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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXVII-2 (2026)

IN MEMORY OF MAREK ZAWADOWSKI (1960–2024)

This special volume of Cahiers de Topologie et Géométrie Différentielle Catégoriques is dedicated to the memory of Marek Zawadowski (1960–2024), whose untimely death deprived our community of a mathematician of rare depth and originality. Zawadowski's work ranged widely across categorical logic, higher category theory, and the semantics of natural language, and it was marked throughout by a characteristic combination of technical mastery and conceptual clarity. This volume gathers two contributions that together reflect both his mathematics and the esteem in which he was held by those who knew and worked with him.

The volume opens with a scientific biography of Zawadowski, tracing his training, his research programs, and his lasting influence on the fields he helped shape. It was prepared collaboratively by a group of his colleagues, collaborators, and former students: Pierre-Louis Curien, Silvio Ghilardi, Martin Hyland, Chris Kapulkin, Jaap van Oosten, and Karol Szumiło. The authors are also glad to acknowledge Justyna Grudzińska, Zawadowski's collaborator on the semantics of natural language, whose recollections and insight informed the biography.

The volume continues with Zawadowski's paper "On positive opetopes, positive opetopic cardinals and positive opetopic sets," published here posthumously. The manuscript was prepared for publication by Pierre-Louis Curien, who undertook its revision with great care. The paper is a fitting representative of Zawadowski's mathematical voice, and we are grateful to be able to bring it before the community in this form.

It is our hope that this volume will serve both as a tribute to Marek Zawadowski and as a lasting record of work that he would have been glad to see in print.

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXVII-2 (2026)

MAREK ZAWADOWSKI: A SCIENTIFIC BIOGRAPHY

Pierre–Louis CURIEN *Silvio GHILARDI*
Martin HYLAND *Krzysztof KAPULKIN*
Jaap van OOSTEN *Karol SZUMIŁO*

Résumé. Marek Zawadowski (1960–2024) était un mathématicien polonais travaillant dans les domaines de la théorie des catégories, de la logique et des fondements, et de la théorie de l'homotopie. Nous esquissons ses nombreuses contributions à ces domaines.

Abstract. Marek Zawadowski (1960–2024) was a Polish mathematician working in the areas of category theory, logic and foundations, and homotopy theory. We outline his manifold contributions to these areas.

Keywords. Marek Zawadowski, category theory, categorical logic, descent, duality, pretopos, model completion, Heyting algebra, sheaf representation, regular theory, analytic monad, operad, opetope, higher-dimensional category, ordered face structure.

Mathematics Subject Classification (2020). 01-02, 18C10, 18N65.

Contents

1. Introduction

— *Krzysztof Kapulkin and Karol Szumiło*

Marek Witold Zawadowski (1960–2024) was a Polish mathematician, specializing in category theory, logic, homotopy theory, and linguistics. Born in Montréal, Que. in 1960, Zawadowski grew up in Warsaw, Poland, where he attended Stanisław Staszic High School, a prestigious and highly selective school specializing in mathematics and computer science, where the mathematics courses were taught by faculty members from the nearby University of Warsaw. Upon graduation in 1978, he enrolled at the University of Warsaw to study mathematics. He graduated in 1983 with a master's degree, having already published his first paper, on the Löwenheim–Skolem theorem in toposes [58]. He joined the faculty of the University of Warsaw on October 1, 1983, and in 1986 returned to Canada for doctoral studies, receiving a Ph.D. in 1989 from the Université de Montréal for the dissertation *Un théorème de descente pour les prétopos* [60], written under the supervision of Gonzalo E. Reyes. Following a postdoctoral appointment in Montréal, he was a faculty member at the University of Warsaw from 1991 onward, where he received his habilitation in 2004 on the basis of the book, written with Silvio Ghilardi, *Sheaves, Games, and Model Completions: A Categorical Approach to Non-Classical Propositional Logics* [21].

Throughout his career, Zawadowski made deep contributions to several areas, both within and outside of mathematics, including logic and foundations, category theory, abstract homotopy theory, type theory, and linguistics. He was an active and valued member of many communities, a supervisor and instructor universally admired by his students, and, above all, a profoundly insightful researcher. This document focuses on the last of these, but let us briefly address the first two.

On December 13, 1981, Zawadowski's undergraduate years coincided with a national crisis when the Polish puppet government, installed by the Soviet Union, declared martial law. In response, the Independent Students' Association had organized a strike in 1981 that included occupying classrooms to demand academic freedom, the removal of mandatory Marxist and Russian-language courses, and the right to free association. This action helped pave the way for the broader democratic movement that eventually led to the fall of the Iron Curtain in Europe. Zawadowski was a central figure in the process, leading the strike activities in the Department of Mathematics. The University of Warsaw's main lecture hall, the Auditorium Maximum, became a scene of this strike, with students sleeping on its floor for four consecutive weeks.

The ancient wisdom of Marcus Aurelius and Epictetus teaches us that during times of external chaos, such as martial law, one should continue to fulfill one's duties. In Albert Camus's *The Plague*, Dr. Rieux insists on "living as normally as possible" as a catastrophe unfolds around him. This lesson stayed with many of Zawadowski's fellow countrymen, including his mentor, the Polish logician Helena Rasiowa, who lived through the Second World War and the difficult post-war years, rebuilding the institutions of the Polish school of mathematics in the 1950s. Zawadowski fits squarely into this tradition: as students occupied the Auditorium Maximum, he initiated numerous seminars and lecture series to ensure that students were still doing what students should be doing. Many of his then-fellow students still remember these lectures and reminisce about learning topos theory from him.

Zawadowski served as the supervisor of numerous master's and doctoral students. His doctoral student, Stanisław Szawiel, received his Ph.D. in 2015 with the dissertation *A Unified Approach to Opetopic Algebra* [53].

His recent M.Sc. students include Jacek Karwowski (M.Sc. thesis *Formal semantics of a reversible language: symmetric groups in Homotopy Type Theory*, and B.Sc. thesis *Products in positive opetopic sets*), Mateusz Zugaj (M.Sc. thesis *Algebraic presentation of sheaves on compact spaces*), Krzysztof Galias (M.Sc. thesis *Kleisli completion of the free adjunction*), and Łukasz Sienkiewicz (M.Sc. thesis *Formal Theory of Monoidal Objects*). The story of Karwowski is of particular interest. He first heard of category theory in his freshman linear algebra class, and found the subject esoteric, if not nonsensical — an opinion he then communicated to Zawadowski, while asking him to supervise his bachelor's thesis. To the surprise of many, Zawadowski agreed, and devised a project well suited to Karwowski's mathematical interests and programming skills. Today, Karwowski is a doctoral student at Oxford University, and he credits his acceptance into the program to Zawadowski's thoughtful supervision and guidance.

Scientifically, Zawadowski's career divides naturally into several periods. Early on, he worked on categorical logic and topos theory. He is best known for his thesis work on *descent and duality*, subsequently published in the *Annals of Pure and Applied Logic* [61]. It grew out of Makkai's duality, which asserts a duality between the category of models of a first-order theory and its syntactic category, a pretopos. Zawadowski extended this duality to the broader setting of pseudoelementary categories. A second striking result is the characterization of the effective descent morphisms of pretoposes and Barr-exact categories as precisely the conservative functors. This period of Zawadowski's career is surveyed in Section 2 by Jaap van Oosten, a fellow categorical logician and a long-time friend of Zawadowski.

Zawadowski's work on duality connects to his subsequent collaboration with Silvio Ghilardi, which began in the early 1990s and culminated in the book *Sheaves, Games, and Model Completions: A Categorical Approach to Non-Classical Propositional Logics* [21], published in 2002. As the title aptly indicates, the collaboration focused on non-classical propositional logics — chiefly modal and intuitionistic systems — by analyzing their categories of (finitely generated) models. It was an interplay between an algebraic question, the existence of model completions, and a proof-theoretic one, concerning the interpretability of second-order propositional logic in ordinary propositional logic. The techniques used to establish the results deserve mention as well: they include Zawadowski's "bread and butter," dualities and sheaf representations, as well as Ehrenfeucht–Fraïssé games and bounded bisimulations. Silvio Ghilardi surveys these developments in Section 3.

In the early 2000s, Zawadowski's interests shifted, following a shift in the interests of his mentor, Michael Makkai. Higher-dimensional category theory had emerged as a central theme in the field, and Zawadowski became an active contributor, focusing in particular on the opetopic approach pioneered by John Baez and James Dolan [3] and developed extensively by Claudio Hermida, Michael Makkai, and John Power [26, 27, 28]. Zawadowski's main contribution was an axiomatic, combinatorial description of opetopes through his framework of *ordered face structures*, contained primarily in [63, 64, 69]. He also pioneered the development of connections between the opetopic approaches to higher category theory and the standard methods of homotopical algebra. These seemingly disparate threads came together in another research program initiated by Zawadowski, which recast his early interests in logic — and specifically in algebraic theories — in operadic and higher-categorical language. Much of this work, though not all, was carried out in collaboration with his Ph.D. student Stanisław Szawiel [52, 54, 55]. It is surveyed by Martin Hyland in Section 4, while Pierre-Louis Curien's overview of Zawadowski's work on the opetopic approach to higher-dimensional category theory is found in Section 5.

In the latter part of the 2010s, Zawadowski's interests shifted once again, as he turned to linguistics. This final strand of his work lies outside the scope of the present article, which concentrates on his mathematical contributions; we mention it here only to convey the full breadth of his curiosity. Indeed, no single document could do justice to the entirety of Zawadowski's work, and we have not attempted to be exhaustive. The contributions that follow are deliberately brief, each offering a high-level overview of one facet of his mathematics, written by a colleague who knew that part of his work intimately. Together, we hope, they convey something of both the depth and the range of what Marek Zawadowski achieved.

2. Categorical logic

— Jaap van Oosten

In this section I would like to discuss to some extent two research themes in Categorical Logic that Marek Zawadowski pursued: the Löwenheim-Skolem theorems in toposes, and his work in the tradition of Makkai and Reyes.

2.1 Löwenheim-Skolem

Already at a very young age, Zawadowski had absorbed the basics of categorical logic, as laid out in Makkai and Reyes' monograph *First Order Categorical Logic* ([40]). His first papers on the subject were probably written in 1982 (they were sent to the journal in February 1983), when he was about 22 years old. The following is a much simplified rendering of his work in this direction.

Expectations were high, in the early 1980's, that the categorical point of view might offer a fruitful perspective for classical Model Theory. But the first thing at hand was to establish facts in arbitrary toposes (Grothendieck or not) which were analogous to the model-theoretic results for the category Set.

Zawadowski set out to develop the Löwenheim-Skolem (curiously, for him Skolem-Löwenheim) theorems in arbitrary Grothendieck toposes. These theorems contain cardinalities as an essential ingredient and since one of these appears as the cardinality of something "external" (the *language*), one thinks of a theory of external *powers* and this is what Zawadowski developed in the papers [58] and [59].

A class G of objects of a topos $\mathcal{E}$ is a *class of* Σ*-generators* if every object X of $\mathcal{E}$ can be written as the union of subobjects $A_i \to X$ with A_i from G. Clearly, every class of Σ-generators is a class of generators; conversely, a class of generators is a class of Σ-generators if it is closed under epimorphic images. A class of Σ-generators is a *class of power generators* if it is closed under subobjects. An important example is the class of subobjects of the terminal object 1; this is a class of generators (equivalently, Σ-generators; note that this class is closed under epimorphic images) if and only if the topos $\mathcal{E}$ is localic over Set ([30], 5.37).

Given a class G of power generators, we define for every object X the *power of* X *relative to* G, $\bar{X}^G$, as the least cardinality $|I|$ of a family $\{A_i \to X \mid i \in I\}$ such that $X = \bigcup_{i\in I} A_i$ and all A_i are from G. Clearly, taking for G the standard class of power generators in Set (the class containing just the terminal object) one gets $\bar{X}_G = |X|$, the usual cardinality of a set X.

So, the power $\bar{X}^G$ depends on G, but in many cases there is a canonical chojoe: the least class G of Σ-generators with the property that $\bar{1}^G = 1$. If the topos $\mathcal{E}$ is sheaves on a complete Heyting algebra then this is simply $\mathrm{Sub}(1)$, the class of subobjects of the terminal object; in the case of a presheaf category $\widehat{M} = \mathrm{Set}^{M^{\mathrm{op}}}$, one takes the closure of the set of representable presheaves under subobjects and epimorphic images; this yields the least class of power generators G. Moreover, $\bar{1}^G = 1$, if and only if the category M has a weak terminal object (an object m such that for every object n there is an arrow $n \to m$).

Now the Löwenheim-Skolem theorem for toposes is formulated by Zawadowski using the framework of *generalized quantifiers*, which I find a little too technical and notation-heavy to review here. Let me restrict myself to the following: for a (one-sorted) language L of predicate logic there is a standard notion of an *interpretation* in a topos: given an underlying universe A (object of $\mathcal{E}$) and interpretations of the non-logical symbols of L, one has an L*-structure* A and for every L-formula $\varphi(x_1,\ldots,x_n)$ a subobject $[\varphi]^A$ of the product A^n. A *substructure* B of A is a subobject B of A which interprets the non-logical symbols of L as the restrictions of those of A, to B; the substructure B is *elementary* if $[\varphi]^B = (\iota^n)^{-1}([\varphi]^A)$ for every formula $\varphi(x_1,\ldots,x_n)$, where $\iota : B \to A$ is the embedding. In other words, if

$$\begin{tikzcd}
{[\varphi]^B} \arrow[r] \arrow[d] & {[\varphi]^A} \arrow[d] \\
B^n \arrow[r] & A^n
\end{tikzcd}$$

is always a pullback.

We have then the following version of the Löwenheim-Skolem theorem:

Theorem 2.1 (Zawadowski). *Let $\mathcal{E}$ be a Grothendieck topos with canonical class G of power generators. Suppose $\omega \le \kappa \le \lambda$ are cardinal numbers, L a language of cardinality $\le \kappa$, A an L-structure in $\mathcal{E}$ with $\bar{A}^G = \lambda$. If C is a* *subobject of A with $\bar{C}^G < \kappa$, then there is an elementary substructure B of A such that C is a subobject of B and $\bar{B}^G \leq \kappa$.*

2.2 Pretoposes

This concerns Marek Zawadowski's thesis work ([60, 61]). It touches on a number of research lines in categorical logic: Makkai and Reyes's *Conceptual Completeness Theorem* for pretoposes ([40]), Pitts' alternative proof using interpolation ([47]), Pitts' generalization of conceptual completeness to Heyting categories and the method of *descent* ([48]). A very clear outline of Conceptual Completeness is in Zawadowski's [62].

Fix a first-order multi-sorted language L. The (finitary) *coherent fragment* of first-order logic over L is the collection of L-formulas built from $\bot, \top, \wedge, \vee, \exists$; a *coherent L-theory* is a collection of *sequents* $\phi \Rightarrow_{\vec{x}} \psi$ where ϕ and ψ belong to the coherent fragment, and $\vec{x}$ is a sequence $x_1, \ldots, x_n$ of variables which contains all variables which occur free in ϕ or ψ.

A *coherent category* (Makkai and Reyes speak of 'logical category') is a regular category which has stable joins of subobjects. In such a category $\mathcal{R}$, an *L-structure* M consists of: for every sort S of the language L, an object $M(S)$; for every function symbol $f : S_1, \ldots, S_n \to T$ a morphism $M(f) : M(S_1) \times \cdots \times M(S_n) \to M(T)$; and for every relation symbol $R : S_1, \ldots, S_n$ a subobject $M(R)$ of the product $M(S_1) \times \cdots \times M(S_n)$. Given such a structure, for every coherent L-formula ϕ and every list of variables $\vec{x}$ containing all free variables of ϕ, a standard inductive construction gives $[\phi]^M_{\vec{x}}$ as subobject of $M(S_1) \times \cdots \times M(S_n)$ (where S_i is the sort of the variable x_i). We now say that the coherent sequent $\phi \Rightarrow_{\vec{x}} \psi$ is *true* in the structure M if $[\phi]^M_{\vec{x}} \leq [\psi]^M_{\vec{x}}$ as subobjects of $M(S_1) \times \cdots \times M(S_n)$. The structure M is a *model* of the L-theory T if every sequent of T is true in M.

Now if M and N are two L-structures in the coherent category $\mathcal{R}$, a *homomorphism* from M to N consists of a morphism $\alpha_S : M(S) \to N(S)$ for each sort S of L, which system has to satisfy the following two conditions:

for each function symbol $f : S_1, \ldots, S_n \to T$ of the language, the diagram

$$\begin{tikzcd}
M(S_1) \times \cdots \times M(S_n) \arrow[r, "M(f)"] \arrow[d, "\alpha_{S_1} \times \cdots \times \alpha_{S_n}"'] & M(T) \arrow[d, "\alpha_T"] \\
N(S_1) \times \cdots \times N(S_n) \arrow[r, "N(f)"'] & N(T)
\end{tikzcd}$$

should commute; and for each relation symbol $R : S_1, \ldots, S_n$ the composition $M(R) \to M(S_1) \times \cdots \times M(S)_n \overset{\alpha_{S_1} \times \cdots \times \alpha_{S_n}}{\longrightarrow} N(S_1) \times \cdots \times N(S_n)$ factors through the subobject $N(R)$ of $N(S_1) \times \cdots \times N(S_n)$.

We make the following remark on the definition of a homomorphism: it follows by a straighforward induction that for such a homomorphism α and a coherent L-formula ϕ with list of free variables $\vec{x}$, that the composition $[\phi]^M_{\vec{x}} \to M(S_1) \times \cdots \times M(S)_n \overset{\alpha_{S_1} \times \cdots \times \alpha_{S_n}}{\longrightarrow} N(S_1) \times \cdots \times N(S_n)$ factors through $[\phi]^N_{\vec{x}}$. So, a homomorphism looks like a natural transformation (and we shall shortly see that it actually is one).

Now we have two interesting constructions, from the point of view of categorical logic: the *canonical language* and the *syntactic category*. Let $\mathcal{R}$ be a coherent category. The canonical language $L_{\mathcal{R}}$ has a sort A for every object A of $\mathcal{R}$, and a function symbol $f : A \to B$ for every arrow $f : A \to B$ in $\mathcal{R}$. We have a straightforward $L_{\mathcal{R}}$-structure $M_{\mathcal{R}}$ in $\mathcal{R}$ which "is the identity", and we can discuss which $L_{\mathcal{R}}$-sequents are true in $M_{\mathcal{R}}$. For example:

a) A diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "h"'] & B \arrow[d, "f"] \\
& C
\end{tikzcd}$$

commutes if and only if the sequent $\top \Rightarrow_x f(g(x)) = h(x)$ is true in $M_{\mathcal{R}}$.

b) The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

is a product diagram if and only if ther sequents

$$\begin{gathered}\top \Rightarrow_{y,z} \exists x(f(x) = y \wedge g(x) = z) \\ f(x) = f(x') \wedge g(x) = g(x') \Rightarrow_{x,x'} x = x'\end{gathered}$$

are both true in $M_{\mathcal{R}}$.

c) The diagram

$$\begin{tikzcd} A \arrow[r, "e"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

is an equalizer if and only if the sequents

$$\begin{gathered}g(y) = h(y) \Rightarrow_{y} \exists x(e(x) = y) \\ \top \Rightarrow_{x} g(e(x)) = h(e(x)) \\ e(x) = e(x') \Rightarrow_{x,x'} x = x'\end{gathered}$$

are all true in $M_{\mathcal{R}}$.

d) Finally, the arrow $f : A \to B$ is a regular epi in $\mathcal{R}$ iff the sequent $\top \Rightarrow \exists x(f(x) = y$ is true in $M_{\mathcal{R}}$.

We see that the whole coherent structure of $\mathcal{R}$ is described by the coherent logic over $L_{\mathcal{R}}$. Makkai and Reyes call this the "first main fact".

The second construction, the *syntactic category* $\mathcal{C}_T$ or, in Zawadowski's words, the *Lindenbaum-Tarski category* of a coherent theory T, is as follows: objects are coherent formulas ϕ up to renaming of free variables. Morphisms from $\phi(\vec{x})$ to $\psi(\vec{y})$ (which by our convention we may assume to have disjoint sets of free variables) are coherent formulas $\theta(\vec{x}, \vec{y})$ which are, T-provably, functional relations: that is, the sequents

$$\begin{gathered}\theta(\vec{x}, \vec{y}) \Rightarrow_{\vec{x},\vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}) \\ \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{u}) \Rightarrow_{\vec{x},\vec{y},\vec{u}} \vec{y} = \vec{u} \\ \phi(\vec{x}) \Rightarrow_{\vec{x}} \exists \vec{y} \theta(\vec{x}, \vec{y})\end{gathered}$$

are true in every model of T.

The category $\mathcal{C}_T$ is coherent. We can now see that a model of T in a coherent category $\mathcal{R}$ is 'nothing but' a coherent funrtor $\mathcal{C}_T \to \mathcal{R}$ and that, modulo this equivalence, a homomorphism between models is 'nothing but' a natural transformation between the corresponding functors.

Finally, we consider *interpretations* between coherent theories. If T_1 is an L_1-theory and T_2 an L_2-theory then an *interpretation* $I : T_1 \to T_2$ sends every sort S of L_1 to a sort $I(S)$ of L_2, every function symbol $f : S_1, \ldots, S_n \to T$ of L_1 to a function symbol $I(f) : I(S_1) \ldots I(S_n) \to I(T)$ and each relation symbol $R : S_1, \ldots, S_n$ to a coherent formula $I(R)$ with free variables of sorts $I(S_1), \ldots, I(S_n)$. This gives a translation I from coherent L_1-formulas to L_2-formulas; and we require that for every axiom of T_1, its I-image is true in every model of T_2. Note that I is also a coherent functor from $\mathcal{C}_{T_1}$ to $\mathcal{C}_{T_2}$. Let $\mathrm{Mod}(T_1)$ be the category of models of T_1, that is: the category of coherent functors $\mathcal{C}_{T_1} \to \mathrm{Set}$ and natural transformations/homomorphisms, and likewise for T_2. Composition with I gives a functor $I^* : \mathrm{Mod}(T_2) \to \mathrm{Mod}(T_1)$. Now suppose that the functor I^* is an equivalence of categories. What can we say about I? Here we have to take into account that there is categorical structure that is recognized by coherent logic, i.e. preserved by every coherent functor, yet not always present in every coherent category. This structure is: *finite coproducts and quotients of equivalence relations*. A coherent category with stable finite coproducts and quotients of equivalence relations is called a *pretopos*. The *Conceptual Completeness Theorem for Pretoposes* (Makkai and Reyes) reads: Let $I : T_1 \to T_2$ be an interpretation between coherent theories. If I^* is an equivalence and $\mathcal{C}_{T_1}$ is a pretopos, then I is an equivalence of categories.

In this theorem, the category Set plays a special role, as category of the models of theories. The completeness for Set-valued models is not quite constructive; it is equivalent to Gödel's Completeness Theorem for first-order logic. The constructivisation was formulated by A.M. Pitts in [47] and consists in taking models in larger families of pretoposes into account. In this version, the special role of Set disappears.

The paper [61] is a quite monumental piece of work. I have, above, dealt in some detail with one of the ingredients of the theory Zawadowski sets up: the Makkai-Reyes Conceptual Completeness Theorem. Two other ingredients are: Makkai's theory of *Ultracategories* and a notion of *descent*. Both are pieces of mathematics that cannot be explained in a few paragraphs. Let me restrict myself to formulate one application (Theorem 11.4). Every pretopos morphism $J : A \to B$ factors through a morphism $J_d : A \to \mathrm{Des}(J)$, the category of descent data. This category turns out to be equivalent to a category of *invariant* subobjects, yielding what the author claims a *global* *invariance theorem.*

The paper [4] claims to offer “a short argument for the descent theorems of Zawadowski ([61] (originally [60])) and Makkai ([41]), which were conjectured by Pitts after the descent theorem of Joyal and Tierney ([31]) for open geometric morphisms of (Grothendieck) toposes”. It uses forcing. Due to (lack of) my university’s permissions I was not able to consult the paper.

3. Model Completeness

— Silvio Ghilardi

In this section, we report the content of the cooperation between Marek Zawadowski and the second co-author: the cooperation started in Montreal 1992 and ended few years later in the publication of a book [21], collecting and extending all results from previous papers [17, 18, 19, 20].

The starting point of this work was a surprising theorem proved by A. Pitts in [49]. The theorem originated from the attempt of solving a well-known and still open question (“is any Heyting algebra the Heyting algebra of global sections of the subobject classifier Ω in an elementary topos?”), but it has an important independent interest. It can be stated as follows (below we use notations like $\theta(\underline{v})$ to say that the formula θ contains at most the variables $\underline{v}$):

Theorem [49, Pitts]. *For each propositional variable x and for each formula $\varphi(x, \underline{y})$ of intuitionistic propositional calculus (IpC), there exist formulas $[\exists^x \varphi]$ and $[\forall^x \varphi]$ (effectively computable from φ and containing at most the variables $\underline{y}$) such that for any formula $\psi(\underline{y}, \underline{z})$, we have*

$$\vdash_{IpC} [\exists^x\varphi] \to \psi \qquad \textit{iff} \qquad \vdash_{IpC} \varphi \to \psi$$

$$\vdash_{IpC} \psi \to [\forall^x\varphi] \qquad \textit{iff} \qquad \vdash_{IpC} \psi \to \varphi.$$

From the proof-theoretic point of view the theorem can be reformulated by saying that there exists a (non conservative) interpretation of second order intuitionistic calculus IpC^2 into IpC or, again, by saying that Craig’s interpolation theorem holds in a very strong (‘uniform’) way for IpC. The starting problem in the above mentioned Marek Zawadowski’s joint work were the following two interrelated problems:

(a) to reinterpret the above theorem in a different mathematical (categorical or model-theoretic) context;

(b) to supply a semantic proof of it.

Answering such questions does not only supply a better understanding of the theorem itself, but paves also the way of investigating extensions to other logical contexts.

The model theoretic interpretation of Pitts' Theorem relies on the following algebraic observations (taken from [21]). Below, we make a systematic identification of formulas of IpC and terms of the first-order theory of Heyting algebras. For such a formula/term $\varphi(x, \underline{y})$, and a matching tuple of elements $\underline{a}$ from a Heyting algebra H, we have that $H \models [\exists^x \varphi](\underline{a}) = 1$ iff H embeds into the Heyting algebra of polynomials $H[\mathrm{x}]$ divided by the congruence generated by the condition $\varphi(\underline{a}, \mathrm{x}) = 1$. Moreover $H \models [\forall^x \varphi](\underline{a}) = 1$ iff $H[\mathrm{x}] \models \varphi(\underline{a}, \mathrm{x}) = 1$. These observations shows that the first order theory of Heyting algebras *admits a model completion*. In fact, it turns out that a system of equations and inequations with parameters $\underline{a}$ from a Heyting algebra H

$$\exists x \begin{pmatrix} \varphi_1(\underline{a}, x) = 1 \ \& \ \cdots \ \& \ \varphi_n(\underline{a}, x) = 1 \ \& \\ \& \ \psi_1(\underline{a}, x) \neq 1 \ \& \ \cdots \ \& \ \psi_m(\underline{a}, x) \neq 1 \end{pmatrix}$$

is solvable in an extension of H iff the quantifier-free formula

$$[\exists^x \bigwedge_{i=1}^{n} \varphi_i](\underline{a}) = 1 \quad \& \ [\forall^x (\textstyle\bigwedge_{i=1}^{n} \varphi_i \to \psi_1)](\underline{a}) \neq 1 \ \& \ \cdots$$
$$\cdots \ \& \ [\forall^x (\textstyle\bigwedge_{i=1}^{n} \varphi_i \to \psi_m)](\underline{a}) \neq 1$$

is true in H; here the formulae/terms

$$[\exists^x \bigwedge_{i=1}^{n} \varphi_i], \ [\forall^x (\bigwedge_{i=1}^{n} \varphi_i \to \psi_1)], \ldots, [\forall^x (\bigwedge_{i=1}^{n} \varphi_i \to \psi_m)]$$

are computed according to Pitts' theorem. This proves that the class of existentially closed Heyting algebras is an elementary class and so *Pitts' theorem implies that the equational theory of Heyting algebras has a model completion.* The interesting fact is that a converse result also holds in an appropriate universal algebraic context.

In order to explain such converse result, Pitts' Theorem needs to be formulated in category-theoretic terms. In this equivalent formulation, the theorem says that *the opposite of the category of finitely presented Heyting algebras is a Heyting category*, where a Heyting category [40] is a coherent category where taking pullbacks on subobjects has not only a left but also a right adjoint. Using this formulation, it is shown in [21] that the first order theory of a variety $\mathbf{V}$ with equationally definable principal congruences and equationally definable principal meets *admits a model completion if and only if* $\mathbf{V}_{fp}^{op}$ (the opposite of the category of finitely presented $\mathbf{V}$-algebras) *is an r-Heyting category* (the notion of an r-Heyting category is a modification of the notion of a Heyting category taking care of the fact that epis might not be regular - a phenomenon that nevertheless does not occur in the logic-algebraic varieties investigated in the book [21], because there Beth definability property holds for the corresponding propositional logics).

Having reformulated Pitts's theorem both in model-theoretic and in category-theoretic terms, it remains the question on how to get a semantic proof of it. This problem was solved in [17] and independently also in [57]. Both [17] and [57] apply combinatorial arguments inspired from previous work on Gödel-Löb logic in [50], the difference lies in the fact that [17] recasts the problem in a sheaf-theoretic environment. Let $\mathbf{P}$ be the opposite of the category of finite Heyting algebras (this category can be equivalently described as the category of finite posets and open maps, i.e. order-preserving maps taking downsets to downsets). Let $\mathbf{J}$ be the canonical topology on $\mathbf{P}$. We have a functor

$$\mathbf{\Phi_H} : HA_{fp}^{op} \longrightarrow Sh(\mathbf{P}, \mathbf{J})$$

sending a finitely presented Heyting algebra H to a sheaf

$$\mathbf{\Phi_H}(H) \;=\; [\, H_0 \mapsto HA(H, H_0)\,]$$

(i.e. $\mathbf{\Phi_H}(H)$ associates to every finite Heyting algebra H_0 the set of all Heyting morphisms from H to H_0 - the functor operates in the obvious way, i.e. by composition, on morphisms of finite and of finitely presented Heyting algebras). $\mathbf{\Phi_H}$ preserves finite limits and is conservative. The strategy to prove Pitts' theorem is now to show that its essential image inherits images and dual images from the ambient sheaf category. To do this, such essential image is described via a duality theorem: in a nutshell, Ehrenfeucht-Fraissé games are introduced in order to be able to recover definable subobjects in the image of $\mathbf{\Phi_H}$. Armed with this extra feature, it is eventually possible to show that HA_{fp}^{op} is a Heyting category and that $\mathbf{\Phi_H}$ preserves such Heyting category structure.

This analysis is extended in [21] to modal logics, where a full classification of logics extending $S4$ (the modal logic arising from interior topology axioms) is obtained; negative examples include $S4$ itself, where Pitts' theorem is shown to fail [18].

The above techniques have further applications: for instance, the duality theorem is exploited in [21] to prove that free Heyting algebras have a dual co-Heyting algebra structure, to supply semantic proofs of Beth definability theorems in certain modal logics, to characterize finitely presented projective Heyting algebras. More recently, the same duality has been used in order to obtain a semantic proof of Ruitenburg's theorem [22] stating that certain endomorphisms of finitely generated free Heyting algebras are ultimately periodic (a remarkable consequence of Ruitenburg's theorem is the definability of least and greatest fixpoints of monotonic formulae in IpC).

4. Algebraic theories

— Martin Hyland

As a byproduct of his research into Opetopes and Opetopic Sets, Marek Zawadowski developed foundations for notions of algebraic theory. High points are his treatment of two special cases: those of *rigid theories* and *regular theories*, where his results are definitive. Alongside these I describe the general setting which he used and indicate where he opened up fresh avenues for investigation. I hope that this will give a glimpse of his fertile mathematical imagination in which there was a productive tension between a preference for concrete presentations and an instinct to support them with abstract theory.

I start with some context. The notion of algebraic theory has three established manifestations. First there is the universal algebra notion of abstract clone, a direct abstraction from the syntax. In modern terms an abstract clone is a (single-sorted) cartesian multicategory. Secondly there is the closely related notion of a Lawvere theory, effectively a category with products generated by a single object. Thirdly there is the notion - abstracted from the semantics - of a finitary monad. A monad is finitary just when its functor preserves filtered colimits: this restriction captures the finiteness of standard syntax. In his work on restricted notions of algebraic theory, Zawadowski provided a general setting underlying the first approach and developed theory involving all three.

What happens when we look at restricted classes of algebraic theory? Typically we have an independent notion of theory presented as some kind of multicategory. From that we obtain a Lawvere theory and a monad. The obvious question is whether we characterise those that arise from our restricted notion amongst the collection of all theories or monads. Typically the Lawvere theory is freely generated from the multicategory and a characterisation expresses that fact. The monad comes from the semantics: for it things are less obvious. It turns out that characterisations can be given in terms of preservation properties of the functor and related properties of the unit and multiplication of the monad. But the results are subtle and we are still far from a full understanding.

4.1 Rigid theories

The original formulation of Opetopic Sets by Baez and Dolan [3] used some form of the theory of Opetopes but in detail it raised concerns as to how exactly to make the definitions precise. The leading suggestion was made by Hermida-Makkai-Power [26, 27, 28] in a series of three papers. They introduced the notion of *multicategories with non-standard amalgamation* and used it as the basis for an approach to higher categories via what they called multitopic sets. However in making things precise they created a new difficulty. The definition given in [27] is perplexing: it is not even obvious that algebras or models within the category of sets are multisorted algebras of some kind. Generally it was not clear how to relate the idea to other more familiar notions of algebraic theories.

Zawadowski was the first properly to understand the situation. One can make sense of multicategories with non-standard amalgamation from two more established and seemingly quite distinct perspectives. On the one hand they can be seen from the point of view of the theory of polynomials (see [16] for a detailed account). They can simply be identified with polynomial monads. On the other hand one can identify multicategories with non-standard amalgamation as special kinds of (many-sorted) operads, namely those for which the permutation actions are in an obvious sense free. Aficionados will call this Σ-free, avoiding confusion as they are not in any sense freely generated as operads. The characterisation as Σ-free operads led Zawadowski to call the corresponding theories *rigid*.

The two equivalences - with polynomial monads and with free or rigid operads - found by Zawadowski clarified what was a very murky state of affairs. The results are not prima facie obvious. I remember Zawadowski telling me some parts of the story in Peter Johnstone's rooms in St. John's College Cambridge at the PSSL meeting in April 2012. Some of the basic facts are already laid out in his [65] which dates from the end of 2009 and in 2012 he was presumably aware of the further aspects which appear in [52] written with his student Stanislas Szawiel.

4.2 Lax monoidal fibrations

The starting point for Zawadowski's treatment of algebraic theories is the paper [65] which he wrote for the 70th birthday of Michael (Mihaly) Makkai. In it, he introduced his notion of a *lax monoidal fibration* which he used consistently in his investigations of notions of algebraic theory. Zawadowski uses the language of fibrations but for purposes of exposition I shall combine it with the terminology of indexed categories. The essential features of the definition are as follows. In the first place we have a fibration $p : \mathbf{E} \to \mathbf{B}$. This is lax monoidal in the sense that for every object $b \in \mathbf{B}$, the fibre $\mathbf{E}_b = p^{-1}(b)$ has the structure of a monoidal category, while the reindexings come with the structure of a lax monoidal functor. The idea is that the fibration captures a flavour of algebraic signature. The monoidal structure corresponds to substitution and so monoids in the fibres correspond to algebraic theories. This picture is supported by a good deal of basic material presented in sections 3 and 4 of [65] and the rest of his paper is devoted to discussion of three main examples.

Burroni's T-categories The idea of a T-category for a monad T appears in an old paper [7] of Albert Burroni. The idea was neglected for many years but came to play a fundamental role in Leinster's work [39] on (non-symmetric) higher operads and categories. Leinster considered the situation which arises when the monad T is cartesian. It is a striking feature of the setting developed by Zawadowski that Burroni's idea falls within it without any restriction whatever. In section 5 of [65], he gives the details and he goes on to rework the definition of opetopes from [39] within his richer setting. All this should perhaps be seen as a proof of concept for lax monoidal fibrations. While this approach to opetopes did not play an explicit role in Zawadowski's work, it seems to me that understanding it fed into Zawadowski's study of algebraic theory.

Nonstandard amalgamation This is the crucial motivating example of [65]: Zawadowski shows how to use his notion to give a mathematically civilised presentation of the Hermida-Makkai-Power multicategories with nonstandard amalgamation [27]. Here I follow Zawadowski in suppressing the 2-dimensional features of the definition.

