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CLASSIFICATION DIAGRAMS OF SIMPLICIAL CATEGORIES

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Résumé. Nous montrons que le diagramme de classification d'une ∞-catégorie relative issu d'une catégorie simpliciale relative est équivalent au nerf degré par degré. Les applications incluent la comparaison de la diagonale du nerf degré par degré et du nerf homotopique cohérent, ainsi qu'un résultat sur les localisations degré par degré des catégories simpliciales.
Abstract. We show that the classification diagram of a relative ∞-category arising from a relative simplicial category is equivalent to the levelwise nerve. Applications include the comparison of the diagonal of the levelwise nerve and the homotopy coherent nerve, and a result on the levelwise localizations of simplicial categories.
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1. Introduction

Given a category $\mathcal{C}$, the **nerve** of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are the set of n composable arrows $X_0 \to \cdots \to X_n$ in $\mathcal{C}$. It is immediate from the definitions that the association $\mathcal{C} \mapsto N(\mathcal{C})$ defines a fully faithful functor

$$N\colon \mathsf{Cat} \to \mathsf{sSet}$$

from the category of small categories to that of simplicial sets. As elementary as this observation may be, it puts forward an amazing fact: A structured object (a 1-category) can be presented as a family of less structured objects (sets, or 0-categories). Experience suggests that the latter is generally easier to work with, and this explains the prevalence of the nerve construction in homotopy theory.

It is natural to look for a higher-categorical analog of the nerve construction. For example, we may ask whether it is possible to present an $(\infty, 1)$-category as a simplicial $(\infty, 0)$-category, or a simplicial space. The answer is yes. Recall that ∞**-categories** (alias quasicateogries [12]) are a certain class of simplicial sets modeling $(\infty, 1)$-categories. Given an ∞-category $\mathcal{C}$, its **classifying diagram** $\mathrm{Cls}(\mathcal{C})$ is the simplicial space (or a bisimplicial set, to be more precise) whose nth space $\mathrm{Cls}(\mathcal{C})_n = \mathrm{Cls}(\mathcal{C})_{n,*}$ is the maximal sub Kan complex of $\mathrm{Fun}(\Delta^n, \mathcal{C})$. Joyal and Tierney showed in [14] that the association $\mathcal{C} \to \mathrm{Cls}(\mathcal{C})$ defines a fully faithful functor from the homotopy category of ∞-categories to that of simplicial spaces, thereby giving an analog of the nerve construction; the essential image of this functor consists of Rezk's **complete Segal spaces** [21].

A generalization of the classifying diagram construction has proved to be especially important in the theory of localizations of ∞-categories (in the sense of [15, Definition 2.4.2]). Given an ∞-category $\mathcal{C}$ and a subcategory $\mathcal{W} \subset \mathcal{C}$ containing all objects, their **classification diagram** $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ [21] is the simplicial ∞-category (hence a bisimplicial set) whose nth ∞-category is given by

$$\mathrm{Cls}(\mathcal{C}, \mathcal{W})_n = \mathrm{Cls}(\mathcal{C}, \mathcal{W})_{n,*} = \mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1}.$$

For example, if $\mathcal{W}$ is the subcategory of equivalences of $\mathcal{C}$, then $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ is nothing but the classifying diagram of $\mathcal{C}$. Inspired by earlier works such as [21, 6, 4], Mazel-Gee showed[1] in [20] (see also [1, 3] for alternative arguments and generalizations) that a functor $f\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories which carries $\mathcal{W}$ into $\mathcal{D}^{\simeq}$ is a localization with respect to $\mathcal{W}$ if and only if the map $\mathrm{Cls}(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}(\mathcal{D}, \mathcal{D}^{\simeq}) = \mathrm{Cls}(\mathcal{D})$ is a weak equivalence of the complete Segal space model structure [21, Theorem 7.2]. In other words, up to fibrant replacement, $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ computes the localization of $\mathcal{C}$. This is useful

[1]To be more precise, Mazel-Gee proved this in the case where $\mathcal{W}$ contains all equivalences (which suffices for most applications). The general case was proved by the author in [1].

because in some cases, the fibrant replacement of $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ have explicit descriptions; see, e.g., [16].

Now recall the **homotopy coherent nerve functor**

$$N_{\mathrm{hc}}\colon \mathsf{Cat}_\Delta \to \mathsf{sSet},$$

due to Cordier [7], is a right Quillen equivalence from Bergner's model structure on the category of simplicial categories [5] to Joyal's model structure on the category of simplicial sets [17, §2.2.5]. Many important ∞-categories arise as the homotopy coherent nerves of simplicial categories, so we frequently want to consider bisimplicial sets of the form $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W}))$, where $\mathcal{C}$ is a fibrant simplicial category and $\mathcal{W}$ is its simplicial subcategory such that $N_{\mathrm{hc}}(\mathcal{W})$ is a subcategory of $N_{\mathrm{hc}}(\mathcal{C})$. However, as is anything constructed from homotopy coherent nerves, this bisimplicial set is somewhat difficult to manipulate by hands. It will therefore be nice if there is an alternative, preferably simpler, presentation of this bisimplicial set. The goal of this paper is to show that this is possible, at least if we consider the *marked* version of classification diagrams.

