CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXVII-1 (2026)

STRICTLY ZERO-DIMENSIONAL BIFRAMES AND RANEY EXTENSIONS

Anna Laura SUAREZ

Resumé. Nous comparons deux catégories qui étendent la catégorie des espaces T_0 à l'aide d'outils issus de la topologie sans points. La catégorie des extensions de Raney est constituée de paires $(L, \mathcal{F})$ où L est un locale et $\mathcal{F} \subseteq \mathsf{S}_\mathfrak{o}(L)$ un sous-locale. Ici, $\mathsf{S}_\mathfrak{o}(L)$ est l'ensemble de toutes les intersections des sous-locaux ouverts de L. Un biframe strictement de dimension zéro est un couple $(L, \mathcal{D})$ où $\mathcal{D} \subseteq \mathsf{S}(L)$ est un sous-local codense. Nous montrons qu'il existe une adjonction entre certains sous-locales de $\mathsf{S}_\mathfrak{o}(L)$ et les sous-locaux codenses de $\mathsf{S}(L)$. Nous montrons que l'adjonction se restreint à un isomorphisme d'ordre entre ce que nous appelons les sous-locaux *admissibles* de $\mathsf{S}_\mathfrak{o}(L)$ et les sous-locaux codenses *essentiels*. En application de notre résultat principal, nous établissons une bijection entre les extensions de Raney admissibles et les biframes strictement de dimension zéro (L_1, L_2, L) telles que L est une extension essentielle de L_2 dans $\mathbf{Frm}$. Nous montrons que cette correspondance ne peut pas être rendue fonctorielle de manière évidente, car un morphisme $f : L \to M$ peut être soulevé en une application $f : (L, \mathcal{F}) \to (L, \mathcal{G})$ d'extensions de Raney sans être soulevé en une application entre les biframes strictement de dimension zéro associés.

Abstract. We compare two categories which extend the category of T_0-spaces using tools from pointfree topology. The category of Raney extensions consists of pairs $(L, \mathcal{F})$ where L is a locale and $\mathcal{F} \subseteq \mathsf{S}_\mathfrak{o}(L)$ a subcolocale. Here, $\mathsf{S}_\mathfrak{o}(L)$ is the collection of all intersections of open sublocales of L. Similarly, a strictly zero-dimensional biframe is a pair $(L, \mathcal{D})$ where $\mathcal{D} \subseteq \mathsf{S}(L)$ is a codense subcolocale. We show that there is an adjunction between certain subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ and codense subcolocales of $\mathsf{S}(L)$. We show that the adjunction maximally restricts to an order-isomorphism between what we call the *admissible* subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ and the *essential* codense subcolocales. As an application of our main result, we establish a bijection between admissible Raney extensions and the strictly zero-dimensional biframes (L_1, L_2, L) such that L is an essential extension of L_2 in $\mathbf{Frm}$. We show that this correspondence cannot be made functorial in the obvious way, as a frame morphism $f : L \to M$ may lift to a map $f : (L, \mathcal{F}) \to (L, \mathcal{G})$ of Raney extensions without lifting to a map between the associated strictly zero-dimensional biframes.

Keywords. Pointfree topology, frame, Raney duality, biframe, sublocale, essential extension.

Mathematics Subject Classification (2010). 06D22 (Primary); 06B10, 06B23 (Secondary)

Introduction

The usual approach in pointfree topology is to consider the adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ between frames and spaces, and to regard frames as pointfree spaces in virtue of this. This is the classical approach found, for example, in [12], [17], [18]. The fixpoints on the $\mathbf{Top}$ side are the sober spaces, so we may view $\mathbf{Frm}$ as a faithful extension of the opposite of the category $\mathbf{Sob}$ of sober spaces. An alternative approach is *T_D-duality*, developed in [6]. This is based on the T_D-axiom, introduced in [2]. The full subcategory of $\mathbf{Top}$ consisting of the T_D-spaces is related via a similar adjunction to a wide subcategory of $\mathbf{Frm}$, the category of frames with *D-morphisms*, called $\mathbf{Frm_D}$. The category $\mathbf{Frm_D}$ is thus shown to faithfully extend the opposite of $\mathbf{Top_D}$. Sobriety and the T_D property are incomparable. This is why the language of frames, under both translations, is not expressive enough to capture certain concepts and constructions. For example, in $\mathbf{Frm}$ we do not have a notion of sobrification. Similarly, in $\mathbf{Frm_D}$, we do not have a notion of T_D-coreflection (which exists for spaces). Refining the language by extending the category $\mathbf{Frm}$ enables us to capture both things in the same category. For example, in [22], the category $\mathbf{Frm}$ is extended to the category $\mathbf{Raney}$ of *Raney extensions*, which also faithfully extends the dual of the category $\mathbf{Top_0}$ of all T_0-spaces. In this category, we have both the notion of *sober coreflection* (Section 6.5), the dual notion of sobrification, and T_D*-reflection* (Section 6.6), dual of the T_D-coreflection.

- Raney extensions are inspired by the work of Raney on completely distributive lattices, see for example [19]. Raney duality, as illustrated in [7], is the result that the dual of $\mathbf{Top_0}$ is equivalent to the category of completely distributive lattices $\mathcal{U}(X)$ consisting of upper sets of some poset X, equipped with an interior operator. A *Raney extension* is a pair (L, C) where C is a coframe and $L \subseteq C$ is a frame with the inherited order, such that the subset inclusion preserves all joins and strongly exact meets. The Raney extension corresponding to a space X is the pair $(\Omega(X), \mathcal{U}(X))$, where $\Omega(X)$ is its frame of opens and $\mathcal{U}(X)$ is the collection of upper sets in its specialization order. The category $\mathbf{Raney}$ of Raney extensions faithfully extends $\mathbf{Top_0}^{op}$.

- Another approach to refining the language of frames is that of *McKinsey-Tarski algebras*, as introduced in [8]. This is based on work by McKinsey and Tarski, who in [16] studied topological spaces in terms of the closure operator they induce on their powerset. McKinsey-Tarski algebras make this approach pointfree, by considering as objects complete Boolean algebras, not necessarily atomic, with interior operators. The category of MT-algebras faithfully extends all of $\mathbf{Top}^{op}$.

- A third approach to faithfully extending the dual of $\mathbf{Top_0}$ is that of *strictly zero-dimensional biframes*, as shown in [13]. Although in [13] the pointfree description of T_0 spaces is not the focus, it is indeed observed (end of Section 4) that there is a dual adjunction between spaces and strictly zero-dimensional biframes, whose fixpoints are the T_0 spaces.

It is then interesting to look at how the three categories interact, and go towards a more unified theory of pointfree T_0 spaces. The connection between Raney extensions and MT-algebras is looked at in [9]. A Raney extension (L, C) is *admissible* if the joins of L distribute over all binary meets in C. In [9] the connection between Raney extensions and MT-algebras is explored, and it is shown that the category of admissible Raney extensions is equivalent to the category $\mathbf{MT_0}$ of T_0 MT-algebras equipped with a notion of morphism based on a proximity-like relation.

With this paper, we add another part to the big picture, connecting explicitly the categories of Raney extensions and that of strictly zero-dimensional biframes. In particular, we show that there is a bijection at the level of objects between admissible Raney extensions and *essential* strictly zero-dimensional biframes. Our approach is by no means the simplest possible way of proving this correspondence. Instead, we want to obtain the correspondence as a byproduct of a more general study of subcolocales of $\mathsf{S}(L)$ and subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$. We point out the following result.

Proposition 0.1. *Let L be a frame. Raney extensions on L are in bijective correspondence with subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ containing all open sublocales. Strictly zero-dimensional biframes whose first component is L are in bijective correspondence with dense subcolocales of $\mathsf{S}(L)$.*

Notice how, in the usual setting, all sublocales of L are sober: for a sober space X, the sober subspaces are exactly the ones of the form

$$\bigcap_i U_i \cup V_i^c$$

where $U_i, V_i \subseteq X$ are opens. Compare this with the fact that every sublocale $S \subseteq L$ of a frame L satisfies

$$S = \bigcap\{\mathfrak{o}(a) \vee \mathfrak{c}(b) \mid S \subseteq \mathfrak{o}(a) \vee \mathfrak{c}(b)\}$$

to see that it is natural to view $\mathsf{S}(L)$ as the collection of all *sober* subspaces of L. But Raney extensions and strictly zero-dimensional biframes tell us that if we want to capture all subspaces of L, not just sober ones, it suffices to look at subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ or of $\mathsf{S}(L)$, rather than sublocales of L. For example, notice that if X is a T_0 space it may be the case that distinct subspaces induce the same sublocale of $\Omega(X)$. However, as Raney extensions faithfully extend $\mathbf{Top_0}$, this means that the two subspace inclusions will induce different surjections from $(\Omega(X), \mathcal{U}(X))$ in $\mathbf{Raney}$. By the equivalence in Proposition 0.1 above, this means that these will induce different subcolocales of $\mathsf{S}_{\mathfrak{o}}(\Omega(X))$.