The basic issue is the treatment of substitution. The idea is that we have typed operations $f : u \to a$ with a string u of inputs and a single output a. If we have $\alpha : u \to a$ and $\beta : v \to b$ and if a appears at a particular point in the list v then we would like to substitute α for a in β. In the standard theory of non-symmetric operads (as developed in [39]) the input of the result is simply the result of inserting the list u for the entry a in the list v; and the output is b. That is standard amalgamation. Motivated by discussion of pasting diagrams in [26], the idea of non-standard amalgamation is systematically to allow the resulting input to be a different ordering of the standard list. This introduces a serious issue. Multiple substitutions for variables can be made in different orders and one wants to ensure that the result is independent of the order in which that is done. Conditions to ensure that are laid out in [27] but the resulting definition of multicategory is offputting.

In this difficult situation the first step taken by Zawadowski was simply to absorb the complications of the non-standard amalgamation into the total category of a monoidal fibration. The objects are signatures: they consist of a set O of types and a set A of operations equipped with a domain map $d : A \to O^*$ and a codomain map $c : A \to O$. (Here I am using the standard notation O^* for finite lists from O.) A map from a signature (O, A) to another (Q, B) consists in the first instance of functions $f : O \to Q$ and $F : A \to B$ where the function F is required to respect the lengths of the domains. This is augmented for each $\alpha \in A$ by a permutation σ_α of the length n say of $d(\alpha)$; and relative to these permutations the domains and codomains are required to match in the following sense. If $\beta : v \to b = F(\alpha : u \to a)$ is the image under F of $\alpha \in A$ then the domains match in the sense that $v_i = f(u_{\sigma_\alpha}(i))$ and the codomains agree, that is, $f(a) = b$. It is straightforward to see that this category is fibred over the category of sets. Moreover there is a clear and intuitive definition of a substitution tensor in each fibre. A multicategory with non-standard amalgamation in the sense of [27] is then simply a monoid in some fibre.

This very substantial reformulation is not merely a matter of clarification as it points to alternative characterisations. Most immediate is the connection with polynomials [16]. There is a lax monoidal fibration of finitary polynomial diagrams or equivalently of finitary polynomial endofunctors.

Theorem 4.1. *The (lax monoidal) fibration of signatures with non-standard amalgamation is equivalent to that of finitary polynomial endofunctors. Thus the fibration of multicategories with non-standard amalgamation is equivalent to that of polynomial monads.*

One can show this using symmetric operads but Zawadowski proves it via a concrete characterisation of the induced monads.

Proposition 4.2. *A monad* $(T, \eta.\mu)$ *on (slices of)* Sets *is polynomial just when the following holds.* T *is finitary, cartesian and preserving wide pullbacks; and* η *and* μ *are cartesian natural transformations.*

The same conditions characterise monads arising from multicategories with non-standard amalgamation and Theorem 4.1 follows. While this proof works, I do not think that Zawadowski was quite happy with the approach as he makes explicit mention of the fact that the characterisation is special to Sets. (Perhaps one should use instead the characterisation in [35]?)

Symmetric operads The third lax monoidal fibration of [65] is that of symmetric signatures or equivalently that of analytic functors. This is a natural many variable extension of Joyal's theory of species [32]. The algebraic theories which arise as monoids in the fibre over 1 are the familiar operads which already appear in May's work [44] on iterated loop spaces. Various characterisations of the resulting monads have been sketched. Zawadowski presents his own version.

Proposition 4.3. *A monad* $(T, \eta.\mu)$ *on (slices of)* Sets *is analytic just when the following holds.* T *is finitary and weakly preserves wide pullbacks; and* η *and* μ *are weakly cartesian natural transformations.*

Comparison with the characterisation of polynomial monads makes it clear that they are special analytic monads. That leads naturally enough to an identification of multicategories with non-standard amalgamation as special (multi-sorted) symmetric operads.

Comparison with Kleisli bicategories There is an alternative approach to algebraic theories in terms of Kleisli bicategories [14]. I give a brief sketch of how this is encompassed by Zawadowski's lax monoidal fibrations.

In the alternative approach one has a bicategory $\mathcal{B}$ equipped with an identity-on-objects, locally fully faithful homomorphism $\mathbf{B} \to \mathcal{B}$ of bicategories from (what is typically) a 2-category $\mathbf{B}$. The images $u_* : b \to c$ in $\mathcal{B}$ of 1-cells $u : b \to c$ from $\mathbf{B}$ are *maps* in the sense that they come with a right adjoint $u^* : c \to b$. (Essentially this is a proarrow equipment in the sense of Wood.) Then $\mathbf{B}$ - or if you prefer its underlying category - is the base for a lax monoidal fibration $\mathbf{E} \to \mathbf{B}$ obtained as follows. For $b \in \mathbf{B}$ we set the fibre $\mathbf{E}_b = \mathcal{B}(b, b)$. The monoidal structure is composition in $\mathcal{B}$. Reindexing uses the fact that 1-cells $u : b \to c$ in $\mathbf{B}$ give maps in $\mathcal{B}$ with $u_* : b \to c$ left adjoint to $u^* : c \to b$. So to reindex along $u : b \to c$ we can take $M \in \mathcal{B}(c, c)$ to the composite $u^*.M.u_* \in \mathcal{B}(b, b)$. This is a lax map of monoidal categories in an evident fashion.

Zawadowski's approach gives added generality, in particular allowing for a general treatment of Burroni's T-multicategories. It also obeys Occam's Razor avoiding excess of structure in the bicategories. (In favour of the bicategorical approach - it involves relatively familiar category theory.)

4.3 Leading examples

The experience of [65] suggested a systematic approach to the study of varieties of algebraic theories. That is the focus of [52] written with Zawadowski's student Stanislaw Szawiel. For purposes of exposition the abstract setting of [65] is set aside in favour of concrete syntactic presentations. There are three leading examples.

- The general notion of an algebraic theory: in [52] these are called *equational theories*.
- Operadic theories: those axiomatised by equations between terms in which exactly the same variables occur exactly once; [52] say *regular-linear theories*. Usually we say *analytic* or *operadic*.)
- What [52] call *rigid theories*. These are special regular-linear theories $\mathbb{T}$. Let t be a term with distinct variables $\mathbf{x}$ and let t^σ be t with the variables permuted by some permutation σ. Then the only circumstances in which $\mathbb{T} \vdash t = t^\sigma$ is when σ is the identity.

Szawiel-Zawadowski treat these examples from the point of view of Lawvere Theories and of monads.

Lawvere Theories The basic perspective of Lawvere's thesis [36] is folklore. From an algebraic theory $\mathbb{T}$ one constructs a Lawvere theory $\mathsf{L}_{\mathbb{T}}$, essentially a category with products generated by a single object. The category of models of a Lawvere theory L is the category $\mathrm{Prod}(\mathsf{L}, \mathbf{Sets})$ of product preserving functors to sets.

For regular-linear and rigid theories $\mathbb{T}$ there is an analogous symmetric monoidal or operadic category $\mathsf{O}_{\mathbb{T}}$ generated by a single object. The category of models of an operadic category O is the category $\mathrm{Mon}(\mathsf{O}, \mathbf{Sets})$ of tensor preserving functors to $\mathbf{Sets}$. The natural question for the restricted classes of theories is to characterise the Lawvere theories $\mathsf{L}_{\mathbb{T}}$ which result from them. Evidently they are freely generated from the operad $\mathsf{O}_{\mathbb{T}}$. Szawiel-Zawadowski show how the resulting Lawvere theories can be characterised very elegantly in terms of a factorisation system together with the requirement that the category has *simple automorphisms*.

Monads The approach to algebraic theories via monads is again folk-lore: algebraic theories correspond to finitary monads.

For regular-linear theories [52] shows in detail that these correspond to analytic monads. So [65] characterised the monadic approach (Proposition

4.3 above). There are many alternative characterisations going back to Joyal [32]. In the same spirit, rigid theories correspond to polynomial monads. Again [65] gives a characterisation (Proposition 4.2 above). One obtains the polynomial condition from the analytic by dropping the 'weak'. The characterisation seems cleaner but that is misleading. There is a problem.

Proposition 4.4. *The problem whether an analytic theory presented by a finite set of regular-linear equations defines a rigid theory is undecidable.*

This is proved by Zawadowski et al in [6]. To my mind this undecidability reflects something a bit riddling about the world of polynomials.

4.4 Regular Theories

It is always good to consider a new and telling example and in [54] Szawiel and Zawadowski do just that. Already on [52] they called a theory $\mathbb{T}$ *regular* when it is axiomatised by equations between terms in which exactly the same variables occur on each side. (The terminology was established long ago: in another logic tradition one would say *relevant*.) The analysis in [54] is parallel to that of [52] but everything is new and Szawiel and Zawadowski adopt a concrete multicategory approach.

In this fresh perspective one gets algebraic theories by taking some (standard skeletal) category F of finite sets: algebraic theories are monoids in the corresponding functor category $[\mathsf{F}, \mathrm{Sets}]$ with substitution tensor product. Operads are obtained by replacing F by B the category of finite sets and bijections. Then regular theories are obtained by taking instead the category S of finite sets and surjections.

Szawiel and Zawadowski look at the corresponding Lawvere theories. They characterise those which arise from regular theories in the same style as their characterisation of operadic theories, that is, in terms of a factorisation system and the condition of simple automorphisms. They then turn to the corresponding monads which they call *semi-analytic*. In terms of the new perspective that means characterising those which arise from monoids in $[\mathsf{S}, \mathbf{Sets}]$ by left Kan extension. That is really not at all obvious and the result is an impressive technical achievement.

Proposition 4.5. *A monad* $(T, \eta.\mu)$ *on (slices of)* $\mathbf{Sets}$ *is semi-analytic just when the following holds.* T *is finitary and semi-analytic in the sense that it* *preserves pullbacks of monomorphisms; and η and μ are semi-cartesian in the sense that their naturality squares for monomorphisms are pullbacks.*

This is subtle. Note for example that it is not obvious from Propositions 4.3 and 4.5 that analytic monads are semi-analytic.

4.5 Other directions

Zawadowski liked working with detailed calculations in specific examples but he had an instinct for abstract theory. It seems worth mentioning some of his ideas which hint at deep structure.

Behind opetopes In their joint paper [51] Szawiel and Zawadowski look again at an idea which has been a puzzle since it first appeared in Baez-Dolan [3]. The idea is to have a theory of theories: theories should be algebras (or models) for some defining theory. This idea plays a role in approaches to the theory of opetopes but [51] is concerned to provide some deeper conceptual foundations. The setting is that of Zawadowski's lax monoidal fibrations and the main ingredient is a remarkable and surprising construction of what is called the *web monoid*. It involves the interplay between two monoidal structures: weirdly the free monoid for one on the unit for the other. This idea is explained in greater detail and with explanatory intuitions in Curien's contribution to this memorial for Zawadowski. I add the thought that Szawiel and Zawadowski found some of the ingredients for their construction in a neglected paper [5] by Baues, Jibladze and Tonks. But the almost completely formal way in which they work things out suggests that something very fundamental is at play. This line of work is surely worth further study.

An abstract setting In a long but unpublished paper [55], Szawiel and Zawadowski returned to the subject of their [52] but from a radically more abstract point of view. The paper starts from one surprising observation. There is an evident monoidal monad on the category of signatures whose Eilenberg-Moore category is essentially the category of finitary functors on $\mathbf{Sets}$. The observation is that the Kleisli category is essentially that of polynomial functors (again with all natural transformations). That is unexpected but perhaps just an oddity. But it seems not as [55] weaves, out of that simple fact, a complicated story covering a range of examples. The paper is rich in detail but it is not easy to obtain a synoptic vision of the material. (For example as in [51] an important role is played by distributive laws but what is going on?) The paper seems not to have reached publication - a pity as there are many suggestive ideas.

Further Questions The papers just discussed are typical of the way in which Zawadowski's work opens up lines of enquiry. To conclude I mention two very general kinds of question suggested by the work.

The first concerns the scope of the general theoretical machinery. Algebraic theories correspond to monoids for the substitution tensor product on the category $[\mathsf{F}, \mathbf{Sets}]$ of set-valued functors on the category of finite sets. Using the categories B of bijections and S of surjections we get operadic and regular theories. Are there other interesting possibilities? Clearly there is I, the category of injections, and it is clear from [67] that Zawadowski contemplated looking at it and the interaction with S. Also there is N with just identities which gives rise to non-symmetric operads. For these there are subtleties laid out in section 6.2 of Leinster [39]. A side remark in [52] shows that Zawadowski was aware of the issue and thought to look more closely. Obviously we would like to know if there are more possibilities of this kind. What subcategories of F admit the substitution tensor product? Then orthogonal to that line of thought further is the special case of the rigid theories. That involves a global restriction. Is that a one-off or are there other similar cases? These kinds of question suggest looking yet more deeply at algebraic theories. One such direction appears in the recent paper Fiore and Ranchod [15].

The second kind of question which I think it worth pursuing concerns the characterisations of the induced monads alluded to above. The obvious puzzle is the contrast between weak properties or the weak preservation of properties and the more familiar preservation in an up to isomorphism sense. At the moment this all seems ad hoc. We have no story which makes sense of the different cases. When looking at the relation between Propositions 4.3 and 4.5, I am reminded of arguments from Gabriel-Ulmer. If C is a category with finite limits, a functor $F : \mathsf{C} \to \mathbf{Sets}$ preserves finite limits just when its category of elements is filtered. Is that part of the same story?

Last thought Zawadowski was a great pioneer in the world of algebraic theories. But that subject is very rich and his work surely shows that there remain many avenues to explore.

5. Opetopic sets

— Pierre-Louis Curien

This chapter reports on works around opetopic sets launched by an inspiring paper of Baez and Dolan published in 1998 [3]. Zawadowski became interested in this line of work in the first years of the present millenium, and since then, over the years and until his last moments, he made so many contributions to their study that he truly deserves the title of master of opetopes. As a milestone in his continuous effort of understanding these notoriously difficult mathematical objects, he organised a workshop on Opetopes, Opetopic Sets and Opetopic Categories in Warsaw in March 2013. When visiting my laboratory in 2022, he proposed the following undebatable definitions of opetopic sets and opetopes, which he called Sèvres definitions (in reference to the location of the Bureau International des Poids et Mesures `https://www.bipm.org/en`).

1. The category $\mathbf{OpeSet}$ of opetopic sets is the category of many-to-one polygraphs.
2. The category $\mathbf{Ope}$ of opetopes is the exponent category of $\mathbf{OpeSet}$, i.e., there is an equivalence of categories $\mathbf{OpeSet} \sim \mathbf{Set}^{\mathbf{Ope}^{op}}$.

Here is a brief dictionary. Polygraphs [2] (also known as computads) are strict infinity categories (or ω-categories) that are freely generated in all dimensions. Generating n-cells have an upper (or source) boundary and a lower (or target) boundary which are both (n-1)-cells. When all generators have a generating cell as target rather than some arbitrary codimension 1 cell, the polygraph is called many-to-one. If moreover the sources of generators are not allowed to be identities (or degeneracies) of codimension 1 cells, then the polygraph is called positive-to-one.

Implicit to item (2) above is a theorem, namely that many-to-one polygraphs form a presheaf category. Alternatively, the category $\mathbf{Ope}$ could be defined first (like in [11]) and then (1) is a theorem to prove. Therefore, (1) and (2) are in fact criteria that have to be met for any of the numerous definitions that flourished since a quarter of century. Of course, like in computability theory, the Sèvres stamp is contagious: any definition equivalent to one that is Sèvres is also Sèvres.

This short and compelling definition is of course anachronic. Opetopic shapes were introduced by Baez and Dolan as a proposal for a theory of weak higher categories. There were some rough corners in their seminal paper though, that triggered works of Hermida, Makkai and Power, who proposed their own version of opetopic sets based on multicategories, which they therefore called *multitopic sets* [26, 27, 28], and works of Leinster [39] and Cheng [8]. Soon after arrived some key contributions in which Zawadowski took an active part. In joint work with Harnik and Makkai, he proved that multitopic sets are the same as many-to-one polygraphs [25] (see also [64]), hitting target (1) above. Target (2) had been already reached in [28]. This altogether made the multitopic setting the first Sèvres definition! Makkai and Zawadowski also proved that general polygraphs do not form a presheaf category [43]. In parallel, Zawadowski also pioneered combinatorial definitions of opetopic shapes, as we shall see.

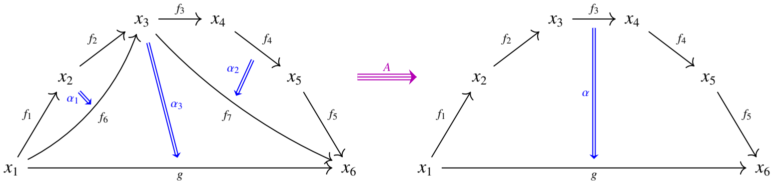
Then came a milestone contribution of Kock, Joyal, Batanin and Mascari [34], that offered both a categorical definition of opetopes based on polynomial diagrams, and a nice combinatorial description of opetopes via zoom complexes. Through its clarity and its appealing combinatorics, this paper made the whole field accessible to a wider audience (including the author of these notes!). In their setting, opetopes are defined through the so-called + construction, similar to the original slice construction of [3]. As we shall see, the mysteries and foundational value of these constructions were uncovered by Szawiel and Zawadowski [51] (see also the more extensive treatment in [53]). Also, in [68], Zawadowski unveiled not only a bijection but a categorical duality between zoom complexes and opetopes (in the positive-to-one case), prolonging a joint work with Makkai on the duality between simple ω-categories and Joyal's disks [42].

During all these years, Zawadowski kept an eye on the applications of opetopes to homotopical algebra and higher category theory. In [66], he showed that positive opetopes (equipped with suitable morphisms) form a test category. His last student Wojciech Dułinski is completing a PhD thesis investigating model structures on opetopic sets (again equipped with suitable morphisms). Let me also mention a recent application of opetopic structures in homotopy type theory: in [1], Allioux, Finster and Sozeau proved the celebrated result that all types form weak ∞-groupoids internally, in an "opetopic" extension of type theory. Yet another study of algebraic structures on opetopic sets is carried out by Ho Thanh and Subramaniam in [29].

The rest of this text proceeds as follows. In Section 5.1, we offer a brief introduction to opetopes. In Section 5.2, we aim at giving an overview of various combinatorial or type-theoretical approaches to opetopes. Finally, in Section 5.3, we zoom on polynomial diagrams and expose (our understanding of) the luminous foundational explanation of Szawiel and Zawadowski alluded to above.

5.1 Pictorial introduction to opetopes

There is a unique opetope ♦ of dimension 0, represented as a point. Starting from dimension 1, opetopes have (a finite set of) sources and one target (cf. "many-to-one" above). There is a unique opetope ■ of dimension 1. It has one source, and both its source and target are (copies of) ♦. It is represented as an arrow. We present 2-opetopes and 3-opetopes at once. The following picture features a 3-opetope A, which I shall explain in terms of what I would like to call geometry of unbiased associativity (just like associahedra express the geometry of biased associativity – we hope to be able to say more on this somewhen). Here, "biased" refers to binary composition as opposed to n-ary.



On the right of the picture is the target of A which is a 2-opetope α, having f_1, f_2, f_3, f_4, f_5 as sources, and g as target; all of them are (copies) of ■. I read α as the witness that g is an unbiased composite of the f_i's, i.e. α is a 2-cell or a homotopy from the formal sequence $f_1 f_2 f_3 f_4 f_5$ (or path, in diagrammatic order) to g. The sources of A are $\alpha_1, \alpha_2, \alpha_3$. Each of these

2-opetopes reads likewise as a composition (of $f_1 f_2$, $f_3 f_4 f_5$ and $f_6 f_7$, respectively). They are pasted together in such a way that the Poincaré dual of this arrangement of sources is a tree whose root node is α_3, with incoming edges f_6 and f_7 to which the nodes α_1 and α_2 are attached. A term notation for this is

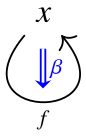
$$T = \alpha_3 \langle f_6 \leftarrow \alpha_1, f_7 \leftarrow \alpha_2 \rangle.$$

Implicitly, T has leaf edges f_1 and f_2 (coming into α_1) and f_3, f_4, f_5 (coming into α_2). The tree T can be read as a "more biased" recipe for composing the f_i's: first compose (in parallel) $f_1 f_2$, yielding f_6 on one hand, and $f_3 f_4 f_5$, yielding f_7 on the other hand, and then compose $f_6 f_7$, which is summarised in the parenthesised expression $((f_1 f_2)(f_3 f_4 f_5))$. Under these glasses, A reads as a witness of associativity (or higher homotopy) between $((f_1 f_2)(f_3 f_4 f_5))$ and $(f_1 f_2 f_3 f_4 f_5)$. One could further refine and build a 4-opetope, where in turn α_2 is refined as, say, $(f_3(f_4 f_5))$. A picture of a 4-opetope can be found in [37]. The term notation can be used for opetopes in all dimensions ≥ 1. Here is the tree for α:

$$t = f_5 \langle x_5 \leftarrow f_4 \langle x_4 \leftarrow f_3 \langle x_3 \leftarrow f_2 \langle x_2 \leftarrow f_1 \rangle \rangle \rangle \rangle.$$

Remarkably, the nodes of (the tree t of sources of) the target α of A are in one-to-one correspondence with the leaves of (the tree T of sources of) A. The expert reader will have recognised here a key feature of zoom complexes, on which we shall say a bit more in the following section.

Note that all opetopes in the picture are positive: there is always at least one source for each of A,α_i, and ■ is positive. Non-positivity starts at dimension 2. Here is the unique non-positive (or degenerate) opetope:



One can think of f as the identity on x, and β as a witness that it is such. As an example of a 3-opetope that is not positive, we can attach above β to α_2 along f_4 (renamed as f), forcing x_4 and x_5 to coincide, and then we have to modify the target α that now features the unbiased composition $(f_1 f_2 f_3 f_5)$ (exit f_4!), and the modified 3-opetope reads as combined associativity-and-unit law: $((f_1 f_2)(f_3(id) f_5)) = (f_1 f_2 f_3 f_5)$.

As a final note, let us stress that 2-opetopes are in bijection with natural numbers (above, β is 0, α is 5, etc.). We refer to [9, 39, 11] for more extensive presentations of opetopes.

5.2 Combinatorial definitions

In [69, 63], Zawadowski gave axiomatic descriptions of opetopic shapes in terms of their face structures, consisting of a graded set $(S_n)_{n\in\omega}$ of faces, and two functions $\mathtt{s}$ and $\mathtt{t}$ assigning to an n-face (or cell) a its (finite) set $\delta(a)$ of sources and its target $\gamma(a)$, where $\delta(a) \subseteq S_{n-1}$ and $\gamma(a) \in S_{n-1}$, subject to a number of axioms. In the positive case, considered in [69], $\delta(a)$ is required to be always non-empty. In this way, Zawadowski defines what he calls *opetopic cardinals* (in the revision of his paper, published posthumously in this volume of the Cahiers). They form a strict ω-category which is the terminal positive-to-one polygraph. Opetopes are defined as the subset of *principal* opetopic cardinals. Jumping over time, in 2023, as part of her Master thesis project on comparing various combinatorial descriptions of opetopes, Louise Leclerc exhibited a simpler axiomatisation for positive opetopes, which she called positive *dendritic face complexes*, and showed that they were equivalent to principal opetopic cardinals [37]. One of Leclerc's axioms is called oriented thinness – a refinement informed by the source/target distinction of a familiar axiom in posets saying that when x, y, z are such that x covers y and y covers z, then there exists a unique other element y' diffferent from x, y, z in the interval $[x, z]$, which hence looks like a lozenge. Louise and I learned this axiom from Amar Hadzihasanovic, who has developped over the recent years a much more general theory of diagrammatic sets in which opetopes sit as special shapes [24, 23].

Zawadowski's axioms were more complicated for a reason: he wanted to capture not only the generators, but all cells of the terminal positive-to-one polygraph. This allowed him to provide a nice definition of their compositions in all dimensions, by means of pushouts. Zawadowski extended his face structures to cover all opetopes in [63].

Independently, Thorsten Palm had proposed a theory of *dendrotopic sets* [45, 46]. In his Bachelor's thesis [33], Chris Kapulkin proved that Palm's axioms were valid in Zawadowski's framework, while whether the converse holds is still uncertain today. Over the years, Marek has been always eager to learn about other approaches, striving to make sure that they were Sèvres.

It is remarkable that opetopes as decribed in the previous section and as face structures of various flavours contain redundancies:

1. The target of an opetope can be derived from its tree of sources, as will be clear from the material of the next section.

2. An n-opetope can be reconstructed from its sequence of targets together with some additional information.

The authors of [34] substantiated (2) by defining *zoom complexes* as sequences of trees featuring the successive targets, with some additional information which in the positive-to-one case consists in matching the leaves of the (n-1)-th target with the nodes of the n-th target (cf. T and t in Section 5.1). Additional book-keeping is needed in the many-to-one case and involves the notion of a tree with subdivisions, allowing some nodes to be only "traversed" by an edge. More precisely, to maintain the tight connection between the (n-1)-th target and the n-th target, the n-th target needs to be complemented by these subdivisions. The nodes of the targets are called black nodes, and the special traversed nodes are called white nodes. For example, the 3-opetope representing the identity law $(f_1(id)f_2) = (f_1f_2)$ has a tree of sources of the form $\gamma\langle f \leftarrow 0\rangle$ – where γ has sources f_1, f, f_2, and 0 is the degenerate 2-opetope –, and its target has $t = f_1\langle x \leftarrow f_2\rangle$ as tree of sources. This information does not suffice to reconstruct where the identity is introduced – here between f_1 and f_2. To record this, the zoom complex comprises a "fattened" version of t, in which a white node representing f is inserted on the edge x of t, If instead we want to describe the opetope for $((id)f_1f_2) = (f_1f_2)$, then we have to insert the white node on the leaf edge of t. We refer to [34] for nice pictures of zoom complexes. A variant of zoom complexes has been implemented by Eric Finster (see `https://ncatlab.org/nlab/show/opetopic+type+theory`). In [38], Leclerc extended her setting to dendritic face complexes (removing the positive restriction), showing them to be equivalent to zoom complexes.

In joint works, I have aimed at substantiating (1), in a type-theoretical style and in a combinatorial style, respectively. With Cédric Ho Thanh and Samuel Mimram [11], we developed a type-theoretical framework for opetopes coded as expressions in the style of the term notation used here, with a typing system guaranteeing that opetopes are exactly the correctly typed expressions. The issues with loops discussed above for zoom complexes were treated by maintaining tables of identifications forced by their presence (cf. $x_4 = x_5$ in the last example of Section 5.1).

With Leclerc [12], we defined *positive epiphytes* and showed them to be equivalent to positive dendritic face complexes. Among all combinatorial definitions of opetopes, epiphytes are arguably the closest to the polynomial definition (see next section): n-epiphytes are trees whose nodes are decorated with (*n*-1)-epiphytes and whose edges are decorated with (*n*-2)-epiphytes, in some compatible way. We refrained to release this paper, as we are still working on extending the picture to all opetopes. For this, the "technology" of trees with subdivision will be needed.

The last combinatorial definition in date, to my knowledge, is due to Taichi Uemura. When reading his work, I could recognise in his axioms ideas present in Palm's dendrotopic setting, and other ideas present in the dendritic setting of Leclerc, both of which he ignored at the time of writing his preprint [56], which makes his work all the more intriguing.

5.3 Opetopes via polynomial diagrams

Polynomial diagrams [16] are triples of maps $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} Q$ in $\mathbf{Set}$. We shall be mostly interested in the case where $O = Q$ (endodiagrams). One should think of O as a set of colours (operadic language), objects (multicategorical language) or sorts (logical language), and of B as a set of operations with an output sort given by $\mathtt{t}$ and input sorts given by $\mathtt{s}$. More precisely, the fiber $\mathtt{E}_b = \mathtt{p}^{-1}(b)$ (usually required to be finite) is the arity (or set of sources) of b and the restriction of $\mathtt{s}$ to $\mathtt{E}_b$, together with the target $\mathtt{t}(b)$, gives the typing of b. A morphism of polynomial (endo)diagrams is given by maps as below:

$$\begin{tikzcd}
O' \arrow[d, "\varphi"] & \mathtt{E}' \arrow[l, "\mathtt{s}'"'] \arrow[r, "\mathtt{p}'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r, "\mathtt{t}"] \arrow[d, "f"] & O' \arrow[d, "\varphi"] \\
O & \mathtt{E} \arrow[l, "\mathtt{s}"] \arrow[r, "\mathtt{p}"'] & B \arrow[r, "\mathtt{t}'"'] & O
\end{tikzcd} \tag{1}$$

The pullback ensures that an operation b is mapped to an operation with equipotent arity.

The category $\mathbf{Poly}$ of polynomial diagrams, together with the projection to $\mathbf{Set}$ mapping a diagram as above to its set of sorts O, forms a fibration. It is in fact a lax monoidal fibration in the sense of [65]. In particular, each fiber $\mathbf{Poly}(O)$ is equipped with a tensor product $\otimes_O$ defined on objects as follows. If $\mathbb{P}$ and $\mathbb{P}'$ are polynomial diagrams as above, then $\mathbb{P} \otimes_O \mathbb{P}' = \mathbb{P}''$ is defined as follows:

- $B'' = \{(b; \{b'_i \mid i \in \mathtt{E}_b\}) \mid b \in B, b'_i \in B'$ and $\mathtt{t}'(b'_i) = \mathtt{s}(i)\}$,
- with target given by the target of b and arity and sources given by the coproduct of the arities $\mathtt{E}_{b'_i}$, and the corresponding restrriction of $\mathtt{s}'$.

Operations of $\mathbb{P}''$ are thus trees of height 2 formed by an operation of $\mathbb{P}$ on all sources of which operations of $\mathbb{P}'$ are grafted. The unit $I_\otimes$ is given by the diagram $O \xleftarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O$.

In [65], it is proved that $\mathbf{Poly}$ is isomorphic to the category $\mathbf{Sig}_a$ of signatures with non standard amalgamation (in the terminology of Hermida, Makkai and Power). When O consists of one sort only, monoids for this tensor product are exactly the Σ-free operads (i.e. symmetric operads in which the action of the symmetric groups is free). Amalgamation here refers to the bijection that relates the arity of $(b; \{b'_i\})$ (which is the set of leaves of the tree described above) to the arity of the composite operation $b \circ \{b'_i\}$ given by the monoid structure. In the case of non-symmetric operads, all sources are (standardly) ordered, and the bijection is the unique order-isomorphism between two equipotent total orders. In the case of symmetric operads, so to say, all bijections are allowed, but equivariance laws governing the commutation of permutations and composition are required. As a matter of fact, these three flavours have led historically to three distinct ways of defining opetopic sets: non-symmetric in Leinster's work [39], Σ-free for the multitopic sets of [28], as well as in most Zawadowski's works and in the present exposition, symmetric in the original setting of [3]. Eugenia Cheng made an extensive comparison work between these three approaches (see e.g. [10] and references therein).

$\mathbf{Poly}(O)$ admits a free monoid construction, left adjoint to the forgetful functor from the category *Mon*$(\mathbf{Poly}(O))$ of $\otimes_O$-monoids to $\mathbf{Poly}(O)$. In fact, in [51], general continuity conditions are given on a monoidal category with coproducts that suffice to prove the existence of free monoid objects;

they are explicitly described as expected as a colimit mimicking the construction of the free monoid X^* in *Mon* $=$ *Mon*$(\mathbf{Set})$. Instantiating this to $\mathbf{Poly}(O)$, we get that the free monoid $\mathbb{P}^*$ over some polynomial diagram $\mathbb{P}$ with data $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} O$ has B-trees as operations. These are rooted trees whose nodes are decorated with operations of B and whose edges – comprising root edge and leaf edges – are decorated with sorts of O, in a source/target respecting way. The target function returns the decoration of the root edge, the arity is the set of *leaves* and the source function returns the decorations of the leaf edges. We write *Tree*(B) for the set of B-trees. The unit of $\mathbb{P}^*$ is specified by assigning to each sort o the node-less tree reduced to its root edge which is also the unique leaf edge, decorated with o. The composition operation $t \circ \{t_l\}$ consists in *grafting* a tree t_l on each leaf of t.

The illuminating point in [34] is the following observation: if $\mathbb{P} = \mathbb{M}$ is moreover equipped with a monoid structure, then the same set of operations *Tree*(B) is also part of a different polynomial diagram, not in $\mathbf{Poly}(O)$, but in $\mathbf{Poly}(B)$. The target function is given by the evaluation, i.e. the effective composition of the tree, viewed as a specification of operations eligible to be composed. The arity of a tree is now its set of *nodes*, and the source function returns the operation decorating each node. It turns out that the resulting polynomial diagram $\mathbb{M}^+$ is also a monoid, so that this construction, known as the + *construction*, can be iterated, leading to the following "four-line" definition of opetopes: for $n \geq 1$, n-opetopes are the operations of the diagram obtained by iterating the + construction (n-1) times on the monoid $\{\blacklozenge\} \xleftarrow{\mathtt{s}} \{*\} \xrightarrow{\mathtt{p}} \{\blacksquare\} \xrightarrow{\mathtt{t}} \{\blacklozenge\}$, where $\mathtt{s}$, $\mathtt{p}$, $\mathtt{t}$ are the obvious bijections, and where $\blacklozenge$ is the unique 0-opetope. One sees in particular that there is a unique 1-opetope $\blacksquare$ with a unique source.

A composite $t \circ \{t_n\}$ in $\mathbb{M}^+$ consists in the *replacement* of each node of t by the tree t_n. The definitions ensure that the leaves of t_n are in bijection with the arity of the operation decorating n in t, ensuring precise rewiring instructions after replacement. We place $\mathbb{M}^*$ and $\mathbb{M}^+$ in perspective to stress the ascension: operations of $\mathbb{M}$ have become sorts of $\mathbb{M}^+$!

$$\begin{array}{ll} \mathbb{M}^* & O \xleftarrow{\mathtt{s}} \mathtt{Leaves} \longrightarrow \textit{Tree}(B) \xrightarrow{\mathtt{t}} O \\ \mathbb{M}^+ & B \xleftarrow{\mathtt{label}} \mathtt{Nodes} \longrightarrow \textit{Tree}(B) \xrightarrow{\mathtt{eval}} B. \end{array}$$

Although quite natural, the + construction as explained above "by hand" lacks a foundational blessing. We have seen that, when B is the set of operations of a monoid in $\mathbf{Poly}(O)$, *Tree*(B) presents the remarkable feature of having two monoid structures, given by grafting and replacement, respectively, but they do not (seem to) live in the same category: the relevant monoidal structures are in $\mathbf{Poly}(O)$ and $\mathbf{Poly}(B)$, respectively. Two natural questions arise:

1. Can we build a (single) category equipped with two monoidal structures (compatible in some way), relatively to which replacement and grafting appear as (compatible) monoid structures?
2. Can we exhibit the + construction as a free construction?

Szawiel and Zawadowski give positive answers to these two questions, which we try to recount below. But before that, I'd like to present an answer to Question (2) obtained by making the link between the + construction and the (idea of) Baez and Dolan's slice construction explicit. Let me start by recalling that any polynomial diagram $\mathbb{P}$ gives rise to a *polynomial functor* $\underline{\mathbb{P}} : \mathbf{Set}/O \to \mathbf{Set}/Q$, mapping a function $f : X \to O$ to $\underline{\mathbb{P}}(f) = \sum_{q \in Q} \sum_{b \in B, \mathtt{t}(b)=q} \prod_{i \in \mathtt{E}_b} f^{-1}(\mathtt{s}(i))$, with the obvious projection to Q. Pictorially, an element of the fiber of $\underline{\mathbb{P}}(f)$ over q is an operation whose target is of type q and whose sources are additionally decorated with elements of X in a compatible way, i.e., if x decorates i, then $f(x) = \mathtt{s}(i)$. The above sum of monomials explains the name polynomial! It is also known that $\underline{\mathbb{P}}$ determines $\mathbb{P}$.

We shall give evidence of an alternative description of $\underline{\mathbb{M}^+}$. For this, I also need to recall the fact that if $\mathbb{C}$ is a monoidal category and M a monoid object in $\mathbb{C}$, then the slice category $\mathbb{C}/M$ comes equipped with a monoidal structure defined by taking I_M to be the unit of the monoid and by setting $f \otimes_M g = \mathtt{m} \circ (f \otimes g)$, where $\mathtt{m}$ is the multiplication of the monoid. We shall apply this to $\mathbf{Poly}(O)$ and $\mathbb{M}$. But before doing that, we need one more observation. It is easily seen that the polynomial diagram underlying $\mathbb{M}$ induces an isomorphism between $\mathbf{Poly}(O)/\mathbb{M}$ and $\mathbf{Set}/B$ (observe that we are in a fiber of $\mathbf{Poly}$, i.e., in the diagram (1) above, we have $\varphi = id$, and all the rest of the data is inferred from the bottom line and from f). Through this isomorphism, the induced tensor product of $f : X \to B$ and $g : Y \to B$

is $h : Z \to B$, where

$$Z = \{((x,b);\{(y_i,b_i) \mid i \in \mathtt{E}_b\}) \mid b, b_i \in B, f(x) = b, g(y_i) = b_i, \mathtt{t}'(b_i) = \mathtt{s}(i)\}$$

and where h is given by the composition $b \circ \{b_i\}$. In plain words, an operation in Z is an operation of $\mathbb{M} \otimes_O \mathbb{M}$ in which each node receives an additional decoration from X or Y, in a compatible way. Then iterating this tensor and applying the general free monoid construction sketched above, we get the following description of the free monoid over $f : X \to O$: its operations are B-trees whose nodes are further decorated with elements of X, respecting the following invariant: if a node is decorated with (x, b), then $f(x) = b$. But this is exactly $\underline{\mathbb{M}^+}(f)$. Therefore $\mathbb{M}^+$ is the polynomial diagram whose underlying polynomial functor is the free monoid functor for the monoidal structure sketched above.

