



FIBRATIONS OF DOUBLE GROUPOIDS, II: CONNECTIONS TO SIMPLICIAL SETS AND TOPOLOGICAL SPACES.

Antonio M. CEGARRA

Résumé. Dans ce deuxième article d'une série en deux parties sur les fibrations de groupoïdes doubles, nous explorons la relation entre les fibrations de groupoïdes doubles et les fibrations de Moerdijk d'ensembles bisimpliciaux ainsi que les fibrations de Hurewicz d'espaces topologiques.

Abstract. In this second paper of a two-part series on double groupoid fibrations, we explore the relationship between double groupoid fibrations and both (Moerdijk) fibrations of bisimplicial sets and (Hurewicz) fibrations of topological spaces.

Keywords. Double groupoid, Fibration, Homotopy groups, Crossed module, Geometric realization.

Mathematics Subject Classification (2020). 18N10, 20L05, 55P15, 55R15.

Introduction and summary.

In this companion paper to [4], we demonstrate the relationship between fibrations of double groupoids and simplicial as well as topological fibrations. We will adhere to the notation introduced in the aforementioned paper.

The construction of a topological space $|\mathcal{A}|$ from a double groupoid $\mathcal{A}$ is well-known and proceeds as follows: the *geometric realization* $|\mathcal{A}| = |\mathrm{diagNN}\mathcal{A}|$ is the space defined by first taking the double nerve, which is

a bisimplicial set, and then geometrically realizing the diagonal simplicial set. A main result in [5] and [3] states that the functor $\mathcal{A} \mapsto |\mathcal{A}|$ induces an equivalence between the homotopy category of double groupoids with the filling condition and the category of homotopy 2-types. The *homotopy double groupoid* construction $X \mapsto \Pi^{(2)}X$, for topological spaces X , provides a quasi-inverse functor.

Moving from double groupoids to spaces, the main result here asserts that a double functor between double groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ is a fibration if and only if the induced simplicial map $\text{diagNN}F : \text{diagNN}\mathcal{A} \rightarrow \text{diagNN}\mathcal{B}$ is a Kan fibration. Hence, each fibration of double groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ induces a Hurewicz fibration on geometric realizations $|F| : |\mathcal{A}| \rightarrow |\mathcal{B}|$. Both the Mayer-Vietoris and homotopy sequences defined by fibrations of double groupoids in [4] provide purely algebraic descriptions of the corresponding Mayer-Vietoris and homotopy sequences defined in the topological context by taking geometric realizations.

Conversely, we prove that if $f : X \rightarrow Y$ is a Hurewicz fibration of topological spaces, then the induced double functor on the homotopy double groupoids $\Pi^{(2)}f : \Pi^{(2)}X \rightarrow \Pi^{(2)}Y$ is a fibration. For applications, we show that both the lower end up to dimension two of the homotopy sequence and the homotopy 2-groupoid of a Hurewicz fibration of spaces can be recovered using a fibration of double groupoids.

1. Relationship between fibrations of double groupoids and Moerdijk bisimplicial fibrations.

Except for explicit references, we refer to [8] for all related to (bi)simplicial sets that we use in the paper.

For a double groupoid $\mathcal{A}$, let $\text{NN}\mathcal{A}$ denote its double nerve; that is, the bisimplicial set where a (p, q) -simplex is a matrix (α_{ij}) of $p \times q$ horizontally

and vertically composable boxes of the form

$$\begin{array}{ccccc}
 a_{pq} & \xleftarrow{f_{pq}} & a_{p-1q} & \cdots & a_{1q} & \xleftarrow{f_{1q}} & a_{0q} \\
 \uparrow x_{pq} & & \uparrow \alpha_{p,q} & & \uparrow x_{p-1q} & & \uparrow \alpha_{1,q} & & \uparrow x_{0q} \\
 a_{pq-1} & \xleftarrow{f_{pq-1}} & a_{p-1q-1} & \cdots & a_{1q-1} & \xleftarrow{f_{1q-1}} & a_{0q-1} \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \\
 & \uparrow & \xleftarrow{f_{p1}} & \uparrow & \cdots & \uparrow & \xleftarrow{f_{11}} & \uparrow \\
 x_{p1} & \uparrow & \alpha_{p,1} & \uparrow & x_{p-11} & \uparrow & \alpha_{1,1} & \uparrow & x_{01} \\
 a_{p0} & \xleftarrow{f_{p0}} & a_{p-10} & \cdots & a_{10} & \xleftarrow{f_{10}} & a_{00}
 \end{array}$$

The bisimplicial face maps are the natural ones, induced by horizontal and vertical composition of boxes in $\mathcal{A}$, and the degeneracy ones by appropriate identity boxes. We picture $\text{NN}\mathcal{A}$ so that the set of (p, q) -simplices is the set in the p -th row and q -th column. Thus, its p -th column, $\text{NN}\mathcal{A}_{p\bullet}$, is the nerve of the “vertical” groupoid whose objects are strings $\cdot \xleftarrow{f_p} \cdots \xleftarrow{f_1} \cdot$ of p composable horizontal arrows in $\mathcal{A}$ and whose arrows are length p sequences of horizontally composable boxes

$$\begin{array}{ccccccc}
 \cdot & \xleftarrow{g_p} & \cdot & \cdots & \cdot & \xleftarrow{g_1} & \cdot \\
 \uparrow & \alpha_p & \uparrow & \cdots & \uparrow & \alpha_2 & \uparrow & \alpha_1 & \uparrow \\
 \cdot & \xleftarrow{f_p} & \cdot & \cdots & \cdot & \xleftarrow{f_1} & \cdot
 \end{array}$$

Analogously, the q -th column, $\text{NN}\mathcal{A}_{\bullet q}$, is the nerve of the “horizontal” groupoid whose objects are length q sequences of composable vertical morphisms in $\mathcal{A}$ and whose arrows are sequences of q vertically composable boxes. In particular, $\text{NN}\mathcal{A}_{0\bullet}$ and $\text{NN}\mathcal{A}_{\bullet 0}$ are, respectively, the nerves of the groupoids of vertical and horizontal morphisms of $\mathcal{A}$.

