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AUTOCATEGORIES: IV. DRAWING, GRADUATION AND COLORATION FOR AUTOGRAPHIC DATA

René GUITART

Abstract. In this fourth paper on autocategories, we add three results regarding autographs. Firstly we show how their representations are special localisations or graduations — with the special example of globular sets of Brown and Higgins, seen as dimensioned autographs. Then we explain how drawings of autographs are ambiguous and are related to virtual knots of Kauffman. And finally we explain how others informations can be add on an autograph by colocalisation or coloration, recovering gractes of Riguet and our 1-regimes. Thus we build bridges between uses of autographs, autograph morphisms, and autocategories

Résumé. Dans ce quatrième volet sur les autocatégories nous ajoutons trois résultats sur les autographes. Tout d'abord nous montrons comment leurs représentations sont des localisations particulière ou graduations — avec le cas particulier des ensembles globulaires de Brown et Higgins, considéré comme autographes dimensionnés. Puis nous expliquons comment le dessin des autographes est ambigu et lié à la question des entrelacs virtuels de Kauffman. Et finalement nous expliquons comment des informations supplémentaires peuvent être ajoutées par colocalisation ou coloration, retrouvant alors les gractes de Riguet et nos 1-régimes. Ainsi nous établissons des transferts entre les usages d'autographes, de morphismes d'autographes, et d'autocatégories.

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Contents

1. Introduction: new aspects of the autographic approach

After some brief reviews of the definitions and results obtained previously (section 2), here we continue our study of autocategories, examining their representations, designs, and uses to describe structures.

Two points — representations and drawings — have to be determined in a more flexible manner. Representations are seen as localisations and/or graduation, and so applied to notion of globular complex (section 3 and section 4). With respect to drawings and ambiguity, we examine relations with virtual knot for drawing of knots and links, and we introduce the notion of virtualized autographs and autosuccessions (section 5). In order to describe structures in a practical way, we look at co-localisations as colorations (section 6).

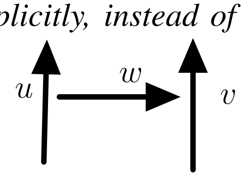
These explorations will therefore allow us to obtain new examples. This will make it possible to highlight the description of numerous autographic structures or algebras by the simple data of a co-relation, or even by a morphism of autographs.

2. Return to previous concepts and examples

2.1 Autograph

Definition 2.1. *(voir [7], p. 66, & p. 76.) An* autograph $A = (\underline{A}, d, c)$ *is an action on a set* $\underline{A}$ *of* $\mathbb{FM}(2) = \{d, c\}^*$ *the free monoid on two generators, i.e. a set* $\underline{A}$ *equipped with two maps "domain" and "codomain"* $d, c : \underline{A} \to \underline{A}$.

Elements in $\underline{A}$ *are considered as arrows in* A. *An element* w *with* $dw = u$ *and* $cw = v$ *being an arrow from* u *to* v, *shortly denoted by* $w : u \to v$.
But w *is not considered as an edge between two vertices* u *and* v *: it is an interaction between two others interactions* $u : p \to q$, $v : r \to s$, *and then* $p : m \to n$, *etc. More explicitly, instead of* $w : u \to v$ *we could write*

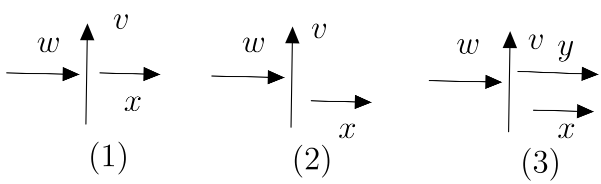


Remarque 2.1. An autograph is not a graph, with vertices and edges: in an autograph there are only arrows. However, a graph determines an autograph

by replacing each vertex X with an arrow called an auto-arrow $X : X \to X$.

In an autograph an auto-arrow $a : a \to a$ could be drawn by a simple circle as $\bigcirc_a$ (see [7]).

Remarque 2.2. In fact often we drew an autograph in the same way as, in knot theory, we draw the representation of a plane projection of an interlacing of oriented strings, with indication at the crossings of the passages of the strings one under the other, the line of the one below being drawn interrupted.



So in the case of an oriented knot or a link, each string (denoted by an arrow w) passing under v is continuing as itself (now denoted by a new arrow w^+) ; but however, in the general case of an autograph it should not be assumed that w is preceded by an arrow w^- passing under u toward w, or that w has to pass under v toward an arrow $x = w^+$ (case (1)). Moreover — see case (2) or (3) — even if an arrow w^- or $x = w^+$, or $y = w^+$ is specified, with $cw^- = dw$ or $dw^+ = cw$, one should not imagine it necessarily draws in the visual prolongation of w, as in the true crossing case (1).

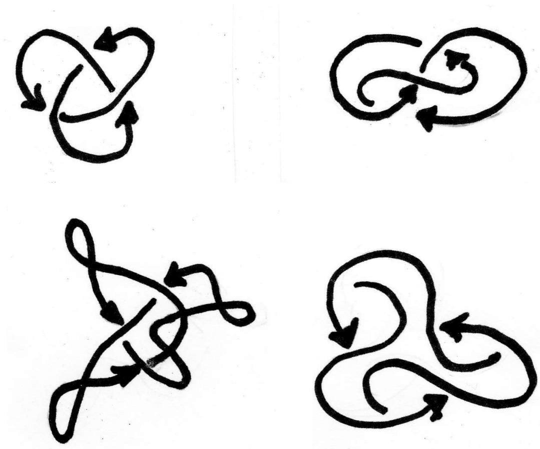
Remarque 2.3. Here we will have to come back to this method of drawing, in particular on the question of virtual crossings, and also on the question of orientation. In any case, we must not confuse the data $A = (\underline{A}, d, c)$ of an autograph with its exhibition by a drawing, particularly because of the ambiguities inherent in these drawings. We will start from various drawings representing the autograph $\mathbb{C}_3$, in the following example 2.2.

2.2 Examples

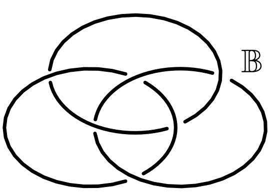
Example 2.2. The autograph $\mathbb{C}_3$ is $\mathbb{C}_3 = (C_3, d_3, c_3)$ with $C_3 = \{x, y, z\}$,

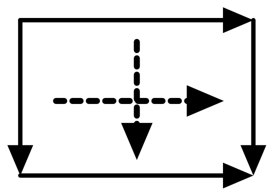
$$d_3(x) = y, c_3(x) = z, d_3(y) = z, c_3(y) = x, d_3(z) = x, c_3(z) = y.$$

$\mathbb{C}_3$ can be drawn in several ways : as planar representation of a knot i.e. as a trefoil or a double eight (first line), or as a virtual knot (see 5.1), or as a triskel (second line, left and right):



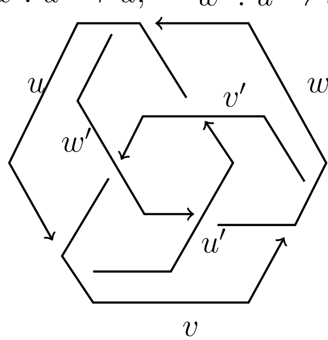
The first reason for introducing autographs was the possibility of a common presentation for interlaces and for 2-categories or double categories, as in the cases of a borromean link or a 2-block in a double category.



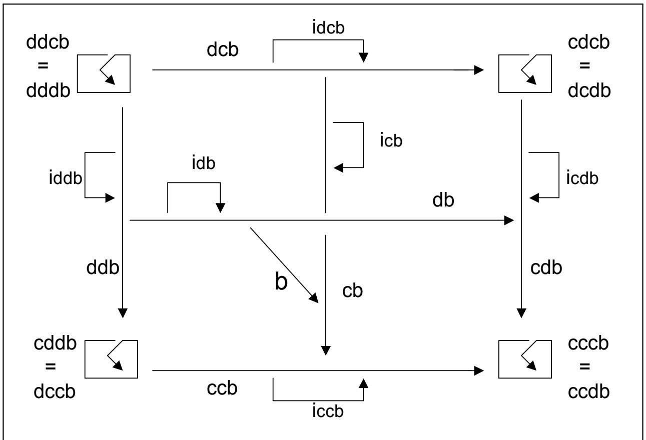


Example 2.3. Our first example is the *Borromean* autograph $\mathrm{As}(\mathbb{B})$, forged from the idea of the Borromean interlacing $\mathbb{B}$ (above on the left):

$$\begin{array}{ll} u : v' \to v, & u' : v \to v', \\ v : w' \to w, & v' : w \to w', \\ w : u' \to u, & w' : u \to u'. \end{array}$$

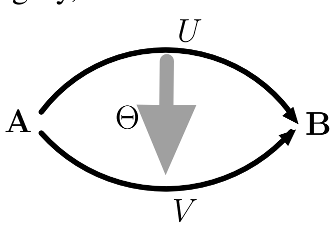


Example 2.4. Our second example is the following presentation as an autograph of a 2-block (as above on the right) b in a double category:



A 2-block b as an autograph

Example 2.5. A special case of example 2.4 is a 2-cell $\Theta : U \Longrightarrow V$ with $U, V : A \to B$, in a 2-category, drawn as



Example 2.6. The free autograph $\mathrm{FA}(3\mathbb{N})$ on $\aleph_0$ generators is explained in [7] (in such a way that $\mathbb{R}$ appears as a "descent-completion" of this $\mathrm{FA}(3\mathbb{N})$). See also hereafter in proposition 5.20.

