CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXVII-1 (2026)

A HOPF FORMULA FOR THE FUNDAMENTAL GROUP IN THE CATEGORY OF PREORDERED GROUPS

Aline MICHEL

Résumé. En nous basant sur la description des extensions centrales et normales récemment obtenue dans le contexte des groupes préordonnés, nous montrons que la formule bien connue de Hopf pour les groupes peut être étendue aux groupes préordonnés. Pour atteindre cet objectif, nous construisons une extension centrale faiblement universelle pour tout groupe préordonné donné. Cette construction nous permet de calculer son groupe fondamental en adaptant le cas classique des groupes au contexte des groupes préordonnés.

Abstract. Based on the description of central and normal extensions recently obtained in the setting of preordered groups, we show that the well-known Hopf formula for groups can be extended to preordered groups. In order to achieve this goal, we build a weak universal central extension for any given preordered group. This construction allows us to compute its fundamental group by adapting the classical case of groups to the setting of preordered groups.

Keywords. Preordered groups, categorical Galois theory, weak universal central extension, regular projective object, Hopf formula, fundamental group.

Mathematics Subject Classification (2010). 06F15, 18E10, 18G50, 18A40, 18C40.

1. Introduction

In the category Grp of groups, there is a well-known categorical Galois theory [7]. Indeed, from the *abelianization functor* $ab\colon \mathsf{Grp} \to \mathsf{Ab}$, which associates to any group X the quotient $X/[X,X]$ by its derived subgroup, we can consider the *Galois structure* $\Gamma_{ab} = (\mathsf{Grp}, \mathsf{Ab}, ab, u, \mathcal{E}_{ab}, \mathcal{Z}_{ab})$ where $u\colon \mathsf{Ab} \to \mathsf{Grp}$ is the inclusion functor, and $\mathcal{E}$ and $\mathcal{Z}$ are the classes of regular epimorphisms in Grp and Ab, respectively. This Galois structure turns out to be admissible, so that a classification theorem of central extensions holds. Furthermore, it has been proven that the central and normal extensions coincide; these are the surjective group morphisms $f\colon A \twoheadrightarrow B$ whose kernel $\mathsf{Ker}(f)$ is in the center $Z(A)$ of A.

Based on this useful information, computing the *fundamental group* [9] of any group X (with respect to the above Galois structure Γ_{ab}) is quite simple. Consider a free presentation of the group X

$$K \overset{k}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} X$$

with kernel (K,k). We can then prove that the induced surjective group morphism $\phi\colon P/[K,P] \twoheadrightarrow X$ in the commutative diagram below is such that $\mathsf{Ker}(\phi) \subseteq Z(P/[K,P])$, i.e. the induced arrow ϕ is a Γ_{ab}-central (or -normal) extension.

$$\begin{tikzcd} P \arrow[r, two heads, "p"] \arrow[d, two heads, "q"'] & X \\ P/[K,P] \arrow[ur, two heads, dotted, "\phi"'] & \end{tikzcd}$$

It is also possible to show that this central extension ϕ is *weak universal*, so that the *Galois group* of ϕ is then an invariant of X, called the *fundamental group* of X. By computing the Galois group of ϕ, we thus obtain a formula for the fundamental group of X:

$$\pi_1(X) = \frac{K \cap [P,P]}{[K,P]}.$$

This is the well-known *Hopf formula* [6], which corresponds to the second homology group $H_2(X, \mathbb{Z})$ of X.

The purpose of this article is to extend this formula to the setting of *preordered groups*. For this, we use the characterization of the central and normal extensions with respect to a "suitable Galois structure" obtained in the paper [5], and somehow imitate what was done in the case of groups. Recall that a *preordered group* $(G, \leq)$ is a (not necessarily abelian) group $G = (G, +, 0)$ endowed with a preorder relation $\leq$ which is compatible with the addition $+$ of the group G: if $a \leq c$ and $b \leq d$ for $a, b, c, d \in G$, then $a + b \leq c + d$. Preordered groups are the objects of a category, the category $\mathsf{PreOrdGrp}$ of preordered groups, whose arrows are given by the monotone group morphisms. The interested reader may find it useful to take a look at the article [3], in which the categorical behaviour of preordered groups was studied.

The present article is structured as follows. We start with a quick review of *categorical Galois theory* and recall the general definitions of the notions of *Galois groupoid*, *Galois group* and *fundamental group*. We then present some properties of the category $\mathsf{PreOrdGrp}$ of preordered groups that will be useful for our purpose. We also recall the main results of the article [5], in particular the description of the central and normal extensions. In the fourth section, we then construct a weak universal central extension for any given preordered group. This construction is used in the last section in order to obtain an explicit formula for the fundamental group (Theorem 5.3).

2. Categorical Galois theory

In this section, we briefly recall the basic notions of categorical Galois theory including, among other things, the definition of the fundamental group. For more details, the reader can refer to [7, 8, 9, 10] for instance.

Definition 2.1. A *Galois structure* is a system $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ in which

- $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, bend left, "U"] \end{tikzcd}$ is an adjunction, with unit η and counit ϵ;

- $\mathcal{E}$ and $\mathcal{Z}$ are classes of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively,

such that

- $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks along morphisms from $\mathcal{E}$ and $\mathcal{Z}$, respectively;
- $\mathcal{E}$ and $\mathcal{Z}$ are closed under composition, contain all isomorphisms and are pullback-stable;
- $F(\mathcal{E}) \subseteq \mathcal{Z}$;
- $U(\mathcal{Z}) \subseteq \mathcal{E}$.

Definition 2.2. Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ be a Galois structure such that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism. Then, the Galois structure Γ is said to be *admissible* when F preserves all pullbacks of the form

$$\begin{tikzcd} B \times_{UF(B)} U(X) \arrow[r] \arrow[d] & U(X) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

where $\phi \in \mathcal{Z}$.

Definition 2.3. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a (Γ-)*trivial extension* when the naturality square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UF(A) \arrow[d, "UF(f)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

is a pullback.

Definition 2.4. A morphism $p\colon E \to B$ in $\mathcal{E}$ is called a *monadic extension* when it is an effective descent morphism [12, 11].

Definition 2.5. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a (Γ-)*central extension* when there exists a monadic extension $p\colon E \to B$ such that $p^*(f)\colon E \times_B A \to E$ is a (Γ-)trivial extension, that is, the following square

$$\begin{tikzcd}
E \times_B A \arrow[r, "\eta_{E\times_B A}"] \arrow[d, "p^*(f)=\pi_1"'] & UF(E \times_B A) \arrow[d, "UF(\pi_1)"] \\
E \arrow[r, "\eta_E"'] & UF(E)
\end{tikzcd}$$

is a pullback, where π_1 is the first projection in the pullback

$$\begin{tikzcd}
E \times_B A \arrow[r, "\pi_2"] \arrow[d, "p^*(f)=\pi_1"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] & B.
\end{tikzcd}$$

Definition 2.6. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a (Γ-)*normal extension* when f is a monadic extension and $f^*(f)$ is a (Γ-)trivial extension.