In conclusion, it may be argued that $\mathsf{S}(L)$ which is not refined enough to capture spaces which are not sober, but that the limitation is overcome by looking at the collections $\mathsf{S}(\mathsf{S}_{\mathfrak{o}}(L))$ and $\mathsf{S}(\mathsf{S}(L))$, instead. In this paper, we show explicitly how to relate the two approaches. We do so by proving an adjunction between codense subcolocales of $\mathsf{S}(L)$ and certain subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ which we call *admissible*. We prove that the adjunction maximally restricts to admissible subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ and a class of subcolocales $\mathcal{D} \subseteq \mathsf{S}(L)$ which we characterize explicitly.

1. Preliminaries

1.1 The categories Frm and Loc

We first recall some background on frames and point-free topology. For more information on the categories of frames and locales, we refer the reader to Johnstone [12] or the more recent [17] and [18]. A *frame* is a complete lattice L satisfying

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\},$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* is a function preserving arbitrary joins, including the bottom element 0, and finite meets, including the top element 1. We call $\mathbf{Frm}$ the category of frames and frame homomorphisms. Frames are complete Heyting algebras, with the Heyting implication computed as

$$x \to y = \bigvee \{z \in L \mid z \wedge x \leq y\}.$$

In particular, the *pseudocomplement* of an $a \in L$ is the element $\neg a = a \to 0$. The archetypal example of frame is the lattice of open sets $\Omega(X)$ for a topological space X. The assignment $X \mapsto \Omega(X)$ is the object part of a functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$. An element p of a frame L is said to be *prime* if it is not 1 and whenever $x \wedge y \leq p$ for some $x, y \in L$, then $x \leq p$ or $y \leq p$. The collection of all primes of L will be denoted by $\mathsf{pt}(L)$, and the assignment $L \mapsto \mathsf{pt}(L)$ is the object part of a functor $\mathsf{pt} : \mathbf{Frm}^{op} \to \mathbf{Top}$, which together with Ω yields an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ with $\Omega \dashv \mathsf{pt}$. A frame is said to be *spatial* if $a \not\leq b$ for $a, b \in L$ implies that there is a prime p with $b \leq p$ and $a \not\leq p$. Spatial frames L are precisely the fixpoints of the adjunction $\Omega \dashv \mathsf{pt}$. This adjunction is idempotent, and so a frame is spatial if $L \cong \Omega(X)$ for some space X. Because of this adjunction, frames are regarded as pointfree spaces, but since the adjunction is contravariant sometimes the category $\mathbf{Loc}$, equivalent to $\mathbf{Frm}^{op}$, is used.

The T_D approach

An alternative to the classical duality is the so-called T_D*-duality* from [6]. For a space X we call a point $x \in X$ a T_D*-point* if it is the intersection of an open and a closed set. We define a space to be T_D if all its points are T_D. The axiom is between T_0 and T_1 and it is introduced in [2]. For a frame L, a prime $p \in L$ is *covered* if $\bigwedge_i x_i = p$ implies $x_i = p$ for some $i \in I$. We call $\mathsf{pt}_D(L)$ the set of covered primes of L. We say that a frame morphisms $f : L \to M$ is a *D-morphism* if its right adjoint $f_* : M \to L$ maps covered primes to covered primes. We call $\mathbf{Frm_D}$ the category of frames and D-morphisms. The assignment $L \mapsto \mathsf{pt}_D(L)$ extends to a functor $\mathsf{pt}_D : \mathbf{Frm_D}^{op} \to \mathbf{Top}$. In [6] it is shown that there is an adjunction $\Omega : \mathbf{Top_D} \leftrightarrows \mathbf{Frm_D}^{op} : \mathsf{pt}_D$ with $\Omega \dashv \mathsf{pt}_D$, where $\mathbf{Top_D}$ is the category of T_D-spaces. We call a frame T_D*-spatial* if it is meet-generated by its covered primes. T_D-spatial frames are exactly the fixpoints of the adjunction above, namely the frames of opens of T_D-spaces.

Sublocales

For a frame L, a *sublocale* of L is a subset inclusion $S \subseteq L$ such that

1. S is closed under arbitrary meets;
2. $a \to s \in S$ for all $a \in L$ and $s \in S$.

Sublocales are frames when equipped with the order inherited from L. In fact, the name comes from the fact that such subset inclusions are, up to isomorphism, the regular monomorphisms in $\mathbf{Loc}$. If $S \subseteq L$ is a sublocale, we call $\bigwedge^S$ and $\wedge^S$ the arbitrary and the binary meets in S, respectively, and we use a similar convention for joins. Whenever L is any lattice and $M \subseteq L$ a lattice with the inherited order, we use analogous notation for lattice operations in M with the inherited order. Sublocales have closure operators associated to them. For a sublocale $S \subseteq L$ the map $\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}$ for all $a \in L$ is called its *nucleus*.

Lemma 1.1. *Let $S \subseteq L$ be a sublocale. Then, for $s_i \in S$;*

1. $\bigwedge_i^S s_i = \bigwedge_i s_i$;
2. $\bigvee_i^S s_i = \nu_S(\bigvee_i s_i)$.

Every sublocale also has an associated frame *congruence*, a binary relation on L which is a subframe of $L \times L$. An alternative approach, which we will adopt, is to consider the bijection between sublocales of L and *precongruences* on L. These are defined to be binary relations R on L such that

1. R is reflexive;
2. R is transitive;
3. $a' \leq a$ and $(a, b) \in R$ and $b \leq b'$ implies $(a', b') \in R$;
4. $(a_i, b) \in R$ implies $(\bigvee_i a_i, b) \in R$;
5. $(a, b_1), (a, b_2) \in R$ implies $(a, b_1 \wedge b_2) \in R$.

This correspondence is introduced in [14]. The sublocale S is associated with the precongruence $\{(a, b) \in L \times L \mid \nu_S(a) \leq \nu_S(b)\}$. Conversely, for a precongruence $R \subseteq L \times L$, the associated sublocale is

$$\bigcap\{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid (x, y) \in R\}.$$

A sublocale of L is *dense* if it contains 0. Dense sublocales, then, are closed under arbitrary intersections. For a frame L, the smallest dense sublocale $\mathfrak{b}(0) \subseteq L$ is always Boolean, and it is called its *Booleanization*.

Subcolocales and the coframe $\mathsf{S}(L)$

Subcolocales

Coframes come equipped with a *co-Heyting operator*, known as the *difference* $x \backslash y$ of two elements $x, y \in C$, computed as

$$x \backslash y = \bigwedge \{z \in C \mid x \leq y \vee z\}.$$

This operator is characterized by the condition that $x \backslash y \leq z$ if and only if $x \leq y \vee z$. In particular, the *supplement* of $c \in C$ is $c^* = 1 \backslash c$. We shall freely use some its properties, listed in the following lemma.

Lemma 1.2. *Let C be a coframe. For elements $c, d, c_i, x \in C$:*

1. *If x is complemented, $c \backslash x = c \wedge x^*$;*
2. $(\bigvee_i c_i) \backslash d = \bigvee_i (c_i \backslash d)$*;*
3. $d \backslash \bigwedge_i c_i = \bigvee_i (d \backslash c_i)$.

For a coframe C, we say that an element $c \in C$ is *linear* if $\bigvee_i (a_i \wedge c) = \bigvee_i a_i \wedge c$ for all $a_i \in C$.

Lemma 1.3. *Complemented elements of a coframe are linear.*

Of particular importance will be the notion dual to that of sublocale. Let us define it explicitly. For a coframe C, a *subcolocale* is an inclusion $D \subseteq C$ such that

1. D is closed under all joins;
2. $d \backslash c \in D$ for all $d \in D$ and $c \in C$.

An inclusion $D \subseteq C$ for a frame D is a subcolocale iff $D^{op} \subseteq C^{op}$ is a sublocale. Dualizing the analogous notion for frames, we see that a subcolocale $D \subseteq C$ determines an interior operator $\nu_D : C \to C$, which we call its *conucleus*.

Lemma 1.4. *Let $D \subseteq C$ be a sublocale. Then, for $d_i \in D$;*

1. $\bigwedge_i^D d_i = \nu_D(\bigwedge_i d_i)$*;*

2. $\bigvee_i^D d_i = \bigvee_i d_i$.

We say that a subcolocale is *codense* if it contains 1. We keep the term for the dual notion and call the smallest codense subcolocale of a coframe its *Booleanization*.

The coframe of sublocales

The family $\mathsf{S}(L)$ of all sublocales of L, ordered by inclusion, is a coframe. Meets are set-theoretical intersections. Because $\mathsf{S}(L)$ is a coframe, it also comes with a difference operation, computed as $S \backslash T = \bigcap\{U \in \mathsf{S}(L) \mid S \subseteq T \vee U\}$. This is studied in [20]. For each $a \in L$, there is an *open sublocale* $\mathfrak{o}(a) = \{b \in L \mid b = a \to b\} = \{a \to b \mid b \in L\}$ and a *closed sublocale* $\mathfrak{c}(a) = {\uparrow} a$. Open and closed sublocales behave like open and closed subspaces in many respects, in the lemma below we list a few.