A **marked bisimplicial set** is a pair (X, S) of a bisimplicial set X and a simplicial subset $S \subset X_1 = X_{1,*}$ which contains the image of the map $X_0 \to X_1$. Equivalently, it is a simplicial object $\{(X_{*,n}, S_n)\}_{n \geq 0}$ in the category of marked simplicial sets. One may argue that the natural place where classification diagrams live is not the category of bisimplicial sets, but the category of marked bisimplicial sets. Indeed, there is a very natural functor

$$\mathrm{Cls}^+\colon \mathsf{sSet}^+ \to \left(\mathsf{sSet}^+\right)^{\mathbf{\Delta}^{\mathrm{op}}} = \mathsf{bsSet}^+$$

from the category of marked simplicial sets to the category of marked bisimplicial sets, defined by $(X, S) \mapsto (X, S)^{(\Delta^\bullet)}$. (Here we wrote $(X, S)^{\Delta^n} = (X, S)^{(\Delta^n)^\sharp}$, where $(\Delta^n)^\sharp$ denotes the standard simplex Δ^n with all edges marked. It is the cotensor of (X, S) by the simplicial set Δ^n with respect to the simplicial enrichment $\mathrm{Map}^\sharp(-,-)$ of [17, §3.1.3].) Unwinding the definitions, we find that $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ is nothing but the underlying bisimplicial set of $\mathrm{Cls}^+(\mathcal{C}, \mathcal{W}_1)$.

In [1], the author showed that marked bisimplicial sets are to complete Segal spaces what marked simplicial sets are to ∞-categories. More precisely, [1, Theorems 2.9 and 3.4] state that bsSet^+ admits a model structure, denoted by $\mathsf{bsSet}^+_{\mathrm{CSS}}$, such that:

- The functor $\mathrm{Cls}^+\colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+$ is a right Quillen equivalence , where sSet^+ carries the model structure for marked simplicial sets [18, Proposition 3.1.3.7, Remark 3.1.4.6].[2]
- The forgetful functor $\mathsf{bsSet}^+_{\mathrm{CSS}} \to \mathsf{bsSet}_{\mathrm{CSS}}$ is also a right Quillen equivalence, where $\mathsf{bsSet}_{\mathrm{CSS}}$ denotes the category of bisimplicial sets equipped with the model structure for complete Segal spaces.

What we will do is to construct a relatively simple marked bisimplicial set which is weakly equivalent to $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W}))$ in $\mathsf{bsSet}^+_{\mathrm{CSS}}$. To give a precise statement of the main theorem, we must introduce some notation and terminology.

Definition 1.1. Let $\mathcal{C}$ be a simplicial category. A simplicial subcategory $\mathcal{W} \subset \mathcal{C}$ is said to be **wide** if it satisfies the following pair of conditions:

- $\mathcal{W}$ contains all objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, the simplicial subset $\mathcal{W}(X,Y) \subset \mathcal{C}(X,Y)$ is a union of components of $\mathcal{C}(X,Y)$.

The pair $(\mathcal{C}, \mathcal{W})$ of a simplicial category and its wide simplicial subcategory will be called a **relative simplicial category**.

Definition 1.2. Let $\mathcal{C}$ be a simplicial category. For each $n \geq 0$, we let $\mathcal{C}_n$ denote the ordinary category constructed from the objects of $\mathcal{C}$ and the n-simplices of the hom-simplicial sets of $\mathcal{C}$. The **binerve** (or the **levelwise nerve**) of $\mathcal{C}$ is the bisimplicial set $N_{\mathrm{bi}}(\mathcal{C})$ whose nth row $N_{\mathrm{bi}}(\mathcal{C})_{*,n}$ is the nerve of $\mathcal{C}_n$. Note that for each $n \geq 1$, the nth column $N_{\mathrm{bi}}(\mathcal{C})_{n,*}$ of $N_{\mathrm{bi}}(\mathcal{C})$ is the disjoint union

$$\coprod_{X_0,\dots,X_n \in \mathcal{C}} \mathcal{C}(X_{n-1}, X_n) \times \cdots \times \mathcal{C}(X_0, X_1).$$

If $\mathcal{W} \subset \mathcal{C}$ is a wide simplicial subcategory, we let $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W})$ denote the marked bisimplicial set $\left(N_{\mathrm{bi}}(\mathcal{C}), \coprod_{X,Y \in \mathcal{C}} \mathcal{W}(X,Y)\right)$.