So we have hit the goal of exhibiting the + construction as a free construction (a fact known to experts). But this was not made internally, in the sense that we had to go back and forth between diagrams and actual functors. Moreover, Question (1) is not yet answered. We repair this now, by extracting and commenting a key construction that we found in [51], expressed there in the language of signatures and translated here in the language of polynomial diagrams. This construction is a sort of more informed version of the slice construction above.

We start again with a monoid $\mathbb{M}$ given by $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} O$ in *Mon*$(\mathbf{Poly}(O))$. We consider the category $\mathbf{Poly}[\mathbb{M}] = \mathbf{Poly}(B)$, which we name differently to stress the fact that we shall take profit of the monoid structure of $\mathbb{M}$ to define a second monoidal product $\odot$ on $\mathbf{Poly}[\mathbb{M}]$, in addition to $\otimes = \otimes_B$ – the "resident" monoidal product in $\mathbf{Poly}(B)$. Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be two objects of $\mathbf{Poly}(B)$, whose data are $B \xleftarrow{\mathtt{s}^1} \mathtt{E}^1 \xrightarrow{\mathtt{p}^1} X \xrightarrow{\mathtt{t}^1} B$ and $B \xleftarrow{\mathtt{s}^2} \mathtt{E}^2 \xrightarrow{\mathtt{p}^2} Y \xrightarrow{\mathtt{t}^2} B$, respectively. We define $\mathbb{P}^1 \odot \mathbb{P}^2 = \mathbb{P}$ as $B \xleftarrow{\mathtt{s}'} \mathtt{E}' \xrightarrow{\mathtt{p}'} Z \xrightarrow{\mathtt{t}'} B$,

• where $\mathtt{t}'$ is the tensor product of $\mathtt{t}^1$ and $\mathtt{t}^2$ in $\mathbf{Set}/B$ as defined above, i.e., Z consists of decorated trees of hight 2 denoted as $((x, b); \{(y_i, b_i)\})$, where $b, b_i \in B$, i ranges over $\mathtt{E}_b$, $x \in X$, $\mathtt{t}^1(x) = b$, $y_i \in Y$, $\mathtt{t}^2(y_i) = b_i$ and $\mathtt{t}(b_i) = \mathtt{s}(i)$, and $\mathtt{t}'$ is the evaluation map returning $b \circ \{b_i\}$;

• the arity is given by taking the coproduct of the arities $(\mathtt{E}^1)_x$ and $(\mathtt{E}^2)_{y_i}$ and finally $\mathtt{s}'$ is given by copairing the relevant restrictions of $\mathtt{s}^1$ and $\mathtt{s}^2$.

Beyond the technicalities, note that the arity is ”node-wise”, as the above coproduct is indexed over all the nodes of $((x,b);\{(y_i,b_i)\})$. The unit $I_\odot$ for this second monoidal structure is given by the diagram $B \longleftarrow \emptyset \longrightarrow O \xrightarrow{\eta} B$, where η is given by the unit of the monoid $\mathbb{M}$.

With this, the goal of endowing $\mathbf{Poly}[\mathbb{M}]$ with two monoidal structures is reached. The two structures interact in a nice way: most notably, there is a distribution of $\otimes$ over $\odot$, and $\mathbf{Poly}[\mathbb{M}]$ satisfies the conditions of Szawiel and Zawadowski's Three tensors theorem – the main theorem in [51] –, which asserts that $\mathcal{F}_\odot(I_\otimes)$, i.e., the free $\odot$-monoid over the $\otimes$ unit, is also a $\otimes$-monoid. To unravel this, applying once more the general recipe for the explicit construction of free monoids, we have to examine $I_\otimes \odot \cdots \odot I_\otimes$. For $n = 2$, replacing above $\mathbb{P}^1$ and $\mathbb{P}^2$ both with $I_\otimes = O \xleftarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O$, we see that x, y_i are determined by b and hence each summand of the coproduct above is a singleton, so that the arity of an operation in $I_\otimes \odot I_\otimes$ is just its set of nodes. Taking the colimit, we recover *Tree*(B) with nodes as sources! In other words, we obtain our by now old friend $\mathbb{M}^+$, and as a bonus, by the Three tensors theorem, we get *for free* that $\mathbb{M}^+ = \mathcal{F}_\odot(I_\otimes)$ is a monoid in $\mathbf{Poly}(B)$, which gives us replacement as $\otimes$-multiplication.

Let us give evidence that the $\odot$-multiplication is the grafting operation: an operation in $\mathbb{M}^+ \odot \mathbb{M}^+$ is of the form $((t,b);\{(t_i,b_i)\})$, where t is B-tree that evaluates to b, which entails that the leaves of t are the sources of b, so that $(t;\{t_i\})$ is an operation of $\mathbb{M}^*$. By the freeness of $\mathbb{M}^+$ as a $\odot$-monoid and the general construction of free monoids, we have that the composite is the “concatenation” of t and the t_i, which here means the tree obtained by grafting the t_i's on the leaves of t. Thus the $\odot$-monoid structure of $\mathbb{M}^+$ in $\mathbf{Poly}[\mathbb{M}]$ litterally transcribes the $\otimes_O$-monoid structure of $\mathbb{M}^*$ in $\mathbf{Poly}(O)$. This all beautifully answers Questions (1) and (2) at once. We summarise the discussion in the following table.

grafting:	$\mathbb{M}^*$ as free $\otimes_O$-monoid in $\mathbf{Poly}(O)$ $\rightsquigarrow$ $\mathbb{M}^+$ as free $\odot$-monoid in $\mathbf{Poly}[\mathbb{M}]$
replacement:	$\mathbb{M}^+$ as $\otimes_B$-monoid in $\mathbf{Poly}(B)$ $\rightsquigarrow$ $\mathbb{M}^+$ as $\otimes$-monoid in $\mathbf{Poly}[\mathbb{M}]$

I would like to end this promenade in the opetopic world of Zawadowski with a mention of the nice connections with the line of work on mathematical accounts of syntax carried out by Marcelo Fiore and coworkers. The Three tensors theorem is an instance of a more general theorem stated by Fiore and Saville in [13]. In yet unpublished work presented in some talks starting in 2017 [1], Fiore has offered an insightful syntactic reading of the two monoid structures on trees : the leaves stand for variables and grafting for first-order substitution, while the nodes stand for function symbols and replacement for second-order substitution (cf. the term notation used in Section 5.1). He has also transposed and generalised the opetopic scene in the world of generalised species. Marek's legacy is very much alive!

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CAHIERS DE TOPOLOGIE ET
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ON POSITIVE OPETOPES, POSITIVE OPETOPIC CARDINALS AND POSITIVE OPETOPIC SETS

Marek ZAWADOWSKI

Abstract. We introduce the notion of a positive opetope and positive opetopic cardinals as certain finite combinatorial structures. The positive opetopic cardinals to positive-to-one polygraphs are like simple ω-graphs to free ω-categories over ω-graphs, c.f. [15]. In particular, they allow us to give an explicit combinatorial description of positive-to-one polygraphs. Using this description we show, among other things, that positive-to-one polygraphs form a presheaf category with the exponent category being the category of positive opetopes. We also show that the category ωCat of ω-categories is monadic over the category $\mathbf{pPoly}$ of positive-to-one polygraphs with the 'free functor' being the inclusion $\mathbf{pPoly} \to \omega Cat$.

Keywords. Opetopes, opetopic sets, polygraphs.
Mathematics Subject Classification (2010). 06A07, 18N30

Foreword by Pierre-Louis Curien

The present paper was submitted to the Cahiers by Marek Zawadowski in April 2023. Marek sadly passed away in early March 2024 without having the chance to revise the paper after he had received my anonymous report. After discussions between the editors-in-chief (Andrée C. Ehresmann, Marino Gran and René Guitart), the handling editor (Clemens Berger) and myself, it was decided to publish the paper in a form very close to the original manuscript submitted (available as arXiv:0708.2658). I chose to implement in the text the most obvious corrections and harmonizations only. Some notes, marked "(PLC)", are also included in the hope of offering some additional guidance through the article. They are written by me, and I am the only one to be blamed if they contain mistakes!

With this publication, we wish to pay a tribute to Marek, our dear friend and outstanding colleague. I'll miss his warm personality, his deep insights, his humour, his kindness, his elegance, his culture, and so will all his numerous colleagues and friends, with a special mention of his last PhD student Wojciech Dulínski, who helped me a lot while preparing this published version.

Contents

1. Introduction

In this paper we present a combinatorial description of the category of the positive-to-one polygraphs $\mathbf{pPoly}$. We show that this category is a presheaf category and we describe its exponent category in a combinatorial way as the category of positive opetopes $\mathbf{pOpe}$, see Section 3. However the proof of that requires some extended studies of the larger category of all positive opetopic cardinals. Intuitively, the (isomorphism classes of) positive opetopic cardinals correspond to the shapes of arbitrary cells in positive-to-one polygraphs. The notion of a positive opetopic cardinal is the main notion introduced in this paper. We describe in a combinatorial way the embedding functor $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$ of the category of positive-to-one polygraphs into the category of ω-categories as the left Kan extension along a suitable functor $\mathbf{j}$, and its right adjoint as the restriction along $\mathbf{j}$. We end by adapting an argument due to Victor Harnik [9] to show that the right adjoint to $\mathbf{e}$ is monadic. This approach does not cover the problem of the cells with empty domains which is important for both Makkai's multitopic categories and Baez-Dolan's opetopic categories. [1] However, it keeps something from the simplicity of Joyal's θ-categories, i.e., the category $\mathbf{pOpeCard}_\omega$ of positive opetopic cardinals with o-omega functors as morphisms is not much more complicated than the category of simple ω-categories, the dual of the

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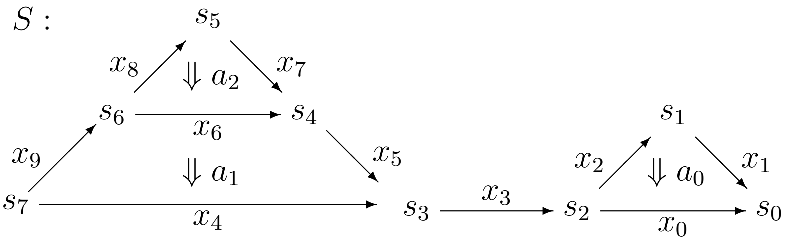
[1] (PLC) The positive restriction is lifted in [19], where Marek proves that many-to-one polygraphs form a presheaf category with the exponent category being the category of all opetopes.

category of disks, c.f. [11], [15], [5]. In this sense this paper may be considered as a step towards a comparison of globular and opetopic approaches.

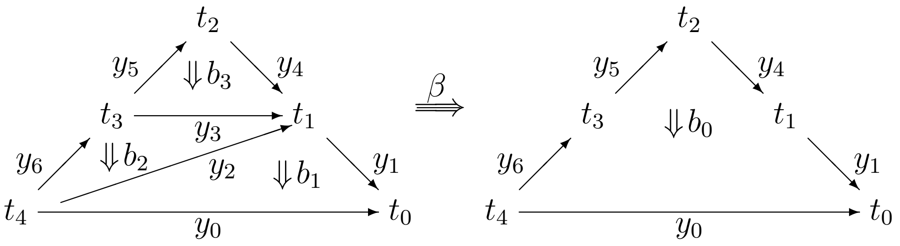
This paper is an extended and improved version of [18]. The terminology, notation, and proofs are changed and adjusted in many cases.

Positive opetopic cardinals

Positive opetopic cardinals represent all possible shapes of cells in positive-to-one polygraphs. A positive opetopic cardinal S of dimension 2 can be pictured as a figure



and a positive opetopic cardinal T of dimension 3 can be pictured as a figure



They have faces of various dimensions that fit together so that it makes sense to compose them in a unique way. By S_n we denote the set of faces of dimension n in S. Each face a has a face $\gamma(a)$ as its codomain and a *non-empty set* of faces $\delta(a)$ as its domain. In S above we have for a_1

$$\gamma(a_1) = x_4 \quad \text{and} \quad \delta(a_1) = \{x_5, x_6, x_9\}$$

and in T we have for β

$$\gamma(\beta) = b_0 \quad \text{and} \quad \delta(\beta) = \{b_1, b_2, b_3\}$$

These are all the data we need. Moreover, these (necessarily finite) data satisfy four conditions (see Section 3 for details). Below we explain them in an intuitive way.

Globularity. This is the main condition. It relates the sets that are obtained by double application of γ and δ. They are

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a) \qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a).$$

Let us look how it works for a_1 and β. In case of the face a_1 we have

$$\gamma\delta(a_1) = \{s_3, s_4, s_6\} \qquad \delta\delta(a_1) = \{s_4, s_6, s_7\}$$

$$\gamma\gamma(a_1) = s_3 \qquad \delta\gamma(a_1) = \{s_7\}.$$

So we have indeed

$$\delta\delta(a_1) - \gamma\delta(a_1) = \{s_4, s_6, s_7\} - \{s_3, s_4, s_6\} = \{s_7\} = \delta\gamma(a_1)$$

$$\gamma\delta(a_1) - \delta\delta(a_1) = \{s_3, s_4, s_6\} - \{s_4, s_6, s_7\} = \{s_3\} = \{\gamma\gamma(a_1)\}.$$

Similarly for the face β we have

$$\gamma\gamma(\beta) = y_0 \qquad \delta\gamma(\beta) = \{y_1, y_4, y_5, y_6\}$$

$$\gamma\delta(\beta) = \{y_0, y_2, y_3\} \qquad \delta\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\}.$$

and hence

$$\gamma\delta(\beta) - \delta\delta(\beta) = \{y_0, y_2, y_3\} - \{y_1, y_2, y_3, y_4, y_5, y_6\} = \{y_0\} = \{\gamma\gamma(\beta)\}$$

$$\delta\delta(\beta) - \gamma\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\} - \{y_0, y_2, y_3\} = \{y_1, y_4, y_5, y_6\} = \delta\gamma(\beta).$$

Using δ's and γ's we can define two binary relations $<^+$ and $<^-$ on faces of the same dimension which are the transitive closures of the relations $\lhd^+$ and $\lhd^-$, respectively, defined as follows: $a \lhd^+ b$ holds iff there is a face α such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$, and $a \lhd^- b$ holds iff $\gamma(a) \in \delta(b)$. We call $<^+$ the *upper order* and $<^-$ the *lower order*. For example, referring to the picture for T above, we have

$$b_3 \lhd^- b_2 \lhd^- b_1 \qquad y_5 \lhd^+ y_3 \lhd^+ y_2 \lhd^+ y_0.$$

The following three conditions refer to these relations.

Strictness. In each dimension, the relation $<^+$ is a strict order. The relation $<^+$ on 0-dimensional faces is required to be a linear order.

Disjointness. This condition says that no two faces can be comparable with respect to both orders $<^+$ and $<^-$.

Pencil linearity. This final condition says that the sets of cells with common codomain (*γ-pencil*) and the sets of cells that have the same distinguished cell in the domain (*δ-pencil*) are linearly ordered by $<^+$.

The morphisms of positive opetopic cardinals are functions that preserve dimensions and operations γ and δ. The size of a positive opetopic cardinal S is defined as an infinite sequence of natural numbers $size(S) = \{size(S)_k\}_{k\in\omega} = \{S_k - \delta(S_{k+1})\}_{k\in\omega}$ (almost all equal 0). We order the sequences lexicographically with higher dimensions being more important. The induction on the size of positive opetopic cardinals provides a convenient way of reasoning about them. The dimension of a positive opetopic cardinal S is the index of the largest non-zero number in the sequence $size(S)$. If for all $k \leq dim(S)$ (resp. for all $k < dim(S)$), $size(S)_k = 1$, then S is *principal* (resp. *normal*). [2] The normal positive opetopic cardinals play the role of the pasting diagrams of [10] and the principal positive opetopic cardinals play the role of the (positive) multitopes.[3] On positive opetopic cardinals we define domain and codomain operations, as well as special pushouts which play the role of composition. With these operations (isomorphisms classes of) the positive opetopic cardinals form the terminal positive-to-one polygraph, and at the same time a monoidal globular category in the sense of Batanin.

Categories and functors

We shall work with the following categories and functors:

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[2] (PLC) As an illustration, in reference to the opetopic cardinals S, T drawn above, T is principal, the left 2-dimensional part of the picture of T (with b_1, b_2, b_3 as top dimensional faces) is normal and not principal, and the opetopic cardinal S is not normal.

[3] (PLC) Optimistically, Marek had added: “the precise connection between these approaches will be described elsewhere”.

$$\begin{tikzcd}
\mathbf{pOpe} \arrow[d, "\mathbf{i}"'] & \mathcal{S} \arrow[d, "\mathbf{k}"] \\
\mathbf{pOpeCard} \arrow[r, "\mathbf{j}"] \arrow[d, "(-)^*"'] & \mathbf{pOpeCard}_\omega \arrow[d] \\
\mathbf{pPoly} \arrow[r, "\mathbf{e}"'] & \omega Cat
\end{tikzcd}$$

where $\mathbf{pOpe}$ is the category of principal positive opetopic cardinals, $\mathbf{pOpeCard}$ is the category of positive opetopic cardinals, $\mathcal{S}$ is the category of simple categories c.f. [15], $(-)^*$ is the embedding functor of positive opetopic cardinals into positive-to-one polygraphs, $\mathbf{e}$ is the inclusion functor, $\mathbf{pOpeCard}_\omega$ is the full image of the composition functor $(-)^*; \mathbf{e}$, with the non-full embedding $\mathbf{j}$, and where also $\mathbf{i}$ and $\mathbf{k}$ are inclusions. Having these functors, we can form the following diagram

$$\begin{tikzcd}
\mathbf{pPoly} \arrow[rr, "\mathbf{e}"] \arrow[d, shift right=3, "\widehat{(-)}"'] & & \omega Cat \arrow[d, shift right=3, "\widehat{(-)}"'] \\
sPb((\mathbf{pOpeCard})^{op}, Set) \arrow[u, shift right=3, "\widetilde{(-)}"'] \arrow[rr, shift left, "Lan_{\mathbf{j}}"] \arrow[d, shift right=3, "\mathbf{i}^*"'] & & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \arrow[u, shift right=3, "\widetilde{(-)}"'] \arrow[ll, shift left, "\mathbf{j}^*"] \arrow[d, shift right=3, "\mathbf{k}^*"'] \\
\widehat{\mathbf{pOpe}} \arrow[u, shift right=3, "Ran_{\mathbf{i}}"'] & & sPb(\mathcal{S}^{op}, Set) \arrow[u, shift right=3, "Ran_{\mathbf{k}}"']
\end{tikzcd}$$

in which all the vertical arrows come in pairs and are adjoint equivalences of categories, and in which the categories $sPb((\mathbf{pOpeCard})^{op}, Set)$, $sPb(\mathcal{S}^{op}, Set)$ and $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ are the full categories of presheaves that preserve special pullbacks. The functors

$$\widehat{(-)} : \mathbf{pPoly} \to sPb((\mathbf{pOpeCard})^{op}, Set)$$

and

$$\widehat{(-)} : \omega Cat \to sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

are defined similarly, using the bottom vertical functors from the previous diagram: for a polygraph Q and an ω-category C, $\widehat{Q}$ and $\widehat{C}$ are presheaves so that for a positive opetopic cardinal S we have

$$\widehat{Q}(S) = \mathbf{pPoly}(S^*, Q) \qquad\qquad \widehat{C}(S) = \omega Cat(S^*, C)\ ^4$$

The adjoint functors $\widetilde{(-)}$ that produce ω-categories are slightly more complicated. They are defined in Sections 13 and 15. The other functors are standard. The functors $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ are the functors $(\mathbf{i};-)$, $(\mathbf{j};-)$, $(\mathbf{k};-)$, respectively. $Ran_{\mathbf{i}}$ and $Ran_{\mathbf{k}}$ are the right Kan extensions along $\mathbf{i}$, $\mathbf{k}$, respectively and $Lan_{\mathbf{j}}$ is the left Kan extension along $\mathbf{j}$.

Since we have $\mathbf{e};\widehat{(-)} = \widehat{(-)};Lan_{\mathbf{j}}$, and since the functors $\widehat{(-)}$ are equivalences of categories, the functor $Lan_{\mathbf{j}}$ is like $\mathbf{e}$ but moved into a more manageable context. In fact, we have a very neat description of this functor.[5]

The content

Since the paper is quite long, I describe below the content of each section to help the reading. Sections 2 and 3 introduce the notion of a positive hypergraph and positive opetopic cardinal. Section 4 is concerned with establishing what kind of inclusions hold between iterated applications of γ's and δ's. Section 5 contains many statements concerning positive opetopic cardinals. All of them are there because they are needed afterwards. But it is not recommended to read the whole section at once. One of the main tools is the so called Path Lemma 5.7. Section 6 describes the embedding $(-)^* : \mathbf{pOpeCard} \to \omega Cat$, i.e., its main goal is to define an ω-category S^* for any positive opetopic cardinal S. Section 7 describes useful properties of normal positive opetopic cardinals. [6] In Section 8 we study a way

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[4] (PLC) Slowly, $\widehat{C}(\mathbf{j}(S)) = \widehat{C}(\mathbf{e}(S^*)) = \omega Cat(\mathbf{e}(S^*), C) = \omega Cat(S^*, C)$.

[5] (PLC) The left vertical column in the last diagram exhibits $\mathbf{pPoly}$ as a presheaf category with $\mathbf{pOpe}$ as exponent category. The right top vertical equivalence is proved by a 3-out-of-2 argument, relying on a direct equivalence between ωCat and $sPb(\mathcal{S}^{op}, Set)$ proved in [15], and on the right bottom equivalence (Proposition 15.1 – this is where Harnik's result 14.3 is invoked). Once the right top equivalence is known, one may concentrate on the top rectangle allowing us to look at $\mathbf{e}$ as $Lan_{\mathbf{j}}$.

[6] (PLC) As a hint for the importance of normal positive opetopic cardinals, we refer to the observation in the proof of Proposition 11.1. Normal positive opetopic cardinals appear also in the definition of principal pushouts (see Proposition 14.1 and Corollary 14.2).

in which we can decompose positive opetopic cardinals if they are at all decomposable. Any positive opetopic cardinal is either principal or decomposable. This provides a way of proving the properties of positive opetopic cardinals by induction on the size. Using this in Section 9 we show that the ω-category S^* and in fact the whole functor $(-)^*$ end up in $\mathbf{pPoly}$. Section 10 describes the inner-outer factorization and its refinements, i.e., a further factorization of inner maps into inner epi and inner monos. These factorizations will play an important role in describing the strongly cartesian monad (c.f. [6]) T_ω on opetopic sets for ω-categories and its decomposition into two simpler monads ($T_\omega = T_\iota \circ T_c$), together with a distributive law combining them.

The next short section 11 describes the terminal positive-to-one polygraph as an ω-category in terms of positive opetopic cardinals. Section 12 gives an explicit description of all the cells in a given positive-to-one polygraph with the help of positive opetopic cardinals. Section 13 establishes the equivalence of categories between $\mathbf{pPoly}$ and the category of presheaves over $\mathbf{pOpe}$. In Section 14, the principal pullbacks are introduced and an adaptation of Harnik's argument to the opetopic context is presented. The original argument was expressed in a different setting and was supposed to show the monadicity of the category of all ω-categories and ω-functors ωCat. However, this original proof contains a gap [9]. Section 15 describes a full nerve functor

$$\widehat{(-)} : \omega Cat \longrightarrow \widehat{\mathbf{pOpeCard}}_\omega$$

and identifies its essential image as the special pullbacks preserving functors. Section 16 describes the inclusion functor $\mathbf{e}$ as the left Kan extension

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

with formulas involving just coproducts (and no other colimits). This gives as a corollary the fact that $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$ preserves connected limits. Then we show that the right adjoint to $Lan_{\mathbf{j}}$

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

(and hence the right adjoint to $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$) is monadic. It is still an open question, c.f. [16], whether the category ωCat is monadic over *all* polygraphs. In the appendix we recall the definition of the category of positive-to-one polygraphs. In Section 17, we describe in detail the strongly cartesian monad T_ω on opetopic sets and its decomposition into two other strongly cartesian monads. We finish the introduction stating an open problem.

Problem. What are the full subcategories of the category of polygraphs $\mathcal{X} \hookrightarrow \mathbf{Poly}$ that are coreflective as subcategories of $\mathcal{X} \hookrightarrow \mathbf{Poly} \hookrightarrow \omega Cat$ with coreflector $\omega Cat \to \mathcal{X}$ being monadic?

This paper shows that positive-to-one polygraphs form one such category.

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I am very grateful to Victor Harnik and Mihaly Makkai for the conversations concerning matters of this paper.

2. Positive hypergraphs

We write ω for the set of natural numbers. For a family $\{X_k\}_{k\in\omega}$ of sets, we write, for all k, $X_{\geq k} = \bigcup_{i\geq k} X_i$ and $X_{\leq k} = \bigcup_{i\leq k} X_i$.

A *positive hypergraph* S is a family $\{S_k\}_{k\in\omega}$ of finite sets of faces, a family of functions $\{\gamma_k : S_{k+1} \to S_k\}_{k\in\omega}$, and a family of total relations $\{\delta_k : S_{k+1} \to S_k\}_{0\leq k<n}$. Moreover, $\delta_0 : S_1 \to S_0$ is a function and only finitely many among sets $\{S_k\}_{k\in\omega}$ are non-empty. As it is always clear from the context, we shall never use the indices of the functions γ and δ. We shall write $\delta(a) = \{b|\ (a,b) \in \delta\}$.

A *morphism of positive hypergraphs* $f : S \longrightarrow T$ is a family of functions $f_k : S_k \longrightarrow T_k$, for $k \in \omega$, such that the diagrams

$$\begin{tikzcd}
S_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\gamma"'] & T_{k+1} \arrow[d, "\gamma"] \\
S_k \arrow[r, "f_k"'] & T_k
\end{tikzcd}
\qquad
\begin{tikzcd}
S_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\delta"'] & T_{k+1} \arrow[d, "\delta"] \\
S_k \arrow[r, "f_k"'] & T_k
\end{tikzcd}$$

commute, for $k \in \omega$. The commutation of the left hand square is the commutation of the diagram of sets of functions but in case of the right hand square we mean more than commutation of a diagram of relations, i.e., we demand that for any $a \in S_{\geq 1}$, $f_a : \delta(a) \longrightarrow \delta(f(a))$ be a bijection, where f_a is the restriction of f to $\delta(a)$. The category of positive hypergraphs is denoted by $\mathbf{pHg}$.

Some notions and notation. Let S be a positive hypergraph.

1. When convenient and not leading to confusions, if $a \in S_k$, i.e., a is k-dimensional face in S, we sometime treat $\gamma(a)$ as an element of S_{k-1} and sometimes as a subset $\{\gamma(a)\}$ of S_{k-1} (following the coercion from functions to relations).

2. The dimension of S is $max\{k \in \omega : S_k \neq \emptyset\}$, and it is denoted by $dim(S)$.

3. The sets of faces of different dimensions are assumed to be disjoint (i.e., $S_k \cap S_l = \emptyset$, for $k \neq l$). S is also used to mean the set of all faces of S, i.e., $\bigcup_{k=0}^{n} S_k$; the notation $A \subseteq S$ means that A is a set of some faces of S; $A_k = A \cap S_k$, for $k \in \omega$.

4. For $a \in S_{\geq 1}$, the set $\partial(a) = \delta(a) \cup \gamma(a)$ is *the boundary of* A, i.e., the set of codimension 1 faces in a.

5. The set $S_{\leq k}$ is closed under δ and γ so it is a sub-hypergraph of S, called k-truncation of S.

6. The image of $A \subseteq S$ under δ and γ will be denoted by
$$\delta(A) = \bigcup_{a \in A} \delta(a), \quad \gamma(A) = \{\gamma(a) \, : \, a \in A\},$$
respectively. In particular, $\delta\delta(a) = \bigcup_{x \in \delta(a)} \delta(x)$, $\gamma\delta(a) = \{\gamma(x) : x \in \delta(a)\}$.

7. $\iota(a) = \delta\delta(a) \cap \gamma\delta(a)$ is *the set of internal faces* of the face $a \in S_{\geq 2}$.

8. On each set S_k we introduce two binary relations $<^{S_k,-}$ and $<^{S_k,+}$, called *lower* and *upper order*, respectively. We usually omit k and even S in the superscript.

   (a) $<^{S_0,-}$ is the empty relation. For $k > 0$, $<^{S_k,-}$ is the transitive closure of the relation $\lhd^{S_k,-}$ on S_k, such that $a \lhd^{S_k,-} b$ iff $\gamma(a) \in \delta(b)$. We write $a \bowtie^{-} b$ iff either $a <^{-} b$ or $b <^{-} a$, and we write $a \leq^{-} b$ iff either $a = b$ or $a <^{-} b$. [7] Of course these notations apply to $<^{+}$, etc. as well.

   (b) $<^{S_k,+}$ is the transitive closure of the relation $\lhd^{S_k,+}$ on S_k, such that $a \lhd^{S_k,+} b$ iff there is $\alpha \in S_{k+1}$, such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We write $a \bowtie^{+} b$ iff either $a <^{+} b$ or $b <^{+} a$, and we write $a \leq^{+} b$ iff either $a = b$ or $a <^{+} b$.

   (c) $a \not\bowtie b$ if both conditions $a \not\bowtie^{+} b$ and $a \not\bowtie^{-} b$ hold.

9. Let $a, b \in S_k$. A *lower path* $a_0, \ldots, a_m$ *from* a *to* b in S is a sequence of faces $a_0, \ldots, a_m \in S_k$ such that $a = a_0$, $b = a_m$ and for $\gamma(a_{i-1}) \in \delta(a_i)$, $i = 1, \ldots, m$.

10. Let $x, y \in S_k$. An *upper path* $x, a_0, \ldots, a_m, y$ *from* x *to* y in S is a sequence of faces $a_0, \ldots, a_m \in S_{k+1}$ such that $x \in \delta(a_0)$, $y = \gamma(a_m)$ and $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

11. The iterations of γ and δ will be denoted in two different ways. By γ^k and δ^k we mean k applications of γ and δ, respectively. By $\gamma^{(k)}$ and $\delta^{(k)}$ we mean the application as many times γ and δ, respectively, to get faces of dimension k. For example, if $a \in S_5$, then $\delta^3(a) = \delta\delta\delta(a) \subseteq S_2$ and $\delta^{(3)}(a) = \delta\delta(a) \subseteq S_3$.

12. For $l \leq k$, $a, b \in S_k$, we define $a <_l b$ iff $\gamma^{(l)}(a) <^{-} \gamma^{(l)}(b)$.

13. A face a is *unary* iff $\delta(a)$ is a singleton.

Lemma 2.1. *If S is a hypergraph and $k \in \omega$, then $<^{S_{k+1},-}$ is a strict partial order iff $<^{S_k,+}$ is a strict partial order.*

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[7] (PLC) In the original submission, the symbol $\perp$ was used in place of $\bowtie$. In January 2024, following referees' recommendations, Marek decided to replace this symbol by the more evocative one here – a change he had the time to implement himself in [20].

3. Positive opetopic cardinals

To simplify the notation, we treat both δ and γ as functions acting on faces as well as on sets of faces, which means that sometimes we confuse elements with singletons. Clearly, both δ and γ, when considered as functions on sets of faces, are monotone.

A positive hypergraph S is a *positive opetopic cardinal* if it is non-empty, i.e., $S_0 \neq \emptyset$, and if the following conditions hold:

1. *Globularity:* for $a \in S_{\geq 2}$:

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a) \qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a).$$

2. *Strictness*: for $k \in \omega$, the relation $<^{S_k,+}$ is a strict order; $<^{S_0,+}$ is linear. [8]

3. *Disjointness*: for $k > 0$,

$$\bowtie^{S_k,-} \cap \bowtie^{S_k,+} = \emptyset.$$

4. *Pencil linearity*: for any $k > 0$ and $x \in S_{k-1}$, the sets

$$\{a \in S_k \mid x = \gamma(a)\} \quad \text{and} \quad \{a \in S_k \mid x \in \delta(a)\}$$

are linearly ordered by $<^{S_k,+}$.

Remarks.

1. The reason why we call the first condition 'globularity' is that it will imply the usual globularity condition in the ω-categories generated by positive opetopic cardinals.

2. Note that if we were to assume that each positive opetopic cardinal has a single cell of dimension -1, then linearity of $<^{S_0,+}$ would become a special case of pencil linearity.

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[8] (PLC) In particular $\lhd^{S_k,+}$ is irreflexive, i.e., for every face a of S, $\gamma(a)$ and $\delta(a)$ are disjoint.

3. The fact that, for $x \in S_{k-1}$, the set $\{a \in S_k \mid x = \gamma(a)\}$ is linearly ordered will sometimes be referred to as *γ-linearity* of $<^{S_k,+}$, and the fact that the set $\{a \in S_k \mid \ x \in \delta(a)\}$ is linearly ordered is sometimes referred to as *δ-linearity* of $<^{S_k,+}$.

4. The *size of positive opetopic cardinal S* is the sequence of natural numbers $size(S) = \{|S_n - \delta(S_{n+1})|\}_{n\in\omega}$, with almost all being equal 0. We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$. This order is well founded and many facts about positive opetopic cardinals will be proved by induction on the size.

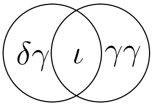
5. The *category of positive opetopic cardinals* is the full subcategory of positive hypergraphs $\mathbf{pHg}$ whose objects are the positive opetopic cardinals and is denoted by $\mathbf{pOpeCard}$.

6. Let S be a positive opetopic cardinal. S is k-principal iff $size(S)_l = 1$, for $l \leq k$. S is a *positive opetope* iff S is $dim(S)$-principal. S is *normal* iff S is $(dim(S) - 1)$-principal. By $\mathbf{pOpe}$ we denote the full subcategory of $\mathbf{pOpeCard}$ whose objects are positive opetopes.

4. Atlas for γ and δ

Lemma 4.1. *Let S be a positive opetopic cardinal, $a \in S_n$, $n > 1$. Then*

1. *the sets $\delta\gamma(a)$, $\iota(a)$, and $\gamma\gamma(a)$ are disjoint;*



2. *$\delta\delta(a) = \delta\gamma(a) \cup \iota(a)$;*

3. *$\gamma\delta(a) = \gamma\gamma(a) \cup \iota(a)$.*

Proof. These are immediate consequences of globularity. $\Box$

Lemma 4.2. *Let S be a positive opetopic cardinal, $a \in S_n$, $n > 2$. Then we have*

1. $\delta\gamma\gamma(a) \subseteq \delta\gamma\delta(a) \subseteq \delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) = \delta\delta\delta(a)$*;*

2. $\gamma\gamma\gamma(a) \subseteq \gamma\gamma\delta(a) \subseteq \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a) = \gamma\delta\delta(a)$*.*

Proof. From globularity we have $\gamma\gamma(\alpha) \subseteq \gamma\delta(\alpha)$. Thus by monotonicity of δ and γ we get

$$\gamma\gamma\gamma(\alpha) \subseteq \gamma\gamma\delta(\alpha) \quad \text{and} \quad \delta\gamma\gamma(\alpha) \subseteq \delta\gamma\delta(\alpha) \quad \text{and} \quad \gamma\gamma\delta(\alpha) \subseteq \gamma\delta\delta(\alpha).$$

Similarly, as we have $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$ by globularity, it follows by monotonicity of δ and γ:

$$\gamma\delta\gamma(\alpha) \subseteq \gamma\delta\delta(\alpha) \quad \text{and} \quad \delta\delta\gamma(\alpha) \subseteq \delta\delta\delta(\alpha) \quad \text{and} \quad \delta\gamma\delta(\alpha) \subseteq \delta\delta\delta(\alpha).$$

The equalities

$$\delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) \quad \text{and} \quad \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a)$$

follow from Lemma 4.1. Thus it remains to show that:

1. $\delta\delta\gamma(a) \supseteq \delta\delta\delta(a)$,
2. $\gamma\delta\gamma(a) \supseteq \gamma\delta\delta(a)$.