Similar to the nerve functor with groupoids and simplicial sets, the double nerve functor embeds the category of double groupoids as a full subcategory into the category of bisimplicial sets.

There is a closed model structure in the category of bisimplicial sets, the so-called *Moerdijk structure* [12], [8, Chap. IV, §3.3], in which a fibration is a bisimplicial map $f : X \rightarrow Y$ which induces a Kan fibration

$\text{diag}f : \text{diag}X \rightarrow \text{diag}Y$ between the associated diagonal simplicial sets. The following theorem generalizes to double groupoids the fact proven by Moerdijk and Svensson for 2-groupoids in [13, Proposition 2.1 (ii)].

Theorem 1.1. *A double functor between double groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ is a fibration if and only if the induced simplicial map*

$$\text{diagNN}F : \text{diagNN}\mathcal{A} \rightarrow \text{diagNN}\mathcal{B}$$

is a fibration of simplicial sets.

Proof. (i) $\Rightarrow$ (ii): We must prove that every lifting-extension problem

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{diagNN}\mathcal{A} \\ \downarrow & \nearrow ? & \downarrow \text{diagNN}F \\ \Delta^n & \longrightarrow & \text{diagNN}\mathcal{B} \end{array}$$

has a solution. For $n \geq 3$, this is a direct consequence Lemma 8 of [5], namely: *If $\mathcal{G}$ is a double groupoid and $n \geq 3$, then every extension problem*

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{diagNN}\mathcal{G} \\ \downarrow & \nearrow ? & \\ \Delta^n & & \end{array}$$

has a solution and it is unique.

For $n = 2$ and $k = 0$, let us consider the lifting-extension problem

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(-, \alpha_1, \alpha_2)} & \text{diagNN}\mathcal{A} \\ \downarrow & \nearrow \begin{pmatrix} \alpha_{22} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{pmatrix} ? & \downarrow \text{diagNN}F \\ \Delta^2 & \xrightarrow{\begin{pmatrix} \beta_{22} & \beta_{12} \\ \beta_{21} & \beta_{11} \end{pmatrix}} & \text{diagNN}\mathcal{B} \end{array} \quad (1)$$

Then, α_1 and α_2 are boxes of $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & \alpha_1 & \uparrow x_1 \\ \cdot & \longleftarrow f_1 & a \end{array} \quad \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & \alpha_2 & \uparrow x_2 \\ \cdot & \longleftarrow f_2 & a \end{array}$$

for some horizontal morphisms f_1 and f_2 and some vertical morphisms x_1 and x_2 , all with the same domain, say a , such that

$$F\alpha_1 = \begin{array}{ccccc} & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \beta_{22} & \uparrow & \beta_{12} & \uparrow & y_{02} \\ & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \beta_{21} & \uparrow & \beta_{11} & \uparrow & y_{01} \\ & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ & g_{20} & & g_{10} & & \end{array} \quad \text{and} \quad F\alpha_2 = \beta_{11}.$$

In particular, $Fx_1 = y_{02} y_{01}$, $Ff_1 = g_{20} g_{10}$, $Fx_2 = y_{01}$, and $Ff_2 = g_{10}$.

Since F is a fibration, we can choose boxes σ and τ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} & \cdot & \xleftarrow{\quad} & \cdot \\ x_1 \uparrow & \sigma & \uparrow & \\ a \downarrow & & f_2^{-1} \downarrow & \cdot \end{array} \quad \begin{array}{ccc} & \cdot & \xleftarrow{f_1} & a \\ \uparrow & \tau & \uparrow & x_2^{-1} \\ & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

such that

$$F\sigma = \begin{array}{ccc} & \cdot & \xleftarrow{g_{12}^{-1}} & \cdot \\ y_{02} \uparrow & \beta_{12}^h & \uparrow & \\ & \cdot & \xleftarrow{\quad} & \cdot \\ y_{01} \uparrow & \beta_{11}^h & \uparrow & \\ & \cdot & \xleftarrow{g_{10}^{-1}} & \cdot \end{array} \quad \text{and} \quad F\tau = \begin{array}{ccc} & \cdot & \xleftarrow{g_{20}} & \cdot & \xleftarrow{g_{10}} & \cdot \\ \uparrow & \beta_{21}^v & \uparrow & \beta_{11}^v & \uparrow & y_{01}^{-1} \\ & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

The boxes α_1 , α_2 , σ , and τ fit together defining a 2-simplex $\begin{pmatrix} \alpha_{22} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{pmatrix}$ of $\text{diagNN}\mathcal{A}$, in which

$$\begin{array}{ccc} \alpha_{11} = & \begin{array}{ccc} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha_2 & \uparrow & x_2 \\ & \cdot & \xleftarrow{f_2} & a \end{array} & \alpha_{21} = \begin{array}{ccc} & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \tau^v & \uparrow & \alpha_2^h & \uparrow & \\ & \cdot & \xleftarrow{f_1} & a & \xleftarrow{f_2^{-1}} & \cdot \end{array} \\ \alpha_{12} = & \begin{array}{ccc} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \sigma^h & \uparrow & x_1 \\ & \cdot & \xleftarrow{\quad} & a \\ \uparrow & \alpha_2^v & \uparrow & x_2^{-1} \\ & \cdot & \xleftarrow{\quad} & \cdot \end{array} & \alpha_{22} = \begin{array}{ccc} & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha_1 & \uparrow & \sigma & \uparrow & \\ & \cdot & \xleftarrow{\quad} & a & \xleftarrow{\quad} & \cdot \\ \uparrow & \tau & \uparrow & \alpha_2^{hv} & \uparrow & \\ & \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array} \end{array}$$

