta = \beta$. Since $Ff_\beta = \mathrm{I}^{\mathrm{h}}_b = G\mathrm{I}^{\mathrm{h}}_{b'}$ and $Fx_\beta = \mathrm{I}^{\mathrm{v}}_b = G\mathrm{I}^{\mathrm{v}}_{b'}$, we have that

$$\big((\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)\big) : (b', a) \curvearrowright (b', a) \tag{8}$$

is a loop in the double groupoid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.

Lemma 4.2. *If the choice of α_β in* (7) *is changed, then the loop* (8) *is changed to a homotopic loop in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.*

Proof. Suppose any other box in $\mathcal{A}$

$$\begin{array}{ccc} a & \overset{f'}{\longleftarrow} & \cdot \\ \| & \alpha' & \uparrow x' \\ a & = & a \end{array}$$

such that $F\alpha' = \beta$. We define a homotopy $\alpha : (f_\beta, x_\beta) \simeq (f', x')$ in $\mathcal{A}$ by

$$\begin{array}{ccc} \cdot & \overset{f_\beta^{-1} f'}{\longleftarrow} & \cdot \\ \| & \alpha & \uparrow x' x_\beta^{-1} \\ \cdot & = & \cdot \end{array} \quad = \quad \begin{array}{ccccc} \cdot & \overset{f_\beta^{-1}}{\longleftarrow} & a & \overset{f'}{\longleftarrow} & \cdot \\ x_\beta \uparrow & \alpha_\beta^{-\mathrm{h}} & \| & \alpha' & \uparrow x' \\ a & = & a & = & a \\ x_\beta^{-1} \uparrow & & \mathrm{I}^{\mathrm{h}} & & \uparrow x_\beta^{-1} \\ \cdot & & = & & \cdot \end{array}$$

Since

$$F\alpha = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta^{-h} & \| & \beta & \| \\ b & = & b & = & b \\ \| & & \mathrm{I}_b & & \| \\ b & & = & & b \end{array} = \mathrm{I}_b = G\mathrm{I}_{b'}$$

$(\mathrm{I}_{b'},\alpha):\big((\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)\big)\simeq\big((\mathrm{I}^{\mathrm{h}}_{b'},f'),(I^{\mathrm{v}}_{b'},x')\big)$ is a homotopy in the pull-back double groupoid $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$. □

Definition 4.3. *The map $\partial_2:\pi_2(\mathcal{B},b)\to\pi_1(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},a')$ is given, on every $\beta\in\pi_2(\mathcal{B},b)$, by*

$$\partial_2\beta=[(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)]:(b',a)\to(b',a),$$

where $(f_\beta,x_\beta):a\curvearrowright a$ is the loop of $\mathcal{A}$ given by the vertical target f_β and the horizontal source x_β of a (any) box α_β, as in (7)*, such that $F\alpha_\beta=\beta$.*

Let us stress that, by Lemma 4.2, ∂_2 is a well-defined map.

Proposition 4.4. $\partial_2:\pi_2(\mathcal{B},b)\to\pi_1(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a))$ *is a homomorphism of groups.*

Proof. Suppose $\beta,\gamma\in\pi_2(\mathcal{B},b)$. Let

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow{\scriptstyle x_\beta} \\ a & = & a \end{array}\qquad\qquad \begin{array}{ccc} a & \xleftarrow{f_\gamma} & \cdot \\ \| & \alpha_\gamma & \uparrow{\scriptstyle x_\gamma} \\ a & = & a \end{array}$$

be boxes of $\mathcal{A}$ such that $F\alpha_\beta=\beta$ and $F\alpha_\gamma=\gamma$. Since F is a fibration, $Fx_\beta=\mathrm{I}^{\mathrm{v}}_b$, and $Ff_\gamma=\mathrm{I}^{\mathrm{h}}_b$, we can choose a box θ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{f'} & \cdot \\ {\scriptstyle x_\beta}\uparrow & \theta & \uparrow{\scriptstyle x'} \\ a & \xleftarrow[f_\gamma]{} & \cdot \end{array}$$

such that $F\theta=\mathrm{I}_b$. Hence $F\theta=G\mathrm{I}_{b'}$ and the diagram in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$

$$\begin{array}{ccccc} (b',a) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta)} & (b',\cdot) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f')} & (b',\cdot) \\ & & {\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)}\uparrow & (\mathrm{I}_{b'},\theta) & \uparrow{\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x')} \\ & & (b',a) & \xleftarrow[(\mathrm{I}^{\mathrm{h}}_{b'},f_\gamma)]{} & (b',\cdot) \\ & & & & \uparrow{\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x_\gamma)} \\ & & & & (b',a) \end{array}$$

tell us that, in the group $\pi_1(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a))$,

$$\begin{aligned}\partial_2\beta \cdot \partial_2\gamma &= [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] \cdot [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\gamma), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\gamma)] \\ &= [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta f'), (\mathrm{I}^{\mathrm{v}}_{b'}, x' x_\gamma)].\end{aligned}$$

Now, we have the box $\alpha_{\beta+\gamma}$ of $\mathcal{A}$ defined by

$$\begin{array}{ccc} a & \xleftarrow{f_\beta f'} & \cdot \\ \| & \alpha_{\beta+\gamma} & \uparrow x' x_\gamma \\ a & = & a \end{array} \quad = \quad \begin{array}{ccccc} a & \xleftarrow{f_\beta} & \cdot & \xleftarrow{f'} & \cdot \\ \| & \alpha_\beta & \uparrow & \theta & \uparrow x' \\ a & = & a & \xleftarrow{f_\gamma} & \cdot \\ \| & & \alpha_\gamma & & \uparrow x_\gamma \\ a & = & = & = & a \end{array}$$

which satisfies that

$$F(\alpha_{\beta+\gamma}) = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta & \| & \mathrm{I}_b & \| \\ b & = & b & = & b \\ \| & & \gamma & & \| \\ b & = & = & = & b \end{array} = \beta + \gamma.$$

Hence, by Lemma 4.2, $\partial_2(\beta + \gamma) = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta f'), (\mathrm{I}^{\mathrm{v}}_{b'}, x' x_\gamma)] = \partial_2\beta \cdot \partial_2\gamma$. $\square$

4.2 The connecting map $\partial_1 : \pi_1(\mathcal{B}, b) \to \pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, [b', a])$.

Let $(\tilde{f}, \tilde{x}) : b \curvearrowright b$ be a loop in $\mathcal{B}$ based at b

$$\begin{array}{ccc} b & \xleftarrow{\tilde{f}} & \cdot \\ & & \uparrow \tilde{x} \\ & & b \end{array}$$

By Proposition 3.1, we can choose a path $(f, x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$ in $\mathcal{A}$

$$\begin{array}{ccc} a_{\tilde{f},\tilde{x}} & \xleftarrow{f} & \cdot \\ & & \uparrow x \\ & & a \end{array} \qquad (9)$$

such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$. Since $Fa_{\tilde{f},\tilde{x}} = b$, the pair $(b', a_{\tilde{f},\tilde{x}})$ is an object of the pullback double groupoid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.

Lemma 4.5. (i) *If the choice of* (f, x) *in* (9) *is changed, then* $(b', a_{\tilde{f},\tilde{x}})$ *is changed to a path-connected object in* $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.