By definition, it is clear that any trivial extension and any normal extension is central. By using the admissibility of the Galois structure Γ, it is also possible to prove that all trivial extensions are normal.

Assume from now on that we have an admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ for which the category $\mathscr{C}$ is pointed. Let $p\colon E \to B$ be a (Γ-)normal extension in $\mathscr{C}$, and consider its kernel pair (seen as an internal groupoid in $\mathscr{C}$):

$$\begin{tikzcd}
\mathsf{Eq}(p) \times_E \mathsf{Eq}(p) \arrow[r, "\tau"] & \mathsf{Eq}(p) \arrow[loop above, "\sigma"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & E. \arrow[l, "\Delta" description]
\end{tikzcd}$$

Definition 2.7.

- The *Galois groupoid* $\mathsf{Gal}(E, p)$ of p is the image under the left adjoint F of the kernel pair of p, despicted as:

$$\begin{tikzcd}
F(\mathsf{Eq}(p)) \times_{F(E)} F(\mathsf{Eq}(p)) \\
F(\mathsf{Eq}(p) \times_E \mathsf{Eq}(p)) \arrow[u, phantom, "\cong" rotate] \arrow[r, "F(\tau)"] & F(\mathsf{Eq}(p)) \arrow[loop above, "F(\sigma)"] \arrow[r, shift left=2, "F(p_1)"] \arrow[r, shift right=2, "F(p_2)"'] & F(E). \arrow[l, "F(\Delta)" description]
\end{tikzcd}$$

- The *Galois group* $\mathsf{Gal}(E, p, 0)$ of p is the kernel of the induced morphism $\langle F(p_1), F(p_2)\rangle$, i.e. is as in the following pullback:

$$\begin{tikzcd}
\mathsf{Gal}(E,p,0) \arrow[rr] \arrow[d, tail] & & 0 \arrow[d] \\
F(\mathsf{Eq}(p)) \arrow[rr, "\langle F(p_1),F(p_2)\rangle"'] & & F(E) \times F(E).
\end{tikzcd}$$

When the normal extension p is *weak universal*, it turns out that the Galois group of p is an invariant of B. In this case, the Galois group of p is called the *fundamental group* of B and is denoted by $\pi_1(B)$. Let us recall that a normal extension $p\colon E \to B$ is said to be *weak universal* when, for any other normal extension $p'\colon E' \to B$, there exists an arrow $v\colon E \to E'$ making the diagram below commutative:

$$\begin{tikzcd}
E \arrow[rr, "p"] \arrow[dr, dotted, "v"'] & & B \\
& E'. \arrow[ur, "p'"'] &
\end{tikzcd}$$

3. Central extensions of preordered groups

Consider the category whose

- objects are pairs (G, P_G), also represented by the inclusion $P_G \rightarrowtail G$, where G is a group and P_G a submonoid of G closed under conjugation in G;

- arrows $(G, P_G) \to (H, P_H)$ are given by group morphisms $f\colon G \to H$ such that $f(P_G) \subseteq P_H$. Alternatively, a morphism from (G, P_G) to (H, P_H) can be seen as a pair $(f, \bar{f})$, with $f\colon G \to H$ a group morphism and $\bar{f}\colon P_G \to P_H$ a monoid morphism making the following square commute:

$$\begin{tikzcd} P_G \arrow[r, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, "f"'] & H. \end{tikzcd} \tag{3.1}$$

It is well known, and not difficult to prove, that this category is actually isomorphic to the category $\mathsf{PreOrdGrp}$ of preordered groups. To any preordered group $(X, \leq)$, we can associate a pair (G, P_G) as above by taking $G = X$ and $P_G = \{x \in X \mid 0 \leq x\}$. Incidentally, the definition of P_G is the reason why we usually call the submonoid P_G the *positive cone* of G. Conversely, any such pair (G, P_G) corresponds, via the isomorphism, to the preordered group $(G, \leq)$ where the preorder relation $\leq$ is defined as follows: $a \leq b$ in G if and only if $b - a \in P_G$. As a result, since the above category is isomorphic to $\mathsf{PreOrdGrp}$, we can use the above description as an alternative definition of the category $\mathsf{PreOrdGrp}$ of preordered groups.

The categorical behaviour of preordered groups was studied in [3] by Clementino, Martins-Ferreira and Montoli. They proved, among other things, that $\mathsf{PreOrdGrp}$ is both complete and cocomplete. It is also a *normal category* in the sense of [13], i.e.

- it is pointed, with zero object $(0, 0)$;
- it is regular;
- any regular epimorphism in it is the cokernel of its kernel.

Note that a morphism $(f, \bar{f})$ as in (3.1) is a monomorphism whenever f (and then $\bar{f}$) is injective, while it is an epimorphism whenever f is an epimorphism. The regular epimorphisms in $\mathsf{PreOrdGrp}$ are given by the arrows $(f, \bar{f})$ as in (3.1) such that both f and $\bar{f}$ are surjective, and coincide (as already mentioned above) with the normal epimorphisms. In particular, the cokernel of $(f, \bar{f})$ is given by

$$\begin{tikzcd} P_H \arrow[r, two heads, "\bar{q}"] \arrow[d, tail] & P_Q \arrow[d, tail] \\ H \arrow[r, two heads, "q"'] & Q \end{tikzcd}$$

where $q\colon H \twoheadrightarrow Q$ is the cokernel of f in the category Grp of groups, and P_Q the direct image $q(P_H)$ of P_H along q. The kernels in $\mathsf{PreOrdGrp}$ are, on the other hand, even easier to compute since they are taken "component-wise": the kernel of $(f, \bar{f})$ is given by $((K, P_K), (k, \bar{k}))$, with (K, k) the kernel of f in Grp and $(P_K, \bar{k})$ the kernel of $\bar{f}$ in the category Mon of monoids (or, alternatively, $P_K = K \cap P_G$).

Note that, since the category $\mathsf{PreOrdGrp}$ of preordered groups is normal, the following lemma is then satisfied in our context:

Lemma 3.1. Let $\mathscr{C}$ be a normal category. Consider a commutative diagram of short exact sequences in $\mathscr{C}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "k"] \arrow[d, "a"'] & B \arrow[r, two heads, "f"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, tail, "k'"'] & B' \arrow[r, two heads, "f'"'] & C' \arrow[r] & 0.
\end{tikzcd}$$

Then the left-hand square is a pullback if and only if the arrow c is a monomorphism.

