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SOMMAIRE

AUTOCATEGORIES: IV. DRAWING, GRADUATION AND COLORATION FOR AUTOGRAPHIC DATA

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Abstract. In this fourth paper on autocategories, we add three results regarding autographs. Firstly we show how their representations are special localisations or graduations — with the special example of globular sets of Brown and Higgins, seen as dimensioned autographs. Then we explain how drawings of autographs are ambiguous and are related to virtual knots of Kauffman. And finally we explain how others informations can be add on an autograph by colocalisation or coloration, recovering gractes of Riguet and our 1-regimes. Thus we build bridges between uses of autographs, autograph morphisms, and autocategories

Résumé. Dans ce quatrième volet sur les autocatégories nous ajoutons trois résultats sur les autographes. Tout d'abord nous montrons comment leurs représentations sont des localisations particulière ou graduations — avec le cas particulier des ensembles globulaires de Brown et Higgins, considéré comme autographes dimensionnés. Puis nous expliquons comment le dessin des autographes est ambigu et lié à la question des entrelacs virtuels de Kauffman. Et finalement nous expliquons comment des informations supplémentaires peuvent être ajoutées par colocalisation ou coloration, retrouvant alors les gractes de Riguet et nos 1-régimes. Ainsi nous établissons des transferts entre les usages d'autographes, de morphismes d'autographes, et d'autocatégories.

Contents

1. Introduction: new aspects of the autographic approach

After some brief reviews of the definitions and results obtained previously (section 2), here we continue our study of autocategories, examining their representations, designs, and uses to describe structures.

Two points — representations and drawings — have to be determined in a more flexible manner. Representations are seen as localisations and/or graduation, and so applied to notion of globular complex (section 3 and section 4). With respect to drawings and ambiguity, we examine relations with virtual knot for drawing of knots and links, and we introduce the notion of virtualized autographs and autosuccessions (section 5). In order to describe structures in a practical way, we look at co-localisations as colorations (section 6).

These explorations will therefore allow us to obtain new examples. This will make it possible to highlight the description of numerous autographic structures or algebras by the simple data of a co-relation, or even by a morphism of autographs.

2. Return to previous concepts and examples

2.1 Autograph

Definition 2.1. *(voir [7], p. 66, & p. 76.) An* autograph $A = (\underline{A}, d, c)$ *is an action on a set $\underline{A}$ of $\mathbb{FM}(2) = \{d, c\}^*$ the free monoid on two generators, i.e. a set $\underline{A}$ equipped with two maps "domain" and "codomain" $d, c : \underline{A} \to \underline{A}$.*

Elements in $\underline{A}$ are considered as arrows in A. An element w with $dw = u$ and $cw = v$ being an arrow from u to v, shortly denoted by $w : u \to v$. But w is not considered as an edge between two vertices u and v : it is an interaction between two others interactions $u : p \to q$, $v : r \to s$, and then $p : m \to n$, etc. More explicitly, instead of $w : u \to v$ we could write
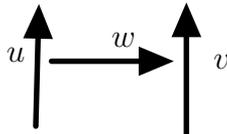
Remarque 2.1. An autograph is not a graph, with vertices and edges: in an autograph there are only arrows. However, a graph determines an autograph

by replacing each vertex X with an arrow called an auto-arrow $X : X \to X$.

In an autograph an auto-arrow $a : a \to a$ could be drawn by a simple circle as $\bigcirc_a$ (see [7]).

Remarque 2.2. In fact often we drew an autograph in the same way as, in knot theory, we draw the representation of a plane projection of an interlacing of oriented strings, with indication at the crossings of the passages of the strings one under the other, the line of the one below being drawn interrupted.

$$(1) \qquad\qquad (2) \qquad\qquad (3)$$
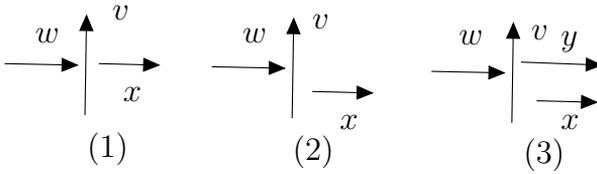
So in the case of an oriented knot or a link, each string (denoted by an arrow w) passing under v is continuing as itself (now denoted by a new arrow w^+) ; but however, in the general case of an autograph it should not be assumed that w is preceded by an arrow w^- passing under u toward w, or that w has to pass under v toward an arrow $x = w^+$ (case (1)). Moreover — see case (2) or (3) — even if an arrow w^- or $x = w^+$, or $y = w^+$ is specified, with $cw^- = dw$ or $dw^+ = cw$, one should not imagine it necessarily draws in the visual prolongation of w, as in the true crossing case (1).

Remarque 2.3. Here we will have to come back to this method of drawing, in particular on the question of virtual crossings, and also on the question of orientation. In any case, we must not confuse the data $A = (\underline{A}, d, c)$ of an autograph with its exhibition by a drawing, particularly because of the ambiguities inherent in these drawings. We will start from various drawings representing the autograph $\mathbb{C}_3$, in the following example 2.2.

2.2 Examples

Example 2.2. The autograph $\mathbb{C}_3$ is $\mathbb{C}_3 = (C_3, d_3, c_3)$ with $C_3 = \{x, y, z\}$,

$$d_3(x) = y, c_3(x) = z, d_3(y) = z, c_3(y) = x, d_3(z) = x, c_3(z) = y.$$

$\mathbb{C}_3$ can be drawn in several ways : as planar representation of a knot i.e. as a trefoil or a double eight (first line), or as a virtual knot (see 5.1), or as a

triskel (second line, left and right):
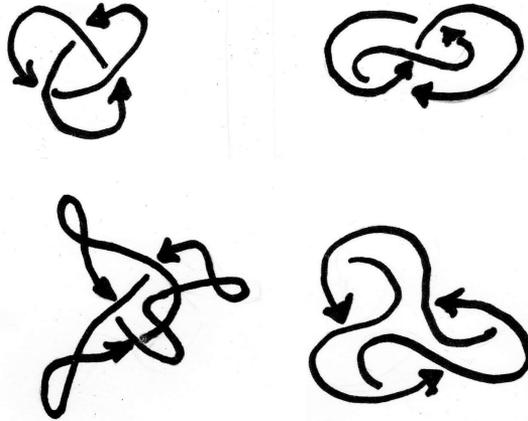
The first reason for introducing autographs was the possibility of a common presentation for interlaces and for 2-categories or double categories, as in the cases of a borromean link or a 2-block in a double category.

$\mathbb{B}$

$2 - block$
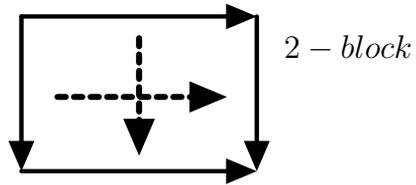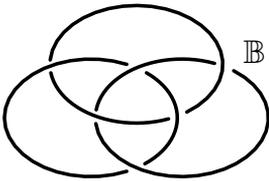
Example 2.3. Our first example is the *Borromean* autograph $\mathrm{As}(\mathbb{B})$, forged from the idea of the Borromean interlacing $\mathbb{B}$ (above on the left):

$$u : v' \to v, \qquad u' : v \to v',$$
$$v : w' \to w, \qquad v' : w \to w',$$
$$w : u' \to u, \qquad w' : u \to u'.$$

u $\quad$ v' $\quad$ w

w'

u'

v
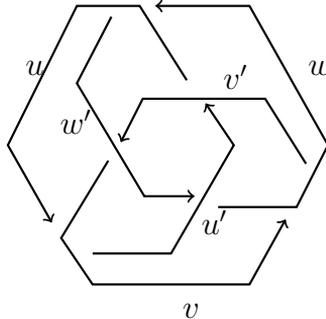
Example 2.4. Our second example is the following presentation as an autograph of a 2-block (as above on the right) b in a double category:

A 2-block **b** as an autograph
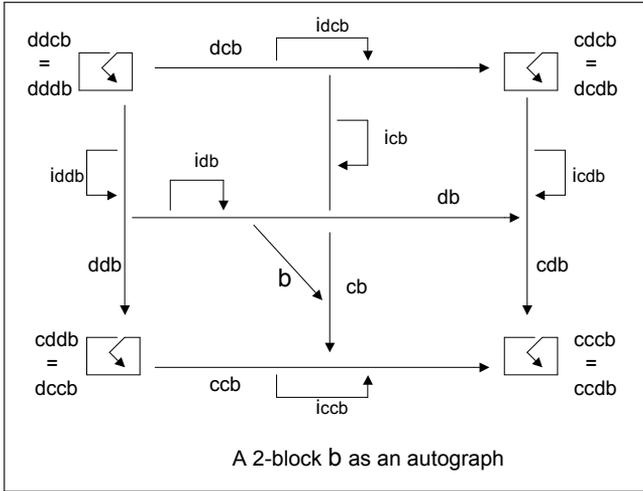
Example 2.5. A special case of example 2.4 is a 2-cell $\Theta : U \Longrightarrow V$ with $U, V : A \to B$, in a 2-category, drawn as
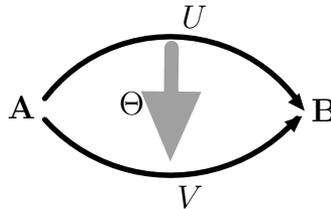
Example 2.6. The free autograph FA($3\mathbb{N}$) on $\aleph_0$ generators is explained in [7] (in such a way that $\mathbb{R}$ appears as a "descent-completion" of this FA($3\mathbb{N}$)). See also hereafter in proposition 5.20.

2.3 Autocategories, autographic algebra

Definition 2.7. *An* autocategory *is an autograph with identifiers (see [7]), equipped with a composition law given by gf for consecutive arrows f and g (with $dg = cf$), which is unitary and associative. As with autographs, autocategories do not have objects like categories do, but an autocategory can be associated with a category, by replacing objects with auto-arrows. An autocategory "is" also a mono-flexicategory, i.e. a category $\mathcal{C}$ equipped*

with a flex $\varphi : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{C})$ which is injective (see proposition 3.4 in [8]).

Examples of autographs and autocategories are given in [7] and [9] : graphs, categories, 2-graphs, 2-categories, doubles categories, but also knots and links.

Definition 2.8. *(voir [8], p. 156.) An* autographic algebra *is an algebra of a monad* $\mathbb{T} = (T, \eta, \mu)$ *on the category* Agraph *of autographs, which of course is a topos So — of course — it is a determined by a morphism* $T(A) \xrightarrow{\theta} A$, *that is to say an object in the category* Agraph$/A$ *of objects over A.*

In [8], we shew that graphs, basic graphic algebras, categories, autocategories, are autographic algebras, and we compared autographic algebras with Burroni's graphic algebras.

Example 2.9 (Free graph on an autograph). The forgetting functor Φ : Graph $\to$ Agraph associates to a graph G the autograph $\Phi(G)$ with the same set of arrows, where for each vertex V the arrow $i(V)$ is now an auto-arrow. Its left adjoint Λ : Agraph $\to$ Graph associates to an autograph A the graph $\Lambda(A)$ with $\Phi(\Lambda(A))$ the quotient of A by $c^2 = c, d^2 = d, cd = d, dc = c$, and hence $\Lambda(\Phi(G)) \simeq G$. (see Proposition 2.2 in [8]).

2.4 Surgery and co-relations

In the papers [7], [8] and [9], we show how surgery can be analyzed with spans or cospans in Agraph, i.e. with relations or co-relations between autographs. This point will have to be developed later in relation to the logic of the topos Agraph. In order to do so, now we allow ourselves from this point to introduce the following terminology by the definition 2.10:

Definition 2.10. *An* autographic co-relation (reps.relation) *between two autographs* A *and* A' *is a co-span (reps. a span) of two morphisms* i *and* i' *(reps.* p *and* p'*) of autographs*

$$A \xrightarrow{i} C \xleftarrow{i'} A'), \ (\text{resp. } A \xleftarrow{p} R \xrightarrow{p'} A').$$

Remarque 2.4. *Intuitive terminology* — A morphism of autograph m : $B \to A$ is seen as a *graduation* of B over A (and an object of the localisation of A), and it is seen as a *coloration* (or a decoration) of A (and an object of the co-localisation of B). So a span or a co-span is a kind of mixed composition of a graduation and a coloration.

3. Localisations and representations among autographs

The purpose of this section is to show how a representation of an autograph A is a localisation relative to A, i.e. a map $B \to A$, and consequently how it is a structuration of A. But let us also say as well that $B \to A$ a structuration of B; this is compatible with the fact that structures can be both algebraic or coalgebraic.

In the paper [9], we described categories of representations of a given autograph A, and that allowed us to introduce the notions of *autorelation* and *automap*. However, this notion of representation will have to be adapted here in order to make obvious its relationship with the localisation in the topos of the autographs. What we do here is therefore the analog in the framework of autographs of Yoneda's lemma and the relationship between actions of categories and fibrations.

3.1 Localisations over an autograph

We revisit the notion of a representation of an autograph (seen [9]) and extend it a little, and this allows us to conceive of representations as localisations.

Definition 3.1. *1* — *The category* $\text{Agraph} = \text{Set}^{\mathbb{FM}(2)}$ *of autographs is introduced in [8, p. 152], a morphism* $m' : A' \to A$ *in this category from* $A' = (\underline{A'}, d', c')$ *to* $A = (\underline{A}, d, c)$ *being a map* $m' = \underline{A'} \to \underline{A}$ *between sets, such that*

$$m'd' = dm', \quad m'c' = cm'.$$

This Agraph *is a topos.*
2 — *Given an autograph* $A = (\underline{A}, d, c)$, *the topos of* autographs over A *is denoted by* Agraph/A, *its objects are* (A', m') *with* $m' : A' \to A$ *in* Agraph, *and a morphism in* Agraph/A *from* (A', m') — *with* $m' : A' \to A$,

to (A'', m'') — with $m'' : A'' \to A$, is a morphism $h : A' \to A''$ in Agraph
such that

$$m'' h = m'.$$

The category Agraph$/A$ *is actually a topos, named the* localisation of *Agraph*
with respect to A. Also if (A', m') is an object of Agraph$/A$*, the autograph*
A' *is said to be* localised over A via m'.
3 — We introduce a forgetful functor associated to each given A:

$$U_A : \text{Agraph}/A \longrightarrow \text{Agraph} : ((A', m') \overset{h}{\to} (A', m')) \mapsto (A' \overset{h}{\to} A').$$

This functor U_A determines A, by $A = U_A(T)$, with T the terminal object in
Agraph$/A$.

3.2 Representations of an autograph

Definition 3.2. *Given an autograph $A = (\underline{A}, d, c)$, a* spanning representation
of A is a data $\varphi = (\Phi, \phi^d, \phi^c)$, with for each $f \in A$, the data of a set $\Phi(f)$
and a span of functions (named the d and the c projections)

$$\Phi(df) \overset{\phi^d(f)}{\longleftarrow} \Phi(f) \overset{\phi^c(f)}{\longrightarrow} \Phi(cf).$$

The category of spanning representations of A is denoted by $\text{Rep}_{\text{span}}(\underline{A}, d, c)$
or $\text{Rep}_{\text{span}}(A)$*, a morphism t in this category from φ to φ' being the data,*
for each $f \in A$ of a map

$$t_f : \Phi(f) \to \Phi'(f),$$

such that

$$t_{df}\phi^d(f) = \phi'^d(f)t_f, \quad t_{cf}\phi^c(f) = \phi'^c(f)t_f.$$

Definition 3.3. *We define some full subcategories of* $\text{Rep}_{\text{span}}(\underline{A}, d, c)$*:*

$$\text{Rep}_{\text{func}}(\underline{A}, d, c) \subset \text{Rep}_{\text{part}}(\underline{A}, d, c) \subset \text{Rep}_{\text{rel}}(\underline{A}, d, c) \subset \text{Rep}_{\text{span}}(\underline{A}, d, c),$$

as those with objects:

- relational representation*: for every $f \in \underline{A}$, the double (d, c) projection
  is a canonical inclusion*

$$[\phi^d(f), \phi^c(f)] : \Phi(f) \subset \Phi(df) \times \Phi(cf).$$

- partially functional representation: *a relational representation where, for every $f \in \underline{A}$, the d projection is a canonical inclusion:*

$$\phi^d(f) : \Phi(f) \subset \Phi(df).$$

- functional representation: *a partially functional representation in which, for every $f \in \underline{A}$, the d projection is an identity:*

$$\phi^d(f) : \Phi(f) = \Phi(df).$$

These sets of objects are denoted by

$$\mathcal{F}(\underline{A}, d, c) \subset \mathcal{P}(\underline{A}, d, c) \subset \mathcal{R}(\underline{A}, d, c) \subset \mathcal{S}(\underline{A}, d, c).$$

Remark 3.4. 1 — A category of functional representations $\text{Rep}(A, d, c)$ is defined in [9, Propos. 1.14]. Our definition here of $\text{Rep}_{\text{func}}(A, d, c)$ can be seen as a slightly modification of this $\text{Rep}(A, d, c)$, by adding the restriction that t is 'strict':

$t_f^d[\text{resp. } t_f^c]$ depends only on $df\,[\text{resp. } cf]$, it could be denoted by $t_{df}[\text{resp. } t_{cf}]$.

2 — In [9, Definition 1.6.] a spanning representation is identical to a relational representation, but now it is different, the fact that the double projection is a canonical inclusion is not yet assumed.

Proposition 3.5. *To each $w \in A$ is associated a functional representation Γ_w^A of A, and for every representation Φ of A we have a natural bijection:*

$$\xi_w : \text{Hom}_{\mathbf{Rep}(A,d,c)}(\Gamma_w^A, \Phi) \simeq \Phi(cw) : t \mapsto \xi_w(t) = t_{cw}(w).$$

Proof. Of course, according to [9, Propos. 1.15], (A, d, c) can be identified with an object of this (new) $\text{Rep}(A, d, c)$ or of $\mathbf{Rep}(A, d, c)$, via its regular representation (Γ^A, γ^A): for each $g \in A$ we consider the set $\Gamma^A(g)$ of (d, c)-paths (cf. [7, Definition 1.4]) with end g, that is to say $z = (z_n)_{0 \leq n < k}$ with

$$cz_0 = dz_1,\ cz_1 = dz_2,\ cz_{k-2} = dz_{k-1},\ cz_{k-1} = g,$$

and for each $f \in A$ the map $\gamma^A(f) : \Gamma^A(df) \to \Gamma^A(cf)$ described by concatenation with f.

Then, for each $w \in A$ and $g \in A$ we define $\Gamma^A_w(g)$ as a subset of $\Gamma^A(g)$:

$$\Gamma^A_w(g) = \{z = (z_n)_{0 \leq n < k} \in \Gamma^A(g); z_0 = w\},$$

and with γ^A_w defined by restriction of γ^A this determines the representation associated to w.

Now, if $t : \Gamma^A_w \to \Phi$, by the naturally of t, for $f : g \to h$ we get

$$\phi(f)t_g(z) = t_h(\gamma^A_w(f)(z)),$$

and as $z = \gamma^A_w(z_{k-1}) \ldots \gamma^A_w(z_1)(w)$, we have, for every $z \in \Gamma^A_w(g)$:

$$t_g(z) = \phi(z_{k-1}) \ldots \phi(z_1)t_{cw}(w),$$

and so t is determined by $\xi_w(t) = t_{cw}(w) \in \Phi(cw)$.

Remark. Up to a bijection, Γ^A_w depends only on cw, but for a given t, the associated $\xi_w(t)$ really depends on w, hence ξ_w depends on w. □

Definition 3.6. *Given an autograph* $A = (\underline{A}, d, c)$, *in [7, Proposition 6.3] we have constructed* $\mathrm{P}(A) = \big(\mathrm{Path}^\iota(A, d, c), D, C\big)$ *the free autocategory of paths in* A, $(z_n)_{0 \leq n < k} = (z_n)_k$, *with* $D((z_n)_k) = (dz_0)_0$, $C((z_n)_k) = (cz_{k-1})_0$, *in which we add, for every* $a \in A$ *of the form* dx *or* cy, *an identity element* I_a *on the path* $(a)_0$ *of length* 1 *determined by* a.

Now we modified this construction to get the category $\mathrm{P}_{lab}(A)$ *of paths with labelled vertices: an object is an element* u *of* A, *seen as a label for* cu, *and a morphism from* u *to* v *is a morphism* $(z_n)_{0 \leq n < k}$ *in* $\mathrm{P}(A)$ *from* $cu = dz_0$ *to* $cv = cz_{k-1}$.

Proposition 3.7. *Given an autograph* A *and two elements* $u, v \in A$, *then, with definition 3.6 and Proposition 3.5, we have:*

$$Hom_{\mathrm{Rep}(A,d,c)}(\Gamma^A_u, \Gamma^A_v) = Hom_{\mathrm{P}_{lab}(A)}(v, u),$$

hence Γ^A *is a functor*

$$\Gamma^A : \mathrm{P}_{lab}(A) \longrightarrow \mathrm{Rep}(A, d, c)^{op}.$$

Proof. Proposition 3.5 for $\Phi = \Gamma^v$ gives $Hom_{\mathrm{Rep}(A,d,c)}(\Gamma^A_u, \Gamma^A_v) = \Gamma^A_v(cu)$. □

Proposition 3.8. *1 —- Given a relational representation $\varphi = (\Phi, \phi^d, \phi^c)$ of an autograph $A = (\underline{A}, d, c)$, as in [9, Definition 1.6.], with for each $f \in A$,*

$$\Phi(df) \overset{\phi^d(f)}{\longleftarrow} \Phi(f) \overset{\phi^c(f)}{\longrightarrow} \Phi(cf),$$

we get a morphism in Agraph *over A,*

$$q_\varphi : \Sigma\varphi \to A,$$

with $\Sigma\varphi = (S_\varphi, d_\varphi, c_\varphi)$, $S_\varphi = \{(f, u); f \in A, u \in \Phi(f)\}$, $q_\varphi(f, u) = f$, $d_\varphi(f, u) = (d_A f, \phi^d(u))$, $c_\varphi(f, u) = (c_A f, \phi^c(u))$.
2 — Conversely, given an arbitrary arrow

$$q : A' = (\underline{A'}, d', c') \to A,$$

we get a relational representation of A given by $\Phi(f) = q^{-1}(f)$, $\phi^d(x) = d'(x)$, $\phi^c(x) = c'(x)$.
3 — As a corollary, being such a $q : A' \to A$, each autographic algebra

$$\theta : T(A) \to A$$

determines a relational representation of A.

Proof. The part 1 is [9, Proposition 1.7.], with some corrections of typos, and the part 2 is the generalization of the end of [9, Proposition 1.2.].   □

4. Dimensioned autographs, the case of globular sets

As a sequel of section 3, in this section we show how globular sets are some autographs structured by $dd = dc$ and $cc = cd$, and a map to $\mathbb{N}^{\text{aug}}$.

4.1 Dimensioned autographs

We introduce the notion of a *dimensioned autographs* as being an autograph equipped with a graduation (see *terminology* at the beginning of section 2.4) toward the autograph of numbers $\mathbb{N}^{\text{aug}}$.

Definition 4.1. *We denoted by* $\underline{\mathbb{N}}^{\text{aug}}$ *the autograph of which the underlying set is the set* $\mathbb{N}$ *of integer augmented with* -1

$$\underline{\mathbb{N}}^{\text{aug}} = \mathbb{N} \cup \{-1\},$$

and where the domain and codomain are equal to the same map "predecessor" pred *given by* $\text{pred}(n+1) = n$, $\text{pred}(0) = \text{pred}(-1) = -1$. *Also we consider* $\underline{\mathrm{D}}_n$ *with*

$$\underline{\mathrm{D}}_n = \{x \in \underline{\mathbb{N}}^{\text{aug}}; x \leq n\},$$

with on it the domain and codomain induced from pred *as in the picture :*

$$\mathrm{D}_n$$
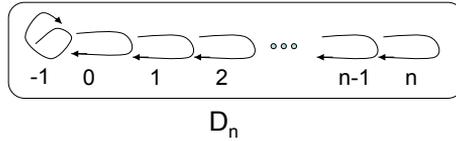
A Dimensioned augmented autograph — *or shortly a* dimensioned autograph — *is an autograph equipped with a "dimension",i.e. a morphism of autograph*

$$D : A \to \underline{\mathbb{N}}^{\text{aug}}.$$

These dimensioned (augmented) autographs constitute a topos over Agraph

$$\text{Agraph}_{\text{dim}} = \text{Agraph}/\underline{\mathbb{N}}^{\text{aug}} \longrightarrow \text{Agraph},$$

of which

$$\text{Agraph}_{n-\text{dim}} = \text{Agraph}/\underline{D}_n$$

is the sub-topos of n*-dimensioned autographs.*

4.2 ∞-graphs and globular sets

The next definition comes — without using the term — from Ronnie Brown and Philip J. Higgins [1].

Definition 4.2. *A* globular set X *is a collection* $(X_n)_{n \geq 0}$ *of sets with maps* $s_n, t_n : X_{n+1} \to X_n$ *such that*

$$s_n s_{n+1} = s_n t_{n+1} \text{ and } t_n s_{n+1} = t_n t_{n+1}.$$

Hence given $m < n$ there are only two maps $s_{m,n}, t_{m,n} : X_n \to X_m$ obtainable by compositions of s_k and t_l:

$$s_{m,n} = s_m...., t_{m,n} = t_m.... : X_n \longrightarrow X_m.$$

Theses globular sets constitute a topos Glob *of presheaves on* $\mathbb{G}$*, where* $\mathbb{G}$ *is the globe category, with objects integers* $n \in \mathbb{N}$*, and whose morphisms are generate by* $\sigma_n, \tau_n : n \to n + 1$*, with relations*

$$\sigma_{n+1}\sigma_n = \tau_{n+1}\sigma_n \text{ and } \sigma_{n+1}\tau_n = \tau_{n+1}\tau_n,$$

$$\text{Glob} = \text{Ens}^{\mathbb{G}^{op}}$$

Proposition 4.3. *A globular set is exactly determined by an autograph* $A = (\underline{A}, d, c)$ *with*

$$dd = dc, \quad cd = cc,$$

equipped with a morphism "dimension" as in Definition 4.1

$$D : A \to \mathbb{N}^{\text{aug}}.$$

Proof. The datum of a globular set X is equivalent to the datum of an autograph $\tilde{X} = (\underline{\tilde{X}}, d, c)$ with $\tilde{X}_n = X_n \times \{n\}$, $\tilde{X}_{-1} = \{*\} \times \{-1\}$,

$$\underline{\tilde{X}} = \bigcup_{n \geq -1} \tilde{X}_n = \bigcup_{n > 0} X_n \times \{n\} \cup (\{*\} \times \{-1\}),$$

$d(x, n) = (s_{n-1}(x), n - 1), \ c(x, n) = (t_{n-1}(x), n - 1)$ for $x \in X_n, n > 0$,

$d(x, 0) = (*, -1) = c(x, 0)$, for $x \in X_0, \ d(*, -1) = (*, -1) = c(*, -1)$,

and consequently we have

$$dd = dc, \quad cd = cc.$$

Given this $\tilde{X}$ we recovered X with: $\tilde{X}_{-1} = \{x \in \tilde{X}; dx = x = cx\}$, $\tilde{X}_0 = \{x \in \tilde{X}; dx, cx \in \tilde{X}_{-1}, x \notin \tilde{X}_{-1}\}$, and, for $n > 1$,

$$\tilde{X}_n = \{x \in \tilde{X}; dx, cx \in \tilde{X}_{n-1}, x \notin \tilde{X}_{n-1}\}.$$

$\square$

5. Observations on drawings of autographs

The following observations could be considered as comments on drawings of $\mathbb{C}_3$ in example 2.2. And along the way we are led to interesting structural enrichements of autographs (virtualizations, transductions, autosuccessions, partial autographs and free autographs, unoriented autographs). They serve as preparations of examples for colorings (section 6) and their uses in the description of structured autographs.

5.1 Virtual drawings, virtualized autographs, auto-transductions

As evocated in section 2, in our previous papers [7], [8] and [9], we gave some planar drawings of (oriented) autographs for alternating knots, for some knots and links. We gave also drawings for 2-categories, double categories, autocategories: let us notice that for a 2-blocks in a double categories, the drawing is not planar, because (see Example 2.4) cb is a broken arrow, passing under $\overline{db}$, going from dcb to ccb, and b is naming a kind of "cushion" between two faces db and cb; so its picture is rather related in fact to virtual link (see 5). Furthermore some case may have planar and not planar (virtual) drawings, as it is the case for $\mathbb{C}_3$ in Example 2.2. Nevertheless, planar drawings are not enough to represent any finite link, or a fortiori any finite autograph — and now here we are reaching such convenient planar representations with oriented *virtual diagram* or *virtual links*, for links, and *planar oriented virtual drawings* for arbitrary finite autographs.

The autograph $\mathbb{C}_3$ in example 2.2 is drawn (second line, left) by a virtual knot, according to definition 5.1. In fact it is an autosuccession (Definition 5.12).

Definition 5.1. *According to [13, Def.1.4, p.8], a* (planar) virtual diagram *or a diagram of a* virtual link *is any image of an immersion of a framed* 4-*valent graph in* $\mathbb{R}^2$ *with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a* virtual crossing *and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph are called* classical crossings *or just* crossing.

Definition 5.2. *To an oriented virtual diagram* Λ *of an oriented link we associate an autograph* $\mathbb{A}[\Lambda]$ *as follows. We neglect virtual crossings — such that any arc* x *arriving at a virtual crossing with an arc* y *(which may be* x *itself !) is continuing its path on the other side of* y, *with the same name* x, *the domain* dx *of* x *being the other arc that* x *meet at its starting classic crossing, the codomain* cx *of* x *being the other arc at its ending classic crossing.*

Definition 5.3. *A* virtual link *is an equivalence class* $[\Lambda]$ *of a virtual diagram* Λ *modulo the generalised Reidmeister moves.*

Proposition 5.4. *A virtual knot* $[K]$ *is determines by* $\mathbb{A}[K]$.

Proof. As observed in [13, p. 9], in order to define a *virtual knot* $[K]$, we need only to know the position of classical crossings and there connections with each others in K. Moreover, positions of paths connecting classical crossings, their intersections and self-intersections, are not important for us.
□

Proposition 5.5. *[Kauffman, [11]] Any link can be drawn as a virtual link, and starting from a virtual link we can decide if it representing a true link.*

Proof. The idea of *virtual knot* and *virtual link* comes from Louis Kauffman. For example, from [11, p. 666] we look to these two unoriented pictures:
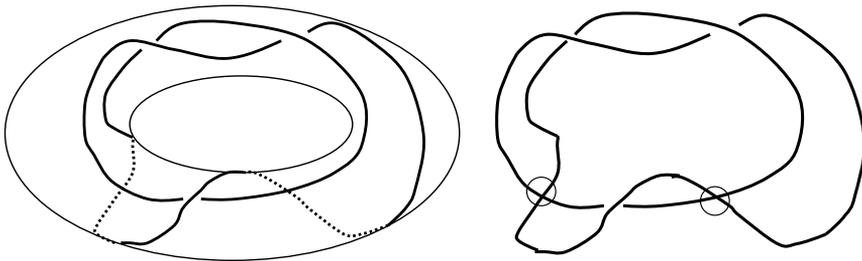
On the left a knot is drawn on a torus, with 3 crossings *on the surface of the torus*, each one with two lines near a same point on the torus, and with also 2 "false crossings", i.e. crossing of a continous line (ahead) and a dashed line (behind) in our perspective, which in fact are not near the same point on the torus. In a planar view (on the right), these "false crossing"" are marked by small circle, wheras now the lines are all continuous; there we get *virtual planar drawing* of a knot.

In our example C_3 in 2.2 the virtual knot on the left down is coming from the knot on the left up, hence to a virtual knot is associated an autograph with 3 arcs x, y, z, which is also the autograph of a knot (the trefoil knot) (cf. [7, Example 4.3., p. 71] where x, y, z are u, v, w). But a virtual link (or virtual knot) is *not* necessarily coming from a link (or a knot) in such a way.

The autograph associated to an oriented virtual diagram determines the associated oriented virtual link. This virtual link could be a classical link, and generally it can be determined by many autographs.

Proposition 5.6. *Any* planar virtual drawing *is determined by at least one such finite autograph.*

Of course, starting with a finite autograph it is easy to draw an oriented virtual link to which it is associated, the use of virtual crossings being unlimited.

The observations above do suggested to introduce the notion of a *virtual drawing of an autograph* and of a *virtualized autograph* (or a strictly increasing *auto-transduction*) as follows in definition 5.10. This notion allows to obtain a more accurate codification of planar virtual drawings of autographs.

Definition 5.7 (Virtual drawing of an autograph). *A virtual drawing of an autograph $A = (\underline{A}, d, c)$ is (similarly to the explanation given in definition 5.1) a picture in the plan made of a collection of arrows $[w]$, $w \in A$, each such arrow being drawn "from" its domain $[dw]$ "toward" its codomain $[cw]$; so we have an* effective crossing *for $[w]$ and $[dw]$, for $[w]$ and $[cw]$; each $[w]$ has also a finite number of intersections with the others, each of them is called a virtual crossing.*

Easily we have the following proposition 5.8.

Proposition 5.8. *Each finite autograph A admits a virtual drawing which is representable by a* planar graph $\Gamma(A)$ *with four types of vertices, as given in the picture below. The first type is "virtual", the third and fourth or "half-crossing" types, are namely "codomain" and "domain", and the second is "(complete) crossing". A complete crossing is to think as a kind of apairing or coupling of a "codomain" and a "domain", a fact explicitely signifiable by $s(w) = x$ [see remarks 2.2, remarks 2.3, subsection 5.2, definition 5.12].*

Vir $Cros$ Cod Dom
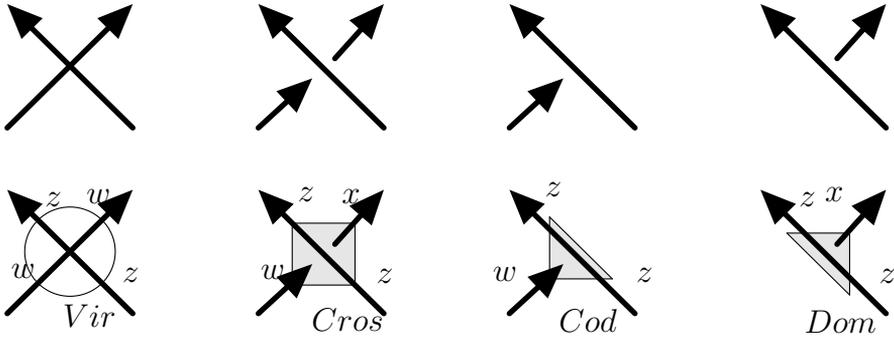
Proof. Of course, starting with a finite autograph $A = (\underline{A}, c, d)$ it is easy to draw a virtual drawing, the use of virtual crossings being unlimited. Furthermore, from such a drawing we can observe the corresponding virtualized autograph $(\underline{A}, c, d, V)$, according to Definition 5.10. Then the graph $\Gamma(A)$ is obtained from A with vertices the various virtual crossings, complete crossings and half-crossings, and with, for each arrow $f \in \underline{A}$, $f : df \to cf$, a sequence of edges $(f, f_1) : df \to f_1$, ..., $(f_i, f_{i+1}) : f_i \to f_{i+1}$, ..., $(f_n, cf) : f_n \to cf$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.9. A finite graph is not necessarily planar, but of course considered as an autograph it admits a finite planar representation by a virtual drawing (with virtual crossings). The reader will not mistake these two facts.

Definition 5.10. *A* virtualized autograph *is the data* $(\underline{A}, c, d, V)$ *of an autograph* $A = (\underline{A}, c, d)$ *structured by the data of a function* $V : \underline{A} \to \underline{A}^*$ — *with* $\underline{A}^*$ *the free monoïd of words in* $\underline{A}$, *where, for every* $f \in \underline{A}$, $V(f)$ *is thought as the sequence of arrows in A which, in a planar drawing, could virtually be crossed (virtually) by f in the order of circulation from cf to df,* $V(f) = f_1 f_2 \dots f_n$ *; but it is not assumed that such a planar drawing do exist for this given V.*
An auto-transduction *is a* $(\underline{A}, W)$ *of a set* $\underline{A}$ *with a function* $W : \underline{A} \to \underline{A}^*$.
Obviously, with $W(f) = d(f)V(f)c(f)$, *a virtualized autograph is equivalent to an auto-transduction where for every* $f \in \underline{A}$, $W(f)$ *is of length at least* 2; *we can say that such a W is strictly increasing.*

Example 5.11. In the case of the virtual drawing of $\mathbb{C}_3$ in 2.2 left down, we get the virtualization $V(x) = yyxx$, $V(y) = zzyy$, $V(z) = zz$.

In the case of example 2.4, we have $V(cb) = db$, $V(db) = cb$, and for others f, $V(f) = ()$ (the empty word).

5.2 The case of an autosuccession

If we want to structure an autograph to get closer to the notion of a knot or a link, we can introduce the notion of an *autosuccession*. In the case of the autograph $(\underline{A}, d, c)$ associated to a link we have $c(A) \subseteq d(A)$, and more precisely, for each w there is a unique $x = s(w)$ such that $cx = dw$. So it is an autosuccession according to Definition 5.12.

Definition 5.12. *An* autosuccession *is a data* $(\underline{A}, d, c; s)$ *of an autograph* $(\underline{A}, d, c)$ *equipped with a map* $s : \underline{A} \to \underline{A}$ *such that*

$$ds = c.$$

If furthermore d, c *and* s *are bijective, the autosuccession is said to be an* alternating autosuccession.

Remark 5.13. Of course an autosuccession is an arbitrary data $((\underline{A}, d, s)$ of an autograph (with s as a codomain map), and $(\underline{A}, d, c; s) = (\underline{A}, d, ds; s)$; so, starting from an autograph $((\underline{A}, d, s)$ we generate a sequence of autosuccessions: $(\underline{A}, d, ds; s)$, $(\underline{A}, d, d^2s; ds)$, $\ldots (\underline{A}, d, d^{n+1}s; d^n s)$.

Proposition 5.14. *In a finite autosuccession each element* u *generates a circular path* $u, su, ssu, \ldots, s^n u = u$, *for* $n = n(u)$ *an integer associated to* u. *These paths can be drawn in a planar virtual diagram of a virtual link (oriented), and conversely any virtual knot (oriented) is determined by at least one such autosuccession. Furthermore finite alternative autosuccessions correspond to alternating knots.*

Proof. It is an immediate consequence of Proposition 5.5. □

Remark 5.15. The analysis of a link as an autograph structured as an autosuccession is not the only possibility. For instance, we could use of quandles or racks (see [10], [14]).

5.3 Two drawings of a reversible autograph.

In fact $\mathbb{C}_3$ in example 2.2, considered as being a planar view of a knot (see on the first line of its given drawings), is a reversible autograph according to the definition 5.18 and so it is possible to draw it in another way.

Definition 5.16. *[standard partial drawing, dual partial drawings] If a set $\underline{A}$ is an autograph with d and c, for each $a \in \underline{A}$, we write it as on the left (standard partial drawing). If a set $\underline{A}'$ is an autograph with d' and c', we would like to have a kind of "dual "drawing process", for each $a \in A'$, as given on the right (dual partial drawing).*
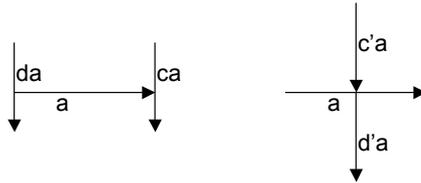
NB. Here "partial " means limited to the only element a.