This solves the lifting-extension problem (1) since, on the one hand, $\alpha_{11} = \alpha_2$ and

$$\begin{array}{c}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\end{array}
=
\begin{array}{c}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\end{array}
= \alpha_1$$

and, on the other hand, $F\alpha_{11} = F\alpha_2 = \beta_{11}$,

$$\begin{array}{c}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\end{array}
= \beta_{12}, \quad
\begin{array}{c}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\end{array}
= \beta_{21},$$

and

$$\begin{array}{c}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\begin{array}{ccc}
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot \\
\uparrow & & \uparrow \\
\cdot & \xleftarrow{\quad} & \cdot
\end{array}
\end{array}
= \beta_{22}.$$

For $n = 2$, the case in which $k = 2$ is dual of the case $k = 0$ above, and the case in which $k = 1$ is easier: let us consider the lifting-extension problem

$$\begin{array}{ccc}
\Lambda_1^2 & \xrightarrow{(\alpha_0, -, \alpha_2)} & \text{diagNN}\mathcal{A} \\
\downarrow & \nearrow \begin{pmatrix} \alpha_{22} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{pmatrix} ? & \downarrow \text{diagNN}F \\
\Delta^2 & \xrightarrow{\begin{pmatrix} \beta_{22} & \beta_{12} \\ \beta_{21} & \beta_{11} \end{pmatrix}} & \text{diagNN}\mathcal{B}
\end{array} \tag{2}$$

Then, α_0 and α_2 are boxes of $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha_0 & \uparrow x_0 \\ \cdot & \xleftarrow{f_0} & a \end{array} \quad \begin{array}{ccc} & & \cdot \\ & a & \xleftarrow{f_2} \\ x_2 & \uparrow & \alpha_2 \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

for some object a , such that $F\alpha_0 = \beta_{22}$ and $F\alpha_2 = \beta_{11}$. Since F is a fibration, we can choose boxes α_{12} and α_{21} in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ x_0 & \uparrow & \alpha_{12} \\ a & \xleftarrow{f_2} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{f_0} & a \\ \uparrow & \alpha_{21} & \uparrow x_2 \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

such that $F\alpha_{12} = \beta_{12}$ and $F\alpha_{21} = \beta_{21}$. Then, the 2-simplex $\begin{pmatrix} \alpha_0 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}$ of $\text{diagNN}\mathcal{A}$ solves the lifting-extension problem (2).

Let us discuss finally the case in which $n = 1$ and $k = 0$ (the case $k = 1$ is parallel). Consider the extension-lifting problem

$$\begin{array}{ccc} \Lambda_0^1 & \xrightarrow{(-, a_1)} & \text{diagNN}\mathcal{A} \\ \downarrow & \nearrow \alpha? & \downarrow \text{diagNN}F \\ \Delta^1 & \xrightarrow{\beta} & \text{diagNN}\mathcal{B} \end{array} \quad (3)$$

given by an object a_1 of $\mathcal{A}$ and a box β of $\mathcal{B}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \beta & \uparrow y \\ \cdot & \xleftarrow{g} & Fa_1 \end{array}$$

Since F is a fibration, we can first choose a vertical morphism x and a horizontal morphism f , both with source a_1 , such that $Fx = y$ and $Ff = g$, and then to find a box α of $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha & \uparrow x \\ \cdot & \xleftarrow{f} & a_1 \end{array}$$

such that $F\alpha = \beta$. This α is a solution to the lifting-extension problem (3).

(ii) $\Rightarrow$ (i): To prove that F restricts to a fibration between the groupoids of horizontal morphisms, let a be an object of $\mathcal{A}$ and let $g : Fa \rightarrow b$ a horizontal morphism of $\mathcal{B}$. Since $\text{diagNN}F : \text{diagNN}\mathcal{A} \rightarrow \text{diagNN}\mathcal{B}$ is a fibration of simplicial sets, there is a solution, say α , to the extension-lifting problem (3) in which $a_1 = a$ and $\beta = I_g^v$. Then α is a box of $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha & \uparrow \\ \cdot & \xleftarrow{\quad} & a \end{array}$$

and satisfies that $F\alpha = I_g^v$. In particular, if $f : a \rightarrow a'$ is the horizontal source of α , then $Ff = g$. Dually, we prove that F restricts to a fibration between the groupoids of vertical morphisms.

Suppose now the lifting problem for F

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \exists? & \uparrow x \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad \xrightarrow{F} \quad \begin{array}{ccc} \cdot & \xleftarrow{Ff} & \cdot \\ \uparrow & \beta & \uparrow Fx \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

Since $\text{diagNN}F$ is a fibration, there is a solution, say $\begin{pmatrix} \alpha_{22} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{pmatrix}$, to the extension-lifting problem (1) in which $\alpha_0 = I_f^v$, $\alpha_2 = I_x^h$, $\beta_{22} = I_{Ff}^v$, $\beta_{12} = I_{Fa}^v$, $\beta_{21} = \beta$, and $\beta_{11} = I_{Fx}^h$. Then, the box α_{21} satisfies that $F\alpha_{21} = \beta_{21} = \beta$ and, since $\alpha_{22} = I_f^v$. Moreover, since $\alpha_{11} = I_x^h$, α_{21} is of the required form

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \alpha_{21} & \uparrow x \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

□

Theorem 1.1 above, for the double functor $\mathcal{A} \rightarrow *$, gives the following corollary. (Cf. [5, Theorem 8].)