(ii) *If* $(\tilde{g}, \tilde{y}) : b \curvearrowright b$ *is a loop in* $\mathcal{B}$ *homotopic to* $(\tilde{f}, \tilde{x})$*, then a suitable selection of the lifting path of* $(\tilde{g}, \tilde{y})$ *leaves the object* $a_{\tilde{f},\tilde{x}}$ *unaltered.*

Proof. (i) Suppose $(g, y) : a \curvearrowright a_1$ other path in $\mathcal{A}$ such that $(Fg, Fy) = (\tilde{f}, \tilde{x})$. Since $Fy = \tilde{x} = Fx$ and F is a fibration, we can choose a box α in $\mathcal{A}$ as in the diagram

$$\begin{array}{ccccc} a_1 & \xleftarrow{\;g\;} & \cdot & \xleftarrow{\;h\;} & \cdot \\ & & {\scriptstyle y}\uparrow & & \uparrow \\ & & a & \alpha & \big|\, {\scriptstyle z} \\ & & {\scriptstyle x^{-1}}\uparrow & & \big| \\ & & \cdot & \xleftarrow[f^{-1}]{} & a_{\tilde{f},\tilde{x}} \end{array}$$

such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{\tilde{f}^{-1}}$. Since $F(gh) = Fg\,Fh = \tilde{f}\,\tilde{f}^{-1} = \mathrm{I}^{\mathrm{h}}_b = G(\mathrm{I}^{\mathrm{h}}_{b'})$ and $Fz = \mathrm{I}^{\mathrm{v}}_b = G(\mathrm{I}^{\mathrm{v}}_{b'})$, the path $\big((\mathrm{I}^{\mathrm{h}}_{b'}, gh), (\mathrm{I}^{\mathrm{v}}_{b'}, z)\big) : (b', a_{\tilde{f},\tilde{x}}) \curvearrowright (b', a_1)$ belongs to the pullback double groupoid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$, so that $[b', a_{f,x}] = [b', a_1]$ in $\pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A})$.

(ii) If $(\tilde{g}, \tilde{y}) : b \curvearrowright b$ is a loop homotopic to $(\tilde{f}, \tilde{x})$ in $\mathcal{B}$, by Proposition 3.2, there is a path $(g, y) : a \curvearrowright a_{\tilde{f},\tilde{x}}$, homotopic to (f, x), such that $(Fg, Fy) = (\tilde{g}, \tilde{y})$. Choosing this lifting path, we have $a_{\tilde{g},\tilde{y}} = a_{\tilde{f},\tilde{x}}$. □

Definition 4.6. *The map* $\partial_1 : \pi_1(\mathcal{B}, b) \to \pi_0\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, [b', a]\big)$ *is given, for every loop* $(\tilde{f}, \tilde{x}) : b \curvearrowright b$ *of* $\mathcal{B}$*, by*

$$\partial_1[\tilde{f}, \tilde{x}] = [b', a_{\tilde{f},\tilde{x}}]$$

where $a_{\tilde{f},\tilde{x}}$ *is the end of a (any) path* $(f, x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$ *in* $\mathcal{A}$*, as in* (9)*, such that* $(Ff, Fx) = (\tilde{f}, \tilde{x})$.

Remark that, by Lemma 4.2, ∂_1 is a well-defined map. Moreover, since $(F\mathrm{I}^{\mathrm{h}}_a, F\mathrm{I}^{\mathrm{v}}_a) = (\mathrm{I}^{\mathrm{h}}_b, \mathrm{I}^{\mathrm{v}}_b)$, we have $\partial_1[\mathrm{I}^{\mathrm{h}}_b, \mathrm{I}^{\mathrm{v}}_b] = [b', a]$, that is, ∂_1 is a pointed map.

4.3 The exactness of the Mayer-Vietoris sequence.

Proposition 4.7. *The sequence of group homomorphisms below is exact.*

$$0 \to \pi_2\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big) \xrightarrow{(F'_*, G'_*)} \pi_2(\mathcal{B}', b') \times \pi_2(\mathcal{A}, a) \xrightarrow{F_* - G_*} \pi_2(\mathcal{B}, b)$$

Proof. Exactness of the sequence above means that the homomorphisms F'_* and G'_* induce an isomorphism

$$\pi_2\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)\cong\pi_2(\mathcal{A},a)\times_{\pi_2(\mathcal{B},b)}\pi_2(\mathcal{B}',b'),$$

which follows directly from the definition of π_2. □

Proposition 4.8. *The following sequence of group homomorphisms is exact.*

$$\pi_2(\mathcal{B}',b')\times\pi_2(\mathcal{A},a)\xrightarrow{F_*-G_*}\pi_2(\mathcal{B},b)\xrightarrow{\partial_2}\pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)$$

Proof. If σ is an element of the group $\pi_2(\mathcal{A},a)$, then we can choose the box $\alpha_{F\sigma}=\sigma$ in (7). Hence, $\partial_2(F\sigma)=[(\mathrm{I}^{\mathrm{h}}_{b'},\mathrm{I}^{\mathrm{h}}_{a}),(\mathrm{I}^{\mathrm{h}}_{b'},\mathrm{I}^{\mathrm{h}}_{a})]=[\mathrm{I}^{\mathrm{h}}_{(b',a)},\mathrm{I}^{\mathrm{v}}_{(b',a)}]$. So $\mathrm{Im}F_*\subseteq\mathrm{Ker}\partial_2$. Let β' be an element of the group $\pi_2(\mathcal{B}',b')$. For any chosen box in $\mathcal{A}$ as in (7)

$$\begin{array}{ccc} a & \overset{f}{\longleftarrow} & \cdot \\ \| & \alpha & \uparrow x \\ a & = & a \end{array}$$

such that $F\alpha=G\beta'$, the box of $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$

$$\begin{array}{ccc} (b',a) & \overset{(\mathrm{I}^{\mathrm{h}}_{b'},f)}{\longleftarrow} & (b',\cdot) \\ \| & (\beta',\alpha) & \uparrow (\mathrm{I}^{\mathrm{v}}_{b'},x) \\ (b',a) & = & (b',a) \end{array}$$

is a homotopy $(\beta',\alpha):(\mathrm{I}^{\mathrm{h}}_{(b',a)},\mathrm{I}^{\mathrm{v}}_{(b',a)})\simeq\big((\mathrm{I}^{\mathrm{h}}_{b'},f),(\mathrm{I}^{\mathrm{v}}_{b'},x)\big)$. Hence,

$$\partial_2(G\beta')=[(\mathrm{I}^{\mathrm{h}}_{b'},f),(\mathrm{I}^{\mathrm{v}}_{b'},x)]=[\mathrm{I}^{\mathrm{h}}_{(b',a)},\mathrm{I}^{\mathrm{v}}_{(b',a)}].$$

So $\mathrm{Im}G_*\subseteq\mathrm{Ker}\partial_2$.

We now prove $\mathrm{Ker}\partial_2\subseteq\mathrm{Im}F_*+\mathrm{Im}G_*$: Suppose $\beta\in\pi_2(\mathcal{B},b)$ such that $\partial_2\beta=[\mathrm{I}^{\mathrm{h}}_{(b',a)},\mathrm{I}^{\mathrm{v}}_{(b',a)}]$. As in (7), let

$$\begin{array}{ccc} a & \overset{f_\beta}{\longleftarrow} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array}$$

be a box of $\mathcal{A}$ such that $F\alpha_\beta = \beta$; so that $\partial_2\beta = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)]$. There is then a homotopy $(\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}) \simeq \big((\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)\big)$; that is, a box in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ of the form

$$\begin{array}{ccc} (b',a) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta)} & (b',\cdot) \\ \| & (\beta',\alpha) & \uparrow{\scriptstyle(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)} \\ (b',a) & = & (b',a) \end{array}$$

for some boxes β' of $\mathcal{B}'$ and α of $\mathcal{A}$ of the form

$$\begin{array}{ccc} b' & = & b' \\ \| & \beta' & \| \\ b' & = & b' \end{array} \qquad \begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha & \uparrow{\scriptstyle x_\beta} \\ a & = & a \end{array}$$

satisfying that $G\beta' = F\alpha$. Define $\sigma = \alpha_\beta \circ_{\mathrm{h}} \alpha^{-\mathrm{h}} \in \pi_2(\mathcal{A}, a)$