Proposition 5.17. *For each autograph (A, d, c) the standard partial drawing is possible for any given $a \in A$. For an autograph (A', d', c') the dual partial drawing is possible for any given $a \in A'$ if and only if d' and c' are injective.*

Proof. For c, d, every arrow a has a unique domain da and a unique codomain ca, and this is physically clear on the standard drawing. For c', d' also we admit that every arrow a has a unique domain $d'a$ and a unique codomain $c'a$, but this is not obvious on the picture of the dual drawing, as we seem to be physically free to introduce several arrows starting from or arriving on the side of a. Furthermore a more serious difficulty is the question to draw a fact as $a = c'u$ and $a = c'v$, meaning that simultaneously a has to arrived on the side of u and on the side of v ! In fact the dual drawing for c', d' is possible if and only if c' and d' are injective. $\square$

Definition 5.18. *An autograph is a* reversible autograph *if the two maps d and c are invertible (bijective maps); the inverses of d and c are denoted by $d^{-1} = d'$ and $c^{-1} = c'$.*

Proposition 5.19. *Given a reversible autograph* $A = (\underline{A}, d, c, d', c')$, *with* $c' = c^{-1}, d' = d^{-1}$, *we can draw it using the standard local drawing for* d, c *and the dual local drawing for* d', c', *at a point* a, *and these drawings are compatible and can be drawn both in the same picture:*

.
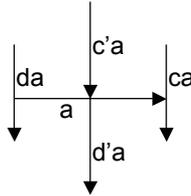
5.4 Partial autograph and associated free autograph

Proposition 5.20. *1 — We can construct the free autograph on one generator* $\mathbb{FA}(1)$ *as being* $\mathbb{FM}(2)$.
2 — We can construct the free reversible autograph on one generator $\mathbb{FRA}(1)$ *as being* $\mathbb{FG}(2) = \mathbb{Z} \star \mathbb{Z}$, *the free group on two generators.*

Proof. 1 — The free autograph on one generator "()" or "f" is denoted by $\mathbb{FA}(())$ or $\mathbb{FA}(\{f\})$, or $\mathbb{FA}(1)$, and in fact we have

$$\mathbb{FA}(1) = \mathbb{FA}(()) = \mathbb{FA}(\{f\}) \simeq \mathbb{FM}(2).$$

A detailed description of $\mathbb{FA}(\{f\})$ is given in the proof of [7, Proposition 3.1], as the sub-autograph generated by 0 in the free autograph on $\aleph_0$ generators

$$\mathbb{FA}(3\mathbb{N}) = (\mathbb{N}, m \mapsto 3m + 1, m \mapsto 3m + 2).$$

If in $\mathbb{FA}(\{f\})$ we cancel everywhere the letter f, or if we replace it by the empty word (), we get the set $\mathbb{FM}(2) = \{d, c\}^*$, with a domain d and a codomain c given by $d(m) = dm, c(m) = cm$, for every word m. As

showded in [7, Proposition 3.1], a picture of this starts with:
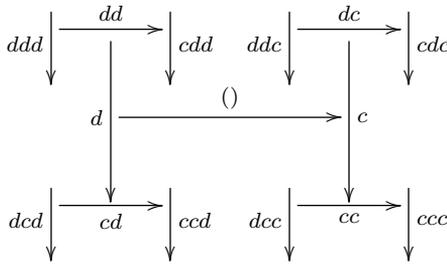
2 — If to the picture for $\mathbb{FA}(\{f\})$ indefinitely we add new arrows in such a way that each old or new arrow g is the domain and the codomain of an unique arrow $u = d'g$ and an unique arrow $v = c'g$, then we get $\mathbb{FRA}(\{f\})$. A picture of this starts with this extension of the begin of the start for $\mathbb{FM}(2)$ given above

□
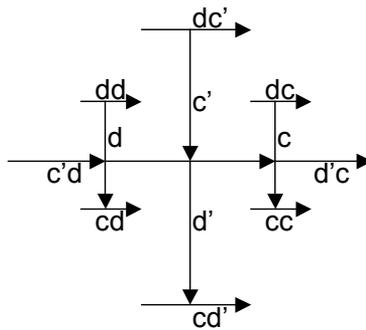
The observations in the subsection 5.3 and this subsection 5.4 , and notably the consideration of $\mathbb{FA}(\{f\})$, leads to the next construction of $FA(X)$ (Proposition 5.24).

Definition 5.21. *A* partial autograph *is a data* $X = (\underline{X}, d, c)$ *of a set* $\underline{X}$ *equipped with two partial maps* $d, c : \underline{X} \to \underline{X}$.

Proposition 5.22. *A finite partial autograph admits a virtual drawing which is representable by a* planar graph *with six types of vertices, namely the four given in proposition 5.8, and two new one* $\star(u)$, *at the source of* u *if* df *is not defined, and* $\bigcirc(u)$ *at the target of* u *if* cf *is not defined.*

Remark 5.23. Given a *fragment of an autograph* as in Example 1.5 in [7], with some "sources" $\bigcirc_j$ and "targets" $\star_i$, if we neglect these sources and target, we get a partial autograph M, which can be transformed into an autograph by substituting to each source or target an auto-arrow (as explained in [7]). But we can also obtain an autograph by using Proposition 5.24, and substituting a "free arrow" to each source or target, with the construction $FA(X)$ for $X = M$.

Proposition 5.24 (Free autograph $FA(X)$ on a partial autograph X)**.** — *We construct $FA(X)$ the free autograph generated by X by the next formal additions: for each $f \in \underline{X}$, if df is not defined, we add to $\underline{X}$ a formal data df and $FA(\{df\})$; if cf is not defined, we add to $\underline{X}$ a formal data cf and $FA(\{cf\})$. Hence this new set $\overline{FA(X)}$ is underlying to an autograph $FA(X)$, in which X is embedded $J_X : X \to FA(X)$ and such that each morphism $U : X \to A$ to an autograph admits a unique factorization $U' : FA(X) \to A$, with $U'J_X = U$.*

Obviously we have:

Proposition 5.25. *Given an autograph $A = (\underline{A}, d, c)$, a subset $E \subseteq A$, and the partial autograph $Z_E = (E, d_{|E}, c_{|E})$, the inclusion $U_E : E \to A$ factorize through $J_{Z_E} : E \to FA(Z_E)$ as $V_E : FA(Z_E) \to A$. The image of V_E is the sub-autograph of A generated by E.*

5.5 Drawings with or without orientations

With respect to this question of drawings (see subsection 2.2), now we have to precise something about "unoriented autographs".

In [8] and [9] we show how graphs "are" autographs, and how algebraicity or monadicity over graphs can be transfer over autographs. At this moment we let on the side the question of the description from the point of view of autographs of some important variant notions of graphs. Here we specially examine the question of orientation.

5.5.1   Equivocal on terminology

In fact, according to the history and tradition in "Graph Theory" and/or in "Category Theory", there are several variants, according to several facts:

vertices are or are not interpretable as arrows, arrows between two vertices are or are not unique, arrows are or are not oriented.

var1. We use the term "*graph*" as in the categoricians'style, for a datum of a span $V \xleftarrow{s} E \xrightarrow{t} V$ with a map $V \xrightarrow{i} E$ such that $si = 1_V = ti$. This is sketched by $\mathbb{G}(2)$ in our Definition 1.2. in [8, p.152], with $G(v_0) = V$, $G(v_1) = E$, $G(\delta_0) = s$, $G(\gamma_0) = t$, $G(\iota) = i$. This definition is named *var1*, and in [8, Proposition 2.2, p. 154] we explained how this notion is algebraic over autographs.

var2. Someones named "graph" the same thing that our "graph" in *var1* excepted they miss i, i.e. considering only a datum $V \xleftarrow{s} E \xrightarrow{t} V$.

var3. For others persons such a graph in our sense in *var1* is named a di-multigraph (or an oriented multigraph) , and for them a "directed graph" or a "digraph" is a datum $E \subset V \times V$ (that we named a binary relation on V), whereas a "graph" is a datum $E \subset V \times V$ which is symmetric (a symmetric binary relation on V). Hence for them a "multigraph" is a dimultigraph without orientations on arrows.

var4. Some other authors named "graph" a datum $V \xleftarrow{s} E \xrightarrow{t} V$ with a map $E \xrightarrow{()^\sigma} E$ such that $(e^\sigma)^\sigma = e$, $s(e) = t(e^\sigma)$, and "oriented graph" such a graph in which in each pair $\{e, e^\sigma\}$ one edge is chosen to be called the positively oriented edge, and denoted e^+, and the other is e^-.

Remark. Of course such a graph in *var4* is equivalent to the datum of an "unoriented (multi)graph" i.e. a map $E \xrightarrow{\delta} \mathcal{P}_{1 \leq 2}(V)$, from E to the set of unordered pairs of elements of V. given by $\delta(\epsilon) = \{se, te\}$; and such an "oriented graph" is equivalent to the datum of a graph in our sense (*var1*).

Precaution. Consequently on reading the mathematical litterature we have to be careful with the meaning of the term "graph" when referring to results or notions in Graph Theory, as: coloured graph, automorphisms of a graph, Cayley's graph, Frucht's theorem, etc. The same precaution will be necessary here with the terms "autograph" and "unoriented autograph", "coloured autograph", etc.

5.5.2   Unorientation or orientation?

When we draw an autograph with arrows, each arrow has to be oriented, according the direction of the move of our pencil. But if we do not write the information of this move, for instance with a hook at the end, then the

visible result is an *unoriented autograph*. So an autograph is an unoriented autograph (definition 5.26) equipped with an orientation.

Definition 5.26. *An* unoriented autograph $U = (\underline{U}, \delta)$ *is a set $\underline{U}$ with a map* $\delta : \underline{U} \to \mathcal{P}_{1 \leq 2}(\underline{U})$ *from $\underline{U}$ to the set of unordered pairs of elements of $\underline{U}$. The fact that $\delta(w) = \{u, v\}$ is drawn as an unoriented arc or "unarrow" $u \overset{w}{\sim} v$.*

Proposition 5.27 (unoriented simulation). *Each autograph $A = (\underline{A}, d, c)$ can be simulated by a pointed unoriented autograph.*

Proof. Each autograph $A = (\underline{A}, d, c)$ determines an underlying unoriented autograph $U(A) = (\underline{A}, \delta)$, given on the set $\underline{A}$ by $\delta(w) = \{dw, cw\}$. We can add to $U(A)$ an auto-unarrow ω and for each $e \in \underline{A}$ from $de = u$ to $ce = v$, two new unarrows f and g with $\delta(f) = \{e, ce\}$ and $\delta(g) = \{\omega, f\}$, according to the picture
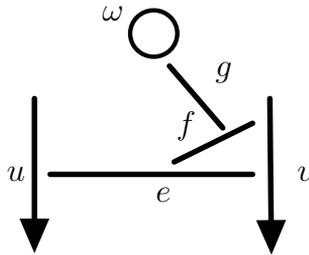
This new unoriented autograph is denoted by $\overline{U}(A)$ and named the unoriented simulation of A. The point is that the knowledge of $\overline{U}(A)$ with the specification of the point ω determine A (equipped with its orientation) : the orientation of e (from u to v) is given by the corresponding data (f, g). So the data A is equivalent to the data

$$U(A) \to \overline{U}(A) \leftarrow \{\omega\}.$$

$\square$

Proposition 5.28. *Any unoriented autograph U can be presented as an involutive autograph A, and also as a bi-pointed autograph.*

Proof. 1 — Given U, according to *var4* in the terminology above, we choose two maps $\phi, \psi : U \to \{s, t\}$ such that $\delta(f) = \{\phi(f), \psi(f)\}$ and we construct $U_{\phi,\psi} = U^+ + U^-$, with $U^+ = \{f^+, f \in U\}$, $U^- = \{f^-, f \in U\}$,

and with d and c given by $d(f^+) = \phi(f)$, $c(f^+) = \psi(f)$, $d(f^-) = \psi(f)$, $c(f^-) = \phi(f)$; furthermore we have an involution σ given by $(f^+)^\sigma = f^-$, $(f^-)^\sigma = f^+$. We recover U by identifying every $g \in A$ (which is an f^+ or an f^-) with g^σ. In fact this identification is simulated as follows.

If now we introduce an autograph with two auto-arrows, namely

$$\mathbb{T} = \{\{+\}, \{-\}\},$$

we add to $U_{\phi,\psi}$ that autograph and, for each $f \in U$ an arrow $p : \{+\} \to f^+$ and an arrow $q : \{-\} \to f^-$, and also an arrow $r : p \to q$, in such a way to express the identification by this r and via p and q of f^+ and f^-, we obtain an autograph $\overline{U_{\phi,\psi}}$. Then the data of U is determined by the data

$$U_{\phi,\psi} \leftarrow \mathbb{T}.$$

$\square$

5.5.3 Automorphisms of autographs or of unoriented autographs

As a sequel of subsubsection 5.5.2, here we show how autographs or unoriented autograph can be used in order to describe groups.

Proposition 5.29. *Any autograph can be re-described as an unoriented autograph, with the same automorphism group. Hence any group is the group of automorphism of an autograph as well as the automorphism group of an unoriented autograph.*

Proof. We start with the case of coloured digraph (i.e. a data $E \subset V \times V$, with a coloured map $\kappa : E \to K$) and of graph (in the sense of a data of a symmetric binary relation $E \subset V \times V$).

The colored Cayley's digraph $\mathcal{C}_\Delta(G)$ of a group G generated by a set Δ is the graph with vertices elements u of G, an with arrow any $u \xrightarrow{h} uh$, with $h \in \Delta$, and h being the color of this arrow. Every $f \in G$ determines a coloured automorphism $\rho_f : u \mapsto fu$ on the coloured graph $\mathcal{C}_\Delta(G)$ (an automorphism preserving the colors), in such a way that G is isomorphic to $\mathrm{Aut}(\mathcal{C}_\Delta(G))$ (If ϕ is a coloured automorphism of $\mathcal{C}_\Delta(G)$, then we have $\phi(uh) = \phi(u)h$, and as $u = 1h_1...h_q$ we get $\phi(u) = \phi(1)u = fu$, with $f = \phi(1)$, $\phi = \rho_f$).

Now this coloured graph can be transformed into a graph, the Frucht's graph $\mathcal{F}_\Delta(G)$ with

$$G = \text{Aut}(\mathcal{C}_\Delta(G)) = \text{Aut}(\mathcal{F}_\Delta(G)).$$

To do that, the orientation of an oriented arrow is simulated in an unoriented way by the adjunction of some "lateral" decoration with "unoriented arrows" or segments;in fact these decorations represent simultaneously the orientation and the color of the arrow. This can be performed as in the next picture with an unsymmetrical tower of high i (this i or h_i being the color) — on the left, or any variant — as on the right
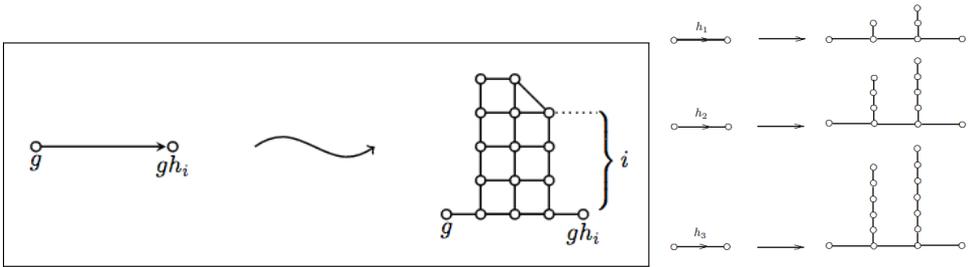
As an example, let us consider as in [18, pp. 295, 299] the colored Cayley's graph of $\mathcal{S}(3)$ generated by a transposition denoted by 1:=(12) and a 3-cycle denoted by 2:= (123) — so we have two colors 1 and 2 —, and conjointly, the corresponding Frucht's unoriented graph, as in the following picture.
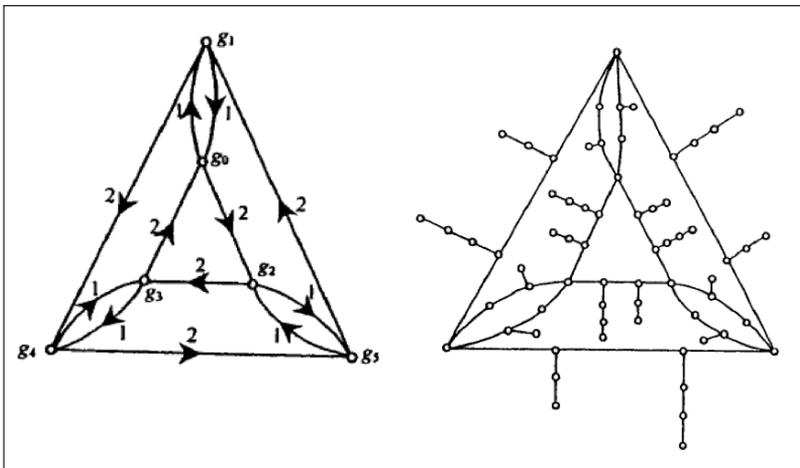
The result for finite groups is in [3], for arbitrary groups in [16] and [4].

Now for an autograph or an unoriented autograph we do the same job.

□

6. Colored autographs and structurations

6.1 Colored autographs

An autograph can be coloured, a color or a label being affected to each arrow. In fact — Proposition 6.2 —colored autographs are autographs under a given fixed colouring autograph. A color can express the name of an action (as with gracts) or the name of a point of view for assimilation (with regimes of assimilations) as in Proposition 6.3.

Definition 6.1. *Given a set K, a K-colored autograph is an autograph $A = (\underline{A}, d, c)$ (a set $\underline{A}$ and two maps $d, c : \underline{A} \to \underline{A}$) equipped with a "coloring map" $\kappa : \underline{A} \to K$, from the set $\underline{A}$ toward the set of colours K. Colors are also named labels.*

Proposition 6.2. *If K is finite, a K-colored autograph (A, κ) can be specified as an ordinary autograph $A[\kappa]$ in which A is included as a sub-autograph, and in which K is represented by an autograph $[K]$, giving a cospan between A and $[K]$:*

$$A \longrightarrow A[\kappa] \longleftarrow [K].$$

So (A, κ) looks like a $[K]$-algebra $A[\kappa]$; that is to say that the category K-Agraph of coloured autograph — of which a map $z : (A, \kappa) \to (B, \lambda)$ is a map of autograph $z : A \to B$ commuting with κ and λ (or $\lambda(z(f)) = \kappa(f)$) — is isomorphic to the category $[K]/$Agraph of autographs under $[K]$.

Proof. If K has n elements, we consider $[K]$ as the disjoint union of n autographs $K_1, \ldots K_n$, each K_i being constituted of i elements $x_1^i, x_2^i, \ldots, x_i^i$, with:

$$dx_1^i = cx_1^i = x_1^i, \; dx_2^i = cx_2^i = x_1^i, \ldots, dx_i^i = cx_i^i = x_{i-1}^i.$$

The construction of K_i ensures that colors i are internally distinguishable in terms of autographs (the K_i are not isomorphic). To construct $A[\kappa]$, to A we

add $[K]$ and, for each $f \in A$ with $\kappa(f) = i$ we put a new arrow $\kappa_f^i : x_1^i \to f$. This arrow says that the colour of f is i.

□
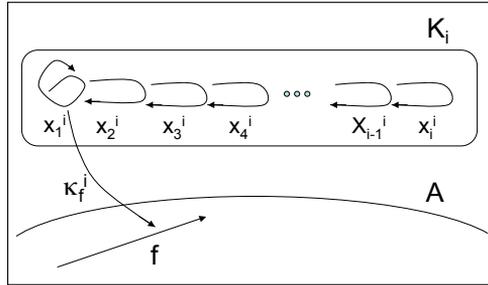
6.2 Gracts and regimes

Proposition 6.3. Gracts *in the sense of [15], and* 1-regimes *in the sense of [6], are examples of coloured autographs.*

Proof. Firstly, let us recall that a graph "is" an autograph, hence a coloured graph is a coloured autograph.

Now, in order to represent an action of a set W of operations on a set S of states, Jacques Riguet [15] introduced the notion of a *gract* or graph-action drawing, symbolizing the fact that the action w acting on s produces t as: $s \xrightarrow{w} t$. There s (resp.t) is the source (resp. target) of $s \xrightarrow{w} t$, or of (t, w, s), but *it is not* the source (resp. target) of the symbol w alone. Especially the symbol w could appears in several arrows as $s \xrightarrow{w} t$ or $u \xrightarrow{w} v$, etc. The gract Γ is the full drawing of all these vertices and arrows. Formally a gract Γ is only determined by a map

$$\gamma : W \times S \to S : (w, s) \mapsto t = w.s = \gamma(w, s),$$

and it can be also represented by a graph $G(\Gamma)$ with vertices $s \in S$, in which an arrow from s to t is a 3-uple (t, w, s) such that $t = w.s$, represented as: $s \xrightarrow{(t,w,s)} t$. The set of arrows of $G(\Gamma)$ is $A(\Gamma) = \{a = (t, w, s); w.s = t\}$ a subset of $S \times W \times S$. The set of arrows from s to t is denoted by $G(s, t)$. Furthermore these arrows have *colored* values in W, given by $\kappa(a) = w$ if and only if $a = (t, w, s)$, the same color w being possibly affected to

different arrows $a = (t, w, s)$ and $a' = (t', w, s')$ in $G(\Gamma)$, but only if a and a' have different source ($s \neq s'$) or different target ($t \neq t'$).

So a gract is nothing else than a graph $G = (A, S)$ equipped with a surjective map $\kappa : A \to W$, injective on each $G(s, t)$; or a graph equipped with an ordinary equivalence relation or a partition on its set of arrows which is discrete on each $G(s, t)$. A gract is also a decomposition

$$A = \sum_{w \in W} A_w, \quad A_w = \{a \in A : \kappa(a) = w\}$$

of a graph G as a sum of binary relations A_w on the set S.

A gract is also a special case of a 1-*regime of assimilation* in the sense of [6], i.e. a map

$$r : W \to \mathcal{P}(S \times S),$$

here given by $r(w) = \{(t, s); w.s = t\}$. It is the special case in which, for all w, $r(w)$ is a function $r(w) : S \to S$, $r : W \to S^S$ is the companion of γ. Of course r is equivalent to the datum

$$\theta : S \times S \to \mathcal{P}(W)$$

transposed from r as $\theta(t, s) = \{w; (t, s) \in r(w)\} = \{w; w.s = t\}$, and that is a binary relation on S coloured in $\mathcal{P}(W)$. $\qquad\square$

6.3 Specifying relations and structures by poly-colorations or glueings

Definition 6.4. *Given two autographs B and A, and given a binary relation $R \subset \underline{B} \times \underline{A}$ we consider the glueing $B \oplus_R A$ or $\mathcal{G}(B, A)(R)$ of B and A with respect to R, which is the autograph obtained from the disjoint sum $B + A$ of B and A by adding a new auto-arrow denote by r (with $dr = r = cr$), and for each $b \in B$, $a \in A$ such that $(b, a) \in R$, two new arrows $s : b \to a$ and $t : s \to r$. We have two canonical injective morphisms $B \xrightarrow{\beta_R} B \oplus_R A \xleftarrow{\alpha_R} A$, and a map $j_R :< \beta_R, \alpha_R >: B + A \to B \oplus_R A = \mathcal{G}(B, A)(R)$.*
If $R = \rho_F$ i.e. if R is the functional relation associated to a map $F : B \to A$, then $B \oplus_{\rho_F} A$ is simply denoted by $B \oplus_F A$.

Proposition 6.5. *The glueing $B \oplus_R A$ with the co-span (β_R, α_R) determines the binary relation R by the fact that $(b, a) \in R$ if and only if in $B \oplus_R A$:*

$$\exists s, \ \exists t \left(ds = \beta_R b, cs = \alpha_R a, dt = s, ct = r \right),$$

or — with $\beta_R b \cong b$ and $\alpha_R a \cong a$ — the picture
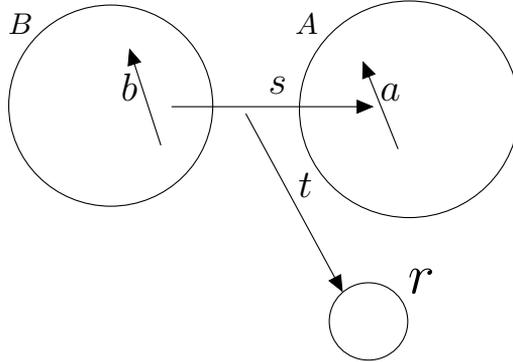
So the binary relation R is specified by the data

$$B + A \to B \oplus_R A \leftarrow \{r\}.$$

Definition 6.6. *Given three autographs C, B and A, and a ternary relation $R \subset \underline{C} \times \underline{B} \times \underline{A}$ we consider the glueing $\mathcal{G}(C, B, A)(R)$ of C, B, A with respect to R, which is the autograph obtained from a disjoint union $C+B+A$ of C, B and A, to which we add a new auto-arrow denote by r (with $dr = r = cr$), and for each $c \in C$, $b \in B$ and $a \in A$ such that $(c, b, a) \in R$, the three following arrows : $s : c \to b$, $t : s \to r$, $u : a \to t$. We have three canonical injective morphisms:*

$$C \xrightarrow{\gamma_R} \mathcal{G}(C, B, A)(R), B \xrightarrow{\beta_R} \mathcal{G}(C, B, A)(R), A \xrightarrow{\alpha_R} \mathcal{G}(C, B, A)(R),$$

and then a map $j_R :< \gamma_R, \beta_R, \alpha_R >: C + B + A \to \mathcal{G}(C, B, A)(R)$.
This works in particular for the ternary relation ρ_F associated to a binary partial law $F : C \times B \to A$.

Proposition 6.7. *The glueing $\mathcal{G}(C, B, A)(R)$ determines the ternary relation R by the fact that $(c, b, a) \in R$ if and only if :*

$$\exists(s, t, u) \left(ds = \gamma c, cs = \beta b, dt = s, ct = r, du = \alpha a, cu = t\right),$$

or — with $\gamma_R c \cong c$, $\beta_R b \cong b$ and $\alpha_R a \cong a$ — the picture
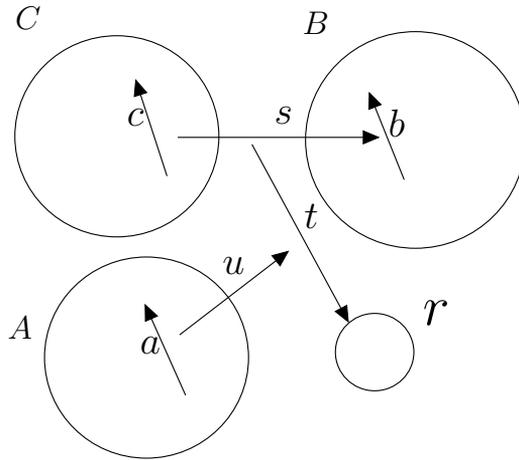
So the ternary relation R is specified by the data

$$C + B + A \to \mathcal{G}(C, B, A)(R) \leftarrow \{r\}.$$

Remark 6.8. The glueing construction $\mathcal{G}(C, B, A)(R)$ extend to three terms C, B and A the construction of proposition 6.4, and also the construction of proposition 6.2 for two terms (A and K). So we considered it as a poly-coloration, specified by three injective morphisms $\gamma : C \to \mathcal{G}(C, B, A)(R)$, $\beta : B \to \mathcal{G}(C, B, A)(R)$, and $\alpha : A \to \mathcal{G}(C, B, A)(R)$, or by the morphism $j = < \gamma, \beta, \alpha >: B + C + A \to \mathcal{G}(C, B, A)(R)$.

7. Autograph, morphism of autograph, or autocategory ?

As a conclusion, we want to emphasize that, during our exploration here, we have established in passing that the three notions of autograph, autograph morphism, autocategory, are equivalent data, and that we can describe each of them by a drawing as a virtual diagram. Indeed :

1. An autograph A is a morphism of autograph.
2. A representation of an autograph is a morphism of autograph.
3. An autographic algebra is a morphism of autograph.
4. An autocategory is an autographic algebra, and so it is a morphism of autograph.
5. A colored autograph or a poly-colored autograph is an autograph.
6. An autocategory is a poly-colored autograph, and so is an autograph.

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A HOPF FORMULA FOR THE FUNDAMENTAL GROUP IN THE CATEGORY OF PREORDERED GROUPS

Aline MICHEL

Résumé. En nous basant sur la description des extensions centrales et normales récemment obtenue dans le contexte des groupes préordonnés, nous montrons que la formule bien connue de Hopf pour les groupes peut être étendue aux groupes préordonnés. Pour atteindre cet objectif, nous construisons une extension centrale faiblement universelle pour tout groupe préordonné donné. Cette construction nous permet de calculer son groupe fondamental en adaptant le cas classique des groupes au contexte des groupes préordonnés.

Abstract. Based on the description of central and normal extensions recently obtained in the setting of preordered groups, we show that the well-known Hopf formula for groups can be extended to preordered groups. In order to achieve this goal, we build a weak universal central extension for any given preordered group. This construction allows us to compute its fundamental group by adapting the classical case of groups to the setting of preordered groups.

Keywords. Preordered groups, categorical Galois theory, weak universal central extension, regular projective object, Hopf formula, fundamental group.
Mathematics Subject Classification (2010). 06F15, 18E10, 18G50, 18A40, 18C40.

1. Introduction

In the category Grp of groups, there is a well-known categorical Galois theory [7]. Indeed, from the *abelianization functor* $ab\colon \mathsf{Grp} \to \mathsf{Ab}$, which associates to any group X the quotient $X/[X,X]$ by its derived subgroup, we can consider the *Galois structure* $\Gamma_{ab} = (\mathsf{Grp}, \mathsf{Ab}, ab, u, \mathcal{E}_{ab}, \mathcal{Z}_{ab})$ where $u\colon \mathsf{Ab} \to \mathsf{Grp}$ is the inclusion functor, and $\mathcal{E}$ and $\mathcal{Z}$ are the classes of regular epimorphisms in Grp and Ab, respectively. This Galois structure turns out to be admissible, so that a classification theorem of central extensions holds. Furthermore, it has been proven that the central and normal extensions coincide; these are the surjective group morphisms $f\colon A \twoheadrightarrow B$ whose kernel $\mathsf{Ker}(f)$ is in the center $Z(A)$ of A.

Based on this useful information, computing the *fundamental group* [9] of any group X (with respect to the above Galois structure Γ_{ab}) is quite simple. Consider a free presentation of the group X

$$K \xrightarrowtail{k} P \xrightarrow{p} X$$

with kernel (K, k). We can then prove that the induced surjective group morphism $\phi\colon P/[K,P] \twoheadrightarrow X$ in the commutative diagram below is such that $\mathsf{Ker}(\phi) \subseteq Z(P/[K,P])$, i.e. the induced arrow ϕ is a Γ_{ab}-central (or -normal) extension.

$$
\begin{array}{ccc}
P & \xrightarrow{\;\;p\;\;} & X \\
{\scriptstyle q}\downarrow & \phi & \\
P/[K,P] & &
\end{array}
$$

It is also possible to show that this central extension ϕ is *weak universal*, so that the *Galois group* of ϕ is then an invariant of X, called the *fundamental group* of X. By computing the Galois group of ϕ, we thus obtain a formula for the fundamental group of X:

$$\pi_1(X) = \frac{K \cap [P,P]}{[K,P]}.$$

This is the well-known *Hopf formula* [6], which corresponds to the second homology group $H_2(X, \mathbb{Z})$ of X.

The purpose of this article is to extend this formula to the setting of *preordered groups*. For this, we use the characterization of the central and normal extensions with respect to a "suitable Galois structure" obtained in the paper [5], and somehow imitate what was done in the case of groups. Recall that a *preordered group* $(G, \leq)$ is a (not necessarily abelian) group $G = (G, +, 0)$ endowed with a preorder relation $\leq$ which is compatible with the addition $+$ of the group G: if $a \leq c$ and $b \leq d$ for $a, b, c, d \in G$, then $a + b \leq c + d$. Preordered groups are the objects of a category, the category PreOrdGrp of preordered groups, whose arrows are given by the monotone group morphisms. The interested reader may find it useful to take a look at the article [3], in which the categorical behaviour of preordered groups was studied.

The present article is structured as follows. We start with a quick review of *categorical Galois theory* and recall the general definitions of the notions of *Galois groupoid*, *Galois group* and *fundamental group*. We then present some properties of the category PreOrdGrp of preordered groups that will be useful for our purpose. We also recall the main results of the article [5], in particular the description of the central and normal extensions. In the fourth section, we then construct a weak universal central extension for any given preordered group. This construction is used in the last section in order to obtain an explicit formula for the fundamental group (Theorem 5.3).

2. Categorical Galois theory

In this section, we briefly recall the basic notions of categorical Galois theory including, among other things, the definition of the fundamental group. For more details, the reader can refer to [7, 8, 9, 10] for instance.

Definition 2.1. A *Galois structure* is a system $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ in which

- $\mathscr{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{F}$ is an adjunction, with unit η and counit ϵ;

- $\mathcal{E}$ and $\mathcal{Z}$ are classes of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively,

such that

- $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks along morphisms from $\mathcal{E}$ and $\mathcal{Z}$, respectively;

- $\mathcal{E}$ and $\mathcal{Z}$ are closed under composition, contain all isomorphisms and are pullback-stable;

- $F(\mathcal{E}) \subseteq \mathcal{Z}$;

- $U(\mathcal{Z}) \subseteq \mathcal{E}$.

Definition 2.2. Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ be a Galois structure such that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism. Then, the Galois structure Γ is said to be *admissible* when F preserves all pullbacks of the form

$$
\begin{array}{ccc}
B \times_{UF(B)} U(X) & \longrightarrow & U(X) \\
\downarrow & & \downarrow{\scriptstyle U(\phi)} \\
B & \xrightarrow{\quad \eta_B \quad} & UF(B)
\end{array}
$$

where $\phi \in \mathcal{Z}$.

Definition 2.3. An arrow $f \colon A \to B$ in $\mathcal{E}$ is a (Γ-)*trivial extension* when the naturality square

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & UF(A) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UF(f)} \\
B & \xrightarrow{\eta_B} & UF(B)
\end{array}
$$

is a pullback.

Definition 2.4. A morphism $p \colon E \to B$ in $\mathcal{E}$ is called a *monadic extension* when it is an effective descent morphism [12, 11].

Definition 2.5. An arrow $f \colon A \to B$ in $\mathcal{E}$ is a (Γ-)*central extension* when there exists a monadic extension $p \colon E \to B$ such that $p^*(f) \colon E \times_B A \to E$ is a (Γ-)trivial extension, that is, the following square

$$E \times_B A \xrightarrow{\eta_{E \times_B A}} UF(E \times_B A)$$

$$p^*(f)=\pi_1 \Big\downarrow \qquad\qquad \Big\downarrow UF(\pi_1)$$

$$E \xrightarrow{\quad\eta_E\quad} UF(E)$$

is a pullback, where π_1 is the first projection in the pullback

$$E \times_B A \xrightarrow{\pi_2} A$$

$$p^*(f)=\pi_1 \Big\downarrow \qquad\qquad \Big\downarrow f$$

$$E \xrightarrow{\quad p\quad} B.$$

Definition 2.6. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a (Γ-)*normal extension* when f is a monadic extension and $f^*(f)$ is a (Γ-)trivial extension.

By definition, it is clear that any trivial extension and any normal extension is central. By using the admissibility of the Galois structure Γ, it is also possible to prove that all trivial extensions are normal.

Assume from now on that we have an admissible Galois structure $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{Z})$ for which the category $\mathcal{C}$ is pointed. Let $p\colon E \to B$ be a (Γ-)normal extension in $\mathcal{C}$, and consider its kernel pair (seen as an internal groupoid in $\mathcal{C}$):

$$\mathsf{Eq}(p) \times_E \mathsf{Eq}(p) \xrightarrow{\tau} \mathsf{Eq}(p) \overset{\overset{p_1}{\longrightarrow}}{\underset{p_2}{\underset{\longrightarrow}{\overset{\longleftarrow}{\Delta}}}} E.$$

with σ a loop on $\mathsf{Eq}(p)$.

Definition 2.7.

- The *Galois groupoid* $\mathsf{Gal}(E, p)$ of p is the image under the left adjoint F of the kernel pair of p, despicted as:

$$F(\mathsf{Eq}(p)) \times_{F(E)} F(\mathsf{Eq}(p))$$

$$\|\mathbb{R}$$

$$F(\mathsf{Eq}(p) \times_E \mathsf{Eq}(p)) \xrightarrow{F(\tau)} F(\mathsf{Eq}(p)) \overset{\overset{F(p_1)}{\longrightarrow}}{\underset{F(p_2)}{\underset{\longrightarrow}{\overset{\longleftarrow}{F(\Delta)}}}} F(E).$$

with $F(\sigma)$ a loop on $F(\mathsf{Eq}(p))$.

- The *Galois group* $\mathsf{Gal}(E, p, 0)$ of p is the kernel of the induced morphism $\langle F(p_1), F(p_2) \rangle$, i.e. is as in the following pullback:

$$
\begin{array}{ccc}
\mathsf{Gal}(E, p, 0) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
F(\mathsf{Eq}(p)) & \xrightarrow[\langle F(p_1), F(p_2) \rangle]{} & F(E) \times F(E).
\end{array}
$$

When the normal extension p is *weak universal*, it turns out that the Galois group of p is an invariant of B. In this case, the Galois group of p is called the *fundamental group* of B and is denoted by $\pi_1(B)$. Let us recall that a normal extension $p \colon E \to B$ is said to be *weak universal* when, for any other normal extension $p' \colon E' \to B$, there exists an arrow $v \colon E \to E'$ making the diagram below commutative:

$$
\begin{array}{ccc}
E & \xrightarrow{\ \ p\ \ } & B \\
& {}_{v} \searrow & \nearrow {}_{p'} \\
& E'. &
\end{array}
$$

3. Central extensions of preordered groups

Consider the category whose

- objects are pairs (G, P_G), also represented by the inclusion $P_G \rightarrowtail G$, where G is a group and P_G a submonoid of G closed under conjugation in G;

- arrows $(G, P_G) \to (H, P_H)$ are given by group morphisms $f \colon G \to H$ such that $f(P_G) \subseteq P_H$. Alternatively, a morphism from (G, P_G) to (H, P_H) can be seen as a pair $(f, \bar{f})$, with $f \colon G \to H$ a group morphism and $\bar{f} \colon P_G \to P_H$ a monoid morphism making the following

square commute:

$$P_G \xrightarrow{\ \bar{f}\ } P_H$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad (3.1)$$

$$G \xrightarrow[\ f\ ]{} H.$$

It is well known, and not difficult to prove, that this category is actually iso-morphic to the category PreOrdGrp of preordered groups. To any preordered group $(X, \leq)$, we can associate a pair (G, P_G) as above by taking $G = X$ and $P_G = \{x \in X \mid 0 \leq x\}$. Incidentally, the definition of P_G is the reason why we usually call the submonoid P_G the *positive cone* of G. Conversely, any such pair (G, P_G) corresponds, via the isomorphism, to the preordered group $(G, \leq)$ where the preorder relation $\leq$ is defined as follows: $a \leq b$ in G if and only if $b - a \in P_G$. As a result, since the above category is isomorphic to PreOrdGrp, we can use the above description as an alternative definition of the category PreOrdGrp of preordered groups.

The categorical behaviour of preordered groups was studied in [3] by Clementino, Martins-Ferreira and Montoli. They proved, among other things, that PreOrdGrp is both complete and cocomplete. It is also a *normal category* in the sense of [13], i.e.

- it is pointed, with zero object $(0,0)$;

- it is regular;

- any regular epimorphism in it is the cokernel of its kernel.

Note that a morphism $(f, \bar{f})$ as in (3.1) is a monomorphism whenever f (and then $\bar{f}$) is injective, while it is an epimorphism whenever f is an epimor-phism. The regular epimorphisms in PreOrdGrp are given by the arrows $(f, \bar{f})$ as in (3.1) such that both f and $\bar{f}$ are surjective, and coincide (as al-ready mentioned above) with the normal epimorphisms. In particular, the cokernel of $(f, \bar{f})$ is given by

$$P_H \xrightarrow{\ \bar{q}\ } P_Q$$

$$\downarrow \qquad\qquad \downarrow$$

$$H \xrightarrow[\ q\ ]{} Q$$

where $q\colon H \twoheadrightarrow Q$ is the cokernel of f in the category Grp of groups, and P_Q the direct image $q(P_H)$ of P_H along q. The kernels in PreOrdGrp are, on the other hand, even easier to compute since they are taken "component-wise": the kernel of $(f, \bar{f})$ is given by $((K, P_K), (k, \bar{k}))$, with (K, k) the kernel of f in Grp and $(P_K, \bar{k})$ the kernel of $\bar{f}$ in the category Mon of monoids (or, alternatively, $P_K = K \cap P_G$).

Note that, since the category PreOrdGrp of preordered groups is normal, the following lemma is then satisfied in our context:

Lemma 3.1. Let $\mathscr{C}$ be a normal category. Consider a commutative diagram of short exact sequences in $\mathscr{C}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{k}{\rightarrowtail} & B & \overset{f}{\twoheadrightarrow} & C & \longrightarrow & 0 \\
& & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\
0 & \longrightarrow & A' & \underset{k'}{\rightarrowtail} & B' & \underset{f'}{\twoheadrightarrow} & C' & \longrightarrow & 0.
\end{array}
$$

Then the left-hand square is a pullback if and only if the arrow c is a monomorphism.

As proved in [3], the category PreOrdGrp of preordered groups is neither protomodular nor Barr-exact. However, the class of *effective descent morphisms* in it coincides with the class of regular epimorphisms, which has a really simple description (as explained above).

We now recall the content of the article [5] on which the present paper is partly based. We first mention the fact that the category PreOrdGrp of preordered groups admits a full subcategory, whose objects are the preordered groups $P_G \rightarrowtail G$ such that both G and P_G are abelian groups. This is the category Mono(Ab) of monomorphisms in the category of abelian groups, which turns out to be the full subcategory of *abelian objects* [1] in PreOrdGrp. Moreover, we can define a functor $F\colon$ PreOrdGrp $\to$ Mono(Ab) from PreOrdGrp to Mono(Ab): given any preordered group (G, P_G),

$$F(G, P_G) = (ab(G), grp(\eta_G(P_G))),$$

where $ab(G) = G/[G, G]$ is the "abelianization" of the group G, $\eta_G\colon G \twoheadrightarrow ab(G)$ the canonical quotient, and $grp\colon$ CMon $\to$ Ab the "group completion functor" associating with any commutative monoid X its group completion

$grp(X)$ (also called "Grothendieck group"). This functor $F\colon$ PreOrdGrp $\rightarrow$ Mono(Ab) turns out to be a reflector:

$$\text{PreOrdGrp} \underset{U}{\overset{F}{\rightleftarrows}} \text{Mono(Ab)}.$$

Taking $\mathcal{E}$ and $\mathcal{Z}$ to be the classes of regular epimorphisms in PreOrdGrp and Mono(Ab) respectively then gives us a Galois structure

$$\Gamma = (\text{PreOrdGrp}, \text{Mono(Ab)}, F, U, \mathcal{E}, \mathcal{Z})$$

which is admissible (see Corollary 5.6 in [5]). As is the case in the setting of groups, normal and central extensions coincide in this context:

Theorem 3.2. [5] Let $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ be a regular epimorphism in PreOrdGrp. Then, the following conditions are equivalent:

1. (a) $\mathsf{Ker}(f) \subseteq Z(G)$;
   (b) for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$.