Corollary 1.2. *A double groupoid $\mathcal{A}$ satisfies the filling condition if and only if $\text{diagNN}\mathcal{A}$ is a Kan complex.*

By [5, Theorem 9], for every object a of a double groupoid $\mathcal{A}$ with the filling property, there are natural isomorphisms

$$\pi_n(\text{diagNN}\mathcal{A}, a) \cong \begin{cases} \pi_n(\mathcal{A}, a) & 0 \leq n \leq 2, \\ 0 & n \geq 3. \end{cases} \quad (4)$$

Then, for every fibration of double groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{B}$ has the filling property and each object a of $\mathcal{A}$, the 9-term exact sequence in [4, Theorem 5.1] gives the exact sequence on homotopy groups of the induced fibre sequence of simplicial sets

$$(\text{diagNN}\mathcal{F}_b, a) \hookrightarrow (\text{diagNN}\mathcal{A}, a) \xrightarrow{\text{diagNN}F} (\text{diagNN}\mathcal{B}, b),$$

where $b = Fa$ and $\mathcal{F}_b$ is the fibre of F over b (see, e.g., [8, Lemma 7.3]).

2. Relationship between fibrations of double groupoids and topological Hurewicz fibrations.

Let $\mathcal{A}$ be a double groupoid. When the bisimplicial set $\text{NN}\mathcal{A}$ is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial space $\Delta^{\text{op}} \rightarrow \mathbf{Top}$, $\mathbf{p} \mapsto |\text{NN}\mathcal{A}_{p,\bullet}|$, whose Segal realization is taken to be $|\mathcal{A}|$, the geometric realization of $\mathcal{A}$. As there are natural homeomorphisms [15, Lemma in page 86]

$$|\mathbf{p} \mapsto |\text{NN}\mathcal{A}_{p,*}|| \cong |\text{diagNN}\mathcal{A}| \cong |\mathbf{p} \mapsto |\text{NN}\mathcal{A}_{*,p}||,$$

one usually takes

$$|\mathcal{A}| = |\text{diagNN}\mathcal{A}|.$$

By [14], [16], and [6], the functor geometric realization $|\cdot|$ carries fibrations of simplicial sets to Hurewicz fibrations of spaces. Hence Theorem 1.1 gives the following.

Theorem 2.1. *If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a fibration of double groupoids, then the induced map $|F| : |\mathcal{A}| \rightarrow |\mathcal{B}|$ is a Hurewicz fibration.*

By [5, Theorem 9], for every object a of a double groupoid $\mathcal{A}$ with the filling property, there are natural isomorphisms

$$\pi_n(|\mathcal{A}|, |a|) \cong \begin{cases} \pi_n(\mathcal{A}, a) & 0 \leq n \leq 2, \\ 0 & n \geq 3. \end{cases} \quad (5)$$

By [11, Corollary 11.6], the functor geometric realization $\mathcal{A} \mapsto |\mathcal{A}|$, from the category of double groupoids to the category of compactly generated Hausdorff spaces, preserves pullbacks. Then, in the situation of double functors between double groupoids

$$\begin{array}{ccc} & \mathcal{A} & \\ & \downarrow F & \\ \mathcal{B}' & \xrightarrow{G} & \mathcal{B} \end{array}$$

where F is a fibration and both $\mathcal{B}'$ and $\mathcal{B}$ have the filling property, the Mayer-Vietoris exact sequences in [4, Theorem 4.1] give the Mayer-Vietoris exact sequences of the pullback of spaces

$$\begin{array}{ccc} |\mathcal{B}'| \times_{|\mathcal{B}|} |\mathcal{A}| & \longrightarrow & |\mathcal{A}| \\ \downarrow & & \downarrow |F| \\ |\mathcal{B}'| & \xrightarrow{|G|} & |\mathcal{B}| \end{array}$$

(Cf. [7, Theorem], [1, Corollary 4.2]).

In particular, for every fibration of double groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{B}$ has the filling property, and every object a of $\mathcal{A}$, the 9-term exact sequence in [4, Theorem 5.1] gives the exact sequence on homotopy groups of the induced fibre sequence

$$(\mathcal{F}_{|b|}, |a|) \hookrightarrow (|\mathcal{A}|, |a|) \xrightarrow{|F|} (|\mathcal{B}|, |b|),$$

where $b = Fa$ and $\mathcal{F}_{|b|} = |F|^{-1}|b|$ is the fibre of $|F|$ over $|b|$.

Transitioning now from topological spaces, every space X has an associated *homotopy double groupoid*, which captures the homotopy 2-type of the space. The data of this double groupoid, denoted by

$$\Pi^{(2)}X,$$

are as follows. (For details, see [5, §4].)

The objects of $\Pi^{(2)}X$ are the paths in X , that is, the continuous maps $u : I \rightarrow X$, from the interval $I = [0, 1]$ to X .

The groupoid of horizontal morphisms in $\Pi^{(2)}X$ is the category with a unique morphism $(u', u) : u \rightarrow u'$ for each pair (u', u) of paths in X such that $u'(1) = u(1)$. Composition is defined by $(u'', u')(u', u) = (u'', u)$. Similarly, the groupoid of vertical morphisms in $\Pi^{(2)}X$ is the category having a unique morphism $(v, u) : u \rightarrow v$ for each pair (v, u) of paths in X such that $v(0) = u(0)$.

A box $[\alpha]$ in $\Pi^{(2)}X$ with boundary as in

$$\begin{array}{ccc} v' & \longleftarrow & v \\ \uparrow & [\alpha] & \uparrow \\ u' & \longleftarrow & u \end{array}$$

is the equivalence class of a square in X of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{v'} & \cdot \\ v \uparrow & \alpha & \downarrow u' \\ \cdot & \xrightarrow{u} & \cdot \end{array}$$

that is, a map $\alpha : I^2 \rightarrow X$, whose effect on the boundary ∂I^2 is

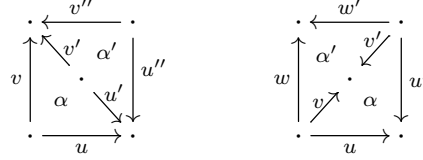
$$\begin{cases} \alpha(t, 0) = u(t), \\ \alpha(0, t) = v(t), \\ \alpha(1, t) = u'(1 - t), \\ \alpha(t, 1) = v'(1 - t). \end{cases}$$