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array} \quad = \quad \begin{array}{ccccc} a & \xleftarrow{f_\beta} & \cdot & \xleftarrow{f_\beta^{-1}} & a \\ \| & \alpha_\beta & \uparrow & \alpha^{-\mathrm{h}} & \| \\ a & = & a & = & a \end{array}$$

Then $F\sigma = F\alpha_\beta - F\alpha = \beta - G\beta'$; so that $\beta = F_*(\sigma) + G_*(\beta') \in \mathrm{Im}F_* + \mathrm{Im}G_*$. □

Proposition 4.9. *The sequence of group homomorphisms below is exact.*

$$\pi_2(\mathcal{B}, b) \xrightarrow{\partial_2} \pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}, (b', a)\big) \xrightarrow{(F'_*,G'_*)} \pi_1(\mathcal{B}', b')\times\pi_1(\mathcal{A}, a)$$

Proof. For every $\beta \in \pi_2(\mathcal{B}, b)$, the box α_β in (7) is actually a homotopy in $\mathcal{A}$, $\alpha_\beta : (\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a) \simeq (f_\beta, x_\beta)$. Hence,

$$\begin{aligned} G'_*(\partial_2\beta) &= G'_*[(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] = [f_\beta, x_\beta] = [\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a], \\ F'_*(\partial_2\beta) &= F'_*[(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] = [\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{v}}_{b'}]. \end{aligned}$$

So $\mathrm{Im}\partial_2 \subseteq \mathrm{Ker}\, F'_* \cap \mathrm{Ker}\, G'_*$. For the opposite inclusion, let

$$\big((f', f), (x', x)\big) : (b', a) \curvearrowright (b', a)$$

be a loop in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ such that $[f', x'] = [\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{v}}_{b'}]$ in $\pi_1(\mathcal{B}', b')$ and $[f, x] = [\mathrm{I}^{\mathrm{h}}_{a}, \mathrm{I}^{\mathrm{v}}_{a}]$ in $\pi_1(\mathcal{A}, a)$. Choose homotopies $\beta' : (\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{v}}_{b'}) \simeq (f', x')$ in $\mathcal{B}'$ and $\alpha : (\mathrm{I}^{\mathrm{h}}_{a}, \mathrm{I}^{\mathrm{v}}_{a}) \simeq (f, x)$ in $\mathcal{A}$; that is, boxes of the form

$$\begin{array}{ccc} b' & \xleftarrow{f'} & \cdot \\ \| & \beta' & \uparrow x' \\ b' & = & b' \end{array} \qquad \begin{array}{ccc} a & \xleftarrow{f} & \cdot \\ \| & \alpha & \uparrow x \\ a & = & a \end{array}$$

Since $Gf' = Ff$ and $Gx' = Fx$, the box $\beta = F\alpha \circ_{\mathrm{h}} G\beta'^{-\mathrm{h}}$

$$\begin{array}{ccc} b & = & b \\ \| & \beta & \| \\ b & = & b \end{array} \quad = \quad \begin{array}{ccccc} b & \xleftarrow{Ff} & \cdot & \xleftarrow{Gf'^{-1}} & b \\ \| & F\alpha & \uparrow & G\beta'^{-\mathrm{h}} & \| \\ b & = & b & = & b \end{array}$$

belongs to $\pi_2(\mathcal{B}, b)$. Let

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array}$$

be a box of $\mathcal{A}$ such that $F\alpha_\beta = \beta$, so that $\partial_2 \beta = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)]$. We can construct a homotopy $\alpha_1 : (f_\beta, x_\beta) \simeq (f, x)$ in $\mathcal{A}$ by

$$\begin{array}{ccc} \cdot & \xleftarrow{f_\beta^{-1} f} & \cdot \\ \| & \alpha_1 & \uparrow x x_\beta^{-1} \\ \cdot & = & \cdot \end{array} \quad = \quad \begin{array}{ccccc} \cdot & \xleftarrow{f_\beta^{-1}} & \cdot & \xleftarrow{f} & \cdot \\ x_\beta \uparrow & \alpha_\beta^{-\mathrm{h}} & \| & \alpha & \uparrow x \\ \cdot & = & \cdot & = & \cdot \\ x_\beta^{-1} \uparrow & & \mathrm{I}^{\mathrm{h}} & & \uparrow x_\beta^{-1} \\ \cdot & & = & & \cdot \end{array}$$

Since

$$F\alpha_1 = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta^{-\mathrm{h}} & \| & F\alpha & \| \\ b & = & b & = & b \\ \| & & \mathrm{I}_b & & \| \\ b & & = & & b \end{array} = G\beta' \circ_{\mathrm{h}} F\alpha^{-\mathrm{h}} \circ_{\mathrm{h}} F\alpha = G\beta',$$

the pair $(\beta', \alpha_1) : \big((\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)\big) \simeq \big((f', f), (x', x)\big)$ is a loop homotopy in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$. Thus, $[(f', f), (x', x)] = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] = \partial_2 \beta = \in \mathrm{Im}\partial_2$. $\square$

Proposition 4.10. *The sequence*

$$\pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)\xrightarrow{(F'_*,G'_*)}\pi_1(\mathcal{B}',b')\times\pi_1(\mathcal{A},a)\xrightarrow{G_*^{-1}\cdot F_*}\pi_1(\mathcal{B},b),$$

where (F'_*,G'_*) is a homomorphism and $G_*^{-1}\cdot F_*$ is a pointed map, is exact.

Proof. Exactness of the sequence above means that the homomorphisms F'_* and G'_* induce an epimorphism

$$\pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)\twoheadrightarrow\pi_1(\mathcal{A},a)\times_{\pi_1(\mathcal{B},b)}\pi_1(\mathcal{B}',b').$$

To prove this, let $(g,y):a\curvearrowright a$ and $(f',x'):b'\curvearrowright b'$ be loops, in $\mathcal{A}$ and $\mathcal{B}'$ respectively, such that $[Fg,Fy]=[Gf',Gx']$ in $\pi_1(\mathcal{B},b)$. By Proposition 3.2, there is a loop $(f,x):a\curvearrowright a$ in $\mathcal{A}$ such that $[f,x]=[g,y]$ and $(Ff,Fx)=(Gf',Gx')$. Then, $\big((f',f),(x',x)\big):(b',a)\curvearrowright(b',a)$ is a loop in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ and $F'_*[(f',f),(x',x)]=[f',x']$ and $G'_*[(f',f),(x',x)]=[f,x]=[g,y]$. □

Proposition 4.11. *The following sequence of pointed maps is exact.*

$$\pi_1(\mathcal{B}',b')\times\pi_1(\mathcal{A},a)\xrightarrow{G_*^{-1}\cdot F_*}\pi_1(\mathcal{B},b)\xrightarrow{\partial_1}\pi_0\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},[b',a]\big)$$

Further, $[\tilde{f}_1,\tilde{x}_1],[\tilde{f}_2,\tilde{x}_2]\in\pi_1(\mathcal{B},b)$ satisfy $\partial_1[\tilde{f}_1,\tilde{x}_2]=\partial_1[\tilde{f}_2,\tilde{f}_2]$ if and only if there are $[f',x']\in\pi_1(\mathcal{B}',b')$ and $[f,x]\in\pi_1(\mathcal{A},a)$ such that

$$[\tilde{f}_2,\tilde{x}_2]=G_*[f',x']^{-1}\cdot[\tilde{f}_1,\tilde{x}_1]\cdot F_*[f,x].\tag{10}$$

Proof. Given $(\tilde{f}_1,\tilde{x}_1),(\tilde{f}_2,\tilde{x}_2):b\curvearrowright b$ loops in $\mathcal{B}$, let us choose paths in $\mathcal{A}$ $(f_1,x_1):a\curvearrowright a_1$ and $(f_2,x_2):a\curvearrowright a_2$ such that $(Ff_1,Fx_1)=(\tilde{f}_1,\tilde{x}_1)$ and $(Ff_2,Fx_2)=(\tilde{f}_2,\tilde{x}_2)$; so that $\partial_1[\tilde{f}_1,\tilde{x}_1]=[b',a_1]$ and $\partial_1[\tilde{f}_2,\tilde{x}_2]=[b',a_2]$.