2. $(f, \bar{f})$ is a (Γ-)normal extension.

3. $(f, \bar{f})$ is a (Γ-)central extension.

4. Construction of a weak universal central extension for any given preordered group

First recall that a *regular projective object* in an arbitrary category $\mathscr{C}$ is an object P such that, for any arrow $\phi\colon P \rightarrow Y$ and any regular epimorphism $\psi\colon X \twoheadrightarrow Y$, there exists a (not necessarily unique) morphism $\alpha\colon P \rightarrow X$ such that $\psi \cdot \alpha = \phi$:

$$\begin{array}{ccc} & & X \\ & \overset{\alpha}{\nearrow} & \downarrow{\psi} \\ P & \underset{\phi}{\longrightarrow} & Y. \end{array}$$

A category $\mathscr{C}$ is said to *have enough regular projectives* when, for any object $C \in \mathscr{C}$, there exists a regular projective object $P \in \mathscr{C}$ as well as a regular epimorphism $P \twoheadrightarrow C$ from P to C.

Proposition 4.1. PreOrdGrp has enough regular projectives.

Proof. Let (G, P_G) be any preordered group. Consider the free group $(P, *)$ on the underlying set $|G|$ of G. Its elements are the "reduced words" in the "alphabet" $G \sqcup G^{-1}$, where G^{-1} is the set of "formal inverses" of the elements in G.

In the following we'll identify each element g_i of G, and in particular of its submonoid P_G, with the corresponding one-letter (reduced) word $g_i \in P$. Let us then define

$$P_P = \{y_1 * y_2 * \cdots * y_m \mid y_i = x_i * g_i * (-x_i),$$
$$x_i \in P, \; g_i \in P_G \; \forall i \in \{1, \cdots, m\}\}.$$

Then, P_P is clearly a submonoid of P, and is closed under conjugation in P. Indeed, for any $x \in P$ and any $y = y_1 * y_2 * \cdots * y_m \in P_P$,

$$x * y * (-x) = x * y_1 * y_2 * \cdots * y_m * (-x)$$
$$= (x * y_1 * (-x)) * (x * y_2 * (-x)) * \cdots * (x * y_m * (-x))$$

where, for any $i \in \{1, \cdots, m\}$,

$$x * y_i * (-x) = x * x_i * g_i * (-x_i) * (-x) = (x * x_i) * g_i * (-(x * x_i))$$

with $x * x_i \in P$ and $g_i \in P_G$, so that $x * y * (-x) \in P_P$. As a consequence, $(P, P_P) \in$ PreOrdGrp.

Define now $p \colon P \to G$, for any $x_1 * \cdots * x_n \in P$, by

$$p(x_1 * \cdots * x_n) = x_1 + \cdots + x_n.$$

It is a group morphism which is surjective. Let us check that the restriction $\bar{p}$ of p to P_P takes its values in P_G. For any $y_1 * \cdots * y_m \in P_P$,

$$p(y_1 * \cdots * y_m) = p(x_1 * g_1 * (-x_1) * \cdots * x_m * g_m * (-x_m))$$
$$= p(x_1) + p(g_1) - p(x_1) + \cdots + p(x_m) + p(g_m) - p(x_m)$$
$$= p(x_1) + g_1 - p(x_1) + \cdots + p(x_m) + g_m - p(x_m)$$

since p is a group morphism. We observe that $g_i \in P_G$ and that $p(x_i) \in G$ for any $i \in \{1, \cdots, m\}$, which implies that $p(x_i) + g_i - p(x_i) \in P_G$ because P_G is closed under conjugation in G. By the closure of P_G under $+$, we conclude that $p(y_1 * \cdots * y_m) \in P_G$. This means that $(p, \bar{p}) \colon (P, P_P) \to (G, P_G)$ is a morphism in PreOrdGrp, which is a regular epimorphism since both p and $\bar{p}$ are surjective.

It remains to show that (P, P_P) is a regular projective object. Consider any regular epimorphism $(\psi, \bar{\psi}) \colon (A, P_A) \twoheadrightarrow (H, P_H)$ and any morphism $(\phi, \bar{\phi}) \colon (P, P_P) \to (H, P_H)$ in PreOrdGrp.
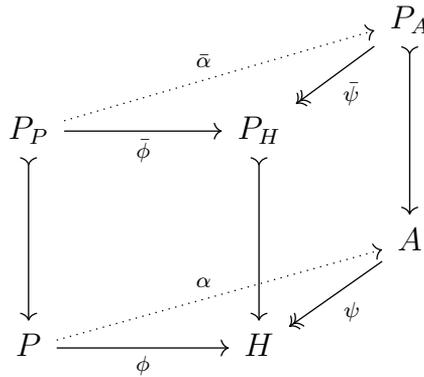
Since $P = F(G)$ is a regular projective object in the category Grp of groups, there exists a group morphism $\alpha \colon P \to A$ such that $\psi \cdot \alpha = \phi$. Indeed, since ψ is surjective, for any $h \in H$, there exists a (not necessarily unique) element a_h in A such that $\psi(a_h) = h$. We then define α, for any $x_1 * \cdots * x_n \in P$, by

$$\alpha(x_1 * \cdots * x_n) = a_{\phi(x_1)} + \cdots + a_{\phi(x_n)},$$

where $a_{\phi(x_i)} \in A$ is such that $\psi(a_{\phi(x_i)}) = \phi(x_i)$ for any $i \in \{1, \cdots, n\}$. Note that this group morphism α is not necessarily unique and that one can choose, in the above construction, $a_h \in P_A$ whenever $h \in P_H$ because $\bar{\psi}$ is surjective. We now prove that the restriction $\bar{\alpha}$ of α to P_P takes its values in P_A. Let $y_1 * \cdots * y_m \in P_P$. Then,

$$\begin{aligned}
\alpha(y_1 * \cdots * y_m) &= \alpha(x_1 * g_1 * (-x_1) * \cdots * x_m * g_m * (-x_m)) \\
&= \alpha(x_1) + \alpha(g_1) - \alpha(x_1) + \cdots + \alpha(x_m) + \alpha(g_m) - \alpha(x_m),
\end{aligned}$$

since α is a group morphism, with $\alpha(x_i) + \alpha(g_i) - \alpha(x_i) \in P_A$ (because $\alpha(g_i) \in P_A$ and P_A is closed under conjugation in A) for any $i \in \{1, \cdots, m\}$,

so that $\alpha(y_1 * \cdots * y_m) \in P_A$. Consequently, there exists a morphism $(\alpha, \bar{\alpha}) \colon (P, P_P) \to (A, P_A)$ in PreOrdGrp such that $(\psi, \bar{\psi}) \cdot (\alpha, \bar{\alpha}) = (\phi, \bar{\phi})$. $\square$

Remark 4.2. While preparing the final version of the article, the referee informed me that Maria Manuel Clementino and Andrea Montoli proved in [2] that the category PreOrdGrp is a finitary quasivariety, from which it follows that it has enough regular projectives. I decided to keep the proof of Proposition 4.1 since it gives an explicit description of the regular projective presentation of any preordered group that could be useful for computations.

We now prove a lemma which will be useful in the construction of a weak universal central extension.

Lemma 4.3. Let $(k, \bar{k}) \colon (K, P_K) \rightarrowtail (P, P_P)$ be a normal monomorphism in PreOrdGrp, and consider the inclusion $(\epsilon, \bar{\epsilon}) \colon ([K, P], [K, P] \cap P_K) \rightarrowtail (K, P_K)$ (which is actually also a normal monomorphism) in PreOrdGrp. Then the composite $(k, \bar{k}) \cdot (\epsilon, \bar{\epsilon}) \colon ([K, P], [K, P] \cap P_K) \to (P, P_P)$ is a normal monomorphism in PreOrdGrp.

Proof. Clearly, $[K, P]$ is a normal subgroup of P. It remains to prove that the external rectangle in the diagram

$$
\begin{array}{ccccc}
[K, P] \cap P_K & \overset{\bar{\epsilon}}{\rightarrowtail} & P_K & \overset{\bar{k}}{\rightarrowtail} & P_P \\
\downarrow & & \downarrow & & \downarrow \\
[K, P] & \overset{\epsilon}{\rightarrowtail} & K & \overset{k}{\rightarrowtail} & P
\end{array}
$$

is a pullback. This is the case since it is made of two squares which are themselves pullbacks. $\square$

Proposition 4.4. One can construct, for any preordered group (G, P_G), a weak universal central extension of (G, P_G).

Proof. Let $(G, P_G) \in$ PreOrdGrp.
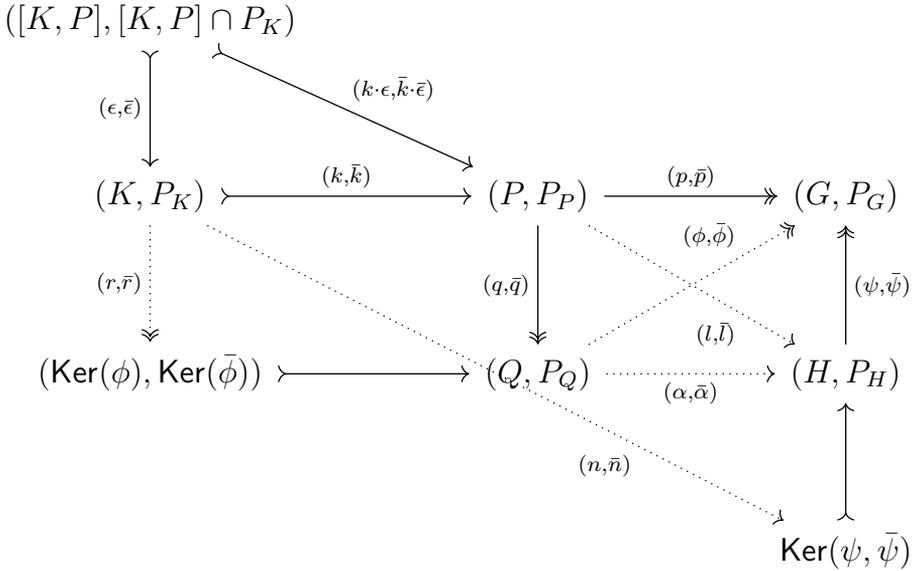
Thanks to Proposition 4.1, we know that we can build a regular epimorphism $(p, \bar{p}) \colon (P, P_P) \twoheadrightarrow (G, P_G)$ with (P, P_P) a regular projective object. Consider the kernel $(k, \bar{k}) \colon (K, P_K) \rightarrowtail (P, P_P)$ of $(p, \bar{p})$. Since $(p, \bar{p})$ is a regular epimorphism in a normal category, it is then the cokernel of its kernel $(k, \bar{k})$. Consider also the morphism $(\epsilon, \bar{\epsilon}) \colon ([K, P], [K, P] \cap P_K) \to (K, P_K)$, where ϵ is the inclusion of $[K, P]$ in K. By Lemma 4.3, the composite $(k \cdot \epsilon, \bar{k} \cdot \bar{\epsilon})$ is a normal monomorphism. By taking its cokernel $(q, \bar{q}) \colon (P, P_P) \twoheadrightarrow (Q, P_Q)$, we thus obtain a short exact sequence. Now, we compute that $(p, \bar{p}) \cdot (k \cdot \epsilon, \bar{k} \cdot \bar{\epsilon}) = (0, 0)$. By the universal property of cokernels, there exists a unique morphism $(\phi, \bar{\phi}) \colon (Q, P_Q) \to (G, P_G)$ such that $(\phi, \bar{\phi}) \cdot (q, \bar{q}) = (p, \bar{p})$. This induced arrow turns out to be a regular epimorphism since so is $(p, \bar{p})$.

Take now the kernel of $(\phi, \bar{\phi})$. Then $(\phi, \bar{\phi})$ is its cokernel. Since

$$(\phi, \bar{\phi}) \cdot (q, \bar{q}) \cdot (k, \bar{k}) = (p, \bar{p}) \cdot (k, \bar{k}) = (0, 0),$$

by the universal property of kernels, there is then a unique morphism $(r, \bar{r}) \colon (K, P_K) \to \mathsf{Ker}(\phi, \bar{\phi}) = (\mathsf{Ker}(\phi), \mathsf{Ker}(\bar{\phi}))$ such that $\mathsf{ker}(\phi, \bar{\phi}) \cdot (r, \bar{r}) = (q, \bar{q}) \cdot (k, \bar{k})$. We observe that the square

$$(K, P_K) \xrightarrow{\;(k,\bar{k})\;} (P, P_P)$$

with vertical maps $(r, \bar{r})$ on the left and $(q, \bar{q})$ on the right to

$$(\mathsf{Ker}(\phi), \mathsf{Ker}(\bar{\phi})) \longmapsto (Q, P_Q)$$

is a pullback in PreOrdGrp since $((k, \bar{k}), (p, \bar{p}))$ and $(\mathsf{ker}(\phi, \bar{\phi}), (\phi, \bar{\phi}))$ are two short exact sequences, $1_{(G, P_G)}$ is a monomorphism, and PreOrdGrp is a normal category (see Lemma 3.1). By pullback-stability of regular epimorphisms in PreOrdGrp, it follows that $(r, \bar{r})$ is a regular epimorphism; it is the restriction of $(q, \bar{q})$ to the kernel (K, P_K) of $(p, \bar{p})$. Another consequence of the above square being a pullback is that the pair $((\epsilon, \bar{\epsilon}), (r, \bar{r}))$ forms a short exact sequence in PreOrdGrp.

Let us show that $\mathsf{Ker}(\phi) \subseteq Z(Q)$. Let $y_1 \in \mathsf{Ker}(\phi)$ and $y_2 \in Q$. By surjectivity of r and q, there exist $x_1 \in K$ and $x_2 \in P$ such that $r(x_1) = y_1$ (in particular, $q(x_1) = y_1$) and $q(x_2) = y_2$. Since $x_1 + x_2 - x_1 - x_2 \in [K, P] = \mathsf{Ker}(q)$, we then have that $q(x_1 + x_2 - x_1 - x_2) = 0$, that is, $y_1 + y_2 - y_1 - y_2 = 0$. In other words, $y_1 + y_2 = y_2 + y_1$, which means that $\mathsf{Ker}(\phi) \subseteq Z(Q)$.

Consider now

$$\tilde{P}_Q = \{x - y + z \mid (x, y) \in \mathsf{Eq}(\bar{\phi}) \text{ and } z \in P_Q\}.$$

It is a submonoid of Q. Indeed, for $x - y + z$ and $x' - y' + z'$ in $\tilde{P}_Q$,

$$(x - y + z) + (x' - y' + z') = x + (x' - y') + (-y + z) + z'$$
$$= (x + x') - (y + y') + (z + z')$$

since $x' - y' \in \mathsf{Ker}(\phi) \subseteq Z(Q)$. By assumption, $x + x', y + y', z + z' \in P_Q$ and $\phi(x + x') = \phi(x) + \phi(x') = \phi(y) + \phi(y') = \phi(y + y')$, and this proves that $(x - y + z) + (x' - y' + z') \in \tilde{P}_Q$, as desired. The submonoid $\tilde{P}_Q$ is also closed under conjugation in Q: for $x - y + z \in \tilde{P}_Q$ and $w \in Q$,

$$w + (x - y + z) - w = w + x + (-w + w) - y + (-w + w) + z - w$$
$$= (w + x - w) - (w + y - w) + (w + z - w)$$

with $w + x - w$, $w + y - w$ and $w + z - w$ in P_Q because P_Q is closed under conjugation in Q, and $\phi(w + x - w) = \phi(w) + \phi(x) - \phi(w) = $

$\phi(w) + \phi(y) - \phi(w) = \phi(w + y - w)$, so that $w + (x - y + z) - w \in \tilde{P}_Q$. From this, we deduce that $(Q, \tilde{P}_Q)$ is a preordered group. Moreover, since $P_Q \subset \tilde{P}_Q$, the arrow

$$(Q, P_Q) \xrightarrow{(1_Q, j)} (Q, \tilde{P}_Q),$$

where j is the inclusion arrow, is a morphism of preordered groups. We now observe that, for any $x - y + z \in \tilde{P}_Q$,

$$\phi(x - y + z) = \phi(x - y) + \phi(z) = \phi(z) \in P_G$$

since $z \in P_Q$ and $(\phi, \bar{\phi})$ is a morphism in PreOrdGrp. This implies that the restriction $\tilde{\phi}$ of ϕ to $\tilde{P}_Q$ takes its values in P_G, which means that

$$(Q, \tilde{P}_Q) \xrightarrow{(\phi, \tilde{\phi})} (G, P_G)$$

is a morphism of preordered groups. It is a regular epimorphism because $(\phi, \tilde{\phi}) \cdot (1_Q, j) = (\phi, \bar{\phi})$ with $(\phi, \bar{\phi})$ a regular epimorphism.

Let us now prove that, for any $(a, b) \in \mathsf{Eq}(\tilde{\phi})$, $b - a \in \tilde{P}_Q$. By assumption, $a = x - y + z$ and $b = x' - y' + z'$, with $x, y, z, x', y', z' \in P_Q$, $\phi(x) = \phi(y)$, $\phi(x') = \phi(y')$ and $\phi(z) = \phi(a) = \phi(b) = \phi(z')$. Then,

$$\begin{aligned}
b - a &= (x' - y' + z') - (x - y + z) \\
&= x' - y' + z' - z + y - x \\
&= z' + x' - y' - z + y - x \\
&= z' + x' + y - x - y' - z \\
&= (z' + x' + y) - (z + y' + x) + 0,
\end{aligned}$$

since $x' - y'$ and $y - x$ belong to $\mathsf{Ker}(\phi) \subseteq Z(Q)$, with $z' + x' + y$, $z + y' + x$ and 0 in P_Q, and

$$\phi(z' + x' + y) = \phi(z') + \phi(x') + \phi(y) = \phi(z) + \phi(y') + \phi(x) = \phi(z + y' + x).$$

This means that $b - a \in \tilde{P}_Q$. According to Theorem 3.2, $(\phi, \tilde{\phi})$ is then a Γ-central extension.

It remains to show that it is indeed weak universal. For this, consider any other central extension $(\psi, \bar{\psi}) \colon (H, P_H) \twoheadrightarrow (G, P_G)$ of (G, P_G), that is, $(\psi, \bar{\psi})$ is a regular epimorphism such that

- $\mathsf{Ker}(\psi) \subseteq Z(H)$;

- for any $(a, b) \in \mathsf{Eq}(\bar{\psi})$, $b - a \in P_H$.

Since (P, P_P) is a regular projective object, there exists a morphism $(l, \bar{l})\colon (P, P_P) \to (H, P_H)$ such that $(\psi, \bar{\psi}) \cdot (l, \bar{l}) = (p, \bar{p})$. We then compute that

$$(\psi, \bar{\psi}) \cdot (l, \bar{l}) \cdot (k, \bar{k}) = (p, \bar{p}) \cdot (k, \bar{k}) = (0, 0).$$

The universal property of kernels then induces a unique arrow $(n, \bar{n})\colon (K, P_K) \to \mathsf{Ker}(\psi, \bar{\psi})$ such that $\mathsf{ker}(\psi, \bar{\psi}) \cdot (n, \bar{n}) = (l, \bar{l}) \cdot (k, \bar{k})$. Now, the composite $(n, \bar{n}) \cdot (\epsilon, \bar{\epsilon})$ is trivial. Indeed, for any $x \in [K, P]$ (that is, $x = a + b - a - b$ with $a \in K$ and $b \in P$),

$$\begin{aligned} n(x) &= n(a + b - a - b) \\ &= n(a) + n(b) - n(a) - n(b) \\ &= n(a) - n(a) + n(b) - n(b) \\ &= 0 \end{aligned}$$

since $n(a)$ and $n(b)$ are in $\mathsf{Ker}(\psi)$, which is an abelian group. This allows us to compute that

$$\begin{aligned} (l, \bar{l}) \cdot (k \cdot \epsilon, \bar{k} \cdot \bar{\epsilon}) &= (l, \bar{l}) \cdot (k, \bar{k}) \cdot (\epsilon, \bar{\epsilon}) \\ &= \mathsf{ker}(\psi, \bar{\psi}) \cdot (n, \bar{n}) \cdot (\epsilon, \bar{\epsilon}) \\ &= (0, 0). \end{aligned}$$

By the universal property of cokernels, there is a unique arrow $(\alpha, \bar{\alpha})\colon (Q, P_Q) \to (H, P_H)$ such that $(\alpha, \bar{\alpha}) \cdot (q, \bar{q}) = (l, \bar{l})$, and so

$$(\psi, \bar{\psi}) \cdot (\alpha, \bar{\alpha}) \cdot (q, \bar{q}) = (\psi, \bar{\psi}) \cdot (l, \bar{l}) = (p, \bar{p}) = (\phi, \bar{\phi}) \cdot (q, \bar{q}).$$

Since $(q, \bar{q})$ is a regular epimorphism, we deduce that $(\psi, \bar{\psi}) \cdot (\alpha, \bar{\alpha}) = (\phi, \bar{\phi})$. Let us now check that the restriction $\tilde{\alpha}$ of α to $\tilde{P}_Q$ takes its values in P_H. Let $x - y + z \in \tilde{P}_Q$. In particular, $x - y \in \mathsf{Ker}(\phi)$, so that $\alpha(x) - \alpha(y) \in \mathsf{Ker}(\psi)$. Since in addition $\alpha(x)$ and $\alpha(y)$ belong to P_H, we then have that $(\alpha(y), \alpha(x)) \in \mathsf{Eq}(\bar{\psi})$. By assumption, it follows that $\alpha(x) - \alpha(y) \in P_H$. We also know that $\alpha(z) \in P_H$, hence

$$\alpha(x - y + z) = (\alpha(x) - \alpha(y)) + \alpha(z) \in P_H$$

as desired.

As a conclusion, $(\phi, \bar{\phi})$ is a weak universal central extension of (G, P_G). $\quad\square$

5. A Hopf formula for the fundamental group in PreOrdGrp

Let us start this section by recalling two results that will be of interest in order to obtain a formula for the fundamental group in PreOrdGrp. For completeness, we remind their proofs here below.

Proposition 5.1. [9] Let $\mathscr{C}$ be a pointed category, and consider $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ an admissible Galois structure where $\mathscr{F}$ is a subcategory of $\mathscr{C}$ and $U \colon \mathscr{F} \to \mathscr{C}$ the inclusion functor. If $p \colon E \to B$ is a normal extension, then
$$\mathsf{Gal}(E, p, 0) \cong \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E)$$
where $\eta_E \colon E \to F(E)$ is the E-component of the unit of the adjunction $F \dashv U$.

Proof. Let us denote by (K, k) the kernel of p, and consider the following commutative diagram

where p_1 (respectively p_2) is the first (respectively the second) projection of the kernel pair of p. The left-hand square is a pullback by definition of the kernel of p_1. The right-hand square is also a pullback since p is a normal

extension. The external rectangle is therefore a pullback, which means that $K = \mathsf{Ker}(F(p_1))$ and $\eta_{\mathsf{Eq}(p)} \cdot \langle 0, k \rangle = \mathsf{ker}(F(p_1))$. Let us now consider the diagram

$$
\begin{array}{ccc}
\mathsf{Ker}(\eta_E \cdot k) & \xdashrightarrow{\ \phi\ } \mathsf{Ker}(\eta_E) \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow \\
K \xrightarrow{\ k\ } & E \xrightarrow{\ \eta_E\ } & F(E)
\end{array}
$$

where the dotted arrow $\phi \colon \mathsf{Ker}(\eta_E \cdot k) \to \mathsf{Ker}(\eta_E)$ is induced by the universal property of kernels, and makes the above diagram commute. Since the right-hand square and the external rectangle are pullbacks, it follows that the left-hand square is a pullback as well, which implies that

$$
\mathsf{Ker}(\eta_E \cdot k) \cong K \cap \mathsf{Ker}(\eta_E) = \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E).
$$

It remains to show that $\mathsf{Ker}(\eta_E \cdot k) \cong \mathsf{Gal}(E, p, 0)$. Consider first the following commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ \eta_E \cdot k\ } F(E) \longrightarrow & 0 \\
{\scriptstyle \eta_{\mathsf{Eq}(p)} \cdot \langle 0,k \rangle} \downarrow \quad (1) & \quad \downarrow {\scriptstyle \langle 0,1 \rangle} \quad (2) & \downarrow \\
F(\mathsf{Eq}(p)) \xrightarrow{\langle F(p_1), F(p_2) \rangle} & F(E) \times F(E) \xrightarrow{\ q_1\ } & F(E)
\end{array}
$$

where q_1 (respectively q_2) is the first (respectively the second) projection of the product of $F(E)$ with itself. Note that this diagram is commutative since

$$
F(p_1) \cdot \eta_{\mathsf{Eq}(p)} \cdot \langle 0, k \rangle = 0 = 0 \cdot \eta_E \cdot k,
$$

and

$$
F(p_2) \cdot \eta_{\mathsf{Eq}(p)} \cdot \langle 0, k \rangle = \eta_E \cdot p_2 \cdot \langle 0, k \rangle = \eta_E \cdot k.
$$

We observe that the square (2) and the external rectangle (1) + (2) are pullbacks, so that the square (1) is also a pullback. Let us now build the following diagram:

The lower square is a pullback by definition of $\mathsf{Gal}(E, p, 0)$. By the universal property of pullbacks, there is then a unique arrow $\psi \colon \mathsf{Ker}(\eta_E \cdot k) \to \mathsf{Gal}(E, p, 0)$ making the above cube commute. The upper square is obviously a pullback, and the right-hand square corresponds to the square (1) in the previous diagram, which has been proven to be a pullback. We deduce that the left-hand square is a pullback. By pullback-stability of isomorphisms, it follows that the induced morphism ψ is an isomorphism, that is,

$$\mathsf{Gal}(E, p, 0) \cong \mathsf{Ker}(\eta_E \cdot k) \cong \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E). \qquad \square$$

Lemma 5.2. [4] Let $\mathscr{C}$ be a normal category and let $\mathscr{F}$ be a (normal epi)-reflective subcategory of $\mathscr{C}$. If $f \colon A \twoheadrightarrow B$ is a normal epimorphism such that $\mathsf{Ker}(f) \leq \mathsf{Ker}(\eta_A)$ where $\eta_A \colon A \twoheadrightarrow F(A)$ is the A-component of the unit of the reflection, then the induced commutative square

$$
\begin{array}{ccc}
\mathsf{Ker}(\eta_A) & \rightarrowtail & A \\
\hat{f} \downarrow & & \downarrow f \\
\mathsf{Ker}(\eta_B) & \rightarrowtail & B
\end{array}
$$

is a pullback.

Proof. First note that, thanks to the universal property of η_A (as the A-component of the unit of the reflection), there exists a unique arrow $g \colon F(A) \to F(B)$ making the diagram below commutative:

$$\mathsf{Ker}(f)$$

This induced arrow g is an epimorphism since so is the composite $\eta_B \cdot f$. Since $\mathsf{Ker}(f) \leq \mathsf{Ker}(\eta_A)$, we have that $\eta_A \cdot k = 0$ where k denotes the kernel of f. By the universal property of cokernels, there is then a unique morphism $\gamma \colon B \to F(A)$ such that $\gamma \cdot f = \eta_A$. Using now the universal property of η_B (as the B-component of the unit of the reflection), we obtain the existence of a unique arrow $\delta \colon F(B) \to F(A)$ satisfying $\delta \cdot \eta_B = \gamma$. We then compute that

$$\delta \cdot g \cdot \eta_A = \delta \cdot \eta_B \cdot f = \gamma \cdot f = \eta_A,$$

which entails that $\delta \cdot g = 1_{F(A)}$ since η_A is an epimorphism. This means that g is a split monomorphism. Being also an epimorphism, it is then an isomorphism. Accordingly, the left-hand square of the diagram above is a pullback. $\qquad\square$

Theorem 5.3. Let (G, P_G) be any preordered group and let $(p, \bar{p}) \colon (P, P_P) \twoheadrightarrow (G, P_G)$ be a regular projective presentation of (G, P_G) with kernel (K, P_K) (as in the proof of Proposition 4.1). Then,

$$\pi_1(G, P_G) \cong \left(\frac{K \cap [P, P]}{[K, P]}, \frac{K \cap [P, P]}{[K, P]} \cap \tilde{P}_Q \right)$$

where $\tilde{P}_Q = \{x - y + z \mid x, y, z \in q(P_P) \text{ and } \phi(x) = \phi(y)\}$ with $q \colon P \twoheadrightarrow P/[K, P]$ the canonical quotient and $\phi \colon P/[K, P] \twoheadrightarrow G$ the induced arrow such that $\phi \cdot q = p$.

Proof. Consider the central extension $(\phi, \tilde{\phi}) \colon (Q, \tilde{P}_Q) \twoheadrightarrow (G, P_G)$ constructed in the proof of Proposition 4.4. Since $(\phi, \tilde{\phi})$ is weak universal, we have that

$$\pi_1(G, P_G) = \mathsf{Gal}((Q, \tilde{P}_Q), (\phi, \tilde{\phi}), 0).$$

Now, knowing that the central and normal extensions coincide in our situation (see Theorem 3.2), we obtain, thanks to Proposition 5.1, that

$$\mathsf{Gal}((Q, \tilde{P}_Q), (\phi, \tilde{\phi}), 0) \cong \mathsf{Ker}(\phi, \tilde{\phi}) \cap \mathsf{Ker}(\eta_Q, \tilde{\eta}_Q),$$

so that

$$
\begin{aligned}
\pi_1(G, P_G) &\cong (\mathsf{Ker}(\phi), \mathsf{Ker}(\tilde{\phi})) \cap (\mathsf{Ker}(\eta_Q), \mathsf{Ker}(\tilde{\eta}_Q)) \\
&\cong (\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q), (\mathsf{Ker}(\phi) \cap \tilde{P}_Q) \cap (\mathsf{Ker}(\eta_Q) \cap \tilde{P}_Q)) \\
&\cong (\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q), \mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cap \tilde{P}_Q) \\
&\cong \left(\frac{K \cap [P, P]}{[K, P]}, \frac{K \cap [P, P]}{[K, P]} \cap \tilde{P}_Q \right).
\end{aligned}
$$

Let us recall the proof of the isomorphism

$$\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cong \frac{K \cap [P, P]}{[K, P]}$$

in the category Grp of groups. Consider for this the following commutative cube (with the same notations as before):
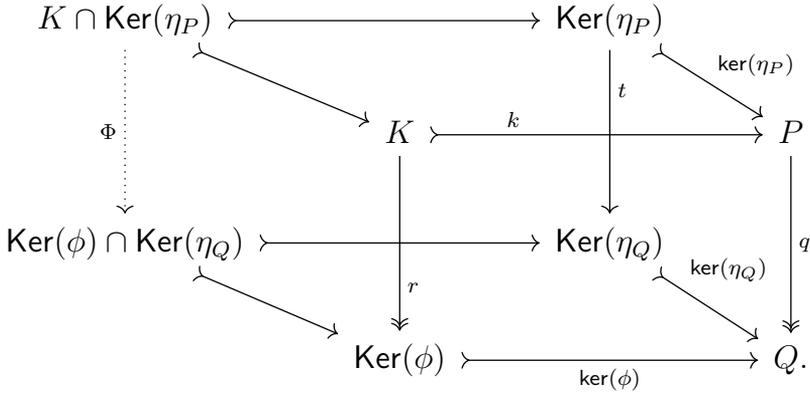
The lower square is clearly a pullback. By the universal property of pullbacks, there exists then a unique morphism $\Phi \colon K \cap \mathsf{Ker}(\eta_P) \to \mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q)$ making the above cube commute. The front square is a pullback as already observed in the proof of Proposition 4.4. The upper square is also obviously a pullback. It follows that the back square is a pullback. According to Lemma 5.2, the right-hand square is also a pullback since

$\mathsf{Ker}(q) = [K, P] \leq [P, P] = \mathsf{Ker}(\eta_P)$. As a conclusion, all the squares involved in the above cube are pullbacks in Grp. By pullback-stability of regular epimorphisms in Grp, it then follows that the induced arrow Φ is a regular epimorphism since so is q. The morphism Φ is then the cokernel of its kernel, so that

$$\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cong \frac{K \cap \mathsf{Ker}(\eta_P)}{\mathsf{Ker}(\Phi)}.$$

But,

$$\mathsf{Ker}(\Phi) \cong \mathsf{Ker}(t) \cong \mathsf{Ker}(q) = [K, P].$$

Knowing that $\mathsf{Ker}(\eta_P) = [P, P]$, we conclude that

$$\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cong \frac{K \cap [P, P]}{[K, P]}. \qquad \square$$

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UNE GÉNÉRALISATION DU LEMME DE YONEDA

Jacques PENON

Résumé. Dans le cadre des catégories mutantes sur une catégorie monoïdale symétrique $\mathbb{V}$, voir [4], on généralise le lemme de Yoneda classique. On montre aussi que le lemme de Yoneda enrichi peut être déduit du cas mutant. On construit encore un plongement de Yoneda qui est un foncteur mutant pleinement fidèle (justifiant ainsi le terme de plongement) vers une categorie mutante de foncteurs mutants. Toutes les constructions necessaires à ces résultats se font sans condition particulière pour la catégorie monoïdale $\mathbb{V}$ contrairement au cas enrichi.

Abstract. In the context of mutant categories over a symetric monoidal category $\mathbb{V}$, see [4], we generalize the classical Yoneda lemma. We also show that the enriched Yoneda lemma can be deduced from the mutant case. We also construct a Yoneda embedding which is a full and faithful mutant functor (thus justifying the term embedding) towards a mutant category of mutant functors. All the constructions necessary for these results are done without any special condition for the monoidal category $\mathbb{V}$ unlike the enriched case.

Keywords. Monoidal category. Enriched category. Bicategory.

Mathematics Subject Classification (2020). 18D20.

Introduction

• Le Lemme de Yoneda , en théorie des catégories, occupe une place centrale. Il est, entre autre, à la base de la théorie des faisceaux pour Grothendieck et son école. Depuis, plusieurs généralisations du Lemme sont venues encore étendre sa portée. Nombre d'entre elles, finalement, se sont avérés des cas particuliers du lemme de Yoneda enrichi. Celle que nous proposons

ici est encore une généralisation du cas enrichi. Outre cette généralisation, elle permet aussi d'éviter les conditions préalables, portant sur la catégorie monoïadale de base (comme d'être fermée et complète) nécessaires à sa formulation.

• Nous nous plaçons ici dans le cadre des catégories mutantes sur une catégorie monoïdale symétrique $\mathbb{V}$ donnée (voir [2] et [4]). Le concept de catégorie mutante est une généralisation de différents concepts d'enrichissement par $\mathbb{V}$ dont : Les catégories enrichies dans $\mathbb{V}$ proprement dites (encore appelées $\mathbb{V}$-catégories), les catégories $\mathbb{V}$-tensorisées (et même les $\mathbb{V}$-prétensorisées) et les catégories $\mathbb{V}$-cotensorisées (ou encore les $\mathbb{V}$-précotensorisées). Toutes ces structures ont étés étudiés dans [3] . On rappelle ici, à la section 1, tout ce qui est à savoir, pour notre généralisation du Lemme, sur les catégories mutantes et ses différentes classes d'exemples.

• Nous aurons encore besoin d'autres outils pour arriver à notre théorème central (la généralisation du Lemme de Y.). Afin de donner un fil conducteur à ces différents outils et leurs propriétés, donnons l'expression du plongement de Yoneda mutant :

$$\mathcal{M} \xrightarrow{yon} \mathbb{V}-\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$$

où ...

- $\mathbb{V}$ est une catégorie monoïdale symétrique fixée,

- $\mathcal{M}$ est une catégorie mutante sur $\mathbb{V}$ quelconque,

- yon est le plongement de Yoneda mutant. C'est un foncteur mutant (voir la définition 1.2) qui est pleinement fidèle (voir la définition 2.2 pour la pleine fidélité). Sa construction est réalisée à la section 6.

- Plus généralement, lorsque $\mathcal{M}$ et $\mathcal{N}$ sont des catégories mutantes sur une catégorie monoïdale symétrique $\mathbb{V}$ on construit la catégorie mutante $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$ dont les objets sont les foncteurs mutants de $\mathcal{M}$ dans $\mathcal{N}$ (voir la section 4). Lorsque $\mathcal{M}$ et $\mathcal{N}$ sont de saveur E (c.a.d. proviennent de catégories enrichies - par exemple $\mathbb{M}$ et $\mathbb{N}$). Mais aussi lorsque $\mathbb{V}$ est fermée, complète et $|\mathbb{M}|$ petit, $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$ est elle-même de saveur E. La catégorie enrichie qu'elle produit n'est autre que $\mathbb{V}[\mathbb{M}, \mathbb{N}]$ c.a.d. celle des $\mathbb{V}$-foncteurs et $\mathbb{V}$-transformations naturelles de $\mathbb{M}$ dans $\mathbb{N}$ (voir [1]).

- $\mathcal{M}^{op}$, la catégorie mutante opposée à $\mathcal{M}$ est définie à la section 3. C'est un cas particulier d'une construction générale appelée "changement de base d'une catégorie mutante".

- $\hat{\mathcal{V}} = \mu c(\hat{\mathbb{V}})$ où $\mu c : \mathbb{V}\text{-}precot \to Cat\mu(\mathbb{V})$ est un 2-foncteur (construit dans [4] et rappelé à la section 1) transformant une catégorie $\mathbb{V}$-précotensorisée en une catégorie mutante (sur $\mathbb{V}$). $\hat{\mathbb{V}}$ est la catégorie des préfaisceaux sur $\mathbb{V}$ munie canoniquement d'une structure de catégorie $\mathbb{V}$-cotensorisée. (voir la section 2 , où $\mathbb{V}$ est remplacée, plus généralement, par une catégorie $\mathbb{V}$-prétensorisée).

- Lorsque $\mathcal{M}$ est de saveur E, on peut construire un foncteur mutant $\mathcal{M} \xrightarrow{yo} \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \mathcal{V})$ où $\mathcal{V}$ est la catégorie mutante canonique associée à $\mathbb{V}$ et on obtient le triangle commutatif à isomorphisme près suivant :

$$
\begin{array}{ccc}
 & \mathcal{M} & \\
{}^{yo}\swarrow & & \searrow^{yon} \\
\mathbb{V}-\mu(\mathcal{M}^{op}, \mathcal{V}) \xrightarrow{\quad can \quad} & & \mathbb{V}-\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})
\end{array}
$$

où la flèche horizontale est un foncteur mutant pleinement fidèle (voir la section 7) construit à partir du foncteur mutant (pleinement fidèle) $\mu t(\mathbb{V}) \xrightarrow{\mu y} \mu c(\hat{\mathbb{V}})$ (voir la section 2). Lorsque $\mathbb{V}$ est fermée et complète et $|\mathcal{M}|$ petit yo peut s'identifier au plongement de Yoneda enrichi habituel (voir la proposition 5.2).

- On arrive enfin au Lemme de Yoneda mutant. En fait celui-ci se rapproche plus du Lemme de Y. classique que de sa version enrichie qui s'exprime de façon interne. Par rapport au cas classique il a surtout un degré de liberté supplémentaire. Précisément, si on note $\hat{\hat{\mathcal{M}}} = \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ et si on considère un foncteur mutant $F : \mathcal{M}^{op} \to \hat{\mathcal{V}}$ quelconque et un objet $X \in |\mathcal{M}|$, il s'écrit : $\hat{\hat{\mathcal{M}}}(yon(X), F) \simeq \mathbb{C}_X$ où $\mathbb{C}_X \xrightarrow{U} \mathbb{V}$ est la fibration discrète associée à $F(X)$ (qui est un préfaisceau). Comme dans le cas classique cela permet de constater que le foncteur mutant $yon : \mathcal{M} \to \hat{\hat{\mathcal{M}}}$ est pleinement fidèle (d'où l'expression "plongement" le désignant).

• Avant de finir cette introduction nous voudrions signaler que lorsque nous nous référons à la théorie, maintenant bien connue, des catégories enrichies, nous avons opté pour nous rapporter de façon précise à un unique ouvrage, à savoir le livre de F. Borceux (voir [1]) et d'utiliser si possible ses notations.

1. Généralités

• On se place dans le contexte des catégories mutantes. Pour cela quelques rappels s'imposent (voir [4]).

Définition 1.1. :(voir aussi [2] sous la dénomination de bicategorie graduée) $\mathbb{V}$ étant une catégorie monoïdale fixée quelconque, une *catégorie mutante sur* $\mathbb{V}$ est la donnée:
- d'une bicatégorie $\mathcal{M}$,
- d'un morphisme strict de bicatégorie $U : \mathcal{M} \to \mathbb{V}^b$ (où $\mathbb{V}^b$ est $\mathbb{V}$ vue comme une bicatégorie à un seul objet, noté $\star$, (alors $Ob(\mathbb{V}^b) = \{\star\}$, $Fl(\mathbb{V}^b) = Ob(\mathbb{V}), 2\text{-}Cell(\mathbb{V}^b) = Fl(\mathbb{V})$).
Ces données satisfont l'unique axiome suivant:
(CM) Pour tout $X, Y \in |\mathcal{M}|$,
$U_{XY} : \mathcal{M}(X, Y) \to \mathbb{V}^b(UX, UY) = \mathbb{V}^b(\star, \star) = \mathbb{V}$ est une fibration discrète.