Both inclusions can be proved similarly. We shall show the first only. Suppose that there is $u \in \delta\delta\delta(a) - \delta\delta\gamma(a)$. Let $x \in \delta(a)$ be $<^-$-minimal element in $\delta(a)$ such that there is $s \in \delta(x)$ with $u \in \delta(s)$. If $s \in \delta\gamma(a)$, then $u \in \delta\delta\gamma(a)$, contrary to the supposition. Thus $s \not\in \delta\gamma(a)$. Since $\delta\gamma(a) = \delta\delta(a) - \gamma\delta(a)$ it follows that $s \in \gamma\delta(a)$. Hence there is $x' \in \delta(a)$ with $\gamma(x') = s$. In particular, $x' <^- x$. Moreover

$$u \in \delta(s) = \delta\gamma(x') \subseteq \delta\delta(x').$$

Then there is $s' \in \delta(x')$ so that $u \in \delta(s')$. This contradicts the $<^-$-minimality of x. □

Corollary 4.3. *Let* S *be a positive opetopic cardinal,* $a \in S_n$*,* $n > 2$*,* $k < n$*. Then, with* ξ^l *and* ξ'^l *being two fixed strings of* γ*'s and* δ*'s of length* l*, we have*

1. $\gamma^k(a) \subseteq \gamma\xi^{k-1}(a)$;
2. $\delta\xi^{k-1}(a) \subseteq \delta^k(a)$;
3. $\delta^k(a) \cap \gamma^k(a) = \emptyset$;
4. $\xi^k(a) \subseteq \gamma^k(a) \cup \delta^k(a)$;
5. $\delta^2\xi^{k-2}(a) = \delta^2\xi'^{k-2}(a)$, *(e.g.* $\delta^k(a) = \delta^2\gamma^{k-2}(a)$*)*;
6. $\gamma\delta\xi^{k-2}(a) = \gamma\delta\xi'^{k-2}(a)$, *(e.g.* $\gamma\delta^{k-1}(a) = \gamma\delta\gamma^{k-2}(a)$*)*;
7. $\xi^{k-2}\delta\gamma(a) = \xi^{k-2}\delta^2(a)$, *for* $k > 2$;
8. $\delta^k(a) = \delta\gamma^{k-1}(a) \cup \iota\gamma^{k-2}(a)$, *for* $k > 1$.

5. Combinatorial properties of positive opetopic cardinals

Local properties

Proposition 5.1. *Let* S *be a positive opetopic cardinal,* $k > 0$ *and* $\alpha \in S_k$, $a_1, a_2 \in \delta(\alpha)$, $a_1 \neq a_2$. *Then we have*

1. $a_1 \not\bowtie^+ a_2$;
2. $\delta(a_1) \cap \delta(a_2) = \emptyset$ *and* $\gamma(a_1) \neq \gamma(a_2)$.

Proof. Ad 1. Suppose on the contrary that there are $a_1, a_2 \in \delta(\alpha)$ such that $a_1 <^+ a_2$. So we have an upper path

$$a_1, \beta_1, \ldots, \beta_r, a_2$$

and hence a lower path

$$\beta_1, \ldots, \beta_r, \alpha.$$

In particular, $\beta_1 <^- \alpha$. As $a_1 \in \delta(\beta_1) \cap \delta(\alpha)$ by δ-linearity, we have $\beta_1 \bowtie^+ \alpha$. But then $(\alpha, \beta_1) \in \bowtie^+ \cap \bowtie^- \neq \emptyset$, i.e., S does not satisfy the disjointness. This shows 1.

Ad 2. This is an immediate consequence of 1. If $a_1, a_2 \in \delta(\alpha)$ and either $\gamma(a_1) = \gamma(a_2)$ or $\delta(a_1) \cap \delta(a_2) \neq \emptyset$, then by pencil linearity we get that $a_1 \bowtie^+ a_2$, contradicting 1. $\square$

We next introduce more notation. Let S be a positive opetopic cardinal, $n \in \omega$.

1. For a face $\alpha \in S_{n+2}$, we shall denote by $\rho(\alpha) \in \delta(\alpha)$ the only face in $\delta(\alpha)$, such that $\gamma(\rho(\alpha)) = \gamma\gamma(\alpha)$.
2. $X \subseteq S_{n+1}$, $a, b \in S_n$ and $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path in S. We say that it is *a path in* X (or X-path) if $\{\alpha_1, \ldots, \alpha_k\} \subseteq X$.

Lemma 5.2. *Let S be a positive opetopic cardinal, $n \in \omega$, $\alpha \in S_{n+2}$, $a, b \in S_{n+1}$, $y \in \delta\delta(\alpha)$. Then*

1. *there is a unique upper $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$;*
2. *there is a unique $x \in \delta\gamma(\alpha)$ and an upper $\delta(\alpha)$-path from x to y such that $\gamma(x) = \gamma(y)$;*
3. *if $t \in \delta(y)$, there are a unique $x \in \delta\gamma(\alpha)$ such $t \in \delta(x)$ and an upper $\delta(\alpha)$-path from x to y;*
4. *If $a <^+ b$, then $\gamma(a) \leq^+ \gamma(b)$.*

Proof. Ad 1. The uniqueness follows from Proposition 5.1.2. To show the existence, let us suppose on the contrary that there is no $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$. We shall construct an infinite upper $\delta(\alpha)$-path from y

$$y, a_1, a_2, \ldots$$

As $y \in \delta\delta(\alpha)$, there is $a_1 \in \delta(\alpha)$ such that $y \in \delta(a_1)$. So now suppose that we have already constructed $a_1, \ldots, a_k$. By assumption $\gamma(a_k) \neq \gamma\gamma(\alpha)$ so, by globularity, $\gamma(a_k) \in \delta\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_k) \in \delta(a_{k+1})$. This ends the construction of the path. This however contradicts strictness and, in fact, there is a $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$.

Ad 2. Suppose that there is no $x \in \delta\gamma(\alpha)$ as claimed. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $\gamma\gamma(a_n) = \gamma(y) = t$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$ such that $\gamma(z) = t$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed.

Ad 3. This case is similar. We prove it for completeness. Suppose that there is no $x \in \delta\gamma(\alpha)$ as above. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $t \in \delta\gamma(a_n)$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$, such that $t \in \delta(z)$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $t \in \delta\gamma(a_{k+1})$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed. The uniqueness again follows from Proposition 5.1.2.

Ad 4. The essential case is when $a \lhd^+ b$. This follows from 1. Then use the induction on the length of the upper path from a to b. □

Lemma 5.3. *Let S be a positive opetopic cardinal, $n > 1$, $\alpha \in S_{n+1}$, and $a, b \in S_n$ such that $a <^+ b$. Then*

1. $\iota\delta(\alpha) = \iota\gamma(\alpha)$ *;*
2. $\iota(a) \subseteq \iota(b)$*;*
3. $\iota(a) \cup \gamma\gamma(a) \subseteq \iota(b) \cup \gamma\gamma(b)$*;*

4. $\iota(a) \cup \delta\gamma(a) \subseteq \iota(b) \cup \delta\gamma(b)$*;*

5. $\partial\partial(a) \subseteq \partial\partial(b)$.

Proof. Ad 1. First we show $\iota\delta(\alpha) \subseteq \iota\gamma(\alpha)$. Fix $a \in \delta(\alpha)$ and $t \in \iota(a)$. Thus there are $x, y \in \delta(a)$ such that $\gamma(x) = t \in \delta(y)$. By Lemma 5.2 2,3 there are $x', y' \in \delta\gamma(\alpha)$ such that $x' \leq^+ x$, $y' \leq^+ y$ and $\gamma(x') = t \in \delta(y')$. Thus $t \in \iota\gamma(\alpha)$ and the first inclusion is proved.

Now we prove the converse inclusion $\iota\delta(\alpha) \supseteq \iota\gamma(\alpha)$. Fix $t \in \iota\gamma(\alpha)$. In particular, there are $x, y \in \delta\gamma(\alpha)$, so that $\gamma(x) = t \in \delta(y)$. Suppose that $t \not\in \iota\delta(\alpha)$. We shall build an infinite $\delta(\alpha)$-path

$$a_1 \lhd^- a_2 \ldots$$

such that $\gamma\gamma(a_i) = t$ for $i \in \omega$.

Since $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$, there is $a_1 \in \delta(\alpha)$ such that $x \in \delta(a_1)$. Since $t \not\in \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_1) = t$. Suppose now that we have already constructed the path

$$a_1 \lhd^- a_2 \lhd^- \ldots \lhd^- a_k$$

with the stated properties. We have $\gamma\gamma(a_k) = t \lhd^+ \gamma(y) \leq^+ \gamma\gamma\gamma(\alpha)$. So, by strictness, $\gamma(a_k) \neq \gamma\gamma(\alpha)$ and $\gamma(a_k) \in \delta\delta(\alpha)$. Then there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_k) \in \delta(a_{k+1})$. Again, as $t \not\in \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. Since, by strictness, such a path cannot exist, we get the other inclusion.

Ad 2. Since the inclusion is transitive, it is enough to consider the case $a \lhd^+ b$, i.e., there is an $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $b = \gamma(\alpha)$. Then by 1. we have

$$\iota(a) \subseteq \iota\delta(\alpha) = \iota\gamma(\alpha) = \iota(b).$$

Ad 3. Again it is enough to consider the case $a \lhd^+ b$, i.e., that there is $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. By 2. we need to show that $\gamma\gamma(a) \in \iota(b) \cup \gamma\gamma(b)$. Using Lemma 4.2.2 we have

$$\gamma\gamma(a) \in \gamma\gamma\delta(\alpha) \subseteq \iota\gamma(\alpha) \cup \gamma\gamma\gamma(\alpha) = \iota(b) \cup \gamma\gamma(b).$$

Ad 4. This is similar to 3 and uses Lemma 4.2.1.

Ad 5. This follows from 3. and 4 and Lemma 4.1. □

Global properties

Lemma 5.4. *Let S be a positive opetopic cardinal, $n \in \omega$, $a, b \in S_n$, $a <^+ b$. Then, there is an upper $(S_{n+1} - \gamma(S_{n+2}))$-path from a to b.*

Proof. Let $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path is S. By Lemma 5.2 we can replace each face α_i in this path which is not in $S - \gamma(S)$ by a sequence of faces which are $<^+$-smaller. Just take $\Gamma \in S_{n+2}$, such that $\gamma(\Gamma) = \alpha_i$ and take instead of α_i a path in $\delta(\Gamma)$ from $\gamma(\alpha_{i-1})$ (if $i = 0$ then from a) to $\gamma(\alpha_i)$. Repeated application of this procedure will eventually yield the required path. □

Lemma 5.5. *Let S be a positive opetopic cardinal, $n > 0$, $a \in S_n$, $\alpha \in S_{n+1}$, such that either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$. Then $a <^+ \gamma(\alpha)$. Moreover, if $\alpha \in S - \gamma(S)$, then there is a unique $a' \in \delta(\alpha)$ such that $a \leq^+ a'$.*

Proof. If $a \in \delta(\alpha)$, there is nothing to prove. So we assume $a \notin \delta(\alpha)$. We begin with the second part of the statement, i.e., we assume $\alpha \in S_{n+1} - \gamma(S_{n+2})$. Let $\gamma(a) \in \iota(\alpha)$. Thus there are $b, c \in \delta(\alpha)$ such that $\gamma(a) = \gamma(b) \in \delta(c)$. In particular, $a <^- c$. By γ-linearity either $b <^+ a$ or $a <^+ b$. Suppose that $b <^+ a$. Then we have an $(S - \gamma(S))$-upper path $b, \beta_0, \ldots, \beta_r, a$. As $b \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $c \in \delta(\alpha) = \delta(\beta_0)$ and hence $c <^+ \gamma(\beta_0) \leq^+ a$. So we get $a <^- c$ and $c <^+ a$, contradicting the disjointness of $\bowtie^+$ and $\bowtie^-$. Thus we can put $a' = b$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(a) = \gamma(a')$.

The case $\delta(a) \cap \iota(\alpha) \neq \emptyset$ is similar and we put it for completeness. There are $b, c \in \delta(\alpha)$ such that $\gamma(b) \in \delta(a) \cap \delta(c)$. In particular, $b <^- a$. By δ-linearity either $c <^+ a$ or $a <^+ c$. Suppose that $c <^+ a$. Then we have an $(S - \gamma(S))$-upper path $c, \beta_0, \ldots, \beta_r, a$. As $c \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $b \in \delta(\alpha) = \delta(\beta_0)$ and hence $b <^+ \gamma(\beta_0) \leq^+ a$. So we get $b <^- a$ and $b <^+ a$, contradicting the disjointness of $\bowtie^+$ and $\bowtie^-$. Thus we can put $a' = c$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(b) \in \delta(a')$ and $a' \in \delta(\alpha)$ and Proposition 5.1.

The first part of the statement follows from the above, Lemma 5.2.4 and the following claim.

Claim. If $\alpha \in S_{n+1}$ and $x \in \iota(\alpha)$, then there is an $\alpha' \in S_{n+1}$ such that $\alpha' \leq^+ \alpha$, $x \in \iota(\alpha')$ and $\alpha' \notin \gamma(S_{n+2})$.

Proof of the claim. Suppose the contrary. To get a contradiction, we shall build an infinite descending $\gamma(S_{n+2})$-path

$$\ldots \lhd^+ \alpha_1 \lhd^+ \alpha_0 = \alpha$$

such that $x \in \iota(\alpha_i)$, for $i \in \omega$.

We put $\alpha_0 = \alpha$. Suppose that we have already constructed $\alpha_0, \ldots, \alpha_k \in \gamma(S_{n+2})$. Hence there is $\beta \in S_{n+2}$ such that $\gamma(\beta) = \alpha_k$. Since, by Lemma 5.3.1, $\iota\delta(\beta) = \iota\gamma(\beta) = \iota(\alpha_k)$, there is $\alpha_{k+1} \in \delta(\beta)$ such that $x \in \iota(\alpha_{k+1})$. This ends the construction of the infinite path and the proof of the claim and the lemma. $\Box$

Corollary 5.6. *Let S be a positive opetopic cardinal. If $a \in S - \delta(S)$, then $\gamma(a) \in S - \iota(S)$ and $\delta(a) \subseteq S - \iota(S)$.*

Proof. Let $a \in S_n$ and $\alpha \in S_{n+2}$. If either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$, then by Lemma 5.5 $a <^+ \gamma(\alpha)$. Thus $a \in \delta(S)$. $\Box$

A lower path $b_0, \ldots, b_m$ is a *maximal path* if $\delta(b_0) \subseteq \delta(S) - \gamma(S)$ and $\gamma(b_m) \in \gamma(S) - \delta(S)$, i.e., if it can't be extended either way.

Lemma 5.7 (Path Lemma). *Let $k \geq 0$, $B = (a_0, \ldots, a_k)$ be a maximal lower S_n-path in a positive opetopic cardinal S, $b \in S_n$, $0 \leq s \leq k$, $a_s <^+ b$. Then there are $0 \leq l \leq s \leq p \leq k$ such that*

1. *$a_i <^+ b$ for $i = l, \ldots, p$;*
2. *$\gamma(a_p) = \gamma(b)$;*
3. *either $l = 0$ and $\delta(a_0) \subseteq \delta(b)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(b)$;*
4. *$\gamma(a_i) \in \iota(S)$, for $l \leq i < p$.*

Proof. Let $0 \leq l \leq p \leq k$ be such that $a_i <^+ b$, for $l \leq i \leq p$ and either $l = 0$ or $a_{l-1} \not<^+ b$ and either $p = k$ or $a_{p+1} \not<^+ b$. We shall show that l and p have the properties stated in the lemma. From the very definition the property 1 holds.

We shall next show 2. Take an upper $(S - \gamma(S))$-path from a_p to b: $a_p, \beta_0, \ldots, \beta_r, b$. If $\gamma(a_p) = \gamma\gamma(\beta_i)$, for $i = 0, \ldots, r$, then $\gamma(a_p) = \gamma\gamma(\beta_r) = \gamma(b)$ and we are done. So suppose the contrary and let

$$i_0 = min\{i : \gamma(a_p) \neq \gamma\gamma(\beta_i)\}.$$

Then there are $a, c \in \delta(\beta_{i_0})$ such that $\gamma(a_p) = \gamma(a) \in \delta(c)$ (NB. $a = a_p$ if $i_0 = 0$ and $a = \gamma(\beta_{i_0-1})$, otherwise). In particular, $\gamma(a_p) \in \iota(\beta_{i_0})$. As $\gamma(a_p) \in \delta(S)$, we have $p < k$. Thus $\gamma(a_p) \in \delta(a_{p+1}) \cap \iota(\beta_{i_0})$, and by Lemma 5.5 $a_{p+1} <^+ c <^+ b$. But this contradicts the choice of p. So the property 2. holds.

Now we shall show 3. Take an upper $(S - \gamma(S))$-path from a_l to b: $a_l, \beta_0, \ldots, \beta_r, b$. We have two cases: $l = 0$ and $l > 0$.

If $l = 0$, then there is no face $a \in S$ such that $\gamma(a) \in \delta(a_l)$. As $\delta(a_l) \subseteq \delta\delta(\beta_0)$, we must have $\delta(a_l) \subseteq \delta\gamma(\beta_i)$, for $i = 0, \ldots, r$. Hence $\delta(a_l) \subseteq \delta\gamma(\beta_r) = \delta(b)$ and 3. holds in this case.

Now suppose that $l > 0$. If $\gamma(a_{l-1}) \in \delta\gamma(\beta_i)$, for $i = 0, \ldots, r$, then $\gamma(a_{l-1}) \in \delta\gamma(\beta_r) = \delta(b)$ and 3. holds again. So suppose the contrary and let

$$i_1 = min\{i : \gamma(a_{l-1}) \notin \delta\gamma(\beta_i)\}.$$

Then there are $a, c \in \delta(\beta_{i_1})$ such that $\gamma(a_{l-1}) = \gamma(a) \in \delta(c)$ (NB: $c = a_l$ if $i_1 = 0$ and $c = \gamma(\beta_{i_1-1})$ otherwise). In particular, $\gamma(a_{l-1}) \in \iota(\beta_{i_1})$, and by Lemma 5.5 we have $a_{l-1} <^+ a <^+ b$ contrary to the choice of l. Thus 3. holds in this case as well.

Finally, we shall show 4. Let $l \leq j < p$ and $a_j, \beta_0, \ldots, \beta_r, b$ be an upper $(S - \delta(S))$-path from a_j to b. As $a_j <^- a_p$ and $a_p <^+ b$, we have $\gamma(a_j) \neq \gamma(b)$. So we can put

$$i_2 = min\{i : \gamma(a_j) \neq \gamma\gamma(\beta_i)\}.$$

But then $\gamma(a_j) \in \gamma\delta(\beta_{i_2}) - \gamma\gamma(\beta_{i_2}) = \iota(\beta_{i_2})$. Therefore $\gamma(a_j) \in \iota(S)$ and 4. holds. $\Box$

Lemma 5.8. *Let S be a positive opetopic cardinal, $n \in \omega$, $x, y \in S_n$, $x <^+ y$. If $x, y \notin \iota(S_{n+2})$, then there is an upper $S_{n+1} - \delta(S_{n+2})$-path from x to y.*

Proof. Assume $x, y \in (S - \iota(S))$ and $x <^+ y$. Let

$$x, b_0, \dots, b_k, y$$

be an upper path from x to y with the longest possible initial segment $b_0, \dots, b_l$ in $S - \delta(S)$. As $x \not\in \iota(S_{n+2})$, such a non-empty path exists. We need to show that $k = l$. Let a be the $<^+$-largest element of the set $\{b \in S : \gamma(b_l) \in \delta(b)\}$. Then $b_{l+1} \leq^+ a$ and $a \not\in \delta(S)$. Since $y \not\in \iota(S)$, by Lemma 5.7.4 there is p such that $l + 1 \leq p \leq k$ such that $\gamma(b_p) = \gamma(a)$. Thus we have an upper path from x to y, $x, b_0, \dots, b_l, a, b_{p+1}, \dots, b_k, y$ with a longer initial segment in $S - \delta(S)$. But this is a contradiction with the choice of the path $x, b_0, \dots, b_k, y$, and it means that in fact $l = k$, as required. □

Order

Lemma 5.9. *Let S be a positive opetopic cardinal, $n \in \omega$, $a, b \in S_n$. Then we have*

1. *if $a <^+ b$, then for any $x \in \delta(a)$ there is $y \in \delta(b)$ such that $y \leq^+ x$;*
2. *if $a <^+ b$ and $\gamma(a) = \gamma(b)$, then for any $y \in \delta(b)$ there is $x \in \delta(a)$ such that $y \leq^+ x$;*
3. *if $\gamma(a) = \gamma(b)$, then either $a = b$ or $a \bowtie^+ b$;*
4. *if $\gamma(a) <^+ \gamma(b)$ then either $a <^+ b$ or $a <^- b$;*
5. *if $a <^+ b$ then $\gamma(a) \leq^+ \gamma(b)$;*
6. *if $a <^- b$ then $\gamma(a) <^+ \gamma(b)$;*
7. *if $\gamma(a) \bowtie^- \gamma(b)$ then $a \not\bowtie^- b$ and $a \not\bowtie^+ b$.*

Proof. Ad 1. Let $a <^+ b$ and $x \in \delta(a)$. We have two cases: either $x \in \gamma(S)$ or $x \not\in \gamma(S)$.

In the first case there is $a' \in S - \gamma(S)$ such that $\gamma(a') = x$. Let $a_0, \dots, a_k$ be a maximal path containing a', a, say $a_{s-1} = a'$ and $a_s = a$, where $0 < s \leq k$. As $a_s <^+ b$, by Lemma 5.7 there is $l \leq s$ and $y \in \delta(a_l) \cap \delta(b)$. Clearly, $y \leq^+ x$.

In the second case consider an upper path from a to b: $a, \beta_0, \ldots, \beta_r, b$. We have $x \in \delta(a) \subseteq \delta\delta(\beta_0)$. As $x \not\in \gamma(S)$ so $x \not\in \gamma\delta(\beta_0)$, and hence $x \in \delta\delta(\beta_0) - \gamma\delta(\beta_0) = \delta\gamma(\beta_0)$. Thus we can define

$$r' = max\{i : x \in \delta\gamma(\beta_i)\}.$$

If we had $r' < r$, then again we would have $x \in \delta\delta(\beta_{r'+1}) - \gamma\delta(\beta_{r'+1}) = \delta\gamma(\beta_{r'+1})$, contrary to the choice of r'. So $r' = r$ and $x \in \delta\gamma(\beta_r) = \delta(b)$. Thus we can put $y = x$.

Ad 2. Fix $a <^+ b$ such that $\gamma(a) = \gamma(b)$ and $y \in \delta(b)$. We need to find $x \in \delta(a)$ with $y \leq^+ x$. Take a maximal $(S - \gamma(S))$-path $a_0, \ldots, a_k$ passing through y, i.e., there is $0 \leq j \leq k$ such that $y \in \delta(a_j)$ and if $y \in \gamma(S)$, then moreover $j > 0$ and $\gamma(a_{j-1}) = y$. Since $a_j \not\in \gamma(S)$ by δ-linearity we have $a_j <^+ b$. Thus by Lemma 5.7 there is $j \leq p \leq k$ such that $\gamma(a_p) = \gamma(b) = \gamma(a)$. Since $a_p \not\in \gamma(S)$ by γ-linearity we have $a_p \leq^+ a$. If $a_p = a$, then we can take as the face x either y if $p = 0$ or $\gamma(a_{p-1})$ if $p > 0$. So assume now $a_p <^+ a$. Again by Lemma 5.7 there is $0 \leq l \leq p$ such that either $l = 0$ and $\delta(a_0) \subseteq \delta(a)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(a)$. As a_l is the first face in the path $a_0, \ldots, a_k$ such that $a_l <^+ a$ and a_j is the first face in the path $a_0, \ldots, a_k$ such that $a_j <^+ b$ and moreover $a <^+ b$, it follows that $j \leq l$. Thus in this case we can take as the face x either y if $l = 0$ or $\gamma(a_{l-1})$ if $l > 0$.

Ad 3. This is an immediate consequence of γ-linearity.

Ad 4. Suppose $\gamma(a) <^+ \gamma(b)$. So there is an upper path

$$\gamma(a), c_1, \ldots, c_k, \gamma(b)$$

with $k > 0$. We put $c_0 = a$. We have $\gamma(c_k) = \gamma(b)$ so by γ-linearity $c_k \bowtie^+ b$ or $c_k = b$. In the latter case $a <^- b$. In the former case, we have two possibilities: either $b <^+ c_k$ or $c_k <^+ b$.

If $b <^+ c_k$, then by Lemma 5.7 for any maximal path that contains b and the face c_k we get that $c_{k-1} <^- b$. Thus we have $a <^- b$.

If $c_k <^+ b$, then by Lemma 5.7 for any maximal path that extends $c_0, c_1, \ldots, c_k$ and face b we get that either there is $0 \leq i < k$ such that $\gamma(c_i) \in \delta(b)$ and then $a <^- b$ or else $a = c_0 <^+ b$.

Ad 5. This is repeated from Lemma 5.2.

Ad 6. Suppose $a <^- b$. Then there is a lower path

$$a = a_0, a_1, \ldots, a_k = b$$

with $k > 0$. Then we have an upper path

$$\gamma(a) = \gamma(a_0), a_1, \ldots, a_k, \gamma(a_k) = \gamma(b).$$

Hence $\gamma(a) <^+ \gamma(b)$.

Ad 7. Easily follows from 5 and 6. $\Box$

Proposition 5.10. *Let S be a positive opetopic cardinal, $a, b \in S_n$, $a \neq b$. Let $\{a_i\}_{0 \leq i \leq n}$, $\{b_i\}_{0 \leq i \leq n}$ be the two sequences of codomains of a and b, respectively, so that*

$$a_i = \gamma^{(i)}(a) \qquad\qquad b_i = \gamma^{(i)}(b)$$

(i.e., $dim(a_i) = i$), for $i = 0, \ldots, n$. Then there are two numbers $0 \leq l \leq k \leq n$ such that

1. *$a_i = b_i$, for $i < l$,*
2. *$a_i <^+ b_i$, for $l \leq i \leq k$,*
3. *$a_i <^- b_i$, for $k + 1 = i \leq n$,*
4. *$a_i \not\bowtie b_i$, for $k + 2 \leq i \leq n$,*

or the roles of a and b are interchanged.

Proof. We can present the above conditions more visually as:

$$a_0 = b_0, \ldots, a_{l-1} = b_{l-1} \quad a_l <^+ b_l, \ldots a_k <^+ b_k$$

$$a_{k+1} <^- b_{k+1} \quad a_{k+2} \not\bowtie b_{k+2}, \ldots, a_n \not\bowtie b_n.$$

We will verify these conditions from the bottom up. Note that by strictness $<^{S_0,+}$ is a linear order. So either $a_0 = b_0$ or $a_0 \bowtie^+ b_0$. In the latter case $l = 0$. As $a \neq b$, then there is $i \leq n$ such that $a_i \neq b_i$. Let l be minimal such, i.e., $l = min\{i : a_i \neq b_i\}$. By Lemma 5.9 3., $a_l \bowtie^+ b_l$. So assume $a_l <^+ b_l$. We put $k = max\{i \leq n : a_i <^+ b_i\}$. If $k = n$, we are done. If $k < n$, then by Lemma 5.9 4., we have $a_{k+1} <^- b_{k+1}$. Then if $k+1<n$, by Lemma 5.9 5. 6. 7., $a_i \not\bowtie b_i$ for $k+2 \leq i \leq n$. This ends the proof. $\Box$

Having Proposition 5.10 we can define a relation $<^-_l$ (or simply $<_l$) on k-faces of any positive opetopic cardinal S, $l<k$, as follows: for $a,b \in S_k$, $a <_l b$ iff $\gamma^{(l)}(a) <^- \gamma^{(l)}(b)$.

Corollary 5.11. *Let S be a positive opetopic cardinal, $a,b \in S_n$, $a \neq b$. Then either $a \bowtie^+ b$ or there is a unique $0 \leq l \leq k$ such that $a \bowtie^-_l b$, but not both.*

The above corollary allows us to define an order $<^S$ (also denoted $<$) on all cells of S as follows: for $a,b \in S_n$,

$$a <^S b \quad \text{iff} \quad a <^+ b \;\text{ or }\; \exists_l \;\; a <^-_l b.$$

Corollary 5.12. *For any positive opetopic cardinal S, and $n \in \omega$, the relation $<^S$ restricted to S_n is a linear order.*

Proof. We need to verify that $<^S$ is transitive.

Let $a,b,c \in S_n$. There are some cases to consider.

If $a <^+ b <^+ c$, then clearly $a <^+ c$.

If $a <^+ b <^-_l c$, then, by Lemma 5.2.4., we have $\gamma^{(l)}(a) \leq^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$, and by transitivity of $<^-$ we have $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$. Hence $a <^-_l c$.

Now assume $a <^-_l b <^+ c$ and consider a lower path in S_l containing $\gamma^{(l)}(a)$ and $\gamma^{(l)}(b)$. By Lemma 5.9.5 $\gamma^{(l)}(b) <^+ \gamma^{(l)}(c)$, and hence by Lemma 5.7, either $\gamma^{(l)}(a) <^+ \gamma^{(l)}(c)$ or $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$. In the latter case, by transitivity of $<_l$ we have $a <_l c$, and we are done. In the former case, by Proposition 5.10, either $a <^+ c$ and we are done, or there is $k>l$ such that $\gamma^{(k)}(a) <^- \gamma^{(k)}(c)$, i.e. $a <_k c$, as required.

The last case $a <^-_k b <^-_l c$ has three subcases.

If $k=l$, then clearly $a <_l c$.

If $k>l$, then $\gamma^{(l)}(a) \leq^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$ and, by the previous argument, $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$, i.e., $a <^-_l c$.

Finally, assume $k<l$. Then $\gamma^{(k)}(a) <^- \gamma^{(k)}(b) <^+ \gamma^{(k)}(c)$. By Path Lemma, either $\gamma^{(k)}(a) <^- \gamma^{(k)}(c)$ or $\gamma^{(k)}(a) <^+ \gamma^{(k)}(c)$. In the former case we are done. In the latter case, by Proposition 5.10, either $a <^+ c$ or there is k', such that $k<k' \leq n$ and $\gamma^{(k')}(a) <^+ \gamma^{(k')}(c)$, as required. $\Box$

Lemma 5.13. *Let S be a positive opetopic cardinal, $a \in S_n$. Then the set*

$$\{b \in S_n \,:\, a \leq^+ b\}$$

is linearly ordered by $\leq^+$.

Proof. Suppose $a \leq^+ b, b'$. If we were to have $b <^-_l b'$ for some $l \leq n$ then, by Corollary 5.12 we would have $a <^-_l b'$ which is a contradiction. □

Corollary 5.14. *Any morphism of positive opetopic cardinals is one-to-one. Moreover, any automorphism of positive opetopic cardinals is an identity.*

Proof. By Corollary 5.12, the (strict, linear in each dimension) order $<^S$ is defined internally using relations $<^-$ and $<^+$ that are preserved by any morphism. Hence $<^S$ must be preserved by any morphism, as well. From this observation the corollary follows. □

Lemma 5.15. *Let S be a positive opetopic cardinal, $a, b \in S_n$. Then*

1. *if $\iota(a) \cap \iota(b) \neq \emptyset$, then $a \bowtie^+ b$;*
2. *if $\emptyset \neq \iota(a) \subset \iota(b) \neq \iota(a)$, then $a <^+ b$;*
3. *if $a \bowtie^- b$, then $\iota(a) \cap \iota(b) = \emptyset$.*

Proof. 2. is an easy consequence of 1. and Lemma 5.3. 3. is an easy consequence of 1. and Disjointness. We shall show 1.

Assume $u \in \iota(a) \cap \iota(b)$. Thus there are $x, y \in \delta(a)$ and $x', y' \in \delta(b)$ such that $\gamma(x) = \gamma(x') = u \in \delta(y) \cap \delta(y')$. If $x = x'$, then by pencil linearity $a \bowtie^+ b$, as required. So assume that $x \neq x'$. Again by pencil linearity $x \bowtie^+ x'$, say $x' <^+ x$. Thus there is an upper $(T - \gamma(T))$-path $x', a_1, \ldots, a_k, x$. As, for $i = 1, \ldots, k$, $\gamma\gamma(a_i) = u$ and $\gamma\gamma(b) \not\in \iota(b) \ni u$, we have that $\gamma(a_i) \neq \gamma(b)$ and $a_i \neq b$. Once again by pencil linearity $a_1 \bowtie^+ b$ and by Path Lemma $a_i < b$, for $i = 1, \ldots, k$ with $\gamma(a_k) \neq \gamma(b)$. As $\gamma(a_k) = x \in \delta(a)$, again by Path Lemma $a <^+ b$, as well. □

Proposition 5.16. *Let S be a positive opetopic cardinal, $a, b \in S_k$, $\alpha \in S_{k+1}$, so that α is a $<^+$-minimal element in S_{k+1}, and $a \in \delta(\alpha)$, $b = \gamma(\alpha)$. Then b is a $<^+$-successor of a.*

Proof. Assume that α is a $<^+$-minimal element in S_{k+1}. Suppose that there is $c \in S_k$ such that $a <^+ c <^+ b$. Thus we have an upper path

$$a, \beta_1, \ldots, \beta_i, c, \beta_{i+1}, \ldots, \beta_l, b.$$

Hence $\beta_1 <^- \beta_l$. Moreover, $a \in \delta(\beta_1) \cap \delta(\alpha)$ and $\gamma(\beta_l) = b = \gamma(\alpha)$. Thus both β_1 and β_l are $<^+$-comparable with α. Since α is $<^+$-minimal, we have $\alpha <^+ \beta_1, \beta_l$. By Lemma 5.13, $\beta_1 \bowtie^+ \beta_l$. But then we have $(\beta_1, \beta_l) \in \bowtie^+ \cap \bowtie^- \neq \emptyset$, contradicting disjointness. □

Proposition 5.17. *Let T be a positive opetopic cardinal and $X \subseteq T$ a subhypergraph of T. Then X is a positive opetopic cardinal iff the relation $<^{X,+}$ is the restriction of $<^{T,+}$ to X.*

Proof. Assume that X is a subhypergraph of a positive opetopic cardinal T. Then X satisfies axioms of globularity, disjointness, and strictness of the relations $<^{X_k,+}$ for $k > 0$.

Clearly, if $<^{X_k,+} = <^{T_k,+} \cap (X_k)^2$, then the relation $<^{X_0,+}$ is linear, the relations $<^{X_k,+}$, for $k > 0$, satisfy pencil linearity, i.e., X is a positive opetopic cardinal.

Now we assume that the subhypergraph X of positive opetopic cardinal T is a positive opetopic cardinal. We shall show that for $k \in \omega$, $a, b \in X_k$, we have $a <^{X_k,+} b$ iff $a <^{T_k,+} b$. Since X is a subhypergraph, $a <^{X_k,+} b$ implies $a <^{T_k,+} b$. Thus it is enough to show that if $a <^{T_k,+} b$ then $a \bowtie^{X_k,+} b$. We shall prove this by induction on k. For $k = 0$, it is obvious, since $<^{X_0,+}$ is linear. So assume that for faces $x, y \in X_l$, with $l < k$, we already know that $x <^{X_l,+} y$ iff $x <^{T_l,+} y$. Fix $a, b \in T_k$ such that $a <^{T_k,+} b$. Then by Lemma 5.9.2 $\gamma(a) \leq^{T_{k-1},+} \gamma(b)$ and hence by inductive hypothesis $\gamma(a) \leq^{X_{k-1},+} \gamma(b)$. Thus we have an upper $(X - \gamma(X))$-path $a = a_r, \gamma(a), a_{r-1} \ldots, a_1, \gamma(b)$, with $r \geq 1$. Since X is a positive opetopic cardinal and $\gamma(a_1) = \gamma(b)$, by pencil linearity we have $a_1 \leq^+ b$. By Path Lemma 5.7, either $a <^{X,-} b$ or $a <^{X,+} b$. Since the first option is impossible, we have $a <^{X,+} b$, as required. □

Lemma 5.18. *Let T be a positive opetopic cardinal, $a, b, \alpha \in T$. If $a \in \delta(\alpha)$ and $a <^+ b <^+ \gamma(\alpha)$, then $b \in \iota(T)$.*

Proof. Assume that $a, b, \alpha \in T$ are as in the assumption of the lemma. Thus we have an upper path $a, \alpha_0, \ldots, \alpha_r, b$. As $a \in \delta(\alpha) \cap \delta(\alpha_0)$, by pencil linearity we have $\alpha \bowtie^+ \alpha_0$. If $\alpha <^+ \alpha_0 <^- \alpha_r$, then $\gamma(\alpha) \leq^+ \gamma(\alpha_r) = b$, contradicting our assumption. Thus $\alpha_0 <^+ \alpha$. Then by Path Lemma 5.7, since $b = \gamma(\alpha_r) <^+ \gamma(\alpha)$, we have $\alpha_r <^+ \alpha$ and $b \in \iota(T)$, as required. $\Box$

Some equations

Proposition 5.19. *Let S be a positive opetopic cardinal $0 < k \in \omega$. Then*

1. $\iota(S_{k+1}) = \iota(S_{k+1} - \delta(S_{k+2}))$ *and* $\iota(S_{k+1}) = \iota(S_{k+1} - \gamma(S_{k+2}))$;
2. $\delta(S_k) = \delta(S_k - \gamma(S_{k+1}))$;
3. $\gamma(S_k) = \gamma(S_k - \gamma(S_{k+1}))$;
4. $\delta(S_k) = \delta(S_k - \iota(S_{k+2}))$;
5. $\delta(S_k) = \delta(S_k - \delta(S_{k+1})) \cup \iota(S_{k+1})$.

Proof. In all the above equations the inclusion $\supseteq$ is obvious. So in each case we need to check the inclusion $\subseteq$ only.