Suppose there are loops $(f',x'):b'\curvearrowright b'$ in $\mathcal{B}'$ and $(f,x):a\curvearrowright a$ in $\mathcal{A}$ such that (10) holds. Choose $(g,y):a_2\curvearrowright a_1$ a path in $\mathcal{A}$ representative of the composite morphism $[f_1,x_1]\cdot[f,x]\cdot[f_2,x_2]^{-1}:a_2\to a_1$ of $\Pi\mathcal{A}$. Since

$$\begin{aligned}[Gf',Gx']&=F_*[f_1,x_1]\cdot F_*[f,x]\cdot F_*[f_2,x_2]^{-1}\\&=F_*\big([f_1,x_1]\cdot[f,x]\cdot[f_2,x_2]^{-1}\big)=[Fg,Fy],\end{aligned}$$

by Proposition 3.2, there is a path $(g',y'):a_2\curvearrowright a_1$ which is homotopic to (g,y) and satisfies $(Fg',Fy')=(Gf',Gx')$. Then,

$$\big((f',g'),(x',y')\big):(b',a_2)\curvearrowright(b',a_1)\tag{11}$$

is a path in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ and therefore $[b', a_1] = [b', a_2]$.

Conversely, assume that $[b', a_1] = [b', a_2]$, so that there is a path in the pullback $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ as (11), for some loop $(f', x') : b' \curvearrowright b'$ in $\mathcal{B}'$ and some path $(g', y') : a_2 \curvearrowright a_1$ in $\mathcal{A}$ such that $(Gf', Gx') = (Fg', Fy')$. The composite morphism in $\Pi\mathcal{A}$

$$[f, x] = [f_1, x_1]^{-1} \cdot [g', y'] \cdot [f_2, x_2] : a \to a$$

is then an element of $\pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a)$ and

$$\begin{aligned}[\tilde{f}_2, \tilde{x}_2] &= F_*[f_2, x_2] = F_*[g'.y']^{-1} \cdot F_*[f_1, x_1] \cdot F_*[f, x] \\ &= G_*[f', x']^{-1} \cdot [\tilde{f}_1, \tilde{x}_1] \cdot F_*[f, x].\end{aligned}$$

□

Proposition 4.12. *The following sequence of pointed maps is exact.*

$$\pi_1(\mathcal{B}, b) \xrightarrow{\partial_1} \pi_0\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, [b', a]\big) \xrightarrow{(F'_*, G'_*)} \big(\pi_0\mathcal{B}' \times_{\pi_0\mathcal{B}} \pi_0\mathcal{A}, ([b'], [a])\big)$$

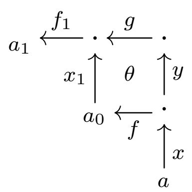
Proof. For every loop $(\tilde{f}, \tilde{x}) : b \curvearrowright b$ in $\mathcal{B}$, the path $(f, x) : a \to a_{\tilde{f},\tilde{x}}$ in (9) tell us that the objects a and $a_{\tilde{f},\tilde{x}}$ are path connected in $\mathcal{A}$. Hence

$$(F'_*, G'_*)\partial_1[\tilde{f}, \tilde{x}] = (F'_*, G'_*)[b', a_{\tilde{f},\tilde{x}}] = ([b'], [a_{\tilde{f},\tilde{x}}]) = ([b'], [a]).$$

So $\mathrm{Im}\partial_1 \subseteq (F'_*, G'_*)^{-1}([b'], [a])$. For the opposite inclusion, let (b'_0, a_0) be an object of $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ such that $[b'_0] = [b']$ in $\pi_0\mathcal{B}'$ and $[a_0] = [a]$ in $\pi_0\mathcal{A}$. Choose paths $(f', x') : b'_0 \curvearrowright b'$ in $\mathcal{B}'$ and $(f, x) : a \curvearrowright a_0$ in $\mathcal{A}$. Since $Gb'_0 = Fa_0$ and F is a fibration, we can select a path $(f_1, x_1) : a_0 \curvearrowright a_1$ such that

$$(Ff_1, Fx_1) = (Gf', Gx') : Fa_0 \curvearrowright b.$$

Further, because of the filling property, we can choose now a box θ in $\mathcal{A}$ as in the diagram



This way, we find the path $(f_1 g, yx) : a \curvearrowright a_1$ in $\mathcal{A}$ which is a lifting of the loop $(\tilde{f}, \tilde{x}) = (F(f_1 g), F(yx)) : b \curvearrowright b$ in $\mathcal{B}$. Hence $\partial_1[\tilde{f}, \tilde{x}] = [b', a_1]$. As

$$\big((f', f_1), (x', x_1)\big) : (b'_0, a_0) \curvearrowright (b', a_1)$$

is a path in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$, $[b'_0, a_0] = [b', a_1]$ in $\pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A})$, and we finally conclude that $\partial_1[\tilde{f}, \tilde{x}] = [b'_0, a_0]$. Thus, $[b'_0, a_0] \in \mathrm{Im}\partial_1$, as claimed. □

Proposition 4.13. *The map* $(F'_*, G'_*) : \pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}) \to \pi_0\mathcal{B}' \times_{\pi_0\mathcal{B}} \pi_0\mathcal{A}$ *is surjective.*

Proof. Suppose objects a_0 of $\mathcal{A}$ and b'_0 of $\mathcal{B}'$ such that $[Fa_0] = [Gb'_0]$ in $\pi_0\mathcal{B}$. Then, we can choose is a loop $(\tilde{f}, \tilde{x}) : Fa_0 \curvearrowright Gb'_0$ in $\mathcal{B}$ and, by Proposition 3.1, we can also choose a loop $(f, x) : a_0 \curvearrowright a_1$ in $\mathcal{A}$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$. Then, since $Fa_1 = Gb'_0$, the pair (b'_0, a_1) is an object of $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ and $F'_*[b'_0, a_1] = [b'_0]$, $G'_*[b'_0, a_1] = [a_1] = [a_0]$. □

5. The homotopy sequence.

Throughout this section, we consider a given fibration of double groupoids $F : \mathcal{A} \to \mathcal{B}$, assuming that $\mathcal{B}$ has the filling property. For an object a in $\mathcal{A}$, let $b = Fa$ and $\mathcal{F}_b = F^{-1}(b)$ be the corresponding double groupoid fibre over b. This setup leads to the following fibre sequence of pointed double groupoids, where Propositions 2.4 and 2.6 ensure that both $\mathcal{A}$ and $\mathcal{F}_b$ have the filling property:

$$(\mathcal{F}_b, a) \overset{i}{\hookrightarrow} (\mathcal{A}, a) \overset{F}{\longrightarrow} (\mathcal{B}, b) \tag{12}$$

Theorem 5.1. *The fibre sequence* (12) *gives rise to an exact sequence (of groups and pointed sets)*

$$\begin{array}{l} 0 \longrightarrow \pi_2(\mathcal{F}_b, a) \xrightarrow{i_*} \pi_2(\mathcal{A}, a) \xrightarrow{F_*} \pi_2(\mathcal{B}, b) \\ \xrightarrow{\partial_2} \pi_1(\mathcal{F}_b, a) \xrightarrow{i_*} \pi_1(\mathcal{A}, a) \xrightarrow{F_*} \pi_1(\mathcal{B}, b) \\ \xrightarrow{\partial_1} \pi_0(\mathcal{F}_b, [a]) \xrightarrow{i_*} \pi_0(\mathcal{A}, [a]) \xrightarrow{F_*} \pi_0(\mathcal{B}, [b]). \end{array} \tag{13}$$