Lemma 1.5. *For every frame L and $a, b, a_i \in L$ we have*

1. $\mathfrak{o}(1) = L$ *and* $\mathfrak{o}(0) = \{1\}$.
2. $\mathfrak{c}(1) = \{1\}$ *and* $\mathfrak{c}(0) = L$.
3. $\mathfrak{o}(a) \cap \mathfrak{c}(a) = \{1\}$ *and* $\mathfrak{o}(a) \vee \mathfrak{c}(a) = L$.
4. $\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i)$ *and* $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.
5. $\bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}(\bigwedge_i a_i)$ *and* $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.

In particular, by 3, open and closed sublocales are complemented, and as such (Lemma 1.3) they are linear. Intersections of open sublocales are called *fitted* sublocales. These form a subcoframe of $\mathsf{S}(L)$, which we denote as $\mathsf{S}_{\mathfrak{o}}(L)$. The closure operator associated with it is called the *fitting*. This is studied in [10]. More explicitly, the corestriction to its fixpoints is:

$$\begin{aligned} \mathit{fit} : & \mathsf{S}(L) \to \mathsf{S}(L) \\ & S \mapsto \bigcap\{\mathfrak{o}(x) \mid x \in L, S \subseteq \mathfrak{o}(x)\}. \end{aligned}$$

Another important class of sublocales is that of the *two-element sublocales*. These are sublocales of the form $\{1, p\}$ for some element $p \in L$, which is then necessarily prime.

Exactness and strong exactness

For a complete lattice L we say that a join (resp. a meet) is *exact* if it distributes over binary meets (resp. joins). These notions are compared in [3]. When L is a frame, we also have the notion of *strongly exact* meet: a meet $\bigwedge_i x_i$ such that $x_i \to y = y$ for all $i \in I$ implies that $\bigwedge_i x_i \to y = y$. A filter $F \subseteq L$ of a frame is called *exact* if it is closed under exact meets, and *strongly exact* if it is closed under strongly exact meets. We call $\mathsf{Filt}_{\mathcal{SE}}(L)$ the ordered collection of strongly exact filters of L, and $\mathsf{Filt}_{\mathcal{E}}(L)$ the collection of the exact filters. The following is Theorem 3.5 in [15].

Lemma 1.6. *There is an isomorphism* $\varphi : \mathsf{S}_{\mathfrak{o}}(L) \cong \mathsf{Filt}_{\mathcal{SE}}(L)$ *given by*

$$\varphi(F) = \{x \in L \mid F \subseteq \mathfrak{o}(x)\}$$

for each $F \in \mathsf{S}_{\mathfrak{o}}(L)$.

We say that a frame map $f : L \to M$ is *exact* if, whenever $\bigwedge_i x_i \in L$ is an exact meet, $\bigwedge_i f(x_i)$ is exact, and $\bigwedge_i f(x_i) = f(\bigwedge_i x_i)$. A sublocale $S \subseteq L$ whose surjection $s : L \to S$ is exact is called an *exact* sublocale. Exact sublocales are precisely those whose surjection preserves all exact meets. The next result is Proposition 7.15 in [22].

Lemma 1.7. *A sublocale* $S \subseteq L$ *is exact if and only if whenever* $\bigwedge_i x_i \in L$ *is an exact meet and* $S \cap \mathfrak{c}(x_i) \subseteq \mathfrak{c}(x)$ *for every* $i \in I$ *then* $S \cap \mathfrak{c}(\bigwedge_i x_i) \subseteq \mathfrak{c}(x)$ *for all* $x \in L$.

Distinguished subcolocales of $\mathsf{S}(L)$

In this work, we look at subcolocales of $\mathsf{S}(L)$. Among these is the collection $\mathsf{S}_b(L)$ of joins of complemented sublocales (see [1]). These also coincide with those sublocales of the form $\bigvee_i \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i)$. We observe that a subcolocale of $\mathsf{S}(L)$ is codense if and only if it contains L. The following is well-known.

Lemma 1.8. *For any frame* L, *the inclusion* $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ *is a codense subcolocale. In particular, it is the Booleanization of* $\mathsf{S}(L)$.

We will consider the map $\varphi \circ fit : \mathsf{S}(L) \to \mathsf{Filt}_{\mathcal{SE}}(L)$. We call this ker, for *kernel*, as it assigns to a sublocale $S \subseteq L$ the set $s^{-1}(1)$, where $s : L \to S$ is its surjection. Theorem 6.6 of [11], together with the fact that φ is an isomorphism, gives the following.

Lemma 1.9. *For every frame L, $ker[\mathsf{S}_b(L)] = \mathsf{Filt}_{\mathcal{E}}(L)$.*

We will need the following result.

Lemma 1.10. *If X is a T_D-space, all sublocales in $\mathsf{S}_b(\Omega(X))$ are T_D-spatial.*

Proof. Proposition 3.2 of [1] states that all sublocales in $\mathsf{S}_b(\Omega(X))$ are induced by some subspace of X, in the sense that, for each of these, the surjection corresponding to them is of the form $\Omega(i_Y) : \Omega(X) \to \Omega(Y)$ for some subspace inclusion $i_Y : Y \subseteq X$. Since subspaces of T_D spaces are T_D, these are T_D-spatial sublocales. □

We will also look at the collection $\mathsf{S}_{sp}(L)$ of spatial sublocales of L, by which we mean the sublocales $S \subseteq L$ where S is a spatial frame. Equivalently, spatial sublocales are characterized as those which are joins of two-element sublocales. For all frames L, the inclusion $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is a subcolocale inclusion, in particular, it is the subcolocale associated with the spatialization surjection of $\mathsf{S}(L)^{op}$ (Proposition 3.14 of [21]). The next result follows by definition of codensity.

Lemma 1.11. *For a frame L, the subcolocale $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is codense if and only if L is spatial.*

The collection of all exact sublocales of L will be denoted as $\mathsf{S}_{\mathcal{E}}(L)$. The coming result follows from Theorem 7.20 of [22] and Proposition 7.12 from the same paper. The second part of the claim follows from $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ being the smallest codense subcolocale.

Lemma 1.12. *The inclusion $\mathsf{S}_{\mathcal{E}}(L) \subseteq \mathsf{S}(L)$ is a codense subcolocale. In particular, $\mathsf{S}_b(L) \subseteq \mathsf{S}_{\mathcal{E}}(L)$.*

Strictly zero-dimensional biframes and Raney extensions

Strictly zero-dimensional biframes

A *biframe* (see [5]) is a triple $\mathcal{L} = (L_1, L_2, L)$ where L is a frame and $L_1, L_2 \subseteq L$ are subframes such that $L_1 \cup L_2$ generates L in the sense that L is the smallest subframe containing $L_1 \cup L_2$. An element $a_1 \in L_1$ is said to be *bicomplemented* if it is complemented in L and its complement is in L_2, and similarly for elements of L_2. A biframe is said to be *zero-dimensional* if both L_1 and L_2 are join-generated by their bicomplemented elements. A biframe is said to be *strictly zero-dimensional* if it is zero-dimensional and all elements of L_1 are bicomplemented. Strictly zero-dimensional biframes are studied in detail in [13]. We will later see (Lemma 1.14) that dense subcolocales of $\mathsf{S}(L)$ contain all closed sublocales. In particular, these are embedded as a subcoframe $\mathfrak{c}[L] \subseteq \mathcal{D}$. We give an equivalent description to the category of strictly zero-dimensional biframes, similar to the one given in Section 3.2 of [13]. The category $\mathbf{SZDBF}$ is the category:

1. Whose objects are pairs $(L, \mathcal{D})$ where L is a frame and $\mathcal{D} \subseteq \mathsf{S}(L)$ a dense subcolocale;
2. Whose morphisms $f : (L, \mathcal{D}) \to (M, \mathcal{E})$ are frame maps $f : L \to M$ such that the anti-isomorphic coframe map $\mathfrak{c}(f) : \mathfrak{c}[L] \to \mathfrak{c}[M]$ extends to a coframe map $\overline{f} : \mathcal{D} \to \mathcal{E}$.

Raney extensions

Raney extensions are introduced in [22]. These structures are inspired by a duality studied in [7], based on work by Raney, see [19]. Raney extensions may be identified with pairs $(L, \mathcal{F})$, where $\mathcal{F} \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)^{op}$ is a subcolocale containing all principal filters (see Theorem 3.9 in [22]). Here, we use the isomorphism in 1.6 to give an equivalent definition. We define $\mathbf{Raney}$ to be the category:

1. Whose objects are pairs $(L, \mathcal{F})$ where L is a frame and $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ a subcolocale containing all open sublocales;
2. Whose morphisms $f : (L, \mathcal{F}) \to (M, \mathcal{G})$ are frame maps $f : L \to M$ such that the isomorphic frame map $\mathfrak{o}(f) : \mathfrak{o}[L] \to \mathfrak{o}[M]$ extends to a coframe map $\overline{f} : \mathcal{F} \to \mathcal{G}$.

The main adjunction

From subcolocales of $\mathsf{S}(L)$ to subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ via fitting

In this subsection, we show that the fitting operator $fit : \mathsf{S}(L) \to \mathsf{S}_{\mathfrak{o}}(L)$ is such that its direct image $fit[-] : \mathcal{P}(\mathsf{S}(L)) \to \mathcal{P}(\mathsf{S}_{\mathfrak{o}}(L))$ sends codense subcolocales to special kinds of subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$. First, we work towards a characterization of codense subcolocales of $\mathsf{S}(L)$.