2.3 Autocategories, autographic algebra

Definition 2.7. *An* autocategory *is an autograph with identifiers (see [7]), equipped with a composition law given by gf for consecutive arrows f and g (with $dg = cf$), which is unitary and associative. As with autographs, autocategories do not have objects like categories do, but an autocategory can be associated with a category, by replacing objects with auto-arrows. An autocategory "is" also a mono-flexicategory, i.e. a category $\mathcal{C}$ equipped* *with a* flex $\varphi : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{C})$ *which is injective (see proposition 3.4 in [8]).*

Examples of autographs and autocategories are given in [7] and [9] : graphs, categories, 2-graphs, 2-categories, doubles categories, but also knots and links.

Definition 2.8. *(voir [8], p. 156.) An* autographic algebra *is an algebra of a monad* $\mathbb{T} = (T, \eta, \mu)$ *on the category* Agraph *of autographs, which of course is a topos So — of course — it is a determined by a morphism* $T(A) \overset{\theta}{\to} A$*, that is to say an object in the category* Agraph/A *of objects over* A*.*

In [8], we shew that graphs, basic graphic algebras, categories, autocategories, are autographic algebras, and we compared autographic algebras with Burroni's graphic algebras.

Example 2.9 (Free graph on an autograph)**.** The forgetting functor Φ : Graph $\to$ Agraph associates to a graph G the autograph $\Phi(G)$ with the same set of arrows, where for each vertex V the arrow $i(V)$ is now an auto-arrow. Its left adjoint Λ : Agraph $\to$ Graph associates to an autograph A the graph $\Lambda(A)$ with $\Phi(\Lambda(A))$ the quotient of A by $c^2 = c, d^2 = d, cd = d, dc = c$, and hence $\Lambda(\Phi(G)) \simeq G$. (see Proposition 2.2 in [8]).

2.4 Surgery and co-relations

In the papers [7], [8] and [9], we show how surgery can be analyzed with spans or cospans in Agraph, i.e. with relations or co-relations between autographs. This point will have to be developed later in relation to the logic of the topos Agraph. In order to do so, now we allow ourselves from this point to introduce the following terminology by the definition 2.10:

Definition 2.10. *An* autographic co-relation (reps.relation) *between two autographs* A *and* A' *is a co-span (reps. a span) of two morphisms* i *and* i' *(reps.* p *and* p'*) of autographs*

$$A \overset{i}{\to} C \overset{i'}{\leftarrow} A'),\ \text{(resp.}\ A \overset{p}{\leftarrow} R \overset{p'}{\to} A').$$

Remarque 2.4. *Intuitive terminology* — A morphism of autograph $m : B \to A$ is seen as a *graduation* of B over A (and an object of the localisation of A), and it is seen as a *coloration* (or a decoration) of A (and an object of the co-localisation of B). So a span or a co-span is a kind of mixed composition of a graduation and a coloration.

3. Localisations and representations among autographs

The purpose of this section is to show how a representation of an autograph A is a localisation relative to A, i.e. a map $B \to A$, and consequently how it is a structuration of A. But let us also say as well that $B \to A$ a structuration of B; this is compatible with the fact that structures can be both algebraic or coalgebraic.

In the paper [9], we described categories of representations of a given autograph A, and that allowed us to introduce the notions of *autorelation* and *automap*. However, this notion of representation will have to be adapted here in order to make obvious its relationship with the localisation in the topos of the autographs. What we do here is therefore the analog in the framework of autographs of Yoneda's lemma and the relationship between actions of categories and fibrations.

3.1 Localisations over an autograph

We revisit the notion of a representation of an autograph (seen [9]) and extend it a little, and this allows us to conceive of representations as localisations.

Definition 3.1. *1 — The category* Agraph $= \mathrm{Set}^{\mathbb{FM}(2)}$ *of autographs is introduced in [8, p. 152], a morphism $m' : A' \to A$ in this category from $A' = (\underline{A'}, d', c')$ to $A = (\underline{A}, d, c)$ being a map $m' = \underline{A'} \to \underline{A}$ between sets, such that*

$$m'd' = dm', \quad m'c' = cm'.$$

This Agraph *is a topos.*
2 — Given an autograph $A = (\underline{A}, d, c)$, the topos of autographs over A *is denoted by* Agraph$/A$*, its objects are (A', m') with $m' : A' \to A$ in* Agraph*, and a morphism in* Agraph$/A$ *from (A', m') — with $m' : A' \to A$,*

to (A'', m'') *— with* $m'' : A'' \to A$*, is a morphism* $h : A' \to A''$ *in* Agraph *such that*

$$m''h = m'.$$

The category Agraph$/A$ *is actually a topos, named the* localisation of *Agraph* with respect to A*. Also if* (A', m') *is an object of* Agraph$/A$*, the autograph* A' *is said to be* localised over A via m'.
3 — We introduce a forgetful functor associated to each given A*:*

$$U_A : \text{Agraph}/A \longrightarrow \text{Agraph} : ((A', m') \xrightarrow{h} (A', m')) \mapsto (A' \xrightarrow{h} A').$$

This functor U_A *determines* A*, by* $A = U_A(T)$*, with* T *the terminal object in* Agraph$/A$*.*

3.2 Representations of an autograph

Definition 3.2. *Given an autograph* $A = (\underline{A}, d, c)$*, a* spanning representation *of* A *is a data* $\varphi = (\Phi, \phi^d, \phi^c)$*, with for each* $f \in A$*, the data of a set* $\Phi(f)$ *and a span of functions (named the* d *and the* c *projections)*

$$\Phi(df) \xleftarrow{\phi^d(f)} \Phi(f) \xrightarrow{\phi^c(f)} \Phi(cf).$$

The category of spanning representations of A *is denoted by* $\text{Rep}_{\text{span}}(\underline{A}, d, c)$ *or* $\text{Rep}_{\text{span}}(A)$*, a morphism* t *in this category from* φ *to* φ' *being the data, for each* $f \in A$ *of a map*

$$t_f : \Phi(f) \to \Phi'(f),$$

such that

$$t_{df}\phi^d(f) = \phi'^d(f)t_f, \quad t_{cf}\phi^c(f) = \phi'^c(f)t_f.$$

Definition 3.3. *We define some full subcategories of* $\text{Rep}_{\text{span}}(\underline{A}, d, c)$*:*

$$\text{Rep}_{\text{func}}(\underline{A}, d, c) \subset \text{Rep}_{\text{part}}(\underline{A}, d, c) \subset \text{Rep}_{\text{rel}}(\underline{A}, d, c) \subset \text{Rep}_{\text{span}}(\underline{A}, d, c),$$

as those with objects:

- relational representation*: for every* $f \in \underline{A}$*, the double* (d, c) *projection is a canonical inclusion*

$$[\phi^d(f), \phi^c(f)] : \Phi(f) \subset \Phi(df) \times \Phi(cf).$$

- partially functional representation*: a relational representation where, for every $f \in \underline{A}$, the* d *projection is a canonical inclusion:*

$$\phi^d(f) : \Phi(f) \subset \Phi(df).$$

- functional representation*: a partially functional representation in which, for every $f \in \underline{A}$, the* d *projection is an identity:*

$$\phi^d(f) : \Phi(f) = \Phi(df).$$

These sets of objects are denoted by

$$\mathcal{F}(\underline{A}, d, c) \subset \mathcal{P}(\underline{A}, d, c) \subset \mathcal{R}(\underline{A}, d, c) \subset \mathcal{S}(\underline{A}, d, c).$$

Remark 3.4. 1 — A category of functional representations Rep(A, d, c) is defined in [9, Propos. 1.14]. Our definition here of Rep$_{\text{func}}(A, d, c)$ can be seen as a slightly modification of this Rep(A, d, c), by adding the restriction that t is 'strict':

$$t^d_f[\text{resp. } t^c_f]\text{depends only on } df[\text{resp. } cf]\text{, it could be denoted by } t_{df}[\text{resp. } t_{cf}].$$

2 — In [9, Definition 1.6.] a spanning representation is identical to a relational representation, but now it is different, the fact that the double projection is a canonical inclusion is not yet assumed.