Furthermore, $[\tilde{f}_1, \tilde{x}_1], [\tilde{f}_2, \tilde{x}_2] \in \pi_1(\mathcal{B}, b)$ *satisfy* $\partial_1[\tilde{f}_1, \tilde{x}_2] = \partial_1[\tilde{f}_2, \tilde{f}_2]$ *if and only if there is an* $[f, x] \in \pi_1(\mathcal{A}, a)$ *such that*

$$[\tilde{f}_2, \tilde{x}_2] = [\tilde{f}_1, \tilde{x}_1] \cdot F_*[f, x].$$

Proof. This result follows from the Mayer-Vietoris sequence stated in Theorem 4.1 above, because $\mathcal{F}_b$ appears in the pullback square of double groupoids depicted below. □

$$\begin{array}{ccc} \mathcal{F}_b \cong * \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{\ i\ } & \mathcal{A} \\ \big\downarrow & & \big\downarrow F \\ * & \xrightarrow{\ b\ } & \mathcal{B} \end{array}$$

If $(\mathcal{F}_b, a) \hookrightarrow (\mathcal{A}, a) \xrightarrow{F} (\mathcal{B}, b)$ is a pointed Moerdijk fibration of 2-groupoids, regarded as double groupoids whose vertical arrows are all identities, then the associated 9-term exact sequence (13) yields the exact sequence constructed by Hardie, Kamps, and Kieboom in [27]. In particular, if $F : \mathcal{A} \to \mathcal{B}$ is a Grothendieck fibration of groupoids, viewed as double groupoids whose vertical morphisms are all identities and whose boxes are all vertical identities, then (13) specializes to the 6-term exact sequence due to Brown [2, Theorem 4.3], [5, 7.2.9].

The following proposition provides further relevant information about the 9-term sequence.

Proposition 5.2. *(i) There is a group action of the group* $\pi_1(\mathcal{A}, a)$ *on the group* $\pi_1(\mathcal{F}_b, a)$ *making the homomorphism* $i_* : \pi_1(\mathcal{F}_b, a) \to \pi_1(\mathcal{A}, a)$ *into a crossed module.*

(ii) There is a canonical action of the group $\pi_1(B, b)$ *on the set* $\pi_0\mathcal{F}_b$ *such that the boundary* ∂_1 *is given by* $\partial_1[\tilde{f}, \tilde{x}] = {}^{[\tilde{f}, \tilde{x}]}[a]$.

(iii) $[a], [a'] \in \pi_0\mathcal{F}_b$ *satisfy* $i_*[a] = i_*[a']$ *if and only if* $[a'] = {}^{[\tilde{f}, \tilde{x}]}[a]$*, for some* $[\tilde{f}, \tilde{x}] \in \pi_1(\mathcal{B}, b)$.

Proof. In the following subsections, these issues are addressed in a more general setting, as detailed in Theorems 5.7 and 5.9, and Proposition 5.10 below. □

If $F : \mathcal{A} \to \mathcal{B}$ is a fibration of 2-groupoids, viewed as double groupoids whose vertical morphisms are all identities, and we consider the equivalence between 2-groupoids and crossed modules over groupoids as established by Brown and Higgins [11], then Proposition 5.2 leads to the analogous statements for fibrations of crossed modules over groupoids, as demostrated by Howie [29, Theorem 3.1] and Brown [6, Theorem 2.11].

5.1 The fundamental crossed module $\pi_1\mathcal{F} \to \pi_1\mathcal{A}$.

We begin by fixing some notations concerning crossed modules over groupoids. If P is a groupoid, a (left) P-*group* is a functor from P to the category Gr of groups. For every P-group $H : \mathsf{P} \to \mathsf{Gr}$, each morphism $\phi : a \to b$ in P, and each $h \in H(a)$, we denote by ${}^{\phi}h$ the value of $H(\phi)$ at h and call it *the action of ϕ on h*. Thus, a P-group H provides groups $H(a)$, one for each $a \in \mathrm{Ob}\mathsf{P}$, and action homomorphisms

$$H(a) \to H(b), \quad h \mapsto {}^{\phi}h,$$

one for each morphism $\phi : a \to b$ in P, satisfying ${}^{\psi}({}^{\phi}h) = {}^{\psi\phi}h$ and ${}^{1}h = h$. For instance, let

$$\pi_1\mathsf{P} : \mathsf{P} \to \mathsf{Gr}, \quad a \mapsto \pi_1(\mathsf{P}, a) = \mathrm{Aut}_{\mathsf{P}}(a), \tag{14}$$

denote the P-group that attaches to each object a the group of its automorphisms in P. The action of a morphism $\phi : a \to b$ on an automorphism $\psi : a \to a$ is given by conjugation in P, that is, ${}^{\phi}\psi = \phi\,\psi\phi^{-1}$. If $\mathsf{P} = G$ is a group regarded as a groupoid with only one object, then $\pi_1 G = G$ with the action on itself by conjugation.

A *morphism of* P-*groups* $\mu : H \to H'$ is a natural transformation, so it consists of homomorphisms $\mu = \mu_a : H(a) \to H'(a)$, one for each object a of P, such that, for every $\phi : a \to b$ in P, $\mu({}^{\phi}h) = {}^{\phi}\mu(h)$.

A P-*crossed module* (or *crossed module of groupoids over* P) is a morphism of P-groups

$$H \xrightarrow{\mu} \pi_1\mathsf{P}$$

such that, for every $h, h' \in H(a)$, $a \in \mathrm{Ob}\mathsf{P}$, the equation below holds:

$$^{\mu(h)}h'\,h = h\,h'. \tag{15}$$

Thus, for every object a of P, $H(a) \xrightarrow{\mu} \pi_1(\mathsf{P}, a)$ is a crossed module over the group of automorphism of a in P.

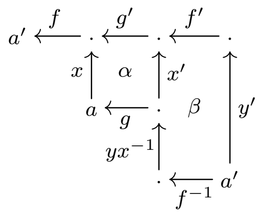
Returning to the fibration $F : \mathcal{A} \to \mathcal{B}$, for $\mathsf{P} = \Pi\mathcal{A}$, the fundamental groupoid of $\mathcal{A}$, we denote the $\Pi\mathcal{A}$-group (14), i.e. $\pi_1(\Pi\mathcal{A})$, simply by

$$\pi_1\mathcal{A} : \Pi\mathcal{A} \to \mathsf{Gr}, \quad a \mapsto \pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a). \tag{16}$$

The assignment $a \mapsto \pi_1(\mathcal{F}_{Fa}, a)$ defines the function on objects of a functor

$$\pi_1\mathcal{F} : \Pi\mathcal{A} \to \mathsf{Gr}$$

whose effect on morphisms is as follows: Suppose $[f, x] : a \to a'$ a morphism in $\Pi\mathcal{A}$, which is represented by a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$, and let $[g, y] \in \pi_1(\mathcal{F}_{Fa}, a)$, represented by a loop $(g, y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$. Since F is a fibration, we can choose boxes α and β in $\mathcal{A}$ as in the diagram



