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ALL SEGAL OBJECTS ARE GENERALISED MONADS IN SPANS

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Résumé. Nous étendons la construction par Barwick et Haugseng d'une ∞-catégorie double de correspondances dans une ∞-catégorie $\mathfrak{C}$ admettant les produits fibrés à des formes plus générales : pour une large classe de patrons algébriques $\mathfrak{P}$, nous définissons une ∞-catégorie $\mathfrak{P}$-monoïdale de correspondances $\mathfrak{P}$-modelées dans $\mathfrak{C}$, et identifions les $\mathfrak{P}$-monades dedans avec les $\mathfrak{P}$-objets de Segal dans $\mathfrak{C}$. Pour le patron cellulaire Θ^{op}, cela recouvre une reformulation homotopique de la définition originale de Batanin des ω-catégories faibles, et en général peut être vu comme une variante des multicatégories généralisées de Burroni, Hermida, Leinster et Cruttwell–Shulman.

Abstract. We extend Barwick's and Haugseng's construction of the double ∞-category of spans in a pullback-complete ∞-category $\mathfrak{C}$ to more general shapes: for a large class of algebraic patterns $\mathfrak{P}$, we define a $\mathfrak{P}$-monoidal ∞-category of $\mathfrak{P}$-shaped spans in $\mathfrak{C}$, and we identify $\mathfrak{P}$-monads in it with Segal $\mathfrak{P}$-objects in $\mathfrak{C}$. For the cell pattern Θ^{op}, this recovers a homotopical reformulation of Batanin's original definition of weak ω-categories, and in general can be seen as a variant of the generalised multicategories of Burroni, Hermida, Leinster and Cruttwell–Shulman.

Keywords. Segal objects, spans, double categories, weak ω-categories, multicategories.

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1. Introduction

1.1 Algebraic structures for higher categories

The various definitions of higher categories come in two families: algebraic definitions specify the minimal amount of shape data (for ℓ-categories, an ℓ-graph, comprised only of elementary cells) and add the structure of all the composition operations and their higher coherences, while geometric definitions start from a bigger shape containing all the possible pasting diagrams of cells and simply impose conditions to ensure that they come from decompositions into compatible elementary cells.

For example, the standard definition of an internal category, in a category $\mathfrak{C}$ admitting finite pullbacks, is as a $\mathbb{\Delta}^{\mathrm{op}}$-shaped object $X_\bullet$ of $\mathfrak{C}$ — where $\mathbb{\Delta}$ is the category of free categories on 1-dimensional pasting diagrams, that is sequences of composable arrows — satisfying the Segal decomposition condition which expresses each value X_n on a pasting of n consecutive arrows as $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. This can be reinterpreted in a more algebraic way as giving a graph $X_\bullet|_{\{[0],[1]\}}$ in $\mathfrak{C}$ and a certain kind of algebra structure on it, subject to the simplicial identities. To make good sense of this algebra structure, it was noticed by [Bén67] that a graph in $\mathfrak{C}$ is nothing but an endomorphism in the bicategory (or better, the double category) of spans in $\mathfrak{C}$, and the required algebra structure is none other than a structure of monad on this endomorphism.

For *strict* higher categories, the situation generalises directly: on the one hand, [Joy97] introduced a category Θ of free ω-categories on ω-categorical pasting diagrams, so that strict ω-categories in any category with fibre products $\mathfrak{C}$ are exactly $\mathfrak{C}$-valued presheaves on Θ satisfying a Segal condition. On the other hand, [Bat98] constructed an internal (strict) ω-category in $\mathfrak{Cat}$ (a globular object in $\mathfrak{Cat}$ equipped with compositions) $\mathbb{Span}_\infty(\mathfrak{C})$ of infinitely iterated spans in $\mathfrak{C}$, so that globular monads in it are exactly strict ω-categories internal to $\mathfrak{C}$.

The key insight of [Bat98] is then that, using the higher structure naturally present in globular categories, one can refine the teminal globular operad to a suitable contractible globular operad $\mathscr{A}_\infty^{\mathbb{G}}$, which contains enough coherence data for $\mathscr{A}_\infty^{\mathbb{G}}$-algebras in $\mathbb{Span}_\infty(\mathfrak{C})$ to be a good definition of weak ω-categories in $\mathfrak{C}$.

While the presence of higher cells in globular sets allows one to make sense of $\mathscr{A}_\infty^{\mathbb{G}}$ as an algebraic resolution of the terminal globular operad, eschewing any homotopical machinery, formulating things in a setting of homotopy theory allows many constructions to become simpler, and more widely applicable. Indeed, the logic of using the higher cells to tame the infinite towers of coherences needed for a resolution only works for full ω-categories, but breaks down if trying to define weak ℓ-categories for some $\ell < \omega$. Nonetheless, [Hau21] showed that the situation for (weak) 1-categories can be dealt with using ∞-categories: category objects in an $(\infty, 1)$-category $\mathfrak{C}$ are identified with homotopy-coherent monads in the double $(\infty, 1)$-category of spans in $\mathfrak{C}$.

In this note, we extend this result (as a direct application of Theorem 5.7) to a characterisation of ℓ-category objects as ℓ-globular monads in ℓ-times iterated spans, which both extends Batanin's definition of weak ω-categories to one for weak ℓ-categories for any $\ell \leq \omega$, and also simplifies it by removing the need to resolve the terminal globular ∞-operad by a more complicated one.

1.2 Multicategories and algebraic patterns

In order to understand how to construct categories of generalised spans, let us switch gears to another categorical structure that can be defined in a similar way: multicategories, or coloured operads. It was noticed by [Bur71, Her04, Lei98] that multicategories can be defined as monads in a double category of Kleisli $\mathscr{M}$-spans, where $\mathscr{M}$ is the "free monoid" monad on $\mathfrak{Set}$, fitting in a more general framework of $\mathscr{T}$-multicategories, for a cartesian monad $\mathscr{T}$, as monads in a double category of Kleisli $\mathscr{T}$-spans, whose morphisms are the spans twisted by $\mathscr{T}$ on their source, and whose composition uses $\mathscr{T}$'s monad structure. In particular, Batanin's globular operads can also be obtained in this way.

Unfortunately, this double category of Keisli $\mathscr{T}$-spans is not characterised by a clear universal property (see [CS10, Remark 4.2]), which makes constructing it in the ∞-categorical world very difficult. Because of this, we will instead use a different kind of structure to organise the generalised spans.

To explain the idea, let us keep focusing of the example of multicategories. An $\mathscr{M}$-span from a set Y_0 to a set X_0 is given by a span $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$, which we interpret as a multispan (as championed by [Baa19] for the study of hyperstructures) of some arbitrary arity $a+1$, one of whose legs (the root) goes to X_0 and the a others (the leaves) to Y_0. To compose it with an $\mathscr{M}$-span $\mathscr{M}Z_0 \leftarrow Y_1 \to Y_0$, one forms the $\mathscr{M}$-span

$$\begin{tikzcd}
 & & & \mathscr{M}Y_1 \underset{\mathscr{M}Y_0}{\times} X_1 \arrow[dl] \arrow[dr] & & \\
 & & \mathscr{M}Y_1 \arrow[dl] \arrow[dr] & & X_1 \arrow[dl] \arrow[dr] & \\
\mathscr{M}Z_0 & \mathscr{M}^2 Z_0 \arrow[l, "\mu"'] & & \mathscr{M}Y_0 & & X_0
\end{tikzcd} \tag{1}$$

expressing that one takes a copies of the multispan corresponding to $\mathscr{M}Z_0 \leftarrow Y_1 \to Y_0$ and glues their distinguished roots to the various leaves of $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$.

As is usual in operad theory, one also, instead of blowing up the situation globally, glue a single new span to one leaf of $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$; the composition operation defined in this way, leaf by leaf, will no longer be a categorical composition, but indeed an operadic one. Thus, multispans can be organised, instead of in a double category, in a categorical operad (internal category in the category of operads).

While there are many different approaches to operadic structures in the 1-categorical setting, in the ∞-categorical one a very convenient and powerful framework is that of the algebraic patterns of [CH21], which extract the necessary data on a category of shapes to speak of Segal decompositions (inert morphisms from elementary objects) and keep additional algebraic operations (active morphisms): in other words, they give a geometric presentation of ∞-operadic structure, while remembering what is the algebraic part. In the approach that we will follow in this note, the choice of an algebraic pattern will play the role of the choice of the cartesian monad $\mathscr{T}$ in the story sketched above.

We will then construct in section 4, for any algebraic pattern $\mathfrak{P}$ (satisfying the very mild condition of soundness — that will be verified in all examples we know of, in particular in section 3 for ω-categories) and any complete enough $(\infty,1)$-category $\mathfrak{C}$, a Segal $\mathfrak{P}$-object $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ in $(\infty,1)$-$\mathfrak{Cat}$

of $\mathfrak{P}$-shaped spans in $\mathfrak{C}$, by adapting the construction of [Hau18a] with the ideas raised in [Str00] and expanded upon in [Web07, Example 4.8]. We will continue in section 5 by showing the result promised above:

Theorem A. *(cf. Theorem 5.7) There is an equivalence between $\mathfrak{P}$-monads in the $\mathfrak{P}$-monoidal ∞-category $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ and Segal $\mathfrak{P}$-objects in $\mathfrak{C}$.*

Then, in section 6, we will spell out the meaning of this result in examples of interest, in particular (∞, ℓ)-categories.

1.3 Aknowledgements

This note was closely inspired by the ideas of [Hau18a, Hau21] and [Str00], and would not exist without the insights developed in these works. Thanks are also due to Damien Calaque for discussions about algebras in iterated spans, to Hugo Pourcelot for conversations about differently-shaped spans, and to Reuben Stern for useful comments about the interpretation of fibrous Θ^{op}-patterns. Many thanks to Jan Steinebrunner and Thomas Blom for pointing out the necessity of soundness in lemma 4.7 and the automaticity of global saturation, and to Jan especially sharing material on sound patterns and a proof of global saturation. I thank the anonymous reviewer for several corrections and improvements.

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2. Algebraic and fibrous patterns

Definition 2.1 (Algebraic pattern)**.** An **algebraic pattern** is a diagram of inclusions of $(\infty, 1)$-categories

$$\begin{tikzcd}
 & & \mathfrak{P} & \\
\mathfrak{P}^{\mathrm{el}} \arrow[r, hook] & \mathfrak{P}^{\mathrm{inrt}} \arrow[ur, hook] & & \mathfrak{P}^{\mathrm{act}} \arrow[ul, hook']
\end{tikzcd} \tag{2}$$

where the wide sub-$(\infty, 1)$-categories $(\mathfrak{P}^{\mathrm{inrt}}, \mathfrak{P}^{\mathrm{act}})$ form an orthogonal factorisation system on $\mathfrak{P}$ and $\mathfrak{P}^{\mathrm{el}} \subset \mathfrak{P}^{\mathrm{inrt}}$ is a full sub-$(\infty, 1)$-category.

The **inert** arrows (those in $\mathfrak{P}^{\mathrm{inrt}}$) are denoted as $\rightarrowtail$ and the **active** ones (those in $\mathfrak{P}^{\mathrm{act}}$) are denoted as $\rightsquigarrow$, while the objects in $\mathfrak{P}^{\mathrm{el}}$ are known as **elementary**.

Notation 2.2. For any $P \in \mathfrak{P}$, we write $\mathfrak{P}^{\mathrm{el}}_{P/} = \mathfrak{P}^{\mathrm{el}} \times_{\mathfrak{P}^{\mathrm{inrt}}} \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

An $(\infty, 1)$-category $\mathfrak{C}$ is said to be **$\mathfrak{P}$-complete** if it admits limits of diagrams of shape $\mathfrak{P}^{\mathrm{el}}_{P/}$ for any $P \in \mathfrak{P}$.

Definition 2.3 (Segal object)**.** Let $\mathfrak{P}$ be an algebraic pattern and $\mathfrak{C}$ a $\mathfrak{P}$-complete $(\infty, 1)$-category. A **Segal $\mathfrak{P}$-object** in $\mathfrak{C}$ is a functor $\mathscr{X}\colon \mathfrak{P} \to \mathfrak{C}$ such that $\mathscr{X}|_{\mathfrak{P}^{\mathrm{inrt}}}$ is the right Kan extension of its restriction to $\mathfrak{P}^{\mathrm{el}}$, which means that for any $P \in \mathfrak{P}$, the canonical arrow

$$\mathscr{X}(P) \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathscr{X}(E) \tag{3}$$

is an equivalence.

The full sub-$(\infty, 1)$-category of the functor $(\infty, 1)$-category $\left\{\mathfrak{P}, \mathfrak{C}\right\}$ on the Segal objects is denoted $\mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{C})$.

Example 2.4 (Product patterns). The $(\infty, 1)$-category of algebraic patterns admits all limits, which can be computed at the level of the underlying $(\infty, 1)$-categories. In particular, it admits products, and these are compatible with currying, in that if $\mathfrak{P}$ and $\mathfrak{Q}$ are two algebraic patterns and $\mathfrak{C}$ is $\mathfrak{P} \times \mathfrak{Q}$-complete, then $\mathfrak{Seg}_{\mathfrak{Q}}(\mathfrak{C})$ is $\mathfrak{P}$-complete and there is an equivalence $\mathfrak{Seg}_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{C}) \simeq \mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{Seg}_{\mathfrak{Q}}(\mathfrak{C}))$.

Example 2.5 ($\mathfrak{P}$-graphs). As observed in [CH21, beginning of §8], any algebraic pattern $\mathfrak{P}$ restricts to a pattern structure on $\mathfrak{P}^{\mathrm{inrt}}$, whose only active morphisms are the equivalences, and further restricts to $\mathfrak{P}^{\mathrm{el}}$. Evidently, the restriction–right Kan extension adjunctions along $\mathfrak{P}^{\mathrm{inrt,el}} = \mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{inrt}}$ and $\mathfrak{P}^{\mathrm{el,el}} = \mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{el}}$ induce equivalences $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}}, \mathfrak{C}\right\}$ and $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}}, \mathfrak{C}\right\}$ for any $\mathfrak{P}$-complete $(\infty, 1)$-category $\mathfrak{C}$. We will refer to (necessarily Segal) $\mathfrak{P}^{\mathrm{el}}$-objects as **$\mathfrak{P}$-graph**, and to the restriction of a Segal $\mathfrak{P}$-object to $\mathfrak{P}^{\mathrm{el}}$ as its **underlying $\mathfrak{P}$-graph**.