Proposition 1.13. *An inclusion $\mathcal{D} \subseteq \mathsf{S}(L)$ is a subcolocale if and only if it is closed under arbitrary joins and stable under both $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ for all $x \in L$.*

Proof. Suppose that $\mathcal{D}$ satisfies the assumptions in the statement. By definition of subcolocale, it suffices to show that $S \in \mathcal{D}$ implies $S \backslash T \in \mathcal{D}$ for all $T \in \mathsf{S}(L)$. Let $S \in \mathcal{D}$ and let $T = \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$. By assumption, for every $i \in I$ we have $S \cap \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i) \in \mathcal{D}$. By 1 of Lemma 1.2, this equals $S \backslash (\mathfrak{o}(x_i) \vee \mathfrak{c}(y_i))$. As $\mathcal{D}$ is closed under all joins and by item 3 of Lemma 1.2, $S \backslash \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i) \in \mathcal{D}$, as desired. For the other direction, if $\mathcal{F} \subseteq \mathsf{S}(L)$ is a subcolocale, indeed it must be stable under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$, as by item 1 of Lemma 1.2 these are the same as $- \backslash \mathfrak{c}(x)$ and $- \backslash \mathfrak{o}(x)$, respectively. □

Lemma 1.14. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale. Then:*

1. *$\mathcal{D}$ contains all open and closed sublocales;*
2. *$\mathcal{D}$ is meet-generated by the elements of the form $\mathfrak{o}(x) \vee \mathfrak{o}(y)$.*

Proof. If $\mathcal{D} \subseteq \mathsf{S}(L)$ is codense, by definition it must contain L. By Lemma 1.13, then, it must contain $L \backslash \mathfrak{o}(x)$ and $L \backslash \mathfrak{c}(x)$ for all $x \in L$, but by 1 of Lemma 1.2 these equal $\mathfrak{c}(x)$ and $\mathfrak{o}(x)$, respectively. Every element of $\mathcal{D}$ is $\nu_{\mathcal{D}}(S)$ for some sublocale $S \in \mathsf{S}(L)$, and $S = \bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$ for some $x_i, y_i \in L$. Then, every element of $\mathcal{D}$ can be written as $\nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i))$ for some $x_i, y_i \in L$. This equals $\bigwedge_i^{\mathcal{D}} \nu_{\mathcal{D}}(\mathfrak{o}(x_i) \vee \mathfrak{c}(x_i))$ by Lemma 1.4. As $\mathcal{D}$ contains all open and closed sublocales by the first item, and as it is closed under joins by 1.13, $\nu_{\mathcal{D}}(\mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)) = \mathfrak{o}(x_i) \vee \mathfrak{c}(y_i)$, and our claim is proven. □

We now characterize the subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ in a similar manner. We call $\bigvee^{fit}$ and $\backslash^{fit}$ the joins and the coframe differences in $\mathsf{S}_{\mathfrak{o}}(L)$, respectively.

Lemma 1.15. *In* $\mathsf{S}_{\mathfrak{o}}(L)$*, for all* F_i, F, G*:*

1. $\bigvee_i^{fit} F_i = fit(\bigvee_i F_i)$;
2. $F\backslash^{fit} G = fit(F\backslash G)$.

Proof. For the joins, it suffices to notice that if $F_i \subseteq F$ is an upper bound in $\mathsf{S}_{\mathfrak{o}}(L)$, as F is a fixpoint of *fit* also $fit(\bigvee_i F_i) \subseteq F$. By its definition, the difference $F\backslash^{fit} G$ is such that $F\backslash^{fit} G \subseteq \mathfrak{o}(x)$ if and only if $F \subseteq G \vee \mathfrak{o}(x)$ for all $x \in L$. But the following are also equivalent:

$$\dfrac{\dfrac{F \subseteq G \vee \mathfrak{o}(x)}{F\backslash G \subseteq \mathfrak{o}(x)}}{fit(F\backslash G) \subseteq \mathfrak{o}(x)).}$$

Since fitted sublocales are intersections of open ones, and we have shown that the fitted sublocales $F\backslash^{fit} G$ and $fit(F\backslash G)$ are contained in the same opens, they must be equal. □

Lemma 1.16. *For a frame* L *and* $F \in \mathsf{S}_{\mathfrak{o}}(L)$*, and* $x \in L$

$$fit(F \cap \mathfrak{c}(x)) = F\backslash^{fit} \mathfrak{o}(x).$$

Proof. By 1 of Lemma 1.2, $F \cap \mathfrak{c}(x) = F\backslash \mathfrak{o}(x)$, and so $fit(F \cap \mathfrak{c}(x)) = fit(F\backslash \mathfrak{o}(x))$. By Lemma 1.15, $fit(F\backslash \mathfrak{o}(x)) = F\backslash^{fit} \mathfrak{o}(x)$. □

Proposition 1.17. *A collection* $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ *is a subcolocale if and only if it is closed under arbitrary joins and is stable under* $fit(- \cap \mathfrak{c}(x))$ *for all* $x \in L$*.*

Proof. Suppose that $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ satisfies the assumptions in the claim. We check that $F \in \mathcal{F}$ implies $F\backslash^{fit} \bigcap_i \mathfrak{o}(x_i) \in \mathcal{F}$ for all families $x_i \in L$. If $F \in \mathcal{F}$, by our assumption, $fit(F \cap \mathfrak{c}(x_i)) \in \mathcal{F}$ for every $i \in I$. By Lemma 1.16, this means that $F\backslash^{fit} \mathfrak{o}(x_i) \in \mathcal{F}$. By 3 of Lemma 1.2, $F\backslash^{fit} \bigcap_i \mathfrak{o}(x_i) = \bigvee_i F\backslash^{fit} \mathfrak{o}(x_i)$, and this is in $\mathcal{F}$ as we assumed this collection is closed under all joins. Conversely, if $\mathcal{F}$ is a subcolocale, it is closed under all joins by definition, and by definition also $F\backslash^{fit} \mathfrak{o}(x) \in \mathcal{F}$ for all $x \in L$. By Lemma 1.16, then, $fit(F \cap \mathfrak{c}(x)) \in \mathcal{F}$ for all $x \in L$. □

Finally, we want to show that for every codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ the inclusion $fit[\mathcal{D}] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is a subcolocale containing all opens. We also look at how the conuclei of these interact.

Lemma 1.18. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale. For every $S \in \mathsf{S}(L)$ and $x \in L$,*

1. $\nu_{\mathcal{D}}(S \cap \mathfrak{o}(x)) = \nu_{\mathcal{D}}(S) \cap \mathfrak{o}(x)$;
2. $\nu_{\mathcal{D}}(S \cap \mathfrak{c}(x)) = \nu_{\mathcal{D}}(S) \cap \mathfrak{c}(x)$.

Proof. The claims holds because $\mathfrak{o}(x), \mathfrak{c}(x) \in \mathcal{D}$ by item 1 of Lemma 1.14, and $\mathcal{D}$ is stable under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ by Proposition 1.13. □

Lemma 1.19. *For a sublocale $S \subseteq L$, $\mathit{fit}(S \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(S) \cap \mathfrak{c}(x))$ for all $x \in L$.*

Proof. The following are equivalent statements for all $y \in L$. At each step we only use basic properties of fitting and of open and closed sublocales.

$$\frac{\frac{\frac{\frac{\mathit{fit}(S \cap \mathfrak{c}(x)) \subseteq \mathfrak{o}(y)}{S \cap \mathfrak{c}(x) \subseteq \mathfrak{o}(y)}}{S \subseteq \mathfrak{o}(y) \vee \mathfrak{o}(x)}}{\mathit{fit}(S) \subseteq \mathfrak{o}(y) \vee \mathfrak{o}(x)}}{\mathit{fit}(S) \cap \mathfrak{c}(x) \subseteq \mathfrak{o}(y).}$$

Indeed, then, $\mathit{fit}(S \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(S) \cap \mathfrak{c}(x))$ as desired. □

Proposition 1.20. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale.*

1. *The collection $\mathcal{F} := \mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is a subcolocale containing all opens;*
2. *$\nu_{\mathcal{F}}(F) = \mathit{fit}(\nu_{\mathcal{D}}(F))$ for all $F \in \mathsf{S}_{\mathfrak{o}}(L)$;*
3. *$\bigwedge^{\mathcal{F}} F_i = \mathit{fit}(\nu_{\mathcal{D}}(\bigcap_i F_i)$ for $D_i \in \mathcal{D}$.*

Proof. Let us prove the three items in turn.

1. We use the characterization in Proposition 1.17. Closure of $\mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ under all joins follows from the fact that fitting is a closure operator on $\mathsf{S}(L)$, and so the map $\mathit{fit} : \mathsf{S}(L) \to \mathsf{S}_{\mathfrak{o}}(L)$ to its fixpoints preserves all joins. Next, we have to show that for $F \in \mathit{fit}[\mathcal{D}]$ the element $\mathit{fit}(F \cap \mathfrak{c}(x))$ is in $\mathit{fit}[\mathcal{D}]$, by Proposition 1.17. This holds because for $D \in \mathcal{D}$ such that $F = \mathit{fit}(D)$ we have $D \cap \mathfrak{c}(x) \in \mathcal{D}$, by

Proposition 1.13 and $\mathit{fit}(D \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(D) \cap \mathfrak{c}(x))$ by Lemma 1.19. Then, $\mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is a subcolocale. Since $\mathfrak{o}(x) \in \mathcal{D}$ for all $x \in L$, by 1 of Lemma 1.14, $\mathit{fit}[\mathcal{D}]$ contains all opens.