[2] In [1], the functor Cls is denoted by N and the functor Cls^+ is denoted by $(t^+)^!$. We changed the notation in the hope that the paper will be more readable, for we will consider many variations of nerves in this paper.

We can now state the main result of this paper.

Theorem 1.3 (Theorem 3.1)**.** *Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. There is a weak equivalence*

$$N^{+}_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}^{+}(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$$

of $\mathsf{bsSet}^{+}_{\mathrm{CSS}}$ *which is natural in* $(\mathcal{C}, \mathcal{W})$*, where* $\mathrm{mor}\, \mathcal{W}_0$ *denotes the set of morphisms of* $\mathcal{W}_0$.

Notice how Theorem 1.3 simplifies the clunky marked bisimplicial set $\mathrm{Cls}^{+}(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$: If we want to directly work $\mathrm{Cls}^{+}(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$, we have to construct coherent higher homotopies governing the homotopy coherent nerve, which is often a hard labor. In contrast, the rows of $N_{\mathrm{bi}}(\mathcal{C})$ are nerves of *ordinary* categories, which has no higher structures.

Remark 1.4. The idea of relating the localization of a relative simplicial category $(\mathcal{C}, \mathcal{W})$ (which corresponds to $\mathrm{Cls}^{+}(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$) to those of $(\mathcal{C}_n, \mathcal{W}_n)$ (which correspond to the rows of $N^{+}_{\mathrm{bi}}(\mathcal{C}, \mathcal{W})$) is reminiscent of the work of Dwyer and Kan: In [8], Dwyer and Kan defined the *simplicial localizations* of relative simplicial categories by first defining them for relative categories, and then applying them levelwise to define them for all cases. We can therefore interpret Theorem 1.3 as another manifestation of Dwyer and Kan's principle that the localization of a relative simplicial category is the totality of the levelwise localization.

Remark 1.5. By the works of Joyal [13] and Joyal and Tierney [14], it has been known that $N_{\mathrm{bi}}(\mathcal{C})$ and $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$ are weakly equivalent in the complete Segal space model structure. Theorem 1.3 may be regarded as a refinement of this.

In addition to simplifying the marked classification diagrams, Theorem 1.3 also has some interesting applications. We list two of them below.

The first one exploits the connection between localizations and levelwise localizations we observed in Remark 1.4:

Corollary 1.6 (Corollary 4.1)**.** *Let* $(\mathcal{C}, \mathcal{W})$ *and* $(\mathcal{C}', \mathcal{W}')$ *be relative simplicial categories, where* $\mathcal{C}$ *and* $\mathcal{C}'$ *are fibrant. Let* $f\colon \mathcal{C} \to \mathcal{C}'$ *be a simplicial functor which carries* $\mathcal{W}$ *into* $\mathcal{W}'$*. Suppose that for each* $n \geq 0$*, the functor*

$$N(\mathcal{C}_n)[N(\mathcal{W}_n)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$$

is a categorical equivalence (i.e., a weak equivalence in the Joyal model structure). Then so is the functor

$$N_{\mathrm{hc}}(\mathcal{C})[N_{\mathrm{hc}}(\mathcal{W})^{-1}] \to N_{\mathrm{hc}}(\mathcal{C}')[N_{\mathrm{hc}}(\mathcal{W}')^{-1}].$$

Corollary 1.6 is especially useful when $\mathcal{C}$ is equal to the ordinary category $\mathcal{C}'_0$, the 0th level of $\mathcal{C}'$. In this case, the corollary says that if the functors $N(\mathcal{C}'_0)[N(\mathcal{W}'_0)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$ are all categorical equivalences, then so is the functor

$$N(\mathcal{C}'_0)N[(\mathcal{W}'_0)^{-1}] \to N_{\mathrm{hc}}(\mathcal{C}')[N(\mathcal{W}')^{-1}].$$

In other words, the localization of a relative ∞-category (the right hand side) is equivalent to a localization of an *ordinary* relative category (the left hand side). This recovers an observation made by Lurie in the proof of [18, Proposition 1.3.4.7]. We also remark that in the same part of loc.cit., Lurie establishes a very useful criterion for the maps $N(\mathcal{C}'_0)[N(\mathcal{W}'_0)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$ to be a categorical equivalence: This happens if $\mathcal{C}'$ admits tensoring by Δ^1 and $\mathcal{W}'$ contains all homotpy equivalences of $\mathcal{C}'$.

The second application relates the homotopy coherent nerve functor with Segal's classical construction of classifying spaces of simplicial categories.

Definition 1.7. [23] Let $\mathcal{C}$ be a simplicial category. We let $B(\mathcal{C})$ denote the diagonal of the bisimplicial set $N_{\mathrm{bi}}(\mathcal{C})$; it is the **classifying space** of $\mathcal{C}$.