(17)

such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. Since $Fy' = \mathrm{I}^{\mathrm{v}}_{Fa'}$ and $F(f\,g'f') = \mathrm{I}^{\mathrm{h}}_{Fa'}$, the loop

$$(f\,g'f', y') : a' \curvearrowright a' \tag{18}$$

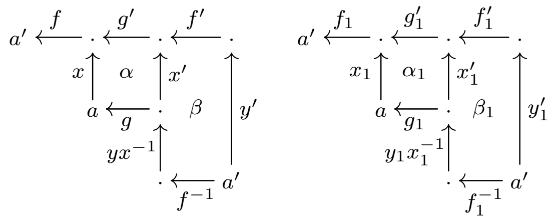
belongs to the double groupoid fibre $\mathcal{F}_{Fa'}$. We define the action of the morphism $[f, x] : a \to a'$ of $\Pi\mathcal{A}$ on $[g, y] \in \pi_1(\mathcal{F}_{Fa}, a)$ by

$$^{[f,x]}[g, y] = [f\,g'f', y'] \in \pi_1(\mathcal{F}_{Fa'}, a'). \tag{19}$$

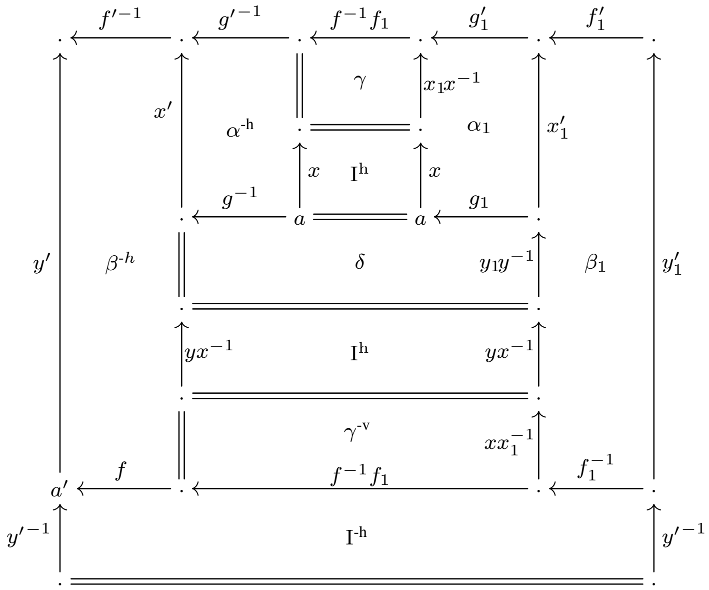
It follows from Lemmas 5.3, 5.4 and 5.5 below that this action is well defined and that $\pi_1\mathcal{F}$ is really a $\Pi\mathcal{A}$-group.

Lemma 5.3. $^{[f,x]}[g, y]$ *is independent of the choices of the representative path of* $[f, x]$ *in* $\mathcal{A}$*, of the representative loop of* $[g, y]$ *in* $\mathcal{F}_{Fa}$*, and of the boxes* α *and* β *in* (17).

Proof. Let $\gamma : (f, x) \simeq (f_1, x_1)$ be a homotopy of paths from a to a' in $\mathcal{A}$ and let $\delta : (g, x) \simeq (g_1, x_1)$ be a homotopy of loops at a in the double groupoid fibre $\mathcal{F}_{Fa}$. Suppose we have selected boxes α, β, α_1, and β_1, as in the diagrams



such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$, $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$, $F\alpha_1 = \mathrm{I}^{\mathrm{h}}_{Fx_1}$, and $F\beta_1 = \mathrm{I}^{\mathrm{v}}_{Ff_1^{-1}}$. Then, we get a homotopy of loops at a' in the double groupoid fibre $\mathcal{F}_{Fa'}$ from $(f\, g' f', y')$ to $(f_1\, g'_1 f'_1, y'_1)$ by pasting the diagram



□

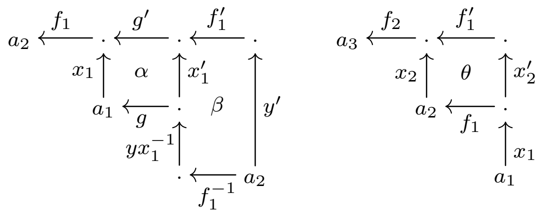
Lemma 5.4. *For every pair of paths in* $\mathcal{A}$

$$(f_1, x_1) : a_1 \curvearrowright a_2,\ (f_2, x_2) : a_2 \curvearrowright a_3$$

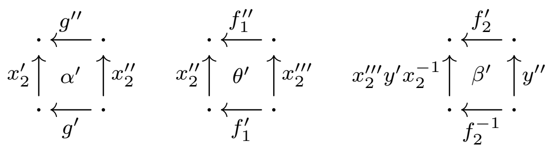
and every loop $(g, y) : a_1 \curvearrowright a_1$ *in the fibre* $\mathcal{F}_{Fa_1}$,

$$^{[f_2,x_2]}\big(^{[f_1,x_1]}[g,y]\big) = {}^{[f_2,x_2]\cdot[f_1,x_1]}[g,y].$$

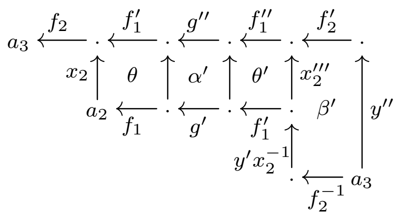
Proof. Let α, β, and θ be boxes of $\mathcal{A}$ as in the diagrams



such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx_1}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff_1^{-1}}$. Hence, $^{[f_1,x_1]}[g,y] = [f_1\, g' f_1', y']$ and $[f_2, x_2] \cdot [f_1, x_1] = [f_2 f_1', x_2' x_1]$. Since F is fibration, we can successively choose boxes α', θ', and β' in $\mathcal{A}$ of the form

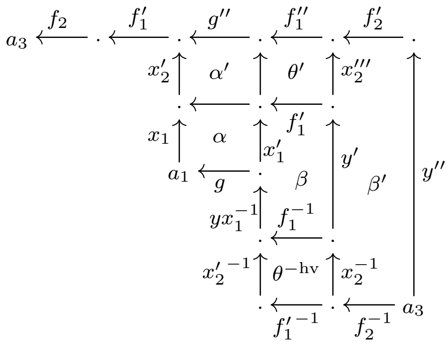


such that $F\alpha' = \mathrm{I}^{\mathrm{h}}_{Fx_2'}$, $F\theta' = \theta^{-\mathrm{h}}$, and $F\beta' = \mathrm{I}^{\mathrm{v}}_{Ff_2^{-1}}$. Then, the pasting diagram



tell us that $^{[f_2,x_2]}\big(^{[f_1,x_1]}[g,y]\big) = [f_2 f_1' g'' f_1'' f_2', y'']$, since $F(\theta \circ_{\mathrm{h}} \alpha' \circ_{\mathrm{h}} \theta') = \mathrm{I}^{\mathrm{h}}_{Fx_2}$

and $F\beta' = \mathrm{I}^{\mathrm{v}}_{Ff_2^{-1}}$, while the pasting diagram

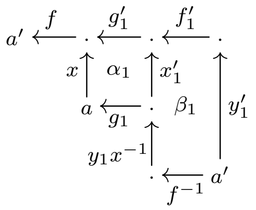


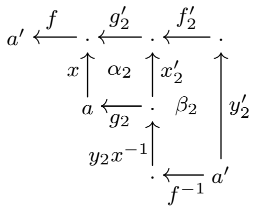
tell us that also ${}^{[f_2,x_2]\cdot[f_1,x_1]}[g,y] = [f_2f_1'g''f_1''f_2', y'']$, since $F(\alpha' \circ_{\mathrm{v}} \alpha) = \mathrm{I}^{\mathrm{v}}_{F(x_2'x_1)}$ and $F\big((\theta' \circ_{\mathrm{v}} \beta \circ_{\mathrm{v}} \theta^{-\mathrm{hv}}) \circ_{\mathrm{h}} \beta'\big) = \mathrm{I}^{\mathrm{v}}_{F(f_2f_1')^{-1}}$. □

Lemma 5.5. *For every pair of loops $(g_1, y_1), (g_2, y_2) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$ and every path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$,*

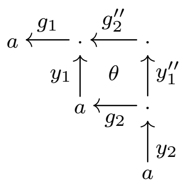
$$
{}^{[f,x]}\big([g_1, y_1] \cdot [g_2, y_2]\big) = {}^{[f,x]}[g_1, y_1] \cdot {}^{[f,x]}[g_2, y_2].
$$