Two such mappings α, α' are equivalent, and then represent the same box in $\Pi^{(2)}X$, whenever they are related by a homotopy relative to the sides of the square. Horizontal and vertical composition of boxes in $\Pi^{(2)}X$

$$\begin{array}{ccc} v'' & \longleftarrow & v' & \longleftarrow & v \\ \uparrow & [\alpha'] & \uparrow & [\alpha] & \uparrow \\ u'' & \longleftarrow & u' & \longleftarrow & u \end{array} \qquad \begin{array}{ccc} w' & \longleftarrow & w \\ \uparrow & [\alpha'] & \uparrow \\ v' & \longleftarrow & v \\ \uparrow & [\alpha] & \uparrow \\ u' & \longleftarrow & u \end{array}$$

are defined in the traditional way by pasting squares in X along a common

pair of sides



The horizontal and vertical identity boxes

$$I_{(v,u)}^h = \begin{array}{c} v \\ \uparrow [C^h] \\ u \end{array} \quad I_{(u',u)}^h = \begin{array}{c} u' \longleftarrow u \\ \parallel [C^v] \parallel \\ u' \longleftarrow u \end{array}$$

are respectively defined by the squares $C^h, C^v : I^2 \rightarrow X$ given by

$$C^h(t, t') = \begin{cases} v(t' - t) & t \leq t', \\ u(t - t') & t \geq t'. \end{cases} \quad C^v(t, t') = \begin{cases} u(t + t') & t + t' \leq 1, \\ u'(2 - t - t') & t + t' \geq 1. \end{cases}$$

For a point $x \in X$, let $c_x : I \rightarrow X$ denote the constant path $c_x(t) = x$. Then, the identity box of c_x in $\Pi^{(2)}X$ is

$$I_{c_x} = \begin{array}{c} c_x \\ \parallel [C_x] \parallel \\ c_x \end{array}$$

where $C_x : I^2 \rightarrow X$ is the constant square $C_x(t, t') = x$.

Theorem 2.2. *If $f : X \rightarrow Y$ is a Hurewicz fibration of spaces, then the induced double functor $f_* : \Pi^{(2)}X \rightarrow \Pi^{(2)}Y$ is a fibration of double groupoids.*

Proof. The fact that the restrictions of f_* to the groupoids of morphisms of $\Pi^{(2)}X$ and $\Pi^{(2)}Y$ are fibrations follows directly from the path-lifting property of f .

Suppose given a lifting problem for f_*

$$\begin{array}{ccc} v' \longleftarrow v & & f v' \longleftarrow f v \\ \uparrow [\alpha]? \quad \uparrow & \xrightarrow{f_*} & \uparrow [\beta] \quad \uparrow \\ \cdot \longleftarrow \cdots \cdots u & & \bar{u} \longleftarrow f u \end{array}$$

defined by paths $u, v, v' : I \rightarrow X$, with $u(0) = v(0)$ and $v(1) = v'(1)$, and a square $\beta : I^2 \rightarrow Y$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{fv'} & \cdot \\ fv \uparrow & \beta & \downarrow \bar{u} \\ \cdot & \xrightarrow{fu} & \cdot \end{array}$$

Let $\alpha' : (0 \times I) \cup (I \times \partial I) \rightarrow X$ be the map given by

$$\begin{cases} \alpha'(0, t) = v(t), \\ \alpha(t, 0) = u(t), \\ \alpha(t, 1) = v'(1 - t). \end{cases}$$

Since f is a fibration and $f\alpha' = \beta|_{(0 \times I) \cup (I \times \partial I)}$, the extension-lifting problem

$$\begin{array}{ccc} (0 \times I) \cup (I \times \partial I) & \xrightarrow{\alpha'} & X \\ \text{in} \downarrow & \nearrow \alpha & \downarrow f \\ I \times I & \xrightarrow{\beta} & Y \end{array}$$

has a solution, say α . Then α is a square in X of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{v'} & \cdot \\ v \uparrow & \alpha & \downarrow u' \\ \cdot & \xrightarrow{u} & \cdot \end{array}$$

which represents a box of $\Pi^{(2)}X$

$$\begin{array}{ccc} v' & \longleftarrow & v \\ \uparrow & [\alpha] & \uparrow \\ u' & \longleftarrow & u \end{array}$$

such that $f_*[\alpha] = [f\alpha] = [\beta]$. $\square$

From here on, we consider a pointed Hurewicz fibration of topological spaces

$$f : (X, x) \rightarrow (Y, y).$$

Let $F_y = f^{-1}(y)$ be the fibre of f over y , so that we have the fibre sequence of pointed spaces

$$(F_y, x) \hookrightarrow (X, x) \xrightarrow{f} (Y, y), \quad (6)$$

and let $\mathcal{F}_{c_y} = f_*^{-1}(c_y)$ be the double groupoid fibre of $f_* : \Pi^{(2)}X \rightarrow \Pi^{(2)}Y$ over c_y , so that we have the fibre sequence of pointed double groupoids

$$(\mathcal{F}_{c_y}, c_x) \hookrightarrow (\Pi^{(2)}X, c_x) \xrightarrow{f_*} (\Pi^{(2)}Y, c_y). \quad (7)$$

Theorem 2.3. (i) *The double groupoid $\mathcal{F}_{c_y}$ has the filling property.*

(ii) *There is an injective on objects equivalence of groupoids*

$$c : \Pi F_y \simeq \Pi \mathcal{F}_{c_y},$$

between the fundamental groupoid of F_y and the fundamental groupoid of $\mathcal{F}_{c_y}$, which carries every point $x \in F_y$ to the constant path c_x and a morphism $[v] : x_0 \rightarrow x_1$ of ΠF_y to the morphism $[(c_{x_1}, v), (v, c_{x_0})] : c_{x_0} \rightarrow c_{x_1}$ of $\Pi \mathcal{F}_{c_y}$ defined by the path

$$((c_{x_1}, v), (v, c_{x_0})) : c_{x_0} \curvearrowright c_{x_1} \quad \begin{array}{ccc} & c_{x_1} & \longleftarrow v \\ & \uparrow & \\ & c_{x_0} & \end{array} \quad (8)$$