In [10, 2.6.1], Hinich constructed a natural transformation $B \to N_{\mathrm{hc}}$. (In fact, there is only one such natural transformation, as we will see in Proposition 2.1.) We then prove the following comparison result:

Corollary 1.8 (Corollary 4.3)**.** *Let $\mathcal{C}$ be a fibrant simplicial category. The map*

$$B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$$

is a weak homotopy equivalence.

In the special case where $\mathcal{C}$ is a simplicial groupoid, Corollary 1.8 is a consequence of [2, Theorem 3.6] and [9, A.5.1]. (It was also announced in [10, Corollary 2.6.3], but its proof relies on 2.6.2 of loc. cit., which has a gap, as pointed out in [2].) Our proof of Corollary 1.8 uses different machinery from these earlier results, and this is why we were able to relax the hypothesis.

Organization of the paper

In Section 2, we will construct the comparison map $B \to N_{\mathrm{hc}}$. In Section 3, we will prove the main result. Section 4 concerns the applications of the main result.

Notation and convention

- If $\mathcal{C}$ is a category, its nerve will be denoted by $N(\mathcal{C})$.
- By a simplicial category, we mean a category enriched over the category of simplicial sets.
- If $\mathcal{C}$ is a simplicial category and $X, Y \in \mathcal{C}$ are its objects, we will write $\mathcal{C}(X, Y)$ or $\mathrm{Map}(X, Y)$ for the hom-simplicial set from X to Y.
- We understand that the category Cat_{Δ} is equipped with the Bergner model structure [5].
- We understand that the category sSet$^+$ is equipped with the model structure for marked simplicial sets [18, proposition 3.1.3.7, Remark 3.1.4.6].
- If X is a bisimplicial set and $n \geq 0$ is an integer, then the nth **column** (resp. nth **row**) of X is the simplicial set $X_{n,*}$ (resp. $X_{*,n}$). The nth column of X is denoted by X_n.
- If $\mathcal{C}$ is a simplicial category, its **homotopy coherent nerve** $N_{\mathrm{hc}}(\mathcal{C})$ is the simplicial set whose n-simplices are the simplicial functors $\mathfrak{C}[\Delta^n] \to \mathcal{C}$. Here $\mathfrak{C}[\Delta^n]$ denotes the simplicial category whose objects are the integers $0, \ldots, n$. If $0 \leq i \leq j \leq n$ are integers, the hom-simplicial set $\mathfrak{C}[\Delta^n](i, j)$ is the nerve of the poset $P_{i,j} = \{S \subset \{0, \ldots, n\} \mid \min S = i, \max S = j\}$, ordered by inclusion. The composition of $\mathfrak{C}[\Delta^n]$ is induced by the operation of union.[3]

[3] It is also common to define $\mathfrak{C}[\Delta^n](i, j)$ to be the nerve of the opposite of $P_{i,j}$. Our convention follows [17]. The only part that will be affected by the choice of conventions is the proof of Proposition 2.1 and the descriptions of maps appearing in Remarks 2.4 and 2.5.

2. The comparison map

In this section, we will construct a comparison map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ (where B is as in Definition 1.7), which will be the source of all the other comparison maps we consider in this note. The comparison map is obtained as the composite of two natural transformation constructed in [10, 2.6.1], but we give a direct approach. In fact, there is only one natural transformation $B \to N_{\mathrm{hc}}$:

Proposition 2.1. *There is a unique natural transformation $B \to N_{\mathrm{hc}}$ of functors* $\mathsf{Cat}_\Delta \to \mathsf{sSet}$.

For the proof of Proposition 2.1 and for later discussions, we introduce a bit of notation.

Notation 2.2. Let $n \geq 0$ and let K be a simplicial set. We define a simplicial category $[n]_K$ as follows: Its objects are the integers $0, \ldots, n$. The hom-simplicial sets are given by

$$[n]_K(i,j) = \begin{cases} \prod_{i<k\leq j} K & \text{if } i \leq j, \\ \emptyset & \text{if } i > j. \end{cases}$$

The composition is defined by concatenation.

Remark 2.3. Let $\mathcal{C}$ be a simplicial category and let K be a simplicial set. We can define an ordinary category $\mathcal{C}_K$ as follows: The objects of $\mathcal{C}_K$ are the objects of $\mathcal{C}$. The hom-sets are given by $\mathcal{C}_K(X,Y) = \mathsf{sSet}(K, \mathcal{C}(X,Y))$. For each $n \geq 0$, a functor $[n] \to \mathcal{C}_K$ can be identified with a simplicial functor $[n]_K \to \mathcal{C}$.