Ad 1. Both equalities follow from Lemma 5.3.

To prove the first equality, let $s \in \iota(S_{k+1})$, i.e., there is $a \in S_{k+1}$ such that $s \in \iota(a)$. By strictness, there is $b \in S_{k+1}$ such that $a \leq^+ b$ and $b \notin \delta(S_{k+1})$. By Lemma 5.3, we have

$$s \in \iota(a) \subseteq \iota(b) \subseteq \iota(S_{k+1} - \delta(S_{k+2}))$$

as required.

To prove the second equality, suppose on the contrary that there is $x \in \iota(S_{n+1})$ such that $x \notin \iota(S_{n+1} - \gamma(S_n))$. Let $a \in S_{n+1}$ be a $<^+$-minimal face such that $x \in \iota(a)$. Since $x \notin \iota(S_{n+1} - \gamma(S_n))$, there is $\alpha \in S_n$ such that $a = \gamma(\alpha)$. By Lemma 5.3 we have

$$\iota(a) = \iota\gamma(\alpha) = \iota\delta(\alpha).$$

Therefore, there is $a' \in \delta(\alpha)$ such that $x \in \iota(a')$. Clearly $a' \lhd^+ a$, and hence a is not $<^+$-minimal, contrary to the supposition. This ends the proof of the first equality above.

Ad 2. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-minimal element in S_k such that $x \in \delta(a)$. We shall show that $a \in S_k - \gamma(S_{k+1})$. Suppose on the contrary that there is an $\alpha \in S_{k+1}$ such that $a = \gamma(\alpha)$. Then by globularity

$$x \in \delta(a) = \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

So there is $b \in \delta(\alpha)$ such that $x \in \delta(b)$. As $b <^+ a$, this contradicts the minimality of a.

Ad 3. This is similar to the previous one but simpler.

Ad 4. Since $\iota(S_{k+2}) \subseteq \gamma(S_{k+1})$ 4. follows from 2.

Ad 5. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-largest element in S_k such that $x \in \delta(a)$. If $a \not\in \delta(S_{k+1})$, then $x \in \delta(S_k - \delta(S_{k+1}))$, as required. So assume that $a \in \delta(S_{k+1})$, i.e., there is $\alpha \in S_{k+1}$ such that $a \in \delta(\alpha)$. Thus $x \in \delta\delta(\alpha)$. As $a <^+ \gamma(\alpha)$, by choice of a we have $x \not\in \delta\gamma(\alpha)$ ($= \delta\delta(\alpha) - \gamma\delta(\alpha)$). So $x \in \gamma\delta(\alpha)$ and hence $x \in \iota(\alpha) \subseteq \iota(S_{k+1})$, as required. □

6. The ω-categories generated by the positive opetopic cardinals

The main objective of this section is to construct an embedding

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$$

of the category of opetopic cardinals into the category of ω-categories. This embedding is not full. In Section 9, we shall show that the image of $(-)^*$ factorizes through $\mathbf{pPoly} \to \omega Cat$ as a full functor.

Let T be a positive opetopic cardinal. By T^*_n we denote the set of all positive opetopic cardinals contained in T of dimension at most n. If one wants to make these sets disjoint, one can think that an element of T^*_n is a pair $\langle n, S\rangle$, where S is a positive opetopic cardinal contained in T. We define below an ω-category, denoted T^*, whose set of n-cells is T^*_n. We introduce operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : T^*_n \longrightarrow T^*_k$$

of the k-th *domain* and the k-th *codomain* (of an m-dimensional cell), where $0 \leq k \leq n$. For S in $(T^*)_m$, the faces of the k-th *domain* $\mathbf{d}^{(k)}(S)$ are:

1. $(\mathbf{d}^{(k)}(S))_l = \emptyset$, for $l > k$,
2. $(\mathbf{d}^{(k)}(S))_k = S_k - \gamma(S_{k+1})$,
3. $(\mathbf{d}^{(k)}(S))_l = S_l$, for $0 \leq l < k$.

and faces of the k-th *codomain* $\mathbf{c}^{(k)}(S)$ are:

1. $(\mathbf{c}^{(k)}(S))_l = \emptyset$, for $l > k$,
2. $(\mathbf{c}^{(k)}(S))_k = S_k - \delta(S_{k+1})$,
3. $(\mathbf{c}^{(k)}(S))_{k-1} = S_{k-1} - \iota(S_{k+1})$, if $k > 0$,
4. $(\mathbf{c}^{(k)}(S))_l = S_l$, for $0 \leq l < k-1$.

Note that the definitions of $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$, for $S \in T^*_n$ do not depend on the ambient opetopic cardinal T, nor even on $dim(S)$. Therefore we can write $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$ without specifying T. If $n \in \omega$ and $S \in T^*_{n+1}$, we write $\mathbf{d}(S)$ for $\mathbf{d}^{(n)}(S)$, and $\mathbf{c}(S)$ for $\mathbf{c}^{(n)}(S)$.

Lemma 6.1. *Let S and T be positive opetopic cardinals and $k \in \omega$. Then*

1. *if $k < dim(S)$, then both $\mathbf{d}^{(k)}(S)$, $\mathbf{c}^{(k)}(S)$ are positive opetopic cardinals of dimension k; if $k \geq dim(S)$, then $\mathbf{d}^{(k)}(S) = S = \mathbf{c}^{(k)}(S)$;*
2. *$\mathbf{d}\mathbf{d}^{(k+1)}(S) = \mathbf{d}^{(k)}(S)$, $\mathbf{c}\mathbf{c}^{(k+1)}(S) = \mathbf{c}^{(k)}(S)$;*
3. *if $S \in T^*_k$ and $k \geq 2$, then $\mathbf{dd}(S) = \mathbf{dc}(S)$, $\mathbf{cd}(S) = \mathbf{cc}(S)$;*
4. *for any $\alpha \in S_k$, the least sub-hypergraph of S containing the face α is again a positive opetopic cardinal of dimension k; it is denoted by $[\alpha]$. Moreover, if $k > 0$, then*

   $$\mathbf{c}[\alpha] = [\gamma(\alpha)], \qquad \mathbf{d}[\alpha] = [\delta(\alpha)]$$

   where $[\delta(\alpha)]$ is the least sub-hypergraph of S containing the set of faces $\delta(\alpha)$. [9]

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[9] (PLC) As usual, parentheses are sometimes omitted: $\mathbf{dd}(S)$ (or even $\mathbf{dd}S$) stands for $\mathbf{d}(\mathbf{d}(S))$, $\mathbf{c}[\alpha]$ stands for $\mathbf{c}([\alpha])$, etc.

Proof. Ad 1. It is obvious that $\mathbf{d}^{(k)}(S)$ is a sub-hypergraph S. By Corollary 5.6 $\mathbf{c}^{(k)}(S)$ is a sub-hypergraph S as well. Any sub-hypergraph T of a positive opetopic cardinal S satisfies the conditions of globularity, strictness (possibly without $<^{T_0,+}$ being linear), and disjointness.

By Lemma 5.4, for $a, b \in \mathbf{d}^{(k)}S_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{d}^{(k)}S_l,+} b$. Moreover, by Lemma 5.8, for $a, b \in \mathbf{c}^{(k)}(S)_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{c}^{(k)}(S)_l,+} b$. Hence by Proposition 5.17 both $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$ are positive opetopic cardinals.

Ad 2. Fix a positive opetopic cardinal S and $k \in \omega$ such that $dim(S) > k$. Then the faces of $\mathbf{c}^{(k+1)}(S)$, $\mathbf{cc}^{(k+1)}(S)$, and $\mathbf{c}^{(k)}(S)$ are as in the table

dim	$\mathbf{c}^{(k+1)}(S)$	$\mathbf{cc}^{(k+1)}(S)$	$\mathbf{c}^{(k)}(S)$
$k+1$	$S_{k+1} - \delta(S_{k+2})$	$\emptyset$	$\emptyset$
k	$S_k - \iota(S_{k+2})$	$(S_k - \iota(S_{k+2}))$ $-\delta(S_{k+1} - \delta(S_{k+2}))$	$S_k - \delta(S_{k+1})$
$k-1$	S_{k-1}	$S_{k-1} - \iota(S_{k+1} - \delta(S_{k+2}))$	$S_{k-1} - \iota(S_{k+1})$
l	S_l	S_l	S_l

where $l < k-1$. Moreover, the faces of $\mathbf{d}^{(k+1)}(S)$, $\mathbf{dd}^{(k+1)}(S)$, and $\mathbf{d}^{(k)}(S)$ are as in the table

dim	$\mathbf{d}^{(k+1)}(S)$	$\mathbf{dd}^{(k+1)}(S)$	$\mathbf{d}^{(k)}(S)$
$k+1$	$S_{k+1} - \gamma(S_{k+2})$	$\emptyset$	$\emptyset$
k	S_k	$S_k - \gamma(S_{k+1} - \gamma(S_{k+2}))$	$S_k - \gamma(S_{k+1})$
l	S_l	S_l	S_l

where $l < k$. Thus the equalities in question all follow from Lemma 5.19.

Ad 3. Let $dim(S) = n > 1$. Note that both $(\mathbf{dd}(S))_{n-2}$ and $(\mathbf{dc}(S))_{n-2}$ are the sets of all $<^+$-minimal elements in S_{n-2}, i.e., they are equal and the equation $\mathbf{dd}(S) = \mathbf{dc}(S)$ holds.

To see that $\mathbf{cd}(S) = \mathbf{cc}(S)$ holds, note first that both $(\mathbf{cd}(S))_{n-2}$ and $(\mathbf{cc}(S))_{n-2}$ are the sets of all $<^+$-maximal elements in S_{n-2}. Moreover

$$(\mathbf{cd}(S))_{n-3} = S_{n-3} - \iota(S_{n-1} - \gamma(S_n))$$

$$(\mathbf{cc}(S))_{n-3} = S_{n-3} - \iota(S_{n-1} - \delta(S_n)).$$

Now the equality $\mathbf{cd}(S) = \mathbf{cc}(S)$ follows from the following equalities

$$\iota(S_{n-1} - \gamma(S_n)) = \iota(S_{n-1}) = \iota(S_{n-1} - \delta(S_n)),$$

that themselves follow from Lemma 5.19.1.

Ad 4. Fix $\alpha \in S_k$. We need to show that $[\alpha]$ is a positive opetopic cardinal. The globularity, strictness (except for linearity of $<^{[\alpha]_0,+}$), and disjointness are clear.

The linearity of $<^{[\alpha]_0,+}$. If $k \leq 2$, it is obvious. Put $a = \gamma^{(k+2)}(\alpha)$. Using Corollary 4.3, we have

$$[\alpha]_k = \delta^{(k)}(\alpha) \cup \gamma^{(k)}(\alpha) =$$

$$= \delta\delta(\gamma^{(k+2)}(\alpha)) \cup \gamma\gamma(\gamma^{(k+2)}(\alpha)) = \delta\delta(a) \cup \gamma\gamma(a).$$

Thus it is enough to check the linearity of $<^{[\alpha]_0,+}$ for α of dimension $k = 2$. But in this case, as we mentioned, the linearity of $<^{[\alpha]_0,+}$ is obvious.

The γ-linearity of $[\alpha]$. The proof proceeds by induction on $k = dim(\alpha)$. For $k \leq 2$, the γ-linearity is obvious. So assume that $k > 2$ and that for $l < k$ and $a \in S_l$ γ-linearity holds in $[a]$.

First we shall show that $\mathbf{c}[\alpha] = [\gamma(\alpha)]$. We have

$$\mathbf{c}[\alpha]_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \delta(\alpha) = \gamma(\alpha) = [\gamma(\alpha)]_{k-1}$$

$$\mathbf{c}[\alpha]_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) - \iota(\alpha) = \gamma\gamma(\alpha) \cup \delta\gamma(\alpha) = [\gamma(\alpha)]_{k-2},$$

and for $l < k - 2$

$$\mathbf{c}[\alpha]_l = \gamma^{(l)}(\alpha) \cup \delta^{(l)}(\alpha) = \gamma^{(l)}(\alpha) \cup \delta^{(l)}\gamma(\alpha) = [\gamma(\alpha)]_l.$$

Note that the definition of $\mathbf{c}(H)$ makes sense for any positive hypergraph H and in the above argument we haven't used the fact (which we don't know yet) that $[\alpha]$ is a positive opetopic cardinal.

Thus, for $l < k - 2$, $[\alpha]_l = [\gamma(\alpha)]_l$. By induction hypothesis, $[\gamma(\alpha)]$ is a positive opetopic cardinal, and hence $[\alpha]_l$ is γ-linear for $l < k - 2$. Clearly $[\alpha]_l$ is γ-linear for $l = k - 1, k$. Thus it remains to show the γ-linearity of $(k-2)$-cells in $[\alpha]$.

Fix $t \in [\alpha]_{k-3}$, and let

$$\Gamma_t = \{x \in [\alpha]_{k-2} : \gamma(x) = t\}.$$

We need to show that Γ_t is linearly ordered by $<^+$. We can assume $t \in \gamma([\alpha]_{n-2}) = \gamma\delta\delta(\alpha) = \gamma\delta\gamma(\alpha)$ (otherwise $\Gamma_t = \emptyset$ is clearly linearly ordered by $<^+$). By Proposition 5.1 there is a unique $x_t \in \delta\gamma(\alpha)$ such that $\gamma(x_t) = t$. From Lemma 5.2.2 we get easily the following claim.

Claim 1. For every $x \in \Gamma_t$ there is a unique upper $\delta(\alpha)$-path from x_t to x.

Now fix $x, x' \in \Gamma_t$. By Claim 1, we have the unique upper $\delta(\alpha)$-path

$$x_t, a_0, \ldots, a_l, x, \quad x_t, a'_0, \ldots, a'_{l'}, x'.$$

Suppose $l \leq l'$. By Proposition 5.1, for $i \leq l$, we have $a_i = a'_i$. Hence either $l = l'$ and $x = x'$ or $l < l'$ and

$$x, a_{l+1}, \ldots, a_{l'}, x'$$

is a $\delta(\alpha)$-upper path. Hence either $x = x'$ or $x \bowtie^+ x'$ and $[\alpha]_{k-2}$ satisfy the γ-linearity, as required.

The proof of the δ-linearity of $[\alpha]$ is very similar to the one above. For the same reasons the only non-trivial thing to check is the condition for $(k-2)$-faces. We pick $t \in \delta\delta(\alpha)$ and consider the set

$$\Delta_t = \{x \in [\alpha]_{k-2} : t \in \delta(x)\}.$$

Then we have a unique $y_t \in \delta\gamma(\alpha)$ such that $t \in \delta(y_t)$. From Lemma 5.2.3 we get the following claim.

Claim 2. For every $y \in \Delta_t$ there is a unique upper $\delta(\alpha)$-path from y_t to y.

The δ-linearity of the $(k-2)$-faces in $[\alpha]$ can be proved from Claim 2 in the same way as the γ-linearity was proved from Claim 1.

It remains to verify the equalities

$$\mathbf{c}[\alpha] = [\gamma(\alpha)] \qquad \mathbf{d}[\alpha] = [\delta(\alpha)].$$

We already checked the first one on the way. To see that the second equality also holds we calculate

$$\mathbf{d}[\alpha]_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \gamma(\alpha) = \delta(\alpha) = [\delta(\alpha)]_{k-1}$$

$$\mathbf{d}[\alpha]_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) = \gamma\delta(\alpha) \cup \delta\delta(\alpha) = [\delta(\alpha)]_{k-2},$$

and for $l < k-2$

$$\mathbf{d}[\alpha]_l = \gamma^{(l)}(\alpha) \cup \delta^{(l)}(\alpha)) = \gamma^{(l)}\delta(\alpha) \cup \delta^{(l)}(\alpha) = [\delta(\alpha)]_l.$$

So the second equality holds as well. □

Remarks.

1. Inspired by the above Lemma 6.1.4 we call S a *weak positive opetopic cardinal* if S satisfies globularity, strictness, and disjointness as a positive hypergraph and if moreover for any face α in S the sub-hypergraph $[\alpha]$ is an opetope. (i.e., pencil linearity is required to hold only 'locally'). The *category of weak positive opetopic cardinals* is the full subcategory of the category of positive hypergraphs $\mathbf{pHg}$ whose objects are the weak positive opetopic cardinals and is denoted by $\mathbf{wpOpeCard}$. For each $k \in \omega$, the k-truncation of a weak positive opetopic cardinal S is again a weak positive opetopic cardinal $S_{\leq k}$. In particular, any k-truncation of a positive opetopic cardinal S is a weak positive opetopic cardinal $S_{\leq k}$, but it does not necessarily satisfy the linearity condition.

2. From Lemma 6.1.1 we know that for any positive opetopic cardinal S the hypergraphs $\mathbf{c}^{(k)}(S)$ and $\mathbf{d}^{(k)}(S)$ are positive opetopic cardinals contained in S. We shall denote these embeddings by

$$\begin{tikzcd}
\mathbf{c}^{(k)}(S) \arrow[r, "\mathbf{c}^{(k)}_S"] & S & \mathbf{d}^{(k)}(S). \arrow[l, "\mathbf{d}^{(k)}_S"']
\end{tikzcd}$$

Lemma 6.2. *Let X and Y be positive opetopic cardinals.*

1. If $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}(Y)$. If we have moreover $\mathbf{c}^{(k)}(X) = X \cap Y$, then the diagram

$$\begin{tikzcd}
Y \cup X & X \arrow[l] \\
Y \arrow[u] & \mathbf{c}^{(k)}(X) \arrow[l, "\mathbf{d}^{(k)}_Y"] \arrow[u, "\mathbf{c}^{(k)}_X"']
\end{tikzcd}$$

of inclusions in $\mathbf{pOpeCard}$ *is a pushout. (Here,* $\cap$ *and* $\cup$ *are level-wise set intersection and union, respectively.)*

2. *If* $\mathbf{c}^{(k)}(X)$ *and* $\mathbf{d}^{(k)}(Y)$ *are isomorphic, then the pushout* $X \oplus_k Y$ *in* $\mathbf{pOpeCard}$ *of* X *and* Y *over* $\mathbf{c}^{(k)}(X)$ *exists.*

3. *If there exists a positive opetopic cardinal* T *such that* $X, Y \in S^*$ *and if* $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$*, then* $\mathbf{c}^{(k)}(X) = X \cap Y$. [10]

Proof. Ad 1. Assume $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}Y$. The fact that $Y \cup X$ is a pushout in $\mathbf{pHg}$ is obvious. Thus the only thing we need to verify is that $Y \cup X$ is a positive opetopic cardinal.

First we write in detail the condition $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}Y$:

1. $X_l \cap Y_l = \emptyset$, for $l > k$,
2. $X_k - \delta(X_{k+1}) \subseteq Y_k - \gamma(Y_{k+1})$,
3. $X_{k-1} - \iota(X_{k+1}) \subseteq Y_{k-1}$,
4. $X_l \subseteq Y_l$, for $l < k - 1$.

Now we describe the orders $<^+$ in $Y \cup X$:

$$<^{(Y\cup X)_l,+} = \begin{cases} <^{X_l,+} + <^{Y_l,+} & \text{for } l > k \\ <^{X_l,+} +_{(X_k - \delta(X_{k+1}))} <^{Y_l,+} & \text{for } l = k \\ <^{X_l,+} +_{(X_{k-1} - \iota(X_{k+1}))} <^{Y_l,+} & \text{for } l = k-1 \\ <^{Y_l,+} & \text{for } l <= k-1. \end{cases}$$

We shall comment on these formulas. For $l > k$, the formulas say that the order $<^+$ in $(Y \cup X)_l$ is the disjoint sum of the orders in X_l and Y_l. This is obvious.

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[10] (PLC) Point 3. was not present in the original submission. It explains why the objects of the ω-category S^* defined at the end of this section can be taken to be opetopic cardinals, while the objects of the terminal polygraph described in Section 11 have to be opetopic cardinals up-to-iso. As a hint, point 3. is an easy consequence of the following properties, that hold for all opetopic cardinals S: (i) for every $x \in S_k \setminus \mathbf{d}^{(k)}S$, there exists $y \in (\mathbf{d}^{(k)}S)_k$ such that $y <^+ x$; (ii) for every of $x \in S_k \setminus \mathbf{c}^{(k)}S$, there exists $y \in (\mathbf{c}^{(k)}S)_k$ such that $x <^+ y$; (iii) for every distinct cells x_1, x_2 of $(\mathbf{d}^{(k)}S)_k$, we have that x_1 and x_2 are incomparable for $<^{S,+}$; (iv) $S_{<k} \subseteq \mathbf{d}^{(k)}S$; (v) for every $x \in S_{>k}$ and every $y \in (\mathbf{c}^{(k)}x)_k$, there exists a cell $z \in (\mathbf{d}^{(k)}x)_k$ such that $z <^+ x$; (vi) if $S \in T^*$ for some T, and if $x, y \in S$ are such that $x <^{T,+} y$, then $x <^{S,+} y$.

For $l < k-1$, the order $<^+$ in $(X\cap Y)_l$ is just the order $<^{Y_l,+}$. The only case that requires an explanation is $l = k-2$. So suppose that $a, b \in Y_{k-2}$ and $a <^{(Y\cup X)_{k-2},+} b$. So we have an upper path

$$a, \alpha_1, \ldots, \alpha_m, b$$

such that $\alpha_i \in (Y\cup X)_{k-1} = \iota(X_{k+1}) \cup Y_{k-1}$. By Lemma 5.4, we can assume that if $\alpha_i \in X_{k-1}$, then $\alpha_i \not\in \gamma(X_k)$. But then $\alpha_i \not\in \iota(X_{k+1})$. So in fact $\alpha_i \in Y_{k-1}$, as required.

The most involved are the formulas for $<^{(X\cap Y)_l,+}$, for $l = k$ and $l = k-1$. In both cases the comparison in $Y\cup X$ involves orders both from X and Y. In the former case, for $a, b \in (Y\cup X)_k$, we have

$$a <^{(Y\cup X)_k,+} b \text{ iff}$$

$$\begin{cases} \text{either } a, b \in Y_k \text{ and } a <^{Y_k,+} b \\ \text{or } a, b \in X_k \text{ and } a <^{X_k,+} b \\ \text{or } a \in \delta(X_{k+1}),\ b \in Y_k \text{ and } \exists_{a' \in X_k - \delta(X_{k+1})} a <^{X_k,+} a' \text{ and } a' \leq^{Y_k,+} b. \end{cases}$$

The orders $<^{X_k,+}$ and $\leq^{Y_k,+}$ are glued together along $X_k - \delta(X_{k+1})$ which is the set of $<^{X_k,+}$-maximal elements in X_k and at the same time it is contained in the set of $<^{Y_k,+}$-minimal elements $Y_k - \gamma(Y_{k+1})$. This is obvious when we realize that $\delta(X_{k+1}) \cap \gamma(Y_{k+1}) = \emptyset$.

In the latter case, for $x, y \in (Y\cup X)_{k-1}$, we have

$$x <^{(Y\cup X)_{k-1},+} y \text{ iff}$$

$$\begin{cases} \text{either } x, y \in X_{k-1} \text{ and } x <^{X_{k-1},+} y \\ \text{or } x, y \in Y_{k-1} \text{ and } x <^{Y_{k-1},+} y \\ \text{or } x \in \iota(X_{k+1}),\ y \in Y_k \text{ and } \exists_{x' \in X_{k-1} - \iota(X_{k+1})} x <^{X_k,+} x' \text{ and } x' \leq^{Y_k,+} y \\ \text{or } x \in Y_k,\ y \in \iota(X_{k+1}) \text{ and } \exists_{x' \in X_{k-1} - \iota(X_{k+1})} x <^{Y_k,+} x' \text{ and } x' \leq^{X_k,+} y. \end{cases}$$

The order $<^{X_{k-1},+}$ is ‘plugged into’ the order $\leq^{Y_{k-1},+}$, along the set $X_k - \iota(X_{k+1})$.

To show that these formulas hold true, we argue by cases. Assume that $x, y \in (Y\cup X)_{k-1}$ and that $x <^{(Y\cup X)_{k-1},+} y$, i.e., there is an upper path

$$x, a_1, \ldots, a_m, y$$

with $a_i \in (Y\cup X)_k$, for $i = 1, \ldots, m$.

First suppose $x, y \in X_{k-1}$ and $\{a_i\}_i \not\subseteq X_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0 \leq i \leq i_1} \subseteq Y_k$. Thus it is an upper path in Y_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1 \\ \gamma(a_{i_0-1}) & \text{otherwise.} \end{cases}$$

Note that it follows from the maximality of the path $a_{i_0}, \ldots, a_{i_1}$ that $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1})$. As we have $\bar{x} <^{Y_{k-1},+} \bar{y}$ from Corollary 5.11, we have $\bar{x} \not\bowtie^{Y_l,-} \bar{y}$, for all $l < k-1$. Clearly $\bowtie^{X_l,-} \subseteq \bowtie^{Y_l,-}$. Thus $\bar{x} \not\bowtie^{X_l,-} \bar{y}$, for all $l < k-1$, as well. But then again by Corollary 5.11 we have that $\bar{x} \bowtie^{X_{k-1},+} \bar{y}$. If we were to have $\bar{y} <^{X_{k-1},+} \bar{x}$, then, as $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1})$, we would have $\bar{y} <^{Y_{k-1},+} \bar{x}$. But this would contradict the strictness of $<^{Y_{k-1},+}$. So we must have $\bar{x} <^{X_{k-1},+} \bar{y}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(Y \cup X)_k$ from x to y by an upper path from x to y in X_k.

Next, suppose $x, y \in Y_{k-1}$ and $\{a_i\}_i \not\subseteq Y_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0 \leq i \leq i_1} \subseteq X_k$. Thus it is an upper path in X_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1 \\ \gamma(a_{i_0-1}) & \text{otherwise.} \end{cases}$$

Note that $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1}) \subseteq Y_{k-1}$ follows from the maximality of the sequence $a_{i_0}, \ldots, a_{i_1}$. Thus by Lemma 5.8 there is an upper path from $\bar{x}$ to $\bar{y}$ in $X_{k-1} - \delta(X_k) \subseteq Y_{k-1}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(Y \cup X)_k$ from x to y by an upper path from x to y in Y_k.

Thus we have justified the first two cases of the above formula. The following two cases are easy consequences of these two. This ends the description of the orders in $Y \cup X$.

From these descriptions follows immediately that $<^{(Y \cup X),+}$ is strict, for all l. It remains to show the pencil linearity. Both γ- and δ-linearity of l-cells, for $l < k-1$ or $l > k$, are obvious.

To see the γ-linearity of k-cells, suppose that we have $a \in X_k$ and $b \in Y_k$ such that $\gamma(a) = \gamma(b)$. Let $\bar{a} \in X_k$ be the $<^{X_k,+}$-maximal k-cells such that $\gamma(a) = \gamma(\bar{a})$. Then $\bar{a} \in \mathbf{c}^{(k)}(X)_k \subseteq \mathbf{d}^{(k)}(Y)_k$. So $\bar{a} \in Y_k$ is a $<^{Y_k,+}$-minimal k-cell such that $\gamma(\bar{a}) = \gamma(b)$. Thus

$$a \leq^{X_k,+} \bar{a} \leq^{Y_k,+} b.$$

Thus the γ-linearity of k-cells holds. The proof of δ-linearity of k-cells is similar.

Finally, we need to establish the γ- and δ-linearity of $(k-1)$-cells in $Y \cup X$.

In order to prove the γ-linearity, let $x \in \iota(X_{k+1})$ and $y \in Y_{k-1}$ such that $\gamma(x) = \gamma(y)$. We need to show that $x \bowtie^{(Y \cup X)_{k-1},+} y$.

Let $\alpha_0 \in X_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$ and let $\alpha_0, \ldots, \alpha_l$ be a lower path in X_{k+1} such that $\gamma(\alpha_l) \in Y_k$. Since $x \in \iota(\alpha_0)$, then $x \in \gamma\delta(\alpha_0)$ and, by Lemma 4.2

$$\gamma(x) \in \gamma\gamma\delta(\alpha_0) \subseteq \iota\gamma(\alpha_0) \cup \gamma\gamma\gamma(\alpha_0).$$

As $\gamma(\alpha_0) \leq^+ \gamma(\alpha_l)$, by Lemma 5.3.3, we have $\gamma(x) \in \iota\gamma(\alpha_l) \cup \gamma\gamma\gamma(\alpha_l)$. Thus we have two cases:

1. $\gamma(x) \in \iota\gamma(\alpha_l)$,
2. $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$.

Case 1: $\gamma(x) \in \iota\gamma(\alpha_l)$. By Lemma 5.2.2, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $\gamma(z) = \gamma(x)$ and $z <^+ x$. As $\gamma(\alpha_l) \in Y_k$, so $z \in Y_{k-1}$. If $y <^{Y_{k-1},+} z$, then indeed $y <^{(Y \cup X)_{k-1},+} z$, as required. By γ-linearity in Y_{k-1}, it is enough to show that it is impossible to have $z <^{Y_{k-1},+} y$.

Suppose on the contrary that there is an upper path $z, b_0, \ldots, b_r, y$ in Y. Since $\gamma(\alpha_l)$ is $<^+$-minimal in Y (as $\alpha_l \in X$) and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity in Y_k we have $\gamma(\alpha_l) <^+ b_0$. By Lemma 5.3.2, we have

$$\gamma(x) \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \iota(b_r).$$

But $\gamma(b_r) = y$ so $\gamma\gamma(b_r) = \gamma(y) = \gamma(x)$. In particular, $\gamma(x) \notin \iota(b_r)$ and we get a contradiction.

Case 2: $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$. By Lemma 5.2.2 there is $z \in \delta\gamma(\alpha_l)$ such that $\gamma(x) = \gamma(z) (= \gamma\gamma\gamma(\alpha_l))$, so that we have

$$z <^{X_{k-1},+} x <^{X_{k-1},+} \gamma\gamma(\alpha_l).$$

As $\gamma(\alpha_l) \in Y_k$ and is $<^+$-minimal in Y_k, by Proposition 5.16, there is no face $y' \in Y_{k-1}$ so that

$$z <^{Y_{k-1},+} y' <^{Y_{k-1},+} \gamma\gamma(\alpha_l).$$

So if $y \in Y_{k-1}$ and $\gamma(y) = \gamma(x)$, then either

$$y \leq^{Y_{k-1},+} z <^{X_{k-1},+} x \quad \text{or} \quad x <^{X_{k-1},+} \gamma\gamma(\alpha_l) \leq^{X_{k-1},+} y.$$

In either case $x \bowtie^{(Y\cup X)_{k-1},+} y$, as required. This ends the proof of γ-linearity of $(k-1)$-faces in $(Y \cup X)$.

Finally, we prove the δ-linearity of $(k-1)$-faces in $Y \cup X$. Let $x \in \iota(X_{k+1})$ and $y \in Y_{k-1}$, $t \in Y_{k-2}$ such that $t \in \delta(x) \cap \delta(y)$. We need to show that $x \bowtie^{(Y\cup X)_{k-1},+} y$.

Let $\alpha_0 \in X_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$, and let $\alpha_0, \ldots, \alpha_l$ be a lower path in X_{k+1} such that $\gamma(\alpha_l) \in Y_k$. As $x \in \iota(\alpha_0)$, using Lemma 4.2 we have

$$t \in \delta(x) \subseteq \delta\gamma\delta(\alpha_0) \subseteq \delta\gamma\gamma(\alpha_0) \cup \iota\gamma(\alpha_0).$$

As $\gamma(\alpha_0) <^+ \gamma(\alpha_l)$, by Lemma 5.3.4, we have two cases:

1. $t \in \iota\gamma(\alpha_l)$,
2. $t \in \delta\gamma\gamma(\alpha_l)$.

Case 1: $t \in \iota\gamma(\alpha_l)$. By Lemma 5.2.3, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $t \in \delta(z)$ and $z <^+ x$. As $\gamma(\alpha_l) \in Y_k$, so $z \in Y_{k-1}$. If $y <^{Y_{k-1},+} z$, then indeed $y <^{(Y\cup X)_{k-1},+} z$, as required. By δ-linearity in Y_k, it is enough to show that it is impossible to have $z <^{Y_{k-1},+} y$.

Suppose on the contrary that there is an upper path in Y

$$z, b_0, \ldots, b_r, y.$$

Since $\gamma(\alpha_l)$ is $<^+$-minimal in Y_k and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity of k-faces in Y we have $\gamma(\alpha_l) <^+ b_0$. By Lemma 5.3.2, we have

$$t \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \ldots \subseteq \iota(b_r).$$

But $\gamma(b_r) = y$, so $t \in \delta(y) \subseteq \delta\gamma(b_r)$. In particular, $t \notin \iota(b_r)$ and we get a contradiction.

Case 2: $t \in \delta\gamma\gamma(\alpha_l)$. By Lemma 5.2.3 there is $z \in \delta\gamma\gamma(\alpha_l)$ such that $t \in \delta(z)$ and we have

$$z <^{X_{k-1},+} x <^{X_{k-1},+} \gamma\gamma(\alpha_l).$$

As $\gamma(\alpha_l) \in Y_k$, and it is $<^+$-minimal face in Y_k, by Lemma 5.16, there is no face $y' \in Y_{k-1}$ such that

$$z <^{Y_{k-1},+} y' <^{Y_{k-1},+} \gamma\gamma(\alpha_l).$$

So if $y \in Y_{k-1}$ and $t \in \delta(y)$, then either

$$y \leq^{Y_{k-1},+} z <^{X_{k-1},+} x \quad \text{or} \quad x <^{X_{k-1},+} \gamma\gamma(\alpha_l) \leq^{X_{k-1},+} y.$$

In either case $x \bowtie^{(Y\cup X)_{k-1},+} y$, as required. This ends the proof of δ-linearity of $(k-1)$-faces in $(Y \cup X)$ and the whole proof that $Y \cup X$ is a positive opetopic cardinal.

Ad 2. We note that we can rename the cells of $Y \setminus (\mathbf{d}^{(k)}Y)$ to ensure disjointness with X and rename the cells of $(\mathbf{d}^{(k)}Y)$ to turn the codomain-domain isomorphism into an equality, and then apply point 1. $\Box$

Let X and Y be positive opetopic cardinals such that $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$. Then the pushout just described

$$\begin{tikzcd}
Y \oplus_k X & X \arrow[l] \\
Y \arrow[u] & \mathbf{c}^{(k)}(X) \arrow[l, "\mathbf{d}^{(k)}_Y"'] \arrow[u, "\mathbf{c}^{(k)}_X"']
\end{tikzcd}$$

is a *special pushout* in $\mathbf{pOpeCard}$ (or a *special pullback* in $\mathbf{pOpeCard}^{op}$).

Now we shall describe an ω-category T^* generated by the positive opetopic cardinal T. The set of m-cells of T^* is T^*_m, i.e., the set of all the positive opetopic cardinals contained in T of dimension at most m, for $m \in \omega$. The *k-th domain* and *k-th codomain* operations in T^* are the operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : T^*_m \longrightarrow T^*_k$$

defined above, with $m \geq k$. The *identity* operations

$$\mathbf{i}^{(m)} : T^*_k \longrightarrow T^*_m$$

are inclusions, and the composition map

$$\mathbf{m}_{m,k,m} : T^*_m \times_{T^*_k} T^*_m \longrightarrow T^*_m,$$

where $k < m$, is the sum, i.e., if X, Y are positive opetopic cardinals contained in T of dimension at most m such that $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$, then

$$\mathbf{m}_{m,k,m}(X,Y) = X \oplus_k Y = X \cup Y.$$

Corollary 6.3. *Let T be a weak positive opetopic cardinal. Then T^* is an ω-category. In fact, we have a functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \omega Cat.$$

Proof. The fact that the operations on T^* are well defined follows from Lemmas 6.1 and 6.2. The satisfaction of the laws of ω-categories is a simple matter of rearrangements of unions.

If $f : S \to T$ is a morphism of positive opetopic cardinals, $X \in S^*$, then the image $f(X) \in T^*$ is isomorphic to X. Then again using Lemmas 6.1 and 6.2, the association $X \mapsto f(X)$ is easily seen to be an ω-functor. □

7. Normal positive opetopic cardinals

Let S be a normal positive opetopic cardinal of dimension k, i.e., S is $(k-1)$-principal. By $\mathbf{p}^S_l$ we denote the unique element of the set $S_l - \delta(S_{l+1})$, for $l < k$. Moreover, as we shall show below, $\mathbf{p}^S_{k-1} \in \gamma(S_k)$ and hence the set $\{x \in S_k : \gamma(x) = \mathbf{p}^S_{k-1}\}$ is not empty. We denote by $\mathbf{p}^S_k$ the $<^+$-largest element of this set. We shall omit the superscript S if it does not lead to a confusion.

Lemma 7.1. *Let S be a $(k-1)$-principal opetope of dimension at least k, $k > 0$. Then*

1. $S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$, *for* $l < k$.
2. $\delta(S_{l+1}) = \delta^{(l)}(S_k)$, *for* $l < k$.
3. $\mathbf{p}_k$ *is the* $<^-$*-largest element in* $S_k - \delta(S_{k+1})$.
4. $\gamma(\mathbf{p}_l) = \mathbf{p}_{l-1}$, *for* $0 < l \leq k$.
5. $\delta(\mathbf{p}_l) = \delta(S_l) - \gamma(S_l)$, *for* $0 < l < k$.