As proved in [3], the category $\mathsf{PreOrdGrp}$ of preordered groups is neither protomodular nor Barr-exact. However, the class of *effective descent morphisms* in it coincides with the class of regular epimorphisms, which has a really simple description (as explained above).

We now recall the content of the article [5] on which the present paper is partly based. We first mention the fact that the category $\mathsf{PreOrdGrp}$ of preordered groups admits a full subcategory, whose objects are the preordered groups $P_G \rightarrowtail G$ such that both G and P_G are abelian groups. This is the category $\mathsf{Mono(Ab)}$ of monomorphisms in the category of abelian groups, which turns out to be the full subcategory of *abelian objects* [1] in $\mathsf{PreOrdGrp}$. Moreover, we can define a functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono(Ab)}$ from $\mathsf{PreOrdGrp}$ to $\mathsf{Mono(Ab)}$: given any preordered group (G, P_G),

$$F(G, P_G) = (ab(G), grp(\eta_G(P_G))),$$

where $ab(G) = G/[G, G]$ is the "abelianization" of the group G, $\eta_G\colon G \twoheadrightarrow ab(G)$ the canonical quotient, and $grp\colon \mathsf{CMon} \to \mathsf{Ab}$ the "group completion functor" associating with any commutative monoid X its group completion

$grp(X)$ (also called "Grothendieck group"). This functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono(Ab)}$ turns out to be a reflector:

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Mono(Ab)}. \arrow[l, bend left, "U"] \end{tikzcd}$$

Taking $\mathcal{E}$ and $\mathcal{Z}$ to be the classes of regular epimorphisms in $\mathsf{PreOrdGrp}$ and $\mathsf{Mono(Ab)}$ respectively then gives us a Galois structure

$$\Gamma = (\mathsf{PreOrdGrp}, \mathsf{Mono(Ab)}, F, U, \mathcal{E}, \mathcal{Z})$$

which is admissible (see Corollary 5.6 in [5]). As is the case in the setting of groups, normal and central extensions coincide in this context:

Theorem 3.2. [5] Let $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ be a regular epimorphism in $\mathsf{PreOrdGrp}$. Then, the following conditions are equivalent:

1. (a) $\mathsf{Ker}(f) \subseteq Z(G)$;
   (b) for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$.
2. $(f, \bar{f})$ is a (Γ-)normal extension.
3. $(f, \bar{f})$ is a (Γ-)central extension.

4. Construction of a weak universal central extension for any given preordered group

First recall that a *regular projective object* in an arbitrary category $\mathscr{C}$ is an object P such that, for any arrow $\phi\colon P \to Y$ and any regular epimorphism $\psi\colon X \twoheadrightarrow Y$, there exists a (not necessarily unique) morphism $\alpha\colon P \to X$ such that $\psi \cdot \alpha = \phi$:

$$\begin{tikzcd} & X \arrow[d, two heads, "\psi"] \\ P \arrow[ur, dotted, "\alpha"] \arrow[r, "\phi"'] & Y. \end{tikzcd}$$

A category $\mathscr{C}$ is said to *have enough regular projectives* when, for any object $C \in \mathscr{C}$, there exists a regular projective object $P \in \mathscr{C}$ as well as a regular epimorphism $P \twoheadrightarrow C$ from P to C.

Proposition 4.1. $\mathsf{PreOrdGrp}$ has enough regular projectives.

Proof. Let (G, P_G) be any preordered group. Consider the free group $(P, *)$ on the underlying set $|G|$ of G. Its elements are the "reduced words" in the "alphabet" $G \sqcup G^{-1}$, where G^{-1} is the set of "formal inverses" of the elements in G.

In the following we'll identify each element g_i of G, and in particular of its submonoid P_G, with the corresponding one-letter (reduced) word $g_i \in P$. Let us then define

$$\begin{aligned} P_P = \{y_1 * y_2 * \cdots * y_m \mid y_i = x_i * g_i * (-x_i), \\ x_i \in P,\ g_i \in P_G\ \forall i \in \{1, \cdots, m\}\}. \end{aligned}$$

Then, P_P is clearly a submonoid of P, and is closed under conjugation in P. Indeed, for any $x \in P$ and any $y = y_1 * y_2 * \cdots * y_m \in P_P$,

$$\begin{aligned} x * y * (-x) &= x * y_1 * y_2 * \cdots * y_m * (-x) \\ &= (x * y_1 * (-x)) * (x * y_2 * (-x)) * \cdots * (x * y_m * (-x)) \end{aligned}$$

where, for any $i \in \{1, \cdots, m\}$,

$$x * y_i * (-x) = x * x_i * g_i * (-x_i) * (-x) = (x * x_i) * g_i * (-(x * x_i))$$

with $x * x_i \in P$ and $g_i \in P_G$, so that $x * y * (-x) \in P_P$. As a consequence, $(P, P_P) \in \mathsf{PreOrdGrp}$.

Define now $p\colon P \to G$, for any $x_1 * \cdots * x_n \in P$, by

$$p(x_1 * \cdots * x_n) = x_1 + \cdots + x_n.$$

It is a group morphism which is surjective. Let us check that the restriction $\bar{p}$ of p to P_P takes its values in P_G. For any $y_1 * \cdots * y_m \in P_P$,

$$\begin{aligned} p(y_1 * \cdots * y_m) &= p(x_1 * g_1 * (-x_1) * \cdots * x_m * g_m * (-x_m)) \\ &= p(x_1) + p(g_1) - p(x_1) + \cdots + p(x_m) + p(g_m) - p(x_m) \\ &= p(x_1) + g_1 - p(x_1) + \cdots + p(x_m) + g_m - p(x_m) \end{aligned}$$

since p is a group morphism. We observe that $g_i \in P_G$ and that $p(x_i) \in G$ for any $i \in \{1, \cdots, m\}$, which implies that $p(x_i) + g_i - p(x_i) \in P_G$ because P_G is closed under conjugation in G. By the closure of P_G under $+$, we conclude that $p(y_1 * \cdots * y_m) \in P_G$. This means that $(p, \bar{p})\colon (P, P_P) \to (G, P_G)$ is a morphism in PreOrdGrp, which is a regular epimorphism since both p and $\bar{p}$ are surjective.