When $\mathfrak{C}$ is $(\infty, 1)$-$\mathfrak{Cat}$, Segal $\mathfrak{P}$-objects $\mathfrak{P} \to (\infty, 1)$-$\mathfrak{Cat}$ can also be seen as categoy objects in the ∞-category $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$ of Segal $\mathfrak{P}$-∞-groupoids, and as such will generally be written as $\mathbb{X}, \mathbb{Y}, \ldots$, in the font reserved for internal categories. Such an object $\mathbb{X}\colon \mathfrak{P} \to (\infty, 1)$-$\mathfrak{Cat}$ can be recast as a cocartesian fibration $\mathfrak{X} = \int^P \mathbb{X} \to \mathfrak{P}$ satisfying the Segal condition for its fibres. We call such fibrations **Segal $\mathfrak{P}$-fibrations**. A certain weakening of this notion turns out to be extremely useful, in particular to define lax morphisms between Segal fibrations.

Definition 2.6 (Fibrous pattern)**.** Let $\mathfrak{P}$ be an algebraic pattern. A **fibrous $\mathfrak{P}$-pattern** is an $(\infty, 1)$-functor $\not{f} \colon \mathfrak{X} \to \mathfrak{P}$ such that:

1. for every object $X \in \mathfrak{X}$, every inert arrow $i \colon \not{f}X \to P$ in $\mathfrak{P}$ admits a $\not{f}$-cocartesian lift $i_! \colon X \to i_! X$;

2. for every $P \in \mathfrak{P}$, the commutative square

$$\begin{tikzcd} \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{P}^{\mathrm{act}}_{/P} \arrow[r] \arrow[d] & \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{P}^{\mathrm{act}}_{/E} \arrow[d] \\ \mathfrak{P}^{\mathrm{act}}_{/P} \arrow[r, "\lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} (P \rightarrowtail E)_!"'] & \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{P}^{\mathrm{act}}_{/E} \end{tikzcd} \tag{4}$$

   is cartesian.

A **morphism of fibrous $\mathfrak{P}$-patterns** from $\mathfrak{X} \to \mathfrak{P}$ to $\mathfrak{Y} \to \mathfrak{P}$ is an ∞-functor $\mathfrak{X} \to \mathfrak{Y}$ over $\mathfrak{P}$ preserving cocartesian arrows over inert arrows of $\mathfrak{P}$.

Morphisms from $\mathfrak{X} \to \mathfrak{P}$ to $\mathfrak{Y} \to \mathfrak{P}$ are also called $\mathfrak{X}$-algebras in $\mathfrak{Y}$, and their $(\infty, 1)$-category is denoted $\mathfrak{Alg}_{\mathfrak{X}}(\mathfrak{Y})$.

Lemma 2.7 ([CH21, Lemma 9.10])**.** *The domain of fibrous pattern $\not{f} \colon \mathfrak{X} \to \mathfrak{P}$ admits a structure of algebraic pattern, where an arrow is active if it is over an active arrow of $\mathfrak{P}$, inert if it is $\not{f}$-cocartesian and lies over an inert arrow, and an object is elementary if it lies over an elementary of $\mathfrak{P}$. In particular, Segal morphisms between (sources of) fibrous $\mathfrak{P}$-patterns are exactly their morphisms of fibrous $\mathfrak{P}$-patterns.*

If $\mathfrak{X} \to \mathfrak{P}$ and $\mathfrak{Y} \to \mathfrak{P}$ are Segal $\mathfrak{P}$-fibrations, with corresponding $\mathfrak{P}$-monoidal $(\infty, 1)$-categories $\mathbb{X}$ and $\mathbb{Y}$, morphisms of fibrous patterns $\mathfrak{X} \to \mathfrak{Y}$ can be seen as the lax morphisms $\mathbb{X} \to \mathbb{Y}$.

Definition 2.8 ($\mathfrak{P}$-Monads)**.** Let $\mathscr{f}\colon \mathfrak{X} \to \mathfrak{P}$ be a fibrous $\mathfrak{P}$-pattern. A **$\mathfrak{P}$-monad** in $\mathfrak{X}$ is a morphism from the terminal (weak) Segal $\mathfrak{P}$-fibration $\mathfrak{P} \xrightarrow{\mathrm{id}} \mathfrak{P}$ to $\mathscr{f}$.

In other words, a $\mathfrak{P}$-monad is a $\mathfrak{P}$-algebra in $\mathfrak{X}$.

Remark 2.9. When $\mathfrak{P}$ is the algebraic pattern $\Delta^{\mathrm{op}\natural}$ for internal categories (recalled in section 6.2), this recovers the usual definition of monads in double ∞-categories. More generally, for enrichable patterns, typically those denoted with a $(-)^{\natural}$ superscript in [CH21], $\mathfrak{P}$-monads can be thought of as a kind of $\mathfrak{P}$-shaped generalisation of monads, as will be explained in examples in section 6. On the other hand, for the associated cartesian patterns $\mathfrak{P}^{\flat}$, then $\mathfrak{P}^{\flat}$-monads correspond rather to a kind of monoids, recovering the etymologically motivating observation of [Bén67, § (5.4.1)].

An important technical condition on algebraic patterns will be that of soundness from [BHS22], which we will introduce with an alternative (equivalent) presentation due to [BS≥25] that is convenient to handle.

Construction 2.10. Let $\mathscr{f}\colon \mathfrak{O} \to \mathfrak{P}$ be a morphism of algebraic patterns. The **inert factorisation** $(\infty,1)$**-category** of $\mathscr{f}$ at $O \in \mathfrak{O}$ and $v\colon \mathscr{f}O \rightarrowtail E \in \mathfrak{P}^{\mathrm{el}}_{\mathscr{f}O/}$ is

$$\mathfrak{Fact}^{\mathrm{inrt}}_{\mathscr{f}}(O,v) \coloneqq \mathfrak{O}^{\mathrm{el}}_{O/} \underset{\mathfrak{P}^{\mathrm{inrt}}_{\mathscr{f}O/}}{\times} \mathfrak{Fact}_{\mathfrak{P}^{\mathrm{inrt}}}(v) \tag{5}$$

where $\mathfrak{Fact}_{\mathfrak{P}^{\mathrm{inrt}}}(v) = \{v\} \times_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \left\{\mathbb{3}, \mathfrak{P}^{\mathrm{inrt}}\right\}$ is the $(\infty,1)$-category of factorisations of v in $\mathfrak{P}^{\mathrm{inrt}}$ (the pullback being defined relative to the functor $\mathbb{2} \to \mathbb{3}$ which is the unique endpoints-preserving one, encoding composition of a composable pair).

Definition 2.11 (Sound patterns)**.** An algebraic pattern $\mathfrak{P}$ is **sound** if for every $f\colon P \rightsquigarrow P'$ in $\mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P})$ and any $h\colon \mathrm{ev}_{\mathrm{o}} f = P \rightarrowtail E$ in $\mathfrak{P}^{\mathrm{el}}_{P/}$ the ∞-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_{\mathrm{o}}\colon \mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P}) \to \mathfrak{P}}(f\colon P \rightsquigarrow P', h\colon P \rightarrowtail E)$ is contractible.

One can see upon examination that the $(\infty,1)$-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_{\mathrm{o}}}(f,h)$, whose objects are diagram of the form

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, squiggly, "f"'] & M \arrow[r, tail] \arrow[d, squiggly] & E, \\ P' \arrow[r, tail] & E' & \end{tikzcd} \tag{6}$$

is equivalent to that denoted $\mathfrak{P}^{\mathrm{el}}_h(f) = \mathfrak{P}^{\mathrm{el}}_{P'/} \times_{\mathfrak{P}^{\mathrm{inrt}}_{P/}} (\mathfrak{P}^{\mathrm{inrt}}_{P/})_{/h}$ of [BHS22, Lemma 3.3.9. (2)], so that this definition recovers the notion of soundness from Definition 3.3.4 of *ibid*.

Remark 2.12. If $\mathfrak{P}$ is a sound algebraic pattern, fibrous $\mathfrak{P}$-patterns coincide with the more familiar (as a generalisation of the ∞-operads of [Lur17]) weak Segal $\mathfrak{P}$-fibrations (also called $\mathfrak{P}$-operads) of [CH21]. In particular, fibrous $\mathfrak{P}$-patterns which are also cocartesian fibrations are then the same thing as Segal $\mathfrak{P}$-fibrations.

Lemma 2.13. *A filtered colimit of sound algebraic patterns is sound.*

Proof. Let $\mathfrak{I}$ be a filtered $(\infty,1)$-category and $\mathscr{P}\colon \mathfrak{I} \to \mathfrak{AlgPatt}$ be a diagram of algebraic patterns all of which are sound. For any pattern $\mathfrak{P}$, the every ∞-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_0\colon \mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P})\to\mathfrak{P}}(f,h)$ are pullbacks of powers of $\mathfrak{P}$ by finite categories, so finite weighted limits, and hence their construction commutes with filtered colimits (in $(\infty,1)$-$\mathfrak{Cat}$, and since limits and filtered colimits of algebraic patterns are computed in $(\infty,1)$-$\mathfrak{Cat}$). Since a filtered colimit of contractible $(\infty,1)$-categories is contractible, as all the terms $\mathscr{P}I$ are sound, we do obtain that the inert factorisation ∞-categories of $\mathrm{colim}_{I\in\mathfrak{I}}\mathscr{P}I$ are contractible. $\square$

Finally, we describe a property of algebraic patterns which will be paramount for the construction and Segality of the categories of spans.

Notation 2.14 (Co-internalisation of a category). For any $(\infty,1)$-category $\mathfrak{E}$, we will let $\mathfrak{E}_{-/}$ denote the ∞-functor $\mathfrak{E}^{\mathrm{op}} \to (\infty,1)$-$\mathfrak{Cat}$ taking an object $E\in\mathfrak{E}$ to the slice $\mathfrak{E}_{E/}$ and an arrow $f\colon E\to E'$ to the **codependent coproduct** (or, plainly, precomposition by f) $\Sigma^f = (-\circ f)\colon \mathfrak{E}_{E'/}\to\mathfrak{E}_{E/}$. We refer to it as the **co-internalisation** of $\mathfrak{E}$, though it differs from the internalisation of $\mathfrak{E}^{\mathrm{op}}$ considered in [Str00] in that the latter, defined for $\mathfrak{E}$ admitting pushouts, has functoriality along f given by the right-adjoint (co-base change) of Σ^f — however, it is related, after passing to presheaves, to the co-internalisation of $\left\{\mathfrak{E},\infty\text{-}\mathfrak{Grpd}\right\}$.

Recall that in [CH21, Proposition 14.16. (2)], an algebraic pattern $\mathfrak{P}$ is said to be **saturated** if the inclusion $\mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}$ is codense[1], and under

[1]It is written there as $\mathfrak{P}^{\mathrm{inrt}} \hookrightarrow \mathfrak{P}$ being codense, but the proof of Proposition 14.20 immediately confirms this as a typo. In addition, extendability is required as part of the definition, but it is not necessary for this characterisation.

the mild assumptions of $\mathfrak{P}$ being slim and extendable, this is equivalent by [CH21, Proposition 14.20] to the more convenient condition of $\mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{inrt}}$ being codense. If the relevant limits $P \simeq \lim_{E\in\mathfrak{P}^{\mathrm{el}}_{P/}} E$ are co-Van Kampen (*i.e.* preserved by the co-internalisation, which again, due to the functoriality used, is different from being Van Kampen colimits in the opposite category, and is in fact rare), taking "global sections" of the co-internalisation $(\mathfrak{P}^{\mathrm{el}}_{P/})_{-/}$ produces for every $P \in \mathfrak{P}$ an equivalence $\mathrm{colim}_{E\in(\mathfrak{P}^{\mathrm{el}}_{P/})^{\mathrm{op}}} \mathfrak{P}^{\mathrm{el}}_{E/} \simeq \mathfrak{P}^{\mathrm{el}}_{P/}$ which we may thus think of as **global saturation** for the pattern $\mathfrak{P}$. As has been observed independently by Thomas Blom and Jan Steinebrunner (who graciously provided the following proof), this weaker property turns out to be satisfied by every algebraic pattern, regardless of saturation and preservation.

Proposition 2.15. *For any $(\infty, 1)$-category $\mathfrak{E}$, the functor* $\mathrm{colim}_{\mathfrak{E}^{\mathrm{op}}} \mathfrak{E}_{-/} \to \mathfrak{E}$ *induced by the projections $\mathfrak{E}_{E/} \to \mathfrak{E}$ is an equivalence.*

Proof [Ste25]. The colimit of the co-internalisation functor $\mathrm{colim}_{E\in\mathfrak{E}^{\mathrm{op}}} \mathfrak{E}_{E/}$ can be computed in two steps: first take its lax colimit, which by [GHN17, Corollary 7.6] is its Grothendieck construction $\mathrm{ev}_0 : \mathcal{A}\mathrm{r}(\mathfrak{E}) \to \mathfrak{E}$ of [Lur09, Corollary 2.4.7.11], and then rectify by localising at the ev_0-cartesian arrows. By [Lur09, Lemma 2.4.7.5] or [RV22, Lemma 7.4.3. (iii)], these are precisely those whose image under ev_1 is an equivalence. Thus, we need only exhibit $\mathrm{ev}_1 : \mathcal{A}\mathrm{r}(\mathfrak{E}) \to \mathfrak{E}$ as such a localisation, which it is by [Cis19, Proposition 7.1.12] because it is a cocartesian fibration (whence smooth) whose fibres $\mathfrak{E}_{/E}$, admitting terminal objects, are contractible. □

Note that, as is seen by unfolding the formulas, this is also the content of [GHN17, Corollary 7.5].

Corollary 2.16. *Any algebraic pattern $\mathfrak{P}$ is globally saturated, that is: for any $P \in \mathfrak{P}$ the canonical map*

$$\underset{E\in(\mathfrak{P}^{\mathrm{el}}_{P/})^{\mathrm{op}}}{\mathrm{colim}} \mathfrak{P}^{\mathrm{el}}_{E/} \to \mathfrak{P}^{\mathrm{el}}_{P/} \tag{7}$$

induced by the Σ^u for each $u : P \rightarrowtail E$ is an equivalence.