Proof. For each $n \geq 0$, set $\mathfrak{B}[\Delta^n] = [n]_{\Delta^n}$. By the Yoneda lemma, the simplicial categories $\{\mathfrak{B}[\Delta^n]\}_{n\geq 0}$ can be organized into a cosimplicial object of Cat_Δ in such a way that the functor B is naturally isomorphic to $\mathsf{Cat}_\Delta(\mathfrak{B}[\Delta^\bullet], -)$. It suffices to show that there is a unique morphism $\mathfrak{C}[\Delta^\bullet] \to \mathfrak{B}[\Delta^\bullet]$ of cosimplicial simplicial categories.

We begin by showing the uniqueness. Suppose there is a map $f\colon \mathfrak{C}[\Delta^\bullet] \to \mathfrak{B}[\Delta^\bullet]$ of cosimplicial simplicial categories. For each $n \geq 0$, the map

$f_n\colon \mathfrak{C}[\Delta^n] \to \mathfrak{B}[\Delta^n]$ must act on the identity maps on objects because f_n is natural in n. If $0 \leq i \leq j \leq n$ are integers, then the map

$$f_n\colon \mathfrak{C}[\Delta^n](i,j) \to \mathfrak{B}[\Delta^n](i,j)$$

is completely determined by its values of the vertices, for both $\mathfrak{C}[\Delta^n](i,j)$ and $\mathfrak{B}[\Delta^n](i,j)$ are nerves of posets. Since every vertex of $\mathfrak{C}[\Delta^n](i,j)$ is a composition of morphisms in the image of maps $\mathfrak{C}[\Delta^1] \to \mathfrak{C}[\Delta^n]$, we deduce that f_n is completely determined by f_1. Now there are exactly two simplicial functors $\mathfrak{C}[\Delta^1] \to \mathfrak{B}[\Delta^1]$ which are bijective on objects (because $\mathfrak{C}[\Delta^1](0,1) = \Delta^0$ and $\mathfrak{B}[\Delta^1](0,1) = \Delta^1$.) If f_1 carried the unique vertex of $\mathfrak{C}[\Delta^1](0,1)$ to the vertex $1 \in \mathfrak{B}[\Delta^1](0,1)$, then the map

$$f_2\colon \mathfrak{C}[\Delta^2](0,2) \to \mathfrak{B}[\Delta^2](0,2)$$

would carry the vertices $\{0,2\}$ and $\{0,1,2\}$ to $(2,2)$ and $(2,1)$, respectively. But since there is no edge $(2,2) \to (2,1)$ in $\mathfrak{B}[\Delta^2](0,2)$, this is impossible. Hence there is only a unique choice for f_1, completing the proof of the uniqueness.

For existence, define $f_n\colon \mathfrak{C}[\Delta^n] \to \mathfrak{B}[\Delta^n]$ as follows: On objects, f_n acts by the identity map. For each $0 \leq i \leq j \leq n$, the map $\mathfrak{C}[\Delta^n](i,j) \to \mathfrak{B}[\Delta^n](i,j)$ is the nerve of the poset map $P_{i,j} \to \underbrace{[n] \times \cdots \times [n]}_{j-i \text{ times}}$ which assigns to each element $\{i = i_0 < \cdots < i_k = j\} \in P_{i,j}$ the element

$$(\underbrace{i_{k-1}, \dots, i_{k-1}}_{i_k - i_{k-1} \text{ times}}, \dots, \underbrace{i_0, \dots, i_0}_{i_1 - i_0 \text{ times}}) \in [n] \times \cdots \times [n].$$

It is easy to check that the simplicial functors $\{f_n\}_{n\geq 0}$ indeed define a map of cosimplicial objects of Cat_Δ. The claim follows. $\square$

Remark 2.4. Recall that the diagonal of a bisimplicial set X is equal to the coend $\int^{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_{*,n} \times \Delta^n$. Therefore, given a simplicial category $\mathcal{C}$, the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ of Proposition 2.1 may equally well be specified by a compatible family of maps $\{\phi_n\colon N(\mathcal{C}_n) \times \Delta^n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$. Unwinding the definitions, the composite $N(\mathcal{C}_n) \times \{i\} \hookrightarrow N(\mathcal{C}_n) \times \Delta^n \to N_{\mathrm{hc}}(\mathcal{C})$ is equal to the composite

$$N(\mathcal{C}_n) \xrightarrow{i^*} N(\mathcal{C}_0) \to N_{\mathrm{hc}}(\mathcal{C}),$$

where the first map is the restriction along the inclusion $[0] \cong \{i\} \hookrightarrow [n]$ and the second map is induced by the simplicial functor $\mathcal{C}_0 \to \mathcal{C}$. In other words, the map ϕ_n is the canonical natural transformation between $n+1$ functors $N(\mathcal{C}_n) \to N_{\mathrm{hc}}(\mathcal{C})$ corresponding to the $n+1$ elements of $[n]$.