It remains to show that (P, P_P) is a regular projective object. Consider any regular epimorphism $(\psi, \bar{\psi})\colon (A, P_A) \twoheadrightarrow (H, P_H)$ and any morphism $(\phi, \bar{\phi})\colon (P, P_P) \to (H, P_H)$ in PreOrdGrp.

$$
\begin{tikzcd}
 & & P_A \arrow[dl, two heads, "\bar{\psi}"] \arrow[dd, tail] \\
P_P \arrow[r, "\bar{\phi}"'] \arrow[urr, dotted, "\bar{\alpha}"] \arrow[dd, tail] & P_H \arrow[dd, tail] & \\
 & & A \arrow[dl, two heads, "\psi"] \\
P \arrow[r, "\phi"'] \arrow[urr, dotted, "\alpha"] & H &
\end{tikzcd}
$$

Since $P = F(G)$ is a regular projective object in the category Grp of groups, there exists a group morphism $\alpha\colon P \to A$ such that $\psi \cdot \alpha = \phi$. Indeed, since ψ is surjective, for any $h \in H$, there exists a (not necessarily unique) element a_h in A such that $\psi(a_h) = h$. We then define α, for any $x_1 * \cdots * x_n \in P$, by

$$\alpha(x_1 * \cdots * x_n) = a_{\phi(x_1)} + \cdots + a_{\phi(x_n)},$$

where $a_{\phi(x_i)} \in A$ is such that $\psi(a_{\phi(x_i)}) = \phi(x_i)$ for any $i \in \{1, \cdots, n\}$. Note that this group morphism α is not necessarily unique and that one can choose, in the above construction, $a_h \in P_A$ whenever $h \in P_H$ because $\bar{\psi}$ is surjective. We now prove that the restriction $\bar{\alpha}$ of α to P_P takes its values in P_A. Let $y_1 * \cdots * y_m \in P_P$. Then,

$$
\begin{aligned}
\alpha(y_1 * \cdots * y_m) &= \alpha(x_1 * g_1 * (-x_1) * \cdots * x_m * g_m * (-x_m)) \\
&= \alpha(x_1) + \alpha(g_1) - \alpha(x_1) + \cdots + \alpha(x_m) + \alpha(g_m) - \alpha(x_m),
\end{aligned}
$$

since α is a group morphism, with $\alpha(x_i) + \alpha(g_i) - \alpha(x_i) \in P_A$ (because $\alpha(g_i) \in P_A$ and P_A is closed under conjugation in A) for any $i \in \{1, \cdots, m\}$, so that $\alpha(y_1 * \cdots * y_m) \in P_A$. Consequently, there exists a morphism $(\alpha, \bar{\alpha})\colon (P, P_P) \to (A, P_A)$ in $\mathsf{PreOrdGrp}$ such that $(\psi, \bar{\psi}) \cdot (\alpha, \bar{\alpha}) = (\phi, \bar{\phi})$. □

Remark 4.2. While preparing the final version of the article, the referee informed me that Maria Manuel Clementino and Andrea Montoli proved in [2] that the category PreOrdGrp is a finitary quasivariety, from which it follows that it has enough regular projectives. I decided to keep the proof of Proposition 4.1 since it gives an explicit description of the regular projective presentation of any preordered group that could be useful for computations.

We now prove a lemma which will be useful in the construction of a weak universal central extension.

Lemma 4.3. Let $(k, \bar{k})\colon (K, P_K) \rightarrowtail (P, P_P)$ be a normal monomorphism in $\mathsf{PreOrdGrp}$, and consider the inclusion $(\epsilon, \bar{\epsilon})\colon ([K, P], [K, P] \cap P_K) \rightarrowtail (K, P_K)$ (which is actually also a normal monomorphism) in $\mathsf{PreOrdGrp}$. Then the composite $(k, \bar{k}) \cdot (\epsilon, \bar{\epsilon})\colon ([K, P], [K, P] \cap P_K) \to (P, P_P)$ is a normal monomorphism in $\mathsf{PreOrdGrp}$.

Proof. Clearly, $[K, P]$ is a normal subgroup of P. It remains to prove that the external rectangle in the diagram

$$\begin{tikzcd}
{[K,P] \cap P_K} \arrow[r, tail, "\bar{\epsilon}"] \arrow[d, tail] & P_K \arrow[r, tail, "\bar{k}"] \arrow[d, tail] & P_P \arrow[d, tail] \\
{[K,P]} \arrow[r, tail, "\epsilon"'] & K \arrow[r, tail, "k"'] & P
\end{tikzcd}$$

is a pullback. This is the case since it is made of two squares which are themselves pullbacks. □

Proposition 4.4. One can construct, for any preordered group (G, P_G), a weak universal central extension of (G, P_G).

Proof. Let $(G, P_G) \in \mathsf{PreOrdGrp}$.

$$\begin{tikzcd}
([K,P],[K,P]\cap P_K) \arrow[d, tail, "{(\epsilon,\bar{\epsilon})}"'] \arrow[dr, tail, "{(k\cdot\epsilon,\bar{k}\cdot\bar{\epsilon})}"] & & \\
(K,P_K) \arrow[r, tail, "{(k,\bar{k})}"] \arrow[d, dotted, two heads, "{(r,\bar{r})}"'] \arrow[ddrr, dotted, tail, "{(n,\bar{n})}"' near end] & (P,P_P) \arrow[r, two heads, "{(p,\bar{p})}"] \arrow[d, two heads, "{(q,\bar{q})}"'] \arrow[dr, dotted, tail, "{(l,\bar{l})}" near end] & (G,P_G) \\
(\mathsf{Ker}(\phi),\mathsf{Ker}(\bar{\phi})) \arrow[r, tail] & (Q,P_Q) \arrow[ur, dotted, two heads, "{(\phi,\bar{\phi})}" near end] \arrow[r, dotted, tail, "{(\alpha,\bar{\alpha})}"'] & (H,P_H) \arrow[u, two heads, "{(\psi,\bar{\psi})}"'] \\
& & \mathsf{Ker}(\psi,\bar{\psi}) \arrow[u, tail]
\end{tikzcd}$$

Thanks to Proposition 4.1, we know that we can build a regular epimorphism $(p,\bar{p})\colon (P,P_P) \twoheadrightarrow (G,P_G)$ with (P,P_P) a regular projective object. Consider the kernel $(k,\bar{k})\colon (K,P_K) \rightarrowtail (P,P_P)$ of $(p,\bar{p})$. Since $(p,\bar{p})$ is a regular epimorphism in a normal category, it is then the cokernel of its kernel $(k,\bar{k})$. Consider also the morphism $(\epsilon,\bar{\epsilon})\colon ([K,P],[K,P]\cap P_K) \to (K,P_K)$, where ϵ is the inclusion of $[K,P]$ in K. By Lemma 4.3, the composite $(k\cdot\epsilon,\bar{k}\cdot\bar{\epsilon})$ is a normal monomorphism. By taking its cokernel $(q,\bar{q})\colon (P,P_P) \twoheadrightarrow (Q,P_Q)$, we thus obtain a short exact sequence. Now, we compute that $(p,\bar{p})\cdot(k\cdot\epsilon,\bar{k}\cdot\bar{\epsilon}) = (0,0)$. By the universal property of cokernels, there exists a unique morphism $(\phi,\bar{\phi})\colon (Q,P_Q) \to (G,P_G)$ such that $(\phi,\bar{\phi})\cdot(q,\bar{q}) = (p,\bar{p})$. This induced arrow turns out to be a regular epimorphism since so is $(p,\bar{p})$.