Proposition 3.5. *To each $w \in A$ is associated a functional representation Γ^A_w of A, and for every representation Φ of A we have a natural bijection:*

$$\xi_w : \mathrm{Hom}_{\mathbf{Rep}(A,d,c)}(\Gamma^A_w, \Phi) \simeq \Phi(cw) : t \mapsto \xi_w(t) = t_{cw}(w).$$

Proof. Of course, according to [9, Propos. 1.15], (A, d, c) can be identified with an object of this (new) Rep(A, d, c) or of **Rep**(A, d, c), via its regular representation (Γ^A, γ^A): for each $g \in A$ we consider the set $\Gamma^A(g)$ of (d, c)-paths (cf. [7, Definition 1.4]) with end g, that is to say $z = (z_n)_{0 \leq n < k}$ with

$$cz_0 = dz_1,\ cz_1 = dz_2,\ cz_{k-2} = dz_{k-1}, cz_{k-1} = g,$$

and for each $f \in A$ the map $\gamma^A(f) : \Gamma^A(df) \to \Gamma^A(cf)$ described by concatenation with f.

Then, for each $w \in A$ and $g \in A$ we define $\Gamma^A_w(g)$ as a subset of $\Gamma^A(g)$:

$$\Gamma^A_w(g) = \{z = (z_n)_{0\leq n<k} \in \Gamma^A(g); z_0 = w\},$$

and with γ^A_w defined by restriction of γ^A this determines the representation associated to w.

Now, if $t : \Gamma^A_w \to \Phi$, by the naturally of t, for $f : g \to h$ we get

$$\phi(f)t_g(z) = t_h(\gamma^A_w(f)(z)),$$

and as $z = \gamma^A_w(z_{k-1})\dots\gamma^A_w(z_1)(w)$, we have, for every $z \in \Gamma^A_w(g)$:

$$t_g(z) = \phi(z_{k-1})\dots\phi(z_1)t_{cw}(w),$$

and so t is determined by $\xi_w(t) = t_{cw}(w) \in \Phi(cw)$.
Remark. Up to a bijection, Γ^A_w depends only on cw, but for a given t, the associated $\xi_w(t)$ really depends on w, hence ξ_w depends on w. □

Definition 3.6. *Given an autograph $A = (\underline{A}, d, c)$, in [7, Proposition 6.3] we have constructed* $\mathrm{P}(A) = \big(\mathrm{Path}^\iota(A, d, c), D, C\big)$ *the* free autocategory of paths *in A, $(z_n)_{0\leq n<k} = (z_n)_k$, with $D((z_n)_k) = (dz_0)_0$, $C((z_n)_k) = (cz_{k-1})_0$, in which we add, for every $a \in A$ of the form dx or cy, an identity element I_a on the path $(a)_0$ of length 1 determined by a.*

Now we modified this construction to get the category $\mathrm{P}_{lab}(A)$ of paths with labelled vertices*: an object is an element u of A, seen as a label for cu, and a morphism from u to v is a morphism $(z_n)_{0\leq n<k}$ in $\mathrm{P}(A)$ from $cu = dz_0$ to $cv = cz_{k-1}$.*

Proposition 3.7. *Given an autograph A and two elements $u, v \in A$, then, with definition 3.6 and Proposition 3.5, we have:*

$$Hom_{\mathrm{Rep}(A,d,c)}(\Gamma^A_u, \Gamma^A_v) = Hom_{\mathrm{P}_{lab}(A)}(v, u),$$

hence Γ^A is a functor

$$\Gamma^A : \mathrm{P}_{lab}(A) \longrightarrow \mathrm{Rep}(A, d, c)^{op}.$$

Proof. Proposition 3.5 for $\Phi = \Gamma^v$ gives $\mathrm{Hom}_{\mathrm{Rep}(A,d,c)}(\Gamma^A_u, \Gamma^A_v) = \Gamma^A_v(cu)$. □

Proposition 3.8. *1 —- Given a relational representation $\varphi = (\Phi, \phi^d, \phi^c)$ of an autograph $A = (\underline{A}, d, c)$, as in [9, Definition 1.6.], with for each $f \in A$,*

$$\Phi(df) \overset{\phi^d(f)}{\leftarrow} \Phi(f) \overset{\phi^c(f)}{\rightarrow} \Phi(cf),$$

we get a morphism in Agraph *over A,*

$$q_\varphi : \Sigma\varphi \to A,$$

with $\Sigma\varphi = (S_\varphi, d_\varphi, c_\varphi)$, $S_\varphi = \{(f, u); f \in A, u \in \Phi(f)\}$, $q_\varphi(f, u) = f$, $d_\varphi(f, u) = (d_A f, \phi^d(u))$, $c_\varphi(f, u) = (c_A f, \phi^c(u))$.
2 — Conversely, given an arbitrary arrow

$$q : A' = (\underline{A'}, d', c') \to A,$$

we get a relational representation of A given by $\Phi(f) = q^{-1}(f)$, $\phi^d(x) = d'(x)$, $\phi^c(x) = c'(x)$.
3 — As a corollary, being such a $q : A' \to A$, each autographic algebra

$$\theta : T(A) \to A$$

determines a relational representation of A.

Proof. The part 1 is [9, Proposition 1.7.], with some corrections of typos, and the part 2 is the generalization of the end of [9, Proposition 1.2.]. □

4. Dimensioned autographs, the case of globular sets

As a sequel of section 3, in this section we show how globular sets are some autographs structured by $dd = dc$ and $cc = cd$, and a map to $\mathbb{N}^{\text{aug}}$.

4.1 Dimensioned autographs

We introduce the notion of a *dimensioned autographs* as being an autograph equipped with a graduation (see *terminology* at the beginning of section 2.4) toward the autograph of numbers $\mathbb{N}^{\text{aug}}$.

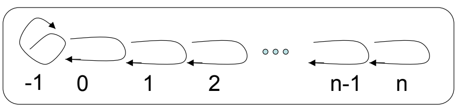
Definition 4.1. *We denoted by* $\mathbb{N}^{\text{aug}}$ *the autograph of which the underlying set is the set* $\mathbb{N}$ *of integer augmented with* -1

$$\underline{\mathbb{N}}^{\text{aug}} = \mathbb{N} \cup \{-1\},$$

and where the domain and codomain are equal to the same map "predecessor" pred *given by* $\text{pred}(n+1) = n$*,* $\text{pred}(0) = \text{pred}(-1) = -1$*. Also we consider* D_n *with*

$$\underline{\text{D}}_n = \{x \in \mathbb{N}^{\text{aug}}; x \leq n\},$$

with on it the domain and codomain induced from pred *as in the picture :*



D_n

A Dimensioned augmented autograph — *or shortly a* dimensioned autograph — *is an autograph equipped with a "dimension",i.e. a morphism of autograph*

$$D : A \to \mathbb{N}^{\text{aug}}.$$

These dimensioned (augmented) autographs constitute a topos over Agraph

$$\text{Agraph}_{\text{dim}} = \text{Agraph}/\mathbb{N}^{\text{aug}} \longrightarrow \text{Agraph},$$

of which

$$\text{Agraph}_{n-\text{dim}} = \text{Agraph}/D_n$$

is the sub-topos of n*-dimensioned autographs.*

4.2 ∞-graphs and globular sets

The next definition comes — without using the term — from Ronnie Brown and Philip J. Higgins [1].