Proof [Ste25]. All the ∞-categories appearing in the colimit can be rewritten as slices $\mathfrak{P}^{\mathrm{el}}_{E/} \simeq (\mathfrak{P}^{\mathrm{el}}_{P/})_{E/}$ of $\mathfrak{P}^{\mathrm{el}}_{P/}$. Therefore, we can simply apply the preceding Proposition to $\mathfrak{E} = \mathfrak{P}^{\mathrm{el}}_{P/}$. □

This argument establishing global saturation for all algebraic patterns is somewhat inexplicit, relying on technical properties of localisation functors, so when applying it to specific algebraic patterns it can often be instructive to verify the reason for the property through direct examination of the pattern so as to get a more thorough understanding of its inner workings.

Example 2.17. [Hau18a, Proposition 5.13] shows explicitly how the algebraic pattern $\Delta^{\mathrm{op}\natural}$ for internal categories is globally saturated, in a way that we will now generalise to internal higher categories.

3. Saturation properties of the cell category Θ

The main example of interest for applying the result on general Segal objects we will establish in section 5 is that of (internal) ω-categories. In order to provide a good understanding of it for readers less familiar with Joyal's cell category, we will here study "by hand" the global saturation — and, along the way, saturation — of the relevant algebraic pattern.

Construction 3.1. Recall that the (non-reflexive) **globe category** $\mathbb{G}$ is generated by objects $\overline{n}$, for all $n \in \mathbb{N}$, and arrows $i_n^{\pm}\colon \overline{n} \to \overline{n+1}$, as presented in the graph

$$\overline{0} \overset{i_0^+}{\underset{i_0^-}{\rightrightarrows}} \overline{1} \overset{i_1^+}{\underset{i_1^-}{\rightrightarrows}} \cdots \overset{i_{n-1}^+}{\underset{i_{n-1}^-}{\rightrightarrows}} \overline{n} \overset{i_n^+}{\underset{i_n^-}{\rightrightarrows}} \cdots, \tag{8}$$

with the relations $i_{n+1}^+ i_n^\varepsilon = i_{n+1}^- i_n^\varepsilon$ for any $n \in \mathbb{N}$ and any $\varepsilon \in \{+,-\}$. A **globular object** in an $(\infty,1)$-category $\mathfrak{C}$ is a $\mathfrak{C}$-valued presheaf on $\mathbb{G}$. A **strict ω-category** is a globular set equipped with units and composition operations satisfying certain equations (spelled out for example in [Str00, p. 300]); such structure is monadic over $\left\{\mathbb{G}^{\mathrm{op}}, \mathfrak{Set}\right\}$, with monad $\mathscr{F}_\omega$.

The **cell category** Θ (first introduced in [Joy97]) has as objects the globular sets that are pastings of appropriately composable globes — a condition encoded precisely as the notion of globular sums in the sense of [Ara10, §2.1.1] or [Lou23, §1.1.2.2] — and as morphisms the morphisms of strict ω-categories between their associated free ω-categories. A morphism f is **inert** (also called an immersion) if it is the image by $\mathscr{F}_\omega$ of a morphism of globular sets, and **active** if in any factorisation $f = ia$ with i inert, i must be an identity (by [Ara10, Proposition 3.3.11], they correspond to the maps also known as algebraic, or covers). By [Ber02, Lemma 1.11] or [Ara10, Proposition 3.3.10], the classes of active and inert morphisms form a unique factorisation system, so in particular an orthogonal one, on Θ.

Notation 3.2 (Generic n-cells). For any $n \in \mathbb{N}$, the representable presheaf $\mathbb{G}(-,\overline{n})$ is canonically endowed with a structure of strict ω-category (which comes from viewing it as the restriction to $\mathbb{G}^{\mathrm{op}}$ of $\Theta(-,\overline{n})$). We denote this ω-category $\mathfrak{D}_n$; it is known as the **n-globe**, or as the generic (or "walking") n-cell.

Definition 3.3. The algebraic pattern $\Theta^{\mathrm{op}\natural}$ is the category Θ^{op}, endowed with the inert–active factorisation system described above, and with elementary objects the ℓ-globes (so that $\Theta^{\mathrm{op}\natural,\mathrm{el}} \simeq \mathbb{G}^{\mathrm{op}}$).

It is an immediate consequence of the definition (and of the fact that all inert maps into a globe in Θ also have to be from a globe) that Segal $\Theta^{\mathrm{op}\natural}$-objects are exactly what are called Θ-models in [Ber02].

Lemma 3.4 ([Ara10, Proposition 2.3.18])**.** *The pattern $\Theta^{\mathrm{op}\natural}$ is saturated.*

Proof. This is essentially a consequence of the definition of globular sums: any such globular set T can be written as an iterated pushout $T \simeq \mathfrak{D}_{i_1} \amalg_{\mathfrak{D}_{i'_1}} \cdots \amalg_{\mathfrak{D}_{i'_{p-1}}} \mathfrak{D}_{i_p}$, and by [Ara10, Lemme 2.3.22], the immersions $\mathfrak{D}_i \rightarrowtail T$ featuring in this pushout define a cofinal subcategory of $\Theta^{\natural,\mathrm{el}}_{/T}$. □

It follows from this that the definition of Segal $\Theta^{\mathrm{op}\natural}$-objects in a complete $(\infty,1)$-category $\mathfrak{C}$ coincides with that of (weak) ω-categories in $\mathfrak{C}$ in the sense of [Lou23], albeit without the Rezk-completeness (or univalent completeness) condition — so that, to be more precise, they correspond to flagged ω-categories as in [AF18].

Remark 3.5. For any $\ell \in \mathbb{N} \cup \{\omega\}$, we let

$$\Theta_\ell = \Theta \cap (\infty,\ell)\text{-}\mathfrak{Cat} \tag{9}$$

be the ℓ-dimensional cell category used in [Rez10]; the pattern structure of Definition 3.3 restricts to one on ${\Theta_\ell}^{\mathrm{op}}$ whose Segal objects are internal flagged ℓ-categories (and we obviously have $\Theta_\omega = \Theta$).

There is an obvious filtration ${\Theta_1}^{\mathrm{op}\natural} \simeq \Delta^{\mathrm{op}\natural} \hookrightarrow {\Theta_2}^{\mathrm{op}\natural} \hookrightarrow {\Theta_3}^{\mathrm{op}\natural} \hookrightarrow \cdots$ and we recover $\Theta^{\mathrm{op}\natural}$ as its colimit. In particular, since each ${\Theta_\ell}^{\mathrm{op}\natural}$ for ℓ finite is known to be sound from [BHS22, Example 3.3.18], it follows from Lemma 2.13 that $\Theta^{\mathrm{op}\natural}$ is sound as well.

Construction 3.6. Since the cells in a pasting diagram are unlabelled, the standard representation of objects of Θ contains redundant information. A more minimal presentation, suggested by [Bat98] and developed more thoroughly in [Ber02] and [Ara10], of these objects is as level trees, functors from some $[\ell]^{\mathrm{op}}$ (ℓ being the categorical dimension) to Δ whose value at the terminal object 0 is $[0]$: the cells in the corresponding pasting diagram can be all recovered as the sectors in the tree.

This description makes it easier to get a handle on the structure of these trees and their categories of inert morphisms: for a tree $T\colon [\ell]^{\mathrm{op}} \to \Delta$, for $k \leq \ell$, we set $|T|(k)$ to be the reunion, over $i \in T(k)$, of the $T(k+1)_i + 1$, where $T(k+1)_i$ is the fibre of $T(k+1) \to T(k)$ at i (and where we decreed $T(\ell+1)$ to be $[-1] = \varnothing$). Note that the assignment $[\ell]^{\mathrm{op}} \ni k \mapsto |T|(k)$ is *not* functorial; however ${\mathbb{G}_{\leq \ell}}^{\mathrm{op}} \ni \mathfrak{D}_k \mapsto |T|(k)$ can be made functorial.

Remark 3.7. The objects of $|T|(k)$ can be understood as the **sectors** at level k as defined by [Ber02] (and, likewise, their ordering is the natural left-to-right ordering of sectors in each fibre), so that $|T|$ coincides with the globular set denoted T^* in [Bat98].

In the dictionary between globular sums and trees, it is the sectors of a tree that correspond to the cells of the corresponding globular sum.

We will now use the decompositions provided by the proof of Lemma 3.4 to understand the categories $\Theta^{\natural,\mathrm{el}}_{/T}$.

Lemma 3.8. *Let $T \in \Theta$ be any globular sum. Then $\Theta^{\natural,\mathrm{el}}_{/T}$ is equivalent to the Grothendieck construction of the globular set $|T|$.*

Proof. We will exhibit an explicit isomorphism between $|T|$ and the underlying $\Theta^\natural$-graph of the presheaf represented by T, since then its Grothendieck construction is indeed the slice. Consider an object of $\Theta^{\natural,\mathrm{el}}_{/T}$, given by a map $\mathfrak{D}_i \rightarrowtail T$, *i.e.* an element of $T(\mathfrak{D}_i)$. Since $\mathfrak{D}_i$ is the free i-cell, this map is uniquely characterised by a choice of an i-cell in T. In terms of the associated trees, D_i is a linear tree and so such a map is characterised by a choice of a branch at level i and a sector around its top point. It follows from Remark 3.7 that these are exactly counted by the elements of $|T|(\mathbb{D}_i)$. $\square$

Example 3.9. For any elementary $\mathbb{D}_\ell$, the category $\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_\ell}$ is freely generated by the graph

$$\begin{tikzcd}
& \ulcorner C_\ell \urcorner & \\
\ulcorner C^-_{\ell-1} \urcorner \arrow[ur] & & \ulcorner C^+_{\ell-1} \urcorner \arrow[ul] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
\ulcorner C^-_1 \urcorner \arrow[u] \arrow[urr] & & \ulcorner C^+_1 \urcorner \arrow[u] \arrow[ull] \\
\ulcorner C^-_0 \urcorner \arrow[u] \arrow[urr] & & \ulcorner C^+_0 \urcorner \arrow[u] \arrow[ull]
\end{tikzcd} \tag{10}$$

where we recall that $\mathbb{D}_\ell$ has a unique ℓ-cell C_ℓ and, for any $0 \leq i < \ell$, two i-cells $C_i^\pm$ serving has source and target for the higher cells, and $\ulcorner C_i^\pm \urcorner \colon \mathbb{D}_i \to \mathbb{D}_\ell$ denotes the (inert) map selecting the corresponding cell. In other words, $\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_\ell}$ is the free-living ℓ-iterated cospan, so that the category we are ultimately interested in, ${\Theta^{\mathrm{op}}}^{\natural,\mathrm{el}}_{\mathbb{D}_\ell/}$, which is its opposite, will be the free-living ℓ-iterated span.

Lemma 3.10. *Let T be a globular sum of the form $\mathbb{D}_m \amalg_{\mathbb{D}_\ell} \mathbb{D}_n$. Then $\Theta^{\natural,\mathrm{inrt}}_{/T}$ is the strict pushout of 1-categories $\Theta^{\natural,\mathrm{inrt}}_{/\mathbb{D}_m} \amalg_{\Theta^{\natural,\mathrm{inrt}}_{/\mathbb{D}_\ell}} \Theta^{\natural,\mathrm{inrt}}_{/\mathbb{D}_n}$*

Proof. Let us call E_i^ε the cells of T in $\mathbb{D}_m$, F_i^ε those in $\mathbb{D}_n$, and C_i^ε those in $\mathbb{D}_\ell$, so that we have $E_i^\varepsilon = C_i^\varepsilon = F_i^\varepsilon$ for $i < \ell$ and $E_\ell^+ = C_\ell = F_\ell^-$. The matter is then that of enumerating the cells and their relations, for which no listing can be as clear as simply drawing a generating graph:

$$
\begin{tikzcd}
& E_m & & & & F_n & \\
E^-_{m-1} \arrow[ur] & & E^+_{m-1} \arrow[ul] & & F^-_{n-1} \arrow[ur] & & F^+_{n-1} \arrow[ul] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] & & \vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
E^-_{\ell+1} \arrow[u] \arrow[urr] & & E^+_{\ell+1} \arrow[u] \arrow[ull] & & F^-_{\ell+1} \arrow[u] \arrow[urr] & & F^+_{\ell+1} \arrow[u] \arrow[ull] \\
E^-_\ell \arrow[u] \arrow[urr] & & & C_\ell \arrow[ul] \arrow[ulll] \arrow[ur] \arrow[urrr] & & & F^+_\ell \arrow[u] \arrow[ull] \\
& C^-_{\ell-1} \arrow[ul] \arrow[urr] \arrow[urrrrr] & & & & C^+_{\ell-1} \arrow[ur] \arrow[ull] \arrow[ulllll] & \\
& \vdots \arrow[u] \arrow[urrrr] & & & & \vdots \arrow[u] \arrow[ullll] & \\
& C^-_0 \arrow[u] \arrow[urrrr] & & & & C^+_0 . \arrow[u] \arrow[ullll] &
\end{tikzcd}
\tag{11}
$$

Since the Grothendieck construction takes colimits of presheaves (of sets) to strict colimits of 1-categories, and our slices are categories of elements as in Lemma 3.8, one can indeed recognise in eq. (11) a strict pushout of three versions of eq. (10). □

Proposition 3.11. *The algebraic pattern $\Theta^{\mathrm{op}\natural}$ is globally saturated.*

Proof. Again, we can use the decomposition $T \simeq \mathfrak{D}_{i_1} \amalg_{\mathfrak{D}_{i'_1}} \cdots \amalg_{\mathfrak{D}_{i'_{p-1}}} \mathfrak{D}_{i_p}$ since it is cofinal, so that all we have to prove is that

$$\Theta^{\natural,\mathrm{el}}_{/T} \simeq \Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i_1}} \coprod_{\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i'_1}}} \cdots \coprod_{\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i'_{p-1}}}} \Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i_p}} . \tag{12}$$

To compute this pushout of $(\infty, 1)$-categories, we will use the Joyal model structure for quasicategories. Letting $N_\bullet\mathfrak{C}$ denote the nerve of an $(\infty, 1)$-category $\mathfrak{C}$, it is clear — since $i'_{j\pm 1} < i_j$ for all j in the decomposition —

that the maps of quasicategories $N_\bullet\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_{i'_{j\pm 1}}} \to N_\bullet\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_{i_j}}$ are injective in every degree, *i.e.* cofibrations in the Joyal model structure, so that the pushout will coincide with the pushout of 1-categories. The result for this strict pushout is then established *via* Lemma 3.10. □

4. Generalised spans

For this section, we fix an algebraic pattern $\mathfrak{P}$ and a $\mathfrak{P}$-complete $(\infty,1)$-category $\mathfrak{C}$. We will adapt to $\mathfrak{P}$ the constructions and arguments of [Hau18a, §5].