2. The following are equivalent statements. Again, for each derivation we are only using basic facts about fitting and the conucleus $\nu_{\mathcal{D}}$.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\mathit{fit}(\nu_{\mathcal{D}}(F)) \subseteq \mathfrak{o}(x)}{\nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(x)}}{\mathit{fit}(D) \subseteq F \text{ implies } \mathit{fit}(D) \subseteq \mathfrak{o}(x) \text{ for all } D \in \mathcal{D}}}{D \subseteq F \text{ implies } D \subseteq \mathfrak{o}(x) \text{ for all } D \in \mathcal{D}}}{\nu_{\mathcal{D}}(\mathit{fit}(F)) \subseteq \nu_{\mathcal{D}}(\mathfrak{o}(x))}}{\nu_{\mathcal{D}}(\mathit{fit}(F)) \subseteq \mathfrak{o}(x).}$$

3. This follows from (2) and from Lemma 1.4. □

We have found a (clearly monotone) map $\mathit{fit}[-] : \mathcal{CD}(\mathsf{S}(L)) \to \mathcal{C}(\mathsf{S}_{\mathfrak{o}}(L))$ from codense subcolocales of $\mathsf{S}(L)$ to subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$ containing all opens. We want to show a concrete example of the assignment $\mathit{fit}[-] : \mathcal{CD}(\mathsf{S}(L)) \to \mathcal{C}(\mathsf{S}_{\mathfrak{o}}(L))$ not being injective in general.

Lemma 1.21. *For a frame L, $\mathit{fit}[\mathsf{S}_b(L)] = \mathit{fit}[\mathsf{S}_{\mathcal{E}}(L)]$.*

Proof. Since $\mathsf{S}_b(L) \subseteq \mathsf{S}_{\mathcal{E}}(L)$ by Lemma 1.12, $\mathit{fit}[\mathsf{S}_b(L)] \subseteq \mathit{fit}[\mathsf{S}_{\mathcal{E}}(L)]$. Let us show the reverse inclusion. By Lemma 1.6, it suffices to show that $\mathit{ker}[\mathsf{S}_{\mathcal{E}}(L)] \subseteq \mathit{ker}[\mathsf{S}_b(L)]$, and since $\mathit{ker}[\mathsf{S}_b(L)] = \mathsf{Filt}_{\mathcal{E}}(L)$, by Lemma 1.9, it suffices to show that if $E \in \mathsf{S}_{\mathcal{E}}(L)$ then $\mathit{ker}(E)$ is an exact filter. Let $\bigwedge x_i \in L$ be an exact meet. If $E \subseteq \mathfrak{o}(x_i)$, then we have $E \cap \mathfrak{c}(x_i) \subseteq \{1\}$. By the characterization of exact sublocales in Lemma 1.7, this implies that $E \cap \mathfrak{c}(\bigwedge_i x_i) \subseteq \{1\}$, that is, $E \subseteq \mathfrak{o}(\bigwedge_i x_i)$, and so, indeed, $\bigwedge_i x_i \in \mathit{ker}(E)$. □

By Lemma 1.21, above, then, to show that $\mathit{fit}[-]$ is not injective, it suffices to find a frame L where $\mathsf{S}_{\mathcal{E}}(L) \not\subseteq \mathsf{S}_b(L)$. Complete sublocales, i.e. sublocales such that their surjection preserves all meets, in particular are exact (Proposition 7.14 in [22]). Then, complete sublocales which are not in $\mathsf{S}_b(L)$ are witnesses of $\mathsf{S}_{\mathcal{E}}(L) \not\subseteq \mathsf{S}_b(L)$. Example 5.12 in [9] presents a concrete example of this. The following class of examples of complete sublocales which are not in $\mathsf{S}_b(L)$ is due to Igor Arrieta, who we thank.

Example 1.22. For every frame L, there is a frame surjection $\varepsilon : \mathcal{D}(L) \to L$ defined as $\varepsilon(D) = \bigvee D$. We claim that if L is completely distributive and is not T_D-spatial, then the sublocale $\varepsilon_*[L] \subseteq \mathcal{D}(L)$ associated with this surjection is exact but not in $\mathsf{S}_b(L)$. An example of such a completely distributive lattice is given, for example, by the interval $[0,1] \subseteq \mathbb{R}$, which has no covered primes. By complete distributivity of L, the surjection $\varepsilon : \mathcal{D}(L) \to L$ preserves all meets, and so it corresponds to an exact sublocale. Let us show that this sublocale is not in $\mathsf{S}_b(L)$. We note that $\mathcal{D}(L)$ is T_D-spatial: the covered primes are exactly the elements of the form $L \backslash {\uparrow} x$ for some $x \in L$, and these meet-generate $\mathcal{D}(L)$. Then, by Lemma 1.10, every sublocale in $\mathsf{S}_b(L)$ is T_D-spatial. As L is not T_D-spatial, by assumption, the sublocale corresponding $\varepsilon_*[L] \subseteq \mathcal{D}(L)$ cannot be in $\mathsf{S}_b(L)$.

1.2 From admissible subcolocales of $\mathsf{S}_\mathfrak{o}(L)$ to subcolocales of $\mathsf{S}(L)$

In this section, we restrict to a particular class of subcolocales of $\mathsf{S}_\mathfrak{o}(L)$ and define for their collection a left order adjoint to (the suitable corestriction of) the monotone map $\mathit{fit}[-] : \mathcal{CD}(\mathsf{S}(L)) \to \mathcal{C}(\mathsf{S}_\mathfrak{o}(L))$. We say that a subcolocale $\mathcal{F} \subseteq \mathsf{S}_\mathfrak{o}(L)$ is *admissible* if it contains all open sublocales and the join $\bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i)$ is exact for each family $x_i \in L$.

Lemma 1.23. *If $\mathcal{D} \subseteq \mathsf{S}(L)$ is a codense subcolocale, then $\mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_\mathfrak{o}(L)$ is admissible.*

Proof. By Proposition 1.20, $\mathit{fit}[\mathcal{D}] \subseteq \mathsf{S}_\mathfrak{o}(L)$ is a subcolocale containing all opens. Let us show that for all $x_i \in L$ and $F \in \mathsf{S}_\mathfrak{o}(L)$:

$$\mathfrak{o}(\bigvee_i x_i) \wedge^{\mathcal{F}} F \subseteq \bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i) \wedge^{\mathcal{F}} F.$$

Suppose that $\mathfrak{o}(x_i) \wedge^{\mathcal{F}} F \subseteq \mathfrak{o}(y)$ for all $i \in I$. Then, by item 2 of Proposition 1.20, $\mathit{fit}(\nu_{\mathcal{D}}(\mathfrak{o}(x_i) \cap F)) \subseteq \mathfrak{o}(y)$, and by Lemma 1.18 this implies $\mathfrak{o}(x_i) \cap \nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y)$. This also means $\nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y) \vee \mathfrak{c}(x_i)$ for all $i \in I$, that is, $\nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y) \vee \mathfrak{c}(\bigvee_i x_i)$. Thus, $\mathfrak{o}(\bigvee_i x_i) \cap \nu_{\mathcal{D}}(F) \subseteq \mathfrak{o}(y)$, and this implies $\mathit{fit}(\nu_{\mathcal{D}}(\mathfrak{o}(\bigvee_i x_i) \cap F)) \subseteq \mathfrak{o}(y)$, where we have used 1.18 again. As the left-hand side is $\mathfrak{o}(\bigvee_i x_i) \wedge^{\mathcal{F}} F$, by item 3 of Proposition 1.20, our claim is proven. □

Corollary 1.24. *The following are all admissible subcolocales.*

1. $\mathsf{S}_{\mathfrak{o}}(L) \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ *for every frame* L*;*
2. $\mathit{fit}[\mathsf{S}_b(L)] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ *for every frame* L*;*
3. $\mathit{fit}[\mathsf{S}_{sp}(L)] \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ *for every spatial frame* L*.*

Proof. We prove each item using 1.20. For the first, we just note $\mathsf{S}_{\mathfrak{o}}(L) = \mathit{fit}[\mathsf{S}(L)]$. The inclusion $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$ is codense by Lemma 1.8. Finally, for a spatial frame L, the inclusion $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$ is codense by Lemma 1.11. □

For every subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ containing all opens, for all $F \in \mathcal{F}$ we define the relation $\leq_F^{\mathcal{F}} \subseteq L \times L$ as:

$$\leq_F^{\mathcal{F}} = \{(x, y) \in L \times L \mid F \wedge^{\mathcal{F}} \mathfrak{o}(x) \subseteq \mathfrak{o}(y)\}.$$

Should $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ be clear from the context, we will sometimes abbreviate this as $\leq_F$.