6. $S_l = \delta^{(l)}(\mathbf{p}_{k-1}) \cup \gamma^{(l)}(\mathbf{p}_{k-1})$, *for* $l < k-2$.

Proof. Ad 1. If H is a hypergraph of dimension greater than l and $\gamma(H_{l+1}) \subseteq \delta(H_{l+1})$, then there is an infinite lower path in H_{l+1}, i.e., $<^{H_l,+}$ is not strict. Thus, if S is a positive opetopic cardinal of dimension greater than l, we have $\delta(S_{l+1}) \subsetneq S_l$. A positive opetopic cardinal is normal iff this difference

$$S_l - \delta(S_{l+1})$$

is a singleton, for $l < k$. Thus, by the above, we must have

$$S_l = \delta(S_{l+1}) \cup \gamma(S_{l+1}). \tag{1}$$

We shall show the first equation of the statement 1. by downward induction on l. Suppose that we have $S_{l+1} = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k)$ (for $l = k-2$, it is true by the above). Then

$$S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) =$$

$$\begin{aligned} &= \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) \cup \gamma(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) = \\ &= \delta\delta^{(l+1)}(S_k) \cup \delta\gamma^{(l+1)}(S_k) \cup \gamma\delta^{(l+1)}(S_k) \cup \gamma\gamma^{(l+1)}(S_k) = \\ &= \delta^{(l)}(S_k) \cup \delta\gamma^{(l)}(S_k) \cup \gamma\delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \\ &= \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k), \end{aligned}$$

where the last equation follows from Corollary 4.3.

The second equation of 1. is obvious, for $l = k-1$. So assume $l < k-1$. We have

$$\begin{aligned} \{\mathbf{p}_l\} &= S_l - \delta(S_{l+1}) = \\ &= S_l - \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) = \\ &= S_l - (\delta^{(l)}(S_k) \cup \delta\gamma^{(l+1)}(S_k)) = \\ & S_l - \delta^{(l)}(S_k). \end{aligned}$$

Thus

$$S_l = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$$

as required.

Ad 2. Let $l < k$. Then using 1. we have

$$\delta^{(l)}(S_k) \subseteq \delta(S_{l+1}) \subsetneq \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}.$$

Hence

$$\delta^{(l)}(S_k) = \delta(S_{l+1}).$$

Ad 3. First we shall show that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$. Suppose on the contrary that there is $\alpha \in S_{k+1}$ such that $\mathbf{p}_k \in \delta(\alpha)$. Then $\gamma(\mathbf{p}_k) \in \gamma\delta(\alpha) = \gamma\gamma(\alpha) \cup \iota(\alpha)$. If $\gamma(\mathbf{p}_k) = \gamma\gamma(\alpha)$, then $\mathbf{p}_k <^+ \gamma(\alpha)$, contradicting the choice of $\mathbf{p}_k$. If $\gamma(\mathbf{p}_k) = \iota(\alpha)$, then there is $a \in \delta(\alpha)$ such that $\gamma(\mathbf{p}_k) \in \delta(a)$. But this means that $\mathbf{p}_{k-1} = \gamma(\mathbf{p}_k) \in \delta(S_k)$, contradicting the choice of $\mathbf{p}_{k-1} \in S_{k-1} - \delta(S_k)$. This shows that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$.

We need to prove that any maximal lower $(S_k - \delta(S_{k+1}))$-path ends at $\mathbf{p}_k$. By strictness, it is enough to show that if $x \in S_k - \delta(S_{k+1})$ and $x \neq \mathbf{p}_k$, then there is $x' \in S_k - \delta(S_{k+1})$ such that $\gamma(x) \in \delta(x')$. So fix $x \in S_k - \delta(S_{k+1})$. If we were to have $\gamma(x) \in \iota(\beta)$, for some $\beta \in S_{k+1}$, then by Lemma 5.5 we would have $x <^+ \gamma(\beta)$, and in particular $x \in \delta(S_{k+1})$, contrary to the assumption. Therefore $\gamma(x) \in S_{k-1} - \iota(S_{k+1})$. As $x, \mathbf{p}_k \in S_k - \delta(S_{k+1})$, by γ-linearity we have $\gamma(x) \neq \gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$. Hence by 1. the set

$$\Delta_{\gamma(x)} = \{y \in S_k : \gamma(x) \in \delta(y)\}$$

is not empty. Let x' be the $<^+$-largest element of this set. It remains to show that $x' \notin \delta(S_{k+1})$. Suppose on the contrary that there is $\alpha \in S_{k+1}$ such that $x' \in \delta(\alpha)$. As $\gamma(x) \notin \iota(S_{k+1})$ and $\gamma(x) \in \delta(x')$, so $\gamma(x) \notin \iota(\alpha)$ and $\gamma(x) \neq \gamma\gamma(\alpha)$. Thus $\gamma(x) \in \delta\gamma(\alpha)$. But this means that $x' <^+ \gamma(\alpha)$ and $\gamma(\alpha) \in \Delta_{\gamma(x)}$. This contradicts the choice of x'. This ends the proof of 3.

Ad 4. $\gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$ by definition. Fix $0 < l < k$. As $S_l = \delta(S_{l+1}) \cup \{\mathbf{p}_l\}$, $\mathbf{p}_l$ is $<^+$-greatest element in S_l. Assume $\gamma(\mathbf{p}_l) \neq \mathbf{p}_{l-1}$. Thus $\gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Let $x \in S_l$. Then $x \leq \mathbf{p}_l$ and, by Lemma 5.9, $\gamma(x) \leq^+ \gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Thus $\mathbf{p}_{l-1} \notin \gamma(S_l)$. So $\gamma(S_l) \subseteq \delta(S_l)$. But this is impossible in a positive opetopic cardinal, as we noticed in the proof of 1. This ends the proof of 4.

Ad 5. Fix $l < k$. First we shall show that

$$\delta(\mathbf{p}_l) \cap \gamma(S_l) = \emptyset. \tag{2}$$

Let $z \in \gamma(S_l)$, i.e., there is $a \in S_l$ such that $\gamma(a) = z$. By 1. $a \leq^+ \mathbf{p}_l$. By Lemma 5.7, there are $x \in \delta(\mathbf{p}_l)$ and $y \in \delta(a)$ such that $x \leq^+ y$. Hence

$x <^+ \gamma(a) = z$. By Proposition 5.1, since $x \in \delta(\mathbf{p}_l)$, it follows that $z \not\in \delta(\mathbf{p}_l)$. This shows (2).

By Lemma 5.19, we have

$$\delta(S_l) = \delta(S_l - \delta(S_{l+1})) \cup \iota(S_{l+1}). \tag{3}$$

Since $\delta(\mathbf{p}_l) = \delta(S_l - \delta(S_{l+1}))$ and $\iota(S_{l+1}) \subseteq \gamma(S_l)$, we have by (2)

$$\delta(S_l - \delta(S_{l+1})) \cap \iota(S_{l+1}) = \emptyset. \tag{4}$$

Next we shall show

$$\iota(S_{l+1}) = \gamma(S_l) \cap \delta(S_l). \tag{5}$$

The inclusion $\subseteq$ is obvious. Let $x \in \gamma(S_l) \cap \delta(S_l)$. Hence there are $a, b \in S_l$ such that $\gamma(a) = x \in \delta(b)$. We can assume that a is $<^+$-maximal with this property. As $a <^- b$, neither a nor b is equal to the $<^+$-greatest element $\mathbf{p}_l \in S_l$. Therefore there is $\alpha \in S_{l+1}$ such that $a \in \delta(\alpha)$. If we were to have $x = \gamma(a) = \gamma\gamma(\alpha)$, then $\gamma(\alpha)$ would be a $<^+$-greater element than a with $\gamma(\gamma(\alpha)) = x$. So $\gamma(a) \neq \gamma\gamma(\alpha)$. Clearly, $x \in \gamma\delta(\alpha)$. By globularity, $x \in \delta\delta(\alpha)$ as well. Thus $x \in \iota(\alpha)$, and (5) is shown.

Using (2), (3), (4), and (5) we have

$$\begin{aligned} \delta(\mathbf{p}_l) = \delta(S_l - \delta(S_{l+1})) = \\ = \delta(S_l) - \iota(S_{l+1}) = \\ = \delta(S_l) - (\gamma(S_l) \cap \delta(S_l)) = \\ = \delta(S_l) - \gamma(S_l) \end{aligned}$$

as required.

Ad 6. By 1. and 2. it is enough to show

$$\delta^{(l)}(S_{k-1}) = \delta^{(l)}(\mathbf{p}_{k-1}),$$

for $l < k-2$. The inclusion $\supseteq$ is obvious. Pick $x \in S_{k-1}$. We have an upper path $x, a_1, \ldots, a_r, \mathbf{p}_{k-1}$. By Corollary 4.3, as $\gamma(a_i) \in \delta(a_{i+1})$, we have

$$\delta^{(l)}(a_i) = \delta^{(l)}\gamma(a_i) \subseteq \delta^{(l)}(\delta(a_{i+1})) = \delta^{(l)}(a_{i+1})$$

for $i = 0, \ldots, r-1$. Then, by transitivity of $\subseteq$ and again Corollary 4.3, we get

$$\delta^{(l)}(x) \subseteq \delta^{(l)}(a_1) \subseteq \delta^{(l)}(a_r) \subseteq \delta^{(l)}(\gamma(a_r)) = \delta^{(l)}(\mathbf{p}_{k-1}).$$

This ends the proof of the inclusion $\subseteq$ and the proof of 6. $\Box$

Lemma 7.2. *Let S be a positive opetopic cardinal of dimension at least k. Then*

1. *S is $(k-1)$-principal iff $\mathbf{d}^{(k)}(S)$ is normal iff $\mathbf{c}^{(k-1)}(S)$ is principal;*
2. *if S is normal, so is $\mathbf{d}(S)$;*
3. *if S is normal, $\mathbf{c}(S)$ is principal.*

Proof. The whole lemma is an easy consequence of Lemma 5.19. We shall show 1., leaving 2. and 3. for the reader. First note that all three conditions in 1. imply that $|S_l - \delta(S_{l+1}| = 1$, for $l < k-2$. In addition, these conditions say:

1. S is $(k-1)$-principal iff $|S_l - \delta(S_{l+1})| = 1$, for $l = k-2, k-1$.
2. $\mathbf{d}^{(k)}S$ is normal iff
   (a) $|S_{k-1} - \delta(S_k - \gamma(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1})| = 1$.
3. $\mathbf{c}^{(k-1)}(S)$ is principal iff
   (a) $|(S_{k-1} - \iota(S_{k+1})) - \delta(S_k - \delta(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1} - \iota(S_{k+1}))| = 1$.

So the equivalence of these conditions follows directly from Lemma 5.19. □

Let N be a normal positive opetopic cardinal of dimension n. We define a $(n+1)$-hypergraph $N^\bullet$ that contains two additional faces: $\mathbf{p}^{N^\bullet}_{n+1}$ of dimension $n+1$, and $\mathbf{p}^{N^\bullet}_n$ of dimension n. We shall drop superscripts if it does not lead to confusions. We also put

$$\delta(\mathbf{p}_{n+1}) = N_n \quad \gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n$$

$$\delta(\mathbf{p}_n) = \delta(N_n) - \gamma(N_n) \quad \gamma(\mathbf{p}_n) = \mathbf{p}_{n-1}(= \gamma(N_n) - \delta(N_n)).$$

As N is normal, $\gamma(N_n) - \delta(N_n)$ has one element, so $\gamma(\mathbf{p}_n)$ is well defined. This determines $N^\bullet$ uniquely. [11] $N^\bullet$ is called *a simple extension of* N.

Example. For a normal positive opetopic cardinal N like this

$$\begin{tikzcd}
 & x_2 \arrow[r, "f_1"] & x_1 \arrow[dr, "f_0"] & \\
x_3 \arrow[ur, "f_2"] & & & x_0
\end{tikzcd}$$

the hypergraph $N^\bullet$ looks like this

$$\begin{tikzcd}
 & x_2 \arrow[r, "f_1"] \arrow[dr, phantom, "\Downarrow \mathbf{p}_2^{N^\bullet}"] & x_1 \arrow[dr, "f_0"] & \\
x_3 \arrow[ur, "f_2"] \arrow[rrr, "\mathbf{p}_1^{N^\bullet}"'] & & & x_0
\end{tikzcd}$$

Proposition 7.3. *Let N be a normal positive opetopic cardinal of dimension n. Then*

1. *$N^\bullet$ is a positive opetope of dimension $n+1$.*
2. *We have $\mathbf{d}(N^\bullet) \cong N$, $\mathbf{c}(N^\bullet) \cong (\mathbf{d}N)^\bullet$.*
3. *If N is a principal, then $N \cong (\mathbf{d}N)^\bullet$.*
4. *If T is a positive opetopic cardinal contained in $N^\bullet$, then either $T = N^\bullet$ or $T = \mathbf{c}(N^\bullet)$ or $T \subseteq N$.*

Proof. Ad 1. We shall check globularity of the new added cells. The other conditions are simple.

For $\mathbf{p}_{n+1}$, we have:

$$\gamma\gamma(\mathbf{p}_{n+1}) = \gamma(\mathbf{p}_n) =$$

$$= \gamma(N_n) - \delta(N_n) = \gamma\delta(\mathbf{p}_{n+1}) - \delta\delta(\mathbf{p}_{n+1})$$

---

[11] (PLC) The uniqueness is meant here with respect to the properties of being principal and having N as domain. Principality is characterized by the addition of the unique top cell p_{n+1}. Having N as domain is taken care of by the equality $\delta(p_{n+1}) = N_n$. Then all the rest is imposed. The cell p_{n+1} must have a (fresh) codomain p_n, which is itself the top cell of $(dN)^\bullet$, as specified by the rest of the definition of $N^\bullet$.

and

$$\delta\gamma(\mathbf{p}_{n+1}) = \delta(\mathbf{p}_n) =$$
$$= \delta(N_n) - \gamma(N_n) = \delta\delta(\mathbf{p}_{n+1}) - \gamma\delta(\mathbf{p}_{n+1}).$$

So globularity holds for $\mathbf{p}_{n+1}$.

For $\mathbf{p}_n$, using Lemmas 7.1, 5.19 and normality of N, we have:

$$\gamma\gamma(\mathbf{p}_n) = \gamma(\mathbf{p}_{n-1}) = \mathbf{p}_{n-2} =$$
$$= \gamma(N_{n-1}) - \delta(N_{n-1}) =$$
$$= \gamma(N_{n-1} - \gamma(N_n)) - \delta(N_{n-1} - \gamma(N_n)) =$$
$$= \gamma(\delta(N_n) - \gamma(N_n)) - \delta(\delta(N_n) - \gamma(N_n)) =$$
$$= \gamma\delta(\mathbf{p}_n) - \delta\delta(\mathbf{p}_n),$$

and similarly

$$\delta\gamma(\mathbf{p}_n) = \delta(\mathbf{p}_{n-1}) =$$
$$= \delta(N_n) - \gamma(N_n) =$$
$$= \delta(\delta(N_n) - \gamma(N_n)) - \gamma(\delta(N_n) - \gamma(N_n)) =$$
$$= \delta\delta(\mathbf{p}_n) - \gamma\delta(\mathbf{p}_n).$$

So globularity holds for $\mathbf{p}_n$, as well.

Ad 2. The first isomorphism is obvious.

The faces of $(N^\bullet)$, $\mathbf{c}(N^\bullet)$, $\mathbf{d}N$, and $(\mathbf{d}N)^\bullet$ are as in the tables

dim	$(N^\bullet)$	$\mathbf{c}(N^\bullet)$
$n+1$	$\{\mathbf{p}^{N^\bullet}_{n+1}\}$	$\emptyset$
n	$N_n \cup \{\mathbf{p}^{N^\bullet}_n\}$	$\{\mathbf{p}^{N^\bullet}_n\}$
$n-1$	N_{n-1}	$N_{n-1} - (\gamma(N_n) \cap \delta(N_n))$
$n-2$	N_{n-2}	N_{n-2}

and

dim	$\mathbf{d}N$	$(\mathbf{d}N)^\bullet$
$n+1$	$\emptyset$	$\emptyset$
n	$\emptyset$	$\{\mathbf{p}^{(\mathbf{d}N)^\bullet}_n\}$
$n-1$	$N_{n-1} - \gamma(N_n)$	$(N_{n-1} - \gamma(N_n)) \cup \{\mathbf{p}^{(\mathbf{d}N)^\bullet}_{n-1}\}$
$n-2$	N_{n-2}	N_{n-2}

We define the isomorphism $f : \mathbf{c}(N^\bullet) \longrightarrow (\mathbf{d}N)^\bullet$ as follows

$$f_n(\mathbf{p}^{N^\bullet}_{n+1}) = \mathbf{p}^{(\mathbf{d}N)^\bullet}_{n+1}$$

$$f_{n-1}(x) = \begin{cases} \mathbf{p}^{(\mathbf{d}N)^\bullet}_{n-1} & \text{if } x = \gamma(\mathbf{p}^{N^\bullet}_n), \\ x & \text{otherwise.} \end{cases}$$

and $f_l = 1_{N_l}$ for $l < n-1$. Clearly, all f_i's are bijective. The preservation of the domains and codomains is left for the reader.

3. is left as an exercise.

Ad 4. If $\mathbf{p}_{n+1} \in T_{n+1}$, then $T = N^\bullet$. If $\mathbf{p}_n \not\in T_n$, then $T \subseteq N$.

Suppose that $\mathbf{p}_{n+1} \not\in T_{n+1}$ but $\mathbf{p}_n \in T_n$. Since $N^\bullet = [\mathbf{p}_{n+1}]$, by Lemma 6.1 it is enough to show that $T = [\mathbf{p}_n]$. Clearly $[\mathbf{p}_n] \subseteq T$. As $[\mathbf{p}_n]_l = N_l$, for $l < n-1$, we have $[\mathbf{p}_n]_l = T_l$, for $l < n-1$, as well.

Fix $x \in N_n$. As $x \in \delta(\mathbf{p}_{n+1})$ and $\gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n$, we have $x <^{N^\bullet,+} \mathbf{p}_n$. So by Corollary 5.11 $x \not\bowtie^{N^\bullet,-}_l \mathbf{p}_n$, for any $l \leq n$. Thus we cannot have $x \bowtie^{T,-}_l \mathbf{p}_n$, for any $l \leq n$ either. As T is a positive opetopic cardinal, again by Corollary 5.11, $x \not\in T$. Since x was an arbitrary element of N_n, we have $T_n = \{\mathbf{p}_n\} = [\mathbf{p}_n]_n$.

It remains to show that $T_{n-1} = [\mathbf{p}_n]_{n-1}$. Suppose that $x \in N_{n-1} - (\delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n))$. Then $x <^{N,+} \gamma(\mathbf{p}_n)$ and hence $x \not\bowtie^{N^\bullet,-}_l \gamma(\mathbf{p}_n)$, for $l \leq n$. So x and $\gamma(\mathbf{p}_n)$ cannot be $<^{T,-}_l$-comparable, for $l \leq n$. Since, as we have shown, $N_n \cap T_n = \emptyset$, it follows that x and $\gamma(\mathbf{p}_n)$ cannot be $<^{T,+}$-comparable. So by Lemma 5.11, $x \not\in T_{n-1}$, i.e., $T_{n-1} = \delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n) = [\mathbf{p}_n]_{n-1}$. $\Box$

8. Decomposition of positive opetopic cardinals

Let T be a positive opetopic cardinal, $X \subseteq T$ a subhypergraph of T, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. We define two subhypergraphs of T, $X^{\downarrow a}$ and $X^{\uparrow a}$, as follows:

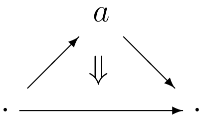
$$X^{\downarrow a}_l = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \leq^+ a\} & \text{for } l > k \\ \{b \in X_k : b \leq^+ a \text{ or } \ b \not\in \gamma(X_{k+1})\} & \text{for } l = k \\ X_l & \text{for } l < k. \end{cases}$$

$$X_l^{\uparrow a} = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \not\leq^{+} a\} & \text{for } l > k \\ \{b \in X_k : b \not<^{+} a \text{ or } b \notin \delta(X_{k+1})\} & \text{for } l = k \\ X_{k-1} - \iota(X_{k+1}^{\downarrow a}) & \text{for } l = k-1 \\ X_l & \text{for } l < k-1. \end{cases}$$

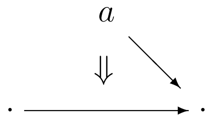
Intuitively, if X is a positive opetopic cardinal contained in T, $X^{\downarrow a}$ is the least positive opetopic cardinal contained in X that contains faces 'smaller or equal' a and can be k-pre-composed with the 'rest' to get X. $X^{\uparrow a}$ is this 'rest' or in other words it is the largest positive opetopic cardinal contained in X that can be k-post-composed with $X^{\downarrow a}$ to get X (or the largest positive opetopic cardinal contained in X that does not contain faces 'smaller' than a). Note that a does not need to be a face in X, in general.

Examples. If X is a hypergraph $a \in T$, then $X^{\downarrow a}$ is a hypergraph, as well. However, this is not the case with $X^{\uparrow a}$, if $a \in \iota(T)$, as we can see below:

X:



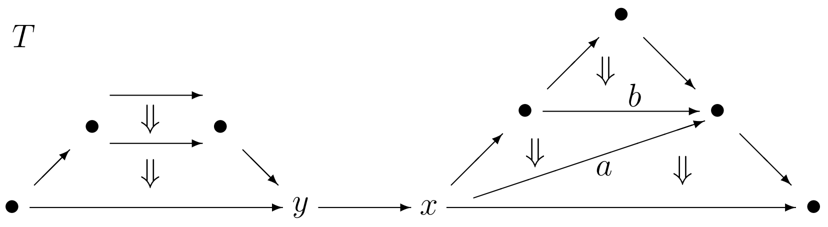
$X^{\uparrow a}$:



Here $X = T$. The faces in the domain of the 2-dimensional face are not in $X^{\uparrow a}$, i.e., $X^{\uparrow a}$ is not closed under δ.

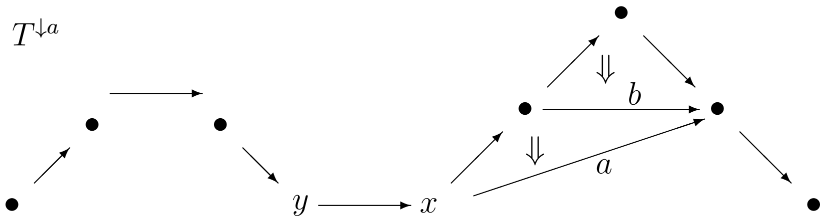
To see some real decompositions, let fix a positive opetopic cardinal T as follows:

T

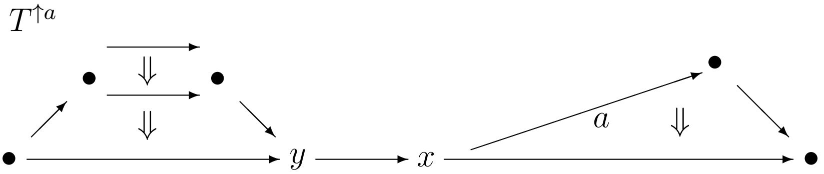


Clearly $x, y, a, b \in T - \iota(T)$. Then

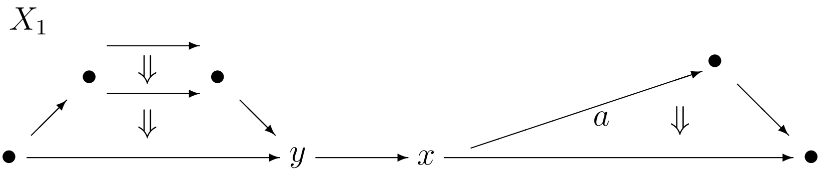
$T^{\downarrow a}$



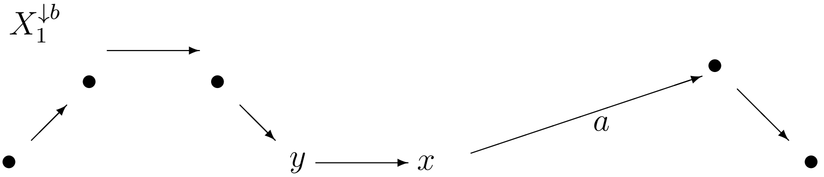
and



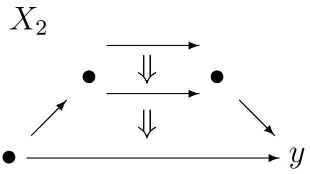
Moreover, with



we have $X_1^{\uparrow b} = X_1$ and



i.e., $X_1^{\downarrow b} = \mathbf{d}^{(1)}(X_1)$. For



we have $X_2^{\downarrow x} = X_2$ and $X_2^{\uparrow x} = \{y\}$.

Lemma 8.1. *Let T be a positive opetopic cardinal, $X \subseteq T$ a subhypergraph of T, $a \in (T - \iota(T))$, $a \in X_k$. Then*

1. *$X^{\downarrow a}$ and $X^{\uparrow a}$ are positive opetopic cardinals;*
2. *$\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a}) = X^{\downarrow a} \cap X^{\uparrow a}$;*
3. *$\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$, $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$;*
4. *$X = X^{\uparrow a} \oplus_k X^{\downarrow a} = X^{\uparrow a} \cup X^{\downarrow a}$.*

Proof. Ad 1. The verification that $X^{\downarrow a}$ and $X^{\uparrow a}$ are closed under γ and δ is routine. For any k, if $x, y \in X^{\downarrow a}_k$, then $x <^{+,X} y$ iff $x <^{+,X^{\downarrow a}} y$. Similarly, for any k, if $x, y \in X^{\uparrow a}_k$, then $x <^{+,X} y$ iff $x <^{+,X^{\uparrow a}} y$. Thus by Lemma 5.17 $X^{\downarrow a}$ and $X^{\uparrow a}$ are positive opetopic cardinals.

Ad 2. Let us spell out $\mathbf{c}^{(k)}(X^{\downarrow a})$ and $\mathbf{d}^{(k)}(X^{\uparrow a})$:
$\mathbf{c}^{(k)}(X^{\downarrow a})$:

1. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{c}^{(k)}(X^{\downarrow a})_k =$ $(\{b \in X_k : b \leq^+ a\} \cup (X_k - \gamma(X_{k+1}))) - \delta(\{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{c}^{(k)}(X^{\downarrow a})_{k-1} = X_{k-1} - \iota(X^{\downarrow a}_{k+1})$;
4. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = X_l$, for $l < k-1$.

$\mathbf{d}^{(k)}(X^{\uparrow a})$:

1. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{d}^{(k)}(X^{\uparrow a})_k =$ $\{b \in X_k : b \not<^+ a \text{ or } b \notin \delta(X_{k+1})\} - \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{d}^{(k)}(X^{\uparrow a})_{k-1} = X_{k-1} - \iota(X^{\downarrow a}_{k+1})$;
4. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = X_l$, for $l < k-1$.

Thus to show that $\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a})$, we need to verify that $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$. As both sets are contained in X_k, we can compare their complements. We have

$$\begin{aligned} X_k - \mathbf{c}^{(k)}(X^{\downarrow a})_k = \{b \in \delta(X_{k+1}) : b <^+ a\} \\ \cup\, \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^+ a\}) \end{aligned}$$

and

$$X_k - \mathbf{d}^{(k)}(X^{\uparrow a})_k = \{b \in \gamma(X_{k+1}) : b \not\leq^+ a\} \cup \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^+ a\}).$$

But it easy to see that

$$\{b \in \delta(X_{k+1}) : b <^{+} a\} = \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^{+} a\})$$

and

$$\gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^{+} a\}) = \{b \in \gamma(X_{k+1}) : b \not\leq^{+} a\}.$$

The second equality uses the fact that $a \not\in \iota(T)$. Thus $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$, as required.

Ad 3. To see that $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$, it is enough to note that $\iota(X_{k+1}) = \iota(X^{\downarrow a}_{k+1}) \cup \iota(X^{\uparrow a}_{k+1})$. The equation $\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$ is even simpler.

Ad 4. Obvious. □

Corollary 8.2. *Let T be a positive opetopic cardinal, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. Then the square*

$$\begin{tikzcd}
T & T^{\downarrow a} \arrow[l] \\
T^{\uparrow a} \arrow[u] & \mathbf{c}^{(k)}(T^{\downarrow a}) \arrow[l, "\mathbf{d}^{(k)}_{T^{\uparrow a}}"'] \arrow[u, "\mathbf{c}^{(k)}_{T^{\downarrow a}}"']
\end{tikzcd}$$

is a special pushout in $\mathbf{pOpeCard}$.

Proof. Follows immediately from Lemmas 6.2 and 8.1. □

We need more notions and notations. Let X, T be positive opetopic cardinals $X \subseteq T$, $a \in (T - \iota(T))$. The decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is said to be *proper* iff $size(X^{\downarrow a}), size(X^{\uparrow a}) < size(X)$. If the decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is proper then a is said to be a *saddle face* of X. $Sd(X)$ is the set of saddle faces of X, $Sd(X)_k = Sd(X) \cap X_k$.

Lemma 8.3. *Let X, S, T be positive opetopic cardinals, $X \subseteq T$, $l \in \omega$. Then*

1. *if $a \in (T_l - \iota(T))$, then $a \in Sd(X)$ iff there are $\alpha, \beta \in X_{l+1}$ such that $\gamma(\alpha) \leq^{+} a$ and $\gamma(\beta) \not\leq^{+} a$;*

2. *if* $\mathbf{c}^{(k)}(S) = \mathbf{d}^{(k)}(T)$, *then*

$$size(S \oplus_k T)_l = \begin{cases} size(S)_l + size(T)_l & \textit{if } l > k \\ size(T)_l & \textit{if } l \leq k; \end{cases}$$

3. $size(S)_k \geq 1$ *iff* $k \leq dim(S)$;

4. *if* $a \in Sd(S)_k$, *then* $size(S)_{k+1} \geq 2$;

5. S *is principal iff* $Sd(S)$ *is empty.*

Proof. We shall show 5. The rest is easy.

If there is $a \in Sd(S)_k$, then by 2., 3. and Lemma 8.1 we have that $size(S)_{k+1} = size(S^{\downarrow a})_{k+1} + size(S^{\uparrow a})_{k+1} \geq 1 + 1 > 1$. So in that case S is not principal.

For the converse, assume that S is not principal. Fix $k \in \omega$ such that $size(S)_{k+1} > 1$. Thus there are $a, b \in S_{k+1}$, such that $a \neq b$. Suppose $\gamma(a) \in \iota(\alpha)$, for some $\alpha \in S_{k+2}$. Then by Lemma 5.5 we get $a <^{+} \gamma(\alpha)$, contrary to the assumption on a. Hence $a \in S - \iota(S)$ and for similar reasons $b \in S - \iota(S)$. We have $a \not\bowtie^{+} b$ and, by pencil linearity, $\gamma(a) \neq \gamma(b)$. Then either $\gamma(a) \not<^{+} \gamma(b)$ and then $\gamma(b) \in Sd(S)_k$, or $\gamma(b) \not<^{+} \gamma(a)$ and then $\gamma(a) \in Sd(S)_k$. In either case $Sd(S)$ is not empty, as required. □

Lemma 8.4. *Let* T, X *be positive opetopic cardinals,* $X \subseteq T$, *and* $a, x \in X - \iota(X)$, $k = dim(x) < dim(a) = m$.

1. *We have the following equations of positive opetopic cardinals:*

$$X^{\downarrow x \downarrow a} = X^{\downarrow a \downarrow x} \quad X^{\downarrow x \uparrow a} = X^{\uparrow a \downarrow x} \quad X^{\uparrow x \downarrow a} = X^{\downarrow a \uparrow x} \quad X^{\uparrow x \uparrow a} = X^{\uparrow a \uparrow x}$$

i.e., 'the decompositions of different dimensions commute'.

2. *If* $x \in Sd(X)$, *then* $x \in Sd(X^{\downarrow a}) \cap Sd(X^{\uparrow a})$.

3. *Moreover, we have the following equations concerning domains and codomains*

$$\mathbf{c}^{(k)}(X^{\downarrow x \downarrow a}) = \mathbf{c}^{(k)}(X^{\downarrow x \uparrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \uparrow a})$$

$$\mathbf{c}^{(m)}(X^{\downarrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\downarrow x \uparrow a}) \quad \mathbf{c}^{(m)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\uparrow x \uparrow a}).$$

4. *Finally, we have the following equations concerning compositions*

$$X^{\downarrow x \uparrow a} \oplus_m X^{\downarrow x \downarrow a} = X^{\downarrow x} \quad X^{\uparrow x \uparrow a} \oplus_m X^{\uparrow x \downarrow a} = X^{\uparrow x}$$

$$X^{\uparrow x \downarrow a} \oplus_k X^{\downarrow x \downarrow a} = X^{\downarrow a} \quad X^{\uparrow x \uparrow a} \oplus_k X^{\downarrow x \uparrow a} = X^{\uparrow a}.$$

Proof. Simple check. □

Lemma 8.5. *Let T, X be positive opetopic cardinals, $X \subseteq T$, and $a, b \in X - \iota(X)$, $dim(a) = dim(b) = m$.*

1. *We have the following equations of positive opetopic cardinals:*

$$X^{\downarrow a \downarrow b} = X^{\downarrow b \downarrow a} \qquad X^{\uparrow a \uparrow b} = X^{\uparrow b \uparrow a},$$

   i.e., 'the decompositions in the same dimension and the same directions commute'.

2. *Assume $a <^{+} b$. Then we have the following further equations of positive opetopic cardinals:*

$$X^{\downarrow a} = X^{\downarrow a \downarrow b} \qquad X^{\uparrow b} = X^{\uparrow a \uparrow b} \qquad X^{\downarrow b \uparrow a} = X^{\uparrow a \downarrow b}.$$

   Moreover, if $a, b \in Sd(X)$, then $a \in Sd(X^{\downarrow b})$ and $b \in Sd(X^{\uparrow a})$.

3. *Assume $a <^{-}_{l} b$, for some $l < m$. Then $X^{\uparrow b \downarrow a}$, $X^{\uparrow a \downarrow b}$, are positive opetopic cardinals, and*

$$X^{\uparrow a \downarrow b} \oplus_m X^{\downarrow a} = X^{\uparrow b \downarrow a} \oplus_m X^{\downarrow b}.$$

   Moreover, if $a, b \in Sd(X)$, then either there is k such that $l - 1 \leq k < m$ and $\gamma^{(k)}(a) \in Sd(X)$ or $a \in Sd(X^{\uparrow b})$ and $b \in Sd(X^{\uparrow a})$.

Proof. Simple check. □

Lemma 8.6. *Let T, X be positive opetopic cardinals, $X \subseteq T$, $dim(X) = n$, $l < n - 1$, $a \in Sd(X)_l$. Then*

1. *$a \in Sd(\mathbf{c}(X)) \cap Sd(\mathbf{d}(X))$;*
2. *$\mathbf{d}(X^{\downarrow a}) = (\mathbf{d}X)^{\downarrow a}$;*

3. $\mathbf{d}(X^{\uparrow a}) = (\mathbf{d}X)^{\uparrow a}$*;*

4. $\mathbf{c}(X^{\downarrow a}) = (\mathbf{c}(X))^{\downarrow a}$*;*

5. $\mathbf{c}(X^{\uparrow a}) = (\mathbf{c}(X))^{\uparrow a}$*.*

Proof. The proof is again by a long and simple check. We shall check part of 5. We should consider separately cases: $l = n-2$, $l = n-3$, and $l < n-3$, but we shall check the case $l = n-3$ only. The other cases can be also shown by similar, but easier, checks.

$(\mathbf{c}(X))^{\uparrow a}$ is:

1. $(\mathbf{c}(X))^{\uparrow a}_l = \emptyset$, for $l \geq n$;
2. $(\mathbf{c}(X))^{\uparrow a}_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a,\ \ x \not\in \delta(X_n)\}$;
3. $(\mathbf{c}(X))^{\uparrow a}_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a,\ \ x \not\in \iota(X_n)\}$;
4. $(\mathbf{c}(X))^{\uparrow a}_{n-3} = \{x \in X_{n-3} : x \not<^+ a$ or $x \not\in \delta(X_{n-2} - \iota(X_n))\}$;
5. $(\mathbf{c}(X))^{\uparrow a}_{n-4} = X_{n-4} - \iota(\{x \in X_{n-2} : x \not\in \iota(X_n),\ \gamma(x) \leq^+ a\})$;
6. $X^{\downarrow a}_l = X_l$, for $l < n-4$.

and $\mathbf{c}(X^{\uparrow a})$ is:

1. $\mathbf{c}(X^{\uparrow a})_l = \emptyset$, for $l \geq n$;
2. $\mathbf{c}(X^{\uparrow a})_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a\} - \delta(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;
3. $\mathbf{c}(X^{\uparrow a})_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a\} - \iota(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;
4. $\mathbf{c}(X^{\uparrow a})_{n-3} = \{x \in X_{n-3} : x \not<^+ a$ or $x \not\in \delta(X_{n-2})\}$;
5. $\mathbf{c}(X^{\uparrow a})_{n-4} = X_{n-4} - \iota(X^{\downarrow a}_{n-2})$;
6. $\mathbf{c}(X^{\uparrow a})_l = X_l$, for $l < n-4$.