(iii) *There are natural isomorphisms*

$$\begin{aligned} \pi_0(\mathcal{F}_{c_y}, [c_x]) &\cong \pi_0(F_y, [x]) \\ \pi_1(\mathcal{F}_{c_y}, c_x) &\cong \pi_1(F_y, x), \\ \pi_2(\mathcal{F}_{c_y}, c_x) &\cong \text{Im}(i_* : \pi_2(F_y, x) \rightarrow \pi_2(X, x)). \end{aligned}$$

Proof. (i) This follows from Theorem 2.2 and [4, Proposition 2.6].

(ii) The objects of the double groupoid $\mathcal{F}_{c_y}$ are the paths $I \rightarrow F_y$. We define the functor $c : \Pi F_y \rightarrow \Pi \mathcal{F}_{c_y}$ on objects by $x \mapsto c_x$. For any two points $x_0, x_1 \in F_y$, a morphism in $\Pi \mathcal{F}_{c_y}$ from c_{x_0} to c_{x_1} is the homotopy class of a path in $\mathcal{F}_{c_y}$ as in (8), which is determined by a path $v : I \rightarrow F_y$ such that $v(0) = x_0$ and $v(1) = x_1$. If v' is any other path in F_y from x_0 to x_1 , then $[(c_{x_1}, v), (v, c_{x_0})] = [(c_{x_1}, v'), (v', c_{x_0})]$ in $\Pi \mathcal{F}_{c_y}$ if and only if there is a box in $\mathcal{F}_{c_y}$ of the form

$$\begin{array}{ccc} v & \longleftarrow & v' \\ \parallel & [\alpha] & \uparrow \\ v & \xlongequal{\quad} & v \end{array}$$

That is, if and only if there is a square $\alpha : I^2 \rightarrow X$ of the form

$$\begin{array}{ccc} x_1 & \xleftarrow{v} & x_0 \\ v' \uparrow & \alpha & \downarrow v \\ x_0 & \xrightarrow{v} & x_1 \end{array}$$

such that there is a homotopy $f\alpha \simeq C_y \text{ rel } \partial I^2$. By Lemma 2.4 below, that happens if and only if there is a similar square $\alpha' : I^2 \rightarrow F_y$ with the same sides. Since this condition simply means that $[v'] = [v]$ in the fundamental groupoid ΠF_y , we conclude with a bijection

$$c : \text{Hom}_{\Pi F_y}(x_0, x_1) \cong \text{Hom}_{\Pi \mathcal{F}_{c_y}}(c_{x_0}, c_{x_1}), \quad [v] \mapsto [(c_{x_1}, v), (v, c_{x_0})].$$

To see that we are in presence of a functor, let $v_1, v_2 : I \rightarrow F_y$ be paths with $v_1(0) = x_0, v_1(1) = x_1 = v_2(0)$, and $v_2(1) = x_2$. Then, in $\Pi \mathcal{F}_{c_y}$,

$$[(c_{x_2}, v_2), (v_2, c_{x_1})] \cdot [(c_{x_1}, v_1), (v_1, c_{x_0})] = [(c_{x_2}, v), (v, c_{x_0})],$$

where v occurs in a configuration such as in

$$\begin{array}{ccccc} & & c_{x_2} & \xleftarrow{v_2} & v \\ & & \uparrow & [\alpha] & \uparrow \\ & & c_{x_1} & \xleftarrow{v_1} & v \\ & & & & \uparrow \\ & & & & c_{x_0} \end{array}$$

for some square $\alpha : I^2 \rightarrow F_y$ with boundary as in

$$\begin{array}{ccc} x_2 & \xleftarrow{v_2} & x_1 \\ v \uparrow & \alpha & \parallel^{c_{x_1}} \\ x_0 & \xrightarrow{v_1} & x_1 \end{array}$$

Hence, in ΠF_y , $[v] = [v_2] \cdot [v_1]$.

Finally, since for any path $u : I \rightarrow F_y$, there is the horizontal morphism

$$(u, c_{x_1}) : c_{u(1)} \rightarrow u,$$

the functor $c : \Pi F_y \rightarrow \Pi \mathcal{F}_{c_y}$ is actually an equivalence of categories.

(iii) Both the bijection $\pi_0 \mathcal{F}_{c_y} \cong \pi_0 F_y$ and the isomorphism $\pi_1(\mathcal{F}_{c_y}, c_x) \cong \pi_1(F_y, x)$ are derived from part (ii) above, which has already been proven. Let us consider the case $n = 2$. The abelian group $\pi_2(\mathcal{F}_{c_y}, c_x)$ consists of boxes $[\alpha]$ in $\Pi^2 X$ of the form

$$\begin{array}{ccc} c_x & \xlongequal{\quad} & c_x \\ \parallel & [\alpha] & \parallel \\ c_x & \xlongequal{\quad} & c_x \end{array}$$

such that $f_*[\alpha] = [C_y]$ or, equivalently, of relative to ∂I^2 homotopy classes of squares $\alpha : I^2 \rightarrow X$ which are constant x along the four sides of the square and such that there is a homotopy $f\alpha \simeq C_y$ relative to ∂I^2 . Then, by Lemma 2.4 below, $\pi_2(\mathcal{F}_{c_y}, c_x)$ can be described as the set of relative to ∂I^2 homotopy classes of squares $\alpha : I^2 \rightarrow X$ which are constant x along the four sides of the square and such that $\alpha(I^2) \subseteq F_y$, that is, the image of $i_* : \pi_2(F_y, x) \rightarrow \pi_2(X, x)$. $\square$