Definition 4.2. *A* globular set X *is a collection* $(X_n)_{n \geq 0}$ *of sets with maps* $s_n, t_n : X_{n+1} \to X_n$ *such that*

$$s_n s_{n+1} = s_n t_{n+1} \text{ and } t_n s_{n+1} = t_n t_{n+1}.$$

Hence given $m < n$ there are only two maps $s_{m,n}, t_{m,n} : X_n \to X_m$ obtainable by compositions of s_k and t_l:

$$s_{m,n} = s_m ...., t_{m,n} = t_m .... : X_n \longrightarrow X_m.$$

Theses globular sets constitute a topos Glob *of presheaves on $\mathbb{G}$, where $\mathbb{G}$ is the globe category, with objects integers $n \in \mathbb{N}$, and whose morphisms are generate by $\sigma_n, \tau_n : n \to n+1$, with relations*

$$\sigma_{n+1}\sigma_n = \tau_{n+1}\sigma_n \text{ and } \sigma_{n+1}\tau_n = \tau_{n+1}\tau_n,$$

$$\text{Glob} = \text{Ens}^{\mathbb{G}^{op}}$$

Proposition 4.3. *A globular set is exactly determined by an autograph $A = (\underline{A}, d, c)$ with*

$$dd = dc, \quad cd = cc,$$

equipped with a morphism "dimension" as in Definition 4.1

$$D : A \to \mathbb{N}^{\text{aug}}.$$

Proof. The datum of a globular set X is equivalent to the datum of an autograph $\tilde{X} = (\underline{\tilde{X}}, d, c)$ with $\tilde{X}_n = X_n \times \{n\}$, $\tilde{X}_{-1} = \{*\} \times \{-1\}$,

$$\underline{\tilde{X}} = \bigcup_{n \geq -1} \tilde{X}_n = \bigcup_{n > 0} X_n \times \{n\} \cup (\{*\} \times \{-1\}),$$

$$d(x,n) = (s_{n-1}(x), n-1),\ c(x,n) = (t_{n-1}(x), n-1) \text{ for } x \in X_n, n > 0,$$

$$d(x,0) = (*,-1) = c(x,0), \text{for } x \in X_0,\ d(*,-1) = (*,-1) = c(*,-1),$$

and consequently we have

$$dd = dc, \quad cd = cc.$$

Given this $\tilde{X}$ we recovered X with: $\tilde{X}_{-1} = \{x \in \tilde{X}; dx = x = cx\}$, $\tilde{X}_0 = \{x \in \tilde{X}; dx, cx \in \tilde{X}_{-1}, x \notin \tilde{X}_{-1}\}$, and, for $n > 1$,

$$\tilde{X}_n = \{x \in \tilde{X}; dx, cx \in \tilde{X}_{n-1}, x \notin \tilde{X}_{n-1}\}.$$

□

5. Observations on drawings of autographs

The following observations could be considered as comments on drawings of $\mathbb{C}_3$ in example 2.2. And along the way we are led to interesting structural enrichements of autographs (virtualizations, transductions, autosuccessions, partial autographs and free autographs, unoriented autographs). They serve as preparations of examples for colorings (section 6) and their uses in the description of structured autographs.

5.1 Virtual drawings, virtualized autographs, auto-transductions

As evocated in section 2, in our previous papers [7], [8] and [9], we gave some planar drawings of (oriented) autographs for alternating knots, for some knots and links. We gave also drawings for 2-categories, double categories, autocategories: let us notice that for a 2-blocks in a double categories, the drawing is not planar, because (see Example 2.4) cb is a broken arrow, passing under db, going from dcb to ccb, and b is naming a kind of "cushion" between two faces db and cb; so its picture is rather related in fact to virtual link (see 5). Furthermore some case may have planar and not planar (virtual) drawings, as it is the case for $\mathbb{C}_3$ in Example 2.2. Nevertheless, planar drawings are not enough to represent any finite link, or a fortiori any finite autograph — and now here we are reaching such convenient planar representations with oriented *virtual diagram* or *virtual links*, for links, and *planar oriented virtual drawings* for arbitrary finite autographs.

The autograph $\mathbb{C}_3$ in example 2.2 is drawn (second line, left) by a virtual knot, according to definition 5.1. In fact it is an autosuccession (Definition 5.12).

Definition 5.1. *According to [13, Def.1.4, p.8], a* (planar) virtual diagram *or a diagram of a* virtual link *is any image of an immersion of a framed* 4*-valent graph in* $\mathbb{R}^2$ *with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a* virtual crossing *and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph are called* classical crossings *or just* crossing.

Definition 5.2. *To an oriented virtual diagram Λ of an oriented link we associate an autograph $\mathbb{A}[\Lambda]$ as follows. We neglect virtual crossings — such that any arc x arriving at a virtual crossing with an arc y (which may be x itself!) is continuing its path on the other side of y, with the same name x, the domain dx of x being the other arc that x meet at its starting classic crossing, the codomain cx of x being the other arc at its ending classic crossing.*

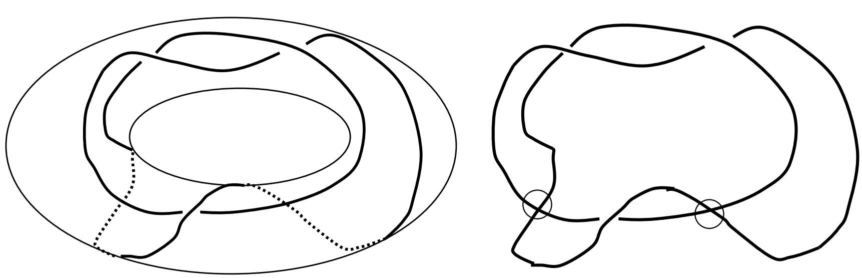
Definition 5.3. *A* virtual link *is an equivalence class $[\Lambda]$ of a virtual diagram Λ modulo the generalised Reidmeister moves.*

Proposition 5.4. *A virtual knot $[K]$ is determines by $\mathbb{A}[K]$.*

Proof. As observed in [13, p. 9], in order to define a *virtual knot* $[K]$, we need only to know the position of classical crossings and there connections with each others in K. Moreover, positions of paths connecting classical crossings, their intersections and self-intersections, are not important for us. □

Proposition 5.5. *[Kauffman, [11]] Any link can be drawn as a virtual link, and starting from a virtual link we can decide if it representing a true link.*

Proof. The idea of *virtual knot* and *virtual link* comes from Louis Kauffman. For example, from [11, p. 666] we look to these two unoriented pictures:



On the left a knot is drawn on a torus, with 3 crossings *on the surface of the torus*, each one with two lines near a same point on the torus, and with also 2 "false crossings", i.e. crossing of a continous line (ahead) and a dashed line (behind) in our perspective, which in fact are not near the same point on the torus. In a planar view (on the right), these "false crossing"" are marked by small circle, wheras now the lines are all continuous; there we get *virtual planar drawing* of a knot.

In our example C_3 in 2.2 the virtual knot on the left down is coming from the knot on the left up, hence to a virtual knot is associated an autograph with 3 arcs x, y, z, which is also the autograph of a knot (the trefoil knot) (cf. [7, Example 4.3., p. 71] where x, y, z are u, v, w). But a virtual link (or virtual knot) is *not* necessarily coming from a link (or a knot) in such a way.

The autograph associated to an oriented virtual diagram determines the associated oriented virtual link. This virtual link could be a classical link, and generally it can be determined by many autographs.

Proposition 5.6. *Any* planar virtual drawing *is determined by at least one such finite autograph.*

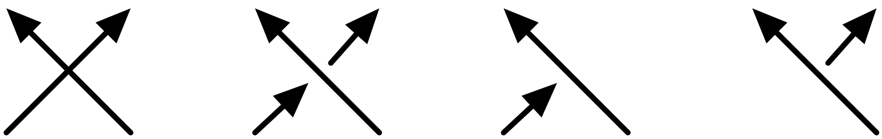
Of course, starting with a finite autograph it is easy to draw an oriented virtual link to which it is associated, the use of virtual crossings being unlimited.

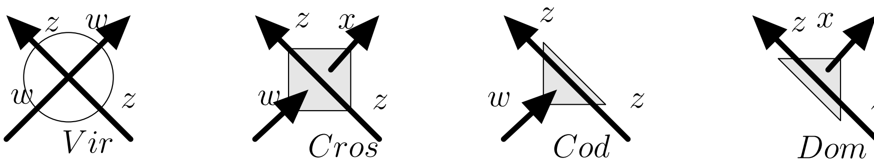
The observations above do suggested to introduce the notion of a *virtual drawing of an autograph* and of a *virtualized autograph* (or a strictly increasing *auto-transduction*) as follows in definition 5.10. This notion allows to obtain a more accurate codification of planar virtual drawings of autographs.

Definition 5.7 (Virtual drawing of an autograph)**.** *A* virtual drawing *of an autograph $A = (\underline{A}, d, c)$ is (similarly to the explanation given in definition 5.1) a picture in the plan made of a collection of arrows $[w]$, $w \in A$, each such arrow being drawn "from" its domain $[dw]$ "toward" its codomain $[cw]$; so we have an* effective crossing *for $[w]$ and $[dw]$, for $[w]$ and $[cw]$; each $[w]$ has also a finite number of intersections with the others, each of them is called a* virtual crossing.

Easily we have the following proposition5.8.