Dans la suite, une catégorie mutante est désignée par sa bicatégorie sous-jacente. On note $\otimes$ la composition horizontale et par un point la composition verticale (en accord avec les compositions dans $\mathbb{V}^b$).

Définition 1.2. : Un *foncteur mutant* $\Phi : \mathcal{M} \to \mathcal{N}$ est la donnée d'un morphisme strict de bicatégorie faisant commuter le triangle suivant:
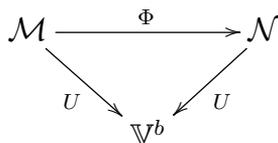
Définition 1.3. : Entre deux foncteurs mutants $\Phi, \Psi : \mathcal{M} \to \mathcal{N}$, *une transformation naturelle mutante* $t : \Phi \to \Psi$ est la donnée d'une famille de flèches $(\Phi(X) \xrightarrow{t_X} \Psi(X))_{X \in |\mathcal{M}|}$ de $\underline{\mathcal{N}}$ (la catégorie sous-jacente à $\mathcal{N}$ - voir la définition 1.5, plus loin) telle que, pour toute flèche $X \xrightarrow{f} Y$ de $\mathcal{M}$ le carré suivant commute strictement dans $\mathcal{N}$:

$$
\begin{array}{ccc}
\Phi(X) & \xrightarrow{\; t_X \;} & \Psi(X) \\
{\scriptstyle \Phi(f)} \downarrow & & \downarrow {\scriptstyle \Psi(f)} \\
\Phi(Y) & \xrightarrow[\; t_Y \;]{} & \Psi(Y)
\end{array}
$$

c.a.d. $\Psi(f)t_X = t_Y\Phi(f)$ (où les compositions ici sont strictes - voir plus loin leurs définitions)

• Catégories mutantes (sur $\mathbb{V}$), foncteurs mutants et transformations naturelles mutantes forment une 2-catégorie notée $Cat\mu(\mathbb{V})$.

Exemples 1.4. : On distingue principalement trois classes d'exemples de catégories mutantes.

- **Les catégories enrichies dans** $\mathbb{V}$:
$\mathbb{C}$ étant une catégorie enrichie dans $\mathbb{V}$, on lui associe la catégorie mutante (sur $\mathbb{V}$) $\mu e(\mathbb{C})$, pour laquelle...
.. $|\mu e(\mathbb{C})| = |\mathbb{C}|$,
.. Pour tout $X, Y \in |\mu e(\mathbb{C})|$, $\mu e(\mathbb{C})(X, Y) = \mathbb{V}/\mathbb{C}(X, Y)$,
.. La composition des flèches de $\mu e(\mathbb{C})$ est donnée, pour un tel couple $X \xrightarrow{(A,f)} Y \xrightarrow{(B,g)} Z$ par $(X \xrightarrow{(B,g)\otimes(A,f)} Z) = (X \xrightarrow{(B\otimes A, g\circ f)} Z)$ où
$g \circ f = (B \otimes A \xrightarrow{g\otimes f} \mathbb{C}(Y,Z) \otimes \mathbb{C}(X,Y) \xrightarrow{comp} \mathbb{C}(X,Z))$,
.. L'identité pour la loi $\otimes$ est donnée, pour $X \in |\mathbb{C}|$, par
$Id_X = (I, I \xrightarrow{id_X} \mathbb{C}(X,X))$,
.. La composition verticale des 2-cellules est la composition dans $\mathbb{V}$,
.. La composition horizontale des 2-cellules est le produit tensoriel dans $\mathbb{V}$,
.. $U : \mu e(\mathbb{C}) \to \mathbb{V}^b$ est donné sur les flèches par $U(A, f) = A$.

- **Les catégories $\mathbb{V}$-tensorisées** (et plus généralement les catégories $\mathbb{V}$-prétensorisées):

où une *catégorie* $\mathbb{V}$-*tensorisées* est la donnée d'une catégorie $\mathbb{E}$ munie d'un foncteur "produit tensoriel extérieur" $\mathbb{V} \times \mathbb{E} \xrightarrow{\wedge} \mathbb{E}$ pour lequel,

.. Pour tout $X \in |\mathbb{E}|, I \wedge X \simeq X$ (cet isomorphisme est noté s_X),

.. Pour tout $X \in |\mathbb{E}|$, pour tout $A, B \in \mathbb{V}, (B \otimes A) \wedge X \simeq B \wedge (A \wedge X)$ (cet isomorphisme est noté $am_{B,A,X}$).

Ces isomorphismes étant naturels et vérifiant des axiomes de cohérence (voir [3])(dans le cas des catégories $\mathbb{V}$-prétensorisées $am_{B,A,X}$ n'est plus nécessairement un isomorphisme).

La catégorie mutante associée, notée $\mu t(\mathbb{E})$, se décrit comme suit :

.. $|\mu t(\mathbb{E})| = |\mathbb{E}|$,

.. Pour $X, Y \in |\mathbb{E}|$, une flèche $X \to Y$ dans $\mu t(\mathbb{E})$, est un couple (A, f) où $A \in |\mathbb{V}|$ et $A \wedge X \xrightarrow{f} Y$ est une flèche de $\mathbb{E}$,

.. Pour deux flèches $(A, f), (A', f') : X \to Y$, une 2-cellule $a : (A, f) \to (A', f')$ dans $\mu t(\mathbb{E})$ est une flèche $a : A \to A'$ de $\mathbb{V}$ rendant commutatif le triangle suivant dans $\mathbb{E}$:

$$A \wedge X \xrightarrow{a \wedge Id} A' \wedge X$$
$$f \searrow \quad \swarrow f'$$
$$Y$$

.. La composition des flèches de $\mu t(\mathbb{E})$ est donnée, pour $X \xrightarrow{(A,f))} Y \xrightarrow{(B,g)} Z$, par $(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où $g \circ f = ((B \otimes A) \wedge X \xrightarrow{am} B \wedge (A \wedge X) \xrightarrow{Id \wedge f} B \wedge Y \xrightarrow{g} Z)$

.. L'identité pour la loi $\otimes$ est donnée, pour $X \in |\mathbb{E}|$, par $Id_X = (I, I \wedge X \xrightarrow{s_X} X)$

.. Le reste de la description de $\mu t(\mathbb{E})$ est essentiellement comme celle de $\mu e(\mathbb{C})$.

.. Dans ([3]) on donne de nombreux exemples de catégories $\mathbb{V}$-tensorisées et même $\mathbb{V}$-prétensorisées.

    - **Les catégories $\mathbb{V}$-cotensorisées** (et plus généralement les catégories $\mathbb{V}$-précotensorisées):

où une *catégorie* $\mathbb{V}$-*cotensorisée* est la donnée d'une catégorie $\mathbb{E}$ munie d'un foncteur "Exponentiel" $\mathbb{E} \times \mathbb{V}^{op} \xrightarrow{Exp} \mathbb{E}$(on note $X^A = Exp(X, A)$) pour lequel...

.. Pour tout $X \in |\mathbb{E}|, X \simeq X^I$ (cet isomorphisme est noté σ_X),

.. Pour tout $X \in |\mathbb{E}|$, pour tout $A, B \in \mathbb{V}, \ (X^A)^B \simeq X^{A \otimes B}$ (cet isomorphisme est noté $\alpha m_{X,A,B}$)

Ces isomorphismes étant naturels et vérifiant des axiomes de cohérence (voir [3]) (Dans le cas des catégories $\mathbb{V}$-précotensorisées $\alpha m_{X,A,B}$ n'est plus nécessairement un isomorphisme).

La catégorie mutante associée, notée $\mu c(\mathbb{E})$, se décrit comme suit :

.. $|\mu c(\mathbb{E})| = |\mathbb{E}|$,

.. Pour $X, Y \in |\mathbb{E}|$, une flèche $X \to Y$ dans $\mu c(\mathbb{E})$, est un couple (A, f) où $A \in |\mathbb{V}|$ et $X \xrightarrow{f} Y^A$ est une flèche de $\mathbb{E}$.

.. Pour deux flèches $(A, f), (A', f') : X \to Y$, une 2-cellule $a : (A, f) \to (A', f')$ dans $\mu c(\mathbb{E})$ est une flèche $a : A \to A'$ de $\mathbb{V}$ rendant commutatif le triangle suivant dans $\mathbb{E}$:

$$
\begin{array}{ccc}
& X & \\
{}^{f'} \swarrow & & \searrow {}^{f} \\
Y^{A'} & \xrightarrow[Id^a]{} & Y^A
\end{array}
$$

.. La composition des flèches de $\mu c(\mathbb{E})$ est donnée, pour

$X \xrightarrow{(A,f)} Y \xrightarrow{(B,g)} Z$, par $(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où

$g \circ f = (X \xrightarrow{f} Y^A \xrightarrow{g^{Id}} (Z^B)^A \xrightarrow{\alpha m} Z^{B \otimes A})$

.. L'identité pour la loi $\otimes$ est donnée, pour $X \in |\mathbb{E}|$, par

$Id_X = (I, X \xrightarrow{\sigma_X} X^I)$

.. Le reste de la description de $\mu c(\mathbb{E})$ est essentiellement comme celle de $\mu e(\mathbb{C})$ et $\mu t(\mathbb{E})$.

.. Au paragraphe suivant nous donnons un exemple général de catégories $\mathbb{V}$-cotensorisée (et même $\mathbb{V}$-précotensorisée).

.. Dans [4] une autre classe d'exemples de catégories mutantes est donnée.

$\bullet$ En plus des catégories enrichies dans $\mathbb{V}$, les catégories $\mathbb{V}$-tensorisées (resp. $\mathbb{V}$-prétensorisées) et les catégories $\mathbb{V}$-cotensorisées (resp. $\mathbb{V}$-précotensorisées) forment des 2-catégories qui sont notées $\mathbb{V}$-$Tens$ (resp. $\mathbb{V}$-$Pretens$) et $\mathbb{V}$-$Cotens$ (resp. $\mathbb{V}$-$Precot$).

Pour la description détaillée de ces 2-catégories nous renvoyons à [4]. La 2-catégorie des catégories enrichies dans $\mathbb{V}$ est notée, comme souvent, $\mathbb{V}$-Cat.

Il y a trois 2-foncteurs canoniques $\mu e : \mathbb{V}\text{-}Cat \to Cat\mu(\mathbb{V})$,
$\mu t : \mathbb{V}\text{-}Pretens \to Cat\mu(\mathbb{V})$, $\mu c : \mathbb{V}\text{-}Precot \to Cat\mu(\mathbb{V})$ qui sont 2-pleinement fidèles.

La composition stricte dans une catégorie mutante

Fixons ici une catégorie mutante $\mathcal{M}$ sur $\mathbb{V}$.

- Étant donné un couple de flèches composables dans $\mathcal{M}$: $X \xrightarrow{f} Y \xrightarrow{g} Z$,

.. Si $U(g) = I$ et $U(f) = A$, on pose $gf = (u_g^{-1})^\star(g \otimes f)$ (où $I \otimes A \xrightarrow{u_g} A$

est la flèche unité à gauche) dans la fibration discrète $\mathcal{M}(X, Z) \xrightarrow{U_{XZ}} \mathbb{V}$.

.. SI $U(g) = A$ et $U(f) = I$, on pose $gf = (u_d^{-1})^\star(g \otimes f)$ (où $A \otimes I \xrightarrow{u_d} A$

est la flèche unité à droite) dans la fibration discrète $\mathcal{M}(X, Z) \xrightarrow{U_{XZ}} \mathbb{V}$.

Dans les deux cas, gf est appelé le *composé strict* de g par f.

- La composition stricte a les propriétés suivantes :

.. Pour $X \xrightarrow{f} Y$ dans $\mathcal{M}$, $fId_X = f = Id_Y f$.

.. Pour $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ dans $\mathcal{M}$, on a $(hg)f = h(gf)$ lorsque deux des trois flèches sont au moins d'image I par U.

.. Dans le cas où seule une des flèches est au moins d'image I par U on a :

... $(h \otimes g)f = h \otimes (gf)$ (quand $U(f) = I$),

... $(hg) \otimes f = h \otimes (gf)$ (quand $U(g) = I$),

... $(hg) \otimes f = h(g \otimes f)$ (quand $U(h) = I$).

Définition 1.5. On appelle *catégorie sous-jacente* à $\mathcal{M}$ (et on la note $\underline{\mathcal{M}}$ la catégorie dont:

- Les objets sont ceux de $\mathcal{M}$,
- Les flèches sont celles de $\mathcal{M}$ s'envoyant sur I par U.

La composition dans $\underline{\mathcal{M}}$ est la composition stricte.

On construit un 2-foncteur canonique $Cat\mu(\mathbb{V}) \xrightarrow{Oub} \mathbb{C}at$ où $Oub(\mathcal{M}) = \underline{\mathcal{M}}$.

Les saveurs d'une catégorie mutante

$\mathcal{M}$ étant une catégorie mutante sur $\mathbb{V}$, on construit un foncteur
$Tri_\mathcal{M} : \mathbb{V}^{op} \times \underline{\mathcal{M}}^{op} \times \underline{\mathcal{M}} \to \mathbb{E}ns$ (où simplement Tri). Il est défini
- sur un objet (A, X, Y) par
$Tri(A, X, Y) = \{f \in |\mathcal{M}(X, Y)|/U_{XY}(f) = A\}$,

- sur une flèche $(a, x, y) : (A, X, Y) \to (A', X', Y')$, $Tri(a, x, y)$ est l'application composée suivante :

$$Tri(A, X, Y) \overset{a^\star(-)}{\to} Tri(A', X, Y) \overset{(-)x}{\to} Tri(A', X', Y) \overset{y(-)}{\to} Tri(A', X', Y')$$

La première application provient de la fibration discrète
$U_{XY} : \mathcal{M}(X, Y) \to \mathbb{V}$. Les deux suivantes résultent de la composition stricte dans $\mathcal{M}$.

Définition 1.6. On dit que $\mathcal{M}$ est de *saveur...*
1) enrichie (ou simplement *de saveur E* si:
Pour tout $X, Y \in |\mathcal{M}|$, $Tri(-, X, Y) : \mathbb{V}^{op} \to \mathbb{E}ns$ est représentable,
2) *tensorisée* (ou simplement *de saveur T*) si :
Pour tout $A \in |\mathbb{V}|$ et $X \in |\mathcal{M}|$, $Tri(A, X, -) : \underline{\mathcal{M}} \to \mathbb{E}ns$ est coreprésentable,
3) *cotensorisée* (ou simplement *de saveur C*) si :
Pour tout $A \in |\mathbb{V}|$ et $Y \in |\mathcal{M}|$, $Tri(A, -, Y) : \underline{\mathcal{M}}^{op} \to \mathbb{E}ns$ est représentable,

Proposition 1.7. 1) Si $\mathcal{M}$ est de saveur E, alors il existe une catégorie enrichie $\mathcal{M}^e$ telle que $\mu e(\mathcal{M}^e) \simeq \mathcal{M}$.
2) Si $\mathcal{M}$ est de saveur T, alors il existe une catégorie prétensorisée $\mathcal{M}^t$ telle que $\mu t(\mathcal{M}^t) \simeq \mathcal{M}$.
3) Si $\mathcal{M}$ est de saveur C, alors il existe une catégorie précotensorisée $\mathcal{M}^c$ telle que $\mu c(\mathcal{M}^c) \simeq \mathcal{M}$.

Remarque 1.8. On se sert de cette proposition pour montrer que les trois 2-foncteurs $\mu e, \mu t$ et μc sont 2-pleinement fidèles.

Critère d'existence des catégories et foncteurs mutants

Remarque 1.9. Ce qui suit n'est pas un rappel mais, les propositions qui y figurent ne présentant pas de difficulté particulières, leur démonstrations ont été omises (elles utilisent principalement la structure de fibration discrète qui, remarquons le, est essentiellement graphique).

Pour construire les 2-foncteurs μe, μt et μc il est utile d'utiliser les propositions suivantes destinées à nous faciliter la tâche.

Proposition 1.10. $\mathbb{V}$ étant fixé, considérons les données suivantes :
1) Un 2-magma $\mathcal{M}$,
2) un morphisme strict de 2-magma $U : \underset{\sim}{\mathcal{M}} \to \underset{\sim}{\mathbb{V}}$ (où $\underset{\sim}{\mathbb{V}}$ désigne le 2-magma sous-jacent à la bicatégorie $\mathbb{V}^b$),
3) trois familles de 2-cellules de $\underset{\sim}{\mathcal{M}}$:

$$((x \otimes y) \otimes z \xrightarrow{Ass_{xyz}} x \otimes (y \otimes z))_{(x,y,z)\in Comp}$$
$$(x \otimes Id_X \xrightarrow{U_{dx}} x)_{x\in Fl} \quad , \quad (Id_{X'} \otimes x \xrightarrow{U_{gx}} x)_{x\in Fl}$$

où Fl est l'ensemble des flèches $X \xrightarrow{x} X'$ de $\underset{\sim}{\mathcal{M}}$ et $Comp$ est l'ensemble des triplets $T \xrightarrow{z} Z \xrightarrow{y} Y \xrightarrow{x} X$ de flèches de $\underset{\sim}{\mathcal{M}}$ composables.

Ces trois données devant satisfaire les conditions suivantes :
(PCM1) Pour tout $X, Y \in |\underset{\sim}{\mathcal{M}}|$, $U_{XY} : \underset{\sim}{\mathcal{M}}(X,Y) \to \underset{\sim}{\mathbb{V}}$ est une fibration discrète,
(PCM2) $U(Ass_{xyz}) = ass_{Ux,Uy,Uz}$, $U(U_{dx}) = u_{dUx}$, $U(U_{gx}) = u_{gUx}$.
Alors il existe une unique structure de catégorie mutante $\mathcal{M}$ (sur $\mathbb{V}$) prolongeant les données précédentes.

Proposition 1.11. Soient $\mathcal{M}$ et $\mathcal{N}$ deux catégories mutantes sur $\mathbb{V}$. Notons $\underset{\sim}{\mathcal{M}}$ et $\underset{\sim}{\mathcal{N}}$ les 2-graphes sous-jacents à $\mathcal{M}$ et $\mathcal{N}$. Soit maintenant $F : \underset{\sim}{\mathcal{M}} \to \underset{\sim}{\mathcal{N}}$ un morphisme de 2-graphe tel que :
1) le triangle suivant commute :

$$
\begin{array}{ccc}
\underset{\sim}{\mathcal{M}} & \xrightarrow{\quad F \quad} & \underset{\sim}{\mathcal{N}} \\
& \searrow{\scriptstyle U} \quad \swarrow{\scriptstyle U} & \\
& \underset{\sim}{\mathbb{V}^b} &
\end{array}
$$

2) Pour tout $X \in |\mathcal{M}|$, $F(Id_X) = Id_{F(X)}$,
3) Pour tout couple de flèches composables $X \xrightarrow{f} Y \xrightarrow{g} Z$ de $\mathcal{M}$,
$F(g \otimes f) = F(g) \otimes F(f)$.
Alors F est un foncteur mutant de $\mathcal{M}$ dans $\mathcal{N}$.

Corollaire 1.12. Soient $\mathcal{M}$ et $\mathcal{N}$ deux catégories mutantes sur $\mathbb{V}$. Alors $\mathcal{M} = \mathcal{N}$ ssi
1) $\underset{\sim}{\mathcal{M}} = \underset{\sim}{\mathcal{N}}$
2) $(\underset{\sim}{\mathcal{M}} \xrightarrow{U} \mathbb{V}^b) = (\underset{\sim}{\mathcal{N}} \xrightarrow{U} \mathbb{V}^b)$
3) Pour tout $X \in |\underset{\sim}{\mathcal{M}}|, Id_X^{\mathcal{M}} = Id_X^{\mathcal{N}}$ (où $Id_X^{\mathcal{M}}$ est l'identité de X pour $\mathcal{M}$)
4) Pour tout couple de flèches composables $X \xrightarrow{f} Y \xrightarrow{g} Z$ de $\mathcal{M}$,
$g \underset{\mathcal{M}}{\otimes} f = g \underset{\mathcal{N}}{\otimes} f$ (où $g \underset{\mathcal{M}}{\otimes} f$ est la composition des flèches dans $\mathcal{M}$).

Proposition 1.13. Soit $F : \mathcal{M} \to \mathcal{N}$ un foncteur mutant. Alors F est un isomorphisme ssi :
1) $|F| : |\mathcal{M}| \to |\mathcal{N}|$ est bijectif,
2) Pour tout $X, X' \in |\mathcal{M}|, |F_{XX'}| : |\mathcal{M}(X, X')| \to |\mathcal{N}(FX, FX')|$, est bijectif.

Proposition 1.14. Soit $t : F \to G : \mathcal{M} \to \mathcal{N}$ une transformation naturelle mutante. Si pour tout $X \in |\mathcal{M}|$, la flèche $F(X) \xrightarrow{t_X} G(X)$ est inversible dans $\underset{\sim}{\mathcal{N}}$ alors t est inversible.

2. Un exemple de catégorie cotensorisée

• L'article [3] manque d'exemple authentique de catégorie cotensorisée. C'est à dire que leur catégorie mutante associée ne provienne pas de $\mathbb{V}$-Cat ou de $\mathbb{V}$-$Pretens$. Nous allons maintenant combler cette lacune en proposant un exemple authentique de catégorie cotensorisée. Il nous sera utile dans la suite.

• Commenons par fixer une catégorie $\mathbb{C}$ et une catégorie $\mathbb{V}$-prétensorisée $\mathbb{E}$. On va maintenant munir la catégorie $\bar{\mathbb{E}} = [\mathbb{E}^{op}, \mathbb{C}]$ d'une structure de catégorie $\mathbb{V}$-précotensorisée.

- Pour $A \in |\mathbb{V}|, F \in |\bar{\mathbb{E}}|$ alors F^A est le foncteur $F(A \wedge -) : \mathbb{E}^{op} \to \mathbb{C}$.

- Pour une flèche $A' \xrightarrow{a} A$ de $\mathbb{V}$, $F^a : F^A \to F^{A'}$ est la transformation naturelle $F(a \wedge -) : F(A \wedge -) \to F(A' \wedge -)$.

- Pour une flèche $F \xrightarrow{t} F'$ de $\bar{\mathbb{E}}$ et $A \in |\mathbb{V}|$, on a la transformation naturelle $(F^A \xrightarrow{t^A} F'^A) = (F(A \wedge -) \xrightarrow{t_{A \wedge -}} F'(A \wedge -))$.

On regroupe tous ces cas particuliers dans le cas général suivant. Pour une

flèche $(t, a) : (F, A) \rightarrow (F', A')$ de $\bar{\mathbb{E}} \times \mathbb{V}^{op}$, $t^a : F^A \rightarrow F'^{A'}$ est la transformation naturelle composée suivante :

$$(F^A \xrightarrow{F^a} F^{A'} \xrightarrow{t^{A'}} F'^{A'}) = (F^A \xrightarrow{t^A} F'^A \xrightarrow{F'^a} F'^{A'})$$

On construit ainsi un foncteur $Exp : \bar{\mathbb{E}} \times \mathbb{V}^{op} \rightarrow \bar{\mathbb{E}}$.
- On construit ensuite :
.. Pour chaque $F \in |\bar{\mathbb{E}}|$, l'isomorphisme $F \xrightarrow{\sigma_F} F^I$ en posant, pour chaque $X \in |\mathbb{E}|$, $(\sigma_F)_X = (F(X) \xrightarrow{F(s_X)} F(I \wedge X))$,
.. Pour tout $F \in |\bar{\mathbb{E}}|$ et $A, B \in |\mathbb{V}|$, la flèche $(F^A)^B \xrightarrow{\alpha m_{F,A,B}} F^{A \otimes B}$ de $\bar{\mathbb{E}}$, en posant, pour tout $X \in |\mathbb{E}|$,
$(\alpha m_{F,A,B})_X = (F(A \wedge (B \wedge X)) \xrightarrow{F(am_{A,B,X})} F((A \otimes B) \wedge X))$
La naturalité de σ et de αm résulte de celle de s et de am. Enfin les axiomes de cohérence souhaités proviennent de ceux de $\mathbb{E}$, ce qui achève d'établir que $\bar{\mathbb{E}}$, muni de Exp, σ et αm a une structure de catégorie précotensorisée sur $\mathbb{V}$. Remarquons que si $\mathbb{E}$ est tensorisée alors $\bar{\mathbb{E}}$ est cotensorisée. Avant de poursuivre, signalons la proposition suivante qui sera utile dans la section 7.

Proposition 2.1. $\mathbb{E}$ étant une catégorie prétensorisée sur une catégorie monoïdale $\mathbb{V}$, soit $F \xrightarrow{t} G$ une flèche de $\bar{\mathbb{E}}$. Notons $\tilde{t} = (I, \underline{t})$ où $\underline{t} = (F \xrightarrow{t} G \xrightarrow{\sigma} G^I)$, la flèche de $\mu c(\bar{\mathbb{E}})$ correspondante. Soit maintenant $F' \xrightarrow{f} F$ et $G \xrightarrow{g} G'$ deux flèches de $\mu c(\bar{\mathbb{E}})$. On écrit $f = (A, \underline{f})$ où $\underline{f} : F' \rightarrow F^A$ est dans $\bar{\mathbb{E}}$ et $g = (B, \underline{g})$ où $\underline{g} : G \rightarrow G'^B$ est dans $\bar{\mathbb{E}}$. Alors on a les compositions strictes suivantes :
1) $g\tilde{t} = (B, \underline{g}.t)$ c.a.d. $F \xrightarrow{t} G \xrightarrow{g} G'^B$.
2) $\tilde{t}f = (A, t^A.\underline{f})$ c.a.d. $F' \xrightarrow{f} F^A \xrightarrow{t^A} G^A$

<u>Preuve</u> : 1) On a $g \otimes \tilde{t} = (B \otimes I, \underline{g} \circ \underline{t})$ où $\underline{g} \circ \underline{t}$ est la flèche composée suivante dans $\bar{\mathbb{E}}$: $F \xrightarrow{t} G \xrightarrow{\sigma} G^I \xrightarrow{\underline{g}^I} (G'^B)^I \xrightarrow{\alpha m} G'^{B \otimes I}$. Pour établir l'identité demandée, il suffit de vérifier que le triangle suivant commute:

$$F$$

$$G'^{B \otimes I} \xrightarrow[Id^u{}_d^{-1}]{} G'^B$$

with arrows $\underline{g \circ t}$ and $\underline{g}.t$ from F.

2) On procède comme au (1).

Un plongement de Yoneda vu comme foncteur mutant.

En particulier la catégorie $\hat{\mathbb{E}} = [\mathbb{E}^{op}, \mathbb{E}ns]$ a canoniquement une structure de cotensorisée. On va voir maintenant que le plongement de Yoneda (classique) $y : \mathbb{E} \to \hat{\mathbb{E}}$ se prolonge en un foncteur mutant (noté μy), $\mu t(\mathbb{E}) \to \mu c(\hat{\mathbb{E}})$. On le construit comme suit :

- Sur les objets, on a $(|\mu t(\mathbb{E})| \xrightarrow{|\mu y|} |\mu c(\hat{\mathbb{E}})|) = (|\mathbb{E}| \xrightarrow{|y|} |\hat{\mathbb{E}}|)$.

- Sur une flèche $(A, f) : X \to X'$ dans $\mu t(\mathbb{E})$ (alors $f : A \wedge X \to X'$ est une flèche de $\mathbb{E}$) on a $\mu y(A, f) = (A, \underline{y}(f))$ où $\underline{y}(f) : y(X) \to y(X')^A$ est la flèche de $\hat{\mathbb{E}}$ qui, appliquée en $Z \in |\mathbb{E}|$, est l'application $\mathbb{E}(Z, X) \to \mathbb{E}(A \wedge Z, X')$ envoyant la flèche $Z \xrightarrow{x} X$ de $\mathbb{E}$, sur le composé suivant : $(A \wedge Z \xrightarrow{Id \wedge x} A \wedge X \xrightarrow{f} X')$.

- Sur une 2-cellule $a : (A, f) \to (A', f') : X \to X'$,
$(\mu y(A, f) \xrightarrow{\mu y(a)} \mu y(A', f')) = ((A, \underline{y}(f)) \xrightarrow{a} (A', \underline{y}(f')))$

C'est une 2-cellule de $\mu c(\hat{\mathbb{E}})$ car le triangle suivant commute dans $\hat{\mathbb{E}}$:

$$y(X)$$

$$y(X')^{A'} \xrightarrow[Id^a]{} y(X')^A$$

with arrows $\underline{y}(f')$ and $\underline{y}(f)$ from $y(X)$.

.. μy est un foncteur mutant car,

... Sur un objet $X \in |\mu t(\mathbb{E})|$, on a $\mu y(Id_X) = Id_{\mu y(X)}$. En effet, $Id_X = (I, s)$ (où $I \wedge X \xrightarrow{s} X$ est la flèche canonique) et $Id_{\mu y(X)} = (I, \sigma)$ (où $y(X) \xrightarrow{\sigma} y(X)^I$ est tel que $\sigma_Z = y(X)(I \wedge Z \xrightarrow{s} Z)$). Mais $\underline{y}(I \wedge X \xrightarrow{s} X) = \sigma$ et donc $Id_{\mu y(X)} = (I, \sigma) = \mu y(I, s) = \mu y(Id_X)$.

... Pour un couple de flèches composables de $\mu t(\mathbb{E})$:
$X_0 \xrightarrow{(A_0, f_0)} X_1 \xrightarrow{(A_1, f_1)} X_2$, montrons que $\mu y((A_1, f_1) \otimes (A_0, f_0)) =$

$\mu y(A_1, f_1) \otimes \mu y(A_0, f_0)$. On a déjà $(A_1, f_1) \otimes (A_0, f_0) = (A_1 \otimes A_0, f_1 \circ f_0)$ où $f_1 \circ f_0 : (A_1 \otimes A_0) \wedge X_0 \to X_2$ est la flèche composée suivante :

$$(A_1 \otimes A_0) \wedge X_0 \xrightarrow{am} A_1 \wedge (A_0 \wedge X_0) \xrightarrow{Id \wedge f_0} A_1 \wedge X_1 \xrightarrow{f_1} X_2$$

Alors $\mu y((A_1, f_1) \otimes (A_0, f_0)) = \mu y(A_1 \otimes A_0, f_1 \circ f_0) = (A_1 \otimes A_0, \underline{y}(f_1 \circ f_0))$, où $\underline{y}(f_1 \circ f_0) = y(X_0) \to y(X_2)^{A_1 \otimes A_0}$ est défini en $Z \in |\mathbb{E}|$ comme étant l'application qui à $Z \xrightarrow{x_0} X_0$ dans $\mathbb{E}$ associe le composé (C) suivant :

$$(A_1 \otimes A_0) \wedge Z \xrightarrow{Id \wedge x_0} (A_1 \otimes A_0) \wedge X_0 \xrightarrow{am} A_1 \wedge (A_0 \wedge X_0) \xrightarrow{Id \wedge f_0} A_1 \wedge X_1 \xrightarrow{f_1} X_2$$

D'un autre côté $\mu y(A_1, f_1) \otimes \mu y(A_0, f_0) = (A_1, \underline{y}(f_1)) \otimes (A_0, \underline{y}(f_0)) = (A_1 \otimes A_0, \underline{y}(f_1) \circ \underline{y}(f_0))$ où $\underline{y}(f_1) \circ \underline{y}(f_0) : y(X_0) \to y(X_2)^{A_1 \otimes A_0}$ est la flèche composée suivante :

$$y(X_0) \xrightarrow{y(f_0)} y(X_1)^{A_0} \xrightarrow{y(f_1)^{Id}} (y(X_2)^{A_1})^{A_0} \xrightarrow{\alpha m} y(X_2)^{A_1 \otimes A_0}$$

Appliquée à $Z \in |\mathbb{E}|$, c'est l'application qui à $Z \xrightarrow{x_0} X_0$ dans $\mathbb{E}$ associe le composé (C') suivant :

$$(A_1 \otimes A_0) \wedge Z \xrightarrow{am} A_1 \wedge (A_0 \wedge Z) \xrightarrow{Id \wedge (Id \wedge x_0)} A_1 \wedge (A_0 \wedge X_0) \xrightarrow{Id \wedge f_0} A_1 \wedge X_1 \xrightarrow{f_1} X_2$$

Or, on voit facilement que les composés (C) et (C') coïncident. D'où l'identité voulue. Finalement, en appliquant le critère de la proposition 1.11 on en déduit que $\mu y : \mu t(\mathbb{E}) \to \mu c(\hat{\mathbb{E}})$ est un foncteur mutant.

• Avant d'aborder la proposition suivante donnons une définition que nous utiliserons à plusieurs reprises dans le courant de cet article.

Définition 2.2. Soit $\Phi : \mathcal{M} \to \mathcal{N}$ un foncteur mutant. On dit qu'il est *pleinement fidèle* si, pour tout $X, Y \in |\mathcal{M}|$, le foncteur $\Phi_{XY} : \mathcal{M}(X, Y) \to \mathcal{N}(\Phi(X), \Phi(Y))$ est un isomorphisme de catégories.

Proposition 2.3. Le foncteur mutant $\mu y : \mu t(\mathbb{E}) \to \mu c(\hat{\mathbb{E}})$ est pleinement fidèle.

Preuve : Notons déjà $\mathcal{E} = \mu t(\mathbb{E})$ et $\hat{\mathcal{E}} = \mu c(\hat{\mathbb{E}})$. Soient maintenant $X, X' \in |\mathcal{E}| = |\mathbb{E}|$. Montrons que le foncteur $\mu y_{XX'} : \mathcal{E}(X, X') \to \hat{\mathcal{E}}(\mu y(X), \mu y(X'))$ est un isomorphisme. Soit (A, t) dans $|\hat{\mathcal{E}}(\mu y(X), \mu y(X'))| = |\hat{\mathcal{E}}(y(X), y(X'))|$. Alors $t : y(X) \to y(X')^A$ est une flèche de $\hat{\mathbb{E}}$. Désignons par $\underline{f}$ l'image de Id_X par

$\mathbb{E}(X, X) \xrightarrow{t_X} \mathbb{E}(A \wedge X, X')$ et posons $f = (A, \underline{f})$. Alors $X \xrightarrow{f} X'$ est une flèche de $\mathcal{E}$. Montrons que $\mu y(f) = (A, t)$. Appliqué en $Z \in |\mathbb{E}|$,

$\mathbb{E}(Z, X) \xrightarrow{\underline{y}(f)_Z} \mathbb{E}(A \wedge Z, X')$ est l'application qui à $Z \xrightarrow{x} X$, dans $\mathbb{E}$, associe le composé $A \wedge Z \xrightarrow{Id \wedge x} A \wedge X \xrightarrow{\underline{f}} X'$. En particulier, pour $x = Id_X$ on a $\underline{y}(\underline{f})_X(Id_X) = \underline{f} = t_X(Id_X)$. Mais, par le Lemme de Yoneda (classique) on sait que l'application $\hat{\mathbb{E}}(y(X), y(X')^A) \to y(X')^A(X), t \mapsto t_X(Id_X)$ est bijective. Donc $\underline{y}(\underline{f}) = t$ et donc $\mu y(f) = (A, t)$. L'unicité de f est immédiate.

- Soit maintenant $f, f' \in |\mathcal{E}(X, X')|$ et $a : \mu y(f) \to \mu y(f')$ une 2-cellule de $\hat{\mathcal{E}}$. Montrons que $f \xrightarrow{a} f'$ est une 2-cellule de $\mathcal{E}$. Écrivons $f = (A, \underline{f})$ et $f' = (A', \underline{f'})$. Dire que $\mu y(f) \xrightarrow{a} \mu y(f')$ est une 2-cellule de $\hat{\mathcal{E}}$ signifie que le triangle suivant commute :

$$
\begin{array}{ccc}
 & y(X) & \\
\underline{y}(\underline{f'}) \swarrow & & \searrow \underline{y}(\underline{f}) \\
y(X')^{A'} & \xrightarrow{\quad y(X')^a \quad} & y(X')^A
\end{array}
$$

Appliqué en X à $Id_X \in \mathbb{E}(X, X)$ on obtient les identitées suivantes :
$\underline{f} = \underline{y}(\underline{f})_X(Id_X) = \mathbb{E}(a \wedge Id_X, Id_{X'}).\underline{y}(\underline{f'}_X)(Id_X) = \mathbb{E}(a \wedge Id_X, Id_{X'})(\underline{f'}) = \underline{f'}.a \wedge Id_X$ ce qui prouve que $f \xrightarrow{a} f'$ est une flèche de $\mathcal{E}(X, X')$. Cela achève de montrer que μy est pleinement fidèle.

3. Changement de base

• Soit $\Phi = (\Phi, \sigma, sor) : \mathbb{V} \to \mathbb{W}$ un foncteur comonoïdal (encore appelé op-monoïdal, colax monoïdal ou oplax monoïdal) où $\sigma : \Phi(I) \to I$ et $sor = (sor_{A,B} : \Phi(A \otimes B) \to \Phi(A) \otimes \Phi(B))_{(A,B) \in |\mathbb{V}|^2}$. À chaque catégorie mutante $\mathcal{N}$ sur $\mathbb{W}$ on va lui faire correspondre une nouvelle catégorie mutante notée $\Phi^\star(\mathcal{N})$ sur $\mathbb{V}$. Mais avant cela considérons la proposition suivante :

Proposition 3.1. On suppose que le foncteur comonoïdal Φ admet un adjoint à droite Ψ. Alors on peut complèter canoniquement Ψ en un foncteur monoïdal.

Preuve : Fixons d'abord une co-unité $\varepsilon : \Phi\Psi \to Id_{\mathbb{W}}$ de l'adjonction $\Phi - |\Psi$. Alors $e : I \to \Psi I$ est l'unique flèche de $\mathbb{V}$ telle que le triangle

suivant commute :

$$\begin{array}{ccc}
\Phi I & \xrightarrow{\ \Phi e\ } & \Phi\Psi I \\
 & {}_{\sigma}\searrow & \downarrow{}^{\varepsilon_I} \\
 & & I
\end{array}$$

On a aussi, pour $A, B \in |\mathbb{W}|$, la flèche $\Psi A \otimes \Psi B \xrightarrow{ent_{A,B}} \Psi(A \otimes B)$ qui est définie comme étant l'unique flèche de $\mathbb{V}$ rendant le carré suivant commutatif :

$$\begin{array}{ccc}
\Phi(\Psi A \otimes \Psi B) & \xrightarrow{\ \Phi(ent_{A,B})\ } & \Phi\Psi(A \otimes B) \\
{}_{sor_{\Psi A,\Psi B}}\downarrow & & \downarrow{}^{\varepsilon_{A\otimes B}} \\
\Phi\Psi A \otimes \Phi\Psi B & \xrightarrow[\varepsilon_A \otimes \varepsilon_B]{} & A \otimes B
\end{array}$$

On vérifie facilement que (ψ, e, ent) (où $ent = (ent_{A,B})_{(A,B)\in|\mathbb{V}|^2}$) est un foncteur monoïdal.

Construction de $\Phi^\star(\mathcal{N})$.

Dans la suite de la construction, $\Phi^\star(\mathcal{N})$, pour simplifier les notations, est simplement désigné par $\mathcal{M}$. On la construit comme suit :

- $|\mathcal{M}| = |\mathcal{N}|$,

- pour $X, Y \in |\mathcal{M}|$, une flèche $X \to Y$ dans $\mathcal{M}$ est un couple (A, f) où $A \in |\mathbb{V}|$ et $X \xrightarrow{f} Y$ une flèche de $\mathcal{N}$ telle que $U(f) = \Phi(A)$.

- Soient $(A, f), (A', f') : X \to Y$ deux flèches de $\mathcal{M}$. Une 2-cellule $(A, f) \to (A', f')$ est une flèche $A \xrightarrow{a} A'$ dans $\mathbb{V}$ telle que $\Phi(a)^\star(f') = f$ (dans $\mathcal{N}(X, Y)$).

.. Le composé des flèches $X \xrightarrow{(A,f)} Y \xrightarrow{(B,g)} Z$ de $\mathcal{M}$ est la flèche $X \xrightarrow{(B\otimes A,g\circ f)} Z$ où $g \circ f = sor^\star_{B,A}(g \otimes f)$ dans $\mathcal{N}(X, Z)$.

.. Remarquons que si $X \in |\mathcal{M}|$, alors $Id_X^{\mathcal{M}} = (I, \sigma^\star(Id_X^{\mathcal{N}}))$ (où $Id_X^{\mathcal{N}}$ désigne l'identité de X dans $\mathcal{N}$).

.. Les composés verticaux et horizontaux des 2-cellules sont dictés par leur pendant dans $\mathbb{V}$.

.. Pour le reste de la construction et la preuve que $\mathcal{M}$, ainsi construit, est bien une catégorie mutante, il suffit d'appliquer le critère 1.10.

Remarques 3.2. 1) Lorsque $A' \xrightarrow{a} A$ est une flèche de $\mathbb{V}$ et $(A, f) : X \to Y$ une flèche de $\mathcal{M} = \Phi^\star(\mathcal{N})$, on a l'identité :

$$a^\star(A, f) = (A', \Phi(a)^\star(f))$$

2) (Caractérisation de la composition stricte dans $\mathcal{M}$). Dans la situation suivante dans $\mathcal{M}$: $X \xrightarrow{(A,f)} Y \xrightarrow{(B,g)} Z$,
.. Si $B = I$, alors $(I, g)(A, f) = (A, g \ltimes f)$, où
$g \ltimes f = (sor_{I,A}.\Phi(u_g)^{-1})^\star(g \otimes f)$
.. Si $A = I$, alors $(B, g)(I, f) = (B, g \rtimes f)$ où
$g \rtimes f = (sor_{B,I}.\Phi(u_d)^{-1})^\star(g \otimes f)$.