Lemma 2.4. *If $\alpha : I^2 \rightarrow X$ be a square in X such that $\alpha(\partial I^2) \subseteq F_y$ and there is a homotopy $H : f\alpha \simeq C_y \text{ rel } \partial I^2$, then there is a square $\alpha' : I^2 \rightarrow F_y$ such that $\alpha'|_{\partial I^2} = \alpha|_{\partial I^2}$ and a homotopy $H' : \alpha \simeq \alpha' \text{ rel } \partial I^2$ such that $fH' = H$.*

Proof. Let $h : (0 \times I^2) \cup (I \times \partial I^2) \rightarrow X$ be the map given by

$$\begin{aligned} h(0, s, t) &= \alpha(s, t), & h(r, 0, t) &= \alpha(0, t), & h(r, 1, t) &= \alpha(1, t), \\ h(r, s, 0) &= \alpha(s, 0), & h(r, s, 1) &= \alpha(s, 1). \end{aligned}$$

Since f is a fibration and $fh = H|_{(0 \times I^2) \cup (I \times \partial I^2)}$, the extension-lifting problem

$$\begin{array}{ccc} (0 \times I^2) \cup (I \times \partial I^2) & \xrightarrow{h} & X \\ \text{\scriptsize in} \downarrow & \nearrow H' & \downarrow f \\ I \times I \times I & \xrightarrow{H} & Y \end{array}$$

has a solution, say H' . Now define $\alpha' : I^2 \rightarrow X$ by $\alpha'(s, t) = H'(1, s, t)$. Since,

$$f\alpha'(s, t) = fH'(1, s, t) = H(1, s, t) = C_y(s, t) = y,$$

α' is actually a square in the fibre F_y . $\square$

If we apply Theorem 2.3 to the fibration $X \rightarrow *$, then obtain the following.

Theorem 2.5. *Let X be a topological space.*

- (i) *The double groupoid $\Pi^{(2)}X$ has the filling property.*
- (ii) *There is an injective on objects equivalence of groupoids*

$$c : \Pi X \simeq \Pi(\Pi^{(2)}X),$$

between the fundamental groupoid of X and the fundamental groupoid of $\Pi^{(2)}X$, which carries every point $x \in X$ to the constant path c_x and a morphism $[u] : x_0 \rightarrow x_1$ of ΠX to the morphism $[(c_{x_1}, u), (u, c_{x_0})] : c_{x_0} \rightarrow c_{x_1}$ of ΠX defined by the path

$$((c_{x_1}, u), (u, c_{x_0})) : c_{x_0} \curvearrowright c_{x_1} \quad \begin{array}{ccc} & \longleftarrow u & \\ & \uparrow & \\ & c_{x_0} & \end{array}$$

- (iii) *For every $x \in X$, there are natural isomorphisms*

$$\begin{aligned} \pi_0(\Pi^{(2)}X, [c_x]) &\cong \pi_0(X, [x]) \\ \pi_1(\Pi^{(2)}X, c_x) &\cong \pi_1(X, x), \\ \pi_2(\Pi^{(2)}X, c_x) &\cong \pi_2(X, x). \end{aligned}$$

Lemma 2.6. *Let $u : I \rightarrow X$ be a path in X with $u(0) = x_0$ and $u(1) = x_1$. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$, and let $v : I \rightarrow F_{y_0}$ be a loop in F_{y_0} based at x_0 . Then, the action [4, (19)] of the morphism*

$$[(c_{x_1}, u), (u, c_{x_0})] : c_{x_0} \rightarrow c_{x_1}$$

of $\Pi(\Pi^{(2)}X)$ on

$$[(c_{x_0}, v), (v, c_{x_0})] \in \pi_1(\mathcal{F}_{c_{y_0}}, c_{x_0})$$

is given by

$$[(c_{x_1}, u), (u, c_{x_0})][[(c_{x_0}, v), (v, c_{x_0})]] = [(c_{x_1}, v'), (v', c_{x_1})] \in \pi_1(\mathcal{F}_{c_{y_1}}, c_{x_1})$$

where $v' : I \rightarrow F_{y_1}$ is a loop in F_{y_1} based at x_1 such that, in ΠX ,

$$[v'] = [u] \cdot [v] \cdot [u]^{-1}.$$

Proof. As an object of $\Pi^{(2)}X$, the path $v' : I \rightarrow X$ with $fv' = c_{y_1}$, i.e. with $Im(v') \subseteq F_{y_1}$, occurs in a diagram

$$\begin{array}{ccccccc}
 & & c_{x_1} & \longleftarrow & u & \longleftarrow & w & \longleftarrow & v' \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & & [\alpha] & & & \\
 & & c_{x_0} & \longleftarrow & v & & & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & u & \longleftarrow & c_{x_1} & &
 \end{array}$$

for some squares $\alpha : I^2 \rightarrow X$ and $\beta : I^2 \rightarrow X$ of the form

$$\begin{array}{ccc}
 x_1 & \xleftarrow{u} & x_0 \\
 w \uparrow & \alpha & \parallel \\
 x_0 & \xrightarrow{v} & x_0
 \end{array}
 \quad
 \begin{array}{ccc}
 x_1 & \xleftarrow{w} & x_0 \\
 v' \uparrow & \beta & \downarrow u \\
 x_1 & \xlongequal{\quad} & x_1
 \end{array}$$