Recall that $\mathfrak{Ar}(\mathfrak{P}) := \left\{\mathbb{2},\mathfrak{P}\right\} \xrightarrow{\mathrm{ev}_0} \left\{\mathbb{1},\mathfrak{P}\right\} \simeq \mathfrak{P}$ is a cartesian fibration classifying the ∞-functor $\mathfrak{P}_{-/}\colon \mathfrak{P}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$. We let $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ be the full sub-$(\infty,1)$-category of $\mathfrak{Ar}(\mathfrak{P})$ on the inert arrows — which, by the dual of [BHS22, Proposition 2.2.2], still defines a cartesian fibration.

Our first goal is to show that $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_0} \mathfrak{P}$ classifies an ∞-functor $\mathfrak{P}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$ whose action on objects is $P \mapsto \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

Construction 4.1. Since the factorisation system of $\mathfrak{P}$ is functorial, projection onto the inert part of an arrow defines a functor $\mathit{inrt}\colon \left\{\mathbb{2},\mathfrak{P}\right\} \to \left\{\mathbb{3},\mathfrak{P}\right\} \to \left\{\mathbb{2},\mathfrak{P}\right\}$, which preserves the image of ev_0 so defines a morphism of categories over $\left\{\mathbb{1},\mathfrak{P}\right\}$ (but not of cartesian fibrations over $\mathfrak{P}$, as it does not preserve cartesian lifts of non-inert morphisms). We let $\mathit{inrt}(\mathfrak{Ar}(\mathfrak{P}))$ denote its essential image, whose objects are then the inert arrows of $\mathfrak{P}$ while morphisms are the squares all of whose edges are inert — so that, in particular, the fibre of $\mathrm{ev}_0|_{\mathit{inrt}(\mathfrak{Ar}(\mathfrak{P}))}$ at $P \in \mathfrak{P}$ is $(\mathfrak{P}^{\mathrm{inrt}})_{P/} = \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

Lemma 4.2. *Consider a commuting triangle of inert arrows below-left*

$$
\begin{tikzcd}
 & Q \arrow[d, tail, "h"] \\
P \arrow[ur, tail, "g"] \arrow[dr, tail, "g'"'] & Q'
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail, "\mathit{inrt}(gf)"'] \arrow[rr, bend left, "\Sigma^f g = gf"] \arrow[d, equal] & M \arrow[r, squiggly] \arrow[d, "\mathit{inrt}(\Sigma^f h)"] & Q \arrow[d, tail, "h"] \\
O \arrow[r, tail, "\mathit{inrt}(g'f)"] \arrow[rr, bend right, "\Sigma^f g' = hgf"'] & M' \arrow[r, squiggly] & Q'
\end{tikzcd}
\tag{13}
$$

defining a morphism in $\mathfrak{P}^{\mathrm{inrt}}_{P/}$, *and let* $O \xrightarrow{f} P$ *be any arrow of* $\mathfrak{P}$, *with inert–active factorisation of* $\Sigma^f h$ *as above-right. Then* $\mathcal{inrt}(\Sigma^f h)$ *is inert.*

Proof. This is a direct application of the left-cancellability property for the left class of an orthogonal factorisation system (see for example [Lur09, Proposition 5.2.8.6. (4)] or [Lou23, Proposition 4.1.2.12]). □

Corollary 4.3. *The projection* $\mathcal{inrt}(\mathfrak{Ar}(\mathfrak{P})) \to \mathfrak{P}$ *is a cartesian fibration, and coincides with* $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \to \mathfrak{P}$. □

We thus obtain an ∞-functor $\mathfrak{P}^{\mathrm{inrt}}_{-/}\colon \mathfrak{P}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$ (whose restriction to $(\mathfrak{P}^{\mathrm{inrt}})^{\mathrm{op}}$ is the co-internalisation of $\mathfrak{P}^{\mathrm{inrt}}$).

Definition 4.4. We denote $\overline{p}\colon \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ the cocartesian fibration classifying the ∞-functor $\left\{\mathfrak{P}^{\mathrm{inrt}}_{-/}, \mathfrak{C}\right\}\colon \mathfrak{P} \to (\infty,1)\text{-}\mathfrak{Cat}$.

Recall that by [Bar22, Proposition 2.37], for any $P \in \mathfrak{P}$ there is an algebraic pattern structure on the slice $\mathfrak{P}_{P/}$, where an object (resp. an arrow) is elementary (resp. inert, resp. active) if and only if its image by ev_1 is so in $\mathfrak{P}$. Furthermore, by [Kos21, Proposition 2.14 and Proposition 2.4], it restricts to an algebraic pattern structure on $\mathfrak{P}^{\mathrm{inrt}}_{P/}$ (which has no non-trivial active morphisms).

Definition 4.5. We call $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ the full sub-$(\infty,1)$-category of $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ on the objects $(P, \mathscr{F}\colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C})$ such that $\mathscr{F}$ is a Segal $\mathfrak{P}^{\mathrm{inrt}}_{P/}$-object.

Remark 4.6. An alternate construction of $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ is provided by [Kos21, Corollary 2.16].

We let $\iota^{\mathrm{inrt}}_{P}\colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{P}_{P/}$ denote the canonical inclusion (induced under slicing by $\mathfrak{P}^{\mathrm{inrt}} \hookrightarrow \mathfrak{P}$). By [Kos21, Proposition 2.15] (which is formulated in the case of $\mathfrak{P} = \Delta^{\mathrm{op}\natural}$ but only relies on the factorisation system), for any arrow $f\colon O \to P$ in $\mathfrak{P}$, the induced ∞-functor $\Sigma^{f,*}\colon \left\{\mathfrak{P}_{O/}, \mathfrak{C}\right\} \to \left\{\mathfrak{P}_{P/}, \mathfrak{C}\right\}$ sends the image of $\iota^{\mathrm{inrt}}_{O,!}$ into the image of $\iota^{\mathrm{inrt}}_{P,!}$.

We can then let $\overline{\mathfrak{PreSpan}}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ denote the Grothendieck construction of the ∞-functor $\left\{\mathfrak{P}_{-/}, \mathfrak{C}\right\}\colon \mathfrak{C} \to (\infty,1)\text{-}\mathfrak{Cat}$, and $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ is the full sub-$(\infty,1)$-category of $\overline{\mathfrak{PreSpan}}_{\mathfrak{P}}(\mathfrak{C})$ on those objects $(P, \mathscr{F}\colon \mathfrak{P}_{P/} \to \mathfrak{C})$ such that $\mathscr{F}$ is in the image of $\iota^{\mathrm{inrt}}_{P,!}$ (so that it is determined by its restriction $\mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C}$).

Lemma 4.7. *Assume the algebraic pattern $\mathfrak{P}$ is sound. The restricted projection $p\colon \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C}) \hookrightarrow \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C}) \xrightarrow{\overline{p}} \mathfrak{P}$ is a cocartesian fibration.*

Proof. As explained in the proof of [Hau18a, Corollary 5.12], since $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ is a full sub-$(\infty,1)$-category of $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$, all we need to do is check that if $(P,\mathscr{F}) \to (Q,\mathscr{G})$ is a $\overline{p}$-cocartesian morphism in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ such that $\mathscr{F}$ is Segal, then $\mathscr{G}$ is Segal as well. Note also that such a cocartesian morphism consists of an arrow $f\colon P \to Q$ in $\mathfrak{P}$ with $\mathscr{G} \simeq \Sigma^{f,*}\mathscr{F} = \mathscr{F} \circ (\Sigma^f)$: in other words, we must show that Segal objects are preserved by composition with codependent coproduct. That is, if $\mathscr{F}\colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C}$ satisfies the Segal condition, then for every $g\colon Q \rightarrowtail Q'$ we must have

$$\mathscr{F}\Big(\mathit{inrt}(P \xrightarrow{f} Q \overset{g}{\rightarrowtail} Q')\Big) \xrightarrow{\simeq} \lim_{(Q \overset{hg}{\rightarrowtail} E)\in(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}} \mathscr{F}\Big(\mathit{inrt}(P \xrightarrow{f} Q \overset{hg}{\rightarrowtail} E)\Big). \quad (14)$$

By the functoriality of the construction $\Sigma^{(-),*}$ and the fact that active and inert morphisms for a factorisation system, we only need to check the comparison in the cases where f is either purely inert or purely active. If f is inert, then inrt acts as the identity on the composition, and the functor $\Sigma^{f,*}$ is even iso-Segal (since both $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}$ and $(\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{gf/}$ are then equivalent to $\mathfrak{P}^{\mathrm{el}}_{Q'/}$ thanks to all maps being inert), which is strictly stronger than preserving Segal objects. The case of f being active is where the soundness assumption comes into play.

Write $P \overset{\mathit{inrt}(hgf)}{\rightarrowtail} M_{hg} \rightsquigarrow E$ the inert-active factorisation of the composition $hgf\colon P \rightsquigarrow Q \rightarrowtail E$, for any $Q \rightarrowtail E$ in $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}$ given by $h\colon Q' \rightarrowtail E$. Then, since $\mathscr{F}$ is Segal, the right-hand side limit in eq. (14) becomes a double limit

$$\lim_{(Q \overset{hg}{\rightarrowtail} E)\in(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}} \; \lim_{(M_{hg} \rightarrowtail E')\in(\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{M_{hg}/}} \mathscr{F}(P \rightarrowtail M_{hg} \rightarrowtail E'), \quad (15)$$

summarised by the dashed arrows in the diagram

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, squiggly, "f"'] & M_g \arrow[r, tail, "\mathit{inrt}(h)"] \arrow[d, squiggly] & M_{hg} \arrow[r, tail, dashed] \arrow[d, squiggly] & E', \\ Q \arrow[r, tail, "g"'] & Q' \arrow[r, tail, "h"'] & E & \end{tikzcd} \quad (16)$$

where the inert transport arrow $\mathit{inrt}(h)$ comes from lemma 4.2. In particular, we can see that $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/} \simeq \mathfrak{P}^{\mathrm{el}}_{Q'/}$ and $(\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{M_{hg}/} \simeq \mathfrak{P}^{\mathrm{el}}_{M_{hg}/}$.

Then, as explained in [BHS22, Observation 3.3.6], soundness of $\mathfrak{P}$ allows this double limit to be computed as a limit over $(M_g \rightarrowtail E') \in \mathfrak{P}^{\mathrm{el}}_{M_g/}$. But this is precisely what we obtain from the Segal decomposition for $\mathscr{F}(P \rightarrowtail M_g)$ in the left-hand side term of eq. (14). $\square$

Proposition 4.8. *Let $\mathfrak{P}$ be a sound algebraic pattern. The cocartesian fibration $p\colon \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ is a Segal fibration, that is the ∞-functor $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})\colon \mathfrak{P} \to (\infty,1)\text{-}\mathfrak{Cat}$ it classifies defines a $\mathfrak{P}$-monoidal $(\infty,1)$-category.*

Proof. To make the definition explicit, we need to show that for any $P \in \mathfrak{P}$,

$$\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{P/}}(\mathfrak{C}) \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{E/}}(\mathfrak{C}) \tag{17}$$

is an equivalence. Since $\mathfrak{P}^{\mathrm{inrt}}_{P/}$ only has inert morphisms, the right Kan extension ∞-functor

$$\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{el}}_{P/}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}}_{P/}, \mathfrak{C}\right\} \to \mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{P/}}(\mathfrak{C}) \tag{18}$$

is an equivalence. Similarly, every factor $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{E/}}(\mathfrak{C})$ in eq. (17) is equivalent to $\left\{\mathfrak{P}^{\mathrm{el}}_{E/}, \mathfrak{C}\right\}$, and so the map of eq. (17) takes the form

$$\left\{\mathfrak{P}^{\mathrm{el}}_{P/}, \mathfrak{C}\right\} \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \left\{\mathfrak{P}^{\mathrm{el}}_{E/}, \mathfrak{C}\right\}. \tag{19}$$

By the property of global saturation from corollary 2.16, and since enriched homs (or cotensors) send colimits in the first variable to limits, this map is an equivalence. $\square$

5. $\mathfrak{P}$-Monads in $\mathfrak{P}$-spans

This section will follow very closely the structure of [Hau21, §3].

Lemma 5.1. *The ∞-functor $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$ admits a right adjoint right inverse.*

Proof. The functor $\ulcorner 1 \urcorner \colon \mathbb{1} \to \mathbb{2}$ has a retraction $\mathbb{2} \xrightarrow{!_{\mathbb{2}}} \mathbb{1}$, which upgrades in fact to a left adjoint left inverse: we clearly have $!_{\mathbb{2}} \circ \ulcorner 1 \urcorner = !_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, while there is a (unique, since $\mathbb{2}$ is posetal) natural transformation $\mathrm{id}_{\mathbb{2}} \Rightarrow \ulcorner 1 \urcorner \circ !_{\mathbb{2}} = \mathcal{const}_1$, and it is easily checked (by unicity of !) that these two transformations satisfy the triangle identities.

Now note that $\mathrm{ev}_1 \colon \mathfrak{Ar}(\mathfrak{P}) = \left\{\mathbb{2}, \mathfrak{P}\right\} \to \left\{\mathbb{1}, \mathfrak{P}\right\}$ is exactly given by $\left\{\ulcorner 1 \urcorner, \mathfrak{P}\right\}$, and so, as powering with $\mathfrak{P}$ is $(\infty, 2)$-functorial (that is, as an ∞-functor $\left\{(-), \mathfrak{P}\right\} \colon (\infty,1)\text{-}\mathfrak{Cat}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$, it is $(\infty,1)$-$\mathfrak{Cat}$-linear, and so upgrades to an $(\infty,2)$-functor), it has a right adjoint right inverse given by $\left\{!_{\mathbb{2}}, \mathfrak{P}\right\}$. The latter ∞-functor can be described very explicitly: it maps an object $P \in \mathfrak{P}$ to its identity arrow $\mathrm{id}_P \in \mathfrak{Ar}(\mathfrak{P})$.