Proof. Let α_1, β_1, α_2, and β_2 be boxes of $\mathcal{A}$ as in the diagrams





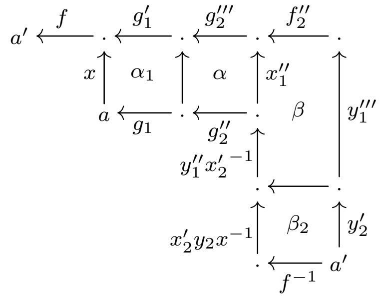
such that $F\alpha_1 = \mathrm{I}^{\mathrm{h}}_{Fx} = F\alpha_2$ and $F\beta_1 = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}} = F\beta_2$, so that ${}^{[f,x]}[g_1, y_1] = [fg_1'f_1', y_1']$ and ${}^{[f,x]}[g_2, y_2] = [fg_2'f_2', y_2']$, and let θ be a box in $\mathcal{F}_{Fa}$ as in the diagram



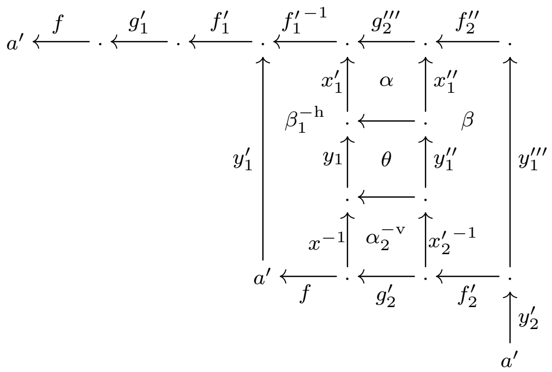
so that, $[g_1, y_1] \cdot [g_2, y_2] = [g_1 g_2'', y_1'' y_2]$ in the group $\pi_1(\mathcal{F}_{Fa}, a)$. Since F is fibration, we can successively choose boxes α and β in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{g_2'''} & \cdot \\ {\scriptstyle x_1'}\uparrow & \alpha & \uparrow{\scriptstyle x_1''} \\ \cdot & \xleftarrow[g_2'']{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{f_2''} & \cdot \\ {\scriptstyle x_1'' y_1'' x_2'^{-1}}\uparrow & \beta & \uparrow{\scriptstyle y_1'''} \\ \cdot & \xleftarrow[f_2']{} & \cdot \end{array}$$

such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. Then, on the one hand, the pasting diagram



tell us that ${}^{[f,x]}\big([g_1, y_1] \cdot [g_2, y_2]\big) = [f g_1' g_2''' f_2'', y_1''' y_2']$, since $F(\alpha_1 \circ_{\mathrm{h}} \alpha) = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F(\beta \circ_{\mathrm{v}} \beta_2) = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. On the other hand, the pasting diagram



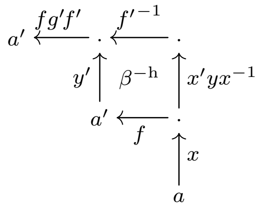
also tell us that ${}^{[f,x]}[g_1, y_1] \cdot {}^{[f,x]}[g_2, y_2] = [f g_1' g_2''' f_2'', y_1''' y_2']$, since the pasted box of the inner boxes belongs to $\mathcal{F}_{Fa'}$. $\square$

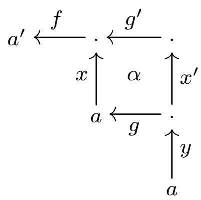
Proposition 5.6. *There is a morphism of* $\Pi\mathcal{A}$*-groups* $i_* : \pi_1\mathcal{F} \to \pi_1\mathcal{A}$ *which, at each object* $a \in \mathrm{Ob}\mathcal{A}$*, consists of the homomorphism induced by the inclusion* $i_* : \pi_1(\mathcal{F}_{Fa}, a) \to \pi_1(\mathcal{A}, a)$*.*

Proof. We must prove that, for every path $(f,x) : a \curvearrowright a'$ in $\mathcal{A}$ and every loop $(g,y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$, the equality

$$i_*\big({}^{[f,x]}[g,y]\big)\cdot[f,x] = [f,x]\cdot i_*[g,y]$$

holds in the fundamental groupoid $\Pi\mathcal{A}$. For, let us choose boxes α and β as in diagram (17) such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$; so that ${}^{[f,x]}[g,y] = [f\,g'f',y']$. We have the diagrams in $\mathcal{A}$





The first of them tell us that

$$i_*\big({}^{[f,x]}[g,y]\big)\cdot[f,x] = [fg'f'f'^{-1},x'yx^{-1}x] = [fg',y'x],$$

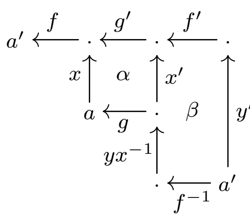
and the second one that also $[f,x]\cdot i_*[g,y] = [fg',x'y]$. $\square$

Theorem 5.7. *The morphism of* $\Pi\mathcal{A}$*-groups* $i_* : \pi_1\mathcal{F}\to\pi_1\mathcal{A}$ *is a crossed module over* $\Pi\mathcal{A}$*.*

Proof. We must prove that, for every pair of loops $(f,x),(g,y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$, the equality

$${}^{i_*[f,x]}[g,y]\cdot[f,x] = [f,x]\cdot[g,y]$$

holds in the fundamental group $\pi_1(\mathcal{F}_{Fa},a)$. To do that, since the double groupoid fibre $\mathcal{F}_{Fa}$ has the filling property, we can choose boxes α and β of $\mathcal{F}_{Fa}$ as in the diagram



Then ${}^{i_*[f,x]}[g,y] = [f\,g'f', y']$, and the diagrams in $\mathcal{F}_{Fa}$

$$
\begin{array}{ccccc}
a' & \xleftarrow{fg'f'} & \cdot & \xleftarrow{f'^{-1}} & \cdot \\
 & y' \uparrow & \beta^{-\mathrm{h}} & \uparrow x'yx^{-1} & \\
 & & a' & \xleftarrow[f]{} & \cdot \\
 & & & & \uparrow x \\
 & & & & a
\end{array}
\qquad\qquad
\begin{array}{ccccc}
a' & \xleftarrow{f} & \cdot & \xleftarrow{g'} & \cdot \\
 & & x \uparrow & \alpha & \uparrow x' \\
 & & a & \xleftarrow[g]{} & \cdot \\
 & & & & \uparrow y \\
 & & & & a
\end{array}
$$

tell us that ${}^{i_*[f,x]}[g,y] \cdot [f,x] = [fg', x'y] = [f,x] \cdot [g,y]$. □

5.2 The $\Pi\mathcal{B}$-set $\pi_0\mathcal{F}$.

If P is a groupoid, a (left) P-*set* is a functor from P to the category Set of sets. For every P-set $H : \mathsf{P} \to \mathsf{Set}$, each morphism $\phi : a \to b$ in P, and each $h \in H(a)$, we denote by ${}^{\phi}h$ the value of $H(\phi)$ at h and call it *the action of* ϕ *on* h. Thus, a P-set H provides of sets $H(a)$, one for each $a \in \mathrm{Ob}\mathsf{P}$, and action homomorphisms

$$H(a) \to H(b), \quad h \mapsto {}^{\phi}h,$$

one for each morphism $\phi : a \to b$ in P, satisfying ${}^{\psi}({}^{\phi}h) = {}^{\psi\phi}h$ and ${}^{1}h = h$.