Take now the kernel of $(\phi,\bar{\phi})$. Then $(\phi,\bar{\phi})$ is its cokernel. Since

$$(\phi,\bar{\phi})\cdot(q,\bar{q})\cdot(k,\bar{k}) = (p,\bar{p})\cdot(k,\bar{k}) = (0,0),$$

by the universal property of kernels, there is then a unique morphism $(r,\bar{r})\colon (K,P_K) \to \mathsf{Ker}(\phi,\bar{\phi}) = (\mathsf{Ker}(\phi),\mathsf{Ker}(\bar{\phi}))$ such that $\mathsf{ker}(\phi,\bar{\phi})\cdot(r,\bar{r}) = (q,\bar{q})\cdot(k,\bar{k})$. We observe that the square

$$\begin{tikzcd}
(K, P_K) \arrow[r, tail, "(k,\bar{k})"] \arrow[d, "(r,\bar{r})"'] & (P, P_P) \arrow[d, two heads, "(q,\bar{q})"] \\
(\mathsf{Ker}(\phi), \mathsf{Ker}(\bar{\phi})) \arrow[r, tail] & (Q, P_Q)
\end{tikzcd}$$

is a pullback in $\mathsf{PreOrdGrp}$ since $((k, \bar{k}), (p, \bar{p}))$ and $(\mathsf{ker}(\phi, \bar{\phi}), (\phi, \bar{\phi}))$ are two short exact sequences, $1_{(G,P_G)}$ is a monomorphism, and $\mathsf{PreOrdGrp}$ is a normal category (see Lemma 3.1). By pullback-stability of regular epimorphisms in $\mathsf{PreOrdGrp}$, it follows that $(r, \bar{r})$ is a regular epimorphism; it is the restriction of $(q, \bar{q})$ to the kernel (K, P_K) of $(p, \bar{p})$. Another consequence of the above square being a pullback is that the pair $((\epsilon, \bar{\epsilon}), (r, \bar{r}))$ forms a short exact sequence in $\mathsf{PreOrdGrp}$.

Let us show that $\mathsf{Ker}(\phi) \subseteq Z(Q)$. Let $y_1 \in \mathsf{Ker}(\phi)$ and $y_2 \in Q$. By surjectivity of r and q, there exist $x_1 \in K$ and $x_2 \in P$ such that $r(x_1) = y_1$ (in particular, $q(x_1) = y_1$) and $q(x_2) = y_2$. Since $x_1 + x_2 - x_1 - x_2 \in [K, P] = \mathsf{Ker}(q)$, we then have that $q(x_1 + x_2 - x_1 - x_2) = 0$, that is, $y_1 + y_2 - y_1 - y_2 = 0$. In other words, $y_1 + y_2 = y_2 + y_1$, which means that $\mathsf{Ker}(\phi) \subseteq Z(Q)$.

Consider now

$$\tilde{P}_Q = \{x - y + z \mid (x, y) \in \mathsf{Eq}(\bar{\phi}) \text{ and } z \in P_Q\}.$$

It is a submonoid of Q. Indeed, for $x - y + z$ and $x' - y' + z'$ in $\tilde{P}_Q$,

$$\begin{aligned}(x - y + z) + (x' - y' + z') &= x + (x' - y') + (-y + z) + z' \\ &= (x + x') - (y + y') + (z + z')\end{aligned}$$

since $x' - y' \in \mathsf{Ker}(\phi) \subseteq Z(Q)$. By assumption, $x + x', y + y', z + z' \in P_Q$ and $\phi(x + x') = \phi(x) + \phi(x') = \phi(y) + \phi(y') = \phi(y + y')$, and this proves that $(x - y + z) + (x' - y' + z') \in \tilde{P}_Q$, as desired. The submonoid $\tilde{P}_Q$ is also closed under conjugation in Q: for $x - y + z \in \tilde{P}_Q$ and $w \in Q$,

$$\begin{aligned}w + (x - y + z) - w &= w + x + (-w + w) - y + (-w + w) + z - w \\ &= (w + x - w) - (w + y - w) + (w + z - w)\end{aligned}$$

with $w + x - w$, $w + y - w$ and $w + z - w$ in P_Q because P_Q is closed under conjugation in Q, and $\phi(w + x - w) = \phi(w) + \phi(x) - \phi(w) =$

$\phi(w) + \phi(y) - \phi(w) = \phi(w + y - w)$, so that $w + (x - y + z) - w \in \tilde{P}_Q$. From this, we deduce that $(Q, \tilde{P}_Q)$ is a preordered group. Moreover, since $P_Q \subset \tilde{P}_Q$, the arrow

$$(Q, P_Q) \xrightarrow{(1_Q, j)} (Q, \tilde{P}_Q),$$

where j is the inclusion arrow, is a morphism of preordered groups. We now observe that, for any $x - y + z \in \tilde{P}_Q$,

$$\phi(x - y + z) = \phi(x - y) + \phi(z) = \phi(z) \in P_G$$

since $z \in P_Q$ and $(\phi, \bar{\phi})$ is a morphism in $\mathsf{PreOrdGrp}$. This implies that the restriction $\tilde{\phi}$ of ϕ to $\tilde{P}_Q$ takes its values in P_G, which means that

$$(Q, \tilde{P}_Q) \xrightarrow{(\phi, \tilde{\phi})} (G, P_G)$$

is a morphism of preordered groups. It is a regular epimorphism because $(\phi, \tilde{\phi}) \cdot (1_Q, j) = (\phi, \bar{\phi})$ with $(\phi, \bar{\phi})$ a regular epimorphism.

Let us now prove that, for any $(a, b) \in \mathsf{Eq}(\tilde{\phi})$, $b - a \in \tilde{P}_Q$. By assumption, $a = x - y + z$ and $b = x' - y' + z'$, with $x, y, z, x', y', z' \in P_Q$, $\phi(x) = \phi(y)$, $\phi(x') = \phi(y')$ and $\phi(z) = \phi(a) = \phi(b) = \phi(z')$. Then,

$$\begin{aligned} b - a &= (x' - y' + z') - (x - y + z) \\ &= x' - y' + z' - z + y - x \\ &= z' + x' - y' - z + y - x \\ &= z' + x' + y - x - y' - z \\ &= (z' + x' + y) - (z + y' + x) + 0, \end{aligned}$$

since $x' - y'$ and $y - x$ belong to $\mathsf{Ker}(\phi) \subseteq Z(Q)$, with $z' + x' + y$, $z + y' + x$ and 0 in P_Q, and

$$\phi(z' + x' + y) = \phi(z') + \phi(x') + \phi(y) = \phi(z) + \phi(y') + \phi(x) = \phi(z + y' + x).$$

This means that $b - a \in \tilde{P}_Q$. According to Theorem 3.2, $(\phi, \tilde{\phi})$ is then a Γ-central extension.