Proposition 3.3. En notant provisoirement $\mathcal{M} = \Phi^\star(\mathcal{N})$, on a les implications suivantes :
1) On suppose que $\Phi I \xrightarrow{\sigma} I$ est inversible, alors :
Si $\mathcal{N}$ est de saveur T (resp. de saveur C) alors $\mathcal{M}$ l'est aussi.
2) On suppose que $\Phi : \mathbb{V} \to \mathbb{W}$ admet un adjoint à droite Ψ, alors :
Si $\mathcal{N}$ est de saveur E il en est de même de $\mathcal{M}$.

Preuve : 1) A) _Le cas de saveur T_. Pour chaque $B \in |\mathbb{W}|$ et $X \in |\mathcal{N}|$ on choisit une représentation de $Tri_\mathbb{W}(B, X, -) : \underline{\mathcal{N}} \to \mathbb{E}ns$. on l'écrit $(B \wedge X, val_{B,X})$. Soit maintenant $A \in |\mathbb{V}|$ et $X \in |\mathcal{M}|$. On montre que $(\Phi(A) \wedge X, (A, val_{\Phi A, X}))$ est une représentation de $Tri_\mathbb{V}(A, X, -)$: $\underline{\mathcal{M}} \to \mathbb{E}ns$. Pour la propriété universelle, on se donne $Y \in |\underline{\mathcal{M}}| = |\mathcal{N}|$ et $(A, f) \in Tri_\mathbb{V}(A, X, Y)$. Comme $f \in Tri_\mathbb{W}(\Phi A, X, Y)$, on utilise la propriété universelle de la représentation de $Tri_\mathbb{W}(\Phi A, X, -)$. Il existe donc une unique flèche $\tilde{f} : \Phi A \wedge X \to Y$ dans $\underline{\mathcal{N}}$ telle que $\tilde{f} val_{\Phi A, X} = f$ (la composition est stricte). On pose ensuite $\hat{f} = \sigma^\star(\tilde{f})$ (dans $\mathcal{N}(\Phi A \wedge X, Y)$). On vérifie alors que $(I, \hat{f})(A, val_{\Phi A, X}) \underset{\star 1}{=} (A, \hat{f} \ltimes val_{\Phi A, X}) \underset{\star 2}{=} (A, f)$.
($\star 1$) Voir la remarque précédente,
($\star 2$) car $\hat{f} \ltimes val_{\Phi A, X} = ((\sigma \otimes Id).sor.\Phi(u_g)^{-1})^\star(\tilde{f} \otimes val_{\Phi A, X}) = (u_g^{-1})^\star(\tilde{f} \otimes val_{\Phi A, X}) = \tilde{f} val_{\Phi A, X} = f$.
Pour l'unicité de (I, f) on procède de façon similaire (en utilisant l'inversibilité de σ).
B)_Le cas de saveur C_ se traite de façon très semblable au (A).
2) _Le cas de saveur E_. Fixons d'abord la co-unité $\varepsilon : \Phi\Psi \to Id_\mathbb{W}$ de l'adjonction $\Phi -|\Psi$. Pour chaque $X, Y \in |\mathcal{N}|$, on choisit une représentation

de $Tri_{\mathbb{W}}(-, X, Y) : \mathbb{W}^{op} \to \mathbb{E}ns$. On l'écrit $(\mathbb{N}(X, Y), ev_{X,Y})$. Il faut maintenant montrer que le préfaisceau $Tri_{\mathbb{V}}(-, X, Y) : \mathbb{V}^{op} \to \mathbb{E}ns$ est représentable. Pour cela, montrons que $(\Psi\mathbb{N}(X, Y), (\Psi\mathbb{N}(X, Y), \varepsilon^\star ev_{X,Y}))$ est une représentation de $Tri_{\mathbb{V}}(-, X, Y)$ (où $\varepsilon = \varepsilon_{\mathbb{N}(X,Y)} : \Phi\Psi\mathbb{N}(X, Y) \to \mathbb{N}(X, Y)$). Soit donc $A \in |\mathbb{V}|$ et $(A, f) \in Tri_{\mathbb{V}}(A, X, Y)$. Comme $f \in Tri_{\mathbb{W}}(\Phi A, X, Y)$, par la propriété universelle de $(\mathbb{N}(X, Y), ev_{X,Y})$, il existe une unique flèche $\tilde{f} : \Phi A \to \mathbb{N}(X, Y)$, dans $\mathbb{W}$, telle que $\tilde{f}^\star(ev_{X,Y}) = f$, dans $\mathcal{N}(X, Y)$. Par l'adjonction $\Phi - |\Psi$, il existe ensuite une unique flèche $\hat{f} : A \to \Psi\mathbb{N}(X, Y)$ dans $\mathbb{V}$ telle que le triangle suivant commute :

$$\Phi A \xrightarrow{\quad \Phi\hat{f} \quad} \Phi\Psi\mathbb{N}(X, Y)$$
$$\tilde{f} \searrow \qquad \swarrow \varepsilon$$
$$\mathbb{N}(X, Y)$$

Alors, dans $\mathcal{M}(X, Y)$, $\hat{f}^\star(\Psi\mathbb{N}(X, Y), \varepsilon^\star ev_{X,Y})) = (A, \Phi(\hat{f})^\star\varepsilon^\star ev_{X,Y})$ (voir la remarque 3.2). Mais $\Phi(\hat{f})^\star\varepsilon^\star ev_{X,Y}) = (\varepsilon.\Phi(\hat{f}))^\star(ev_{X,Y}) = \tilde{f}^\star(ev_{X,Y}) = f$. Ainsi $\hat{f}^\star(\Psi\mathbb{N}(X, Y), \varepsilon^\star ev_{X,Y}) = (A, f)$. L'unicité de $\hat{f}$ pour cette propriété vient de celle de sa définition et de l'unicité de $\tilde{f}$.

Remarque 3.4. • Dans [1], à partir d'un foncteur monoïdal $\Psi = (\Psi, e, ent)$ de $\mathbb{W}$ dans $\mathbb{V}$, on transporte les catégories enrichies de $\mathbb{W}$ dans $\mathbb{V}$ (le terme de "changement de base" est alors employé). Plus précisément, si $\mathbb{C}$ est enrichi dans $\mathbb{W}$, on construit une catégorie enrichie $\mathbb{C}'$ dans $\mathbb{V}$. Elle a les mêmes objets que $\mathbb{C}$. Pour $X, Y \in |\mathbb{C}'|, \mathbb{C}'(X, Y) = \Psi\mathbb{C}(X, Y)$. On a aussi $(I \xrightarrow{id'_X} \mathbb{C}'(X, X)) = (I \xrightarrow{e} \Psi I \xrightarrow{\Psi id_X} \Psi\mathbb{C}(X, X))$ et encore $(\mathbb{C}'(Y, Z) \otimes \mathbb{C}'(X, Y) \xrightarrow{comp'} \mathbb{C}'(X, Z)) =$
$(\Psi\mathbb{C}(Y, Z) \otimes \Psi\mathbb{C}(X, Y) \xrightarrow{ent} \Psi(\mathbb{C}(Y, Z) \otimes \mathbb{C}(X, Y)) \xrightarrow{\Psi comp} \Psi\mathbb{C}(X, Z))$.
Notons $\Psi_\star(\mathbb{C}) = \mathbb{C}'$.
• Si on reprend la partie (2) de la proposition précédente, en notant $\mathbb{N}$ la catégorie enrichie dans $\mathbb{W}$ obtenue à partir des représentations $(\mathbb{N}(X, Y), ev_{XY})$ et $\mathbb{M}$ la catégorie enrichie dans $\mathbb{V}$ obtenue à partir des représentations $(\Psi\mathbb{N}(X, Y), (\Psi\mathbb{N}(X, Y), \varepsilon^\star ev_{XY}))$ (voir [4]), on constate que $\Psi_\star(\mathbb{N}) = \mathbb{M}$ (Il suffit de vérifier que $(id'_X)^\star(\mathbb{M}(X, X), \varepsilon^\star ev_{X,X}) =$
$(I, s^\star(Id_X)))$ et, pour $X, Y, Z \in |\mathbb{M}|, comp'^\star_{XYZ}(\mathbb{M}(X, Z), \varepsilon^\star ev_{X,Z}) =$

$(\mathbb{M}(Y,Z), \varepsilon^\star ev_{Y,Z}) \otimes (\mathbb{M}(X,Y), \varepsilon^\star ev_{X,Y})$ où id'_X et $comp'_{XYZ}$ proviennent de $\Psi_\star(\mathbb{N})$).

La catégorie mutante opposée

• La catégorie mutante opposée est un exemple de changement de base le long d'un foncteur comonoïdal. Ce foncteur comonoïdal est donné par la proposition suivante :

Proposition 3.5. Soit $\mathbb{V}$ une catégorie monoïdale symétrique. Alors il existe canoniquement un foncteur comonoïdal $\Phi : \mathbb{V} \to \mathbb{V}^\star$ où $\mathbb{V}^\star$ est la catégorie monoïdale duale de $\mathbb{V}$ (voir [3] - Elle a la même catégorie sous-jacente et $A \otimes^\star B = B \otimes A$ etc...)

$\underline{Preuve}$: - Le foncteur sous-jacent à Φ est l'identité dans $\mathbb{V}$.
- $\sigma = Id_I$
- $(A \otimes B \xrightarrow{sor_{A,B}} A \otimes^\star B) = (A \otimes B \xrightarrow{sym_{A,B}} B \otimes A)$

• $\mathbb{V}$ étant supposée être une catégorie monoïdale symétrique, à toute catégorie mutante $\mathcal{M}$ sur $\mathbb{V}$ on associe la catégorie mutante (dite opposée), notée $\mathcal{M}^{op}$. On l'obtient en posant $\mathcal{M}^{op} = \Phi^\star(\mathcal{M}^\star)$ (attention ! Il y a un risque de confusion avec les étoiles), où $\mathcal{M}^\star$ est la catégorie mutante duale de $\mathcal{M}$ sur $\mathbb{V}^\star$ (voir [3] pour $\mathbb{V}^\star$ et [4] pour $\mathcal{M}^\star$) et $\Phi : \mathbb{V} \to \mathbb{V}^\star$ a été défini plus haut. Concrètement :
- $|\mathcal{M}^{op}| = |\mathcal{M}|$,
- Pour $X, Y \in |\mathcal{M}^{op}|, \mathcal{M}^{op}(X,Y) = \mathcal{M}(Y,X)$ et $U^{op}_{XY} = U_{YX}$,
- Pour $X \xrightarrow{f} Y \xrightarrow{g} Z$ dans $\mathcal{M}^{op}$ on a $g \otimes^{op} f = sym^\star(f \otimes g)$ (où $sym : U(g) \otimes U(f) \to U(f) \otimes U(g)$).

Proposition 3.6. : Soit $\mathbb{M}$ une catégorie enrichie dans $\mathbb{V}$. Alors :

$$\mu e(\mathbb{M}^{op}) = \mu e(\mathbb{M})^{op}$$

$\underline{Preuve}$: Soit $Z \xrightarrow{g} Y \xrightarrow{f} X$ des flèches de $\mu e(\mathbb{M})$. On pose $f = (A, \underline{f})$ et $g = (B, \underline{g})$ où $A \xrightarrow{\underline{f}} \mathbb{M}(Y,X)$ et $B \xrightarrow{\underline{g}} \mathbb{M}(Z,Y)$. Leur composé dans $\mu e(\mathbb{M}^{op})$ est $(B \otimes A, h)$ où h est la flèche composée
$B \otimes A \xrightarrow{\underline{g} \otimes \underline{f}} \mathbb{M}(Z,Y) \otimes \mathbb{M}(Y,X) \xrightarrow{sym} \mathbb{M}(Y,X) \otimes \mathbb{M}(Z,Y) \xrightarrow{comp} \mathbb{M}(Z,X)$.
Leur composé dans $\mu e(\mathbb{M})^{op}$ est $(B \otimes A, k)$ où k est la flèche composée

$B \otimes A \xrightarrow{sym} A \otimes B \xrightarrow{f \otimes g} \mathbb{M}(Y, X) \otimes \mathbb{M}(Z, Y) \xrightarrow{comp} \mathbb{M}(Z, X)$.

Or $h = k$. D'où l'identité des deux compositions. Le reste de la preuve est sans difficulté.

4. La catégorie mutante $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$

Convention: A partir de cette section, on se fixe une catégorie monoïdale symétrique $\mathbb{V}$.

Construction de $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$ (où $\mathcal{M}$ et $\mathcal{N}$ sont des catégories mutantes sur $\mathbb{V}$):

La catégorie mutante $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$ est définie comme suit :

- $|\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})| = |Cat\mu(\mathbb{V})(\mathcal{M}, \mathcal{N})|$ (voir la section1).

- Si $F, G \in |\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})|$, une flèche $F \to G$ est un couple (A, f) où $A \in |\mathbb{V}|$ et $f = (FX \xrightarrow{f_X} GX)_{X \in |\mathcal{M}|}$ une famille de flèches de $\mathcal{N}$, vérifiant les conditions suivantes :

(TNV 1) Pour tout $X \in |\mathcal{M}|, U(f_X) = A$,

(TNV 2) Pour toute flèche $X \xrightarrow{x} X'$ de $\mathcal{M}$, $sym^\star(Gx \otimes f_X) = f_{X'} \otimes Fx$, où $sym : A \otimes U(x) \to U(x) \otimes A$ (ou encore, il existe une flèche $f_{X'} \otimes Fx \xrightarrow{\alpha} Gx \otimes f_X$ dans $\mathcal{N}(FX, GX')$ telle que $U(\alpha) = sym$).

- $(A, f), (A', f') : F \to G$ étant deux flèches parallèles dans $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$, une 2-cellule $(A, f) \to (A', f')$ est une flèche $A \xrightarrow{a} A'$ dans $\mathbb{V}$ telle que $\forall X \in |\mathcal{M}|, a^\star(f'_X) = f_X$.

.. La composition des flèches de $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$ est donnée, pour

$F \xrightarrow{(A,f)} G \xrightarrow{(B,g)} H$ par $(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où $g \circ f$ est la famille $(FX \xrightarrow{g_X \otimes f_X} HX)_{X \in |\mathcal{M}|}$ (pour la justifier on utilise l'axiome hexagonal de la symétrie dans les catégories monoïdales symétriques).

.. Si $F \in |\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})|$, alors $Id_F = (I, \underline{Id}_F)$ où $\underline{Id}_F = (Id_{FX})_{X \in |\mathcal{M}|}$ (un autre axiome des catégories monoïdales symétrique est utile ici).

.. Pour la composition verticale des 2-cellules

$(A_0, f_0) \xrightarrow{a_0} (A_1, f_1) \xrightarrow{a_1} (A_2, f_2)$ dans $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})(F, G)$ il suffit de remarquer que $a_1.a_0 : (A_0, f_0) \to (A_2, f_2)$ est une flèche de $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})(F, G)$

.. Pour la composition horizontale des 2-cellules $a_0 : (A_0, f_0) \to (A'_0, f'_0) : F_0 \to F_1$ avec $a_1 : (A_1, f_1) \to (A'_1, f'_1) : F_1 \to F_2$, on montre que

$(A_1, f_1) \otimes (A_0, f_0) \xrightarrow{a_1 \otimes a_0} (A'_1, f'_1) \otimes (A'_0, f'_0)$ est une flèche de $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})(F_0, F_2)$ (immédiat).

.. Pour $(A, f) : F \to G$, alors $Id_{(A,f)} = ((A, f) \xrightarrow{Id_A} (A, f))$.

.. La construction du morphisme d'oubli $U : \mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N}) \to \mathbb{V}$ et le fait qu'il commute avec les compositions et les identités précédentes est tout aussi immédiat.

.. Pour $F, G \in \mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$, le fait que $U_{FG} : \mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N}) \to \mathbb{V}$ soit une fibration discrète se montre sans difficulté particulière.

.. $F \xrightarrow{(A,f)} G$ étant une flèche de $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$ on vérifie que

$$(A, f) \otimes Id_F \xrightarrow{u_d} (A, f) \quad \text{et} \quad Id_G \otimes (A, f) \xrightarrow{u_g} (A, f)$$

sont des 2-cellules de $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$ (cela provient de la structure bicatégorique de $\mathcal{N}$ préservée par U)

.. De même, dans la situation suivante $F_0 \xrightarrow{(A_0, f_0)} F_1 \xrightarrow{(A_1, f_1)} F_2 \xrightarrow{(A_2, f_2)} F_3$, on vérifie que

$ass : ((A_2, f_2) \otimes (A_1, f_1)) \otimes (A_0, f_0) \to (A_2, f_2) \otimes ((A_1, f_1) \otimes (A_0, f_0))$

est une 2-cellule de $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$ (cela résulte, la encore, de la structure de bicatégorie de $\mathcal{N}$ préservée par U).

.. Pour achever de prouver que $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$ est une catégorie mutante sur $\mathbb{V}$ il suffit d'appliquer le critère 1.10.

• Le lemme suivant est utile pour les compositions strictes dans $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$.

Lemme 4.1. Soient E, F, G, H quatre objets de $\mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N})$ et trois flèches composables :

$$E \xrightarrow{(I,f)} F \xrightarrow{(A,g)} G \xrightarrow{(I,h)} H$$

On a les compositions strictes suivantes :
1) $(A, g)(I, f) = (A, gf)$ où $gf = (g_X f_X : EX \to GX)_{X \in |\mathcal{M}|}$.
2) $(I, h)(A, g) = (A, hg)$ où $hg = (h_X g_X : FX \to HX)_{X \in |\mathcal{M}|}$.

Preuve : Sans difficulté.

Proposition 4.2. On a l'isomorphisme $\underline{\mathbb{V} - \mu(\mathcal{M}, \mathcal{N})} \simeq Cat\mu(\mathbb{V})(\mathcal{M}, \mathcal{N})$.

Preuve : L'isomorphisme $\gamma : Cat\mu(\mathbb{V})(\mathcal{M}, \mathcal{N}) \to \underline{\mathbb{V} - \mu(\mathcal{M}, \mathcal{N})}$ est défini sur les objets par $|\gamma| = Id$ et sur les flèches, pour $F \xrightarrow{t} G$ on pose

$\gamma(t) = (I, t)$. On vérifie facilement que γ est un foncteur puis que c'est un isomorphisme.

• Donnons nous maintenant deux foncteurs mutants :

$$\mathcal{M}' \xrightarrow{\phi} \mathcal{M} \quad \text{et} \quad \mathcal{N} \xrightarrow{\psi} \mathcal{N}'$$

Grâce à eux, on construit de nouveaux foncteurs mutants :

$$\mathbb{V}-\mu(\mathcal{M},\mathcal{N}) \xrightarrow{\mathbb{V}-\mu(\phi, Id)} \mathbb{V}-\mu(\mathcal{M}',\mathcal{N}) \quad \text{et} \quad \mathbb{V}-\mu(\mathcal{M},\mathcal{N}) \xrightarrow{\mathbb{V}-\mu(Id,\psi)} \mathbb{V}-\mu(\mathcal{M},\mathcal{N}')$$

On les obtient comme suit :

Pour $\mathbb{V}$**-**$\mu(\phi, Id)$ (Écrivons le Φ pour simplifier).
- Pour $F \in |\mathbb{V}$-$\mu(\mathcal{M},\mathcal{N})|$, $\Phi(F) = F.\phi$ (composition des foncteurs mutants. Voir [4].)
- Sur une flèche $F \xrightarrow{(A,f)} G$, on a $\Phi(A, f) = (A, f.\phi)$ où
$f.\phi = (F\phi(X') \xrightarrow{f_{\phi X'}} G\phi(X'))_{X' \in |\mathcal{M}'|}$. On vérifie facilement que
$F.\phi \xrightarrow{(A, f.\phi)} G.\phi$ est une flèche de $\mathbb{V}$-$\mu(\mathcal{M}',\mathcal{N})$.
- Sur une 2-cellule $(A, f) \xrightarrow{a} (A', f') : F \to G$ de $\mathbb{V}$-$\mu(\mathcal{M}',\mathcal{N})$, on vérifie
facilement que $a : \Phi(A, f) \to \Phi(A', f')$ est une 2-cellule de $\mathbb{V}$-$\mu(\mathcal{M}',\mathcal{N})$.
Pour achever de prouver que $\Phi = \mathbb{V}$-$\mu(\phi, Id)$ est un foncteur mutant il suffit
d'appliquer le critère 1.11.

Pour $\mathbb{V}$**-**$\mu(Id, \psi)$ (Écrivons le Ψ pour simplifier).
- Pour $F \in |\mathbb{V}$-$\mu(\mathcal{M},\mathcal{N})|$, $\Psi(F) = \psi.F$ (composition des foncteurs mutants)
- Sur une flèche $F \xrightarrow{(A,f)} G$, on a $\Psi(A, f) = (A, \psi.f)$ où
$\psi.f = (\psi(FX) \xrightarrow{\psi(f_X)} \psi(GX))_{X \in |\mathcal{M}|}$.
- Sur une 2-cellule $(A, f) \xrightarrow{a} (A', f') : F \to G$ de $\mathbb{V}$-$\mu(\mathcal{M},\mathcal{N})$ de $\mathbb{V}$-$\mu(\mathcal{M},\mathcal{N})$, on vérifie
facilement que $a : \Psi(A, f) \to \Psi(A', f')$ est une 2-cellule de $\mathbb{V}$-$\mu(\mathcal{M},\mathcal{N}')$.
On applique enfin le critère **??** pour achever de prouver que $\Psi = \mathbb{V}$-$\mu(Id, \psi)$
est un foncteur mutant.

• La proposition suivante nous sera utile à la section 9.

Proposition 4.3. Lorsque $\Psi : \mathcal{N} \to \mathcal{N}'$ est pleinement fidèle (voir la définition 2.2) il en va de même pour
$\mathbb{V}\text{-}\mu(Id, \psi) : \mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N}) \to \mathbb{V}\text{-}\mu(\mathcal{M}, \mathcal{N}')$.

<u>Preuve</u> : Se montre sans difficulté .

5. La catégorie mutante $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$

• $\mathbb{M}$ et $\mathbb{N}$ étant des catégories enrichies dans $\mathbb{V}$ on cherche à décrire plus simplement la catégorie mutante $\mathbb{V}\text{-}\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$. Notre but est d'arriver à retrouver la catégorie enrichie $\mathbb{V}[\mathbb{M}, \mathbb{N}]$ (voir [1] prop. 6.3.1) lorsque $\mathbb{V}$ a suffisamment de propriétés pour y parvenir.

• **Construction de $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$:**
La catégorie mutante $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$ est définie comme suit :
- $|\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})| = |\mathbb{V}\text{-}Cat(\mathbb{M}, \mathbb{N})|$
- Pour $F, G \in |\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})|$, une flèche $F \to G$ est un couple (A, f) où $A \in |\mathbb{V}|$ et $f = (A \xrightarrow{f_X} \mathbb{N}(FX, GX))_{X \in |\mathbb{M}|}$ une famille de flèches de $\mathbb{V}$ vérifiant la condition suivante :
(TNVE) Pour tout $(X, X') \in |\mathbb{M}|^2$ le carré suivant commute dans $\mathbb{V}$:

$$
\begin{array}{ccc}
A \otimes \mathbb{M}(X, X') & \xrightarrow{f_X \otimes Id} & \mathbb{N}(FX, GX) \otimes \mathbb{M}(X, X') \\
{\scriptstyle f_{X'} \otimes Id} \downarrow & & \downarrow {\scriptstyle v_{XX'}} \\
\mathbb{N}(FX', GX') \otimes \mathbb{M}(X, X') & \xrightarrow{u_{XX'}} & \mathbb{N}(FX, GX')
\end{array}
$$

où $u_{XX'}$ et $v_{XX'}$ sont les flèches composées suivantes :

$u_{XX'} = [\mathbb{N}(FX', GX') \otimes \mathbb{M}(X, X') \xrightarrow{Id \otimes F_{XX'}} \mathbb{N}(FX', GX') \otimes \mathbb{N}(FX, FX') \xrightarrow{comp} \mathbb{N}(FX, GX')]$
et $v_{XX'} =$

$\mathbb{N}(FX, GX) \otimes \mathbb{M}(X, X') \xrightarrow{s} \mathbb{M}(X, X') \otimes \mathbb{N}(FX, GX) \xrightarrow{\underline{G} \otimes Id} \mathbb{N}(GX, GX') \otimes \mathbb{N}(FX, GX) \xrightarrow{c} \mathbb{N}(FX, GX')$

où $s = sym$, $c = comp$, $\underline{G} = G_{XX'}$
- $(A, f), (A', f') : F \to G$ étant deux flèches dans $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$, une 2-cellule $(A, f) \to (A', f')$ est une flèche $A \xrightarrow{a} A'$ dans $\mathbb{V}$ telle que, pour tout

$X \in |\mathbb{M}|$ le triangle suivant commute :

.. La composition des flèches de $\mathbb{V}$-$e(\mathbb{M}, \mathbb{N})$ est donnée, pour
$F \xrightarrow{(A,f)} G \xrightarrow{(B,g)} H$ par $(B, g) \otimes (A, f) = (B \otimes A, g \circ f)$ où
$g \circ f = (g_X \circ f_X)_{X \in |\mathbb{M}|}$ avec

$$g_X \circ f_X = (B \otimes A \xrightarrow{g_X \otimes f_X} \mathbb{N}(GX, HX) \otimes \mathbb{N}(FX, GX) \xrightarrow{comp} \mathbb{N}(FX, HX))$$

(voir la composition dans $\mu e(\mathbb{N})$).
.. Si $F \in |\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})|$, alors $Id_F = (I, \underline{Id}_F)$ où
$\underline{Id}_F = (I \xrightarrow{id_{FX}} \mathbb{N}(FX, FX))_{X \in |\mathbb{M}|}$.
.. Pour le reste de la construction de $\mathbb{V}$-$e(\mathbb{M}, \mathbb{N})$ on procède comme pour
celle de $\mathbb{V}$-$\mu(\mathcal{M}, \mathcal{N})$.

Proposition 5.1. On a $\mathbb{V}$-$e(\mathbb{M}, \mathbb{N}) \simeq \mathbb{V}$-$\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$.

Preuve : L'isomorphisme $\varepsilon : \mathbb{V}$-$e(\mathbb{M}, \mathbb{N}) \to \mathbb{V}$-$\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$ se con-
struit comme suit :
- Sur un objet $F \in |\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})|$, $\varepsilon(F) = \mu e(F)$ (voir la définition de μe à
la section1).
- sur une flèche $F \xrightarrow{(A,f)} G$, $\varepsilon(A, f) = (A, \bar{f})$ où pour $X \in |\mathbb{M}|$,
$\bar{f}_X = (A, f_X) : FX \to GX$ dans $\mu e(\mathbb{N})$. (Pour montrer que $\varepsilon(A, f) :$
$\varepsilon(F) \to \varepsilon(G)$ est une flèche de $\mathbb{V}$-$\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$ il faut montrer que pour
tout $X \xrightarrow{x} X'$ dans $\mu e(\mathbb{M})$ on a :
$sym^\star(\mu e(G)(x) \otimes \bar{f}_X) = \bar{f}_{X'} \otimes (\mu e(F)(x)$ (où $A \otimes U(x) \xrightarrow{sym} U(x) \otimes A)$.
Écrivons déjà $x = (B, \underline{x})$ où $B \xrightarrow{x} \mathbb{M}(X, X')$ est une flèche de $\mathbb{V}$. Alors
$\mu e(G)(x) \otimes \bar{f}_X = (B \otimes A, c_d)$ et $\bar{f}_{X'} \otimes \mu e(F)(x) = (A \otimes B, c_g)$ où c_d et
c_g sont définis par :
$c_d = [B \otimes A \xrightarrow{(G_{XX'}.\underline{x}) \otimes f_X} \mathbb{N}(GX, GX') \otimes \mathbb{N}(FX, GX) \xrightarrow{comp} \mathbb{N}(FX, GX')]$
$c_g = [A \otimes B \xrightarrow{f_{X'} \otimes (F_{XX'}.\underline{x})} \mathbb{N}(FX', GX') \otimes \mathbb{N}(FX, FX') \xrightarrow{comp} \mathbb{N}(FX, GX')]$
Or $sym^\star(B \otimes A, c_d) = (A \otimes B, c_d.sym)$. Il reste donc à montrer que

$c_d.sym = c_g$. Mais cela résulte des identités suivantes :

$c_d.sym = v_{XX'}.(f_X \otimes Id).(Id_A \otimes \underline{x}) = u_{XX'}.(f_{X'} \otimes Id).(Id \otimes \underline{x}) = c_g$.

- Sur une 2-cellule $(A, f) \xrightarrow{a} (A', f')$, il faut simplement vérifier que a : $\varepsilon(A, f) \to \varepsilon(A', f')$ est une 2-cellule de $\mathbb{V}\text{-}\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$ ce qui est immédiat.

.. Le reste de la preuve que ε est un foncteur mutant est sans difficulté.

.. Montrons que ε est un isomorphisme. Sur les objets on utilise le fait que $\mu e : \mathbb{V}\text{-}Cat \to Cat\mu(\mathbb{V})$ est pleinement fidèle.

Sur les flèches : Soit $(A, f) : \varepsilon(F) \to \varepsilon(G)$ dans $\mathbb{V}\text{-}\mu(\mu e(\mathbb{M}), \mu e(\mathbb{N}))$. On pose $\underline{f} = (A \xrightarrow{f_X} \mathbb{N}(FX, GX))_{X \in |\mathbb{M}|}$ où $f_X = (A, \underline{f}_X)$ (flèche de $\mu e(\mathbb{N})$) . On montre ensuite que $(A, \underline{f})$ est dans $|\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})(F, G)|$. Pour cela, soit $(X, X') \in |\mathbb{M}|^2$. Il faut montrer que le carré (C) (dans l'expression de l'axiome (TNVE)) commute. On considère la flèche $X \xrightarrow{x} X'$ de $\mu e(\mathbb{M})$ définie par $x = (\mathbb{M}(X, X'), Id_{\mathbb{M}(X,X')})$. Par hypothèse on a

$sym^\star(Gx \otimes f_X) = f_{X'} \otimes Fx$. En explicitant cette identité on obtient la commutation du carré (C) ce qui répond à la question. On a $\varepsilon(A, \underline{f}) = (A, f)$. l'unicité de $(A, \underline{f})$ est immédiate. ε est donc bijectif sur les flèches. On n'a plus qu'à appliquer le critère 1.13 pour achever de prouver que ε est un isomorphisme.

Proposition 5.2. Sous les hypothèses suivantes :
- $\mathbb{V}$ est fermée (en plus d'être symétrique),
- $\mathbb{V}$ est complète,
- $|\mathbb{M}|$ est petit,

Alors $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$ est de saveur E.

Preuve : Fixons $F, G \in |\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})|$. On va montrer que $Tri(-, F, G) :$ $\mathbb{V}^{op} \to \mathbb{E}ns$ est représentable. $\mathbb{V}$ étant fermée, pour chaque $A \in |\mathbb{V}|$, notons $\mathbb{V}^e(A, -)$ un adjoint à droite de $(-) \otimes A$ et $Ev^A : \mathbb{V}^e(A, -) \otimes A \to (-)$ une co-unité de cette adjonction. Notons encore $\mathbb{V}^e$ la catégorie enrichie sur $\mathbb{V}$ obtenue grâce à ces choix. Notons encore pour $(A_i)_{i \in I}$, une petite famille d'objets de $\mathbb{V}$, $\prod_{i \in I} A_i \xrightarrow{p_i} A_i$ les projections "canoniques". A partir des flèches $u_{XX'}$ et $v_{XX'}$ données pendant la construction de $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$ (où $X, X' \in |\mathbb{M}|$) on construit de nouvelles flèches

$\tilde{u}_{XX'} : \mathbb{N}(FX', GX') \to \mathbb{V}^e(\mathbb{M}(X, X'), \mathbb{N}(FX, GX'))$ et
$\tilde{v}_{XX'} : \mathbb{N}(FX, GX) \to \mathbb{V}^e(\mathbb{M}(X, X'), \mathbb{N}(FX, GX'))$

telles que $Ev.(\tilde{u}_{XX'} \otimes Id) = u_{XX'}$ et $Ev.(\tilde{v}_{XX'} \otimes Id) = v_{XX'}$. Puis des flèches $u, v : \prod_{X \in |\mathbb{M}|} \mathbb{N}(FX, GX) \to \prod_{(X,X') \in |\mathbb{M}|^2} \mathbb{V}^e(\mathbb{M}(X, X'), \mathbb{N}(FX, GX'))$ telles que $\forall X, X' \in |\mathbb{M}|^2 \ p_{(X,X')}.u = \tilde{u}_{XX'}.p_{X'}$ et $p_{(X,X')}.v = \tilde{v}_{XX'}.p_X$. Enfin notons $[F, G] \xrightarrow{i} \prod_{X \in |\mathbb{M}|} \mathbb{N}(FX, GX)$ l'égalisateur de (u, v). D'un autre côté, pour chaque $X \in |\mathbb{M}|$, notons π_X le composé suivant

$$[F, G] \xrightarrow{i} \prod_{X \in |\mathbb{M}|} \mathbb{N}(FX, GX) \xrightarrow{p_X} \mathbb{N}(FX, GX)$$

$\underline{ev}_{FG}$ la famille $([F, G] \xrightarrow{\pi_X} \mathbb{N}(FX, GX))_{X \in |\mathbb{M}|}$ et $ev_{FG} = ([F, G], \underline{ev}_{FG})$. Constatons déjà que $ev_{FG} \in Tri([F, G], F, G)$ et montrons que le couple $([F, G], ev_{FG})$ est une représentation de $Tri(-, F, G)$. Soit $(A, f) : F \to G$ une flèche de $\mathbb{V}\text{-}e(\mathbb{M}, \mathbb{N})$ (c.a.d. $(A, f) \in Tri(A, F, G)$). Posons $h : A \to \prod_{X \in |\mathbb{M}|} \mathbb{N}(FX, GX)$ l'unique flèche telle que $\forall X \in |\mathbb{M}|, p_X.h = f_X$. On vérifie que h égalise (u, v). Soit alors $k : A \to [F, G]$ l'unique flèche telle que $i.k = h$. Elle vérifie $k^\star(ev_{FG}) = (A, f)$. Elle est même unique pour cette propriété. Cela achève de prouver que $Tri(-, F, G)$ est représentable. D'où la conclusion voulue.

Remarque 5.3. La catégorie enrichie (dans $\mathbb{V}$) canonique associée au choix de représentation donné dans la proposition précédente est notée $\mathbb{V}[\mathbb{M}, \mathbb{N}]$. C'est la notation donnée dans [1], prop.6.3.1, pour cette même catégorie enrichie.

Proposition 5.4. On a $\underline{\mathbb{V}[\mathbb{M}, \mathbb{N}]} \simeq \mathbb{V}\text{-}Cat(\mathbb{M}, \mathbb{N})$.

Preuve : Cet isomorphisme est le composé des isomorphismes suivants :

$$\underline{\mathbb{V}[\mathbb{M}, \mathbb{N}]} \simeq \underline{\mathbb{V} - e(\mathbb{M}, \mathbb{N})} \simeq \underline{\mathbb{V} - Cat(\mathbb{M}, \mathbb{N})}$$

6. Le plongement de Yoneda mutant.

• Rappelons que $\mathbb{V}$ est supposée symétrique.
• On se donne maintenant une catégorie mutante $\mathcal{M}$ sur $\mathbb{V}$ telle que : $\forall A \in |\mathbb{V}|, \forall X, Y \in |\mathcal{M}|, Tri(A, X, Y) \in |\mathbb{E}ns|$. On va construire un foncteur mutant $\mathcal{M} \xrightarrow{yon} \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ où $\mathcal{M}^{op}$ a été défini à la section 3 et

où $\hat{\mathcal{V}} = \mu c(\hat{\mathbb{V}})$, $\hat{\mathbb{V}}$ étant un cas particulier de la catégorie cotensorisée $\hat{\mathbb{E}}$ définie à la section 2.

- Pour cela, on part du foncteur $Tri : \mathbb{V}^{op} \times \underline{\mathcal{M}}^{op} \times \underline{\mathcal{M}} \to \mathbb{E}ns$. Grâce à fermeture de $\mathbb{C}AT$, on obtient un nouveau foncteur $hom : \underline{\mathcal{M}}^{op} \times \underline{\mathcal{M}} \to \hat{\mathbb{V}}$ ($\hat{\mathbb{V}}$ étant la catégorie des préfaisceaux sur $\mathbb{V}$).

- Soit maintenant $X \in |\mathcal{M}|$. On construit un foncteur mutant $yon(X) : \mathcal{M}^{op} \to \hat{\mathcal{V}}$.

.. Sur un objet $Y \in |\mathcal{M}^{op}| = |\mathcal{M}|$, $yon(X)(Y) = hom(Y, X)$.

.. Sur une flèche $Y \xrightarrow{f} Z$ de $\mathcal{M}^{op}$ (alors $f : Z \to Y$ est une flèche de $\mathcal{M}$).(on pose $A = U(f)$) considérons la flèche $h(f, X) : hom(Y, X) \to hom(Z, X)^A$ (pour cette dernière notation voir la section 2) dans $\hat{\mathbb{V}}$ définie en $B \in |\mathbb{V}|$ par $h(f, X)_B(g) = f \otimes^{op} g$ (on a bien $U(f \otimes^{op} g) = A \otimes B$ et donc $f \otimes^{op} g \in Tri(A \otimes B, Z, X) = hom(Z, X)^A(B)$). On peut alors poser $yon(X)(f) = (A, h(f, X)) : yon(X)(Y) \to yon(X)(Z)$.

.. Sur une 2-cellule $\alpha : f \to f' : Y \to Z$ de $\mathcal{M}^{op}$ (alors $\alpha : f \to f' : Z \to Y$ dans $\mathcal{M}$) (notons $(A \xrightarrow{a} A') = U(f \xrightarrow{\alpha} f')$). On vérifie que $a : (A, h(f, X)) \to (A', h(f', X))$ est une flèche de $\hat{\mathcal{V}}$, ce qui résulte de la commutation du triangle suivant dans $\hat{\mathbb{V}}$:

$$
\begin{array}{ccc}
 & hom(Y, X) & \\
{\scriptstyle h(f', X)} \swarrow & & \searrow {\scriptstyle h(f, X)} \\
hom(Z, X)^{A'} & \xrightarrow[Id^a]{} & hom(Z, X)^A
\end{array}
$$

On peut alors poser $yon(X)(\alpha) = (yon(X)(f) \xrightarrow{a} yon(X)(f'))$.

.. Montrons qu'on obtient ainsi un foncteur mutant $\mathcal{M}^{op} \to \hat{\mathcal{V}}$.

- Pour tout $Y \in |\mathcal{M}^{op}|$, $yon(X)(Id_Y) = Id_{yon(X)(Y)}$ (En effet $yon(X)(Id_Y) = (I, h(Id_Y, X))$ et $Id_{yon(X)(Y)} = (I, \sigma)$ où, pour $B \in |\mathbb{V}|$, et $g \in Tri(B, Y, X)$, $\sigma_B(g) = u_g^\star(g) = h(Id_Y, X)_B(g)$ d'où l'identité voulue).

- Pour un couple de flèches composables $Y \xrightarrow{g} Z \xrightarrow{k} T$ dans $\mathcal{M}^{op}$, on a $yon(X)(k \otimes^{op} g) = yon(X)(k) \otimes yon(X)(g)$ (En effet, si on pose $B = U(g)$ et $C = U(k)$, alors $yon(X)(k \otimes^{op} g) = (C \otimes B, h(k \otimes^{op} g, X))$ où $h(k \otimes^{op} g, X) : hom(Y, X) \to hom(T, X)^{C \otimes B}$ est la flèche de $\hat{\mathbb{V}}$ définie en $D \in |\mathbb{V}|$ par $h(k \otimes^{op} g, X)_D(f) = (k \otimes^{op} g) \otimes^{op} f$. D'un autre côté $yon(X)(k) \otimes yon(X)(g) = (C, h(k, X)) \otimes (B, h(g, X)) = (C \otimes B, \delta)$ où

$\delta = h(k, X) \circ h(g, X)$ est la flèche composée suivante:

$hom(Y, X) \overset{h(g,X)}{\to} hom(Z, X)^B \overset{h(k,X)^B}{\to} (hom(T, X)^C)^B \overset{\alpha m}{\to} hom(T, X)^{C \otimes B}$

Appliquée en $D \in |\mathbb{V}|$ à $f \in Tri(D, Y, X)$ donne $\delta_D(f) = ass^\star(k \otimes^{op} (g \otimes^{op} f)) = (k \otimes^{op} g) \otimes^{op} f = h(k \otimes^{op} g, X)_D(f)$. D'où l'égalité voulue).

Il reste à appliquer le critère 1.11 pour conclure que $yon(X)$ est un foncteur mutant.