We need to verify the equality $(\mathbf{c}(X))_l^{\uparrow a} = \mathbf{c}(X^{\uparrow a})_l$, for $l = n-1, \ldots, n-4$.

In dimension $n-1$, it is enough to show that if $x \in X_{n-1}$ and $z \in X_n$ so that $\gamma^{(n-3)}(x) \not\leq^+ a$ and $x \in \delta(z)$, then $\gamma^{(n-3)}(z) \not\leq^+ a$.

So assume that we have $x \in X_{n-1}$, $\gamma^{(n-3)}(x) \not\leq^+ a$, $z \in X_n$ such that $x \in \delta(z)$. Hence $x \lhd^+ \gamma(z)$. By Lemma 5.9.5, $\gamma^{(n-3)}(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$ (otherwise we would have $\gamma^{(n-3)}(x) \not\leq^+ a$), as required.

In dimension $n-2$, it is enough to show that if $x \in X_{n-2}$ and $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$, then $\gamma^{(n-3)}(z) \not\leq^+ a$.

So assume that $x \in X_{n-2}$, $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$. Hence $x \leq^+ \gamma\gamma(z)$. By Lemma 5.9.5, $\gamma(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$, as required.

The equality in dimension $n-3$ follows immediately from Lemma 5.19.4.

To show that in dimension $n-4$, the above equation also holds, we shall show that

$$\iota(X_{n-2}^{\downarrow a}) \subseteq \iota(\{x \in X_{n-2} : x \not\in \iota(X_n),\ \gamma(x) \leq^+ a\}).$$

Note that, by Lemma 5.3.1, if $t \in X_{n-4}$ and $x \in X_{n-2}$, $y \in X_{n-1}$, $t \in \iota(x)$ and $\gamma(x) \leq^+ a$ and $x = \gamma(y)$, then there is $x' \in \delta(y)$ (i.e., $x' \lhd^+ x$ and hence $\gamma(x') \leq^+ a$) such that $t \in \iota(x')$.

Thus, as $<^+$ is well founded, it follows from the above observation that, for any $t \in X_{n-4}$ and $x \in X_{n-2}$ such that $t \in \iota(x)$ and $\gamma(x) \leq^+ a$, there is $x'' \leq^+ x$ such that $t \in \iota(x'')$ and $x'' \not\in \gamma(X)$. Then we clearly have that $x'' \not\in \iota(X)$ and $\gamma(x'') \leq^+ a$, as required. □

The following lemma describes how one can express decompositions of a special pushout in terms of decompositions of its components.

Lemma 8.7. *Let T, T_1, T_2 be positive opetopic cardinals, and suppose that $dim(T_1), dim(T_2) > k$, $\mathbf{c}^{(k)}(T_1) = \mathbf{d}^{(k)}(T_2)$ and $T = T_2 \oplus_k T_1$. Then $\mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1) \neq \emptyset$. For any $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$, we have $a \in Sd(T)_k$ and*

- *either $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$*
- *or $a \in Sd(T_1)_k$, $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_2 \oplus_k T_1^{\uparrow a}$.*

Proof. By assumption, $(T_1)_{k+1} \neq \emptyset$ and $(T_2)_{k+1} \neq \emptyset$. So $\mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Fix $a \in \mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Then $T^{\downarrow a}_{k+1} \neq \emptyset$. As $T^{\downarrow a}_{k+1} \cap (T_2)_{k+1} = \emptyset$, we must have $a \in Sd(T)_k$.

Assume $T_1 \neq T^{\downarrow a}$. Then $T^{\downarrow a} \subsetneq T_1$. Hence $(T_1) - (T^{\downarrow a}) \neq \emptyset$. But this means that $a \in Sd(T_1)_k$. The verification that the equalities $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_2 \oplus_k T_1^{\uparrow a}$ hold in this case is left as an exercise. □

9. Positive opetopic cardinals as positive-to-one polygraphs

For the definition of positive-to-one polygraphs and related notation see Appendix. In this section we show that the image of the embedding defined in Section 6

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$$

is in fact contained in the category of polygraphs.

Proposition 9.1. *Let S be a weak positive opetopic cardinal. Then S^* is a positive-to-one polygraph whose k-indeterminates correspond to faces in S_k.*

Proof. The proof is by induction on the dimension n of the weak positive opetopic cardinal S. For $n = 0, 1$, the proposition is obvious.

So assume that for any weak positive opetopic cardinal T of dimension n, T^* is a positive-to-one polygraph of dimension n, generated by faces in T. Suppose that S is a weak positive opetopic cardinal of dimension $n+1$. We shall show that S^* is a polygraph generated by faces in S. Since $S_{\leq n}$ is a weak positive opetopic cardinal, by inductive assumption, $S^*_{\leq n}$ is the polygraph generated by faces in $S_{\leq n}$. So we need to verify that, for any ω-functor $f : S^*_{\leq n} \longrightarrow C$ to any ω-category C and any function $|f| : S_{n+1} \longrightarrow C_{n+1}$ such that for $a \in S_{n+1}$

$$d_C(|f|(a)) = f(\mathbf{d}([a])), \qquad c_C(|f|(a)) = f(\mathbf{c}([a])),$$

there is a unique ω-functor $F : S^* \longrightarrow C$ such that

$$F_{n+1}([a]) = |f|(a), \qquad F_{\leq n} = f$$

as in the diagram

$$\begin{tikzcd}
 & & C \\
S_{n+1} \arrow[urr, "|f|"] \arrow[r, "{[-]}"'] \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\gamma"] & S^*_{n+1} \arrow[ur, "F_{n+1}"'] \arrow[d, shift right, "\mathbf{d}"'] \arrow[d, shift left, "\mathbf{c}"] & \\
S_{\leq n} \arrow[r, "{[-]}"'] & S^*_{\leq n} \arrow[uur, "f"'] &
\end{tikzcd}$$

We define F_{n+1} as follows. For $X \in S^*_{n+1}$:

$$F_{n+1}(X) = \begin{cases} id_{f(X)} \text{ if } dim(X) \leq n \\ |f|(a) \text{ if } dim(X) = n+1, X \text{ is principal and } X = [a], \\ F_{n+1}(X^{\uparrow a}) \circ_l F_{n+1}(X^{\downarrow a}) \text{ if } dim(X) = n+1,\, a \in Sd(X)_l, \end{cases}$$

where $\circ_l$ refers to the composition in the ω-category C. Clearly $F_k = f_k$, for $k \leq n$. The above morphism, if well defined, clearly preserves identities. Uniqueness is also clear by construction. We need to verify three conditions, for $X \in S^*_{n+1}$ and $dim(X) = n+1$:

I F is well defined, i.e., $F_{n+1}(X) = F_{n+1}(X^{\uparrow a}) \circ_l F_{n+1}(X^{\downarrow a})$ does not depend on the choice of the saddle face $a \in Sd(X)$;

II F preserves the domains and codomains, that is to say, $F_n(d(X)) = d(F_{n+1}(X))$ and $F_n(c(X)) = c(F_{n+1}(X))$;

III F preserves compositions, i.e., $F_{n+1}(X) = F_{n+1}(X_2) \circ_k F_{n+1}(X_1)$ whenever $X = X_2 \oplus_k X_1$ and $dim(X_1), dim(X_2) > k$.

We have an embedding $[-] : S_{\leq n} \longrightarrow S^*_{\leq n}$. So let us assume that for positive opetopic cardinals of S of size less than $size(X)$ the above assumption holds. If $size(X)_{n+1} = 0$ or X is principal, all three conditions are obvious. So assume that X is not principal and $dim(X) = n+1$. To save on notation we write F for F_{n+1}.

Ad **I**. First we will consider two saddle faces $a, x \in Sd(X)$ of different dimension $k = dim(x) < dim(a) = m$. Using Lemma 8.4 we have

$$F(X^{\uparrow a}) \circ_m F(X^{\downarrow a}) = \qquad \textit{ind. hyp. I}$$

$$\begin{aligned}
&= (F(X^{\uparrow a\uparrow x}) \circ_k F(X^{\uparrow a\downarrow x})) \circ_m (F(X^{\downarrow a\uparrow x}) \circ_k F(X^{\downarrow a\downarrow x})) = \\
&= (F(X^{\uparrow a\uparrow x}) \circ_m F(X^{\downarrow a\uparrow x})) \circ_k (F(X^{\uparrow a\downarrow x}) \circ_m F(X^{\downarrow a\downarrow x})) = \\
&= (F(X^{\uparrow x\uparrow a}) \circ_m F(X^{\uparrow x\downarrow a})) \circ_k (F(X^{\downarrow x\uparrow a}) \circ_m F(X^{\downarrow x\downarrow a})) = \quad \textit{ind. hyp. III} \\
&\qquad = F(X^{\uparrow x}) \circ_m F(X^{\downarrow x}).
\end{aligned}$$

Now we will consider two saddle faces $a, b \in Sd(X)$ of the same dimension $dim(a) = dim(b) = m$. We shall use Lemma 8.5. Assume $a <^-_l b$, for some $l < m$. If $\gamma^{(k)}(a) \in Sd(X)$, for some $k < m$, then this case reduces to the previous one for two pairs $a, \gamma^{(k)}(a) \in Sd(X)$ and $b, \gamma^{(k)}(a) \in Sd(X)$. Otherwise $a \in Sd(X^{\uparrow b})$ and $a \in Sd(X^{\uparrow b})$ and we have

$$\begin{aligned}
F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) &= \quad \textit{ind. hyp. I} \\
= (F(X^{\uparrow a\uparrow b}) \circ_k F(X^{\uparrow a\downarrow b})) \circ_k F(X^{\downarrow a}) &= \\
= F(X^{\uparrow a\uparrow b}) \circ_k (F(X^{\uparrow a\downarrow b}) \circ_k F(X^{\downarrow a})) &= \quad \textit{ind. hyp. III} \\
= F(X^{\uparrow b\uparrow a}) \circ_k F(X^{\uparrow a\downarrow b} \oplus_k X^{\downarrow a}) &= \\
= F(X^{\uparrow b\uparrow a}) \circ_k F(X^{\uparrow b\downarrow a} \oplus_k X^{\downarrow b}) &= \quad \textit{ind. hyp. III} \\
= F(X^{\uparrow b\uparrow a}) \circ_k (F(X^{\uparrow b\downarrow a}) \circ_k F(X^{\downarrow b})) &= \\
= (F(X^{\uparrow b\uparrow a}) \circ_k F(X^{\uparrow b\downarrow a})) \circ_k F(X^{\downarrow b}) &= \\
= F(X^{\uparrow b}) \circ_k F(X^{\downarrow b}).&
\end{aligned}$$

Finally, we consider the case $a <^+ b$. We have

$$\begin{aligned}
F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) &= \quad \textit{ind. hyp. I} \\
= (F(X^{\uparrow a\uparrow b}) \circ_k F(X^{\uparrow a\downarrow b})) \circ_k F(X^{\downarrow a}) &= \\
= (F(X^{\uparrow b}) \circ_k F(X^{\downarrow b\uparrow a})) \circ_k F(X^{\downarrow b\downarrow a}) &= \\
= F(X^{\uparrow b}) \circ_k (F(X^{\downarrow b\uparrow a}) \circ_k F(X^{\downarrow b\downarrow a})) &= \quad \textit{ind. hyp. I} \\
= F(X^{\uparrow b}) \circ_k F(X^{\downarrow b}).&
\end{aligned}$$

This shows that $F(X)$ is well defined.

Ad **II**. We shall show that the domains are preserved. The proof that the codomains are preserved is similar.

The fact that if $Sd(X) = \emptyset$, then F preserves domains and codomains follows immediately from the assumption on f and $|f|$. So assume $Sd(X) \neq \emptyset$ and let $a \in Sd(X)$, $dim(a) = k$. We use Lemma 8.6. We have to consider two cases $k < n$, and $k = n$.

If $k < n$, then

$$\begin{aligned} F_n(d(X)) = F_n(d(X^{\uparrow a} \oplus_k X^{\downarrow a})) &= \\ = F_n(d((X^{\uparrow a}) \oplus_k d(X^{\downarrow a})) &= \\ = F_n(d(X)^{\uparrow a} \oplus_k d(X)^{\downarrow a}) &= \quad \mathit{ind.\ hyp.\ III} \\ = F_n(d(X)^{\uparrow a}) \circ_k F_n(d(X)^{\downarrow a}) &= \\ = F_n(d(X^{\uparrow a})) \circ_k F_n(d(X^{\downarrow a})) &= \quad \mathit{ind.\ hyp.\ II} \\ = d(F_{n+1}(X^{\uparrow a})) \circ_k d(F_{n+1}(X^{\downarrow a})) &= \\ = d(F_{n+1}(X^{\uparrow a}) \circ_k F_{n+1}(X^{\downarrow a})) &= \quad \mathit{ind.\ hyp.\ I} \\ = d(F_{n+1}(X)). & \end{aligned}$$

If $k = n$, then

$$\begin{aligned} F_n(d(X)) = F_n(d(X^{\uparrow a} \oplus_n X^{\downarrow a})) &= \\ = F_n(d(X^{\downarrow a})) &= \quad \mathit{ind.\ hyp.\ II} \\ = d(F_{n+1}(X^{\downarrow a})) &= \\ = d(F_{n+1}(X^{\uparrow a})) \circ_n F_{n+1}(X^{\downarrow a})) &= \quad \mathit{ind.\ hyp.\ I} \\ = d(F_{n+1}(X)). & \end{aligned}$$

Ad **III**. Suppose that $X = X_2 \oplus_k X_1$ and $dim(X) \leq n+1$. We shall show that F preserves this composition. If $dim(X_1) = k$, then $X = X_2$, $X_1 = \mathbf{d}^{(k)}(X_2)$. We have

$$F_{n+1}(X) = F_{n+1}(X_2) =$$

$$= F_{n+1}(X_2) \circ_k 1^{(n+1)}_{F_k(\mathbf{d}^{(k)}(X_2))} =$$

$$= F_{n+1}(X_2) \circ_k 1^{(n+1)}_{F_k(X_1)} =$$

$$= F_{n+1}(X_2) \circ_k F_{n+1}(X_1).$$

The case $dim(X_2) = k$ is similar. So now assume $dim(X_1), dim(X_2) > k$. We shall use Lemma 8.7. Fix $a \in \mathbf{c}^{(k)}(X_1)_k \cap \gamma(X_1)$. So $a \in Sd(X)_k$. If $X_1 = X^{\downarrow a}$ and $X_2 = X^{\uparrow a}$, then we have

$$F(X) = F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) = F(X_2) \circ_k F(X_1).$$

If $a \in Sd(X_1)_k$, then

$$\begin{array}{rl} F(X) = F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) = & \textit{ind. hyp. II} \\ = (F(X_2) \circ_k F(X_1^{\uparrow a})) \circ_k F(X^{\downarrow a}) = & \\ = F(X_2) \circ_k (F(X_1^{\uparrow a}) \circ_k F(X_1^{\downarrow a})) = & \textit{ind. hyp. II} \\ F(X_2) \circ_k F(X_1). & \end{array}$$

So in any case the composition is preserved. This ends the proof of the lemma. □

Let $\mathbf{wpOpeCard}_n$ be the full subcategory of $\mathbf{wpOpeCard}$ whose objects have dimension at most $n \geq 0$. For $n \in \omega$, we have a functor

$$(-)^{\sharp,n} : \mathbf{wpOpeCard}_n \longrightarrow Set \downarrow D_{n-1}$$

such that, for S in $\mathbf{wpOpeCard}_n$

$$S^{\sharp,n} = (S_n, S^*_{<n}, [\delta], [\gamma])$$

and, for $f : S \to T$ in $\mathbf{wpOpeCard}_n$, we have

$$f^{\sharp,n} = (f_n, (f_{<n})^*).$$

By construction, we have

$$\overline{(-)}^n \circ (-)^{\sharp,n} = (-)^{*,n},$$

where $(-)^{*,n} : \mathbf{wpOpeCard}_n \longrightarrow \mathbf{pPoly}_n$ is the n-dimensional version of $(-)^*$.

Corollary 9.2. *For every $n \in \omega$, the functor $(-)^{\sharp,n}$ is full and faithful, and it preserves special pushouts (i.e., it maps special pushouts to pushouts).*

Proof. Fullness and faithfulness of $(-)^{\sharp,n}$ is left for the reader. We shall show that for every $n \in \omega$, $(-)^{\sharp,n}$ preserves special pushouts. For $n = 0$, there is nothing to prove. For $n = 1$, this is obvious. So assume that $n \geq 1$ and that $(-)^{\sharp,n}$ preserves special pushouts. Let

$$\begin{tikzcd}
S \arrow[r] & S +_R T \\
R \arrow[u] \arrow[r] & T \arrow[u]
\end{tikzcd}$$

be a special pushout in $\mathbf{wpOpeCard}_{n+1}$. Clearly its n-truncation is a pushout in $\mathbf{wpOpeCard}_n$. Hence by inductive hypothesis it is preserved by $(-)^{*,n}$. In dimension $n+1$, the functor $(-)^{\sharp,n+1}$ is an inclusion. Hence, in dimension $n+1$, this square is a pushout (of monos) in Set. So the whole square

$$\begin{tikzcd}
S^{\sharp,n+1} \arrow[r] & (S +_R T)^{\sharp,n+1} \\
R^{\sharp,n+1} \arrow[u] \arrow[r] & T^{\sharp,n+1} \arrow[u]
\end{tikzcd}$$

is a pushout in $Set \downarrow D_{n+1}$, i.e., $(-)^{\sharp,n+1}$ preserves special pushouts. □

Corollary 9.3. *The functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \mathbf{pPoly}$$

is full and faithful and preserves special pushouts. In particular, it is conservative.

Proof. This follows from the previous corollary and the fact that the functor $\overline{(-)}^n : Set \downarrow D_{n-1} \longrightarrow \mathbf{pPoly}_n$ is an equivalence of categories. [12]□

Corollary 9.4. *The functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \omega Cat$$

is faithful, conservative and preserves special pushouts.

---

[12] (PLC) In reference to the description given in the appendix, $\overline{(-)}^n$ is full and essentially surjective by construction. Here is a hint on how faithfulness can be established. The morphisms of the free category are equivalence classes of formal composites in various dimensions of generators of dimension $\leq n$. One can associate to such a formal expression s the multiset $top(s)$ of the generating n-morphisms occurring in it, and one can show that this is an invariant: if two expressions s and t can be proved equal by the laws of ω-categories, then $top(s) = top(t)$. For a generator a we have $top(a) = \{a\}$, hence if two generators a, b where equated, we would have $\{a\} = top(a) = top(b) = \{b\}$, proving faithfulness.

Proof. The faithfulness and preservation of special pushout follows from the previous corollary and the fact that the functor $F_n : \mathbf{pPoly}_n \longrightarrow n\mathbf{Cat}$ (see Appendix) is faithful and a left adjoint. Conservativity follows from the previous corollary and the fact that any isomorphic ω-functor between polygraphs preserves and reflects indeterminates, i.e., is a map of polygraphs.[13]

Let P be a positive-to-one polygraph, a a k-cell in P. A *description of the cell* a is a pair[14]

$$< T_a, \tau_a : T_a^* \longrightarrow P >$$

where T_a is a positive opetopic cardinal and τ_a is a polygraph map such that

$$\tau_a(T_a) = a.$$

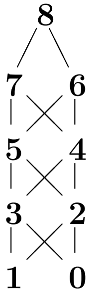
In the remainder of this section we shall define some specific positive opetopic cardinals that will be used later. First we define the globes α^n, for $n \in \omega$. We put

$$\alpha^n_l \;=\; \begin{cases} \emptyset & \text{if } l > n \\ \{2n\} & \text{if } l = n \\ \{2l+1,\, 2l\} & \text{if } 0 \leq l < n \end{cases}$$

$$d,\, c : \alpha^n_l \longrightarrow \alpha^n_{l-1}$$

$$d(x) = \{2l-1\} \qquad c(x) = 2l-2$$

for $x \in \alpha^n_l$, and $1 \leq l \leq n$. For example, α^4 can be pictured as follows:



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[13] (PLC) For a proof of this, see [1, Proposition 16.6.3].

[14] (PLC) The existence and uniqueness of descriptions is proved below (Proposition 12.2).

i.e., 8 is the unique cell of dimension 4 in α^4 that has 7 as its domain and 6 as its codomain, 7 and 6 have 5 as its domain and 4 as its codomain, and so on. Note that, for any $k \leq n$, we have

$$\mathbf{d}^{(k)}\alpha^n = \alpha^k = \mathbf{c}^{(k)}\alpha^n.$$

Let $n_1 < n_0, n_2$ and $n_3 < n_2, n_4$. We define the positive opetopic cardinals α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ as the following colimits in $\mathbf{pOpeCard}$:

$$\begin{tikzcd}
\alpha^{n_0}, \arrow[r, "\kappa_1"] & \alpha^{n_0,n_1,n_2} \\
\alpha^{n_1} \arrow[u, "\mathbf{c}^{(n_1)}_{\alpha^{n_0}}"] \arrow[r, "\mathbf{d}^{(n_1)}_{\alpha^{n_2}}"'] & \alpha^{n_2} \arrow[u, "\kappa_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha^{n_0}, \arrow[r, "\kappa_1"] & \alpha^{n_0,n_1,n_2,n_3,n_4} & \alpha^{n_4} \arrow[l, "\kappa_3"'] \\
\alpha^{n_1} \arrow[u, "\mathbf{c}^{(n_1)}_{\alpha^{n_0}}"] \arrow[r, "\mathbf{d}^{(n_1)}_{\alpha^{n_2}}"'] & \alpha^{n_2} \arrow[u, "\kappa_2"'] & \alpha^{n_3} \arrow[l, "\mathbf{c}^{(n_3)}_{\alpha^{n_2}}"] \arrow[u, "\mathbf{d}^{(n_3)}_{\alpha^{n_4}}"']
\end{tikzcd}$$

Proposition 9.5. *The above colimits are preserved by the functor*

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \mathbf{pPoly}.$$

Moreover, for any ω-category C, we have bijective correspondences

$$\omega Cat((\alpha^n)^*, C) = C_n$$

$$\omega Cat((\alpha^{n_0,n_1,n_2})^*, C) = \{(x,y) \in C_{n_0} \times C_{n_2} : c^{(n_1)}(x) = d^{(n_1)}(y)\}$$

$$\omega Cat((\alpha^{n_0,n_1,n_2,n_3,n_4})^*, C) =$$

$$\{(x,y,z) \in C_{n_0} \times C_{n_2} \times C_{n_4} : c^{(n_1)}(x) = d^{(n_1)}(y) \text{ and } c^{(n_3)}(y) = d^{(n_3)}(z)\}$$

which are natural in C.

Proof. As both positive opetopic cardinals α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ are obtained via special pushout (in the second case applied twice), these colimits are preserved by $(-)^*$. $\square$

Let T be a positive opetopic cardinal. We have a functor

$$\Sigma^T : \mathbf{pOpe} \downarrow T \longrightarrow \mathbf{pOpeCard}$$

such that

$$\Sigma^T(f : B \to T) = B$$

and a cocone

$$\sigma^T : \Sigma^T \longrightarrow T$$

such that

$$\sigma^T_{(f:B\to T)} = f : \Sigma^T(f : B \to T) = B \longrightarrow T.$$

Lemma 9.6. *The cocone $\sigma^T : \Sigma^T \stackrel{\cdot}{\longrightarrow} T$ is a colimiting cocone in* $\mathbf{pOpeCard}$. *Such colimiting cocones are called* special colimits. *Any functor from* $\mathbf{pOpeCard}^{op}$ *which preserves special limits preserves special pullbacks as well.*

Proof. To see that $\sigma^T : \Sigma^T \stackrel{\cdot}{\longrightarrow} T$ is a colimiting cocone we proceed by induction on the size of T. If T is a positive opetope, then the category $\mathbf{pOpe} \downarrow T$ has terminal object id_T which is sent by Σ^T to T. Thus in this case T is the colimit of Σ^T. If T is not a positive opetope, then, by Lemma 8.3.5, it can be presented as a special pushout $T = T_2 \oplus_k T_1$

$$\begin{tikzcd}
T_2 \oplus_k T_1 & T_1 \arrow[l, "\kappa_1"'] \\
T_2 \arrow[u, "\kappa_2"] & \mathbf{d}^{(k)}T_2 \arrow[l, "\mathbf{d}^{(k)}_{T_2}"] \arrow[u, "\mathbf{c}^{(k)}_{T_1}"']
\end{tikzcd}$$

with both T_1 and T_2 of dimension larger than k and size smaller than the size of T, for some $k \in \omega$. By inductive assumption, the limits of Σ^T_1, Σ^T_2, and $\Sigma^{\mathbf{d}^{(k)}T_2}$ are T_1, T_2 and $\Sigma^{\mathbf{d}^{(k)}T_2}$, respectively. Each object $f : B \to T$ of $\mathbf{pOpe} \downarrow T$ factorises (as a morphism) via either κ_1 or κ_2. [15] If it factorises by both, it factorises by $\mathbf{d}^{(k)}T_2$. From this description it is easy to see that indeed in this case T is also the colimit of the functor Σ^T. Moreover, if the limit of Σ^T is preserved, then this special pushout is also preserved. □

Remarks and notation. The full image of the functor $(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$ will be denoted by $\mathbf{pOpeCard}_\omega$. The objects of $\mathbf{pOpeCard}_\omega$ are ω-categories isomorphic to those of form S^* for S being positive opetopic cardinal and the morphism in $\mathbf{pOpeCard}_\omega$ are all ω-functors. In fact, when convenient, we shall think about positive opetopic cardinals S as if they where ω-categories and talk about ω-functors between them. As the above embedding $(-)^*$ is conservative (Corollary 9.4), this will not lead to any confusions.

---

[15] (PLC) Here we use the fact that B is principal!

10. The inner-outer factorization in $\mathbf{pOpeCard}_\omega$

Let $f : S^* \longrightarrow T^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. We say that f is *outer*[16] if there is a map of positive opetopic cardinals $g : S \longrightarrow T$ such that $g^* = f$. We say that f is *inner* iff $f_{dim(S)}(S) = T$. From Corollary 9.3 we have

Lemma 10.1. *An ω-functor $f : S^* \longrightarrow T^*$ is outer iff it is a polygraph map.* □

Proposition 10.2. *Let $f : S^* \longrightarrow T^*$ be an inner map, $dim(S) = dim(T) > 0$. The maps $\mathbf{d}(f) : \mathbf{d}(S) \longrightarrow \mathbf{d}(T)$ and $\mathbf{c}(f) : \mathbf{c}(S) \longrightarrow \mathbf{c}(T)$, being the restrictions of f, are well defined, inner and the squares*

$$\begin{tikzcd}
(\mathbf{d}(T))^* \arrow[r, "\mathbf{d}^*_T"] & T^* & (\mathbf{c}(T))^* \arrow[l, "\mathbf{c}^*_T"'] \\
(\mathbf{d}(S))^* \arrow[r, "\mathbf{d}^*_S"'] \arrow[u, "\mathbf{d}(f)"] & S^* \arrow[u, "f"'] & (\mathbf{c}(S))^* \arrow[l, "\mathbf{c}^*_S"] \arrow[u, "\mathbf{c}(f)"']
\end{tikzcd}$$

commute.

Proof. So suppose that $f : S^* \to T^*$ is an inner map. So $f(S) = T$. Since f is an ω-functor, we have

$$f(\mathbf{d}(S)) = \mathbf{d}(f(S)) = \mathbf{d}(T) \quad \text{and} \quad f(\mathbf{c}(S)) = \mathbf{c}(f(S)) = \mathbf{c}(T).$$

This shows the proposition. □

Proposition 10.3. *The inner and outer morphisms form a factorization system in $\mathbf{pOpeCard}_\omega$. So any ω-functor $f : S^* \longrightarrow T^*$ can be factored essentially uniquely by inner map $\overset{\bullet}{f}$ followed by outer map $\overset{\circ}{f}$:*

$$\begin{tikzcd}
S^* \arrow[rr, "f"] \arrow[dr, "\overset{\bullet}{f}"'] & & T^* \\
& f(S)^* \arrow[ur, "\overset{\circ}{f}"'] &
\end{tikzcd}$$

---

[16] The names ‘inner’ and ‘outer’ are introduced in analogy with the morphisms with the same name and role in the category of disks in [11]. It is also customary nowadays to replace the terminology ‘inner’ and ‘outer’ face operator with the terminology ‘active’ and ‘inert’ monomorphism.

Proof. The factorization is almost tautological. [17] □

The inner maps between positive opetopic cardinals can be further factorized into inner epi and inner mono.

Let P and Q be positive opetopic cardinals, $f : P^* \to Q^*$ an ω-functor between ω-categories generated by them. The *kernel of* f is the set $\ker(f)$ of faces of P sent by f to identities on cells of Q^* of lower dimension. We say that a set I of faces of $P_{>0}$ is an *ideal in* P iff, for any $b \in P_{>0}$:

1. if $\gamma(b) \in I$, then $b \in I$;
2. if $\delta(b) \subseteq I$, then $b \in I$;
3. if $b \in I$, then $|\delta(b) \setminus I| = |\gamma(b)| = 1$.

The proof of the following lemma is left to the reader.

Lemma 10.4. *Let $f : P^* \to Q^*$ be an ω-functor between ω-categories generated by positive opetopic cardinals. Then* $\ker(f)$ *is an ideal.* □

We shall prove the converse of the above lemma, i.e., that any ideal is a kernel of an ω-functor, in fact an inner epi.

Recall that a face $u \in P_{>0}$ is unary if $\delta(u)$ contains one element. Let $U(P)$ be the set of unary faces in P, $I \subseteq P$ an ideal in P, and $I \neq \emptyset$. The face $u \in P$ is called *safe for* P iff $u \in U(P) - \gamma(P) - \delta(U(P)$, i.e., u is a unary face in P that is not a codomain of any other face in P and it is not in the domain of a unary face in P.

The following lemma says that we can always divide any opetopic cardinal by its set of safe faces.

Lemma 10.5. *Let P be an opetopic cardinal, and u a safe face for P. Then we can divide P by u, i.e., we have a quotient ω-functor $q_u : P^* \longrightarrow (P_{/u})^*$ whose kernel is $\{u\}$. q_u is an inner epi.*

---

[17] (PLC) The essential uniqueness is an easy consequence of Lemma 10.9 below, which implies in particular that outer maps are mono. So, if $g : S^* \to U^*$ and $h : U^* \to T^*$ give another factorisation, without loss of generality we wan assume h to be the inclusion, which forces g to be a corestriction of f, $U = f(S)$ and $g = \overset{\bullet}{f}$.

Proof. The opetope $P_{/u}$ is obtained by gluing together $\delta(u)$ and $\gamma(u)$ (to a face $\{\delta(u),\gamma(u)\}$) and dropping u. The map q_u is defined as follows (we describe it on faces of P only)

$$q_u(a) = \begin{cases} \{\delta(u),\gamma(u)\} & \text{if } a = \delta(u),\ \gamma(u),\ u \\ a & \text{otherwise.} \end{cases}$$

i.e., $q_u(\delta(u)) = q_u(\gamma(u)) = q_u(u) = \{\delta(u),\gamma(u)\}$, i.e., the equivalence class containing $\delta(u)$ and $\gamma(u)$. $q_u(a) = a$, for other faces a in P.

Then γ and δ on $P_{/u}$ are so defined to make the quotient map $q_u : P^* \longrightarrow (P_{/u})^*$ preserve both of them.

The only cell sent by q_u to a(n identity on a) cell of a lower dimension is u. Thus $\ker(q_u) = \{u\}$. $\Box$

The following two lemmas show that in any non-empty ideal I in P there is always a safe face for P.

Lemma 10.6. *Let I be a non-empty ideal in P. There is always a unary face $u \in I - \gamma(P)$.*

Proof. Suppose not, and let c be a cell of minimal dimension and $<^+$-minimal in I. If c is not unary then this contradicts condition 3., as c is of minimal dimension in I and hence $\delta(c) \cap I = \emptyset$. If $c \in \gamma(P)$, then this contradicts the choice of c as, if $\gamma(b) = c$, then $\delta(b)$ contains only unary faces, and as c is in I, by 3., $\delta(b) \subseteq I$. Thus $c \in (I \cap U(P)) - \gamma(P)$. $\Box$

Lemma 10.7. *Let I be a non-empty ideal in P. There is always a face $u \in I$ safe for P.*

Proof. Take a unary face u of maximal dimension in I. If $u \in \delta(v)$ with $v \in U(P)$, then, since I satisfies 2., $v \in I \cap U(P)$. This contradicts the choice of u, as $dim(u) < dim(v)$. $\Box$

Theorem 10.8. *If $I \subseteq P_{\geq 1}$ is an ideal, then there is an inner epi map $q_I : P^* \to (P_{/I})^*$ such that it has I as its kernel and q_I is moreover a universal map with this property, i.e., whenever there is an ω-functor $f : P^* \to Q^*$ such that $I \subseteq \ker(f)$, then there is a unique map $f' : (P_{/I})^* \to Q^*$ such that $f = f' \circ q_I$.*

Proof. We can divide the opetope P by a unary cell $u \in I - \gamma(I)$ of maximal dimension, getting the map

$$q_u : P^* \to (P_{/u})^*.$$

Then $q_u(I - \{u\})$, the image of $I - \{u\}$ in $P_{/u}$, is an ideal in $P_{/u}$. Thus we can iterate the construction until the resulting ideal will be empty.

To see that q_I has the stated universal property, it is enough to notice that if $u \in \ker(f)$ is a safe face for P, then we have a factorization

$$\begin{tikzcd}
P^* \arrow[rr, "f"] \arrow[dr, "q_u"'] & & Q^* \\
& (P_{/u})^* \arrow[ur, "f'"'] &
\end{tikzcd}$$

Using the description of q_u, this is clear. □

Note that for an ω-functor $f : P^* \to Q^*$ it might seem that to say 'that it is mono (or epi)' is ambiguous, since this can be applied to either just faces of P or all the cells of P^*. However, in both cases the notions of epi and of mono coincide. Thus, in fact, they are not ambiguous no matter how these notions are interpreted.

Lemma 10.9. *Let $f : P^* \to Q^*$ an ω-functor in* $\mathbf{pOpeCard}_\omega$. *Then* $\ker(f) = \emptyset$ *iff f is mono.*

Proof. Suppose that f is not a mono. Let $a, b \in P_m$, $m \in \omega$, be two different cells of minimal dimension such that $f(a) = f(b)$. If $m = 0$, then $a \bowtie^+ b$ by linearity of $<^+$ on P_0, and if $m > 0$, then since

$$f(\gamma(a)) = \gamma(f(a)) = \gamma(f(b)) = f(\gamma(b)),$$

and by minimality of m, we have that $\gamma(a) = \gamma(b)$. Then by pencil linearity we have $a \bowtie^+ b$, as well. Suppose $a <^+ b$. Thus there is an upper path $a, \alpha_1, \ldots, \alpha_k, b$ in P. As $f(a) = f(b)$, we have $f(\alpha_i) = f(a)$ [18], and hence, $\alpha_i \in \ker(f)$ for $i =, 1, \ldots, k$, i.e., $\ker(f) \neq \emptyset$. □

---

[18] (PLC) It is a peculiarity of the polygraphs of the form P^* that non-identity cells have non-intersecting domains and codomains.

Theorem 10.10. *The inner epis and inner monos form a factorization system on the category* $\mathbf{pOpeCard}_{inn}$ *of opetopic cardinals with inner maps.* [19]

Proof. Let $f : P^* \to Q^*$ be an inner map. Then, by Lemma 10.4, $\ker(f) = I$ is an ideal. By the universal property of q_I stated in Theorem 10.8 we have a factorization

$$\begin{tikzcd}
P^* \arrow[rr, "f"] \arrow[dr, "q_I"'] & & Q^* \\
& (P_{/I})^* \arrow[ur, "f'"'] &
\end{tikzcd}$$

with q_I inner epi. Since $\ker(f) = I = \ker(q_I)$, it follows that $\ker(f') = \emptyset$. Moreover, f' is inner since f is. $\Box$

11. The terminal positive-to-one polygraph

In this section we shall describe the terminal positive-to-one polygraph $\mathcal{T}$ as an ω-category.

The set of n-cells $\mathcal{T}_n$ consists of (isomorphisms classes of) positive opetopic cardinals of dimension less than or equal to n. For $n > 0$, the operations of domain and codomain $d^{\mathcal{T}}, c^{\mathcal{T}} : \mathcal{T}_n \to \mathcal{T}_{n-1}$ are given, for $S \in \mathcal{T}_n$, by [20]

$$d(S) = \begin{cases} S & \text{if } dim(S) < n \\ \mathbf{d}(S) & \text{if } dim(S) = n, \end{cases}$$

and

$$c(S) = \begin{cases} S & \text{if } dim(S) < n \\ \mathbf{c}(S) & \text{if } dim(S) = n. \end{cases}$$

---

[19] (PLC) Putting together the two factorisations, we have thus that an ω-functor f in $\mathbf{pOpeCard}_\omega$ decomposes as $f_3 \circ f_2 \circ f_1$ with f_1 inner epi, f_2 inner mono and f_3 an outer map, and where f_1(resp. f_2) is non trivial iff f_1 maps at least one generator to an identity (resp. f_2 maps at least one generator to a cell that is neither an identity nor a generator).