Returning to the fibration $F : \mathcal{A} \to \mathcal{B}$, the assignment $b \mapsto \pi_0\mathcal{F}_b$ defines the function on objects of a functor from the fundamental groupoid of $\mathcal{B}$

$$\pi_0\mathcal{F} : \Pi\mathcal{B} \to \mathsf{Set}$$

whose effect on morphisms is described as follows: Suppose $[\tilde{f}, \tilde{x}] : b \to b'$ a morphism in the fundamental groupoid $\Pi\mathcal{B}$ of $\mathcal{B}$, represented by a path $(\tilde{f}, \tilde{x}) : b \curvearrowright b'$ in $\mathcal{B}$, and let $[a] \in \pi_0\mathcal{F}_b$, represented by an object a of $\mathcal{F}_b$. Since F is a fibration, we can choose a path in $\mathcal{A}$

$$(f, x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$$

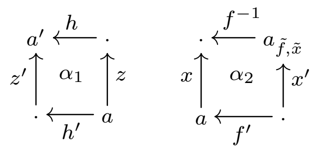
such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$. Then $Fa_{\tilde{f},\tilde{x}} = b'$, so that the object $a_{\tilde{f},\tilde{x}}$ belongs to fibre double groupoid $\mathcal{F}_{b'}$. We define the action of the morphism $[\tilde{f}, \tilde{x}] : b \to b'$ of $\Pi\mathcal{B}$ on $[a] \in \pi_0\mathcal{F}_b$ by

$${}^{[\tilde{f},\tilde{x}]}[a] = [a_{\tilde{f},\tilde{x}}] \in \pi_0\mathcal{F}_{b'}. \tag{20}$$

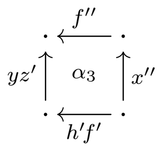
It follows from the lemma below that this action is well-defined.

Lemma 5.8. $[a_{\tilde f,\tilde x}]$ *is independent of the choices of the representative object* a *of* $[a]$ *in* $\mathcal{F}_b$*, of the representative path* $(\tilde f,\tilde x): b \curvearrowright b'$ *of* $[\tilde f,\tilde x]$ *in* $\mathcal{B}$*, and of its lifted path* $(f,x): a \curvearrowright a_{\tilde f,\tilde x}$ *in* $\mathcal{A}$*.*

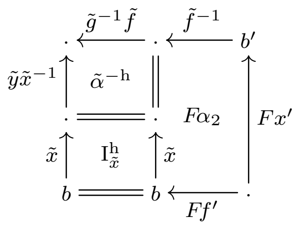
Proof. Let $(h,z): a \curvearrowright a'$ be a path in $\mathcal{F}_b$ and let $\tilde\alpha : (\tilde f,\tilde x) \simeq (\tilde g,\tilde y)$ be a homotopy of paths in $\mathcal{B}$ from b to b'. Suppose we have chosen paths $(f,x): a \curvearrowright a_{\tilde f,\tilde x}$ and $(g,y): a' \curvearrowright a'_{\tilde g,\tilde y}$ in $\mathcal{A}$ such that $(Ff,Fx) = (\tilde f,\tilde x)$ and $(Fg,Fy) = (\tilde g,\tilde y)$. We must prove that there is a path $a_{\tilde f,\tilde x} \curvearrowright a'_{\tilde g,\tilde y}$ in $\mathcal{F}_{b'}$. For, we can proceed as follows: By the filling property, let us choose boxes α_1 of $\mathcal{F}_b$ and $\alpha_2 \in \mathcal{A}$ of the form



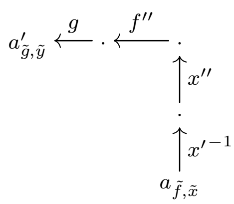
Then, since F is a fibration, we can select a box α_3 in $\mathcal{A}$ of the form



such that

$F\alpha_3 =$ 

This way, we have the path $(gf'', x''x'^{-1}) : a_{\tilde f,\tilde x} \curvearrowright a'_{\tilde g,\tilde y}$



which actually belongs to $\mathcal{F}_{b'}$, since $F(gf'') = \tilde{g}\,\tilde{g}^{-1} = \mathrm{I}^{\mathrm{h}}_{b'}$ and $F(x''x'^{-1}) = Fx'\,Fx'^{-1} = \mathrm{I}^{\mathrm{v}}_{b'}$. □

Theorem 5.9. *$\pi_0\mathcal{F}$ is a $\Pi\mathcal{B}$-set.*

Proof. We must prove that, for every pair of paths $(\tilde{f},\tilde{x}) : b \curvearrowright b'$ and $(\tilde{g},\tilde{y}) : b' \curvearrowright b''$ in $\mathcal{B}$ and every object a in $\mathcal{F}_b$,

$$^{[\tilde{g},\tilde{y}]}\big(^{[\tilde{f},\tilde{x}]}[a]\big) = {}^{[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}]}[a].$$

For, using that F is a fibration and $\mathcal{A}$ has the filling property, let us construct a diagram in $\mathcal{A}$ of the form

$$\begin{array}{ccccc} a'' & \xleftarrow{g} & \cdot & \xleftarrow{f'} & \cdot \\ & & {\scriptstyle y}\uparrow & \theta & \uparrow{\scriptstyle y'} \\ & & a' & \xleftarrow[f]{} & \cdot \\ & & & & \uparrow{\scriptstyle x} \\ & & & & a \end{array}$$

such that $(Ff,Fx) = (\tilde{f},\tilde{x})$ and $(Fg,Fy) = (\tilde{g},\tilde{y})$. Thus, on the one hand, $^{[\tilde{f},\tilde{x}]}[a] = [a']$ and $^{[\tilde{g},\tilde{y}]}\big(^{[\tilde{f},\tilde{x}]}[a]\big) = [a'']$. On the other hand, the induced diagram in $\mathcal{B}$

$$\begin{array}{ccccc} b'' & \xleftarrow{\tilde{g}} & \cdot & \xleftarrow{Ff'} & \cdot \\ & & {\scriptstyle \tilde{y}}\uparrow & F\theta & \uparrow{\scriptstyle Fy'} \\ & & b' & \xleftarrow[\tilde{f}]{} & \cdot \\ & & & & \uparrow{\scriptstyle \tilde{x}} \\ & & & & b \end{array}$$

tell us that, in the fundamental groupoid of $\mathcal{B}$, $[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}] = [\tilde{g}\,Ff', Fy'\,\tilde{x}] = [F(gf'), F(y'x)]$. So that $(gf', y'x) : a \curvearrowright a''$ is a lifting in $\mathcal{A}$ of a representative path in $\mathcal{B}$ of the composite morphism $[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}] : b \to b''$ of $\Pi\mathcal{B}$. Hence, $^{[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}]}[a] = [a'']$. □

Proposition 5.10. *Let b, b' be objects of $\mathcal{B}$ and let*

$$\pi_0\mathcal{F}_b \xrightarrow{i_*} \pi_0\mathcal{A} \xleftarrow{i'_*} \pi_0\mathcal{F}_{b'}$$

be the induced maps by the inclusions $i : \mathcal{F}_b \hookrightarrow \mathcal{A}$ and $i' : \mathcal{F}_{b'} \hookrightarrow \mathcal{A}$. Then, $[a] \in \pi_0\mathcal{F}_b$ and $[a'] \in \pi_0\mathcal{F}_{b'}$ satisfy $i_*[a] = i'_*[a']$ if and only if

$$[a'] = {}^{[\tilde{f},\tilde{x}]}[a]$$

for some morphism $[\tilde{f},\tilde{x}] : b \to b'$ of $\Pi\mathcal{B}$.

Proof. Suppose $i_*[a] = i'_*[a']$, so that there is a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$. Then, if $(\tilde{f}, \tilde{x}) = (Ff, Fx) : b \curvearrowright b'$, we have ${}^{[\tilde{f},\tilde{x}]}[a] = [a']$.

Conversely, suppose $(\tilde{f}, \tilde{x}) : b \curvearrowright b'$ is a path in $\mathcal{B}$ such that ${}^{[\tilde{f},\tilde{x}]}[a] = [a']$. If $(f, x) : a \curvearrowright a'_0$ is a path in A such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$, then we have $i_*[a] = i'_*[a'_0]$ in $\pi_0\mathcal{A}$ and also $[a'_0] = [a']$ in $\pi_0\mathcal{F}_{b'}$. Hence, $i_*[a] = i'_*[a']$. □

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