Remark 2.5. Recall that the diagonal of a bisimplicial set X is equal to the coend $\int^{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Delta^n \times X_{n,*}$. Therefore, given a simplicial category $\mathcal{C}$, the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ of Proposition 2.1 may equally well be specified by a compatible family of maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$. Unwinding the definitions, the map ψ_n admits the following description:

1. If $n = 0$, then ψ_n is the inclusion $N_{\mathrm{bi}}(\mathcal{C})_0 \cong \operatorname{ob}\mathcal{C} \hookrightarrow N_{\mathrm{hc}}(\mathcal{C})$.

2. Let $n \geq 1$ and let $\sigma \colon \Delta^m \to N_{\mathrm{bi}}(\mathcal{C})_n$ be an m-simplex of $N_{\mathrm{bi}}(\mathcal{C})_n$, corresponding to a simplicial functor $\sigma' \colon [n]_{\Delta^m} \to \mathcal{C}$. Then the composite
$$\Delta^n \times \Delta^m \xrightarrow{\mathrm{id}\times\sigma} \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \xrightarrow{\psi_n} N_{\mathrm{hc}}(\mathcal{C})$$
is adjoint to the composite
$$\mathfrak{C}[\Delta^n \times \Delta^m] \xrightarrow{\chi} [n]_{\Delta^m} \xrightarrow{\sigma'} \mathcal{C}.$$
Here χ is defined on objects by $\chi(i,j) = i$. Given integers $0 \leq i \leq i' \leq n$ and $0 \leq j \leq j' \leq m$, let $P_{(i,j),(i',j')}$ denote the poset of linearly ordered subsets $S \subset [n] \times [m]$ such that $\min S = (i,j)$ and $\max S = (i',j')$. Then the simplicial set $\mathfrak{C}[\Delta^n \times \Delta^m]((i,j),(i',j'))$ is the nerve of $P_{(i,j),(i',j')}$, and the map $\mathfrak{C}[\Delta^n \times \Delta^m]((i,j),(i',j')) \to [n]_{\Delta^m}(i,i')$ is induced by the poset map
$$\begin{aligned} P_{(i,j),(i',j')} \to \prod_{i<p\leq i'} [m], \\ S \mapsto (\max\{q \in [m] \mid (i'',q) \in S \text{ for some } i'' < p\})_{i<p\leq i'}. \end{aligned}$$

3. Main result

In this section, we state and prove the main result of this paper.

Here is the statement of the main result.

Theorem 3.1. *Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. The maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$ of Remark 2.5 induces a weak equivalence*

$$\theta \colon N_{\mathrm{bi}}^+(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$$

of $\mathsf{bsSet}^+_{\mathrm{CSS}}$.

The remainder of this section is devoted to the proof of Theorem 3.1. We begin by establishing the unmarked version of Theorem 3.1:

Proposition 3.2. *Let $\mathcal{C}$ be a fibrant simplicial category. The maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$ of Remark 2.5 induces a weak equivalence*

$$N_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$$

of $\mathsf{bsSet}_{\mathrm{CSS}}$.

Proof. Observe that the Reedy fibrant replacement $N^f_{\mathrm{bi}}(\mathcal{C})$ of $N_{\mathrm{bi}}(\mathcal{C})$ is a Segal space; indeed, for each $n \geq 2$, the square

$$\begin{array}{ccc} N_{\mathrm{bi}}(\mathcal{C})_n & \longrightarrow & N_{\mathrm{bi}}(\mathcal{C})_{n-1} \\ \downarrow & & \downarrow \\ N_{\mathrm{bi}}(\mathcal{C})_1 & \longrightarrow & N_{\mathrm{bi}}(\mathcal{C})_0 \end{array}$$

induced by the inclusions $[1] \cong \{n-1 < n\} \hookrightarrow [n] \hookleftarrow [n-1]$ is homotopy cartesian. Therefore, by [21, Theorem 7.7], it suffices to show that the induced map $N^f_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$ is a Dwyer–Kan equivalence. Since the map $N_{\mathrm{bi}}(\mathcal{C})_{0,0} \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_{0,0}$ is bijective, it suffices to show that the square

$$\begin{array}{ccc} N_{\mathrm{bi}}(\mathcal{C})_1 & \longrightarrow & \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_1 \\ \downarrow & & \downarrow \\ N_{\mathrm{bi}}(\mathcal{C})_0 \times N_{\mathrm{bi}}(\mathcal{C})_0 & \longrightarrow & \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_0 \times \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_0 \end{array}$$

is homotopy cartesian. For each pair of objects $X, Y \in \mathcal{C}$, the induced map between the fibers of the vertical arrows over (X, Y) can be identified with the homotopy equivalence

$$\mathcal{C}(X, Y) \to \mathrm{Hom}_{N_{\mathrm{hc}}(\mathcal{C})}(X, Y)$$

of [19, Tag 01LF] (see Remark 2.5). Hence the square is homotopy cartesian, as required. □

Before we proceed, we recall a few facts on marked bisimplicial sets.