- Construisons maintenant yon sur une flèche. Soit $X \overset{f}{\to} Y$ une flèche de $\mathcal{M}$. On construit $yon(f) : yon(X) \to yon(Y)$ dans $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$. Posons $A = U(f)$. On pose aussi $yon(f) = (A, \underline{yon(f)})$ où $\underline{yon(f)} = (yon(f)_Z)_{Z \in |\mathcal{M}^{op}|}$ avec $yon(f)_Z = (A, h(Z, f)))$ et $\overline{h(Z, f)}$: $hom(Z, X) \to hom(Z, Y)^A$ est défini en B sur $g \in Tri(B, Z, X)$, par $h(Z, f)_B(g) = f \otimes g$ (comme $U(f \otimes g) = A \otimes B$, $f \otimes g$ est bien dans $Tri(A \otimes B, Z, Y) = h(Z, Y)^A(B)$). $yon(f)$ est une flèche de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ car $\forall Z \in |\mathcal{M}^{op}|, U(yon(f)_Z) = A$ et pour toute flèche $Z \overset{k}{\to} T$ de $\mathcal{M}^{op}$, si on pose $C = U(k)$, on a

$$sym^\star(yon(Y)(k) \otimes yon(f)_Z) \underset{(I)}{=} yon(f)_T \otimes yon(X)(k)$$

(où $C \otimes A \overset{sym}{\to} A \otimes C$). En effet $yon(Y)(k) \otimes yon(f)_Z = (C, h(k, Y)) \otimes (A, h(Z, f)) = (C \otimes A, h(k, Y) \circ h(Z, f))$ et $yon(f)_T \otimes yon(X)(k) = (A, h(T, f)) \otimes (C, h(k, X)) = (A \otimes C, h(T, f) \circ h(k, X))$. On aura alors l'identité (I) si le triangle (T) suivant commute dans $\hat{\mathbb{V}}$:

Or $h(k, Y) \circ h(Z, f)$ est le composé suivant :

$hom(Z, X) \overset{h(Z,f)}{\to} hom(Z, Y)^A \overset{h(k,Y)^A}{\to} (hom(T, Y)^C)^A \overset{\alpha m}{\to} hom(T, Y)^{C \otimes A}$

et $h(T, f) \circ h(k, X)$ est le composé :

$hom(Z, X) \overset{h(k,X)}{\to} hom(T, X)^C \overset{h(T,f)^C}{\to} (hom(T, Y)^A)^C \overset{\alpha m}{\to} hom(T, Y)^{A \otimes C}$

En appliquant en $B \in |\mathbb{V}|$ à $g \in Tri(B, Z, X)$ les deux composés possibles

on obtient $(h(k, Y) \circ h(Z, f))_B(g) = ass^\star(k \otimes^{op} (f \otimes g))$ où ici
$ass : (C \otimes A) \otimes B \to C \otimes (A \otimes B)$ et $(h(T, f) \circ h(k, X))_B(g) = ass'^\star(f \otimes (k \otimes^{op} g))$ où cette fois $ass' : (A \otimes C) \otimes B \to A \otimes (C \otimes B)$.
Ainsi, pour montrer la commutation du triangle (T) il nous suffit de montrer l'identité suivante pour tout g :

$$(sym \otimes Id)^\star ass'^\star(f \otimes (k \otimes^{op} g)) = ass^\star(k \otimes^{op} (f \otimes g))$$

ou encore, en utilisant la définition de $\otimes^{op}$, on est ramené à l'identité :
$F_g^\star(f \otimes (g \otimes k)) = F_d^\star(f \otimes (g \otimes k))$ où F_g et F_d sont les composés suivants :
$((C \otimes A) \otimes B \xrightarrow{sym \otimes Id} (A \otimes C) \otimes B \xrightarrow{ass'} A \otimes (C \otimes B) \xrightarrow{Id \otimes sym} A \otimes (B \otimes C))$
$((C \otimes A) \otimes B \xrightarrow{ass} C \otimes (A \otimes B) \xrightarrow{sym} (A \otimes B) \otimes C \xrightarrow{ass} A \otimes (B \otimes C))$.
Mais $F_g = F_d$ (par définition d'une symétrie dans une catégorie monoïdale symétrique). D'où l'identité (I) annoncée.
- Pour construire yon sur une 2-cellule, par exemple $\alpha : f \to f' : X \to Y$, (on pose $(A \xrightarrow{a} A') = U(f \xrightarrow{\alpha} f')$) il nous faut montrer que $a : yon(f) \to yon(f')$ est une 2-cellule de $\mathbb{V}$-$\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$, c'est-à-dire que, pour tout Z dans $|\mathcal{M}^{op}|$, $a^\star(yon(f')_Z) = yon(f)_Z$, ou encore, que le triangle suivant commute dans $\hat{\mathbb{V}}$:

$$
\begin{array}{ccc}
& hom(Z, X) & \\
{\scriptstyle h(Z,f')} \swarrow & & \searrow {\scriptstyle h(Z,f)} \\
hom(Z, Y)^{A'} \xrightarrow[Id^a]{\hspace{3cm}} & & hom(Z, Y)^A
\end{array}
$$

ce que l'on vérifie facilement. On pose alors $yon(\alpha) = (yon(f) \xrightarrow{a} yon(f'))$.
- Pour finir de montrer que $\mathcal{M} \xrightarrow{yon} \mathbb{V}$-$\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ est un foncteur mutant il nous reste à montrer essentiellement les identités suivantes :
1) Pour tout $X \in |\mathcal{M}|$, $yon(Id_X) = Id_{yon(X)}$.
2) Pour tout $X \xrightarrow{f} Y \xrightarrow{g} Z$ dans $\mathcal{M}$, on a $yon(g \otimes f) = yon(g) \otimes yon(f)$.
(1) On a $Id_{yon(X)} = (I, \underline{Id}_{yon(X)})$ où $\underline{Id}_{yon(X)} = (Id_{hom(Z,X)})_{Z \in |\mathcal{M}|}$ et où $Id_{hom(Z,X)}$ est une identité dans $\hat{\mathcal{V}}$ ce qui s'écrit $Id_{hom(Z,X)} = (I, \sigma)$ où $hom(Z, X) \xrightarrow{\sigma} hom(Z, X)^I$. D'un autre côté $yon(Id_X) = (I, \underline{yon}(Id_X))$ où $\underline{yon}(Id_X) = (hom(Z, X) \xrightarrow{(I, h(Z, Id_X))} hom(Z, X))_{Z \in |\mathcal{M}|})$. Or on vérifie facilement que $h(Z, Id_X) = \sigma$. D'où l'identité voulu.

(2) Il nous suffit de vérifier que pour tout $S \in |\mathcal{M}|$ on a $h(S, g \otimes f) = h(S, g) \circ h(S, f)$ ou encore que le diagramme suivant commute :

$$\begin{array}{ccc} & hom(S, Y)^A \xrightarrow{h(S,g)^A} (hom(S, Z)^B)^A & \\ {\scriptstyle h(S,f)} \nearrow & & \searrow {\scriptstyle \alpha m} \\ hom(S, X) \xrightarrow{\hspace{2cm} h(S,g \otimes f) \hspace{2cm}} & & hom(S, Z)^{B \otimes A} \end{array}$$

ce que l'on vérifie facilement.

En utilisant le critère 1.11 on achève de prouver que $\mathcal{M} \xrightarrow{yon} \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ est un foncteur mutant.

7. Le plongement en saveur E.

• On suppose maintenant que la catégorie mutante $\mathcal{M}$ est de saveur E. Cela va nous permettre de construire un foncteur mutant $\mathcal{M} \xrightarrow{yo} \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \mathcal{V})$ où $\mathcal{V} = \mu t(\mathbb{V})$, $\mathbb{V}$ étant vue comme une catégorie tensorisée sur elle-même. Puis on vérifiera que le triangle suivant commute à isomorphisme près :

$$\begin{array}{ccc} \mathcal{M} \xrightarrow{\hspace{2cm} yo \hspace{2cm}} & \mathbb{V}-\mu(\mathcal{M}^{op}, \mathcal{V}) \\ {\scriptstyle yon} \searrow & \swarrow {\scriptstyle \mathbb{V}-\mu(Id, \mu y)} \\ & \mathbb{V}-\mu(\mathcal{M}^{op}, \hat{\mathcal{V}}) \end{array}$$

(pour les définitions des foncteurs mutants μy et $\mathbb{V}\text{-}\mu(Id, \mu y)$ voir les sections 2 et 4).

• $\mathcal{M}$ étant supposé de saveur E, cela signifie que pour tout X, Y dans $|\mathcal{M}|$, le préfaisceau $hom(X, Y)$ est représentable. Choisissons donc une représentation $(\mathbb{M}(X, Y), ev_{XY})$ de ce préfaisceau. Nous pouvons maintenant construire le foncteur mutant yo.

- Sur un objet $X \in |\mathcal{M}|$, on construit le foncteur mutant $yo(X) : \mathcal{M}^{op} \to \mathcal{V}$ en posant :

.. sur un objet $Y \in |\mathcal{M}^{op}|$, $yo(X)(Y) = \mathbb{M}(Y, X)$,

.. sur une flèche $Y \xrightarrow{f} Y'$ de $\mathcal{M}^{op}$ (donc $Y' \xrightarrow{f} Y$ est une flèche de $\mathcal{M}$) après avoir posé $B = U(f)$, on définit $yo(X)(Y) \xrightarrow{yo(X)(f)} yo(X)(Y')$ en posant

$yo(X)(f) = (B, \underline{yo(X)(f)})$ où $\underline{yo(X)(f)} : B \otimes \mathbb{M}(Y, X) \to \mathbb{M}(Y', X)$ est l'unique flèche de $\overline{\mathbb{V}}$ définie par $\underline{yo(X)(f)}^\star(ev_{Y'X}) = f \otimes^{op} ev_{YX}$,

.. sur une 2-cellule $\beta : f \to f' : Y \to Y'$ de $\mathcal{M}^{op}$, si on pose $(B \xrightarrow{b} B') = U(f \xrightarrow{\beta} f')$, on voit que $b : yo(X)(f) \to yo(X)(f')$ est une 2-cellule de $\mathcal{V}$ car $\underline{yo(X)(f')}.(b \otimes Id) = \underline{yo(X)(f)}$ (en effet $(b \otimes Id)^\star \underline{yo(X)(f')}^\star(ev_{Y'X}) = (b \otimes Id)^\star(f' \otimes^{op} ev_{YX}) = \overline{(b \otimes Id)^\star sym^\star(ev_{YX} \otimes f')} = sym^\star(Id \otimes b)^\star(ev_{YX} \otimes f') = sym^\star(ev_{YX} \otimes f) = f \otimes^{op} ev_{YX}$). On peut donc poser :

$yo(X)(f \xrightarrow{\beta} f') = (yo(X)(f) \xrightarrow{b} yo(X)(f'))$.

- Pour établir que $yo(X)$ est un foncteur mutant $\mathcal{M}^{op} \to \mathcal{V}$, on montre que :

.. Pour tout $Y \in |\mathcal{M}^{op}|$, on a $yo(X)(Id_Y) = Id_{yo(X)(Y)}$ (en effet $yo(X)(Id_Y) = (I, \underline{yo(X)(Id_Y)}) = (I, u_g) = Id_{\mathbb{M}(Y, X)} = Id_{yo(X)(Y)}$),

.. Pour tout couple de flèches composables $Y \xrightarrow{f} Y' \xrightarrow{f'} Y''$ dans $\mathcal{M}^{op}$ on a $yo(X)(f' \otimes^{op} f) = yo(X)(f') \otimes yo(X)(f)$ (En effet, posons $B = U(f), B' = U(f')$. On a $yo(X)(f') \otimes yo(X)(f) = (B' \otimes B, \underline{yo(X)(f')} \circ \underline{yo(X)(f)})$ où $\underline{yo(X)(f')} \circ \underline{yo(X)(f)})$ est la flèche composée suivante :

$$(B' \otimes B) \otimes \mathbb{M}(Y, X) \xrightarrow{ass} B' \otimes (B \otimes \mathbb{M}(Y, X)) \xrightarrow{Id \otimes \underline{yo(X)(f)}} B' \otimes \mathbb{M}(Y', X) \xrightarrow{\underline{yo(X)(f')}} \mathbb{M}(Y'', X).$$

Donc $(\underline{yo(X)(f')} \circ \underline{yo(X)(f)})^\star(ev_{Y''X}) = ass^\star(Id \otimes \underline{yo(X)(f)})^\star \underline{yo(X)(f')}^\star(ev_{Y''X}) = ass^\star(Id \otimes \underline{yo(X)(f)})^\star(f' \otimes^{op} ev_{Y'X}) = ass^\star(f' \otimes^{op} (f \otimes^{op} ev_{YX})) = (f' \otimes^{op} f) \otimes^{op} ev_{YX} = \underline{yo(X)(f' \otimes^{op} f)}^\star(ev_{Y''X})$. D'où l'identité voulue).

.. Pour le reste de la preuve, essentiellement, on applique le critère 1.11.

- Sur une flèche $X \xrightarrow{x} X'$ de $\mathcal{M}$ (où $A = U(x)$), on construit $yo(x) : yo(X) \to yo(X')$ en posant $yo(x) = (A, \underline{yo(x)})$ où $\underline{yo(x)} = (\mathbb{M}(Y, X) \xrightarrow{yo(x)_Y} \mathbb{M}(Y, X'))_{Y \in |\mathcal{M}^{op}|}$ et où $yo(x)_Y = (A, \underline{yo(x)}_Y)$, $\underline{yo(x)}_Y : A \otimes \mathbb{M}(Y, X) \to \mathbb{M}(Y, X')$ étant l'unique flèche de $\mathbb{V}$ telle que $\underline{yo(x)}_Y^\star(ev_{YX'}) = x \otimes ev_{YX}$. $yo(x)$ est bien une flèche $yo(X) \to yo(X')$ dans $\mathbb{V}$-$\mu(\mathcal{M}^{op}, \mathcal{V})$ (En effet, $Y \xrightarrow{f} Y'$ étant une flèche quelconque de $\mathcal{M}^{op}$ (on pose $B = U(f)$) on veut montrer que $sym_{A,B} : yo(x)_{Y'} \otimes yo(X)(f) \to yo(X')(f) \otimes yo(x)_Y$ est une 2-cellule de $\mathcal{V}$. Mais $yo(X')(f) \otimes yo(x)_Y = (B \otimes A, \underline{yo(X')(f)} \circ \underline{yo(x)}_Y)$ où $\underline{yo(X')(f)} \circ \underline{yo(x)}_Y$ est le composé F suivant :

$$(B \otimes A) \otimes \mathbb{M}(Y, X) \xrightarrow{ass} B \otimes (A \otimes \mathbb{M}(Y, X)) \xrightarrow{Id \otimes yo(x)_Y} B \otimes \mathbb{M}(Y, X') \xrightarrow{yo(X')(f)} \mathbb{M}(Y', X')$$

et $yo(x)_{Y'} \otimes yo(X)(f) = (A \otimes B, \underline{yo(x)}_{Y'} \circ \underline{yo(X)}(f))$ où
$\underline{yo(x)}_{Y'} \circ \underline{yo(X)}(f)$ est le composé G suivant:

$$(A \otimes B) \otimes \mathbb{M}(Y, X) \xrightarrow{ass} A \otimes (B \otimes \mathbb{M}(Y, X)) \xrightarrow{Id \otimes yo(X)(f)} A \otimes \mathbb{M}(Y', X) \xrightarrow{yo(x)_{Y'}} \mathbb{M}(Y', X')$$

. Il nous faut montrer que le triangle suivant commute :

Or $F^\star(ev_{Y'X'}) = (ass.sym.ass)^\star(x \otimes (ev_{YX} \otimes f))$
et $G^\star(ev_{Y'X'}) = ((Id \otimes sym).ass)^\star(x \otimes (ev_{YX} \otimes f))$ et donc
$(F.(sym \otimes Id))^\star(ev_{Y'X'}) = (ass.sym.ass.(sym \otimes Id))^\star(x \otimes (ev_{YX} \otimes f))$.
Mais $ass.sym.ass.(sym \otimes Id) = (Id \otimes sym).ass$ (axiome de symétrie)
donc $(F.(sym \otimes Id))^\star(ev_{Y'X'}) = G^\star(ev_{Y'X'})$ et ainsi $F.(sym \otimes Id)) = G$.
D'où la conclusion voulue.
- Sur une 2-cellule $\alpha : x \to x' : X \to X'$ de $\mathcal{M}$ (on pose $(A \xrightarrow{a} A') = U(x \xrightarrow{\alpha} x')$). Il nous faut montrer que $a : yo(x) \to yo(x') : yo(X) \to yo(X')$ est une 2-cellule de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \mathcal{V})$ ou encore que, pour tout Y dans $|\mathcal{M}^{op}|$, $a^\star(yo(x')_Y) = yo(x)_Y$ c.a.d. que le triangle suivant commute :

ce que l'on vérifie facilement.
- Montrons qu'on a ainsi construit un foncteur mutant $\mathcal{M} \to \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \mathcal{V})$.
.. Pour $X \in |\mathcal{M}|$, on a $yo(Id_X) = Id_{yo(X)}$ (En effet $Id_{yo(X)} = (I, \underline{Id}_{yo(X)})$
où $\underline{Id}_{yo(X)} = (Id_{\mathbb{M}(Y,X)})_{Y \in |\mathcal{M}^{op}|}$ et
$Id_{\mathbb{M}(Y,X)} = (I, I \otimes \mathbb{M}(Y, X) \xrightarrow{u_g} \mathbb{M}(Y, X))$. D'un autre côté $yo(Id_X) = (I, \underline{yo(Id_X)})$ où $\underline{yo(Id_X)} = (yo(Id_X)_Y)_{Y \in |\mathcal{M}^{op}|}$ avec $yo(Id_X)_Y = (I, \underline{yo(Id_X)}_Y)$ et $\underline{yo(Id_X)}_Y : I \otimes \mathbb{M}(Y, X) \to \mathbb{M}(Y, X)$ est l'unique flèche de $\mathbb{V}$ telle que $\underline{yo(Id_X)}_Y^\star(ev_{YX}) = Id_X \otimes ev_{YX}$. Mais alors $\underline{yo(Id_X)}_Y = u_g$,

d'où l'identité voulue).

.. Pour un couple de flèches composables $X \xrightarrow{x} X' \xrightarrow{x'} X"$ dans $\mathcal{M}$ on a $yo(x' \otimes x) = yo(x') \otimes yo(x)$ (En effet, posons $A = U(x), A' = U(x')$, alors $yo(x') \otimes yo(x) = (A' \otimes A, \underline{yo(x')} \circ \underline{yo(x)})$ où, pour tout $Y \in |\mathcal{M}^{op}|$, $(\underline{yo(x')} \circ \underline{yo(x)})_Y = \underline{yo(x')}_Y \otimes \underline{yo(x)}_Y = (A' \otimes A, \underline{yo(x')}_Y \circ \underline{yo(x)}_Y)$ et où $\underline{yo(x')}_Y \circ \underline{yo(x)}_Y$ est le composé suivant :

$$(A' \otimes A) \otimes \mathbb{M}(Y, X) \xrightarrow{ass} A' \otimes (A \otimes \mathbb{M}(Y, X)) \xrightarrow{Id \otimes \underline{yo(x)}_Y} A' \otimes \mathbb{M}(Y, X') \xrightarrow{\underline{yo(x')}_Y} \mathbb{M}(Y, X")$$

Or $ass^\star (Id \otimes \underline{yo(x)}_Y)^\star \underline{yo(x')}_Y^\star (ev_{YX"}) = (x' \otimes x) \otimes ev_{YX} = \underline{yo(x' \otimes x)}_Y^\star (ev_{YX"})$ et donc $\underline{yo(x')}_Y \circ \underline{yo(x)}_Y = \underline{yo(x' \otimes x)}_Y$. D'où l'identité voulue).

.. Pour finir de prouver que yo est un foncteur mutant on applique essentiellement le critère 1.11.

• Il nous faut maintenant établir que $\mathbb{V}\text{-}\mu(Id, \mu y).yo \simeq yon$. Notons $\bar{\mu} = \mathbb{V}\text{-}\mu(Id, \mu y)$ et $\bar{y} = \bar{\mu}.yo$. Alors, pour chaque $X \in |\mathcal{M}|$, $\bar{y}(X) = (\mathcal{M}^{op} \xrightarrow{yo(X)} \mu t(\mathbb{V}) \xrightarrow{\mu y} \mu c(\hat{\mathbb{V}}))$.

On construit maintenant une transformation naturelle mutante $\gamma : \bar{y} \to yon : \mathcal{M} \to \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$. Pour cela, on commence par considérer pour X, Y dans $|\mathcal{M}|$ et $A \in |\mathbb{V}|$ l'application $\Gamma_{XY,A} : \mathbb{V}(A, \mathbb{M}(Y, X)) \to Tri(A, Y, X)$ définie par $\Gamma_{XY,A}(f) = f^\star(ev_{YX})$ dans $\mathcal{M}(Y, X)$. On vérifie que $\Gamma_{XY,A}$ est naturel en A. On obtient ainsi une flèche $\Gamma_{XY} : y(\mathbb{M}(Y, X)) \to hom(Y, X)$ dans $\hat{\mathbb{V}}$. Remarquons au passage que $y(\mathbb{M}(Y, X)) = \bar{y}(X)(Y)$. On définit ensuite la flèche composée suivante :

$$(\underline{\gamma_X})_Y = (\bar{y}(X)(Y) \xrightarrow{\Gamma_{XY}} hom(Y, X) \xrightarrow{\sigma} hom(Y, X)^I)$$

puis $(\gamma_X)_Y = (I, (\underline{\gamma_X})_Y)$. Ainsi $(\gamma_X)_Y : \bar{y}(X)(Y) \to yon(X)(Y)$ est une flèche de $\underline{\hat{\mathcal{V}}}$. On montre ensuite que $\underline{\gamma_X} =$

$(\bar{y}(X)(Y) \xrightarrow{(\gamma_X)_Y} yon(X)(Y))_{Y \in |\mathcal{M}^{op}|}$ est une transformation naturelle mutante $\bar{y}(X) \to yon(X)$. Pour cela on se donne une flèche $Y \xrightarrow{g} Y'$ de $\mathcal{M}^{op}$ et on montre que le carré suivant commute strictement dans $\hat{\mathcal{V}}$:

$$\bar{y}(X)(Y) \xrightarrow{(\gamma_X)_Y} yon(X)(Y)$$

$$\bar{y}(X)(g) \downarrow \qquad\qquad\qquad \downarrow yon(X)(g)$$

$$\bar{y}(X)(Y') \xrightarrow[(\gamma_X)_{Y'}]{} yon(X)(Y')$$

Grâce à la prop.2.1 cela revient à montrer que le carré suivant commute dans $\hat{\mathbb{V}}$ (où $B = U(g)$) :

$$y(\mathbb{M}(Y,X)) \xrightarrow{\Gamma_{XY}} hom(Y,X)$$

$$\underline{y(yo(X)(g))} \downarrow \qquad\qquad\qquad \downarrow h(g,X)$$

$$y(\mathbb{M}(Y',X))^B \xrightarrow[\Gamma^B_{XY'}]{} hom(Y',X)^B$$

Mais pour tout $A \in |\mathbb{V}|$ et $A \xrightarrow{f} \mathbb{M}(Y,X)$, on a : $h(g,X)_A \Gamma_{XY,A}(f) = g \otimes^{op} f^\star(ev_{YX}) = (Id \otimes f)^\star(g \otimes^{op} ev_{YX}) = \Gamma_{XY',B\otimes A}\bar{y}(yo(x)g)_A(f)$. D'où les commutations voulues. Ainsi $\underline{\gamma}_X : \bar{y}(X) \to yon\underline{(X)}$ est une transformation naturelle mutante. Posons maintenant $\gamma_X = (I, \underline{\gamma}_X)$. On a $\gamma_X : \bar{y}(X) \to yon(X)$ qui est une flèche de $\underline{\mathbb{V} - \mu(\mathcal{M}^{op}, \hat{\mathcal{V}})}$ (voir la prop.4.2).

- Montrons maintenant que $\gamma = (\gamma_X)_{X\in|\mathcal{M}|}$ est une transformation naturelle mutante $\bar{y} \to yon$. Soit donc $X \xrightarrow{x} X'$ une flèche de $\mathcal{M}$. On va montrer que le carré suivant commute strictement dans $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$:

$$\bar{y}(X) \xrightarrow{\gamma_X} yon(X)$$

$$\bar{y}(x) \downarrow \qquad\qquad\qquad \downarrow yon(x)$$

$$\bar{y}(X') \xrightarrow[\gamma_{X'}]{} yon(X')$$

Posons déjà $A = U(x)$. Il faut donc montrer que pour tout $Y \in |\mathcal{M}^{op}|$ on a, dans $\hat{\mathcal{V}}$, $yon(x)_Y(\gamma_X)_Y = (\gamma_{X'})_Y \bar{y}(x)_Y$. (Les compositions sont strictes) (voir le lemme 4.1). Mais grâce, à nouveau, à la prop.2.1 cela revient à montrer que le carré suivant commute dans $\hat{\mathbb{V}}$:

$$y(\mathbb{M}(Y,X)) \xrightarrow{\Gamma_{XY}} hom(Y,X)$$

$$\underline{y}(\underline{yo(x)}_Y) \downarrow \qquad\qquad\qquad \downarrow h(Y,x)$$

$$y(\mathbb{M}(Y,X'))^A \xrightarrow[\Gamma^A_{X'Y}]{} hom(Y,X')^A$$

Or, appliqué en $B \in |\mathbb{V}|$, sur $B \xrightarrow{b} \mathbb{M}(Y,X)$, on obtient :

$\Gamma_{X'Y,A\otimes B}.\underline{y}(\underline{yo(x)}_Y)_B(b) = (\underline{yo(x)}_Y.Id \otimes b)^\star(ev_{YX'}) =$
$(Id \otimes b)^\star(x \otimes ev_{YX}) = x \otimes \overline{b}^\star(ev_{YX}) = h(Y,X)_B.\Gamma_{XY,B}(b)$, ce qui achève
de prouver que $\gamma : \bar{y} \to yon$ est une transformation naturelle mutante. Enfin
γ est inversible. En effet, pour tout $X, Y \in |\mathcal{M}|$ et $A \in |\mathbb{V}|, \Gamma_{XY,A}$ est
bijectif car $hom(Y,X)$ est représentable. Donc Γ_{XY} est inversible dans $\hat{\mathbb{V}}$.
Alors $(\underline{\gamma}_X)_Y$ est aussi inversible dans $\hat{\underline{\mathcal{V}}}$ (car $\underline{\hat{\mathcal{V}}} \simeq \hat{\mathbb{V}}$). Ceci étant vrai pour
tout Y, $\underline{\gamma}_X$ est une transformation naturelle mutante inversible (on applique
la prop.1.14). Alors γ_X est lui aussi inversible (voir la prop.4.2). Et ceci
pour tout X. Enfin, en appliquant à nouveau la prop.1.14 , la transformation
naturelle mutante γ est elle même inversible.

8. Le Lemme de Yoneda mutant.

• Notre but ici est de montrer le théorème suivant :

Théorème 8.1. (le lemme de Yoneda mutant) : Soit $\mathcal{M}$ une catégorie mu-
tante sur une catégorie monoïdale symétrique $\mathbb{V}$. Soit aussi $F : \mathcal{M}^{op} \to \hat{\mathcal{V}}$
un foncteur mutant (où $\hat{\mathcal{V}} = \mu c(\hat{\mathbb{V}})$ et où la structure cotensorisée de $\hat{\mathbb{V}}$ est
défini à la section 2 - Pour μc voir la section 1). On fixe aussi un objet X de
$\mathcal{M}$. Notons :
1) $\mathbb{C}_X \xrightarrow{U} \mathbb{V}$ la fibration discrète associée au préfaisceau $F(X) \in |\hat{\mathcal{V}}| = |\hat{\mathbb{V}}|$,
2) $(\mathbb{C}'_X \xrightarrow{U} \mathbb{V}) = (\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})(yon(X), F) \xrightarrow{U_{yon(X),F}} \mathbb{V})$ (autre fibration
discrète).
Alors $(\mathbb{C}'_X, U) \simeq (\mathbb{C}_X, U)$ (un isomorphisme de fibration discrète).
Plus précisément l'isomorphisme $\theta_X : (\mathbb{C}'_X, U) \to (\mathbb{C}_X, U)$ est donné sur
$(A, t) \in |\mathbb{C}'_X|$, par $\theta_X(A,t) = (A,x)$ où x est l'image de Id_X par
l'application composée :
$Tri(I,X,X) \xrightarrow{(\underline{t}_X)_I} F(X)(A \otimes I) \xrightarrow{F(X)(u_d^{-1})} F(X)(A)$ où $t_X = (A, \underline{t}_X)$.

<u>Preuve</u> : • θ_X est bien défini sur les flèches. Par exemple sur $(A', t') \xrightarrow{a} (A, t)$ de $\mathbb{C}'_X$, on voit facilement que $F(X)(a)(x) = x'$ où $(A, x) = \theta_X(A, t)$, $(A', x') = \theta_X(A', t')$. θ_X est clairement fonctoriel. C'est même un morphisme de fibration discrète.

• On construit maintenant un nouveau morphisme $\theta'_X : (\mathbb{C}_X, U) \to (\mathbb{C}'_X, U)$.

- Fixons $(A, x) \in |\mathbb{C}_X|$ (donc $x \in F(X)(A)$), et soit $Y \in |\mathcal{M}|$ et $B \in |\mathbb{V}|$, on considère l'application $T_{Y,B} : Tri(B, Y, X) \to F(Y)(A \otimes B)$ où $T_{Y,B}(f)$ est l'image de x par l'application composée :

$$
F(X)(A) \xrightarrow{\underline{F}(f)_A} F(Y)(B \otimes A) \xrightarrow{F(Y)(sym)} F(Y)(A \otimes B)
$$

$\underline{F}(f) : F(X) \to F(Y)^B$ provenant de $F(f) = (B, \underline{F}(f)) : F(X) \to F(Y)$, flèche de $\hat{\mathcal{V}}$.

.. $T_{Y,B}$ est naturel en B. (En effet $B' \xrightarrow{b} B$ étant une flèche de $\mathbb{V}$, il nous faut montrer la commutation du carré suivant :

$$
\begin{array}{ccc}
Tri(B, Y, X) & \xrightarrow{T_{Y,B}} & F(Y)(A \otimes B) \\
b^\star \downarrow & & \downarrow F(Y)(Id \otimes b) \\
Tri(B', Y, X) & \xrightarrow[T_{Y,B'}]{} & F(Y)(A \otimes B')
\end{array}
$$

Or $F(Y)(Id_A \otimes b).T_{Y,B}(f) = F(Y)(sym.Id_A \otimes b).\underline{F}(f)_A(x)$ où $sym = sym_{A,B}$ et $T_{Y,B'}.b^\star(f) = F(Y)(sym').\underline{F}(f')_A(x)$ où $sym' = sym_{A,B'}$ et $f' = b^\star(f)$. Mais $\underline{F}(f') = (F(X) \xrightarrow{\underline{F}(f)} F(Y)^B \xrightarrow{F(Y)^b} F(Y)^{B'})$, car F est un foncteur mutant, et donc $T_{Y,B'}.b^\star(f) = F(Y)(b \otimes Id_A.sym').\underline{F}(f)_A(x) = F(Y)(sym.Id_A \otimes b)\underline{F}(f)_A(x) = F(Y)(Id_A \otimes b)T_{Y,B}(f)$. Notons $\underline{t}_Y : hom(Y, X) \to F(Y)^A$ la transformation naturelle obtenue et $t_Y = (A, \underline{t}_Y) : yon(X)(Y) \to F(Y)$ la flèche de $\hat{\mathcal{V}}$ correspondante.

.. Après avoir posé $t = (t_Y)_{Y \in |\mathcal{M}^{op}|}$, montrons que $(A, t) : yon(X) \to F$ est une flèche de $\mathbb{V}$-$\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$. Pour cela, montrons que pour toute flèche $Y \xrightarrow{y} Y'$ de $\mathcal{M}^{op}$, $t_{Y'} \otimes yon(X)(y) \xrightarrow{sym} F(y) \otimes t_Y$ (où $sym = sym_{A,B}$) est une 2-cellule de $\hat{\mathcal{V}}$. On pose $B = U(y)$. Alors $F(y) \otimes t_Y = (B \otimes A, \underline{F}(y) \circ \underline{t}_Y)$ où $\underline{F}(y) \circ \underline{t}_Y$ est le composé suivant dans $\hat{\mathcal{V}}$:

$$
hom(Y, X) \xrightarrow{\underline{t}_Y} F(Y)^A \xrightarrow{\underline{F}(y)^A} (F(Y')^B)^A \xrightarrow{\alpha m} F(Y')^{B \otimes A}
$$

et $t_{Y'} \otimes yon(x)(y) = (A \otimes B, \underline{t}_{Y'} \circ h(y, X))$ où $\underline{t}_{Y'} \circ h(y, X)$ est le composé suivant :

$$hom(Y, X) \xrightarrow{h(y,X)} hom(Y', X)^B \xrightarrow{t^B_{Y'}} (F(Y')^A)^B \xrightarrow{\alpha m} F(Y')^{A \otimes B}.$$

Il nous faut donc montrer que le triangle suivant commute :

Appliqué en $C \in |\mathbb{V}|$ sur $f \in Tri(C, Y, X)$, cela revient à montrer que le diagramme suivant commute en x :

ou encore, après avoir utilisé l'axiome hexagonal de symétrie, transporté par $F(Y')$, il nous reste à montrer que le diagramme suivant commute en x :

$$F(X)(A)$$

$$\underline{F}(f)_A \qquad \underline{F}(y\otimes^{op}f)_A$$

$$F(Y)(C \otimes A) \qquad\qquad F(Y')((B \otimes C) \otimes A)$$

$$F(Y)(sym)\Big\downarrow \qquad\qquad \Big\downarrow F(Y')(ass^{-1})$$

$$F(Y)(A \otimes C) \qquad\qquad F(Y')(B \otimes (C \otimes A))$$

$$\underline{F}(y)_{A\otimes C} \qquad\qquad F(Y')(Id\otimes sym)$$

$$F(Y')(B \otimes (A \otimes C))$$

Mais $\underline{F}(y\otimes^{op}f) = (F(X) \xrightarrow{\underline{F}(f)} F(Y)^C \xrightarrow{\underline{F}(y)^C} (F(Y')^B)^C \xrightarrow{\alpha m} F(Y')^{B\otimes C})$.
Le diagramme précédant se simplifie encore. Finalement, il nous suffit de montrer que le carré suivant commute :

$$
\begin{array}{ccc}
F(Y)(C \otimes A) & \xrightarrow{\underline{F}(y)_{C\otimes A}} & F(Y')(B \otimes (C \otimes A)) \\
{\scriptstyle F(Y)(sym)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(Y')(Id\otimes sym)} \\
F(Y)(A \otimes C) & \xrightarrow[\underline{F}(y)_{A\otimes C}]{} & F(Y')(B \otimes (A \otimes C))
\end{array}
$$

ce qui est immédiat car $\underline{F}(y) : F(Y) \to F(Y')^B$ est une transformation naturelle. Ainsi $(A,t) : yon(X) \to F$ est une flèche de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$. On peut donc poser $\theta'_X(A,x) = (A,t)$.
- Sur une flèche $(A',x') \xrightarrow{a} (A,x)$ de $\mathbb{C}_X$, il nous faut montrer que $a :$ $\theta'_X(A',x') \to \theta'_X(A,x)$ est une 2-cellule de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$. Posons $(A,t) =$ $\theta'_X(A,x)$ et $(A',t') = \theta'_X(A',x')$. On doit donc montrer que pour tout $Y \in |\mathcal{M}^{op}|$, $a^\star(t_Y) = t'_Y$, ou encore que le triangle suivant commute :

$$hom(Y,X)$$

$$\underline{t}_Y \qquad\qquad \underline{t}'_Y$$

$$F(Y)^A \xrightarrow[Id^a]{} F(Y)^{A'}$$

ce que l'on vérifie aisément.
- Le fait que θ'_X est fonctoriel est immédiat.
$\bullet$ Il nous reste à montrer que θ_X et θ'_X sont inverses l'un de l'autre.

L'identité $\theta_X.\theta'_X = Id_{\mathbb{C}_X}$: Soit $(A, x) \in |\mathbb{C}_X|$. Posons $(A, t) = \theta'_X(A, x)$. On a $(\underline{t}_X)_I(Id_X)$ qui est l'image de x par l'application composée suivante :

$$F(X)(A) \xrightarrow{\underline{F}(Id_X)_A} F(X)(I \otimes A) \xrightarrow{F(X)(sym)} F(X)(A \otimes I)$$

Or $\underline{F}(Id_X)_A = F(X)(u_g)$ et $F(X)(sym).F(X)(u_g) = F(X)(u_d)$. Alors, si on pose $\theta_X(A, t) = (A, x')$, on sait que x' est l'image de Id_X par :

$$Tri(I, X, X) \xrightarrow{(\underline{t}_X)_I} F(X)(A \otimes I) \xrightarrow{F(X)(u_d^{-1})} F(X)(A)$$

Mais $(\underline{t}_X)_I(Id_X) = F(X)(u_d)(x)$. Ainsi $x' = x$. Finalement $\theta_X.\theta'_X = Id_{\mathbb{C}_X}$.

L'identité $\theta'_X.\theta_X = Id_{\mathbb{C}'_X}$: Soit $(A, t) \in |\mathbb{C}'_X|$. Posons $(A, x) = \theta_X(A, t)$ et $(A, t') = \theta'_X(A, x)$. Montrons que $t' = t$. On peut écrire, lorsque $Y \in |\mathcal{M}|, B \in |\mathbb{V}|$ et $f \in Tri(B, Y, X)$,
$(\underline{t}'_Y)_B(f) = [F(Y)(sym_{A,B}).\underline{F}(f)_A.F(X)(u_{dA}^{-1}).(\underline{t}_X)_I](Id_X)$ ou encore
$(\underline{t}'_Y)_B(f) = [F(Y)(u_{dA}^{-1} \otimes Id_B).F(Y)(sym_{A\otimes I,B}).\underline{F}(f)_{A\otimes I}.(\underline{t}_X)_I](Id_X)$.
D'autre part, comme $(A, t) : yon(X) \to F$ est une flèche de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$, alors $sym : t_Y \otimes yon(X)(f) \to F(f) \otimes t_X : yon(X)(Y) \to F(Y)$ est une 2-cellule de $\hat{\mathcal{V}}$. Ce qui aboutit au diagramme (D) suivant dans $\mathbb{E}ns$:

On peut donc écrire :
$(\underline{t}'_Y)_B(f) = [F(Y)(u_d^{-1} \otimes Id).F(Y)(sym).\underline{F}(f)_{A\otimes I}.(\underline{t}_X)_I](Id_X)$
$= [F(Y)(u_d^{-1}).F(Y)(sym \otimes Id).F(Y)(ass).\underline{F}(f)_{A\otimes I}.(\underline{t}_X)_I](Id_X)$
$\underset{(D)}{=} [F(Y)(u_d^{-1}).F(Y)(ass).(\underline{t}_Y)_{B\otimes I}.h(f, X)_I](Id_X)$

$= [F(Y)(Id \otimes u_d^{-1}).(\underline{t}_Y)_{B \otimes I}.h(f, X)_I](Id_X)$

$= [(\underline{t}_Y)_B.Tri(u_g^{-1}, Id, Id).Tri(sym, Id, Id).h(f, X)_I](Id_X)$.

Mais $[Tri(u_g^{-1}, Id, Id).Tri(sym, Id, Id).h(f, X)_I](Id_X) =$

$(u_g^{-1})^\star.(sym)^\star(f \otimes^{op} Id_X) = (u_g^{-1})^\star sym^\star sym^\star (Id_X \otimes f) =$

$(u_g^{-1})^\star(Id_X \otimes f) = f$. Donc $(\underline{t}'_Y)_B(f) = (\underline{t}_Y)_B(f)$. Finalement on obtient

$t' = t$ et donc $\theta'_X.\theta_X = Id$.

Au final on a montré que $\theta_X : (\mathbb{C}'_X, U) \to (\mathbb{C}_X, U)$ est inversible.

Corollaire 8.2. Soit $\mathcal{M}$ une catégorie mutante sur une catégorie monoïdale symétrique $\mathbb{V}$. Alors le foncteur mutant $yon : \mathcal{M} \to \mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})$ (voir sa construction à la section 6) est pleinement fidèle.

Preuve : Soient $X, Y \in |\mathcal{M}|$. On applique le lemme de Yoneda mutant au foncteur mutant $F = yon(Y) : \mathcal{M}^{op} \to \hat{\mathcal{V}}$, en remarquant que $\mathbb{C}_X \simeq \mathcal{M}(X, Y)$. Si on note $\gamma : \mathcal{M}(X, Y) \to \mathbb{C}_X$ cet isomorphisme on montre que le triangle suivant commute :

$$\mathcal{M}(X, Y) \xrightarrow{\quad yon_{XY} \quad} \mathbb{V}-\mu(\mathcal{M}^{op}, \hat{\mathcal{V}})(yon(X), yon(Y))$$

avec γ et θ_X vers $\mathbb{C}_X$.

θ_X et γ étant des isomorphismes, yon_{XY} en est un aussi. D'où la conclusion voulue.