Proposition 5.8. *Each finite autograph A admits a virtual drawing which is representable by a* planar graph $\Gamma(A)$ *with four types of vertices, as given in the picture below. The first type is "virtual", the third and fourth or "half-crossing" types, are namely "codomain" and "domain", and the second is "(complete) crossing". A complete crossing is to think as a kind of apairing or coupling of a "codomain" and a "domain", a fact explicitely signifiable by $s(w) = x$ [see remarks 2.2, remarks 2.3, subsection 5.2, definition 5.12].*





Proof. Of course, starting with a finite autograph $A = (\underline{A}, c, d)$ it is easy to draw a virtual drawing, the use of virtual crossings being unlimited. Furthermore, from such a drawing we can observe the corresponding virtualized autograph $(\underline{A}, c, d, V)$, according to Definition 5.10. Then the graph $\Gamma(A)$ is obtained from A with vertices the various virtual crossings, complete crossings and half-crossings, and with, for each arrow $f \in \underline{A}$, $f : df \to cf$, a sequence of edges $(f, f_1) : df \to f_1, \ldots, (f_i, f_{i+1}) : f_i \to f_{i+1}, \ldots, (f_n, cf) : f_n \to cf$. □

Remark 5.9. A finite graph is not necessarily planar, but of course considered as an autograph it admits a finite planar representation by a virtual drawing (with virtual crossings). The reader will not mistake these two facts.

Definition 5.10. *A* virtualized autograph *is the data* $(\underline{A}, c, d, V)$ *of an autograph* $A = (\underline{A}, c, d)$ *structured by the data of a function* $V : \underline{A} \to \underline{A}^*$ *— with* $\underline{A}^*$ *the free monoïd of words in* $\underline{A}$ *, where, for every* $f \in \underline{A}$*,* $V(f)$ *is thought as the sequence of arrows in* A *which, in a planar drawing, could virtually be crossed (virtually) by* f *in the order of circulation from* cf *to* df*,* $V(f) = f_1 f_2 \ldots f_n$ *; but it is not assumed that such a planar drawing do exist for this given* V*.*
An auto-transduction *is a* $(\underline{A}, W)$ *of a set* $\underline{A}$ *with a function* $W : \underline{A} \to \underline{A}^*$*.*
Obviously, with $W(f) = d(f)V(f)c(f)$*, a virtualized autograph is equivalent to an auto-transduction where for every* $f \in \underline{A}$*,* $W(f)$ *is of length at least* 2*; we can say that such a* W *is strictly increasing.*

Example 5.11. In the case of the virtual drawing of $\mathbb{C}_3$ in 2.2 left down, we get the virtualization $V(x) = yyxx$, $V(y) = zzyy$, $V(z) = zz$.

In the case of example 2.4, we have $V(cb) = db$, $V(db) = cb$, and for others f, $V(f) = ()$ (the empty word).

5.2 The case of an autosuccession

If we want to structure an autograph to get closer to the notion of a knot or a link, we can introduce the notion of an *autosuccession*. In the case of the autograph $(\underline{A}, d, c)$ associated to a link we have $c(A) \subseteq d(A)$, and more precisely, for each w there is a unique $x = s(w)$ such that $cx = dw$. So it is an autosuccession according to Definition5.12.

Definition 5.12. *An* autosuccession *is a data* $(\underline{A}, d, c; s)$ *of an autograph* $(\underline{A}, d, c)$ *equipped with a map* $s : \underline{A} \to \underline{A}$ *such that*

$$ds = c.$$

If furthermore d*,* c *and* s *are bijective, the autosuccession is said to be an* alternating autosuccession.

Remark 5.13. Of course an autosuccession is an arbitrary data $((\underline{A}, d, s)$ of an autograph (with s as a codomain map), and $(\underline{A}, d, c; s) = (\underline{A}, d, ds; s)$; so, starting from an autograph $((\underline{A}, d, s)$ we generate a sequence of autosuccessions: $(\underline{A}, d, ds; s)$,$(\underline{A}, d, d^2s; ds)$, $\ldots (\underline{A}, d, d^{n+1}s; d^ns)$.

Proposition 5.14. *In a finite autosuccession each element* u *generates a circular path* $u, su, ssu, \ldots, s^nu = u$*, for* $n = n(u)$ *an integer associated to* u*. These paths can be drawn in a planar virtual diagram of a virtual link (oriented), and conversely any virtual knot (oriented) is determined by at least one such autosuccession. Furthermore finite alternative autosuccessions correspond to alternating knots.*

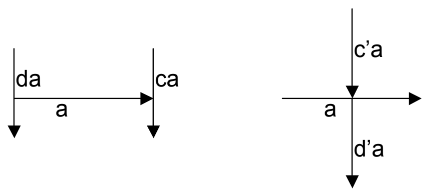
Proof. It is an immediate consequence of Proposition 5.5. □

Remark 5.15. The analysis of a link as an autograph structured as an autosuccession is not the only possibility. For instance, we could use of quandles or racks (see [10], [14]).

5.3 Two drawings of a reversible autograph.

In fact $\mathbb{C}_3$ in example 2.2, considered as being a planar view of a knot (see on the first line of its given drawings), is a reversible autograph according to the definition 5.18 and so it is possible to draw it in another way.

Definition 5.16. *[standard partial drawing, dual partial drawings] If a set $\underline{A}$ is an autograph with d and c, for each $a \in \underline{A}$, we write it as on the left (standard partial drawing). If a set $\underline{A'}$ is an autograph with d' and c', we would like to have a kind of "dual "drawing process", for each $a \in A'$, as given on the right (dual partial drawing).*



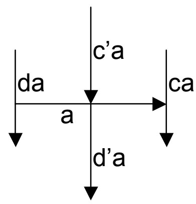
NB. Here "partial " means limited to the only element a.

Proposition 5.17. *For each autograph (A, d, c) the standard partial drawing is possible for any given $a \in A$. For an autograph (A', d', c') the dual partial drawing is possible for any given $a \in A'$ if and only if d' and c' are injective.*

Proof. For c, d, every arrow a has a unique domain da and a unique codomain ca, and this is physically clear on the standard drawing. For c', d' also we admit that every arrow a has a unique domain $d'a$ and a unique codomain $c'a$, but this is not obvious on the picture of the dual drawing, as we seem to be physically free to introduce several arrows starting from or arriving on the side of a. Furthermore a more serious difficulty is the question to draw a fact as $a = c'u$ and $a = c'v$, meaning that simultaneously a has to arrived on the side of u and on the side of v ! In fact the dual drawing for c', d' is possible if and only if c' and d' are injective. □

Definition 5.18. *An autograph is a* reversible autograph *if the two maps d and c are invertible (bijective maps); the inverses of d and c are denoted by $d^{-1} = d'$ and $c^{-1} = c'$.*

Proposition 5.19. *Given a reversible autograph* $A = (\underline{A}, d, c, d', c')$*, with* $c' = c^{-1}, d' = d^{-1}$*, we can draw it using the standard local drawing for* d, c *and the dual local drawing for* d', c'*, at a point* a*, and these drawings are compatible and can be drawn both in the same picture:*



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5.4 Partial autograph and associated free autograph

Proposition 5.20. *1 — We can construct the free autograph on one generator* $\mathbb{FA}(1)$ *as being* $\mathbb{FM}(2)$.
2 — We can construct the free reversible autograph on one generator $\mathbb{FRA}(1)$ *as being* $\mathbb{FG}(2) = \mathbb{Z} \star \mathbb{Z}$*, the free group on two generators.*

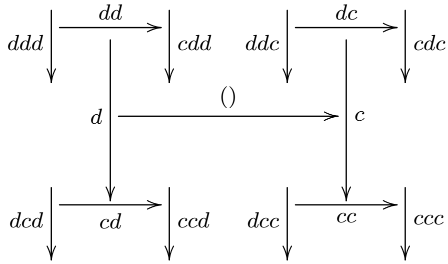
Proof. 1 — The free autograph on one generator "()" or "f" is denoted by $\mathbb{FA}(())$ or $\mathbb{FA}(\{f\})$, or $\mathbb{FA}(1)$, and in fact we have

$$\mathbb{FA}(1) = \mathbb{FA}(()) = \mathbb{FA}(\{f\}) \simeq \mathbb{FM}(2).$$

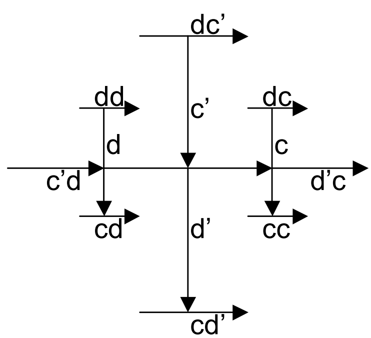
A detailed description of $\mathbb{FA}(\{f\})$ is given in the proof of [7, Proposition 3.1], as the sub-autograph generated by 0 in the free autograph on $\aleph_0$ generators

$$\mathbb{FA}(3\mathbb{N}) = (\mathbb{N}, m \mapsto 3m+1, m \mapsto 3m+2).$$

If in $\mathbb{FA}(\{f\})$ we cancel everywhere the letter f, or if we replace it by the empty word $()$, we get the set $\mathbb{FM}(2) = \{d, c\}^*$, with a domain d and a codomain c given by $d(m) = dm, c(m) = cm$, for every word m. As showded in [7, Proposition 3.1], a picture of this starts with:



2 — If to the picture for $\mathbb{FA}(\{f\})$ indefinitely we add new arrows in such a way that each old or new arrow g is the domain and the codomain of an unique arrow $u = d'g$ and an unique arrow $v = c'g$, then we get $\mathbb{FRA}(\{f\})$. A picture of this starts with this extension of the begin of the start for $\mathbb{FM}(2)$ given above



□

The observations in the subsection 5.3 and this subsection 5.4 , and notably the consideration of $\mathbb{FA}(\{f\})$, leads to the next construction of $FA(X)$ (Proposition 5.24).