In particular, it factors through $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ — as identity arrows are inert — and since this sub-$(\infty,1)$-category of $\mathfrak{Ar}(\mathfrak{P})$ is full, the astriction of $\left\{!_{\mathbb{2}}, \mathfrak{P}\right\}$ to it furnishes the desired right adjoint right inverse to $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$. □

Given its description, we will denote $\ulcorner \mathrm{id} \urcorner \colon \mathfrak{P} \to \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ the right adjoint right inverse to ev_1. The unit will simply be known as $\eta \colon \mathrm{id}_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \Rightarrow \ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$; its component at $(P \rightarrowtail Q) \in \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ is the square

$$\eta_{(P \rightarrowtail Q)} \colon \begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & Q \arrow[d, equal] \\ Q \arrow[r, equal] & Q. \end{tikzcd} \tag{20}$$

Proposition 5.2. *The ∞-functor $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$ exhibits $\mathfrak{P}$ as the localisation of $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ at the set $\mathcal{I}$ of* ev_0*-cartesian morphisms lying over inert arrows of $\mathfrak{P}$.*

Proof. Let $\mathcal{W}$ be the set of morphisms in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ inverted by ev_1; by [Lur09, Corollary 2.4.7.11 and Lemma 2.4.7.] (*cf.* also [BHS22, Proposition 2.2.2.(2)]), W consists exactly of the ev_0-cartesian morphisms, so that we do have $\mathcal{I} \subset \mathcal{W}$. If $(f, g) \colon (P \rightarrowtail Q) \to (P' \rightarrowtail Q')$ is a morphism in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ lifting $f \colon P \to P'$ in $\mathfrak{P}$, it is in $\mathcal{W}$ if and only if $g \colon Q \to Q'$ is an equivalence so that we have a commutative square

$$\begin{tikzcd} (P \rightarrowtail Q) \arrow[r, "(f,g)"] \arrow[d] & (P' \rightarrowtail Q') \arrow[d] \\ (Q = Q) \arrow[r, "\simeq"] & (Q' = Q') \end{tikzcd} \tag{21}$$

in which the two vertical morphisms are in $\mathcal{I}$. Any ∞-functor from $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to some $(\infty,1)$-category $\mathfrak{C}$ inverting the morphisms in $\mathcal{I}$ will then send the square of eq. (21) to a square whose veritcal arrows (in addition to the lower horizontal one) are equivalences, whence its upper horizontal is one as well since equivalences always satisfy the 2-of-3 property. This means that such an ∞-functor automatically inverts all the morphisms in $\mathcal{W}$, and we only need to show that ev_1 is a localisation, along $\mathcal{W}$. This follows readily from the fact that it has a right adjoint right inverse (in fact it is equivalent to it), but in our specific situation it can be seen in a more explicit way.

Let $\mathfrak{C}$ be again any $(\infty,1)$-category and let us consider the comparison ∞-functor $\left\{\mathrm{ev}_1, \mathfrak{C}\right\} \colon \left\{\mathfrak{P}, \mathfrak{C}\right\} \to \left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathcal{W})}$, where the target denotes the full sub-$(\infty,1)$-category of $\left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}$ on the ∞-functors inverting the morphisms in $\mathcal{W}$ (through which $\left\{\mathrm{ev}_1, \mathfrak{C}\right\}$ does factor by definition of $\mathcal{W}$). The crux of the matter is that the components of the unit transformation η all belong to $\mathcal{I}$ — as can be seen in eq. (20) — and so *a fortiori* to $\mathcal{W}$. Hence, the adjunction $\left\{\mathrm{ev}_1, \mathfrak{C}\right\} \dashv \left\{\ulcorner \mathrm{id} \urcorner, \mathfrak{C}\right\}$ restricts on $\left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathcal{W})}$ to an equivalence (as its counit was already an identity, and its unit becomes one after this restriction), which means that ev_1 is a localisation along $\mathcal{W}$. $\square$

Construction 5.3. Let $\not{f} \colon \mathfrak{X} \to \mathfrak{P}$ be an ∞-functor such that $\mathfrak{X}$ admits $\not{f}$-

cocartesian lifts of inert morphisms. Consider the solid pullback

$$
\begin{tikzcd}
 & \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ddl, bend left, "\mathrm{ev}_0"] & & & \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[lll, "\mathrm{ev}_0^* \ell"'] \arrow[ddl, bend left, "\ell^* \mathrm{ev}_0"] \\
\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ur, bend left, "\mathrm{id}"] \arrow[ur, dashed, bend left=60, "\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] \arrow[ur, phantom, "\Leftarrow \eta"] \arrow[d, phantom, shift left=4ex, "\Rightarrow \mathrm{ev}_0 \eta"] & & & \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[lll, "\mathrm{ev}_0^* \ell"] \arrow[ur, bend right, "\ell^* \mathrm{id} = \mathrm{id}"'] \arrow[ur, dotted, bend left] \arrow[ur, phantom, "\Leftarrow \eta_!"] \arrow[d, "\ell^* \mathrm{ev}_0"] \arrow[dlll, phantom, very near start, "\llcorner"] & \\
\mathfrak{P} & & & \mathfrak{X} \arrow[lll, "\ell"] &
\end{tikzcd}
\tag{22}
$$

which is a (strongly) commutative diagram in the $(\infty, 2)$-category $(\infty, 1)$-$\mathfrak{Cat}$. Adding $\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$, represented as a dashed arrow, the induced back-left triangle does not commute; however, adding as well the unit cell η and its whiskering $\mathrm{ev}_0 \eta\colon\ \mathrm{ev}_0 = \mathrm{ev}_0 \circ \mathrm{id}_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \Rightarrow \mathrm{ev}_0 \circ \ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$ we obtain a "2-commutative" pasting diagram.

Now as ℓ admits cocartesian lifts of inert arrows so does its base-change $\mathrm{ev}_0^* \ell$ (since cocartesian lifts are stable by pullback, by the co-dual of [RV22, Proposition 5.2.4]), and so, using the formulation of cocartesian lifts from [RV22, Definition 5.4.2], the transformation $\eta(\mathrm{ev}_0 \ell^*)$, whose components are inert, admits an $\mathrm{ev}_0 \ell^*$-cocartesian dotted lift $\mathrm{id}_{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \overset{\eta_!}{\Longrightarrow} (\mathrm{id}_{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})})_\eta =: \ell^\eta(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1)$.

We can finally define

$$
\ell^* \mathrm{ev}_1 \coloneqq (\ell^* \mathrm{ev}_0) \circ \ell^\eta(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1). \tag{23}
$$

Explicitly, $\ell^* \mathrm{ev}_1$ sends an object $(X, \jmath\colon \ell X \rightarrowtail Q) \in \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to $(X_Q, Q = Q)$ where $X \xrightarrow{\jmath_!} X_Q$ is a cocartesian lift of $\jmath$. By construction it comes equipped with a natural transformation that we will call $\ell^* \alpha \coloneqq$

$(f^*\mathrm{ev}_0)\eta_!\colon f^*\mathrm{ev}_0 \Rightarrow f^*\mathrm{ev}_1$, sitting in the diagram

$$\begin{tikzcd}
& \mathfrak{P} & & \mathfrak{X}\times_{\mathfrak{P}}\mathfrak{P}\simeq\mathfrak{X} \arrow[ll] \arrow[dl, bend left, "f^*\mathrm{id}_{\mathfrak{P}}=\mathrm{id}_{\mathfrak{X}}"] \\
\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ur, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] \arrow[ur, phantom, "\Uparrow\alpha"] & & \mathfrak{X}\underset{\mathfrak{P}}{\times}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ll] \arrow[ur, "f^*\mathrm{ev}_1"] \arrow[d, "f^*\mathrm{ev}_0"'] \arrow[ur, phantom, "\Uparrow f^*\alpha"'] \arrow[dll, phantom, "\llcorner"] & \\
\mathfrak{P} & & \mathfrak{X} \arrow[ll, "f"] &
\arrow[from=1-2, to=3-1, bend left]
\end{tikzcd} \tag{24}$$

whose front and back squares are cartesian, but whose top square is not — and where the natural transformation α comes from cotensoring with $\mathfrak{P}$ the canonical 2-cell $\ulcorner 0<1\urcorner\colon \ulcorner 0\urcorner \Rightarrow \ulcorner 1\urcorner\colon \mathbb{1}\to\mathbb{2}$ (in particular, it is easily checked that the adjunction $!_{\mathbb{2}}\dashv\ulcorner 1\urcorner$ lives under $\mathbb{1}$ so that $\mathrm{ev}_1\dashv\ulcorner\mathrm{id}\urcorner$ lives over $\mathfrak{P}$). Conjecturally, the right face of eq. (24) could be seen in terms of the $(\infty,3)$-topos of $(\infty,2)$-categories as the strong base change, along f admitting enough cocartesian lifts, between fibrational lax slice $(\infty,2)$-categories, justifying our notation, though since the conditions for its construction are rather specific we will not pursue this point of view in further generality.

Lemma 5.4. *The ∞-functor $f^*\mathrm{ev}_1$ admits a right adjoint right inverse.*

Proof. Note that in addition to being a right adjoint right inverse to ev_1, the map $\ulcorner\mathrm{id}\urcorner$ is also a left adjoint right inverse to ev_0. We will denote the counit of this adjunction κ. Since the identity unit exhibits $\mathrm{ev}_0\circ\ulcorner\mathrm{id}\urcorner=\mathrm{id}_{\mathfrak{P}}$, the ∞-functor $\ulcorner\mathrm{id}\urcorner$ lifts strongly to $f^*\ulcorner\mathrm{id}\urcorner\colon\mathfrak{X}\to\mathfrak{X}\times_{\mathfrak{P}}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$: the equivalent of eq. (24) with ev_1 replaced by $\ulcorner\mathrm{id}\urcorner$ (and α replaced by the identity unit, *mutatis mutandis*) is a strongly commutative diagram, and fully cartesian. We claim that $f^*\ulcorner\mathrm{id}\urcorner$ is the sought-after right adjoint right inverse to $f^*\mathrm{ev}_1$.

To see this, we will show that the transformation $\eta_!$ constructed in eq. (22) works as a unit with identity counit; it requires first identifying its target $f^\eta(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1)$ as $f^*\ulcorner\mathrm{id}\urcorner\circ f^*\mathrm{ev}_1$. This is in fact trivial, because the transformations $\eta_!\colon\mathrm{id}\Rightarrow f^\eta(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1)$ and $(f^*\ulcorner\mathrm{id}\urcorner)(f^*\mathrm{ev}_0)\eta_!\colon\mathrm{id}\Rightarrow f^*\ulcorner\mathrm{id}\urcorner\circ f^*\mathrm{ev}_1$ are both f-cocartesian lifts of $\eta\colon\mathrm{id}\Rightarrow\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1$, but there is another interesting way of seeing it, that we detail in the next paragraph.

Since the unit of the adjunction $\ulcorner\mathrm{id}\urcorner\dashv\mathrm{ev}_0$ is an equivalence, the triangle identities imply that the whiskering $\kappa\ulcorner\mathrm{id}\urcorner$ is the identity transformation of $\ulcorner\mathrm{id}\urcorner$, and also $\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1\simeq\mathrm{id}_{\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1}$. There are now two things we can do: since $\mathrm{id}_{\not{\ell}^{\eta}(\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)}$ is a cocartesian lift of $\mathrm{id}_{\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1}$, the transformation $\mathrm{id}_{\not{\ell}^{\eta}(\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)}$ factors through a unique lift $\not{\ell}^{\eta}(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)$ of $\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1$, which because of the factorisation has to be an identity. At the same time, one can take a cocartesian lift of $\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1$, which is easily seen to coincide with $\not{\ell}^{\eta}(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)$; as a cocartesian lift of an identity, it is, again, an identity. We thus have an equivalence

$$\begin{aligned}(\not{\ell}^*\ulcorner\mathrm{id}\urcorner)\circ(\not{\ell}^*\mathrm{ev}_1) = (\not{\ell}^*\ulcorner\mathrm{id}\urcorner)\circ(\not{\ell}^*\mathrm{ev}_0)\circ\not{\ell}^{\eta}(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1)& \\ \xrightarrow[\not{\ell}^{\eta}(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)]{\simeq}\not{\ell}^{\eta}(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1),&\end{aligned} \tag{25}$$

expressing the decomposition we needed.

Furthermore, constructing the equivalent of eq. (22) but with $\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner$ in place of $\mathrm{id}_{\mathfrak{P}}$ (so with structure map to $\mathfrak{P}$ given by $\mathrm{id}_{\mathfrak{P}}$ instead of ev_0), and with the identity counit $\varepsilon\colon \mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner \overset{=}{\Longrightarrow} \mathrm{id}_{\mathfrak{P}}$ instead of η, we obtain, after strongly pulling back $\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner$, an $\not{\ell}$-cocartesian transformation

$$\varepsilon_!\colon \not{\ell}^*(\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner)\Rightarrow(\not{\ell}^*(\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner))_{\varepsilon} = \mathrm{id}_{\mathfrak{X}}, \tag{26}$$

which as a cocartesian lift of ε which is an identity, is itself an equivalence.