Then, v' is a path in F_{y_1} based at x_1 and, in ΠX ,

$$[v'] = [w] \cdot [u]^{-1} = ([u] \cdot [v]) \cdot [u]^{-1}.$$

□

By Theorems 2.3 and 2.5, the 9-term exact sequence in [4, Theorem 5.1] defined by the fibre sequence of pointed double groupoids (7),

$$(\mathcal{F}_{c_y}, c_x) \hookrightarrow (\Pi^{(2)}X, c_x) \xrightarrow{f_*} (\Pi^{(2)}Y, c_y),$$

gives the 9-term exact sequence

$$\begin{array}{ccccc}
 \pi_2(F_y, x) & \longrightarrow & \pi_2(X, x) & \longrightarrow & \pi_2(Y, y) \\
 & & \swarrow & & \\
 \pi_1(F_y, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \pi_1(Y, y) \\
 & & \swarrow & & \\
 \pi_0(F_y, [x]) & \longrightarrow & \pi_0(X, [x]) & \longrightarrow & \pi_0(Y, [y]),
 \end{array}$$

which is the bottom end up to dimension 2 of the long exact sequence on homotopy groups defined by the fibre sequence of pointed spaces (6)

$$(F_y, x) \hookrightarrow (X, x) \xrightarrow{f} (Y, y).$$

(See, for example, [17, Chap. IV, (8.6)]). Further, by the equivalence between crossed modules and cat^1 -groups, the fundamental crossed module in [4, Proposition 5.2 (i)] gives Loday's fundamental crossed module [10, 2.7]

$$\pi_1(F_y, x) \rightarrow \pi_1(X, x).$$

(See [2, §2.6].) Even more, by the equivalence between crossed modules over groupoids and 2-groupoids, we can see that the fundamental crossed module in [4, Theorem 5.7] for the fibration $f_* : \Pi^{(2)}X \rightarrow \Pi^{(2)}Y$ correspond, up to equivalence, to the homotopy 2-groupoid of the fibration $f : X \rightarrow Y$ by Kamps and Porter [9]. In effect, the fundamental $\Pi(\Pi^{(2)}X)$ -crossed module of the fibration $f_* : \Pi^{(2)}X \rightarrow \Pi^{(2)}Y$,

$$\pi_1\mathcal{F} \rightarrow \pi_1(\Pi^{(2)}X), \quad u \mapsto (\pi_1(\mathcal{F}_{fu}, u) \rightarrow \pi_1(\Pi^{(2)}X, u)),$$

by the equivalence $c : \Pi X \hookrightarrow \Pi(\Pi^{(2)}X)$, gives the ΠX -crossed module

$$c^*\pi_1\mathcal{F} \rightarrow c^*\pi_1(\Pi^{(2)}X), \quad x \mapsto (\pi_1(\mathcal{F}_{cx}, c_x) \rightarrow \pi_1(\Pi^{(2)}X, c_x)).$$

This, by Theorems 2.3 and 2.5 and Lemma 2.6 above, is isomorphic to the crossed module over the fundamental groupoid ΠX ,

$$\pi_1 F \rightarrow \pi_1 X, \quad x \mapsto (\pi_1(F_{fx}, x) \rightarrow \pi_1(X, x)),$$

which justly defines the homotopy 2-groupoid of the fibration $f : X \rightarrow Y$ by Kamps and Porter.

References

- [1] R. Brown, P.R. Heath and K.H. Kamps, Groupoids and the Mayer-Vietoris sequence. *J. Pure Appl. Algebra* 30 (1983), 109-129.
- [2] R. Brown, P.J. Higgins and R. Sivera, Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by C. D. Wensley and S.V. Soloviev. EMS Tracts in Mathematics, 15. European Mathematical Society (EMS), Zürich, 2011.
- [3] A.M. Cegarra, Double groupoids and Postnikov invariants, *Cahiers Topologie Géom. Différentielle Catég.* LXIV (2023), 125-175.

- [4] A.M. Cegarra, Fibrations of Double Groupoids, I: Algebraic Properties and Homotopy Sequences. *Cahiers Topologie Géom. Différentielle Catég.* LXVI (2025), 79-119.
- [5] A.M. Cegarra, B.A. Heredia and J. Remedios, Double Groupoids and Homotopy 2-types. *Appl. Categ. Structures* 20 (2012), 323-378.
- [6] R. Cauty, Sur les ouverts des CW-complexes et les fibrés de Serre. *Colloq. Math.* 63 (1992), 1-7.
- [7] E. Dyer and J. Roitberg, Note on sequences of Mayer-Vietoris type. *Proc. Amer. Math. Soc.* 80 (1980), 660-662.
- [8] P.G. Goerss and J.F. Jardine, Simplicial homotopy theory. *Modern Birkhäuser Classics*, Birkhäuser Verlag, Basel, 200
- [9] K.H. Kamps and T. Porter, A homotopy 2-groupoid from a fibration. *Homology Homotopy Appl.* 1 (1999), 79-93.
- [10] J.-L. Loday, Spaces with finitely many nontrivial homotopy groups, *J. Pure Appl. Algebra* 24 (1982), 179-202.
- [11] J.P. May, The geometry of iterated loop spaces. *Lecture Notes in Mathematics*, Vol. 271. Springer-Verlag, Berlin-New York, 1972.
- [12] I. Moerdijk, Bisimplicial sets and the group completion theorem. In: *Algebraic K-Theory: Connections with Geometry and Topology*, 225-240. Kluwer, Dordrecht, 1989.
- [13] I. Moerdijk and J.-A. Svensson, Algebraic classification of equivariant homotopy 2-types. I. *J. Pure Appl. Algebra* 89 (1993), 187-216.
- [14] D.G. Quillen, The geometric realization of a Kan fibration is a Serre fibration. *Proc. Amer. Math. Soc.* 19 (1968), 1499-1500.
- [15] D.G. Quillen, Higher algebraic K-theory: I, in *Algebraic K-theory I*, Springer LNM 341 (1973), 85-147.
- [16] M. Steinberger and J. West, Covering homotopy properties of maps between C.W. complexes or ANRs. *Proc. Amer. Math. Soc.* 92 (1984) 573-577.

- [17] G.W. Whitehead, Elements of homotopy theory. *Graduate Texts in Mathematics*, 61. Springer-Verlag, New York-Berlin, 1978.

Antonio M. Cegarra
Department of Algebra, Faculty of Sciences
University of Granada
18071 Granada, Spain
acegarra@ugr.es