Proposition 1.25. *A subcolocale* $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ *containing all opens is admissible if and only if for every* $F \in \mathcal{F}$ *the relation* $\leq_F^{\mathcal{F}}$ *is a frame precongruence.*

Proof. The only condition which is not shown with routine computations is stability under arbitrary joins. If $x_i \leq_F y$ for $F \in \mathcal{F}$, then $\bigvee_i^{\mathcal{F}} (F \wedge^{\mathcal{F}} \mathfrak{o}(x_i)) \subseteq \mathfrak{o}(y_i)$. The desired result follows by exactness of the join $\bigvee_i^{\mathcal{F}} \mathfrak{o}(x_i) = \mathfrak{o}(\bigvee_i x_i)$. For the converse, suppose that there is a subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ containing all opens, which is not admissible. Let $F \in \mathcal{F}$ and $x_i \in L$ with $F \wedge^{\mathcal{F}} \mathfrak{o}(\bigvee_i x_i) \nsubseteq \bigvee_i^{\mathcal{F}} (F \wedge^{\mathcal{F}} \mathfrak{o}(x_i))$. So, there is $y \in L$ with $F \wedge^{\mathcal{F}} \mathfrak{o}(x_i) \subseteq \mathfrak{o}(y)$ for all $i \in I$ but $F \wedge^{\mathcal{F}} \mathfrak{o}(\bigvee_i x_i) \nsubseteq \mathfrak{o}(y)$. The first set inclusion means $x_i \leq_F y$ for each $i \in I$, and the second means $\bigvee_i x_i \nleq_F y$. Then, the relation $\leq_F$ is not stable under joins, and so it is not a frame precongruence. □

To our ends, the following characterization of admissible subcolocales will be more useful.

Corollary 1.26. *A subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ containing all opens is admissible if and only if for every $F \in \mathcal{F}$ there is $\sigma(F) \in \mathsf{S}(L)$ such that*

$$\sigma(F) \subseteq \mathfrak{c}(x) \vee \mathfrak{o}(y) \text{ if and only if } x \leq_F y, \tag{1}$$

which is necessarily unique. Equivalently, $\sigma(F)$ is such that

$$fit(\sigma(F) \cap \mathfrak{o}(x)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \text{ for each } x \in L. \tag{2}$$

Proof. The first claim follows from 1.25, and by the correspondence between precongruences and sublocales. To see that the second condition is equivalent to the first, we note that the first condition amounts to having $\sigma(F) \cap \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$ if and only if $x \leq_F y$. In turn, this is equivalent to having $\sigma(F) \cap \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$ if and only if $F \wedge^{\mathcal{F}} \mathfrak{o}(x) \subseteq \mathfrak{o}(y)$. □

For each admissible subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ we may then define a map

$$\begin{aligned} \sigma_{\mathcal{F}} :& \mathcal{F} \to \mathsf{S}(L) \\ & F \mapsto \bigcap \{\mathfrak{o}(x) \vee \mathfrak{c}(y) \mid x, y \in L, x \leq_F y\}. \end{aligned}$$

When $\mathcal{F}$ is clear from the context, we simply call this map σ.

Let us see a few basic facts about this map.

Lemma 1.27. *Let $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ be an admissible subcolocale. Then, for each $F \in \mathcal{F}$ and $x \in L$:*

1. *$fit(\sigma(F)) = F$;*
2. *$\sigma(\mathfrak{o}(x)) = \mathfrak{o}(x)$;*
3. *$\sigma(F \wedge^{\mathcal{F}} \mathfrak{o}(x)) = \sigma(F) \cap \mathfrak{o}(x)$.*

Proof. Suppose that $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is an admissible codense subcolocale.

1. By Corollary 1.26, $fit(\sigma(F)) = fit(\sigma(F) \cap \mathfrak{o}(1)) = F \wedge^{\mathcal{F}} \mathfrak{o}(1) = F$.
2. We use Corollary 1.26. We have to check $fit(\mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y)$ for all $y \in L$. Indeed, $fit(\mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathfrak{o}(x) \cap \mathfrak{o}(y) = \mathfrak{o}(x \wedge y)$, and this equals $\mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y)$ as $\mathcal{F}$ contains all open sublocales.

3. We use Corollary 1.26 again. We have to show that

$$\mathit{fit}(\sigma(F) \cap \mathfrak{o}(x) \cap \mathfrak{o}(y)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y) \tag{1}$$

for all $y \in L$. By the characterization of $\sigma(F)$ from 1.26.

$$\mathit{fit}(\sigma(F) \cap \mathfrak{o}(x) \cap \mathfrak{o}(y)) = \mathit{fit}(\sigma(F) \cap \mathfrak{o}(x \wedge y)) = F \wedge^{\mathcal{F}} \mathfrak{o}(x \wedge y), \tag{2}$$

As $\mathcal{F}$ contains all open sublocales,

$$F \wedge^{\mathcal{F}} \mathfrak{o}(x \wedge y) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \cap \mathfrak{o}(y) = F \wedge^{\mathcal{F}} \mathfrak{o}(x) \wedge^{\mathcal{F}} \mathfrak{o}(y). \tag{3}$$

By combining 2 and 3, we obtain 1 as desired. □

We call $\Delta(\mathcal{F})$ the subcolocale $\mathcal{S}(\sigma[\mathcal{F}])$. For a complete lattice C and a collection $X \subseteq C$ we call $\mathcal{J}(X)$ the closure of X under arbitrary joins. If C is a coframe, we call $\mathcal{S}(X)$ the smallest subcolocale containing X.

Lemma 1.28. *Let $\mathcal{X} \subseteq \mathsf{S}(L)$ be any subset. Then $\mathcal{S}(\mathcal{X})$ is*

$$\mathcal{J}(\{X \cap \mathfrak{o}(a) \cap \mathfrak{c}(b) \mid a, b \in L, X \in \mathcal{X}\}).$$

Proof. By Proposition 1.13, any subcolocale $\mathcal{S} \subseteq \mathsf{S}(L)$ containing $\mathcal{X}$ must also contain the collection in the claim. To show the desired result, then, it suffices to show that this is a subcolocale. We use the characterization in 1.13. Closure under joins is clear, and stability under $- \cap \mathfrak{o}(x)$ and $- \cap \mathfrak{c}(x)$ follows from linearity of open and closed sublocales. □

Lemma 1.29. *Let $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ be an admissible subcolocale. Then*

$$\Delta(\mathcal{F}) = \{\bigvee_i \sigma(F_i) \cap \mathfrak{c}(x_i) : F_i \in \mathcal{F}, x_i \in L\}.$$

Proof. By item 3 of Lemma 1.27, $\sigma(F) \cap \mathfrak{o}(x) \in \sigma[\mathcal{F}]$ for every $F \in \mathcal{F}$ and $x \in L$. Then, by Lemma 1.28, $\mathcal{S}(\sigma[\mathcal{F}])$ is as desired. □

Proposition 1.30. *If $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is an admissible subcolocale then $\mathit{fit}[\Delta(\mathcal{F})] = \mathcal{F}$.*

Proof. The inclusion $\mathcal{F} \subseteq \mathit{fit}[\Delta(\mathcal{F})]$ holds by item 1 of Lemma 1.27. Let us show the reverse inclusion. We notice that the map $\mathit{fit} : \Delta(\mathcal{F}) \to \mathsf{S}_{\mathfrak{o}}(L)$ preserves all joins, as joins in $\Delta(\mathcal{F})$ are computed as in $\mathsf{S}(L)$. Then, it suffices to prove the claim for basic elements $\sigma(G) \cap \mathfrak{c}(x)$. We now claim that $\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) \in \mathcal{F}$. Note that, by Lemma 1.19,

$$\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) = \mathit{fit}(\mathit{fit}(\sigma(G)) \cap \mathfrak{c}(x)),$$

and $\mathit{fit}(\sigma(G)) = G$ by item 1 of Lemma 1.27. We have then shown

$$\mathit{fit}(\sigma(G) \cap \mathfrak{c}(x)) = \mathit{fit}(G \cap \mathfrak{c}(x)).$$

By Proposition 1.17, this is in $\mathcal{F}$. □

Corollary 1.31. *A subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is admissible if and only if it is of the form $\mathit{fit}[\mathcal{D}]$ for some codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$.*

Proof. If $\mathcal{F}$ is admissible then $\mathit{fit}[\Delta(\mathcal{F})] = \mathcal{F}$ by Proposition 1.30. If $\mathcal{D} \subseteq \mathsf{S}(L)$ is codense, $\mathit{fit}[\mathcal{D}]$ is admissible by Lemma 1.23. □

Remark 1.32. Note that one could define $\sigma(F) = \bigcap\{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid x \leq_F y\}$ even when $\mathcal{F}$ is not admissible, but in that case $\sigma(F) \subseteq \mathfrak{c}(x) \vee \mathfrak{o}(y)$ does not necessarily imply $x \leq_F y$. Item 1 of Lemma 1.27, stating $\mathit{fit}(\sigma(F)) = F$ for $F \in \mathcal{F}$, relies on this direction of the implication. This is why one cannot extend the definition of the adjoint Δ to non-admissible subcolocales using this more general definition of σ. If F is not admissible, we cannot prove in a similar way that $\mathit{fit}(\sigma(F)) = F$, and so our proof of the $\mathcal{F} \subseteq \mathit{fit}[\Delta(\mathcal{F})]$ half of the adjointness condition does not go through.