Finally, the fact that $\not{\ell}^*\eta := \eta_!$ and $\varepsilon_!$ satisfy the triangle identities is a consequence of the triangle identities for η and ε, to which is applied the same reasoning we used to obtain eq. (25). □

It is worthwhile to note that the component of $\not{\ell}^*\eta\colon \mathrm{id}\Rightarrow\not{\ell}^*\ulcorner\mathrm{id}\urcorner\circ\not{\ell}^*\mathrm{ev}_1$ at an object $(X,\not{\ell}X\overset{\jmath}{\rightarrowtail}Q)\in\mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})$ is

$$\not{\ell}^*\eta_{(X,\not{\ell}X\rightarrowtail Q)}\colon \left(X,\ \begin{matrix}\not{\ell}X\\ \downarrow\\ Q\end{matrix}\right)\ \begin{matrix}\xrightarrow{(X,\jmath)}\\ \\ \xrightarrow[(X,\mathrm{id}_Q)]{}\end{matrix}\ \left(X,\ \begin{matrix}Q\\ \|\\ Q\end{matrix}\right). \tag{27}$$

Proposition 5.5. *Let $\not{\ell}\colon\mathfrak{X}\to\mathfrak{P}$ be an ∞-functor such that $\mathfrak{X}$ admits $\not{\ell}$-cocartesian lifts of inert morphisms. The ∞-functor $\not{\ell}^*\mathrm{ev}_1\colon\mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})\to\mathfrak{X}$ exhibits $\mathfrak{X}$ as the localisation of $\mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})$ at the set $\mathfrak{I}_{\mathfrak{X}}$ of morphisms $(X;(\not{\ell}(X)\rightarrowtail Q))\to(X';(\not{\ell}(X')\rightarrowtail Q'))$ such that*

- *$X \to X'$ is $\not\!\!f$-cocartesian and*
- *$(\not\!\!f(X) \rightarrowtail Q) \to (\not\!\!f(X') \rightarrowtail Q')$ is ev_0-cartesian and ev_0-over an inert arrow.*

Proof. The proof follows the lines of that of Proposition 5.2. Let $\mathcal{W}_{\mathfrak{X}}$ be the class of morphisms inverted by $\not\!\!f^* \mathrm{ev}_1$. A morphism of $\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$, of the form (ξ, θ) where $\xi : X \to Y$ in $\mathfrak{X}$ and θ sits is a commutative square

$$\begin{tikzcd} \not\!\!f X \arrow[r, "\not\!\!f\xi"] \arrow[d, tail, "\jmath"'] & \not\!\!f X' \arrow[d, tail, "\jmath'"] \\ Q \arrow[r, "\theta"'] & Q' \end{tikzcd} \tag{28}$$

in $\mathfrak{P}$, is in $\mathcal{W}_{\mathfrak{X}}$ if and only if θ is an equivalence $Q \simeq Q'$, so that it induces a commutative square

$$\begin{tikzcd} (X, \not\!\!f X \rightarrowtail Q) \arrow[r, "(\xi,\theta)"] \arrow[d, "\jmath_!"'] & (X', \not\!\!f X' \rightarrowtail Q') \arrow[d, "\jmath'_!"] \\ (X_Q, \not\!\!f(X_Q) = Q) \arrow[r, "(\jmath_!\xi,\theta)"'] & (X'_{Q'}, \not\!\!f(X'_{Q'} = Q')) \end{tikzcd} \tag{29}$$

where $X \xrightarrow{\jmath_!} X_Q$ and $X' \xrightarrow{\jmath'_!} X'_{Q'}$ are cocartesian lifts of $\jmath$ and $\jmath'$, and $\jmath_!\xi$ is the arrow $X_Q \to X'_{Q'}$ uniquely induced by cartesianity, which is invertible since it lifts the isomorphism $Q \simeq Q'$. The vertical morphisms are in $\mathcal{I}_{\mathfrak{X}}$ by construction, so it follows from the 2-of-3 property of equivalences that any ∞-functor that inverts the morphisms in $\mathcal{I}_{\mathfrak{X}}$ will invert the morphisms in $\mathcal{W}_{\mathfrak{X}}$, and that the localisations along $\mathcal{I}_{\mathfrak{X}}$ and $\mathcal{W}_{\mathfrak{X}}$ coincide.

But again, it can be seen in eq. (27) that the components of $\not\!\!f^*\eta$ are in $\mathcal{I}_{\mathfrak{X}}$ whence in $\mathcal{W}_{\mathfrak{X}}$, so $\not\!\!f^* \mathrm{ev}_1$ is indeed a localisation along $\mathcal{W}_{\mathfrak{X}}$. $\square$

From now on, we assume that $\mathfrak{P}$ is a sound pattern, and we let $\mathfrak{C}$ be a $\mathfrak{P}$-complete $(\infty, 1)$-category.

Corollary 5.6. *Let $\not\!\!f : \mathfrak{X} \to \mathfrak{P}$ be an ∞-functor such that $\mathfrak{X}$ admits $\not\!\!f$-cocartesian lifts of inert morphisms. There is a fully faithful ∞-functor $\left\{\mathfrak{X}, \mathfrak{C}\right\} \hookrightarrow \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}$ whose essential image is spanned by the ∞-functors preserving cocartesian morphisms over inert morphisms of $\mathfrak{P}$.*

Proof. Direct application of [GHN17, Proposition 7.3] shows that for any $(\infty,1)$-category $\mathfrak{X}$ over $\mathfrak{P}$ there is an equivalence

$$\left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}} \simeq \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}, \tag{30}$$

in which an ∞-functor $\mathcal{S}\colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ over $\mathfrak{P}$ (so mapping $X \in \mathfrak{X}$ to $\mathcal{S}(X)\colon \mathfrak{P}^{\mathrm{inrt}}_{\not{p}X/} \to \mathfrak{C}$) corresponds to $\widetilde{\mathcal{S}}\colon \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \to \mathfrak{C}$ mapping

$$(X, \not{p}X \rightarrowtail Q) \mapsto \mathcal{S}(X)(\not{p}X \rightarrowtail Q). \tag{31}$$

In addition, by the description of $\overline{\not{p}}$-cocartesian morphisms in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ provided by [Lur09, Corollary 3.2.2.13], one sees that an ∞-functor $\mathcal{S}\colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ takes an arrow $\xi\colon X \to X'$ to a cocartesian arrow in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ if and only if the corresponding $\widetilde{\mathcal{S}}$ takes all morphisms (ξ,θ) where θ is ev_0-cartesian in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to equivalences in $\mathfrak{C}$.

By Proposition 5.5, $\left\{\mathfrak{X}, \mathfrak{C}\right\}$ identifies as the full sub-$(\infty,1)$-category of $\left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}$ on those ∞-functors inverting all morphisms in $\mathfrak{I}_{\mathfrak{X}}$. More precisely, the equivalence of eq. (30) sits in the sequence

$$\left\{\mathfrak{X}, \mathfrak{C}\right\} \simeq \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathfrak{I}_{\mathfrak{X}})} \hookrightarrow \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\} \simeq \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}. \tag{32}$$

One then only needs to observe that an ∞-functor $\widetilde{\mathcal{S}} \in \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}$, corresponding to $\mathcal{S} \in \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}$, is in $\left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathfrak{I}_{\mathfrak{X}})}$ if and only if for any $\xi\colon X \to X'$ in $\mathfrak{X}$ that is $\not{p}$-cocartesian and any θ as in eq. (28) that is ev_0-cartesian and ev_0-over an inert arrow, $\widetilde{\mathcal{S}}(\xi,\theta)$ is an equivalence, which is exactly the description given above of $\mathcal{S}$ taking $\not{p}$-cocartesian morphisms $\not{p}$-over (since $\mathrm{ev}_0(\theta) = \not{p}(\xi)$) an inert arrow to $\overline{\not{p}}$-cocartesian arrows. □

We can now arrive at our main result.

Theorem 5.7. *Let $\mathfrak{X} \to \mathfrak{P}$ be a fibrous $\mathfrak{P}$-pattern, with $\mathfrak{P}$ sound, and $\mathfrak{C}$ a $\mathfrak{P}$-complete $(\infty,1)$-category. There is an equivalence of $(\infty,1)$-categories*

$$\mathfrak{Seg}_{\mathfrak{X}}(\mathfrak{C}) \simeq \mathfrak{Alg}_{\mathfrak{X}}(\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})). \tag{33}$$

In particular, taking $\mathfrak{X} = \mathfrak{P}$ to be the terminal fibrous $\mathfrak{P}$-pattern, we obtain theorem A.

Proof. Let $\mathscr{S}\colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ be an ∞-functor over $\mathfrak{P}$ that preserves cocartesian morphisms over inert arrows (so corresponds to $\widetilde{\mathscr{S}}\colon \mathfrak{X} \to \mathfrak{C}$). It factors through $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ if and only if for every $X \in \mathfrak{X}$, the $\mathfrak{P}^{\mathrm{inrt}}_{\ell X/}$-object $\mathscr{S}(X)$ in $\mathfrak{C}$ is Segal.

By [Bar22, Lemma 2.39], for any map of algebraic patterns $\mathfrak{O} \to \mathfrak{P}$ and any $P \in \mathfrak{P}$, the projection $\mathfrak{O} \times_{\mathfrak{P}} \mathfrak{P}_{P/} \to \mathfrak{P}$ is an iso-Segal morphism. Applying this to $\mathfrak{O} = \mathfrak{P}^{\mathrm{inrt}}$ and $P = \ell X$ (for any X), we find that the above condition is equivalent to $\widetilde{\mathscr{S}}$ being a Segal $\mathfrak{X}$-object. □

6. Some examples: flavours of generalised multicategories

Remark 6.1 (Graphs and endomorphisms). Since the Segal condition for a pattern $\mathfrak{P}^{\mathrm{el}}$ with only elementary objects and inert morphisms is trivial, the underlying $\mathfrak{P}$-graph of the $\mathfrak{P}$-monoidal $(\infty,1)$-category $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ is $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{C})$, which is directly given by the ∞-functor $\left\{\mathfrak{P}^{\mathrm{el}}_{-/}, \mathfrak{C}\right\}$. Since $\mathfrak{P}^{\mathrm{el}}_{E/}$, for any elementary E, generally has a simple form, this will make the underlying $\mathfrak{P}$-graph of $\mathfrak{P}$-spans easy to describe.

Furthermore, since the "algebraic operations" in Segal $\mathfrak{P}$-objects come from active morphisms, a $\mathfrak{P}^{\mathrm{el}}$-monad carries no algebraic structure and can simply be seen as a $\mathfrak{P}$-endomorphism. The statement of Theorem 5.7 thus restricts to saying that $\mathfrak{P}$-endomorphisms in $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ are exactly $\mathfrak{P}$-graphs in $\mathfrak{C}$.

6.2 Categories and multiple categories

Take $\mathfrak{P}$ to be the pattern $\Delta^{\mathrm{op}\natural}$, consisting of the simplicial indexing category Δ^{op} with its usual inert-active factorisation system (where a map $[n] \to [m]$ in Δ is inert if it is a subinterval inclusion and active if it is endpoints-preserving), and $[0]$ and $[1]$ as elementary objects. Its Segal objects are internal categories.

Remark 6.2.1. Direct comparison shows that for any $[n] \in \Delta^{\mathrm{op}}$, the category $(\Delta^{\mathrm{op}\natural})^{\mathrm{inrt}}_{[n]/}$ is equivalent (in fact isomorphic) to the twisted arrow category of $\mathbb{n} + \mathbb{1} = [n]$, as has been previously noticed in [Hau18a, Remark 5.4] and implicitly used in [Kos21, Remark 2.18]: more precisely, a morphism in

$\mathfrak{Tw}(\mathfrak{m}+\mathbb{1})$ represented by a factorising square in $\mathfrak{m}+\mathbb{1}$ below-left

$$\begin{tikzcd} i & i' \arrow[l, "i'\leq i"'] \arrow[d, "i'\leq j'"] \\ j \arrow[u, leftarrow, "i\leq j"'] \arrow[r, "j\leq j'"'] & j' \end{tikzcd} \qquad \begin{tikzcd} {[n]} \arrow[rr, equal] & & {[n]} \\ {[j-i]\simeq\{i,\dots,j\}} \arrow[u, hook] \arrow[rr, hook] & & {[j'-i']\simeq\{i',\dots,j'\}} \arrow[u, hook] \end{tikzcd} \tag{34}$$

corresponds to the morphism in $(\Delta^{\mathrm{op}\natural})^{\mathrm{inrt}}_{[n]/}$ represented as the commutative square (in Δ) above-right.

Thus for any $(\infty,1)$-category $\mathfrak{C}$ admitting finite fibre products, $\mathbb{Span}_{\Delta^{\mathrm{op}\natural}}(\mathfrak{C})$ is the double $(\infty,1)$-category of spans in $\mathfrak{C}$ constructed in [Bar13] and [Hau18a] (and denoted $\mathfrak{SPAN}_1^+(\mathfrak{C})$ there).

Now, we also note that fibrous $\Delta^{\mathrm{op}\natural}$-patterns are virtual double ∞-categories (also referred to as generalised non-symmetric ∞-operads in [GH15]) so that morphisms of fibrous $\Delta^{\mathrm{op}\natural}$-patterns correspond to "lax double functors", and in particular $\Delta^{\mathrm{op}\natural}$-monads recover the usual notion of monad in a virtual double $(\infty,1)$-category. In conclusion, Theorem 5.7 applied to the pattern $\Delta^{\mathrm{op}\natural}$ recovers the main theorem of [Hau21], that monads (or algebras) in spans are internal categories.

Example 6.2.2. More generally, using products of algebraic patterns (*cf.* Example 2.4), one sees that for any $d\in\mathbb{N}$, the Segal $\Delta^{\mathrm{op}\natural,d}$-$(\infty,1)$-category $\mathbb{Span}_{\Delta^{\mathrm{op}\natural,d}}(\mathfrak{C})$ is the $(d+1)$-uple $(\infty,1)$-category $\mathfrak{SPAN}_d^+(\mathfrak{C})$ of iterated spans also constructed in [Hau18a].

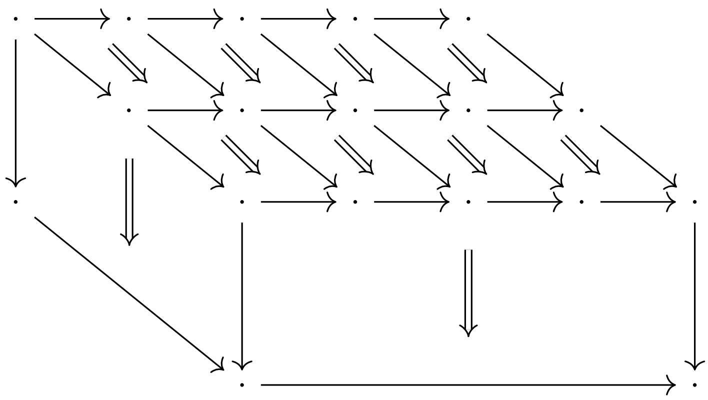
We now explain how lax Segal $\Delta^{\mathrm{op}\natural,d}$-fibrations should be seen as virtual $(d+1)$-uple ∞-categories. When viewing (strong) Segal $\Delta^{\mathrm{op}\natural,d}$-fibrations as $(d+1)$-uple categories, one should separate the d directions coming from Δ^d, which we dub the **algebraic** directions, from the last one coming from straightening the cocartesian fibration, which we will know as the categorical, or **transversal**, direction. A lax Segal $\Delta^{\mathrm{op}\natural,d}$-fibration $\mathfrak{X}$ is then virtual in all the algebraic directions: it has, for all $n\leq d$, algebraic n-cells in the usual directions for d-uple categories, and it has transversal cells from any n-dimensional grid of n-cells to a single n-cell. We stress that, for the domains of the transversal n-cells, we only require grids rather than the more general n-uple pasting diagrams of [Rui22], as the grids are the objects of Δ^d.