It remains to show that it is indeed weak universal. For this, consider any other central extension $(\psi, \bar{\psi})\colon (H, P_H) \twoheadrightarrow (G, P_G)$ of (G, P_G), that is, $(\psi, \bar{\psi})$ is a regular epimorphism such that

- $\mathsf{Ker}(\psi) \subseteq Z(H)$;
- for any $(a,b) \in \mathsf{Eq}(\bar{\psi})$, $b-a \in P_H$.

Since (P, P_P) is a regular projective object, there exists a morphism $(l,\bar{l})\colon (P,P_P) \to (H,P_H)$ such that $(\psi,\bar{\psi})\cdot(l,\bar{l}) = (p,\bar{p})$. We then compute that

$$(\psi,\bar{\psi})\cdot(l,\bar{l})\cdot(k,\bar{k}) = (p,\bar{p})\cdot(k,\bar{k}) = (0,0).$$

The universal property of kernels then induces a unique arrow $(n,\bar{n})\colon (K,P_K) \to \mathsf{Ker}(\psi,\bar{\psi})$ such that $\mathsf{ker}(\psi,\bar{\psi})\cdot(n,\bar{n}) = (l,\bar{l})\cdot(k,\bar{k})$. Now, the composite $(n,\bar{n})\cdot(\epsilon,\bar{\epsilon})$ is trivial. Indeed, for any $x \in [K,P]$ (that is, $x = a+b-a-b$ with $a \in K$ and $b \in P$),

$$\begin{aligned} n(x) &= n(a+b-a-b) \\ &= n(a)+n(b)-n(a)-n(b) \\ &= n(a)-n(a)+n(b)-n(b) \\ &= 0 \end{aligned}$$

since $n(a)$ and $n(b)$ are in $\mathsf{Ker}(\psi)$, which is an abelian group. This allows us to compute that

$$\begin{aligned} (l,\bar{l})\cdot(k\cdot\epsilon,\bar{k}\cdot\bar{\epsilon}) &= (l,\bar{l})\cdot(k,\bar{k})\cdot(\epsilon,\bar{\epsilon}) \\ &= \mathsf{ker}(\psi,\bar{\psi})\cdot(n,\bar{n})\cdot(\epsilon,\bar{\epsilon}) \\ &= (0,0). \end{aligned}$$

By the universal property of cokernels, there is a unique arrow $(\alpha,\bar{\alpha})\colon (Q,P_Q) \to (H,P_H)$ such that $(\alpha,\bar{\alpha})\cdot(q,\bar{q}) = (l,\bar{l})$, and so

$$(\psi,\bar{\psi})\cdot(\alpha,\bar{\alpha})\cdot(q,\bar{q}) = (\psi,\bar{\psi})\cdot(l,\bar{l}) = (p,\bar{p}) = (\phi,\bar{\phi})\cdot(q,\bar{q}).$$

Since $(q,\bar{q})$ is a regular epimorphism, we deduce that $(\psi,\bar{\psi})\cdot(\alpha,\bar{\alpha}) = (\phi,\bar{\phi})$. Let us now check that the restriction $\tilde{\alpha}$ of α to $\tilde{P}_Q$ takes its values in P_H. Let $x-y+z \in \tilde{P}_Q$. In particular, $x-y \in \mathsf{Ker}(\phi)$, so that $\alpha(x)-\alpha(y) \in \mathsf{Ker}(\psi)$. Since in addition $\alpha(x)$ and $\alpha(y)$ belong to P_H, we then have that $(\alpha(y),\alpha(x)) \in \mathsf{Eq}(\bar{\psi})$. By assumption, it follows that $\alpha(x)-\alpha(y) \in P_H$. We also know that $\alpha(z) \in P_H$, hence

$$\alpha(x-y+z) = (\alpha(x)-\alpha(y))+\alpha(z) \in P_H$$

as desired.

$$
\begin{tikzcd}
(Q, P_Q) \arrow[rr, two heads, "(\phi,\bar{\phi})"] \arrow[d, "(1_Q{,}j)"'] \arrow[drr, "(\alpha{,}\bar{\alpha})"] & & (G, P_G) \\
(Q, \tilde{P}_Q) \arrow[rr, dotted, "(\alpha{,}\tilde{\alpha})"'] \arrow[urr, dotted, two heads, "(\phi{,}\tilde{\phi})"] & & (H, P_H) \arrow[u, two heads, "(\psi{,}\bar{\psi})"']
\end{tikzcd}
$$

As a conclusion, $(\phi, \tilde{\phi})$ is a weak universal central extension of (G, P_G). $\square$

5. A Hopf formula for the fundamental group in $\mathsf{PreOrdGrp}$

Let us start this section by recalling two results that will be of interest in order to obtain a formula for the fundamental group in $\mathsf{PreOrdGrp}$. For completeness, we remind their proofs here below.

Proposition 5.1. [9] Let $\mathscr{C}$ be a pointed category, and consider $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ an admissible Galois structure where $\mathscr{F}$ is a subcategory of $\mathscr{C}$ and $U: \mathscr{F} \to \mathscr{C}$ the inclusion functor. If $p: E \to B$ is a normal extension, then

$$\mathsf{Gal}(E, p, 0) \cong \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E)$$

where $\eta_E: E \to F(E)$ is the E-component of the unit of the adjunction $F \dashv U$.