Notation 3.3. [1, Definition 3.1] We let $\mathrm{diag}^+\colon \mathsf{bsSet}^+ \to \mathsf{sSet}^+$ denote the functor

$$\mathrm{diag}^+(X, S) = (\mathrm{diag}(X), S_1),$$

where $\mathrm{diag}(X) = \{X_{n,n}\}_{n\geq 0}$ denotes the diagonal of X. The functor diag^+ is the left adjoint of the functor $\mathrm{Cls}^+\colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+$.

Theorem 3.4. *There is a simplicial model structure on* bsSet^+*, denoted by* $\mathsf{bsSet}^+_{\mathrm{CSS}}$*, which has the following properties:*

1. *The cofibrations are the monomorphisms.*
2. *The fibrant objects are the marked bisimplicial sets of the form* $\mathcal{X}^\natural = (\mathcal{X}, \mathcal{X}_{\mathrm{hoeq}})$*, where* $\mathcal{X}$ *is a complete Segal space and* $\mathcal{X}_{\mathrm{hoeq}} \subset \mathcal{X}_1$ *is the union of components spanned by homotopy equivalences of* $\mathcal{X}$*.*

Moreover, the model structure has the following additional properties:

3. *If* $\mathcal{X}$ *is a complete Segal space and* (A, S) *is a marked bisimplicial set, then* $\mathrm{Map}\big((A, S), \mathcal{X}^\natural\big)$ *is the component of* $\mathrm{Map}(A, \mathcal{X})$ *spanned by the maps* $A \to \mathcal{X}$ *which carries* S *into* $\mathcal{X}_{\mathrm{hoeq}}$*. Here* $\mathrm{Map}(A, \mathcal{X})$ *denotes the simplicial enrichment of* bsSet *adapted to the complete Segal space model structure [21, §2.3].*
4. *The adjunction* $\mathrm{diag}^+\colon \mathsf{bsSet}^+_{\mathrm{CSS}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{sSet}^+\colon \mathrm{Cls}^+$ *is a Quillen equivalence.*
5. *The model structure* $\mathsf{bsSet}^+_{\mathrm{CSS}}$ *is a Bousfield localization of the Reedy model structure on* $\mathsf{bsSet}^+ = \big(\mathsf{sSet}^+\big)^{\mathbf{\Delta}^{\mathrm{op}}}$*.*
6. *The functor* $\mathrm{Cls}^+\colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+_{\mathrm{CSS}}$ *preserves and reflects all weak equivalences.*

Proof. The first half is established in [1, Theorem 2.9]. Point (3) is proved in [1, Remark 2.7], points (4) and (5) are proved in [1, Theorem 3.4], and point (6) is proved in [1, Theorem 4.2]. □

Notation 3.5. Let $(\mathcal{C}, \mathcal{W})$ be a relative simplicial category. We define a marked bisimplicial set $B^+(\mathcal{C}, \mathcal{W})$ by

$$B^+(\mathcal{C}, \mathcal{W}) = \operatorname{diag}^+ N_{\mathrm{bi}}^+(\mathcal{C}, \mathcal{W}).$$

We now arrive at the proof of Theorem 3.1.

Proof of Theorem 3.1. We prove the theorem in four steps.

(Step 1) Assume first that $\mathcal{W} \subset \mathcal{C}$ is the smallest wide simplicial subcategory containing all homotopy equivalences of $\mathcal{C}$. If $\mathcal{X}$ is a complete Segal space, then every morphism $N_{\mathrm{bi}}(\mathcal{C}) \to \mathcal{X}$ of bisimplicial sets induces a map $N_{\mathrm{bi}}^+(\mathcal{C}, \mathcal{W}) \to \mathcal{X}^\natural$ of marked bisimplicial sets. Indeed, we only have to show that the induced map $N(\mathcal{C}_0) \to \mathcal{X}_{*,0}$ between the 0th row respects the markings (because $\mathcal{X}_{\mathrm{hoeq}} \subset \mathcal{X}$ is a union of components), which is obvious. Likewise, any map $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C})) \to \mathcal{X}$ lifts to a morphism $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W})) \to \mathcal{X}^\natural$. Therefore, by properties (1) through (3) of Theorem 3.4, it suffices to show that the underlying map

$$N_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$$

of θ is a weak equivalence of $\mathsf{bsSet}_{\mathrm{CSS}}$. This is nothing but Proposition 3.2.

(Step 2) According to Theorem 3.4, the functor $\mathrm{Cls}^+ \colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+_{\mathrm{CSS}}$ is a right Quillen equivalence and preserves all weak equivalences. Therefore, θ is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$ if and only if the map

$$B^+(\mathcal{C}, \mathcal{W}) \to (N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0)$$

which is adjoint to θ is a weak equivalence of sSet^+.