Definition 5.21. *A* partial autograph *is a data* $X = (\underline{X}, d, c)$ *of a set* $\underline{X}$ *equipped with two partial maps* $d, c : \underline{X} \to \underline{X}$.

Proposition 5.22. *A finite partial autograph admits a virtual drawing which is representable by a* planar graph *with six types of vertices, namely the four given in proposition 5.8, and two new one* $\star(u)$*, at the source of* u *if* df *is not defined, and* $\bigcirc(u)$ *at the target of* u *if* cf *is not defined.*

Remark 5.23. Given a *fragment of an autograph* as in Example 1.5 in [7], with some "sources" $\bigcirc_j$ and "targets" $\star_i$, if we neglect these sources and target, we get a partial autograph M, which can be transformed into an autograph by substituting to each source or target an auto-arrow (as explained in [7]). But we can also obtain an autograph by using Proposition 5.24, and substituting a "free arrow" to each source or target, with the construction $FA(X)$ for $X = M$.

Proposition 5.24 (Free autograph $FA(X)$ on a partial autograph X)**.** — *We construct $FA(X)$ the free autograph generated by X by the next formal additions: for each $f \in \underline{X}$, if df is not defined, we add to $\underline{X}$ a formal data df and $FA(\{df\})$; if cf is not defined, we add to $\underline{X}$ a formal data cf and $FA(\{cf\})$. Hence this new set $\overline{FA(X)}$ is underlying to an autograph $FA(X)$, in which X is embedded $J_X : X \to FA(X)$ and such that each morphism $U : X \to A$ to an autograph admits a unique factorization $U' : FA(X) \to A$, with $U'J_X = U$.*

Obviously we have:

Proposition 5.25. *Given an autograph $A = (\underline{A}, d, c)$, a subset $E \subseteq A$, and the partial autograph $Z_E = (E, d_{|E}, c_{|E})$, the inclusion $U_E : E \to A$ factorize through $J_{Z_E} : E \to FA(Z_E)$ as $V_E : FA(Z_E) \to A$. The image of V_E is the sub-autograph of A generated by E.*

5.5 Drawings with or without orientations

With respect to this question of drawings (see subsection 2.2), now we have to precise something about "unoriented autographs".

In [8] and [9] we show how graphs "are" autographs, and how algebraicity or monadicity over graphs can be transfer over autographs. At this moment we let on the side the question of the description from the point of view of autographs of some important variant notions of graphs. Here we specially examine the question of orientation.

5.5.1 Equivocal on terminology

In fact, according to the history and tradition in "Graph Theory" and/or in "Category Theory", there are several variants, according to several facts:

vertices are or are not interpretable as arrows, arrows between two vertices are or are not unique, arrows are or are not oriented.
var1. We use the term "*graph*" as in the categoricians'style, for a datum of a span $V \overset{s}{\leftarrow} E \overset{t}{\rightarrow} V$ with a map $V \overset{i}{\rightarrow} E$ such that $si = 1_V = ti$. This is sketched by $\mathbb{G}(2)$ in our Definition 1.2. in [8, p.152], with $G(v_0) = V$, $G(v_1) = E$, $G(\delta_0) = s$, $G(\gamma_0) = t$, $G(\iota) = i$. This definition is named *var1*, and in [8, Proposition 2.2, p. 154] we explained how this notion is algebraic over autographs.
var2. Someones named "graph" the same thing that our "graph" in *var1* excepted they miss i, i.e. considering only a datum $V \overset{s}{\leftarrow} E \overset{t}{\rightarrow} V$.
var3. For others persons such a graph in our sense in *var1* is named a dimultigraph (or an oriented multigraph) , and for them a "directed graph" or a "digraph" is a datum $E \subset V \times V$ (that we named a binary relation on V), whereas a "graph" is a datum $E \subset V \times V$ which is symmetric (a symmetric binary relation on V). Hence for them a "multigraph" is a dimultigraph without orientations on arrows.
var4. Some other authors named "graph" a datum $V \overset{s}{\leftarrow} E \overset{t}{\rightarrow} V$ with a map $E \overset{()^\sigma}{\rightarrow} E$ such that $(e^\sigma)^\sigma = e$, $s(e) = t(e^\sigma)$, and "oriented graph" such a graph in which in each pair $\{e, e^\sigma\}$ one edge is chosen to be called the positively oriented edge, and denoted e^+, and the other is e^-.
Remark. Of course such a graph in *var4* is equivalent to the datum of an "unoriented (multi)graph" i.e. a map $E \overset{\delta}{\rightarrow} \mathcal{P}_{1\leq 2}(V)$, from E to the set of unordered pairs of elements of V. given by $\delta(\epsilon) = \{se, te\}$; and such an "oriented graph" is equivalent to the datum of a graph in our sense (*var1*).
Precaution. Consequently on reading the mathematical litterature we have to be careful with the meaning of the term "graph" when referring to results or notions in Graph Theory, as: coloured graph, automorphisms of a graph, Cayley's graph, Frucht's theorem, etc. The same precaution will be necessary here with the terms "autograph" and "unoriented autograph", "coloured autograph", etc.

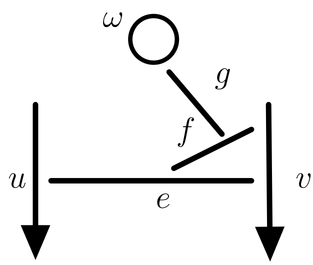
5.5.2 Unorientation or orientation?

When we draw an autograph with arrows, each arrow has to be oriented, according the direction of the move of our pencil. But if we do not write the information of this move, for instance with a hook at the end, then the visible result is an *unoriented autograph.* So an autograph is an unoriented autograph (definition 5.26) equipped with an orientation.

Definition 5.26. *An* unoriented autograph $U = (\underline{U}, \delta)$ *is a set* $\underline{U}$ *with a map* $\delta : \underline{U} \to \mathcal{P}_{1\leq 2}(\underline{U})$ *from* $\underline{U}$ *to the set of unordered pairs of elements of* $\underline{U}$*. The fact that* $\delta(w) = \{u, v\}$ *is drawn as an unoriented arc or "unarrow"* $u \overset{w}{\sim} v$*.*

Proposition 5.27 (unoriented simulation)**.** *Each autograph* $A = (\underline{A}, d, c)$ *can be simulated by a pointed unoriented autograph.*

Proof. Each autograph $A = (\underline{A}, d, c)$ determines an underlying unoriented autograph $U(A) = (\underline{A}, \delta)$, given on the set $\underline{A}$ by $\delta(w) = \{dw, cw\}$. We can add to $U(A)$ an auto-unarrow ω and for each $e \in \underline{A}$ from $de = u$ to $ce = v$, two new unarrows f and g with $\delta(f) = \{e, ce\}$ and $\delta(g) = \{\omega, f\}$, according to the picture



This new unoriented autograph is denoted by $\overline{U}(A)$ and named the unoriented simulation of A. The point is that the knowledge of $\overline{U}(A)$ with the specification of the point ω determine A (equipped with its orientation) : the orientation of e (from u to v) is given by the corresponding data (f, g). So the data A is equivalent to the data

$$U(A) \to \overline{U}(A) \leftarrow \{\omega\}.$$

□

Proposition 5.28. *Any unoriented autograph* U *can be presented as an involutive autograph* A*, and also as a bi-pointed autograph.*

Proof. 1 — Given U, according to *var4* in the terminology above, we choose two maps $\phi, \psi : U \to \{s, t\}$ such that $\delta(f) = \{\phi(f), \psi(f)\}$ and we construct $U_{\phi,\psi} = U^+ + U^-$, with $U^+ = \{f^+, f \in U\}$, $U^- = \{f^-, f \in U\}$, and with d and c given by $d(f^+) = \phi(f)$, $c(f^+) = \psi(f)$, $d(f^-) = \psi(f)$, $c(f^-) = \phi(f)$; furthermore we have an involution σ given by $(f^+)^\sigma = f^-$, $(f^-)^\sigma = f^+$. We recover U by identifying every $g \in A$ (which is an f^+ or an f^-) with g^σ. In fact this identification is simulated as follows.