Proof. Let us denote by (K, k) the kernel of p, and consider the following commutative diagram

$$
\begin{tikzcd}
K \arrow[r, "\langle 0{,}k\rangle"] \arrow[d] & \mathsf{Eq}(p) \arrow[r, "\eta_{\mathsf{Eq}(p)}"] \arrow[d, "p_1"] & F(\mathsf{Eq}(p)) \arrow[d, "F(p_1)"] \\
0 \arrow[r] & E \arrow[r, "\eta_E"'] & F(E)
\end{tikzcd}
$$

where p_1 (respectively p_2) is the first (respectively the second) projection of the kernel pair of p. The left-hand square is a pullback by definition of the kernel of p_1. The right-hand square is also a pullback since p is a normal extension. The external rectangle is therefore a pullback, which means that $K = \mathsf{Ker}(F(p_1))$ and $\eta_{\mathsf{Eq}(p)} \cdot \langle 0, k\rangle = \mathsf{ker}(F(p_1))$. Let us now consider the diagram

$$\begin{tikzcd}
\mathsf{Ker}(\eta_E \cdot k) \arrow[r, dotted, "\phi"] \arrow[d] & \mathsf{Ker}(\eta_E) \arrow[r] \arrow[d] & 0 \arrow[d] \\
K \arrow[r, "k"'] & E \arrow[r, "\eta_E"'] & F(E)
\end{tikzcd}$$

where the dotted arrow $\phi\colon \mathsf{Ker}(\eta_E \cdot k) \to \mathsf{Ker}(\eta_E)$ is induced by the universal property of kernels, and makes the above diagram commute. Since the right-hand square and the external rectangle are pullbacks, it follows that the left-hand square is a pullback as well, which implies that

$$\mathsf{Ker}(\eta_E \cdot k) \cong K \cap \mathsf{Ker}(\eta_E) = \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E).$$

It remains to show that $\mathsf{Ker}(\eta_E \cdot k) \cong \mathsf{Gal}(E, p, 0)$. Consider first the following commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\eta_E \cdot k"] \arrow[d, "\eta_{\mathsf{Eq}(p)} \cdot \langle 0,k\rangle"'] \arrow[dr, phantom, "(1)"] & F(E) \arrow[r] \arrow[d, "\langle 0,1\rangle"] \arrow[dr, phantom, "(2)"] & 0 \arrow[d] \\
F(\mathsf{Eq}(p)) \arrow[r, "\langle F(p_1), F(p_2)\rangle"'] & F(E) \times F(E) \arrow[r, "q_1"'] & F(E)
\end{tikzcd}$$

where q_1 (respectively q_2) is the first (respectively the second) projection of the product of $F(E)$ with itself. Note that this diagram is commutative since

$$F(p_1) \cdot \eta_{\mathsf{Eq}(p)} \cdot \langle 0, k\rangle = 0 = 0 \cdot \eta_E \cdot k,$$

and

$$F(p_2) \cdot \eta_{\mathsf{Eq}(p)} \cdot \langle 0, k\rangle = \eta_E \cdot p_2 \cdot \langle 0, k\rangle = \eta_E \cdot k.$$

We observe that the square (2) and the external rectangle (1) + (2) are pullbacks, so that the square (1) is also a pullback. Let us now build the following diagram:

$$\begin{tikzcd}
\mathsf{Ker}(\eta_E \cdot k) \arrow[rr, "\mathsf{ker}(\eta_E \cdot k)"] \arrow[dr] \arrow[dd, dotted, "\psi"'] & & K \arrow[dr, "\eta_E \cdot k"] \arrow[dd, "\eta_{\mathsf{Eq}(p)} \cdot \langle 0,k \rangle"] & \\
& 0 \arrow[rr] \arrow[dd, equal] & & F(E) \arrow[dd, "\langle 0,1 \rangle"] \\
\mathsf{Gal}(E,p,0) \arrow[rr] \arrow[dr] & & F(\mathsf{Eq}(p)) \arrow[dr, "\langle F(p_1),F(p_2) \rangle"'] & \\
& 0 \arrow[rr] & & F(E) \times F(E)
\end{tikzcd}$$

The lower square is a pullback by definition of $\mathsf{Gal}(E, p, 0)$. By the universal property of pullbacks, there is then a unique arrow $\psi\colon \mathsf{Ker}(\eta_E \cdot k) \to \mathsf{Gal}(E, p, 0)$ making the above cube commute. The upper square is obviously a pullback, and the right-hand square corresponds to the square (1) in the previous diagram, which has been proven to be a pullback. We deduce that the left-hand square is a pullback. By pullback-stability of isomorphisms, it follows that the induced morphism ψ is an isomorphism, that is,

$$\mathsf{Gal}(E, p, 0) \cong \mathsf{Ker}(\eta_E \cdot k) \cong \mathsf{Ker}(p) \cap \mathsf{Ker}(\eta_E). \qquad \square$$

Lemma 5.2. [4] Let $\mathscr{C}$ be a normal category and let $\mathscr{F}$ be a (normal epi)-reflective subcategory of $\mathscr{C}$. If $f\colon A \twoheadrightarrow B$ is a normal epimorphism such that $\mathsf{Ker}(f) \leq \mathsf{Ker}(\eta_A)$ where $\eta_A\colon A \twoheadrightarrow F(A)$ is the A-component of the unit of the reflection, then the induced commutative square

$$\begin{tikzcd}
\mathsf{Ker}(\eta_A) \arrow[r, tail] \arrow[d, "\hat{f}"'] & A \arrow[d, two heads, "f"] \\
\mathsf{Ker}(\eta_B) \arrow[r, tail] & B
\end{tikzcd}$$

is a pullback.

Proof. First note that, thanks to the universal property of η_A (as the A-component of the unit of the reflection), there exists a unique arrow $g\colon F(A) \to F(B)$ making the diagram below commutative:

$$\begin{tikzcd}
 & \mathsf{Ker}(f) \arrow[dl, tail] \arrow[d, tail, "k"] & \\
\mathsf{Ker}(\eta_A) \arrow[r, tail] \arrow[d, "\hat{f}"'] & A \arrow[r, two heads, "\eta_A"] \arrow[d, two heads, "f"'] & F(A) \arrow[d, dotted, shift right, "g"'] \\
\mathsf{Ker}(\eta_B) \arrow[r, tail] & B \arrow[r, two heads, "\eta_B"'] \arrow[ur, dotted, "\gamma"] & F(B). \arrow[u, dotted, shift right, "\delta"']
\end{tikzcd}$$

This induced arrow g is an epimorphism since so is the composite $\eta_B \cdot f$. Since $\mathsf{Ker}(f) \leq \mathsf{Ker}(\eta_A)$, we have that $\eta_A \cdot k = 0$ where k denotes the kernel of f. By the universal property of cokernels, there is then a unique morphism $\gamma \colon B \to F(A)$ such that $\gamma \cdot f = \eta_A$. Using now the universal property of η_B (as the B-component of the unit of the reflection), we obtain the existence of a unique arrow $\delta \colon F(B) \to F(A)$ satisfying $\delta \cdot \eta_B = \gamma$. We then compute that

$$\delta \cdot g \cdot \eta_A = \delta \cdot \eta_B \cdot f = \gamma \cdot f = \eta_A,$$

which entails that $\delta \cdot g = 1_{F(A)}$ since η_A is an epimorphism. This means that g is a split monomorphism. Being also an epimorphism, it is then an isomorphism. Accordingly, the left-hand square of the diagram above is a pullback. $\square$

Theorem 5.3. Let (G, P_G) be any preordered group and let $(p, \bar{p}) \colon (P, P_P) \twoheadrightarrow (G, P_G)$ be a regular projective presentation of (G, P_G) with kernel (K, P_K) (as in the proof of Proposition 4.1). Then,

$$\pi_1(G, P_G) \cong \left(\frac{K \cap [P, P]}{[K, P]}, \frac{K \cap [P, P]}{[K, P]} \cap \tilde{P}_Q \right)$$

where $\tilde{P}_Q = \{x - y + z \mid x, y, z \in q(P_P) \text{ and } \phi(x) = \phi(y)\}$ with $q \colon P \twoheadrightarrow P/[K, P]$ the canonical quotient and $\phi \colon P/[K, P] \twoheadrightarrow G$ the induced arrow such that $\phi \cdot q = p$.