1.3 Essential subcolocales of $\mathsf{S}(L)$

We look at what subcolocales of $\mathsf{S}(L)$ are $\Delta(\mathcal{F})$ for some admissible $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$. For a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ we call an element *saturated* if it is of the form $\bigwedge_i^{\mathcal{D}} \mathfrak{o}(x_i) = \nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i))$ for some family $x_i \in L$. We call $\mathsf{Sat}(\mathcal{D})$ their ordered collection, and note that $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is a subcoframe inclusion. Let us define a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ to be *essential* if it is $\mathcal{S}(\mathsf{Sat}(\mathcal{D}))$.

Lemma 1.33. *For a codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$*

$$\mathcal{S}(\mathsf{Sat}(\mathcal{D})) = \mathcal{J}(\{F \cap \mathfrak{c}(z) \mid F \in \mathsf{Sat}(\mathcal{D}), z \in L\}).$$

Proof. This follows from the characterization in 1.28. □

We recall that in a category an *essential extension* is a monomorphism $j : A \to B$ such that whenever $m \circ f : A \to C$ is a monomorphism m, too, is a monomorphism. An essential extension $n : L \to N$ is *maximal* if for every essential extension $m : L \to M$ there is a unique morphism $f : M \to N$ with $f \circ m = n$. Essential extensions for frames are studied in [4]. We now justify the terminology and show that a subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ is essential if and only if $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is an essential extension in **CoFrm**.

Theorem 1.34. *Let L be a frame and $\mathcal{D} \subseteq \mathsf{S}(L)$ a codense subcolocale. Then, $\mathcal{D}$ is essential if and only if $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is essential in* **CoFrm**.

Proof. Suppose that $\mathcal{D} \subseteq \mathsf{S}(L)$ is an essential subcolocale. Suppose, now, that there is a coframe map $f : \mathcal{D} \to C$ such that it is injective when restricted to $\mathsf{Sat}(\mathcal{D})$. The elements of the form $\mathfrak{o}(x) \vee \mathfrak{c}(y)$ meet-generate $\mathcal{D}$, by Lemma 1.14, and as it is essential the elements of the form $F \cap \mathfrak{c}(z)$ with $F \in \mathsf{Sat}(\mathcal{D})$ join-generate it, by Lemma 1.33. Then, to show injectivity it suffices to show that $F \cap \mathfrak{c}(z) \nsubseteq \mathfrak{o}(x) \vee \mathfrak{c}(y)$ implies that $f(F \cap \mathfrak{c}(z)) \nleq f(\mathfrak{o}(x) \vee \mathfrak{c}(y))$. Our assumption means that $F \cap \mathfrak{o}(y) \nsubseteq \mathfrak{o}(x \vee z)$. By assumption on f, we have that $f(F \cap \mathfrak{o}(y)) \nleq f(\mathfrak{o}(x \vee z))$, and as f is a coframe map this also implies that $f(F) \wedge f(\mathfrak{o}(y)) \nleq f(\mathfrak{o}(x)) \vee f(\mathfrak{o}(z))$. As f also preserves complements, this implies $f(F) \wedge f(\mathfrak{c}(z)) \nleq f(\mathfrak{o}(x)) \vee f(\mathfrak{c}(y))$, and again as f preserves the lattice operations, $f(F \cap \mathfrak{c}(z)) \nleq f(\mathfrak{o}(x) \vee \mathfrak{c}(y))$ as desired. If $\mathsf{Sat}(\mathcal{D}) \subseteq \mathcal{D}$ is essential in **CoFrm**, consider the coframe quotient given by the subcolocale $\mathcal{S}(\mathsf{Sat}(\mathcal{D}))$. This is clearly injective when restricted to $\mathsf{Sat}(\mathcal{D})$, and by essentiality it is also injective on all of $\mathcal{D}$. But then it is a bijective coframe map, hence an isomorphism, so $\mathcal{S}(\mathsf{Sat}(\mathcal{D})) = \mathcal{D}$. □

We show some concrete examples of essential subcolocales.

Lemma 1.35. *Let L be a frame. Then $\mathfrak{b}(p) = \mathfrak{c}(p) \cap fit(\mathfrak{b}(p))$ for every $p \in \mathsf{pt}(L)$.*

Proof. We show $\mathfrak{c}(p) \cap fit(\mathfrak{b}(p)) \subseteq \mathfrak{b}(p)$. Suppose, then, that $x \in \mathfrak{c}(p) \cap fit(\mathfrak{b}(p))$ and $x \neq 1$. Since $x \in \mathfrak{c}(p)$, $p \leq x$. Since $x \in fit(\mathfrak{b}(p))$, whenever $p \in \mathfrak{o}(y)$ then $x \in \mathfrak{o}(y)$. As p is prime, $x \to p = p$ if and only if $x \nleq p$, and so our condition means $y \nleq p$ implies $y \to x = x$. As $x \neq 1$, $x \to x \neq x$, and so $x \leq p$. Then $x = p \in \mathfrak{b}(p)$, as desired. □

Proposition 1.36. *For a frame L, the following are essential subcolocales of $\mathsf{S}(L)$.*

1. *The subcolocale $\mathsf{S}_{sp}(L)$ of spatial sublocales;*
2. *The subcolocale $\mathsf{S}_b(L)$ of joins of complemented sublocales.*

Proof. For the first item, we recall that $\mathsf{S}_{sp}(L)$ is join-generated by $\{\mathfrak{b}(p) \mid p \in \mathsf{pt}(L)\}$. Then, it suffices to show that every $\mathfrak{b}(p)$ is $F \cap \mathfrak{c}(p)$ for some $F \in \mathsf{Sat}(\mathsf{S}_{sp}(L))$. We let $\nu_{sp} : \mathsf{S}(L) \to \mathsf{S}(L)$ be the conucleus associated with $\mathsf{S}_{sp}(L) \subseteq \mathsf{S}(L)$. By 1.18, and using Lemma 1.35,

$$\nu_{sp}(\mathit{fit}(\mathfrak{b}(p)) \cap \mathfrak{c}(p)) = \nu_{sp}(\mathit{fit}(\mathfrak{b}(p)) \cap \mathfrak{c}(p)) = \mathfrak{b}(p).$$

By definition of saturated element, $\nu_{sp}(\mathit{fit}(\mathfrak{b}(p))) \in \mathsf{Sat}(\mathsf{S}_{sp}(L))$, and so the desired claim holds. For the second item, we note that all elements of $\mathsf{S}_b(L)$ are joins of elements of the form $\mathfrak{o}(x) \cap \mathfrak{c}(y)$, and $\mathfrak{o}(x) \in \mathsf{Sat}(\mathsf{S}_b(L))$ for every $x \in L$. □

We now want to characterize essential sublocales as those codense subcolocales of the form $\Delta(\mathcal{F})$ for some admissible subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$.

Lemma 1.37. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense sublocale. Then, $\nu_{\mathcal{D}}(\mathit{fit}(D)) = \sigma(\mathit{fit}(D))$ for all $D \in \mathcal{D}$.*

Proof. Let $D \in \mathcal{D}$. We use the characterization in 1.26. We have to show

$$\mathit{fit}(\nu_{\mathcal{D}}(\mathit{fit}(D)) \cap \mathfrak{o}(x)) = \mathit{fit}(D) \wedge^{\mathcal{F}} \mathfrak{o}(x)$$

for all $x \in L$. By Lemma 1.18, the left-hand side is $\mathit{fit}(\nu_{\mathcal{D}}(\mathit{fit}(D) \cap \mathfrak{o}(x)))$. By item 3 of Proposition 1.20, this equals the right-hand side. □

Lemma 1.38. *Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a codense subcolocale.*

1. *The maps $\sigma : \mathit{fit}[\mathcal{D}] \leftrightarrows \mathcal{D} : \mathit{fit}$ are order adjoints, with $\mathit{fit} \dashv \sigma$;*
2. *The map $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ is a subcoframe embedding;*
3. *$\sigma[\mathcal{F}] = \mathsf{Sat}(\mathcal{D})$.*

Proof. Let $\mathcal{D} \subseteq \mathsf{S}(L)$ be a dense subcolocale.

1. For every $D \in \mathcal{D}$ and $F \in \mathit{fit}[\mathcal{D}]$, the following are equivalent.

$$\frac{\dfrac{\mathit{fit}(D) \subseteq F}{D \subseteq F}}{D \subseteq \nu_{\mathcal{D}}(F).}$$

   As $\nu_{\mathcal{D}}(F) = \sigma(F)$ for all $F \in \mathit{fit}[\mathcal{D}]$, by Lemma 1.37, the result follows.

2. As we have just shown, σ is a right adjoint, and so it preserves all meets. For binary joins, it suffices to show it preserves joins of the form $\mathfrak{o}(x) \vee \mathfrak{o}(y)$, but indeed $\sigma(\mathfrak{o}(x \vee y)) = \mathfrak{o}(x \vee y)$ by item 2 of Lemma 1.27. Injectivity of σ follows from its left adjoint being surjective.