Let us represent the low dimensions; for ease of viewing we shall draw the transversal direction vertically, from top to bottom (since drawing it transversally would hide the face with the most information in the back).

For $d = 1$, the description — of virtual double ∞-categories — is well-known: there are objects and algebraic arrows, and in addition there are transversal arrows between objects and transversal cells from any pasting diagram (*i.e.* composable sequence) of algebraic arrows to one algebraic arrow, drawn as 2-cells in

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \arrow[d] \\ \cdot \arrow[rrr] & & & \cdot \end{tikzcd} \tag{35}$$

For $d = 2$, we similarly have objects, two kinds of algebraic 1-arrows, and algebraic squares or 2-arrows, and in addition transversal 1-arrows between objects, two kinds of transversal 2-cells, corresponding to the two directions of algebraic arrows, and finally transversal cubes or 3-cells for any grid of composable squares, as represented in



(36)

where the 3-cell is not visible but fills the cube.

A $\Delta^{\mathrm{op}\natural,d}$-monad then consists of monads (whose structure cells are transversal) in all possible algebraic directions and throughout the different dimensions, resembling (a less lax version of) the intermonads of [GP17, §7.1].

Example 6.2.3. If one takes instead the pattern $\Delta^{\mathrm{op}^\flat}$, which has the same underlying category and factorisation system but only $[1]$ as elementary object — whose Segal objects are internal categories $X_\bullet$ with trivial object

X_0 of objects, so internal associative monoids — then the monoidal $(\infty,1)$-category $\mathbb{Span}_{\Delta^{\mathrm{op}\flat}}(\mathfrak{C})$, for $\mathfrak{C}$ admitting finite products (for this is what $\Delta^{\mathrm{op}\flat}$-completeness means) is $\mathfrak{C}$ itself seen with its cartesian monoidal structure.

Generalising to $\Delta^{\mathrm{op}\flat,n}$ (whose Segal objects are n-iterated associative monoids, so $\mathscr{E}_n$-monoids), we have that $\mathbb{Span}_{\Delta^{\mathrm{op}\flat,n}}(\mathfrak{C})$ is $\mathfrak{C}$ seen with its cartesian structure as an $\mathscr{E}_n$-monoidal structure. In this case, Theorem 5.7 simply recovers the fact that Segal $\Delta^{\mathrm{op}\flat,n}$-objects in a cartesian $(\infty,1)$-category $\mathfrak{C}$ are n-uply commutative (meaning $\mathscr{E}_n$-) algebras in the cartesian monoidal $(\infty,1)$-category $\mathfrak{C}^\times$ (*i.e.* $\mathscr{E}_n$-monoids in $\mathfrak{C}$).

6.3 Commutative monoids

Take $\mathfrak{P}$ to be the pattern $\Gamma^{\mathrm{op}\flat}$ where $\Gamma^{\mathrm{op}} \simeq \mathfrak{Fin}_*$ is the opposite of Segal's category, which is equivalent to the category of pointed finite sets, with its usual inert-active factorisation system, and $\langle 1\rangle$ as the only elementary object. Its Segal objects are commutative (or $\mathscr{E}_\infty$) monoids. As explained in [CH21, Example 14.22], this algebraic pattern is not saturated; however its global saturation is easily seen from the fact that $\Gamma^{\mathrm{op}\flat,\mathrm{el}}_{\langle n\rangle/}$ is a set of n elements.

It also follows that for any $\Gamma^{\mathrm{op}\flat}$-complete (*i.e.* admitting finite products) $(\infty,1)$-category $\mathfrak{C}$, $\mathbb{Span}_{\Gamma^{\mathrm{op}\flat}}(\mathfrak{C})$ is again $\mathfrak{C}$ itself equipped with its cartesian symmetric monoidal structure. Since fibrous $\Gamma^{\mathrm{op}\flat}$-patterns are ∞-operads in the sense of [Lur17] and $\Gamma^{\mathrm{op}\flat}$-monads are commutative algebras, we recover that Segal $\Gamma^{\mathrm{op}\flat}$-objects in $\mathfrak{C}$ are commutative monoids in $\mathfrak{C}$ (where it is again understood that the term "monoid" refers to an algebra in a cartesian monoidal ∞-category).

Remark 6.3.1. For the product pattern $\mathfrak{P} = \Gamma^{\mathrm{op}\flat} \times \Delta^{\mathrm{op}\natural}$, whose Segal objects are internal symmetric monoidal categories, we recover as $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ the double $(\infty,1)$-category of spans in $\mathfrak{C}$, endowed with its symmetric monoidal structure coming from the cartesian product in $\mathfrak{C}$.

Example 6.3.2. As a further variant, one may consider the algebraic pattern $\Gamma^{\mathrm{op}\natural}$, which is like $\Gamma^{\mathrm{op}\flat}$ but also has $\langle 0\rangle$ as an additional elementary object. Its Segal objects in a $\Gamma^{\mathrm{op}\natural}$-complete $(\infty,1)$-category $\mathfrak{C}$ are commutative monoids in a slice of $\mathfrak{C}$, which it is convenient to interpret as families of commutative monoids in $\mathfrak{C}$ indexed by an object of $\mathfrak{C}$. In the same spirit, fibrous $\Gamma^{\mathrm{op}\natural}$-patterns are generalised ∞-operads of [Lur17], which are the same thing as families of ∞-operads.

For any object $\langle n\rangle$, the category $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle n\rangle/}$ is

$$\begin{tikzcd} \rho_1 \arrow[drr] & \rho_2 \arrow[dr] & \cdots & \rho_{n-1} \arrow[dl] & \rho_n \arrow[dll] \\ & & \left(\langle n\rangle \xrightarrow{!} \langle 0\rangle\right) & & \end{tikzcd} \tag{37}$$

where $\rho_i \colon \langle n\rangle \to \langle 1\rangle$ sends i to 1 and all the other elements of $\langle n\rangle$ to 0, from which it is seen that the pattern $\Gamma^{\mathrm{op}\natural}$ is globally saturated. More generally, any inert map $\langle n\rangle \to \langle k\rangle$ (with necessarily $k \leq n$) determines and is uniquely determined by a k-element subset of $n = \langle n\rangle \setminus \{0\}$, so that writing $\wp(n)$ for the powerset of n (equipped with its natural order), we have $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle n\rangle/} \simeq \wp(n)^{\mathrm{op}}$. For example, for $n = 3$ the poset $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 3\rangle/}$ is

$$\begin{tikzcd} & & \{1,2,3\} \arrow[dl] \arrow[d] \arrow[dr] & & \\ & \{1,2\} \arrow[dl] \arrow[dr] & \{1,3\} \arrow[dll] \arrow[drr] & \{2,3\} \arrow[dl] \arrow[dr] & \\ \{1\} \arrow[drr] & & \{2\} \arrow[d] & & \{3\} \arrow[dll] \\ & & \{0\} & & \end{tikzcd} \tag{38}$$

(containing copies of $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 2\rangle/}$, $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 1\rangle/}$, and $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 0\rangle/}$ on the left). We can thus see that $\mathbb{Span}_{\Gamma^{\mathrm{op}\natural}}(\mathfrak{C})$ is the family of slices of $\mathfrak{C}$, each equipped with its monoidal structure given by the pullbacks in $\mathfrak{C}$, and Theorem 5.7 recovers the description of Segal $\Gamma^{\mathrm{op}\natural}$-objects given above.

6.4 Higher categories and iterated spans

We now take $\mathfrak{P}$ to be the pattern $\Theta_\ell{}^{\mathrm{op}\natural}$ of Remark 3.5, for some $\ell \in \mathbb{N} \cup \{\omega\}$.

It follows from the description given in eq. (10) that $\mathbb{Span}_{\Theta_\ell^{\mathrm{op}\natural}}(\mathfrak{C})$ is a cellular $(\infty, 1)$-category of ℓ-times iterated spans: for any $k \leq \ell$, the $(\infty, 1)$-category of k-cells has as objects the spans between the apices of two $(k-1)$-iterated spans, and as morphisms the morphisms between spans. In other words, it is a categorical enhancement of the $(\infty, \ell+1)$-category $\mathfrak{Span}_\ell^+(\mathfrak{C})$ of ℓ-iterated spans from [Hau18a, Definition 5.16, Remark 5.17], obtained by discarding all the extraneous "algebraic" directions of the $(\ell+1)$-uple one as in [*ibid.*] but still retaining the transversal one.

Remark 6.4.1. At the level of the underlying $\Theta^{\mathrm{op}\natural}$-graph, the fact that our construction of the globular category of iterated spans through slices of $\Theta^{\mathrm{op}\natural,\mathrm{el}} = \mathbb{G}^{\mathrm{op}}$ recovers the combinatorial one given in [Bat98, Definition 3.2] was already observed in [Str00].

Example 6.4.2. For any $k \leq \ell$, we can also define the pattern $({\Theta_\ell}^{\mathrm{op}})^{\Sigma^k\natural}$ to consist of the same structure as $\Theta^{\mathrm{op}\natural}$ but only the globes $\mathfrak{D}_n$ with $n \geq k$ as elementaries. For example, if ℓ is finite, taking $k = \ell$ recovers the pattern denoted $\Theta_\ell^{\mathrm{op}\flat}$ in [CH21]. Segal objects for $({\Theta_\ell}^{\mathrm{op}})^{\Sigma^k\natural}$ are $\mathscr{E}_k$-monoidal internal $(\ell-k)$-categories.

As noted in [CH21, Example 9.8. (iv)], fibrous ${\Theta_\ell}^{\mathrm{op}\natural}$-patterns are an ∞-categorical version of the ℓ-globular multicategories or many-sorted ℓ-globular operads of [Lei04, p. 273] and [CS10, Example 4.11], themselves a many-sorted, or coloured, version of the ℓ-globular operads of [Bat98]. They are similar to the fibrous $\Delta^{\mathrm{op}\natural,\ell}$-patterns described in EXAMPLE 6.2.2, but where the domain of a transversal n-cell is an n-categorical pasting diagram instead of an n-dimensional grid (and its codomain is a single n-globe rather than an n-cube).

Warning 6.4.3. Despite their name of "ℓ-operads" in [Bat98], fibrous $\Delta^{\mathrm{op}\natural,\ell}$-patterns should not be thought of as a kind of (∞, ℓ)-operads, meaning ∞-operads enriched in $(\infty, \ell-1)$-$\mathfrak{Cat}$. Indeed, as seen from the description above, they contain more data and structure than (∞, ℓ)-operads.

Likewise, the strong Segal ${\Theta_\ell}^{\mathrm{op}\natural}$-fibrations, known as "monoidal ℓ-globular categories" in [*ibid.*], are really categorical (∞, ℓ)-categories. In particular, ${\Theta_\ell}^{\mathrm{op}\natural}$-monads are very different from any kind of usual ℓ-categorical monads that could be made sense of (for example following the philosophy of [Hau18b] identifying Segal ${\Theta_{\ell+1}}^{\mathrm{op}\natural}$-objects with reduced categorical ${\Theta_\ell}^{\mathrm{op}\natural}$-objects) in categorical (∞,ℓ)-categories: the ${\Theta_\ell}^{\mathrm{op}\natural}$-monad structure associates to any configuration of (algebraic) n-cells a *transversal* cell, so is really independent of the (∞,ℓ)-categorical structure.

As such, we will only refer to ${\Theta_\ell}^{\mathrm{op}\natural}$-monads as ℓ**-globular monads**.

We then obtain by applying Theorem 5.7 that ${\Theta_\ell}^{\mathrm{op}\natural}$-monads in the categorical (∞,ℓ)-category $\mathbb{S}\mathrm{pan}_{{\Theta_\ell}^{\mathrm{op}\natural}}(\mathfrak{C})$ are Segal ${\Theta_\ell}^{\mathrm{op}\natural}$-objects in $\mathfrak{C}$, or in more evocative language:

Corollary 6.4.4. *There is an equivalence of $(\infty,1)$-categories between ℓ-globular monads in the categorical (∞,ℓ)-category $\mathbb{S}\mathrm{pan}_\ell^+(\mathfrak{C})$ of ℓ-iterated spans in $\mathfrak{C}$, and internal (∞,ℓ)-categories in $\mathfrak{C}$.*

For $\ell=\omega$, this recovers a homotopical formulation of the definition of weak ω-categories given by [Bat98] as well as that of [Lei04] (*cf.* [CL04] for an explanation of the different definitions of ω-categories).

6.5 Multicategories and multispans

We finish by considering the algebraic pattern $\Omega^{\mathrm{op}\natural}$ (resp. $\Omega^{\mathrm{op}\natural}_{\mathrm{pl.}}$) whose Segal objects are internal coloured operads (resp. internal coloured planar operads). Here, Ω is the dendroidal category, whose objects are rooted trees (resp. with planar structure), henceforth referred to as dendrices to avoid confusion with the objects of Θ, and whose morphisms express the grafting of dendrices — in contrast with the morphisms of Θ which express the pasting of trees. The algebraic pattern structure is given by having the inert morphisms be the sub-dendrex inclusions, the active morphisms the boundary-preserving maps, and the elementary objects be the corollas $\star_a$ (determined by their arities $a\in\mathbb{N}$) and the nodeless edge η. As noted in [CH21, Examples 14.21], the pattern $\Omega^{\mathrm{op}\natural}$ is saturated because any dendrex can be decomposed as a gluing of corollas along edges, and the same argument shows that it is also globally saturated.