Proof. Consider the central extension $(\phi, \tilde{\phi}) \colon (Q, \tilde{P}_Q) \twoheadrightarrow (G, P_G)$ constructed in the proof of Proposition 4.4. Since $(\phi, \tilde{\phi})$ is weak universal, we have that

$$\pi_1(G, P_G) = \mathsf{Gal}((Q, \tilde{P}_Q), (\phi, \tilde{\phi}), 0).$$

Now, knowing that the central and normal extensions coincide in our situation (see Theorem 3.2), we obtain, thanks to Proposition 5.1, that

$$\mathsf{Gal}((Q,\tilde{P}_Q),(\phi,\tilde{\phi}),0) \cong \mathsf{Ker}(\phi,\tilde{\phi}) \cap \mathsf{Ker}(\eta_Q,\tilde{\eta}_Q),$$

so that

$$\begin{aligned}
\pi_1(G,P_G) &\cong (\mathsf{Ker}(\phi),\mathsf{Ker}(\tilde{\phi})) \cap (\mathsf{Ker}(\eta_Q),\mathsf{Ker}(\tilde{\eta}_Q)) \\
&\cong (\mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q), (\mathsf{Ker}(\phi)\cap\tilde{P}_Q)\cap(\mathsf{Ker}(\eta_Q)\cap\tilde{P}_Q)) \\
&\cong (\mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q), \mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q)\cap\tilde{P}_Q) \\
&\cong \left(\frac{K\cap[P,P]}{[K,P]}, \frac{K\cap[P,P]}{[K,P]}\cap\tilde{P}_Q\right).
\end{aligned}$$

Let us recall the proof of the isomorphism

$$\mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q) \cong \frac{K\cap[P,P]}{[K,P]}$$

in the category Grp of groups. Consider for this the following commutative cube (with the same notations as before):

$$\begin{tikzcd}
K\cap\mathsf{Ker}(\eta_P) \arrow[rr, tail] \arrow[dr, tail] \arrow[dd, dotted, "\Phi"'] & & \mathsf{Ker}(\eta_P) \arrow[dr, tail, "\mathsf{ker}(\eta_P)"] \arrow[dd, "t" near start] & \\
& K \arrow[rr, tail, "k" near start] \arrow[dd, two heads, "r" near end] & & P \arrow[dd, two heads, "q"] \\
\mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q) \arrow[rr, tail] \arrow[dr, tail] & & \mathsf{Ker}(\eta_Q) \arrow[dr, tail, "\mathsf{ker}(\eta_Q)"] & \\
& \mathsf{Ker}(\phi) \arrow[rr, tail, "\mathsf{ker}(\phi)"'] & & Q.
\end{tikzcd}$$

The lower square is clearly a pullback. By the universal property of pullbacks, there exists then a unique morphism $\Phi\colon K\cap\mathsf{Ker}(\eta_P) \to \mathsf{Ker}(\phi)\cap\mathsf{Ker}(\eta_Q)$ making the above cube commute. The front square is a pullback as already observed in the proof of Proposition 4.4. The upper square is also obviously a pullback. It follows that the back square is a pullback. According to Lemma 5.2, the right-hand square is also a pullback since

$\mathsf{Ker}(q) = [K, P] \leq [P, P] = \mathsf{Ker}(\eta_P)$. As a conclusion, all the squares involved in the above cube are pullbacks in Grp. By pullback-stability of regular epimorphisms in Grp, it then follows that the induced arrow Φ is a regular epimorphism since so is q. The morphism Φ is then the cokernel of its kernel, so that

$$\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cong \frac{K \cap \mathsf{Ker}(\eta_P)}{\mathsf{Ker}(\Phi)}.$$

But,

$$\mathsf{Ker}(\Phi) \cong \mathsf{Ker}(t) \cong \mathsf{Ker}(q) = [K, P].$$

Knowing that $\mathsf{Ker}(\eta_P) = [P, P]$, we conclude that

$$\mathsf{Ker}(\phi) \cap \mathsf{Ker}(\eta_Q) \cong \frac{K \cap [P, P]}{[K, P]}.$$ □

References

[1] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[2] M. M. Clementino and A. Montoli, *Right-preordered groups from a categorical perspective*, Algebra Univers. 86:8 (2025).

[3] M. M. Clementino, N. Martins-Ferreira, and A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra 223 (2019), 4226-4225.

[4] M. Duckerts, T. Everaert, and M. Gran, *A description of the fundamental group in terms of commutators and closure operators*, J. Pure Appl. Algebra 216 (2012), 1837-1851.

[5] M. Gran and A. Michel, *Central extensions of preordered groups*, Bulletin de la Société Mathématique de France, 151 (4) (2023), 659-686.

[6] H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. 14 (1942), 257-309.

[7] G. Janelidze, *Pure Galois theory in categories*, J. Algebra 132 (2) (1990), 270-286.

[8] G. Janelidze, *Categorical Galois theory: revision and some recent developments*, Galois connections and applications, Math. Appl., vol. 565, Kluwer Acad. Publ. (2004), 139-171.

[9] G. Janelidze, *Galois groups, abstract commutators, and Hopf formula*, Appl. Categor. Struct. 16 (2008), 653-668.

[10] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra 97 (1994), 135-161.

[11] G. Janelidze, M. Sobral, and W. Tholen, *Beyond Barr exactness: effective descent morphisms*, In: M. C. Pedicchio, W. Tholen (eds.), Categorical Foundations, Encyclopedia Math. Appl., vol. 97, Cambridge Univ. Press (2004), 359-405.

[12] G. Janelidze and W. Tholen, *Facets of Descent, II*, Appl. Categ. Struct. (1997), 5:229-248.

[13] Z. Janelidze, *The pointed subobject functor,* 3×3 *lemmas, and subtractivity of spans*, Theory Appl. Categ. 23 (11) (2010), 221-242.

Aline Michel
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
Chemin du Cyclotron 2
1348 Louvain-la-Neuve (Belgium)
michelaline96@gmail.com