If now we introduce an autograph with two auto-arrows, namely

$$\mathbb{T} = \{\{+\}, \{-\}\},$$

we add to $U_{\phi,\psi}$ that autograph and, for each $f \in U$ an arrow $p : \{+\} \to f^+$ and an arrow $q : \{-\} \to f^-$, and also an arrow $r : p \to q$, in such a way to express the identification by this r and via p and q of f^+ and f^-, we obtain an autograph $\overline{U_{\phi,\psi}}$. Then the data of U is determined by the data

$$U_{\phi,\psi} \leftarrow \mathbb{T}.$$

□

5.5.3 Automorphisms of autographs or of unoriented autographs

As a sequel of subsubsection 5.5.2, here we show how autographs or unoriented autograph can be used in order to describe groups.

Proposition 5.29. *Any autograph can be re-described as an unoriented autograph, with the same automorphism group. Hence any group is the group of automorphism of an autograph as well as the automorphism group of an unoriented autograph.*

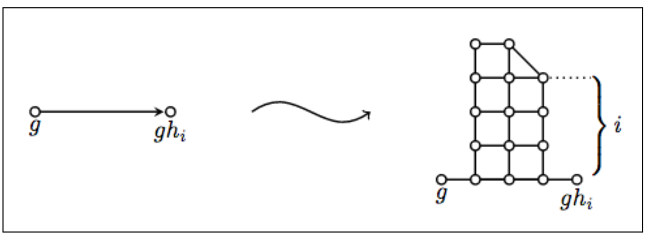
Proof. We start with the case of coloured digraph (i.e. a data $E \subset V \times V$, with a coloured map $\kappa : E \to K$) and of graph (in the sense of a data of a symmetric binary relation $E \subset V \times V$).

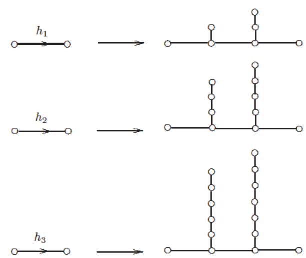
The colored Cayley's digraph $\mathcal{C}_\Delta(G)$ of a group G generated by a set Δ is the graph with vertices elements u of G, an with arrow any $u \xrightarrow{h} uh$, with $h \in \Delta$, and h being the color of this arrow. Every $f \in G$ determines a coloured automorphism $\rho_f : u \mapsto fu$ on the coloured graph $\mathcal{C}_\Delta(G)$ (an automorphism preserving the colors), in such a way that G is isomorphic to $\mathrm{Aut}(\mathcal{C}_\Delta(G))$ (If ϕ is a coloured automorphism of $\mathcal{C}_\Delta(G)$, then we have $\phi(uh) = \phi(u)h$, and as $u = 1h_1...h_q$ we get $\phi(u) = \phi(1)u = fu$, with $f = \phi(1)$, $\phi = \rho_f$).

Now this coloured graph can be transformed into a graph, the Frucht's graph $\mathcal{F}_\Delta(G)$ with

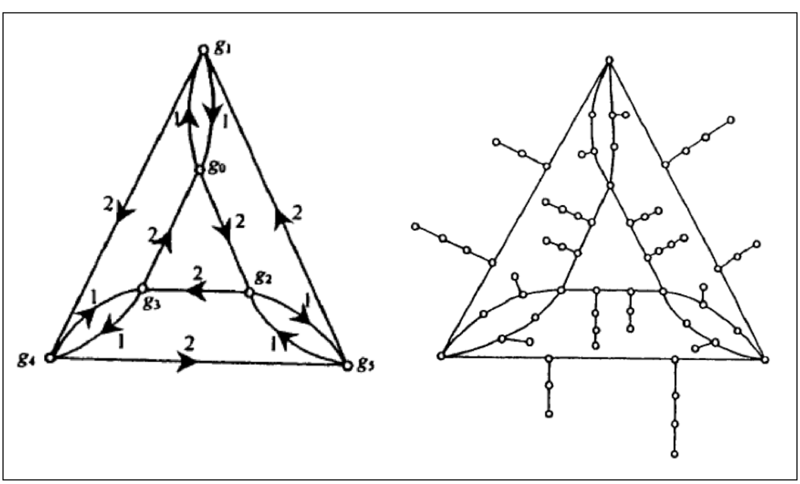
$$G = \text{Aut}(\mathcal{C}_\Delta(G)) = \text{Aut}(\mathcal{F}_\Delta(G)).$$

To do that, the orientation of an oriented arrow is simulated in an unoriented way by the adjunction of some "lateral" decoration with "unoriented arrows" or segments;in fact these decorations represent simultaneously the orientation and the color of the arrow. This can be performed as in the next picture with an unsymmetrical tower of high i (this i or h_i being the color) — on the left, or any variant — as on the right





As an example, let us consider as in [18, pp. 295, 299] the colored Cayley's graph of $\mathcal{S}(3)$ generated by a transposition denoted by 1:=(12) and a 3-cycle denoted by 2:= (123) — so we have two colors 1 and 2 —, and conjointly, the corresponding Frucht's unoriented graph, as in the following picture.



The result for finite groups is in [3], for arbitrary groups in [16] and [4].

Now for an autograph or an unoriented autograph we do the same job. □

6. Colored autographs and structurations

6.1 Colored autographs

An autograph can be coloured, a color or a label being affected to each arrow. In fact — Proposition 6.2 —colored autographs are autographs under a given fixed colouring autograph. A color can express the name of an action (as with gracts) or the name of a point of view for assimilation (with regimes of assimilations) as in Proposition 6.3.

Definition 6.1. *Given a set K, a K-*colored autograph *is an autograph $A = (\underline{A}, d, c)$ (a set $\underline{A}$ and two maps $d, c : \underline{A} \to \underline{A}$) equipped with a "coloring map" $\kappa : \underline{A} \to K$, from the set $\underline{A}$ toward the set of colours K. Colors are also named labels.*

Proposition 6.2. *If K is finite, a K-colored autograph (A, κ) can be specified as an ordinary autograph $A[\kappa]$ in which A is included as a sub-autograph, and in which K is represented by an autograph $[K]$, giving a cospan between A and $[K]$:*

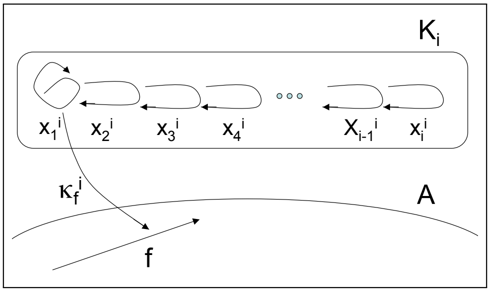
$$A \longrightarrow A[\kappa] \longleftarrow [K].$$

*So (A, κ) looks like a $[K]$-algebra $A[\kappa]$; that is to say that the category K-Agraph of coloured autograph — of which a map $z : (A, \kappa) \to (B, \lambda)$ is a map of autograph $z : A \to B$ commuting with κ and λ (or $\lambda(z(f)) = \kappa(f)$) — is isomorphic to the category $[K]/$*Agraph *of autographs under $[K]$.*

Proof. If K has n elements, we consider $[K]$ as the disjoint union of n autographs $K_1, \ldots K_n$, each K_i being constituted of i elements $x^i_1, x^i_2, \ldots, x^i_i$, with:

$$dx^i_1 = cx^i_1 = x^i_1,\ dx^i_2 = cx^i_2 = x^i_1, \ldots, dx^i_i = cx^i_i = x^i_{i-1}.$$

The construction of K_i ensures that colors i are internally distinguishable in terms of autographs (the K_i are not isomorphic). To construct $A[\kappa]$, to A we add $[K]$ and, for each $f \in A$ with $\kappa(f) = i$ we put a new arrow $\kappa^i_f : x^i_1 \to f$. This arrow says that the colour of f is i.