9. Retour au Lemme de Yoneda enrichi.

• Reprenons l'hypothèse de la section 7, c.a.d. $\mathcal{M}$ est de saveur E. On peut même supposer que $\mathcal{M} = \mu e(\mathbb{M})$ où $\mathbb{M}$ est enrichi dans $\mathbb{V}$. Supposons aussi que $\mathbb{V}$ soit fermé (ou encre, de façon équivalente, qu'en tant que catégorie tensorisée - On la note encore $\mathbb{V}$- elle soit enrichissable). Pour chaque objet $A \in |\mathbb{V}|$, notons $\tilde{\mathbb{V}}(A, -)$ un adjoint à droite de $(-) \otimes A$ et $Ev^A : \tilde{\mathbb{V}}(A, -) \otimes A \to (-)$ une co-unité de cette adjonction. Notons aussi $\tilde{\mathbb{V}}$ la catégorie enrichie (dans $\mathbb{V}$) canonique produite par cette adjonction.

Proposition 9.1. $\mu e(\tilde{\mathbb{V}}) \simeq \mu t(\mathbb{V})$.

Preuve : L'isomorphisme $\gamma : \mu e(\tilde{\mathbb{V}}) \to \mu t(\mathbb{V})$ est défini, sur les objets par $|\gamma| = Id$, sur une flèche $(A, f) : B \to C$ par $\gamma(A, f) = (A, \underset{\sim}{f})$ où $\underset{\sim}{f} = (A \otimes B \xrightarrow{f \otimes Id} \tilde{\mathbb{V}}(B, C) \otimes B \xrightarrow{Ev} C)$. Le reste de la preuve est sans difficulté.

• Fixons maintenant un objet $X \in |\mathbb{M}|$ et considérons le foncteur enrichi $Y(X) : \mathbb{M}^{op} \to \tilde{\mathbb{V}}$ défini,

- sur un objet $S \in |\mathbb{M}|$ par $Y(X)(S) = \mathbb{M}(S, X)$,

- Pour $S, S' \in |\mathbb{M}^{op}|$, $Y(X)_{SS'} : \mathbb{M}^{op}(S, S') \to \tilde{\mathbb{V}}(Y(X)(S), Y(X)(S'))$ est l'unique flèche de $\mathbb{V}$ rendant le carré suivant commutatif :

$$
\begin{array}{ccc}
\mathbb{M}(S', S) \otimes \mathbb{M}(S, X) & \xrightarrow{Y(X)_{SS'} \otimes Id} & \tilde{\mathbb{V}}(\mathbb{M}(S, X), \mathbb{M}(S', X)) \otimes \mathbb{M}(S, X) \\
{\scriptstyle sym} \downarrow & & \downarrow {\scriptstyle Ev} \\
\mathbb{M}(S, X) \otimes \mathbb{M}(S', S) & \xrightarrow[comp]{} & \mathbb{M}(S', X)
\end{array}
$$

Le fait que $Y(X)$ soit un foncteur enrichi : $\mathbb{M}^{op} \to \tilde{\mathbb{V}}$ est bien connu et sans difficulté particulière à démontrer.

Proposition 9.2. Le carré suivant commute :

$$
\begin{array}{ccc}
\mu e(\mathbb{M}^{op}) & \xrightarrow{\mu e(Y(X))} & \mu e(\tilde{\mathbb{V}}) \\
{\scriptstyle Id} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mu e(\mathbb{M})^{op} & \xrightarrow[yo(X)]{} & \mu t(\mathbb{V})
\end{array}
$$

Preuve : - Sur un objet $S \in |\mu e(\mathbb{M}^{op})| = |\mathbb{M}|$, $\gamma.\mu e(Y(X))(S) = \mathbb{M}(S, X) = yo(X)(S)$.

- Sur une flèche $S \xrightarrow{(A,f)} S'$ de $\mu e(\mathbb{M}^{op}) = \mu e(\mathbb{M})^{op}$ (voir la proposition 3.6). Alors on voit que $(\gamma.\mu e(Y(X)))(A, f) = (A, c)$ où $A \otimes \mathbb{M}(S, X) \xrightarrow{c} \mathbb{M}(S', X)$ est la flèche composée suivante :

$$
A \otimes \mathbb{M}(S, X) \xrightarrow{f \otimes Id} \mathbb{M}(S', S) \otimes \mathbb{M}(S, X) \xrightarrow{sym} \mathbb{M}(S, X) \otimes \mathbb{M}(S', S) \xrightarrow{comp} \mathbb{M}(S', X).
$$

D'un autre côté $yo(X)(A, f) = (A, \overline{yo(X)(A, f)})$ où $\overline{yo(X)(A, f)} : A \otimes \mathbb{M}(S, X) \to \mathbb{M}(S', X)$ est l'unique flèche de $\mathbb{V}$ telle que

$\underline{yo(X)(A, f)^{\star}(ev_{S'X})} = (A, f) \otimes^{op} ev_{SX}$ (voir la section 7). Or ici $ev_{SX} = \overline{(\mathbb{M}(S, X), Id_{\mathbb{M}(S,X)})}$ et d'une façon générale, pour un couple de flèches composables $S_0 \xrightarrow{(A_0, f_0)} S_1 \xrightarrow{(A_1, f_1)} S_2$ de $\mu e(\mathbb{M})^{op}$ on a
$(A_1, f_1) \otimes^{op} (A_0, f_0) = (A_1 \otimes A_0, f_1 \circ^{op} f_0)$ où $f_1 \circ^{op} f_0 =$

$$[A_1 \otimes A_0 \xrightarrow{sym} A_0 \otimes A_1 \xrightarrow{f_0 \otimes f_1} \mathbb{M}(S_1, S_0) \otimes \mathbb{M}(S_2, S_1) \xrightarrow{comp} \mathbb{M}(S_2, S_0)].$$

On obtient ici $(A, f) \otimes^{op} ev_{SX} = (A \otimes \mathbb{M}(S, X), \bar{f})$ où $\bar{f} =$

$$[A \otimes \mathbb{M}(S, X) \overset{sym}{\to} \mathbb{M}(S, X) \otimes A \overset{Id \otimes f}{\to} \mathbb{M}(S, X) \otimes \mathbb{M}(S', S) \overset{comp}{\to} \mathbb{M}(S', X)]$$

et donc $\underline{yo(X)(A, f)} = \bar{f}$. Enfin, comme $\bar{f} = c$, on a l'identité
$\gamma.\mu e(Y(\overline{X})(A, f) = yo(X)(A, f)$. D'où la commutation du carré proposé.
• Considérons maintenant le foncteur mutant
$\mathbb{V}\text{-}e(\mathbb{M}^{op}, \tilde{\mathbb{V}}) \overset{\Phi}{\to} \mathbb{V}\text{-}\mu(\mu e(\mathbb{M})^{op}, \mu c(\hat{\mathbb{V}}))$ obtenu en composant les quatre foncteurs mutants suivants :

$$\mathbb{V} - e(\mathbb{M}^{op}, \tilde{\mathbb{V}})$$
$$\downarrow \varepsilon \quad (\star 1)$$
$$\mathbb{V} - \mu(\mu e(\mathbb{M}^{op}), \mu e(\tilde{\mathbb{V}}))$$
$$\downarrow \mathbb{V} - \mu(Id, \gamma) \quad (\star 2)$$
$$\mathbb{V} - \mu(\mu e(\mathbb{M}^{op}), \mu t(\mathbb{V}))$$
$$\downarrow \mathbb{V} - \mu(Id,, \mu y) \quad (\star 3)$$
$$\mathbb{V} - \mu(\mu e(\mathbb{M})^{op}, \mu c(\hat{\mathbb{V}}))$$

$(\star 1)$ Voir la proposition 5.1, $(\star 2)$ voir la section 4 et la proposition 9.1 $(\star 3)$ voir les sections 4 et 2

Proposition 9.3. Φ est pleinement fidèle.

    <u>Preuve</u> : Résulte du lemme suivant et des propositions 2.3 et 4.3.

Lemme 9.4. Le composé de foncteurs mutants pleinement fidèles est lui même pleinement fidèle.

Preuve : du lemme : Immédiat.

• Soit $X \in |\mathbb{M}|$ et $F : \mathbb{M}^{op} \to \tilde{\mathbb{V}}$ un foncteur enrichi. Alors F et $Y(X)$ sont dans $|\mathbb{V}\text{-}e(\mathbb{M}^{op}, \tilde{\mathbb{V}})|$. Comme Φ est pleinement fidèle, on a l'isomorphisme suivant :

$$\mathbb{V}\text{-}e(\mathbb{M}^{op}, \tilde{\mathbb{V}})(Y(X), F) \xrightarrow{\Phi_{Y(X),F}} \mathbb{V}\text{-}\mu(\mu e(\mathbb{M})^{op}, \mu c(\hat{\mathbb{V}}))(\Phi(Y(X)), \Phi(F))$$

où $\Phi(Y(X)) \underset{\star 1}{=} \mu y.yo(X) \underset{\star 2}{\simeq} yon(X), (\star 1)$ par la prop.9.2 et $(\star 2)$ voir la section 7 (l'isomorphisme $\mu y.yo(X) \to yon(X)$ est noté γ_X) et $\Phi(F) = \mu y.\hat{F}$ où $\hat{F} = (\mathbb{V}\text{-}\mu(Id, \gamma).\varepsilon)(F)$. Et donc $\Phi(F)(X) = y(\hat{F}(X)) = y(F(X))$. En d'autres termes, $\Phi(F)(X)$ est représentable de représentation $(F(X), Id_{F(X)})$.

Utilisons maintenant les deux lemmes suivants:

Lemme 9.5. Soient $\mathcal{M}$ une catégorie mutante, $X \in |\mathcal{M}|$ et $F : \mathcal{M}^{op} \to \mu c(\hat{\mathbb{V}})$ un foncteur mutant. On suppose que $F(X)$, qui est un préfaisceau, est représentable. Notons (R, r) une représentation de $F(X)$ et posons $(R, \rho) = \theta_X^{-1}(R, r)$ (voir le théorème 8.1). Alors (R, ρ) est un objet final de $\mathbb{V}\text{-}\mu(\mathcal{M}^{op}, \mu c(\hat{\mathbb{V}}))(yon(X), F)$.

Preuve : du lemme : Car (R, r) est un objet final de $\mathbb{C}_X$ (voir la notation dans le théorème 8.1)

Lemme 9.6. Soient $\mathbb{A}, \mathbb{B}$ deux catégories enrichies dans $\mathbb{V}$ telles que $\mathbb{V}\text{-}e(\mathbb{A}, \mathbb{B})$ soit de saveur E. Soient aussi $F, G : \mathbb{A} \to \mathbb{B}$ des foncteurs enrichis. Si on note $([F, G], ev_{FG})$ une représentation de $Tri(-, F, G) : \mathbb{V}^{op} \to \mathbb{E}ns$ où $Tri = Tri_{\mathbb{V}\text{-}e(\mathbb{A},\mathbb{B})}$. Alors ev_{FG} est un objet final de $\mathbb{V}\text{-}e(\mathbb{A}, \mathbb{B})(F, G)$.

Preuve : du lemme : Immédiat.

Dans la catégorie $\mathbb{C}'_X = \mathbb{V} - \mu(\mathcal{M}^{op}, \mu c(\hat{\mathbb{V}}))(yon(X), \Phi(F))$ nous disposons donc de deux objets finaux.
- Le premier, par le lemme 9.5, étant $(F(X), \rho) = \theta_X^{-1}(F(X), Id_{F(X)})$.
- Le second, par le lemme 9.6 , étant la flèche composée stricte suivante : $([Y(X), F], \rho') =$

$$[yon(X) \xrightarrow{\gamma_X^{-1}} \mu y.yo(X) \xrightarrow{Id} \Phi(Y(X)) \xrightarrow{\Phi(ev)} \Phi(F)]$$

(puisque $\Phi_{Y(X),F}$ est un isomorphisme). Voir aussi le lemme suivant :

Lemme 9.7. Soit $\mathcal{M}$ une catégorie mutante, $X, X', S \in |\mathcal{M}|$ et $f : S \to X$ une flèche inversible de $\underline{\mathcal{M}}$. Alors la composition stricte par f produit un isomorphisme $\mathcal{M}(X, X') \to \mathcal{M}(S, X')$.

<u>*Preuve*</u> : du lemme : Sans difficulté.

• Ainsi $(F(X), \rho) \simeq ([Y(X), F], \rho')$ et donc $F(X) \simeq [Y(X), F]$. Encore noté $\mathbb{V}\text{-}Nat(Y(X), F)$. On retrouve donc la conclusion du lemme de Yoneda enrichi.

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STRICTLY ZERO-DIMENSIONAL BIFRAMES AND RANEY EXTENSIONS

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Resumé. Nous comparons deux catégories qui étendent la catégorie des espaces T_0 à l'aide d'outils issus de la topologie sans points. La catégorie des extensions de Raney est constituée de paires $(L, \mathcal{F})$ où L est un locale et $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ un sous-locale. Ici, $\mathsf{S}_{\mathfrak{o}}(L)$ est l'ensemble de toutes les intersections des sous-locaux ouverts de L. Un biframe strictement de dimension zéro est un couple $(L, \mathcal{D})$ où $\mathcal{D} \subseteq \mathsf{S}(L)$ est un sous-local codense. Nous montrons qu'il existe une adjonction entre certains sous-locales de $\mathsf{S}_{\mathfrak{o}}(L)$ et les sous-locaux codenses de $\mathsf{S}(L)$. Nous montrons que l'adjonction se restreint à un isomorphisme d'ordre entre ce que nous appelons les sous-locaux *admissibles* de $\mathsf{S}_{\mathfrak{o}}(L)$ et les sous-locaux codenses *essentiels*. En application de notre résultat principal, nous établissons une bijection entre les extensions de Raney admissibles et les biframes strictement de dimension zéro (L_1, L_2, L) telles que L est une extension essentielle de L_2 dans **Frm**. Nous montrons que cette correspondance ne peut pas être rendue fonctorielle de manière évidente, car un morphisme $f : L \to M$ peut être soulevé en une application $f : (L, \mathcal{F}) \to (L, \mathcal{G})$ d'extensions de Raney sans être soulevé en une application entre les biframes strictement de dimension zéro associés.

Abstract. We compare two categories which extend the category of T_0-spaces using tools from pointfree topology. The category of Raney extensions consists of pairs $(L, \mathcal{F})$ where L is a locale and $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ a sub-colocale. Here, $\mathsf{S}_{\mathfrak{o}}(L)$ is the collection of all intersections of open sublocales

of L. Similarly, a strictly zero-dimensional biframe is a pair $(L, \mathcal{D})$ where $\mathcal{D} \subseteq \mathsf{S}(L)$ is a codense subcolocale. We show that there is an adjunction between certain subcolocales of $\mathsf{S}_\mathfrak{o}(L)$ and codense subcolocales of $\mathsf{S}(L)$. We show that the adjunction maximally restricts to an order-isomorphism between what we call the *admissible* subcolocales of $\mathsf{S}_\mathfrak{o}(L)$ and the *essential* codense subcolocales. As an application of our main result, we establish a bijection between admissible Raney extensions and the strictly zero-dimensional biframes (L_1, L_2, L) such that L is an essential extension of L_2 in **Frm**. We show that this correspondence cannot be made functorial in the obvious way, as a frame morphism $f : L \to M$ may lift to a map $f : (L, \mathcal{F}) \to (L, \mathcal{G})$ of Raney extensions without lifting to a map between the associated strictly zero-dimensional biframes.

Keywords. Pointfree topology, frame, Raney duality, biframe, sublocale, essential extension.

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Introduction

The usual approach in pointfree topology is to consider the adjunction $\Omega :$ **Top** $\leftrightarrows$ **Frm**^{op} : pt between frames and spaces, and to regard frames as pointfree spaces in virtue of this. This is the classical approach found, for example, in [12], [17], [18]. The fixpoints on the **Top** side are the sober spaces, so we may view **Frm** as a faithful extension of the opposite of the category **Sob** of sober spaces. An alternative approach is T_D-*duality*, developed in [6]. This is based on the T_D-axiom, introduced in [2]. The full subcategory of **Top** consisting of the T_D-spaces is related via a similar adjunction to a wide subcategory of **Frm**, the category of frames with *D-morphisms*, called **Frm$_D$**. The category **Frm$_D$** is thus shown to faithfully extend the opposite of **Top$_D$**. Sobriety and the T_D property are incomparable. This is why the language of frames, under both translations, is not expressive enough to capture certain concepts and constructions. For example, in **Frm** we do not have a notion of sobrification. Similarly, in **Frm$_D$**, we do not have a notion of T_D-coreflection (which exists for spaces). Refining the language by extending the category **Frm** enables us to capture both things in the same category. For example, in [22], the category **Frm** is extended to the cate-

gory **Raney** of *Raney extensions*, which also faithfully extends the dual of the category **Top₀** of all T_0-spaces. In this category, we have both the notion of *sober coreflection* (Section 6.5), the dual notion of sobrification, and T_D-*reflection* (Section 6.6), dual of the T_D-coreflection.

- Raney extensions are inspired by the work of Raney on completely distributive lattices, see for example [19]. Raney duality, as illustrated in [7], is the result that the dual of **Top₀** is equivalent to the category of completely distributive lattices $\mathcal{U}(X)$ consisting of upper sets of some poset X, equipped with an interior operator. A *Raney extension* is a pair (L, C) where C is a coframe and $L \subseteq C$ is a frame with the inherited order, such that the subset inclusion preserves all joins and strongly exact meets. The Raney extension corresponding to a space X is the pair $(\Omega(X), \mathcal{U}(X))$, where $\Omega(X)$ is its frame of opens and $\mathcal{U}(X)$ is the collection of upper sets in its specialization order. The category **Raney** of Raney extensions faithfully extends $\mathbf{Top_0}^{op}$.

- Another approach to refining the language of frames is that of *McKinsey-Tarski algebras*, as introduced in [8]. This is based on work by McKinsey and Tarski, who in [16] studied topological spaces in terms of the closure operator they induce on their powerset. McKinsey-Tarski algebras make this approach pointfree, by considering as objects complete Boolean algebras, not necessarily atomic, with interior operators. The category of MT-algebras faithfully extends all of $\mathbf{Top}^{op}$.

- A third approach to faithfully extending the dual of **Top₀** is that of *strictly zero-dimensional biframes*, as shown in [13]. Although in [13] the pointfree description of T_0 spaces is not the focus, it is indeed observed (end of Section 4) that there is a dual adjunction between spaces and strictly zero-dimensional biframes, whose fixpoints are the T_0 spaces.

It is then interesting to look at how the three categories interact, and go towards a more unified theory of pointfree T_0 spaces. The connection between Raney extensions and MT-algebras is looked at in [9]. A Raney extension (L, C) is *admissible* if the joins of L distribute over all binary meets in C. In [9] the connection between Raney extensions and MT-algebras is explored,

and it is shown that the category of admissible Raney extensions is equivalent to the category $\mathbf{MT_0}$ of T_0 MT-algebras equipped with a notion of morphism based on a proximity-like relation.

With this paper, we add another part to the big picture, connecting explicitly the categories of Raney extensions and that of strictly zero-dimensional biframes. In particular, we show that there is a bijection at the level of objects between admissible Raney extensions and *essential* strictly zero-dimensional biframes. Our approach is by no means the simplest possible way of proving this correspondence. Instead, we want to obtain the correspondence as a byproduct of a more general study of subcolocales of $\mathsf{S}(L)$ and subcolocales of $\mathsf{S_o}(L)$. We point out the following result.

Proposition 0.1. *Let L be a frame. Raney extensions on L are in bijective correspondence with subcolocales of $\mathsf{S_o}(L)$ containing all open sublocales. Strictly zero-dimensional biframes whose first component is L are in bijective correspondence with dense subcolocales of $\mathsf{S}(L)$.*

Notice how, in the usual setting, all sublocales of L are sober: for a sober space X, the sober subspaces are exactly the ones of the form

$$\bigcap_i U_i \cup V_i^c$$

where $U_i, V_i \subseteq X$ are opens. Compare this with the fact that every sublocale $S \subseteq L$ of a frame L satisfies

$$S = \bigcap \{\mathfrak{o}(a) \vee \mathfrak{c}(b) \mid S \subseteq \mathfrak{o}(a) \vee \mathfrak{c}(b)\}$$

to see that it is natural to view $\mathsf{S}(L)$ as the collection of all *sober* subspaces of L. But Raney extensions and strictly zero-dimensional biframes tell us that if we want to capture all subspaces of L, not just sober ones, it suffices to look at subcolocales of $\mathsf{S_o}(L)$ or of $\mathsf{S}(L)$, rather than sublocales of L. For example, notice that if X is a T_0 space it may be the case that distinct subspaces induce the same sublocale of $\Omega(X)$. However, as Raney extensions faithfully extend $\mathbf{Top_0}$, this means that the two subspace inclusions will induce different surjections from $(\Omega(X), \mathcal{U}(X))$ in $\mathbf{Raney}$. By the equivalence in Proposition 0.1 above, this means that these will induce different subcolocales of $\mathsf{S_o}(\Omega(X))$.

In conclusion, it may be argued that $S(L)$ which is not refined enough to capture spaces which are not sober, but that the limitation is overcome by looking at the collections $S(S_o(L))$ and $S(S(L))$, instead. In this paper, we show explicitly how to relate the two approaches. We do so by proving an adjunction between codense subcolocales of $S(L)$ and certain subcolocales of $S_o(L)$ which we call *admissible*. We prove that the adjunction maximally restricts to admissible subcolocales of $S_o(L)$ and a class of subcolocales $\mathcal{D} \subseteq S(L)$ which we characterize explicitly.

1. Preliminaries

1.1 The categories Frm and Loc

We first recall some background on frames and point-free topology. For more information on the categories of frames and locales, we refer the reader to Johnstone [12] or the more recent [17] and [18]. A *frame* is a complete lattice L satisfying

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\},$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* is a function preserving arbitrary joins, including the bottom element 0, and finite meets, including the top element 1. We call **Frm** the category of frames and frame homomorphisms. Frames are complete Heyting algebras, with the Heyting implication computed as

$$x \to y = \bigvee \{z \in L \mid z \wedge x \leq y\}.$$

In particular, the *pseudocomplement* of an $a \in L$ is the element $\neg a = a \to 0$. The archetypal example of frame is the lattice of open sets $\Omega(X)$ for a topological space X. The assignment $X \mapsto \Omega(X)$ is the object part of a functor $\Omega : \textbf{Top} \to \textbf{Frm}^{op}$. An element p of a frame L is said to be *prime* if it is not 1 and whenever $x \wedge y \leq p$ for some $x, y \in L$, then $x \leq p$ or $y \leq p$. The collection of all primes of L will be denoted by $\mathsf{pt}(L)$, and the assignment $L \mapsto \mathsf{pt}(L)$ is the object part of a functor $\mathsf{pt} : \textbf{Frm}^{op} \to \textbf{Top}$, which together with Ω yields an adjunction $\Omega : \textbf{Top} \leftrightarrows \textbf{Frm}^{op} : \mathsf{pt}$ with $\Omega \dashv \mathsf{pt}$. A frame is said to be *spatial* if $a \not\leq b$ for $a, b \in L$ implies that there is a prime p with $b \leq p$ and $a \not\leq p$. Spatial frames L are precisely the fixpoints

of the adjunction $\Omega \dashv$ pt. This adjunction is idempotent, and so a frame is spatial if $L \cong \Omega(X)$ for some space X. Because of this adjunction, frames are regarded as pointfree spaces, but since the adjunction is contravariant sometimes the category **Loc**, equivalent to $\mathbf{Frm}^{op}$, is used.

The T_D approach

An alternative to the classical duality is the so-called T_D-*duality* from [6]. For a space X we call a point $x \in X$ a T_D-*point* if it is the intersection of an open and a closed set. We define a space to be T_D if all its points are T_D. The axiom is between T_0 and T_1 and it is introduced in [2]. For a frame L, a prime $p \in L$ is *covered* if $\bigwedge_i x_i = p$ implies $x_i = p$ for some $i \in I$. We call $\mathsf{pt}_D(L)$ the set of covered primes of L. We say that a frame morphisms $f : L \to M$ is a *D-morphism* if its right adjoint $f_* : M \to L$ maps covered primes to covered primes. We call $\mathbf{Frm_D}$ the category of frames and D-morphisms. The assignment $L \mapsto \mathsf{pt}_D(L)$ extends to a functor $\mathsf{pt}_D : \mathbf{Frm_D}^{op} \to \mathbf{Top}$. In [6] it is shown that there is an adjunction $\Omega : \mathbf{Top_D} \leftrightarrows \mathbf{Frm_D}^{op} : \mathsf{pt}_D$ with $\Omega \dashv \mathsf{pt}_D$, where $\mathbf{Top_D}$ is the category of T_D-spaces. We call a frame T_D-*spatial* if it is meet-generated by its covered primes. T_D-spatial frames are exactly the fixpoints of the adjunction above, namely the frames of opens of T_D-spaces.

Sublocales

For a frame L, a *sublocale* of L is a subset inclusion $S \subseteq L$ such that

1. S is closed under arbitrary meets;

2. $a \to s \in S$ for all $a \in L$ and $s \in S$.

Sublocales are frames when equipped with the order inherited from L. In fact, the name comes from the fact that such subset inclusions are, up to isomorphism, the regular monomorphisms in **Loc**. If $S \subseteq L$ is a sublocale, we call $\bigwedge^S$ and $\wedge^S$ the arbitrary and the binary meets in S, respectively, and we use a similar convention for joins. Whenever L is any lattice and $M \subseteq L$ a lattice with the inherited order, we use analogous notation for lattice operations in M with the inherited order. Sublocales have closure

operators associated to them. For a sublocale $S \subseteq L$ the map $\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}$ for all $a \in L$ is called its *nucleus*.

Lemma 1.1. *Let $S \subseteq L$ be a sublocale. Then, for $s_i \in S$;*

1. $\bigwedge_i^S s_i = \bigwedge_i s_i$;

2. $\bigvee_i^S s_i = \nu_S(\bigvee_i s_i)$.

Every sublocale also has an associated frame *congruence*, a binary relation on L which is a subframe of $L \times L$. An alternative approach, which we will adopt, is to consider the bijection between sublocales of L and *precongruences* on L. These are defined to be binary relations R on L such that

1. R is reflexive;

2. R is transitive;

3. $a' \leq a$ and $(a, b) \in R$ and $b \leq b'$ implies $(a', b') \in R$;

4. $(a_i, b) \in R$ implies $(\bigvee_i a_i, b) \in R$;

5. $(a, b_1), (a, b_2) \in R$ implies $(a, b_1 \wedge b_2) \in R$.

This correspondence is introduced in [14]. The sublocale S is associated with the precongruence $\{(a, b) \in L \times L \mid \nu_S(a) \leq \nu_S(b)\}$. Conversely, for a precongruence $R \subseteq L \times L$, the associated sublocale is

$$\bigcap\{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid (x, y) \in R\}.$$

A sublocale of L is *dense* if it contains 0. Dense sublocales, then, are closed under arbitrary intersections. For a frame L, the smallest dense sublocale $\mathfrak{b}(0) \subseteq L$ is always Boolean, and it is called its *Booleanization*.

Subcolocales and the coframe $\mathsf{S}(L)$

Subcolocales

Coframes come equipped with a *co-Heyting operator*, known as the *difference* $x \backslash y$ of two elements $x, y \in C$, computed as

$$x \backslash y = \bigwedge \{z \in C \mid x \le y \vee z\}.$$

This operator is characterized by the condition that $x \backslash y \le z$ if and only if $x \le y \vee z$. In particular, the *supplement* of $c \in C$ is $c^* = 1 \backslash c$. We shall freely use some its properties, listed in the following lemma.

Lemma 1.2. *Let C be a coframe. For elements $c, d, c_i, x \in C$:*

   1. If x is complemented, $c \backslash x = c \wedge x^*$;

   2. $(\bigvee_i c_i) \backslash d = \bigvee_i (c_i \backslash d)$;

   3. $d \backslash \bigwedge_i c_i = \bigvee_i (d \backslash c_i)$.

For a coframe C, we say that an element $c \in C$ is *linear* if $\bigvee_i (a_i \wedge c) = \bigvee_i a_i \wedge c$ for all $a_i \in C$.

Lemma 1.3. *Complemented elements of a coframe are linear.*

Of particular importance will be the notion dual to that of sublocale. Let us define it explicitly. For a coframe C, a *subcolocale* is an inclusion $D \subseteq C$ such that

   1. D is closed under all joins;

   2. $d \backslash c \in D$ for all $d \in D$ and $c \in C$.

An inclusion $D \subseteq C$ for a frame D is a subcolocale iff $D^{op} \subseteq C^{op}$ is a sublocale. Dualizing the analogous notion for frames, we see that a subcolocale $D \subseteq C$ determines an interior operator $\nu_D : C \to C$, which we call its *conucleus*.

Lemma 1.4. *Let $D \subseteq C$ be a sublocale. Then, for $d_i \in D$;*

   1. $\bigwedge_i^D d_i = \nu_D(\bigwedge_i d_i)$;

2. $\bigvee_i^D d_i = \bigvee_i d_i$.

We say that a subcolocale is *codense* if it contains 1. We keep the term for the dual notion and call the smallest codense subcolocale of a coframe its *Booleanization*.

The coframe of sublocales

The family $\mathsf{S}(L)$ of all sublocales of L, ordered by inclusion, is a coframe. Meets are set-theoretical intersections. Because $\mathsf{S}(L)$ is a coframe, it also comes with a difference operation, computed as $S \backslash T = \bigcap \{U \in \mathsf{S}(L) \mid S \subseteq T \vee U\}$. This is studied in [20]. For each $a \in L$, there is an *open sublocale* $\mathfrak{o}(a) = \{b \in L \mid b = a \to b\} = \{a \to b \mid b \in L\}$ and a *closed sublocale* $\mathfrak{c}(a) = \uparrow a$. Open and closed sublocales behave like open and closed subspaces in many respects, in the lemma below we list a few.

Lemma 1.5. *For every frame L and $a, b, a_i \in L$ we have*

1. $\mathfrak{o}(1) = L$ *and* $\mathfrak{o}(0) = \{1\}$.

2. $\mathfrak{c}(1) = \{1\}$ *and* $\mathfrak{c}(0) = L$.

3. $\mathfrak{o}(a) \cap \mathfrak{c}(a) = \{1\}$ *and* $\mathfrak{o}(a) \vee \mathfrak{c}(a) = L$.

4. $\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i)$ *and* $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.

5. $\bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}(\bigwedge_i a_i)$ *and* $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.

In particular, by 3, open and closed sublocales are complemented, and as such (Lemma 1.3) they are linear. Intersections of open sublocales are called *fitted* sublocales. These form a subcoframe of $\mathsf{S}(L)$, which we denote as $\mathsf{S}_{\mathfrak{o}}(L)$. The closure operator associated with it is called the *fitting*. This is studied in [10]. More explicitly, the corestriction to its fixpoints is:

$$\mathit{fit} : \mathsf{S}(L) \to \mathsf{S}(L)$$
$$S \mapsto \bigcap \{\mathfrak{o}(x) \mid x \in L, S \subseteq \mathfrak{o}(x)\}.$$

Another important class of sublocales is that of the *two-element sublocales*. These are sublocales of the form $\{1, p\}$ for some element $p \in L$, which is then necessarily prime.

Exactness and strong exactness

For a complete lattice L we say that a join (resp. a meet) is *exact* if it distributes over binary meets (resp. joins). These notions are compared in [3]. When L is a frame, we also have the notion of *strongly exact* meet: a meet $\bigwedge_i x_i$ such that $x_i \to y = y$ for all $i \in I$ implies that $\bigwedge_i x_i \to y = y$. A filter $F \subseteq L$ of a frame is called *exact* if it is closed under exact meets, and *strongly exact* if it is closed under strongly exact meets. We call $\mathsf{Filt}_{\mathcal{SE}}(L)$ the ordered collection of strongly exact filters of L, and $\mathsf{Filt}_{\mathcal{E}}(L)$ the collection of the exact filters. The following is Theorem 3.5 in [15].

Lemma 1.6. *There is an isomorphism* $\varphi : \mathsf{S}_{\mathfrak{o}}(L) \cong \mathsf{Filt}_{\mathcal{SE}}(L)$ *given by*

$$\varphi(F) = \{x \in L \mid F \subseteq \mathfrak{o}(x)\}$$

for each $F \in \mathsf{S}_{\mathfrak{o}}(L)$.

We say that a frame map $f : L \to M$ is *exact* if, whenever $\bigwedge_i x_i \in L$ is an exact meet, $\bigwedge_i f(x_i)$ is exact, and $\bigwedge_i f(x_i) = f(\bigwedge_i x_i)$. A sublocale $S \subseteq L$ whose surjection $s : L \to S$ is exact is called an *exact* sublocale. Exact sublocales are precisely those whose surjection preserves all exact meets. The next result is Proposition 7.15 in [22].

Lemma 1.7. *A sublocale* $S \subseteq L$ *is exact if and only if whenever* $\bigwedge_i x_i \in L$ *is an exact meet and* $S \cap \mathfrak{c}(x_i) \subseteq \mathfrak{c}(x)$ *for every* $i \in I$ *then* $S \cap \mathfrak{c}(\bigwedge_i x_i) \subseteq \mathfrak{c}(x)$ *for all* $x \in L$.

Distinguished subcolocales of $\mathsf{S}(L)$

In this work, we look at subcolocales of $\mathsf{S}(L)$. Among these is the collection $\mathsf{S}_b(L)$ of joins of complemented sublocales (see [1]). These also coincide with those sublocales of the form $\bigvee_i \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i)$. We observe that a subcolocale of $\mathsf{S}(L)$ is codense if and only if it contains L. The following is well-known.

Lemma 1.8. *For any frame* L, *the inclusion* $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ *is a codense subcolocale. In particular, it is the Booleanization of* $\mathsf{S}(L)$.

We will consider the map $\varphi \circ \mathit{fit} : \mathsf{S}(L) \to \mathsf{Filt}_{\mathcal{S}\mathcal{E}}(L)$. We call this ker, for *kernel*, as it assigns to a sublocale $S \subseteq L$ the set $s^{-1}(1)$, where $s : L \to S$ is its surjection. Theorem 6.6 of [11], together with the fact that φ is an isomorphism, gives the following.

Lemma 1.9. *For every frame L, $\mathit{ker}[\mathsf{S}_b(L)] = \mathsf{Filt}_{\mathcal{E}}(L)$.*

We will need the following result.

Lemma 1.10. *If X is a T_D-space, all sublocales in $\mathsf{S}_b(\Omega(X))$ are T_D-spatial.*

Proof. Proposition 3.2 of [1] states that all sublocales in $\mathsf{S}_b(\Omega(X))$ are induced by some subspace of X, in the sense that, for each of these, the surjection corresponding to them is of the form $\Omega(i_Y) : \Omega(X) \to \Omega(Y)$ for some subspace inclusion $i_Y : Y \subseteq X$. Since subspaces of T_D spaces are T_D, these are T_D-spatial sublocales. $\qquad\qquad\square$

We will also look at the collection $\mathsf{S}_{sp}(L)$ of spatial sublocales of L, by which we mean the sublocales $S \subseteq L$ where S is a spatial frame. Equivalently, spatial sublocales are characterized as those which are joins of two-element sublocales. For all frames L, the inclusion $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is a subcolocale inclusion, in particular, it is the subcolocale associated with the spatialization surjection of $\mathsf{S}(L)^{op}$ (Proposition 3.14 of [21]). The next result follows by definition of codensity.

Lemma 1.11. *For a frame L, the subcolocale $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is codense if and only if L is spatial.*

The collection of all exact sublocales of L will be denoted as $\mathsf{S}_{\mathcal{E}}(L)$. The coming result follows from Theorem 7.20 of [22] and Proposition 7.12 from the same paper. The second part of the claim follows from $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ being the smallest codense subcolocale.

Lemma 1.12. *The inclusion $\mathsf{S}_{\mathcal{E}}(L) \subseteq \mathsf{S}(L)$ is a codense subcolocale. In particular, $\mathsf{S}_b(L) \subseteq \mathsf{S}_{\mathcal{E}}(L)$.*

Strictly zero-dimensional biframes and Raney extensions

Strictly zero-dimensional biframes

A *biframe* (see [5]) is a triple $\mathcal{L} = (L_1, L_2, L)$ where L is a frame and $L_1, L_2 \subseteq L$ are subframes such that $L_1 \cup L_2$ generates L in the sense that L is the smallest subframe containing $L_1 \cup L_2$. An element $a_1 \in L_1$ is said to be *bicomplemented* if it is complemented in L and its complement is in L_2, and similarly for elements of L_2. A biframe is said to be *zero-dimensional* if both L_1 and L_2 are join-generated by their bicomplemented elements. A biframe is said to be *strictly zero-dimensional* if it is zero-dimensional and all elements of L_1 are bicomplemented. Strictly zero-dimensional biframes are studied in detail in [13]. We will later see (Lemma 1.14) that dense subcolocales of $\mathsf{S}(L)$ contain all closed sublocales. In particular, these are embedded as a subcoframe $\mathfrak{c}[L] \subseteq \mathcal{D}$. We give an equivalent description to the category of strictly zero-dimensional biframes, similar to the one given in Section 3.2 of [13]. The category **SZDBF** is the category:

1. Whose objects are pairs $(L, \mathcal{D})$ where L is a frame and $\mathcal{D} \subseteq \mathsf{S}(L)$ a dense subcolocale;

2. Whose morphisms $f : (L, \mathcal{D}) \to (M, \mathcal{E})$ are frame maps $f : L \to M$ such that the anti-isomorphic coframe map $\mathfrak{c}(f) : \mathfrak{c}[L] \to \mathfrak{c}[M]$ extends to a coframe map $\overline{f} : \mathcal{D} \to \mathcal{E}$.

Raney extensions

Raney extensions are introduced in [22]. These structures are inspired by a duality studied in [7], based on work by Raney, see [19]. Raney extensions may be identified with pairs $(L, \mathcal{F})$, where $\mathcal{F} \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)^{op}$ is a subcolocale containing all principal filters (see Theorem 3.9 in [22]). Here, we use the isomorphism in 1.6 to give an equivalent definition. We define **Raney** to be the category:

1. Whose objects are pairs $(L, \mathcal{F})$ where L is a frame and $\mathcal{F} \subseteq \mathsf{S}_\mathfrak{o}(L)$ a subcolocale containing all open sublocales;

2. Whose morphisms $f : (L, \mathcal{F}) \to (M, \mathcal{G})$ are frame maps $f : L \to M$ such that the isomorphic frame map $\mathfrak{o}(f) : \mathfrak{o}[L] \to \mathfrak{o}[M]$ extends to a coframe map $\overline{f} : \mathcal{F} \to \mathcal{G}$.

The main adjunction

From subcolocales of $\mathsf{S}(L)$ to subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ via fitting

In this subsection, we show that the fitting operator $fit : \mathsf{S}(L) \to \mathsf{S}_{\mathfrak{o}}(L)$ is such that its direct image $fit[-] : \mathcal{P}(\mathsf{S}(L)) \to \mathcal{P}(\mathsf{S}_{\mathfrak{o}}(L))$ sends codense subcolocales to special kinds of subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$. First, we work towards a characterization of codense subcolocales of $\mathsf{S}(L)$.

Proposition 1.13. *An inclusion $\mathcal{D} \subseteq \mathsf{S}(L)$ is a subcolocale if and only if it is closed under arbitrary joins and stable under both $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ for all $x \in L$.*

Proof. Suppose that $\mathcal{D}$ satisfies the assumptions in the statement. By definition of subcolocale, it suffices to show that $S \in \mathcal{D}$ implies $S \backslash T \in \mathcal{D}$ for all $T \in \mathsf{S}(L)$. Let $S \in \mathcal{D}$ and let $T = \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$. By assumption, for every $i \in I$ we have $S \cap \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i) \in \mathcal{D}$. By 1 of Lemma 1.2, this equals $S \backslash (\mathfrak{o}(x_i) \vee \mathfrak{c}(y_i))$. As $\mathcal{D}$ is closed under all joins and by item 3 of Lemma 1.2, $S \backslash \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i) \in \mathcal{D}$, as desired. For the other direction, if $\mathcal{F} \subseteq \mathsf{S}(L)$ is a subcolocale, indeed it must be stable under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$, as by item 1 of Lemma 1.2 these are the same as $- \backslash \mathfrak{c}(x)$ and $- \backslash \mathfrak{o}(x)$, respectively. $\square$

Lemma 1.14. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale. Then:*

1. *$\mathcal{D}$ contains all open and closed sublocales;*

2. *$\mathcal{D}$ is meet-generated by the elements of the form $\mathfrak{o}(x) \vee \mathfrak{o}(y)$.*

Proof. If $\mathcal{D} \subseteq \mathsf{S}(L)$ is codense, by definition it must contain L. By Lemma 1.13, then, it must contain $L \backslash \mathfrak{o}(x)$ and $L \backslash \mathfrak{c}(x)$ for all $x \in L$, but by 1 of Lemma 1.2 these equal $\mathfrak{c}(x)$ and $\mathfrak{o}(x)$, respectively. Every element of $\mathcal{D}$ is $\nu_{\mathcal{D}}(S)$ for some sublocale $S \in \mathsf{S}(L)$, and $S = \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$ for some $x_i, y_i \in L$. Then, every element of $\mathcal{D}$ can be written as $\nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i))$ for some $x_i, y_i \in L$. This equals $\bigwedge_i^{\mathcal{D}} \nu_{\mathcal{D}}(\mathfrak{o}(x_i) \vee \mathfrak{c}(x_i))$ by Lemma 1.4. As $\mathcal{D}$ contains all open and closed sublocales by the first item, and as it is closed under joins by 1.13, $\nu_{\mathcal{D}}(\mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)) = \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$, and our claim is proven. $\square$

We now characterize the subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ in a similar manner. We call $\bigvee^{fit}$ and $\backslash^{fit}$ the joins and the coframe differences in $\mathsf{S}_{\mathfrak{o}}(L)$, respectively.