To understand the dendroidal $(\infty,1)$-category $\mathbb{S}\mathrm{pan}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$, let us first describe its underlying categorical $\Omega^{\mathrm{op}\natural}$-graph $\mathbb{S}\mathrm{pan}_{\Omega^{\mathrm{op}\natural,\mathrm{el}}}(\mathfrak{C})$. At the level of colours, we just have $\Omega^{\natural,\mathrm{el}}_{/\eta}=\{\mathrm{id}_\eta\}$. At the level of operations, writing

$e_1, \ldots, e_a$ the leaves of the corolla $\bigstar_a$ and r its root, we find that $\Omega^{\natural,\mathrm{el}}_{/\bigstar_a}$ is the category

$$\begin{tikzcd}
(\eta \xrightarrow{\ulcorner e_1 \urcorner} \bigstar_a) \arrow[dr] & \cdots & (\eta \xrightarrow{\ulcorner e_a \urcorner} \bigstar_a) \arrow[dl] \\
& \mathrm{id}_{\bigstar_a} & \\
& (\eta \xrightarrow{\ulcorner r \urcorner} \bigstar_a) \arrow[u] &
\end{tikzcd} \tag{39}$$

of $(a+1)$-ary multicospans.

The structure of category objects in multicategories (coloured non-symmetric operads) was studied in [CGR14, Definition 3.9]. In our case, we get for $\mathbb{S}\mathrm{pan}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$ an operadic composition of multispans by fibre products along the relevant legs, where each multispan has a distinguished root as seen in eq. (39).

Example 6.5.1. It is also possible to replace $\Omega^{\mathrm{op}\natural}$ by the pattern $\Xi^{\mathrm{op}\natural}$ of [HRY19], whose Segal objects are cyclic operads. We obtain for $\mathbb{S}\mathrm{pan}_{\Xi^{\mathrm{op}\natural}}(\mathfrak{C})$ the same structure as above, except that the $(a+1)$-ary spans come without a choice of root. Note also that in Ξ, the nodeless edge η is equipped with an involution, which for Segal objects becomes a "duality" operation on colours. In our case, it acts as the identity.

Going further, we may also use the pattern $\Upsilon^{\mathrm{op}\natural}$ of [HRY20] (denoted $\mathbb{U}$ there), whose Segal objects are modular operads. The categorical modular ∞-operad $\mathbb{S}\mathrm{pan}_{\Upsilon^{\mathrm{op}\natural}}(\mathfrak{C})$ works much as $\mathbb{S}\mathrm{pan}_{\Xi^{\mathrm{op}\natural}}(\mathfrak{C})$, but with additional contraction operations that turn the abstract self-duality of objects into an actual self-duality (in the usual monoidal, or rather properadic, sense).

The fibrous $\Omega^{\mathrm{op}\natural}$-patterns were identified in [Ber22] as "tree-hyperoperads", which are cumbersome to describe in detail (*cf.* [GK98, §4.1] or [MSS02, Definition 5.45] for the modular generalisation, simply called hyperoperads). Nevertheless, we still obtain from Theorem 5.7 that dendroidal monads in the categorical ∞-operad of multispans in $\mathfrak{C}$ are internal operads in $\mathfrak{C}$.

Remark 6.5.2. The definition of operads, and more general multicategorical structures, as monads in multispans is well-known: it dates to [Bur71], and was independently rediscovered by both [Her04] and [Lei98], and then further systematised by [Lei04] and [CS10]. From a cartesian monad $\mathscr{T}$ on a category $\mathfrak{C}$, one constructs a double category of Kleisli $\mathscr{T}$-spans, whose objects are those of $\mathfrak{C}$ and morphisms from C to D are spans from $\mathscr{T}C$ to D, composition of spans using the monad structure.

For example, taking $\mathscr{T}$ to be the monad $\mathscr{F}_{\mathbb{\Gamma}^{\mathrm{op}\flat}}$ for free monoids, a Kleisli $\mathscr{T}$-span is a multispan of arbitrary arity, and monads (in the double-categorical sense) are coloured operads. Generally speaking, if $\mathscr{T}$ is the monad $\mathscr{F}_{\mathfrak{P}}$ for free Segal $\mathfrak{P}$-objects on $\mathfrak{P}$-graphs for some appropriate algebraic pattern $\mathfrak{P}$, we expect that monads in Kleisli $\mathscr{F}_{\mathfrak{P}}$-spans should be fibrous $\mathfrak{P}$-patterns, obtained as the Segal objects for a plus construction $\mathfrak{P}^{+}$ of $\mathfrak{P}$ (as in [Ker23, Proposition 3.2.10]), so as $\mathfrak{P}^{+}$-monads in $\mathfrak{P}^{+}$-spans.

However, Kleisli $\mathscr{F}_{\mathbb{\Gamma}^{\mathrm{op}\flat}}$-spans and $\Omega^{\mathrm{op}\natural}$-spans, while both admitting a natural interpretation as multispans, form markedly different structures. On the one hand, $\mathbb{Span}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$ is a categorical ∞-operad, whose operadic composition is given (leg by leg) by simple pullbacks. On the other hand, Keisli $\mathscr{F}_{\mathbb{\Gamma}^{\mathrm{op}\flat}}$-spans only form a double category, but its composition is more complex and makes full use of the monad structure on $\mathscr{F}_{\mathbb{\Gamma}^{\mathrm{op}\flat}}$. For a general algebraic pattern $\mathfrak{P}$, the difference will be similar: we think of it as moving the structure from the microcosm (on the Keisli $\mathscr{F}_{\mathfrak{P}}$-spans side) to the macrocosm (on the $\mathfrak{P}^{+}$-spans side).

It is nonetheless unclear what the precise relation between the two constructions is, if there even is one: the Keisli-type construction can be abstracted away from a span setting by using general monads acting on virtual double categories, but it is unlikely to be able to handle non-directed structures such as the cyclic and modular ∞-operads of EXAMPLE 6.5.1.

7. Conclusion: A fibrational perspective

Notation 7.1. In this section, we will identify $(\infty,1)$-categories with internal categories in ∞-$\mathfrak{Grpd}$, where internal categories are by definition Segal $\mathbb{\Delta}^{\mathrm{op}\natural}$-objects satisfying Rezk's univalence-completeness condition. It will also be convenient to see Segal $\mathfrak{P}$-objects in $(\infty,1)$-$\mathfrak{Cat}$ — such as, in particular, the $\mathfrak{P}$-$(\infty,1)$-categories of $\mathfrak{P}$-spans — as internal categories in $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$.

Recall that, for any regular cardinal κ, the $(\infty,1)$-category $\infty\text{-}\mathfrak{Grpd}^{(\kappa)} \subset$

∞-$\mathfrak{Grpd}$ is the base of the universal discrete cocartesian fibration with κ-small fibres $\infty\text{-}\mathfrak{Grpd}_{\bullet}^{(\kappa)} \to \infty\text{-}\mathfrak{Grpd}^{(\kappa)}$ in the $(\infty,2)$-topos $(\infty,1)$-$\mathfrak{Cat}$, just as $\mathfrak{Set}^{(\kappa)} \subset \mathfrak{Set}$ is the universal κ-small discrete cocartesian fibration in the $(2,2)$-topos $\mathfrak{Cat}$. In [Web07, Examples 4.7 and 4.8], it is explained that, for an algebraic pattern $\mathfrak{P}^{\mathrm{el}}$ in which all objects are elementary and all morphisms inert, the construction $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(-)$ preserves classifying discrete fibrations, so that the 2-topos $\mathfrak{Cat}\big(\{\mathfrak{P}^{\mathrm{el}}, \mathfrak{Set}\}\big)$ has a sufficient family of classifying discrete cocartesian fibrations given by $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{Set}_{\bullet}^{(\kappa)}) \to \mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{Set}^{(\kappa)})$ (where "sufficient" means that every discrete cocartesian fibration is classified by one in the family).

In the ∞-categorical setting, the properties of universal (or "classifying") fibrations are captured by the notion of univalence, which we restate from [GK16] (see also [Ras21b, Theorem 4.4]) in the internal setting.

Construction 7.2. Let $\mathfrak{C}$ be a finitely complete $(\infty,1)$-category. Recall that a **discrete cocartesian fibration** in $\mathfrak{C}$ is an internal functor $f\colon \mathbb{E} \to \mathbb{B}$ such that $(d_1, f_1)\colon \mathbb{E}_1 \to \mathbb{E}_0 \times_{\mathbb{B}_0} \mathbb{B}_1$ is an equivalence.

Lifting the construction of [GK16, Theorem 2.10] to the cartesian closed $(\infty,2)$-category $\mathfrak{Cat}(\mathfrak{C})$, one can construct for any discrete cocartesian fibration $f\colon \mathbb{E} \to \mathbb{B}$ in $\mathfrak{C}$ an internal category $\mathbb{Eq}_{/\mathbb{B}\times\mathbb{B}}(\varpi_1^*\mathbb{E}, \varpi_2^*\mathbb{E})$ over $\mathbb{B}\times\mathbb{B}$ (where $\varpi_1, \varpi_2\colon \mathbb{B}\times\mathbb{B} \to \mathbb{B}$ are the two projections) characterising equivalences between fibres of f.

Definition 7.3 (Univalent fibration)**.** A discrete cocartesian fibration $\mathbb{E} \to \mathbb{B}$ internal to $\mathfrak{C}$ is **univalent** if $\mathbb{B} \to \mathbb{Eq}_{/\mathbb{B}\times\mathbb{B}}(\varpi_1^*\mathbb{E}, \varpi_2^*\mathbb{E})$ is an equivalence.

Example 7.4. It is shown in [Cis19, Proposition 5.3.13] that the universal discrete cocartesian fibration $\infty\text{-}\mathfrak{Grpd}_{\bullet} \to \infty\text{-}\mathfrak{Grpd}$ (in the $(\infty,2)$-category $(\infty,1)\text{-}\mathfrak{Cat} = \mathfrak{Cat}(\infty\text{-}\mathfrak{Grpd})$) is univalent.

Using this characterisation, one can show (though we omit the proof here as this result is only used for motivation) that for any sound algebraic pattern $\mathfrak{P}^{\mathrm{inrt}}$ all of whose morphisms are inert, the construction

$$\mathbb{Span}_{\mathfrak{P}^{\mathrm{inrt}}}(-)\colon (\infty,1)\text{-}\mathfrak{Cat}^{(\mathfrak{P}\text{-cplt})} \to \mathfrak{Cat}(\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\infty\text{-}\mathfrak{Grpd})) \tag{40}$$

preserves classifying discrete cocartesian fibrations:

Proposition 7.5. *Let $\mathfrak{G}_\bullet \to \mathfrak{G}$ be a univalent discrete cocartesian fibration. Then $\mathbb{Span}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G}_\bullet) \to \mathbb{Span}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G})$ is a univalent discrete cocartesian fibration internally to $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G})$.* $\square$

For an algebraic pattern with non-trivial active morphisms, the situation becomes richer and goes beyond the $(\infty,2)$-topos theory of internal categories in presheaf $(\infty,1)$-topoi. Indeed, Theorem 5.7 shows that, even when restricting our attention as we are doing here from fibrous patters to Segal fibrations, the morphisms of interest will be the lax morphisms, the maps of underlying fibrous patterns. We will thus use a notion of lax univalence, obtained by replacing strong morphisms by general lax morphisms of categorical Segal $\mathfrak{P}$-∞-groupoids in the definition of univalence for fibrations in $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$.

Conjecture 7.6. Let $\mathfrak{G}_\bullet \to \mathfrak{G}$ be a univalent discrete cocartesian fibration. Then $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{G}_\bullet) \to \mathbb{Span}_{\mathfrak{P}}(\mathfrak{G})$ is a lax-univalent discrete cocartesian fibration internally to $\mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{G})$.

This Conjecture states that $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{G}_\bullet) \to \mathbb{Span}_{\mathfrak{P}}(\mathfrak{G})$ classifies a class of discrete cocartesian fibrations. It remains to see that *every* such class is classified by a universal fibration of this form.

Conjecture 7.7. Suppose $\left(\mathfrak{G}_\bullet^{(\kappa)} \to \mathfrak{G}^{(\kappa)}\right)_{\kappa\in K}$ is a sufficient family of univalent fibrations for $(\infty,1)$-$\mathfrak{Cat}$. Then $\left(\mathbb{Span}_{\mathfrak{P}}(\mathfrak{G}_\bullet^{(\kappa)}) \to \mathbb{Span}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})\right)_{\kappa\in K}$ provides enough lax-univalent fibrations for $\mathfrak{Cat}(\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd}))$.

Corollary 7.8. *Let $\mathfrak{P}$ be a sound algebraic pattern and $\mathfrak{X} \to \mathfrak{P}$ be a Segal $\mathfrak{P}$-fibration, and assume that Conjecture 7.6 and Conjecture 7.7 hold. Then Segal $\mathfrak{X}$-objects in ∞-$\mathfrak{Grpd}$ are internal discrete cocartesian fibrations over the straightening $\mathbb{X}$ of $\mathfrak{X}$.*

Proof. The key point is that, by [GK16, Proposition 3.8], if $\mathfrak{G}_\bullet^{(\kappa)} \to \mathfrak{G}^{(\kappa)}$ is univalent then $\mathfrak{G}^{(\kappa)}$ is a full sub-$(\infty,1)$-category of ∞-$\mathfrak{Grpd}$, from which it follows that lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})$ can be seen as lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$. By the two conjectures, discrete cocartesian fibrations over $\mathbb{X}$ are the same thing as lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$ (factoring through some $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})$). At the same time, by Theorem 5.7, the latter are the same thing as Segal $\mathfrak{X}$-objects in ∞-$\mathfrak{Grpd}$ (or, to be precise, in some $\mathfrak{G}^{(\kappa)}$), which proves the result. $\square$

Example 7.9 (Double fibrations and Segal fibrations). For the algebraic pattern, Corollary 7.8 says explicitly that discrete cocartesian fibrations of double ∞-categories correspond biunivocally to lax double ∞-functors to the double ∞-category of spans (of ∞-groupoids). This is precisely an ∞-categorical version of the main construction of [Lam21]. This use of internal discrete fibrations is also very similar to how [Ras21a] deals with fibrations of Segal spaces (see also [Lou23, §6.1.1] for the version for fibrations of ω-categories).

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