(Step 3) Suppose that $\mathcal{W}$ contains all homotopy equivalences of $\mathcal{C}$. We observe that:

- The marked edges of $B^+(\mathcal{C}, \mathcal{W})$ are precisely the inverse image of $\operatorname{mor} \mathcal{W}_0$ under the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$; this is because $(\mathcal{C}, \mathcal{W})$ is a relative simplicial category.
- The map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ induces a surjection between the set of edges; this follows by inspection.

Since $\mathrm{mor}\,\mathcal{W}_0$ contains all equivalences of $N_{\mathrm{hc}}(\mathcal{C})$, the claim now follows by combining the above observations, Steps 1 and 2, and the definition of weak equivalences of marked simplicial sets.

(Step 4) We prove the theorem in the general case. Let $\mathcal{W}' \subset \mathcal{C}$ denote the smallest wide simplicial subcategory containing $\mathcal{W}$ and all homotopy equivalences of $\mathcal{C}$. The map $N^+_{\mathrm{bi}}(\mathcal{C},\mathcal{W}) \to N^+_{\mathrm{bi}}(\mathcal{C},\mathcal{W}')$ is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$, and the map $(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}'_0)$ is a weak equivalence of sSet^+. The claim now follows from part (6) of Theorem 3.4.

□

Remark 3.6. Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. As we saw in Step 2 of the proof of Theorem 3.1, Theorem 3.1 implies that the map

$$B^+(\mathcal{C},\mathcal{W}) \to (N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0)$$

is a weak equivalence of marked simplicial sets. Since the geometric realization functor models homotopy colimits in simplicial model categories ([22, Theorem 5.2.3], [11, Lemma 15.3.9]), and since $\mathrm{diag}^+\colon \mathsf{bsSet}^+ \to \mathsf{sSet}^+$ is nothing but the geometric realization functor, we may interpret Theorem 3.1 as saying that the localization $N_{\mathrm{hc}}(\mathcal{C})[N_{\mathrm{hc}}(\mathcal{W})^{-1}]$ is a homotopy colimit of the simplicial ∞-category $[n] \mapsto N(\mathcal{C}_n)[N(\mathcal{W}_n)^{-1}]$. This point of view was articulated by Lurie in [18, Proposition 1.3.4.14]; in fact, Theorem 3.1 can also be proved using Lurie's result (and Theorem 3.4), though the proof will be a little longer.

4. Applications

We now list two applications of Theorem 3.1.

Corollary 4.1. *Let $(\mathcal{C},\mathcal{W})$ and $(\mathcal{C}',\mathcal{W}')$ be relative simplicial categories, where $\mathcal{C}$ and $\mathcal{C}'$ are fibrant. Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a simplicial functor which carries $\mathcal{W}$ into $\mathcal{W}'$. If for each $n \geq 0$, the map*

$$(N(\mathcal{C}_n), \mathrm{mor}\,\mathcal{W}_n) \to (N(\mathcal{C}'_n), \mathrm{mor}\,\mathcal{W}'_n)$$

is a weak equivalence of marked simplicial sets, then the map

$$(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}'_0)$$

is also a weak equivalence.

Proof. By hypothesis, the map $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to N^+_{\mathrm{bi}}(\mathcal{C}', \mathcal{W}')$ induces a weak equivalence of marked simplicial sets in each row. Thus, by part (5) of Theorem 3.4, it is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$. It follows from Theorem 3.1 that the map $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to \mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}'_0)$ is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$. Using part (6) of Theorem 3.4, we deduce that the map $(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}'_0)$ is also a weak equivalence of sSet^+, and we are done. □

Remark 4.2. We do not know if the converse of Corollary 4.1 holds. We expect that this is false, given that the proof of the corollary relies on point (5) of Theorem 3.4, which only gives a *sufficient* condition for a map of $\mathsf{bsSet}^+_{\mathrm{CSS}}$ to be a weak equivalence. However, we are not aware of explicit counterexamples.

Corollary 4.3. *Let $\mathcal{C}$ be a fibrant simplicial category. The map*

$$B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$$

of Proposition 2.1 is a weak homotopy equivalence.

Proof. By definition, every weak equivalence of marked simplicial sets induces a weak homotopy equivalence between the underlying simplicial sets [17, Proposition 3.1.3.3]. It will therefore suffice to show that the map $B^+(\mathcal{C}, \mathcal{C}^{\simeq}) \to (N_{\mathrm{hc}}(\mathcal{C}), (\mathcal{C}^{\simeq})_0) = N_{\mathrm{hc}}(\mathcal{C})^\natural$ is a weak equivalence of marked simplicial sets, where $\mathcal{C}^{\simeq} \subset \mathcal{C}$ denotes the smallest wide simplicial subcategory containing all equivalences of $\mathcal{C}$. This is immediate from Theorem 3.1 (and part (4) of Theorem 3.4). □

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