3. By item 2, of 1.27, $\sigma(\mathfrak{o}(x)) = \mathfrak{o}(x)$ for all $x \in L$. Additionally, we have just shown that $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ is a coframe map. Hence, for all collections $x_i \in L$, $\sigma(\bigwedge_i^{\mathcal{F}} \mathfrak{o}(x_i))$ is $\nu_{\mathcal{D}}(\bigcap_i \mathfrak{o}(x_i))$, and so it is saturated. Since this holds for all families $x_i \in L$, all saturated elements of $\mathcal{D}$ are of this form. □

Lemma 1.39. *For every codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$, and every admissible subcolocale $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$, $\Delta(\mathcal{F}) \subseteq \mathcal{D}$ if and only if $\mathcal{F} \subseteq \mathit{fit}[\mathcal{D}]$.*

Proof. If $\Delta(\mathcal{F}) \subseteq \mathcal{D}$, then $\mathit{fit}[\Delta(\mathcal{F})] \subseteq \mathit{fit}[\mathcal{D}]$, but the left-hand side is $\mathcal{F}$, by Proposition 1.30. Suppose that $\mathcal{F} \subseteq \mathit{fit}[\mathcal{D}]$. By definition of $\Delta(\mathcal{F})$, it suffices to show $\sigma(F) \in \mathcal{D}$, for all $F \in \mathcal{F}$. By our assumption, every such F is $\mathit{fit}(D)$ for some $D \in \mathit{fit}(D)$, and, as $\sigma(\mathit{fit}(D)) = \nu_{\mathcal{D}}(\mathit{fit}(D))$ by Lemma 1.37, this is indeed in $\mathcal{D}$. □

Proposition 1.40. *Let L be a frame. A codense subcolocale $\mathcal{D} \subseteq \mathsf{S}(L)$ is essential if and only if $\mathcal{D} \subseteq \Delta(\mathit{fit}[\mathcal{D}])$.*

Proof. By definition, $\Delta(\mathit{fit}[\mathcal{D}]) = \mathcal{S}(\sigma[\mathit{fit}[\mathcal{D}]])$. By Lemma 1.38, $\sigma[\mathit{fit}[\mathcal{D}]] = \mathsf{Sat}(\mathcal{D})$. Then, it suffices to show that $\mathcal{D}$ is essential if and only if $\mathcal{D} \subseteq \mathcal{S}(\mathsf{Sat}(\mathcal{D}))$, but this just the definition of essentiality. □

We have obtained the main theorem. We call $\mathcal{CD}_{ess}(\mathsf{S}(L))$ the collection of essential codense subcolocales, and $\mathcal{PC}(\mathsf{S}_{\mathfrak{o}}(L))$ the collection of admissible subcolocales of $\mathsf{S}_{\mathfrak{o}}(L)$.

Theorem 1.41. *There is an order adjunction $\Delta : \mathcal{PC}(\mathsf{S}_{\mathfrak{o}}(L)) \leftrightarrows \mathcal{CD}(\mathsf{S}(L)) : \mathit{fit}[-]$ with $\Delta \dashv \mathit{fit}[-]$, and which maximally restricts to an isomorphism $\mathcal{PC}(\mathsf{S}_{\mathfrak{o}}(L)) \cong \mathcal{CD}_{ess}(\mathsf{S}(L))$.*

Proof. This follows from Lemmas 1.39 and 1.30. □

We want to use the characterization in 1.40 to provide an example of a subcolocale of $\mathsf{S}(L)$ which is not essential.

Example 1.42. As $\mathsf{S}_b(L)$ is essential, by Proposition 1.36, $\Delta(\mathit{fit}[\mathsf{S}_b(L)]) = \mathsf{S}_b(L)$ by Proposition 1.40. By Lemma 1.21, $\mathit{fit}[\mathsf{S}_b(L)] = \mathit{fit}[\mathsf{S}_{\mathcal{E}}(L)]$. Thus, if for some frame we had $\mathsf{S}_{\mathcal{E}}(L) \nsubseteq \mathsf{S}_b(L)$, this would imply $\mathsf{S}_{\mathcal{E}}(L) \nsubseteq \Delta(\mathit{fit}[\mathsf{S}_{\mathcal{E}}(L)])$, giving the desired counterexample as $\mathsf{S}_{\mathcal{E}}(L)$ would not be essential by Proposition 1.40. For sublocales witnessing $\mathsf{S}_{\mathcal{E}}(L) \nsubseteq \mathsf{S}_b(L)$, once again we refer to Example 5.12 in [9] and Example 1.22.

2. The categories Raney and SZDBF

2.1 Objects

We say that a strictly zero-dimensional biframe $(L, \mathcal{D})$ is *essential* if $\mathcal{D} \subseteq \mathsf{S}(L)$ is an essential subcolocale. We say that a Raney extension $(L, \mathcal{F})$ is *admissible* if $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is an admissible subcolocale. We will use the results of the previous section to establish a bijection between admissible Raney extensions and essential strictly zero-dimensional biframes. For an admissible Raney extension $(L, \mathcal{F})$ and for a strictly zero-dimensional biframe $(M, \mathcal{D})$ we define:

$$\Delta(L, \mathcal{F}) = (L, \Delta(\mathcal{F})), \qquad \mathit{fit}(M, \mathcal{D}) = (M, \mathit{fit}[\mathcal{D}]).$$

Theorem 1.41, then, amounts to the following.

Theorem 2.1. *The assignments fit and Δ are mutually inverse bijections between admissible Raney extensions and essential strictly zero-dimensional biframes.*

2.2 Morphisms

The assignment fit can be easily extended to morphisms. By Lemma 1.38, there is a subcoframe inclusion $\sigma : \mathit{fit}[\mathcal{D}] \to \mathcal{D}$ for every codense subcolocale $\mathcal{D}$. Then, every morphism $f : (L, \mathcal{D}) \to (M, \mathcal{E})$ determines a coframe map $\mathit{fit}(f) : \mathit{fit}[\mathcal{D}] \to \mathit{fit}[\mathcal{E}]$, which further restricts to the open sublocales to yield a frame map isomorphic to $f : L \to M$.

Proposition 2.2. *There is a functor* $\mathit{fit} : \mathbf{SZDBF} \to \mathbf{Raney}$*, whose essential image consists of the admissible Raney extensions.*

On the other hand, the assignment $(L, \mathcal{F}) \mapsto \Delta(L, \mathcal{F})$ cannot be extended to morphisms in a similar fashion. We will show that there are Raney morphisms

$$f : (L, \mathcal{F}) \to (M, \mathcal{G})$$

such that the frame map $f : L \to M$ does not lift to a map

$$f : (L, \Delta(\mathcal{F})) \to (M, \Delta(\mathcal{G})).$$

Lemma 2.3. $\Delta(L, \mathit{fit}[\mathsf{S}_b(L)]) = (L, \mathsf{S}_b(L))$ *for all frames* L.

Proof. The strictly zero-dimensional biframe $(L, \mathsf{S}_b(L))$ is essential, by Proposition 1.36. By Proposition 1.40, then, $\Delta(\mathit{fit}[\mathsf{S}_b(L)]) = \mathsf{S}_b(L)$. □

Proposition 2.4. *For a frame* L*, and a sublocale* $S \subseteq L$*:*

1. S *is smooth if and only if it lifts to a map* $f : (L, \mathsf{S}_b(L)) \to (S, \mathsf{S}_b(S))$ *of strictly zero-dimensional biframes.*
2. S *is exact if and only if it lifts to a map* $f : (L, \mathit{fit}[\mathsf{S}_b(L)]) \to (S, \mathit{fit}[\mathsf{S}_b(L)])$ *of Raney extensions.*

Proof. We prove the two items in turn.

1. This follows immediately from both Lemma 3.39 in [13] and Corollary 4.2 of [1].
2. In [11] it is proven that there is an isomorphism $\mathit{fit}[\mathsf{S}_b(L)] \cong \mathsf{S}_\mathfrak{c}(L)^{op}$. This is also an isomorphism $(L, \mathit{fit}[\mathsf{S}_b(L)]) \cong (L, \mathsf{S}_\mathfrak{c}(L)^{op})$ of Raney extensions. In Proposition 6.6 of [22] the frame morphisms $f : L \to M$ that lift to Raney extensions are characterized as the exact maps. Finally, in Proposition 7.14 of [22] it is shown that surjections that preserve all exact meets are exact maps. □

We are ready to give the desired counterexample.

Example 2.5. Suppose there is a frame L with a sublocale $S \subseteq L$ which is exact but not smooth. Proposition 2.4 and Lemma 2.3 imply that the corresponding frame surjection $s : L \to S$ lifts to a Raney morphism $f : (L, \mathit{fit}[\mathsf{S}_b(L)]) \to (S, \mathit{fit}[\mathsf{S}_b(S)])$ which does not in turn lift to a morphism $f : \Delta(L, \mathit{fit}[\mathsf{S}_b(L)]) \to \Delta(S, \mathit{fit}[\mathsf{S}_b(S)])$ in $\mathbf{SZDBF}$. Therefore, once again a counterexample is provided by both 5.12 of [9] and Example 1.22.

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Anna Laura Suarez
University of the Western Cape
Private Bag X17 Bellville 7535
Cape Town (South Africa)
annalaurasuarez993@gmail.com