□

6.2 Gracts and regimes

Proposition 6.3. Gracts *in the sense of [15], and* 1-regimes *in the sense of [6], are examples of coloured autographs.*

Proof. Firstly, let us recall that a graph "is" an autograph, hence a coloured graph is a coloured autograph.
Now, in order to represent an action of a set W of operations on a set S of states, Jacques Riguet [15] introduced the notion of a *gract* or graph-action drawing, symbolizing the fact that the action w acting on s produces t as: $s \overset{w}{\to} t$. There s (resp.t) is the source (resp. target) of $s \overset{w}{\to} t$, or of (t, w, s), but *it is not* the source (resp. target) of the symbol w alone. Especially the symbol w could appears in several arrows as $s \overset{w}{\to} t$ or $u \overset{w}{\to} v$, etc. The gract Γ is the full drawing of all these vertices and arrows. Formally a gract Γ is only determined by a map

$$\gamma : W \times S \to S : (w, s) \mapsto t = w.s = \gamma(w, s),$$

and it can be also represented by a graph $G(\Gamma)$ with vertices $s \in S$, in which an arrow from s to t is a 3-uple (t, w, s) such that $t = w.s$, represented as: $s \overset{(t,w,s)}{\longrightarrow} t$. The set of arrows of $G(\Gamma)$ is $A(\Gamma) = \{a = (t, w, s); w.s = t\}$ a subset of $S \times W \times S$. The set of arrows from s to t is denoted by $G(s, t)$. Furthermore these arrows have *colored* values in W, given by $\kappa(a) = w$ if and only if $a = (t, w, s)$, the same color w being possibly affected to different arrows $a = (t, w, s)$ and $a' = (t', w, s')$ in $G(\Gamma)$, but only if a and a' have different source ($s \neq s'$) or different target ($t \neq t'$).

So a gract is nothing else than a graph $G = (A, S)$ equipped with a surjective map $\kappa : A \to W$, injective on each $G(s, t)$; or a graph equipped with an ordinary equivalence relation or a partition on its set of arrows which is discrete on each $G(s, t)$. A gract is also a decomposition

$$A = \sum_{w \in W} A_w, \quad A_w = \{a \in A : \kappa(a) = w\}$$

of a graph G as a sum of binary relations A_w on the set S.

A gract is also a special case of a 1-*regime of assimilation* in the sense of [6], i.e. a map

$$r : W \to \mathcal{P}(S \times S),$$

here given by $r(w) = \{(t, s); w.s = t\}$. It is the special case in which, for all w, $r(w)$ is a function $r(w) : S \to S$, $r : W \to S^S$ is the companion of γ. Of course r is equivalent to the datum

$$\theta : S \times S \to \mathcal{P}(W)$$

transposed from r as $\theta(t, s) = \{w; (t, s) \in r(w)\} = \{w; w.s = t\}$, and that is a binary relation on S coloured in $\mathcal{P}(W)$. □

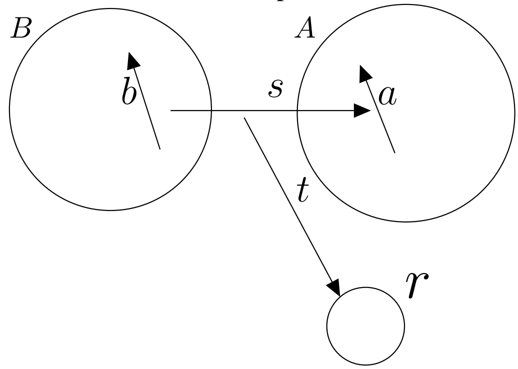
6.3 Specifying relations and structures by poly-colorations or glueings

Definition 6.4. *Given two autographs B and A, and given a binary relation $R \subset \underline{B} \times \underline{A}$ we consider the* glueing *$B \oplus_R A$ or $\mathcal{G}(B, A)(R)$ of B and A with respect to R, which is the autograph obtained from the disjoint sum $B + A$ of B and A by adding a new auto-arrow denote by r (with $dr = r = cr$), and for each $b \in B$, $a \in A$ such that $(b, a) \in R$, two new arrows $s : b \to a$ and $t : s \to r$. We have two canonical injective morphisms $B \overset{\beta_R}{\to} B \oplus_R A \overset{\alpha_R}{\leftarrow} A$, and a map $j_R :< \beta_R, \alpha_R >: B + A \to B \oplus_R A = \mathcal{G}(B, A)(R)$.*
If $R = \rho_F$ i.e. if R is the functional relation associated to a map $F : B \to A$, then $B \oplus_{\rho_F} A$ is simply denoted by $B \oplus_F A$.

Proposition 6.5. *The glueing $B \oplus_R A$ with the co-span (β_R, α_R) determines the binary relation R by the fact that $(b, a) \in R$ if and only if in $B \oplus_R A$:*

$$\exists s,\ \exists t\ \big(ds = \beta_R b, cs = \alpha_R a, dt = s, ct = r\big),$$

or — with $\beta_R b \cong b$ and $\alpha_R a \cong a$ — the picture



So the binary relation R is specified by the data

$$B + A \to B \oplus_R A \leftarrow \{r\}.$$

Definition 6.6. *Given three autographs C, B and A, and a ternary relation $R \subset \underline{C} \times \underline{B} \times \underline{A}$ we consider the glueing $\mathcal{G}(C, B, A)(R)$ of C, B, A with respect to R, which is the autograph obtained from a disjoint union $C+B+A$ of C, B and A, to which we add a new auto-arrow denote by r (with $dr = r = cr$), and for each $c \in C$, $b \in B$ and $a \in A$ such that $(c, b, a) \in R$, the three following arrows : $s : c \to b$, $t : s \to r$, $u : a \to t$. We have three canonical injective morphisms:*

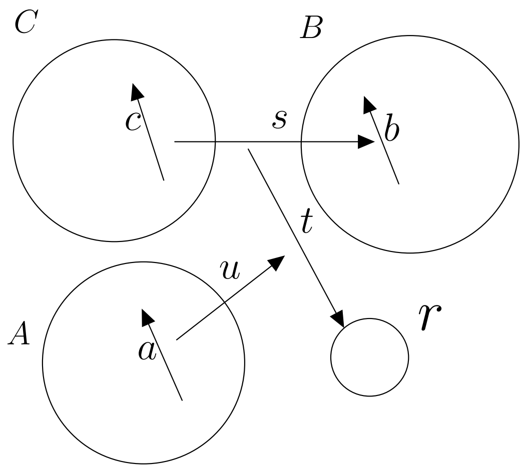
$$C \overset{\gamma_R}{\to} \mathcal{G}(C, B, A)(R), B \overset{\beta_R}{\to} \mathcal{G}(C, B, A)(R), A \overset{\alpha_R}{\to} \mathcal{G}(C, B, A)(R),$$

and then a map $j_R :< \gamma_R, \beta_R, \alpha_R >: C + B + A \to \mathcal{G}(C, B, A)(R)$.
This works in particular for the ternary relation ρ_F associated to a binary partial law $F : C \times B \to A$.

Proposition 6.7. *The glueing $\mathcal{G}(C, B, A)(R)$ determines the ternary relation R by the fact that $(c, b, a) \in R$ if and only if :*

$$\exists (s, t, u) \big(ds = \gamma c, cs = \beta b, dt = s, ct = r, du = \alpha a, cu = t\big),$$

or — with $\gamma_R c \cong c$, $\beta_R b \cong b$ and $\alpha_R a \cong a$ — the picture



So the ternary relation R is specified by the data

$$C + B + A \to \mathcal{G}(C, B, A)(R) \leftarrow \{r\}.$$

Remark 6.8. The glueing construction $\mathcal{G}(C, B, A)(R)$ extend to three terms C, B and A the construction of proposition 6.4, and also the construction of proposition 6.2 for two terms (A and K). So we considered it as a poly-coloration, specified by three injective morphisms $\gamma : C \to \mathcal{G}(C, B, A)(R)$, $\beta : B \to \mathcal{G}(C, B, A)(R)$, and $\alpha : A \to \mathcal{G}(C, B, A)(R)$, or by the morphism $j =< \gamma, \beta, \alpha >: B + C + A \to \mathcal{G}(C, B, A)(R)$.

7. Autograph, morphism of autograph, or autocategory ?

As a conclusion, we want to emphasize that, during our exploration here, we have established in passing that the three notions of autograph, autograph morphism, autocategory, are equivalent data, and that we can describe each of them by a drawing as a virtual diagram. Indeed :

1. An autograph A is a morphism of autograph.
2. A representation of an autograph is a morphism of autograph.
3. An autographic algebra is a morphism of autograph.
4. An autocategory is an autographic algebra, and so it is a morphism of autograph.
5. A colored autograph or a poly-colored autograph is an autograph.
6. An autocategory is a poly-colored autograph, and so is an autograph.

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René Guitart
Université Paris Diderot Paris 7. IMJ-PRG. UMR 7586
Bâtiment Sophie Germain. Case 7012
75205 Paris Cedex 13
rene.guitart@orange.fr