Lemma 1.15. *In* $\mathsf{S_o}(L)$*, for all* F_i, F, G*:*

1. $\bigvee_i^{fit} F_i = fit(\bigvee_i F_i)$;

2. $F\backslash^{fit} G = fit(F\backslash G)$.

Proof. For the joins, it suffices to notice that if $F_i \subseteq F$ is an upper bound in $\mathsf{S_o}(L)$, as F is a fixpoint of *fit* also $fit(\bigvee_i F_i) \subseteq F$. By its definition, the difference $F\backslash^{fit} G$ is such that $F\backslash^{fit} G \subseteq \mathfrak{o}(x)$ if and only if $F \subseteq G \vee \mathfrak{o}(x)$ for all $x \in L$. But the following are also equivalent:

$$\frac{\dfrac{F \subseteq G \vee \mathfrak{o}(x)}{F\backslash G \subseteq \mathfrak{o}(x)}}{fit(F\backslash G) \subseteq \mathfrak{o}(x)).}$$

Since fitted sublocales are intersections of open ones, and we have shown that the fitted sublocales $F\backslash^{fit} G$ and $fit(F\backslash G)$ are contained in the same opens, they must be equal. □

Lemma 1.16. *For a frame* L *and* $F \in \mathsf{S_o}(L)$*, and* $x \in L$

$$fit(F \cap \mathfrak{c}(x)) = F\backslash^{fit} \mathfrak{o}(x).$$

Proof. By 1 of Lemma 1.2, $F \cap \mathfrak{c}(x) = F\backslash\mathfrak{o}(x)$, and so $fit(F \cap \mathfrak{c}(x)) = fit(F\backslash\mathfrak{o}(x))$. By Lemma 1.15, $fit(F\backslash\mathfrak{o}(x)) = F\backslash^{fit} \mathfrak{o}(x)$. □

Proposition 1.17. *A collection* $\mathcal{F} \subseteq \mathsf{S_o}(L)$ *is a subcolocale if and only if it is closed under arbitrary joins and is stable under* $fit(- \cap \mathfrak{c}(x))$ *for all* $x \in L$.

Proof. Suppose that $\mathcal{F} \subseteq \mathsf{S_o}(L)$ satisfies the assumptions in the claim. We check that $F \in \mathcal{F}$ implies $F\backslash^{fit} \bigcap_i \mathfrak{o}(x_i) \in \mathcal{F}$ for all families $x_i \in L$. If $F \in \mathcal{F}$, by our assumption, $fit(F \cap \mathfrak{c}(x_i)) \in \mathcal{F}$ for every $i \in I$. By Lemma 1.16, this means that $F\backslash^{fit} \mathfrak{o}(x_i) \in \mathcal{F}$. By 3 of Lemma 1.2, $F\backslash^{fit} \bigcap_i \mathfrak{o}(x_i) = \bigvee_i F\backslash^{fit} \mathfrak{o}(x_i)$, and this is in $\mathcal{F}$ as we assumed this collection is closed under all joins. Conversely, if $\mathcal{F}$ is a subcolocale, it is closed under all joins by definition, and by definition also $F\backslash^{fit} \mathfrak{o}(x) \in \mathcal{F}$ for all $x \in L$. By Lemma 1.16, then, $fit(F \cap \mathfrak{c}(x)) \in \mathcal{F}$ for all $x \in L$. □

Finally, we want to show that for every codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ the inclusion $fit[\mathcal{D}] \subseteq \mathsf{S_o}(L)$ is a subcolocale containing all opens. We also look at how the conuclei of these interact.

Lemma 1.18. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale. For every $S \in \mathsf{S}(L)$ and $x \in L$,*

1. *$\nu_{\mathcal{D}}(S \cap \mathfrak{o}(x)) = \nu_{\mathcal{D}}(S) \cap \mathfrak{o}(x)$;*

2. *$\nu_{\mathcal{D}}(S \cap \mathfrak{c}(x)) = \nu_{\mathcal{D}}(S) \cap \mathfrak{c}(x)$.*

Proof. The claims holds because $\mathfrak{o}(x), \mathfrak{c}(x) \in \mathcal{D}$ by item 1 of Lemma 1.14, and $\mathcal{D}$ is stable under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ by Proposition 1.13. □

Lemma 1.19. *For a sublocale $S \subseteq L$, $\mathit{fit}(S \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(S) \cap \mathfrak{c}(x))$ for all $x \in L$.*

Proof. The following are equivalent statements for all $y \in L$. At each step we only use basic properties of fitting and of open and closed sublocales.

$$\frac{\mathit{fit}(S \cap \mathfrak{c}(x)) \subseteq \mathfrak{o}(y)}{\frac{S \cap \mathfrak{c}(x) \subseteq \mathfrak{o}(y)}{\frac{S \subseteq \mathfrak{o}(y) \vee \mathfrak{o}(x)}{\frac{\mathit{fit}(S) \subseteq \mathfrak{o}(y) \vee \mathfrak{o}(x)}{\mathit{fit}(S) \cap \mathfrak{c}(x) \subseteq \mathfrak{o}(y).}}}}$$

Indeed, then, $\mathit{fit}(S \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(S) \cap \mathfrak{c}(x))$ as desired. □

Proposition 1.20. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale.*

1. *The collection $\mathcal{F} := \mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_\mathfrak{o}(L)$ is a subcolocale containing all opens;*

2. *$\nu_{\mathcal{F}}(F) = \mathit{fit}(\nu_{\mathcal{D}}(F))$ for all $F \in \mathsf{S}_\mathfrak{o}(L)$;*

3. *$\bigwedge^{\mathcal{F}} F_i = \mathit{fit}(\nu_{\mathcal{D}}(\bigcap_i F_i)$ for $D_i \in \mathcal{D}$.*

Proof. Let us prove the three items in turn.

1. We use the characterization in Proposition 1.17. Closure of $\mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_\mathfrak{o}(L)$ under all joins follows from the fact that fitting is a closure operator on $\mathsf{S}(L)$, and so the map $\mathit{fit} : \mathsf{S}(L) \to \mathsf{S}_\mathfrak{o}(L)$ to its fixpoints preserves all joins. Next, we have to show that for $F \in \mathit{fit}[\mathcal{D}]$ the element $\mathit{fit}(F \cap \mathfrak{c}(x))$ is in $\mathit{fit}[\mathcal{D}]$, by Proposition 1.17. This holds because for $D \in \mathcal{D}$ such that $F = \mathit{fit}(D)$ we have $D \cap \mathfrak{c}(x) \in \mathcal{D}$, by

Proposition 1.13 and $fit(D \cap \mathfrak{c}(x)) = fit(fit(D) \cap \mathfrak{c}(x))$ by Lemma 1.19. Then, $fit[\mathcal{D}] \subseteq S_\mathfrak{o}(L)$ is a subcolocale. Since $\mathfrak{o}(x) \in \mathcal{D}$ for all $x \in L$, by 1 of Lemma 1.14, $fit[\mathcal{D}]$ contains all opens.

2. The following are equivalent statements. Again, for each derivation we are only using basic facts about fitting and the conucleus $\nu_\mathcal{D}$.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{fit(\nu_\mathcal{D}(F)) \subseteq \mathfrak{o}(x)}{\nu_\mathcal{D}(F) \subseteq \mathfrak{o}(x)}}{fit(D) \subseteq F \text{ implies } fit(D) \subseteq \mathfrak{o}(x) \text{ for all } D \in \mathcal{D}}}{D \subseteq F \text{ implies } D \subseteq \mathfrak{o}(x) \text{ for all } D \in \mathcal{D}}}{\nu_\mathcal{D}(fit(F)) \subseteq \nu_\mathcal{D}(\mathfrak{o}(x))}}{\nu_\mathcal{D}(fit(F)) \subseteq \mathfrak{o}(x).}$$

3. This follows from (2) and from Lemma 1.4. $\qquad\square$

We have found a (clearly monotone) map $fit[-] : \mathcal{CD}(S(L)) \to \mathcal{C}(S_\mathfrak{o}(L))$ from codense subcolocales of $S(L)$ to subcolocales of $S_\mathfrak{o}(L)$ containing all opens. We want to show a concrete example of the assignment $fit[-] :$ $\mathcal{CD}(S(L)) \to \mathcal{C}(S_\mathfrak{o}(L))$ not being injective in general.

Lemma 1.21. *For a frame L, $fit[S_b(L)] = fit[S_\mathcal{E}(L)]$.*

Proof. Since $S_b(L) \subseteq S_\mathcal{E}(L)$ by Lemma 1.12, $fit[S_b(L)] \subseteq fit[S_\mathcal{E}(L)]$. Let us show the reverse inclusion. By Lemma 1.6, it suffices to show that $ker[S_\mathcal{E}(L)] \subseteq ker[S_b(L)]$, and since $ker[S_b(L)] = \mathsf{Filt}_\mathcal{E}(L)$, by Lemma 1.9, it suffices to show that if $E \in S_\mathcal{E}(L)$ then $ker(E)$ is an exact filter. Let $\bigwedge x_i \in L$ be an exact meet. If $E \subseteq \mathfrak{o}(x_i)$, then we have $E \cap \mathfrak{c}(x_i) \subseteq \{1\}$. By the characterization of exact sublocales in Lemma 1.7, this implies that $E \cap \mathfrak{c}(\bigwedge_i x_i) \subseteq \{1\}$, that is, $E \subseteq \mathfrak{o}(\bigwedge_i x_i)$, and so, indeed, $\bigwedge_i x_i \in ker(E)$. $\qquad\square$

By Lemma 1.21, above, then, to show that $fit[-]$ is not injective, it suffices to find a frame L where $S_\mathcal{E}(L) \nsubseteq S_b(L)$. Complete sublocales, i.e. sublocales such that their surjection preserves all meets, in particular are exact (Proposition 7.14 in [22]). Then, complete sublocales which are not in $S_b(L)$ are witnesses of $S_\mathcal{E}(L) \nsubseteq S_b(L)$. Example 5.12 in [9] presents a concrete example of this. The following class of examples of complete sublocales which are not in $S_b(L)$ is due to Igor Arrieta, who we thank.

Example 1.22. For every frame L, there is a frame surjection $\varepsilon : \mathcal{D}(L) \to L$ defined as $\varepsilon(D) = \bigvee D$. We claim that if L is completely distributive and is not T_D-spatial, then the sublocale $\varepsilon_*[L] \subseteq \mathcal{D}(L)$ associated with this surjection is exact but not in $\mathsf{S}_b(L)$. An example of such a completely distributive lattice is given, for example, by the interval $[0, 1] \subseteq \mathbb{R}$, which has no covered primes. By complete distributivity of L, the surjection $\varepsilon : \mathcal{D}(L) \to L$ preserves all meets, and so it corresponds to an exact sublocale. Let us show that this sublocale is not in $\mathsf{S}_b(L)$. We note that $\mathcal{D}(L)$ is T_D-spatial: the covered primes are exactly the elements of the form $L\backslash{\uparrow}x$ for some $x \in L$, and these meet-generate $\mathcal{D}(L)$. Then, by Lemma 1.10, every sublocale in $\mathsf{S}_b(L)$ is T_D-spatial. As L is not T_D-spatial, by assumption, the sublocale corresponding $\varepsilon_*[L] \subseteq \mathcal{D}(L)$ cannot be in $\mathsf{S}_b(L)$.

1.2 From admissible subcolocales of $\mathsf{S}_o(L)$ to subcolocales of $\mathsf{S}(L)$

In this section, we restrict to a particular class of subcolocales of $\mathsf{S}_o(L)$ and define for their collection a left order adjoint to (the suitable corestriction of) the monotone map $fit[-] : \mathcal{CD}(\mathsf{S}(L)) \to \mathcal{C}(\mathsf{S}_o(L))$. We say that a subcolocale $\mathcal{F} \subseteq \mathsf{S}_o(L)$ is *admissible* if it contains all open sublocales and the join $\bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i)$ is exact for each family $x_i \in L$.

Lemma 1.23. *If $\mathcal{D} \subseteq \mathsf{S}(L)$ is a codense subcolocale, then $fit[\mathcal{D}] \subseteq \mathsf{S}_o(L)$ is admissible.*

Proof. By Proposition 1.20, $fit[\mathcal{D}] \subseteq \mathsf{S}_o(L)$ is a subcolocale containing all opens. Let us show that for all $x_i \in L$ and $F \in \mathsf{S}_o(L)$:

$$\mathfrak{o}(\bigvee_i x_i) \wedge^{\mathcal{F}} F \subseteq \bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i) \wedge^{\mathcal{F}} F.$$

Suppose that $\mathfrak{o}(x_i) \wedge^{\mathcal{F}} F \subseteq \mathfrak{o}(y)$ for all $i \in I$. Then, by item 2 of Proposition 1.20, $fit(\nu_{\mathcal{D}}(\mathfrak{o}(x_i) \cap F)) \subseteq \mathfrak{o}(y)$, and by Lemma 1.18 this implies $\mathfrak{o}(x_i) \cap \nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y)$. This also means $\nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y) \vee \mathfrak{c}(x_i)$ for all $i \in I$, that is, $\nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y) \vee \mathfrak{c}(\bigvee_i x_i)$. Thus, $\mathfrak{o}(\bigvee_i x_i) \cap \nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y)$, and this implies $fit(\nu_{\mathcal{D}}(\mathfrak{o}(\bigvee_i x_i) \cap F)) \subseteq \mathfrak{o}(y)$, where we have used 1.18 again. As the left-hand side is $\mathfrak{o}(\bigvee_i x_i) \wedge^{\mathcal{F}} F$, by item 3 of Proposition 1.20, our claim is proven. $\square$

Corollary 1.24. *The following are all admissible subcolocales.*

1. $\mathsf{S_o}(L) \subseteq \mathsf{S_o}(L)$ *for every frame L;*

2. $fit[\mathsf{S}_b(L)] \subseteq \mathsf{S_o}(L)$ *for every frame L;*

3. $fit[\mathsf{S}_{sp}(L)] \subseteq \mathsf{S_o}(L)$ *for every spatial frame L.*

Proof. We prove each item using 1.20. For the first, we just note $\mathsf{S_o}(L) = fit[\mathsf{S}(L)]$. The inclusion $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ is codense by Lemma 1.8. Finally, for a spatial frame L, the inclusion $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is codense by Lemma 1.11. $\square$

For every subcolocale $\mathcal{F} \subseteq \mathsf{S_o}(L)$ containing all opens, for all $F \in \mathcal{F}$ we define the relation $\leq_F^{\mathcal{F}} \subseteq L \times L$ as:

$$\leq_F^{\mathcal{F}} = \{(x, y) \in L \times L \mid F \wedge^{\mathcal{F}} \mathfrak{o}(x) \subseteq \mathfrak{o}(y)\}.$$

Should $\mathcal{F} \subseteq \mathsf{S_o}(L)$ be clear from the context, we will sometimes abbreviate this as $\leq_F$.

Proposition 1.25. *A subcolocale $\mathcal{F} \subseteq \mathsf{S_o}(L)$ containing all opens is admissible if and only if for every $F \in \mathcal{F}$ the relation $\leq_F^{\mathcal{F}}$ is a frame precongruence.*

Proof. The only condition which is not shown with routine computations is stability under arbitrary joins. If $x_i \leq_F y$ for $F \in \mathcal{F}$, then $\bigvee_i^{\mathcal{F}}(F \wedge^{\mathcal{F}} \mathfrak{o}(x_i)) \subseteq \mathfrak{o}(y_i)$. The desired result follows by exactness of the join $\bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i) = \mathfrak{o}(\bigvee_i x_i)$. For the converse, suppose that there is a subcolocale $\mathcal{F} \subseteq \mathsf{S_o}(L)$ containing all opens, which is not admissible. Let $F \in \mathcal{F}$ and $x_i \in L$ with $F \wedge^{\mathcal{F}} \mathfrak{o}(\bigvee_i x_i) \nsubseteq \bigvee_i^{\mathcal{F}}(F \wedge^{\mathcal{F}} \mathfrak{o}(x_i))$. So, there is $y \in L$ with $F \wedge^{\mathcal{F}} \mathfrak{o}(x_i) \subseteq \mathfrak{o}(y)$ for all $i \in I$ but $F \wedge^{\mathcal{F}} \mathfrak{o}(\bigvee_i x_i) \nsubseteq \mathfrak{o}(y)$. The first set inclusion means $x_i \leq_F y$ for each $i \in I$, and the second means $\bigvee_i x_i \nleq_F y$. Then, the relation $\leq_F$ is not stable under joins, and so it is not a frame precongruence. $\square$

To our ends, the following characterization of admissible subcolocales will be more useful.

Corollary 1.26. *A subcolocale $\mathcal{F} \subseteq S_o(L)$ containing all opens is admissible if and only if for every $F \in \mathcal{F}$ there is $\sigma(F) \in S(L)$ such that*

$$\sigma(F) \subseteq \mathfrak{c}(x) \vee \mathfrak{o}(y) \text{ if and only if } x \leq_F y, \tag{1}$$

which is necessarily unique. Equivalently, $\sigma(F)$ is such that

$$\mathit{fit}(\sigma(F) \cap \mathfrak{o}(x)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \text{ for each } x \in L. \tag{2}$$

Proof. The first claim follows from 1.25, and by the correspondence between precongruences and sublocales. To see that the second condition is equivalent to the first, we note that the first condition amounts to having $\sigma(F) \cap \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$ if and only if $x \leq_F y$. In turn, this is equivalent to having $\sigma(F) \cap \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$ if and only if $F \wedge^{\mathcal{F}} \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$. □

For each admissible subcolocale $\mathcal{F} \subseteq S_o(L)$ we may then define a map

$$\sigma_{\mathcal{F}} : \mathcal{F} \to S(L)$$
$$F \mapsto \bigcap \{\mathfrak{o}(x) \vee \mathfrak{c}(y) \mid x, y \in L, x \leq_F y\}.$$

When $\mathcal{F}$ is clear from the context, we simply call this map σ.
Let us see a few basic facts about this map.

Lemma 1.27. *Let $\mathcal{F} \subseteq S_o(L)$ be an admissible subcolocale. Then, for each $F \in \mathcal{F}$ and $x \in L$:*

1. $\mathit{fit}(\sigma(F)) = F$;

2. $\sigma(\mathfrak{o}(x)) = \mathfrak{o}(x)$;

3. $\sigma(F \wedge^{\mathcal{F}} \mathfrak{o}(x)) = \sigma(F) \cap \mathfrak{o}(x)$.

Proof. Suppose that $\mathcal{F} \subseteq S_o(L)$ is an admissible codense subcolocale.

1. By Corollary 1.26, $\mathit{fit}(\sigma(F)) = \mathit{fit}(\sigma(F) \cap \mathfrak{o}(1)) = F \wedge^{\mathcal{F}} \mathfrak{o}(1) = F$.

2. We use Corollary 1.26. We have to check $\mathit{fit}(\mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y)$ for all $y \in L$. Indeed, $\mathit{fit}(\mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathfrak{o}(x) \cap \mathfrak{o}(y) = \mathfrak{o}(x \wedge y)$, and this equals $\mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y)$ as $\mathcal{F}$ contains all open sublocales.

3. We use Corollary 1.26 again. We have to show that

$$\mathit{fit}(\sigma(F) \cap \mathfrak{o}(x) \cap \mathfrak{o}(y)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y) \qquad (1)$$

for all $y \in L$. By the characterization of $\sigma(F)$ from 1.26.

$$\mathit{fit}(\sigma(F) \cap \mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathit{fit}(\sigma(F) \cap \mathfrak{o}(x \wedge y)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x \wedge y), \quad (2)$$

As $\mathcal{F}$ contains all open sublocales,

$$F \wedge^{\mathcal{F}} \mathfrak{o}(x \wedge y) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \cap \mathfrak{o}(y) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y). \qquad (3)$$

By combining 2 and 3, we obtain 1 as desired. □

We call $\Delta(\mathcal{F})$ the subcolocale $\mathcal{S}(\sigma[\mathcal{F}])$. For a complete lattice C and a collection $X \subseteq C$ we call $\mathcal{J}(X)$ the closure of X under arbitrary joins. If C is a coframe, we call $\mathcal{S}(X)$ the smallest subcolocale containing X.

Lemma 1.28. *Let $\mathcal{X} \subseteq \mathsf{S}(L)$ be any subset. Then $\mathcal{S}(\mathcal{X})$ is*

$$\mathcal{J}(\{X \cap \mathfrak{o}(a) \cap \mathfrak{c}(b) \mid a, b \in L, X \in \mathcal{X}\}).$$

Proof. By Proposition 1.13, any subcolocale $\mathcal{S} \subseteq \mathsf{S}(L)$ containing $\mathcal{X}$ must also contain the collection in the claim. To show the desired result, then, it suffices to show that this is a subcolocale. We use the characterization in 1.13. Closure under joins is clear, and stability under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ follows from linearity of open and closed sublocales. □

Lemma 1.29. *Let $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ be an admissible subcolocale. Then*

$$\Delta(\mathcal{F}) = \{\bigvee_i \sigma(F_i) \cap \mathfrak{c}(x_i) : F_i \in \mathcal{F}, x_i \in L\}.$$

Proof. By item 3 of Lemma 1.27, $\sigma(F) \cap \mathfrak{o}(x) \in \sigma[\mathcal{F}]$ for every $F \in \mathcal{F}$ and $x \in L$. Then, by Lemma 1.28, $\mathcal{S}(\sigma[\mathcal{F}])$ is as desired. □

Proposition 1.30. *If $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is an admissible subcolocale then $\mathit{fit}[\Delta(\mathcal{F})] = \mathcal{F}$.*

Proof. The inclusion $\mathcal{F} \subseteq \mathit{fit}[\Delta(\mathcal{F})]$ holds by item 1 of Lemma 1.27. Let us show the reverse inclusion. We notice that the map $\mathit{fit} : \Delta(\mathcal{F}) \to \mathsf{S_o}(L)$ preserves all joins, as joins in $\Delta(\mathcal{F})$ are computed as in $\mathsf{S}(L)$. Then, it suffices to prove the claim for basic elements $\sigma(G) \cap \mathfrak{c}(x)$. We now claim that $\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) \in \mathcal{F}$. Note that, by Lemma 1.19,

$$\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(\sigma(G)) \cap \mathfrak{c}(x)),$$

and $\mathit{fit}(\sigma(G)) = G$ by item 1 of Lemma 1.27. We have then shown

$$\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) = \mathit{fit}(G \cap \mathfrak{c}(x)).$$

By Proposition 1.17, this is in $\mathcal{F}$. □

Corollary 1.31. *A subcolocale $\mathcal{F} \subseteq \mathsf{S_o}(L)$ is admissible if and only if it is of the form $\mathit{fit}[\mathcal{D}]$ for some codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$.*

Proof. If $\mathcal{F}$ is admissible then $\mathit{fit}[\Delta(\mathcal{F})] = \mathcal{F}$ by Proposition 1.30. If $\mathcal{D} \subseteq \mathsf{S}(L)$ is codense, $\mathit{fit}[\mathcal{D}]$ is admissible by Lemma 1.23. □

Remark 1.32. Note that one could define $\sigma(F) = \bigcap\{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid x \leq_F y\}$ even when $\mathcal{F}$ is not admissible, but in that case $\sigma(F) \subseteq \mathfrak{c}(x) \vee \mathfrak{o}(y)$ does not necessarily imply $x \leq_F y$. Item 1 of Lemma 1.27, stating $\mathit{fit}(\sigma(F)) = F$ for $F \in \mathcal{F}$, relies on this direction of the implication. This is why one cannot extend the definition of the adjoint Δ to non-admissible subcolocales using this more general definition of σ. If F is not admissible, we cannot prove in a similar way that $\mathit{fit}(\sigma(F)) = F$, and so our proof of the $\mathcal{F} \subseteq \mathit{fit}[\Delta(\mathcal{F})]$ half of the adjointness condition does not go through.

1.3 Essential subcolocales of $\mathsf{S}(L)$

We look at what subcolocales of $\mathsf{S}(L)$ are $\Delta(\mathcal{F})$ for some admissible $\mathcal{F} \subseteq \mathsf{S_o}(L)$. For a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ we call an element *saturated* if it is of the form $\bigwedge_i^{\mathcal{D}} \mathfrak{o}(x_i) = \nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i))$ for some family $x_i \in L$. We call $\mathsf{Sat}(\mathcal{D})$ their ordered collection, and note that $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is a subcoframe inclusion. Let us define a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ to be *essential* if it is $\mathcal{S}(\mathsf{Sat}(\mathcal{D}))$.

Lemma 1.33. *For a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$*

$$\mathcal{S}(\mathsf{Sat}(\mathcal{D})) = \mathcal{J}(\{F \cap \mathfrak{c}(z) \mid F \in \mathsf{Sat}(\mathcal{D}), z \in L\}).$$

Proof. This follows from the characterization in 1.28.   □

We recall that in a category an *essential extension* is a monomorphism $j : A \to B$ such that whenever $m \circ f : A \to C$ is a monomorphism m, too, is a monomorphism. An essential extension $n : L \to N$ is *maximal* if for every essential extension $m : L \to M$ there is a unique morphism $f : M \to N$ with $f \circ m = n$. Essential extensions for frames are studied in [4]. We now justify the terminology and show that a subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ is essential if and only if $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is an essential extension in **CoFrm**.

Theorem 1.34. *Let L be a frame and $\mathcal{D} \subseteq \mathsf{S}(L)$ a codense subcolocale. Then, $\mathcal{D}$ is essential if and only if $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is essential in* **CoFrm**.

Proof. Suppose that $\mathcal{D} \subseteq \mathsf{S}(L)$ is an essential subcolocale. Suppose, now, that there is a coframe map $f : \mathcal{D} \to C$ such that it is injective when restricted to $\mathsf{Sat}(\mathcal{D})$. The elements of the form $\mathfrak{o}(x) \vee \mathfrak{c}(y)$ meet-generate $\mathcal{D}$, by Lemma 1.14, and as it is essential the elements of the form $F \cap \mathfrak{c}(z)$ with $F \in \mathsf{Sat}(\mathcal{D})$ join-generate it, by Lemma 1.33. Then, to show injectivity it suffices to show that $F \cap \mathfrak{c}(z) \not\subseteq \mathfrak{o}(x) \vee \mathfrak{c}(y)$ implies that $f(F \cap \mathfrak{c}(z)) \not\leq f(\mathfrak{o}(x) \vee \mathfrak{c}(y))$. Our assumption means that $F \cap \mathfrak{o}(y) \not\subseteq \mathfrak{o}(x \vee z)$. By assumption on f, we have that $f(F \cap \mathfrak{o}(y)) \not\leq f(\mathfrak{o}(x \vee z))$, and as f is a coframe map this also implies that $f(F) \wedge f(\mathfrak{o}(y)) \not\leq f(\mathfrak{o}(x)) \vee f(\mathfrak{o}(z))$. As f also preserves complements, this implies $f(F) \wedge f(\mathfrak{c}(z)) \not\leq f(\mathfrak{o}(x)) \vee f(\mathfrak{c}(y))$, and again as f preserves the lattice operations, $f(F \cap \mathfrak{c}(z)) \not\leq f(\mathfrak{o}(x) \vee \mathfrak{c}(y))$ as desired. If $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is essential in **CoFrm**, consider the coframe quotient given by the subcolocale $\mathcal{S}(\mathsf{Sat}(\mathcal{D}))$. This is clearly injective when restricted to $\mathsf{Sat}(\mathcal{D})$, and by essentiality it is also injective on all of $\mathcal{D}$. But then it is a bijective coframe map, hence an isomorphism, so $\mathcal{S}(\mathsf{Sat}(\mathcal{D})) = \mathcal{D}$.   □

We show some concrete examples of essential subcolocales.

Lemma 1.35. *Let L be a frame. Then $\mathfrak{b}(p) = \mathfrak{c}(p) \cap \mathit{fit}(\mathfrak{b}(p))$ for every $p \in \mathsf{pt}(L)$.*

Proof. We show $\mathfrak{c}(p) \cap \mathit{fit}(\mathfrak{b}(p)) \subseteq \mathfrak{b}(p)$. Suppose, then, that $x \in \mathfrak{c}(p) \cap \mathit{fit}(\mathfrak{b}(p))$ and $x \neq 1$. Since $x \in \mathfrak{c}(p)$, $p \leq x$. Since $x \in \mathit{fit}(\mathfrak{b}(p))$, whenever $p \in \mathfrak{o}(y)$ then $x \in \mathfrak{o}(y)$. As p is prime, $x \to p = p$ if and only if $x \not\leq p$, and so our condition means $y \not\leq p$ implies $y \to x = x$. As $x \neq 1$, $x \to x \neq x$, and so $x \leq p$. Then $x = p \in \mathfrak{b}(p)$, as desired.   □

Proposition 1.36. *For a frame L, the following are essential subcolocales of* $\mathsf{S}(L)$.

    1. The subcolocale $\mathsf{S}_{sp}(L)$ *of spatial sublocales;*

    2. The subcolocale $\mathsf{S}_b(L)$ *of joins of complemented sublocales.*

Proof. For the first item, we recall that $\mathsf{S}_{sp}(L)$ is join-generated by $\{\mathfrak{b}(p) \mid p \in \mathsf{pt}(L)\}$. Then, it suffices to show that every $\mathfrak{b}(p)$ is $F \cap \mathfrak{c}(p)$ for some $F \in \mathsf{Sat}(\mathsf{S}_{sp}(L))$. We let $\nu_{sp} : \mathsf{S}(L) \to \mathsf{S}(L)$ be the conucleus associated with $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$. By 1.18, and using Lemma 1.35,

$$\nu_{sp}(\mathit{fit}(\mathfrak{b}(p)) \cap \mathfrak{c}(p)) = \nu_{sp}(\mathit{fit}(\mathfrak{b}(p)) \cap \mathfrak{c}(p)) = \mathfrak{b}(p).$$

By definition of saturated element, $\nu_{sp}(\mathit{fit}(\mathfrak{b}(p))) \in \mathsf{Sat}(\mathsf{S}_{sp}(L))$, and so the desired claim holds. For the second item, we note that all elements of $\mathsf{S}_b(L)$ are joins of elements of the form $\mathfrak{o}(x) \cap \mathfrak{c}(y)$, and $\mathfrak{o}(x) \in \mathsf{Sat}(\mathsf{S}_b(L))$ for every $x \in L$. $\square$

    We now want to characterize essential sublocales as those codense subcolocales of the form $\Delta(\mathcal{F})$ for some admissible subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$.

Lemma 1.37. *Let* $\mathcal{D} \subseteq \mathsf{S}(L)$ *be a codense sublocale. Then,* $\nu_{\mathcal{D}}(\mathit{fit}(D)) = \sigma(\mathit{fit}(D))$ *for all* $D \in \mathcal{D}$.

Proof. Let $D \in \mathcal{D}$. We use the characterization in 1.26. We have to show

$$\mathit{fit}(\nu_{\mathcal{D}}(\mathit{fit}(D)) \cap \mathfrak{o}(x)) = \mathit{fit}(D) \wedge^{\mathcal{F}} \mathfrak{o}(x)$$

for all $x \in L$. By Lemma 1.18, the left-hand side is $\mathit{fit}(\nu_{\mathcal{D}}(\mathit{fit}(D) \cap \mathfrak{o}(x)))$. By item 3 of Proposition 1.20, this equals the right-hand side. $\square$

Lemma 1.38. *Let* $\mathcal{D} \subseteq \mathsf{S}(L)$ *be a codense subcolocale.*

    1. The maps $\sigma : \mathit{fit}[\mathcal{D}] \leftrightarrows \mathcal{D} : \mathit{fit}$ *are order adjoints, with* $\mathit{fit} \dashv \sigma$;

    2. The map $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ *is a subcoframe embedding;*

    3. $\sigma[\mathcal{F}] = \mathsf{Sat}(\mathcal{D})$.

Proof. Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a dense subcolocale.

1. For every $D \in \mathcal{D}$ and $F \in \mathit{fit}[\mathcal{D}]$, the following are equivalent.

$$\frac{\dfrac{\mathit{fit}(D) \subseteq F}{D \subseteq F}}{D \subseteq \nu_{\mathcal{D}}(F).}$$

   As $\nu_{\mathcal{D}}(F) = \sigma(F)$ for all $F \in \mathit{fit}[\mathcal{D}]$, by Lemma 1.37, the result follows.

2. As we have just shown, σ is a right adjoint, and so it preserves all meets. For binary joins, it suffices to show it preserves joins of the form $\mathfrak{o}(x) \vee \mathfrak{o}(y)$, but indeed $\sigma(\mathfrak{o}(x \vee y)) = \mathfrak{o}(x \vee y)$ by item 2 of Lemma 1.27. Injectivity of σ follows from its left adjoint being surjective.

3. By item 2, of 1.27, $\sigma(\mathfrak{o}(x)) = \mathfrak{o}(x)$ for all $x \in L$. Additionally, we have just shown that $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ is a coframe map. Hence, for all collections $x_i \in L$, $\sigma(\bigwedge_i^{\mathcal{F}} \mathfrak{o}(x_i))$ is $\nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i))$, and so it is saturated. Since this holds for all families $x_i \in L$, all saturated elements of $\mathcal{D}$ are of this form. $\qquad\square$

Lemma 1.39. *For every codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$, and every admissible subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$, $\Delta(\mathcal{F}) \subseteq \mathcal{D}$ if and only if $\mathcal{F} \subseteq \mathit{fit}[\mathcal{D}]$.*

Proof. If $\Delta(\mathcal{F}) \subseteq \mathcal{D}$, then $\mathit{fit}[\Delta(\mathcal{F})] \subseteq \mathit{fit}[\mathcal{D}]$, but the left-hand side is $\mathcal{F}$, by Proposition 1.30. Suppose that $\mathcal{F} \subseteq \mathit{fit}[\mathcal{D}]$. By definition of $\Delta(\mathcal{F})$, it suffices to show $\sigma(F) \in \mathcal{D}$, for all $F \in \mathcal{F}$. By our assumption, every such F is $\mathit{fit}(D)$ for some $D \in \mathit{fit}(D)$, and, as $\sigma(\mathit{fit}(D)) = \nu_{\mathcal{D}}(\mathit{fit}(D))$ by Lemma 1.37, this is indeed in $\mathcal{D}$. $\qquad\square$

Proposition 1.40. *Let L be a frame. A codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ is essential if and only if $\mathcal{D} \subseteq \Delta(\mathit{fit}[\mathcal{D}])$.*

Proof. By definition, $\Delta(\mathit{fit}[\mathcal{D}]) = \mathcal{S}(\sigma[\mathit{fit}[\mathcal{D}]])$. By Lemma 1.38, $\sigma[\mathit{fit}[\mathcal{D}]] = \mathsf{Sat}(\mathcal{D})$. Then, it suffices to show that $\mathcal{D}$ is essential if and only if $\mathcal{D} \subseteq \mathcal{S}(\mathsf{Sat}(\mathcal{D}))$, but this just the definition of essentiality. $\qquad\square$

We have obtained the main theorem. We call $\mathcal{CD}_{ess}(\mathsf{S}(L))$ the collection of essential codense subcolocales, and $\mathcal{PC}(\mathsf{S}_{\mathfrak{o}}(L))$ the collection of admissible subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$.

Theorem 1.41. *There is an order adjunction* $\Delta : \mathcal{PC}(\mathsf{S}_\mathfrak{o}(L)) \leftrightarrows \mathcal{CD}(\mathsf{S}(L)) :$ *fit*$[-]$ *with* $\Delta \dashv$ *fit*$[-]$, *and which maximally restricts to an isomorphism* $\mathcal{PC}(\mathsf{S}_\mathfrak{o}(L)) \cong \mathcal{CD}_{ess}(\mathsf{S}(L))$.

Proof. This follows from Lemmas 1.39 and 1.30. □

We want to use the characterization in 1.40 to provide an example of a subcolocale of $\mathsf{S}(L)$ which is not essential.

Example 1.42. As $\mathsf{S}_b(L)$ is essential, by Proposition 1.36, $\Delta(\textit{fit}[\mathsf{S}_b(L)]) = \mathsf{S}_b(L)$ by Proposition 1.40. By Lemma 1.21, $\textit{fit}[\mathsf{S}_b(L)] = \textit{fit}[\mathsf{S}_\mathcal{E}(L)]$. Thus, if for some frame we had $\mathsf{S}_\mathcal{E}(L) \not\subseteq \mathsf{S}_b(L)$, this would imply $\mathsf{S}_\mathcal{E}(L) \not\subseteq \Delta(\textit{fit}[\mathsf{S}_\mathcal{E}(L)])$, giving the desired counterexample as $\mathsf{S}_\mathcal{E}(L)$ would not be essential by Proposition 1.40. For sublocales witnessing $\mathsf{S}_\mathcal{E}(L) \not\subseteq \mathsf{S}_b(L)$, once again we refer to Example 5.12 in [9] and Example 1.22.

2. The categories **Raney** and **SZDBF**

2.1 Objects

We say that a strictly zero-dimensional biframe $(L, \mathcal{D})$ is *essential* if $\mathcal{D} \subseteq \mathsf{S}(L)$ is an essential subcolocale. We say that a Raney extension $(L, \mathcal{F})$ is *admissible* if $\mathcal{F} \subseteq \mathsf{S}_\mathfrak{o}(L)$ is an admissible subcolocale. We will use the results of the previous section to establish a bijection between admissible Raney extensions and essential strictly zero-dimensional biframes. For an admissible Raney extension $(L, \mathcal{F})$ and for a strictly zero-dimensional biframe $(M, \mathcal{D})$ we define:

$$\Delta(L, \mathcal{F}) = (L, \Delta(\mathcal{F})), \qquad \textit{fit}(M, \mathcal{D}) = (M, \textit{fit}[\mathcal{D}]).$$

Theorem 1.41, then, amounts to the following.

Theorem 2.1. *The assignments fit and* Δ *are mutually inverse bijections between admissible Raney extensions and essential strictly zero-dimensional biframes.*

2.2 Morphisms

The assignment *fit* can be easily extended to morphisms. By Lemma 1.38, there is a subcoframe inclusion $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ for every codense subcolocale $\mathcal{D}$. Then, every morphism $f : (L, \mathcal{D}) \to (M, \mathcal{E})$ determines a coframe map $\mathit{fit}(f) : \mathit{fit}[\mathcal{D}] \to \mathit{fit}[\mathcal{E}]$, which further restricts to the open sublocales to yield a frame map isomorphic to $f : L \to M$.

Proposition 2.2. *There is a functor* $\mathit{fit} : \mathbf{SZDBF} \to \mathbf{Raney}$, *whose essential image consists of the admissible Raney extensions.*

On the other hand, the assignment $(L, \mathcal{F}) \mapsto \Delta(L, \mathcal{F})$ cannot be extended to morphisms in a similar fashion. We will show that there are Raney morphisms

$$f : (L, \mathcal{F}) \to (M, \mathcal{G})$$

such that the frame map $f : L \to M$ does not lift to a map

$$f : (L, \Delta(\mathcal{F})) \to (M, \Delta(\mathcal{G})).$$

Lemma 2.3. $\Delta(L, \mathit{fit}[\mathsf{S}_b(L)]) = (L, \mathsf{S}_b(L))$ *for all frames* L.

Proof. The strictly zero-dimensional biframe $(L, \mathsf{S}_b(L))$ is essential, by Proposition 1.36. By Proposition 1.40, then, $\Delta(\mathit{fit}[\mathsf{S}_b(L)]) = \mathsf{S}_b(L)$.   □

Proposition 2.4. *For a frame* L, *and a sublocale* $S \subseteq L$:

1. S *is smooth if and only if it lifts to a map* $f : (L, \mathsf{S}_b(L)) \to (S, \mathsf{S}_b(S))$ *of strictly zero-dimensional biframes.*

2. S *is exact if and only if it lifts to a map* $f : (L, \mathit{fit}[\mathsf{S}_b(L)]) \to (S, \mathit{fit}[\mathsf{S}_b(L)])$ *of Raney extensions.*

Proof. We prove the two items in turn.

1. This follows immediately from both Lemma 3.39 in [13] and Corollary 4.2 of [1].

2. In [11] it is proven that there is an isomorphism $\mathit{fit}[\mathsf{S}_b(L)] \cong \mathsf{S}_{\mathfrak{c}}(L)^{op}$. This is also an isomorphism $(L, \mathit{fit}[\mathsf{S}_b(L)]) \cong (L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ of Raney extensions. In Proposition 6.6 of [22] the frame morphisms $f : L \to M$ that lift to Raney extensions are characterized as the exact maps. Finally, in Proposition 7.14 of [22] it is shown that surjections that preserve all exact meets are exact maps.   □

We are ready to give the desired counterexample.

Example 2.5. Suppose there is a frame L with a sublocale $S \subseteq L$ which is exact but not smooth. Proposition 2.4 and Lemma 2.3 imply that the corresponding frame surjection $s : L \to S$ lifts to a Raney morphism $f : (L, \mathit{fit}[\mathsf{S}_b(L)]) \to (S, \mathit{fit}[\mathsf{S}_b(S)])$ which does not in turn lift to a morphism $f : \Delta(L, \mathit{fit}[\mathsf{S}_b(L)]) \to \Delta(S, \mathit{fit}[\mathsf{S}_b(S)])$ in **SZDBF**. Therefore, once again a counterexample is provided by both 5.12 of [9] and Example 1.22.

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