[20] (PLC) An equivalent description consists in setting $d^{\mathcal{T}} = \mathbf{d}^{(n-1)}$ and $c^{\mathcal{T}} = \mathbf{c}^{(n-1)}$ (cf. Section 6).

and, for $S, S' \in \mathcal{T}_n$ such that $c^{(k)}(S) = d^{(k)}(S')$, the composition in $\mathcal{T}$ is just the special pushout

$$S' \circ_k S = S' \oplus_k S,$$

i.e.,

$$\begin{tikzcd}
S' \oplus_k S & S \arrow[l] \\
S' \arrow[u] & \mathbf{c}^{(k)}(S) \arrow[l, "\mathbf{d}^{(k)}"] \arrow[u, "\mathbf{c}^{(k)}"']
\end{tikzcd}$$

The identity $id_{\mathcal{T}} : T_{n-1} \to \mathcal{T}_n$ is the inclusion map. The n-indeterminates in $\mathcal{T}$ are positive opetopic cardinals of dimension n.

Proposition 11.1. *$\mathcal{T}$ just described is the terminal positive-to-one polygraph.*

Proof. The fact that $\mathcal{T}$ is an ω-category is easy. The fact that $\mathcal{T}$ is free with free n-indeterminates being opetopes of dimension n can be shown much like the freeness of S^* in the proof of Proposition 9.1. The fact that $\mathcal{T}$ is terminal relies on the following observation.

Observation. For every pair of parallel positive opetopic cardinals of dimension n, N and B (i.e., $\mathbf{d}(N) = \mathbf{d}(B)$ and $\mathbf{c}(N) = \mathbf{c}(B)$) such that B is principal, it follows that N is normal and there is a unique (up to an iso) principal positive opetopic cardinal $N^\bullet$ of dimension $n+1$ such that $\mathbf{d}(N)^\bullet = N$ and $\mathbf{c}(N)^\bullet = B$. [21]

The universal property of $\mathcal{T}$ is then established by induction on the dimension. Let P be a positive-to-one polygraph and let f be a morphism from P to $\mathcal{T}$. Then $f(\alpha)$ has to be a generator, which is uniquely determined by induction and by the observation. □

Lemma 11.2. *Let S be a positive opetopic cardinal and $! : S^* \longrightarrow \mathcal{T}$ the unique map from S^* to $\mathcal{T}$. Then, for $T \in S^*_k$, we have*

$$!_k(T) = T.$$

---

[21] (PLC) That N is normal follows from point 1. of Lemma 7.2 and from the fact that the codomain of N is principal, being the codomain of B. Also, following up with the comment on the uniqueness of $N^\bullet$ (Section 7), we also have, for parallel N and B as above, that in fact $B = (\mathbf{d}N)^\bullet$.

Proof. The proof is by induction on $k \in \omega$ and the size of T in S^*_k. For $k = 0, 1$, the lemma is obvious. Let $k > 1$ and assume that the lemma holds for $i < k$.

If $dim(T) = l < k$, then, using the inductive hypothesis and the fact that ! is an ω-functor, we have

$$!_k(T) = !_k(1^{(k)}_T) = 1^{(k)}_{!_l(T)} = 1^{(k)}_T = T.$$

Suppose that $dim(T) = k$ and T is principal. As ! is a polygraph map, $!_k(T)$ is an indeterminate, and thus principal. Using again the inductive hypothesis and the fact that ! is an ω-functor, we obtain

$$d(!_k(T)) = !_{k-1}(\mathbf{d}(T)) = \mathbf{d}(T)$$

$$c(!_k(T)) = !_{k-1}(\mathbf{c}(T)) = \mathbf{c}(T).$$

As T is the only (up to a unique iso) positive opetopic cardinal with the domain $\mathbf{d}(T)$ and the codomain $\mathbf{c}(T)$, it follows that $!_k(T) = T$, as required.

Finally, suppose that $dim(T) = k$, T is not principal, and that the lemma holds for all positive opetopic cardinals of size smaller than the size of T. Thus there are $l \in \omega$ and $a \in Sd(T)_l$ so that

$$!_k(T) = !_k(T^{\downarrow a} \oplus_l T^{\uparrow a}) = !_k(T^{\downarrow a}) \oplus_l !_k(T^{\uparrow a}) = T^{\downarrow a} \oplus_l T^{\uparrow a} = T,$$

as required $\Box$

12. A description of the positive-to-one polygraphs

In this section we shall describe all the cells in positive-to-one polygraphs using positive opetopic cardinals, in other words we shall describe in concrete terms the functor:

$$\overline{(-)}^n : Set \downarrow D_{n-1} \longrightarrow \mathbf{pPoly}_n.$$

More precisely, the positive-to-one polygraphs of dimension 1 (and all polygraphs, as well) are free polygraphs over graphs and are well understood. So suppose that $n > 1$, and we are given an object of $Set \downarrow D_{n-1}$, i.e., a quadruple $(|P|_n, P, d, c)$ such that

1. a positive-to-one $(n-1)$-polygraph P;
2. a set $|P|_n$ with two functions $c : |P|_n \longrightarrow |P|_{n-1}$ and $d : |P|_n \longrightarrow P_{n-1}$ such that, for $x \in |P|_n$, $cc(x) = cd(x)$ and $dc(x) = dd(x)$; we assume that $d(x)$ is not an identity, for any $x \in |P|_n$.

If the maps d and c in the object $(|P|_n, P, d, c)$ are understood from the context, we can abbreviate the notation to $(|P|_n, P)$.

Recall that for a positive opetopic cardinal S, with $dim(S) \leq n$, we denote by $S^{\sharp,n}$ the object $(S_n, (S_{<n})^*, [\delta], [\gamma])$ in $Set \downarrow D_{n-1}$. In fact, we have an obvious functor

$$(-)^{\sharp,n} : \mathbf{pOpeCard} \longrightarrow Set \downarrow D_{n-1}$$

$$S \mapsto (S_n, (S_{<n})^*, [\delta], [\gamma]).$$

We shall describe the positive-to-one n-polygraph $\overline{P} = \overline{(|P|_n, P, d, c)}^n$ whose $(n-1)$-truncation is P and whose n-indeterminates are $|P|_n$ with the domains and codomains given by maps c and d.

n-cells of $\overline{P}$. An n-cell in $\overline{P}_n$ is a(n equivalence class of) pair(s) (S, f), which we shall call *auxiliary descriptions*, where

1. S is a positive opetopic cardinal, $dim(S) \leq n$;
2. $f : (S_n, (S_{<n})^*, [\delta], [\gamma]) \longrightarrow (|P|_n, P, d, c)$ is a morphism in $Set \downarrow D_{n-1}$, i.e., $f = (|f|_n, f_{<n})$, and

$$\begin{tikzcd}
S_n \arrow[r, "|f|_n"] \arrow[d, shift right, "{[\delta]}"'] \arrow[d, shift left, "{[\gamma]}"] & {|P|_n} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
(S_{<n})^* \arrow[r, "f_{<n}"'] & P
\end{tikzcd}$$

commutes.

We identify two pairs (S, f), (S', f') if there is an isomorphism $h : S \longrightarrow S'$ such that the triangles of sets and of $(n-1)$-polygraphs

$$\begin{tikzcd}
S_n \arrow[rr, "h_n"] \arrow[dr, "|f|_n"'] & & S'_n \arrow[dl, "|f'|_n"] \\
& |P|_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
(S_{<n})^* \arrow[rr, "(h_{<n})^*"] \arrow[dr, "f_{<n}"'] & & (S'_{<n})^* \arrow[dl, "f'_{<n}"] \\
& P &
\end{tikzcd}$$

commute. Clearly, if such an h exists, it is unique. Even if formally cells in P_n are equivalence classes of triples, we will work on triples themselves as if they were cells understanding that equality between such cells is an isomorphism in the sense defined above.

Domains and codomains. The domain and codomain functions

$$d^{(k)}, c^{(k)} : \overline{P}_n \longrightarrow \overline{P}_k$$

are defined for an n-cell (S, f) as follows:

$$d^{(k)}(S, f) = (\mathbf{d}^{(k)}(S), \mathbf{d}^{(k)} f)$$

where, for $x \in (\mathbf{d}^{(k)}(S))_k$

$$(\mathbf{d}^{(k)} f)_k(x) = f_k([x])(x)$$

(i.e., we take the positive opetopic cardinals $[x]$ contained in S, then the value of f on it, and then we evaluate the map in $Set \downarrow D_{n-1}$ on x, the only element of $[x]_k$),

$$(\mathbf{d}^{(k)} f)_l = f_l$$

for $l < k$;

$$c^{(k)}(S, f) = (\mathbf{c}^{(k)}(S), \mathbf{c}^{(k)}(f))$$

where, for $x \in (\mathbf{c}^{(k)}(S))_k$

$$(\mathbf{c}^{(k)}(f))_k(x) = f_k([x])(x)$$

and

$$(\mathbf{d}^{(k)}(f))_l = f_l$$

for $l < k$, i.e., we calculate the k-th domain and k-th codomain of an n-cell (S, f) by taking $\mathbf{d}^{(k)}$ and $\mathbf{c}^{(k)}$ of the domain S of the cell f, respectively, and by restricting the maps f accordingly.

Identities. The identity function

$$\mathbf{i} : \overline{P}_{n-1} \longrightarrow \overline{P}_n$$

is defined, for an $(n-1)$-cell $((S,f)$ in P_{n-1}, as follows:

$$\mathbf{i}(S,f) = \begin{cases} (S,f) & \text{if } dim(S) < n-1 \\ (S,\overline{f}) & \text{if } dim(S) = n-1. \end{cases}$$

Note that $\overline{f}$ is the map $\mathbf{pPoly}_{n-1}$ which is the value of the functor $\overline{(-)}$ on a map f from $Set \downarrow D_{n+1}$. So it is in fact defined as 'the same $(n-1)$-cell' but considered as an n-cell.

Compositions. Suppose that (S^i, f^i) are n-cells for $i = 0, 1$ such that

$$c^{(k)}(S^0, f^0) = d^{(k)}(S^1, f^1).$$

Then their composition is defined, via pushout in $Set \downarrow D_{n-1}$, as

$$(S^1, f^1) \circ_k (S^0, f^0) = (S^1 \oplus_k S^0, [f^1, f^0]),$$

i.e.,

$$\begin{tikzcd}
S^0_n \sqcup S^1_n \arrow[rr, "{[f^0_n, f^1_n]}"] \arrow[d, shift right=2, "{[\delta]}"'] \arrow[d, shift left=2, "{[\gamma]}"] & & {|P|_n} \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
((S^0 \oplus_k S^1)_{\leq n-1})^*_{n-1} \arrow[rr, "{[f^0_{n-1}, f^1_{n-1}]}"'] & & P_{n-1}
\end{tikzcd}$$

This ends the description of the polygraph $\overline{P}$.

Now let $h : P \to Q$ be a morphism in $Set \downarrow D_{n-1}$, i.e., a function $h_n : |P|_n \longrightarrow |Q|_n$ and a $(n-1)$-polygraph morphism $h_{<n} : P_{<n} \longrightarrow Q_{<n}$ such that the square

$$\begin{tikzcd}
{|P|_n} \arrow[r, "h_n"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & {|Q|_n} \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
P_{n-1} \arrow[r, "h_{n-1}"'] & Q_{n-1}
\end{tikzcd}$$

commutes serially. We define

$$\bar{h} : \bar{P} \longrightarrow \bar{Q}$$

by putting $\bar{h}_k = h_k$, for $k < n$, and, for $(S, f) \in \bar{P}_n$, we put

$$\bar{h}(S, f) = (S, h \circ f).$$

Notation. Let $x = (S, f)$ be a cell in $\bar{P}_n$ as above, and $a \in Sd(S)$. Then we denote by $x^{\downarrow a} = (S^{\downarrow a}, f^{\downarrow a})$ and $x^{\uparrow a} = (S^{\uparrow a}, f^{\uparrow a})$ the cells in $\overline{P}_n$ that are the obvious restrictions of x. Clearly, we have $c^{(k)}(x^{\downarrow a}) = d^{(k)}((x^{\uparrow a})$ and $x = x^{\uparrow a} \circ_k x^{\downarrow a}$, where $k = dim(a)$.

In the following proposition, we collect several statements concerning the above construction. This includes that the above construction is correct. We need to prove them together, that is, by simultaneous induction.

Proposition 12.1. *Let $n \in \omega$. We have*

1. *Let P be an object of $Set \downarrow D_n$. We define the function*

$$\eta_P : |P|_n \longrightarrow \overline{P}_n$$

as follows. Let $x \in |P|_n$. By induction, there is a unique description (as defined in Section 9)

$$< T_{d(x)}, \tau_{d(x)} : T^*_{d(x)} \longrightarrow P_{<n} >$$

of the cell $d(x)$, where $T_{d(x)}$ is a normal positive opetopic cardinal. [22] *Then we have a unique auxiliary description in $\overline{P}$*

$$\bar{x} = ((T_{d(x)})^\bullet, (|\overline{\tau}_x|_n : \{(T_{d(x)})^\bullet\} \to |P|_n, (\overline{\tau}_x)_{<n} : ((T_{d(x)})^\bullet)^*_{<n} \to P_{<n}))$$

(note: $|(T_{d(x)})^\bullet|_n = \{(T_{d(x)})^\bullet\}$) such that

$$|\overline{\tau}_x|_n((T_{d(x)})^\bullet) = x$$

---

[22] (PLC) This follows from the observation in the proof of Proposition 11.1 and from the following one: since by definition of positive-to-one polygraphs $d(x)$ is not an identity cell, it follows that $T_{d(x)}$ has the same dimension as $T_{c(x)}$ which is principal and parallel to it.

and

$$(\overline{\tau}_x)_{n-1}(S) = \begin{cases} c(x) & \textit{if } S = \mathbf{c}((T_{d(x)})^\bullet) \\ (\tau_{dx})_{n-1}(S) & \textit{if } S \subseteq T_{dx} \end{cases}$$

and $(\overline{\tau}_x)_{<(n-1)} = (\tau_{dx})_{<(n-1)}$. *We put* $\eta_P(x) = \bar{x}$.

Then $\overline{P}$ *is a positive-to-one polygraph with* η_P *the inclusion of* n*-indeterminates. Then any positive-to-one* n*-polygraph* Q *is equivalent to a polygraph* $\overline{P}$*, for some* P *in* $Set \downarrow D_{n-1}$.

2. *Let* P *be an object of* $Set \downarrow D_{n-1}$*,* $! : \overline{P} \longrightarrow \mathcal{T}$ *the unique morphism into the terminal object* $\mathcal{T}$ *and* $f : S^{\sharp,n} \to P$ *a cell in* $\overline{P}_n$*. Then*

$$!_n(f : S^{\sharp,n} \to P) = S.$$

3. *Let* $h : P \to Q$ *be an object of* $Set \downarrow D_{n-1}$*. Then* $\bar{h} : \overline{P} \longrightarrow \overline{Q}$ *is a polygraph morphism.*

4. *Let* S *be a positive opetopic cardinal of dimension at most* n*. For a morphism* $f : S^{\sharp,n} \longrightarrow P$ *in* $Set \downarrow D_{n-1}$*, we have that*

$$\overline{f}_k(T) = f \circ (i_T)^{\sharp,n}$$

where $k \leq n$*,* $T \in S^*_k$ *and* $i_T : T \longrightarrow S$ *is the inclusion.*

5. *Let* S *be a positive opetopic cardinal of dimension* n*,* P *a positive-to-one polygraph,* $g, h : S^* \longrightarrow P$ *polygraph maps. Then*

$$g = h \qquad \text{iff} \qquad g_n(S) = h_n(S).$$

6. *Let* S *be a positive opetopic cardinal of dimension at most* n*,* P *be an object in* $Set \downarrow D_{n-1}$*. Then we have a bijective correspondence*

$$\frac{f : S^{\sharp,n} \longrightarrow P \quad \in Set \downarrow D_{n-1}}{\overline{f} : S^* \longrightarrow \overline{P} \quad \in \mathbf{pPoly}_n}$$

such that $\overline{f}_n(S) = f$*, and, for* $g : S^* \longrightarrow \overline{P}$*, we have* $g = \overline{g_n(S)}$*.* [23]

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[23] (PLC) The correspondence $f \mapsto \overline{f}$ shows the equivalence between descriptions and auxiliary descriptions (a terminology introduced in the revision). Systematically unfolding an auxiliary description yields the following informal third description: a cell of a positive-to-one polygraph is a positive opetopic cardinal all of whose faces are (consistently) decorated by generators.

7. *We have a bijection*

$$\kappa^P_n : \coprod_S \mathbf{pPoly}(S^*, \overline{P}) \longrightarrow \overline{P}_n$$

$$g : S^* \to \overline{P} \quad \mapsto \quad g_n(S)$$

where the coproduct is taken over all (up to iso) positive opetopic cardinals S of dimension at most n. In other words, any cell in $\bar{P}$ has a unique description.

Proof. Ad 1. We have to verify that $\overline{P}$ satisfies the laws of ω-categories and that it is free in the appropriate sense.

The laws for ω-categories are left for the reader, as they easily follow from the fact that S^* is a positive to one-polygraph for any positive opetopic cardinal S. We shall show that $\overline{P}$ is free in the appropriate sense.

Let C be an ω-category, $g_{<n} : P_{<n} \to C_{<n}$ and $(n-1)$-functor and $g_n : |P|_n \to C_n$ a function so that the diagram

$$\begin{tikzcd}
{|P|_n} \arrow[r, "g_n"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & C_n \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
P_{n-1} \arrow[r, "g_{n-1}"'] & C_{n-1}
\end{tikzcd}$$

commutes serially. We shall define an n-functor $\overline{g} : \overline{P} \to C$ extending $g_{<n}$ and g_n. For $x = (S, f) \in \overline{P}_n$, we put

$$\overline{g}_n(x) = \begin{cases} 1_{g_{n-1}\circ f_{n-1}(S)} & \text{if } dim(S) < n \\ g_n \circ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ is principal, } S_n = \{m_S\} \\ \overline{g}_n(x^{\uparrow a}) \circ_k \overline{g}_n(x^{\downarrow a}) & \text{if } dim(S) = n,\ a \in Sd(S)_k. \end{cases}$$

We need to check that $\overline{g}$ is well defined, that it is unique extending g, and that it preserves domains, codomains, compositions and identities.

All these calculations are similar, and they are very much like those in the proof of Proposition 9.1 and use facts from Section 8. We shall check, assuming that we already know that $\overline{g}$ is well defined and preserves identities, that compositions are preserved. The proof is by induction on the size of the composition and uses Lemma 8.7. So let T, T_1, T_2 be positive opetopic cardinals such that $T = T_2 \oplus_k T_1$. Since $\overline{g}$ preserves identities, we can restrict our attention to the case $dim(T_1), dim(T_2) > k$.

Fix $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$. So $a \in Sd(T)_k$. If $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$, then we have

$$\overline{g}(T) = \overline{g}(T^{\uparrow a}) \circ_k \overline{g}(T^{\downarrow a}) = \overline{g}(T_2) \circ_k \overline{g}(T_1).$$

If $a \in Sd(T_1)_k$, then

$$\begin{aligned} \overline{g}(T) = \overline{g}(T^{\uparrow a}) \circ_k \overline{g}(T^{\downarrow a}) = \\ = (\overline{g}(T_2) \circ_k \overline{g}(T_1^{\uparrow a})) \circ_k \overline{g}(T^{\downarrow a}) = \\ = \overline{g}(T_2) \circ_k (\overline{g}(T_1^{\uparrow a}) \circ_k \overline{g}(T_1^{\downarrow a})) = \\ \overline{g}(T_2) \circ_k \overline{g}(T_1). \end{aligned}$$

The remaining verifications are similar.

Ad 2. Let $! : \overline{P} \longrightarrow \mathcal{T}$ be the unique polygraph map into the terminal object, S a positive opetopic cardinal such that $dim(S) = l \leq n$, $f : S^{\sharp,n} \longrightarrow P$ a cell in $\overline{P}_n$.

If $l < n$, then by induction we have $!_n(f) = S$. If $l = n$ and S is principal, then we have, by induction

$$!_n(d(f) : (\mathbf{d}(S))^{\sharp,n} \to P) = \mathbf{d}(S) \qquad !_n(c(f) : (\mathbf{c}(S))^{\sharp,n} \to P) = \mathbf{c}(S).$$

As f is an indeterminate in $\overline{P}$, $!_n(f)$ is a positive opetope. But the only (up to an iso) positive opetope B such that

$$\mathbf{d}(B) = \mathbf{d}(S) \qquad \mathbf{d}(B) = \mathbf{d}(S)$$

is S itself. Thus, in this case, $!_n(f) = S$.

Now assume that $l = n$, and S is not principal, and that for positive opetopic cardinals T of smaller size than S the statement holds. Let $a \in Sd(S)_k$. We have

$$!_n(f) = f^{\uparrow a}) \circ_k !_n(f^{\downarrow a} = !_n(f^{\uparrow a}) \oplus_k !_n(f^{\downarrow a}) = S^{\uparrow a} \oplus_k S^{\downarrow a} = S$$

where $f^{\downarrow a} = f \circ (\kappa^{\downarrow a})^{\sharp,n}$ and $f^{\uparrow a} = f \circ (\kappa^{\uparrow a})^{\sharp,n}$ and $\kappa^{\downarrow a}$ and $\kappa^{\uparrow a}$ are the maps as in the following pushout:

$$\begin{tikzcd}
S & S^{\downarrow a} \arrow[l, "\kappa^{\downarrow a}"'] \\
S^{\uparrow a} \arrow[u, "\kappa^{\uparrow a}"] & \mathbf{c}^{(k)}(S) \arrow[l] \arrow[u]
\end{tikzcd}$$

Ad 3. The main thing is to show that $\overline{h}$ preserves compositions. This follows from the fact that the functor

$$(-)^{\sharp,n} : \mathbf{pOpeSet}_n \longrightarrow Set \downarrow D_{n-1}$$

preserves special pullbacks.

Ad 4. This is an immediate consequence of 3.

Ad 5. Let S be a positive opetopic cardinal S of dimension at most n. To prove 5., we are going to use the auxiliary description of the n-cells in positive-to-one polygraphs given in 1. Moreover, note that by 3. and Lemma 11.2 we have that for $T \in S^*_k$, the value of g at T is a map in $Set \downarrow D_k$ such that $g_k(T) : T^{\sharp,k} \longrightarrow U_k(P)$, i.e., the domain of $g_k(T)$ is necessarily $T^{\sharp,k}$.

The implication $\Rightarrow$ is obvious. So assume that $g, h : S^* \longrightarrow P$ are different polygraph maps. Then there is $k \leq n$ and $x \in S_k$ such that $g_k([x]) \neq h_k([x])$. We shall show, by induction on the size of T, that for any $T \in S^*_l$ such that $x \in T$, we have

$$g_k(T) \neq h_k(T). \tag{6}$$

If $T = [x]$, then T has the least size among those positive opetopic cardinals that contain x, and (6) holds in this case by assumption.

Suppose that (6) holds for all $U \in S^*_{l'}$ whenever $l' < l$ and $x \in U$. Suppose that $T = [y]$, for some $y \in S_l$, and $x \in [y]$. Then either $x \in \mathbf{d}[y]$ or $x \in \mathbf{c}([y])$. In the former case we have, by inductive hypothesis, that $g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T)$. Thus

$$d(g_k(T)) = g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T) = d(g_k(T)).$$

But then (6) holds as well. The latter case ($x \in \mathbf{c}([y])$) is similar.

Now suppose that T is not principal $x \in T$ and that, for U of a smaller size with $x \in U$, the condition (6) holds. Let $a \in Sd(T)_r$. Then either $x \in T^{\downarrow a}$ or $x \in T^{\uparrow a}$. Both cases are similar, so we will consider the first one only. Thus, as $T^{\downarrow a}$ has a smaller size than T, by inductive hypothesis we have

$$g_k(T^{\downarrow a}) \neq h_k(T^{\downarrow a}). \tag{7}$$

As the compositions in P are calculated via pushouts, we have that

$$g_l(T^{\uparrow a}) \circ_r g_l(T^{\downarrow a}) = [g_l(T^{\uparrow a}), g_l(T^{\downarrow a})],$$

where $[g_l(T^{\downarrow a}), g_l(T^{\uparrow a})]$ is the unique morphism from the pushout as in the following diagram:

$$\begin{tikzcd}
 & & (|P|_n, P, d, c) \\
(T^{\uparrow a})^{\sharp,l} \arrow[r] \arrow[urr, bend left, "g_l(T^{\downarrow a})"] & T^{\sharp,l} \arrow[ur, "{[g_l(T^{\uparrow a}), g_l(T^{\downarrow a})]}"] & \\
(\mathbf{c}(T^{\uparrow a}))^{\sharp,l} \arrow[r] \arrow[u] & (T^{\downarrow a})^{\sharp,l} \arrow[u] \arrow[uur, "g_l(T^{\uparrow a})"'] &
\end{tikzcd}$$

Similarly

$$h_l(T^{\uparrow a}) \circ_r h_l(T^{\downarrow a}) = [h_l(T^{\uparrow a}), h_l(T^{\downarrow a})].$$

As morphisms from the pushout are equal if and only if both of their components are equal we have

$$g_l(T) = g_l(T^{\uparrow a} \oplus_r T^{\downarrow a}) = g_l(T^{\uparrow a}) \circ_r g_l(T^{\downarrow a}) =$$

$$[g_l(T^{\uparrow a}), g_l(T^{\downarrow a})] \neq [h_l(T^{\uparrow a}), h_l(T^{\downarrow a})] =$$

$$= h_l(T^{\uparrow a}) \circ_r h_l(T^{\downarrow a}) = h_l(T^{\uparrow a} \oplus_r T^{\downarrow a}) = h_l(T).$$

Thus (6) holds for all $T \in S^*$ such that $x \in T$. As $x \in S$, we get that

$$g_n(S) \neq h_n(S),$$

as required.

Ad 6. Fix a positive opetopic cardinal S of dimension at most n.

Let $f : S^{\sharp,n} \longrightarrow P$ be a cell in $\overline{P}_n$. By 4., we have

$$\overline{f}_n(S) = f \circ (i_S)^{\sharp,n} = f \circ (1_S)^{\sharp,n} = f \circ (1_S^{\sharp,n}) = f.$$

Let $g : S^* \longrightarrow \overline{P}$ be a polygraph map. To show that $g = \overline{g_n(S)}$, by 5., it is enough to show that

$$(\overline{g_n(S)})_n(S) = g_n(S).$$

Using 4. again, we have

$$(\overline{g_n(S)})_n(S) = g_n(S) \circ (i_S)^{\sharp,n} =$$

$$= g_n(S) \circ i_{S^{\sharp,n}} = g_n(S) \circ 1_{S^{\sharp,n}} = g_n(S).$$

Thus, by 5., $(\overline{g_n(S)}) = g$.

Ad 7. It follows immediately from 6. □

From Proposition 12.1.7 we know that each cell in a positive-to-one polygraph has (up to an isomorphism) a unique description. The following proposition is a bit more specific.

Proposition 12.2. *Let P be a positive-to-one polygraph, $n \in \omega$, and $a \in P_n$. Let T_a be $!^P_n(a)$ (where $!^P : P \longrightarrow \mathcal{T}$ is the unique morphism into the terminal polygraph). Then there is a unique polygraph map $\tau_a : T_a^* \longrightarrow P$ such that $(\tau_a)_n(T_a) = a$, i.e., each cell has an essentially unique description. Moreover:*

1. *For any $a \in P$, we have*

$$\tau_{da} = d(\tau_a) = \tau_{da} = \tau_a \circ (\mathbf{d}_{T_a})^* \qquad \tau_{c(a)} = c(\tau_a) = \tau_{c(a)} = \tau_a \circ (\mathbf{c}_{T_a})^*$$

$$\tau_{1_a} = \tau_a.$$

2. *For any $a, b \in P$ such that $c^{(k)}(a) = d^{(k)}(b)$, we have*

$$\tau_{a;_k b} = [\tau_a, \tau_b] : T_a^* +_{\mathbf{c}^{(k)}T_a^*} T_b^* \longrightarrow P.$$

3. *For any positive opetopic cardinal S, for any polygraph map $f : S^* \longrightarrow P$,*

$$\overline{\tau_{f_n(S)}} = f.$$

4. *For any positive opetopic cardinal S, any ω-functor $f : S^* \longrightarrow P$ can be essentially uniquely factorized as*

$$\begin{tikzcd}
S^* \arrow[rr, "f"] \arrow[dr, "f^{in}"'] & & P \\
& T^*_{f(S)} \arrow[ur, "\tau_{f(S)}"'] &
\end{tikzcd}$$

where f^{in} is an inner map and $(\tau_{f(S)}, T_{f(S)})$ is the description of the cell $f(S)$.

Proof. Using the above description of the positive-to-one polygraph P, we have that $a : (T_a)^{\sharp,n} \longrightarrow (|P|_n, P, d, c)$. We put $\tau_a = \overline{a}$. By Proposition 12.1.6, we have that $(\tau_a)_n(T_a) = \overline{a}_n(T_a) = a$, as required.

The uniqueness of (T_a, τ_a) follows from Proposition 12.1 point 5.

The remaining part is left for the reader. □

13. Positive-to-one polygraphs form a presheaf category

In this section we want to prove that the category $\mathbf{pPoly}$ is equivalent to the presheaf category $\widehat{\mathbf{pOpe}}$. In fact, we will show that both categories are equivalent to the category $sPb((\mathbf{pOpeCard})^{op}, Set)$ of special pullbacks preserving functors from $(\mathbf{pOpeCard})^{op}$ to Set.

First note that the inclusion functor $\mathbf{i} : \mathbf{pOpe} \longrightarrow \mathbf{pOpeCard}$ induces the adjunction

$$\begin{tikzcd}
\widehat{\mathbf{pOpe}} \arrow[r, shift left, "Ran_{\mathbf{i}}"] & \widehat{\mathbf{pOpeCard}} \arrow[l, shift left, "\mathbf{i}^*"]
\end{tikzcd}$$

where $\mathbf{i}^*$ is the functor of composing with $\mathbf{i}$ and $Ran_{\mathbf{i}}$ is the right Kan extension along $\mathbf{i}$. Recall that for F in $\widehat{\mathbf{pOpe}}$, S in $\mathbf{pOpeCard}$, it is defined as the following limit

$$(Ran_{\mathbf{i}}F)(S) = Lim(F \circ \Sigma^{S,op})$$

where $\Sigma^{S,op}$ is the dual of the functor Σ^S defined before Lemma 9.6. Note that, as $(\mathbf{pOpe} \downarrow S)^{op} = S \downarrow (\mathbf{pOpe})^{op}$, we have

$$\Sigma^{S,op} : S \downarrow (\mathbf{pOpe})^{op} \longrightarrow (\mathbf{pOpe})^{op}.$$

As $\mathbf{i}$ is full and faithful, the right Kan extension $Ran_{\mathbf{i}}(F)$ is an extension. Therefore the counit of this adjunction

$$\varepsilon_F : (Ran_{\mathbf{i}}\, F) \circ \mathbf{i} \longrightarrow F$$

is an isomorphism. The functor $Ran_{\mathbf{i}} F$ is so defined that it preserves special limits. Hence, by Lemma 9.6, it preserves special pullbacks. As any positive opetopic cardinal can be constructed from positive opetopes via special pushouts, for G in $\widehat{\mathbf{pOpeCard}}$, the unit

$$\eta_G : G \longrightarrow Ran_{\mathbf{i}}(G \circ \mathbf{i})$$

is an isomorphism iff G preserves special pullbacks. This establishes the following proposition.

Proposition 13.1. *The above adjunction restricts to the following equivalence of categories*

$$\begin{tikzcd} \widehat{\mathbf{pOpe}} \arrow[r, shift left, "Ran_{\mathbf{i}}"] & sPb((\mathbf{pOpeCard})^{op}, Set). \arrow[l, shift left, "\mathbf{i}^*"] \end{tikzcd}$$

□

Now we will set up the adjunction

$$\begin{tikzcd} sPb((\mathbf{pOpeCard})^{op}, Set) \arrow[r, shift left, "\widetilde{(-)}"] & \mathbf{pPoly} \arrow[l, shift left, "\widehat{(-)} = \mathbf{pPoly}((\simeq)^*, -)"] \end{tikzcd}$$

which will be later proved to be an equivalence of categories. The functor $\widehat{(-)}$ sends a positive-to-one polygraph P to the functor

$$\widehat{P} = \mathbf{pPoly}((-)^*, P) : (\mathbf{pOpeCard})^{op} \longrightarrow Set.$$

The action of $\widehat{(-)}$ on morphisms is defined in the obvious way, by composition.

Lemma 13.2. *Let P be a positive-to-one polygraph. Then $\widehat{P}$ defined above is a special pullbacks preserving functor.*

Proof. This is an immediate consequence of the fact that the functor $(-)^*$ preserves special pushouts. $\Box$

Now suppose we have a special pullbacks preserving functor $F : (\mathbf{pOpeCard})^{op} \longrightarrow Set$. We shall define a positive-to-one polygraph $\widetilde{F}$.

As n-cells of $\widetilde{F}$ we put

$$\widetilde{F}_n = \coprod_S F(S)$$

where the coproduct is taken over all[24] (up to iso) positive opetopic cardinals S of dimension at most n.

If $k \leq n$, the identity map

$$1^{(n)} : \widetilde{F}_k \longrightarrow \widetilde{F}_n$$

is the obvious embedding induced by identity maps on the components of the coproducts.

Now we shall describe the domains and codomains in $\widetilde{F}$. Let S be a positive opetopic cardinal of dimension at most n, $a \in F(S) \hookrightarrow \widetilde{F}_n$. We have in $\mathbf{pOpeCard}$ the k-th domain and the k-th codomain morphisms:

$$\begin{tikzcd}
 & S & \\
\mathbf{d}^{(k)}S \arrow[ur, "\mathbf{d}_S^{(k)}"] & & \mathbf{c}^{(k)}(S) \arrow[ul, "\mathbf{c}_S^{(k)}"']
\end{tikzcd}$$

We put

$$\mathbf{d}^{(k)}(a) = F(\mathbf{d}_S^{(k)})(a) \in F(\mathbf{d}^{(k)}(S)) \hookrightarrow \widetilde{F}_k$$

$$\mathbf{c}^{(k)}(a) = F(\mathbf{c}_S^{(k)})(a) \in F(\mathbf{c}^{(k)}(S)) \hookrightarrow \widetilde{F}_k.$$

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[24] In fact, we think about such a coproduct $\coprod_S F(S)$ as if it were to be taken over a sufficiently large (so that each isomorphism type of positive opetopic cardinals is represented) set of positive opetopic cardinals S of dimension at most n. Then, if positive opetopic cardinals S and S' are isomorphic via (necessarily unique) isomorphism h, then the cells $x \in F(S)$ and $x' \in F(S')$ are considered equal iff $F(h)(x) = x'$.

Finally, we define the compositions in $\widetilde{F}$. Let $n_1, n_2 \in \omega$, $n = max(n_1, n_2)$, $k < min(n_1, n_2)$, and

$$a \in F(S) \hookrightarrow \widetilde{F}_{n_1} \qquad b \in F(T) \hookrightarrow \widetilde{F}_{n_2},$$

such that

$$\mathbf{c}^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) = F(\mathbf{d}^{(k)}_T)(b) = \mathbf{d}^{(k)}(b).$$

We shall define the cell $b \circ_k a \in \widetilde{F}_n$. We take a special pushout in $\mathbf{pOpeCard}$:

$$\begin{tikzcd}
T \oplus_k S & S \arrow[l, "\kappa_1"'] \\
T \arrow[u, "\kappa_2"] & \mathbf{c}^{(k)}(S) \arrow[l, "\mathbf{d}^{(k)}_T"] \arrow[u, "\mathbf{c}^{(k)}_S"']
\end{tikzcd}$$

As F preserves special pullbacks (in $(\mathbf{pOpeCard})^{op}$), it follows that the square

$$\begin{tikzcd}
F(T \oplus_k S) \arrow[r, "F(\kappa_1)"] \arrow[d, "F(\kappa_2)"'] & F(S) \arrow[d, "F(\mathbf{c}^{(k)}_S)"] \\
F(T) \arrow[r, "F(\mathbf{d}^{(k)}_T)"'] & F(\mathbf{c}^{(k)}(S))
\end{tikzcd}$$

is a pullback in Set. Thus there is a unique element

$$x \in F(S \oplus_k T) \hookrightarrow \widetilde{F}_n$$

such that

$$F(\kappa_1)(x) = a \qquad F(\kappa_2)(x) = b.$$

We put

$$b \circ_k a = x.$$

This ends the definition of $\widetilde{F}$.

For a morphism $\alpha : F \longrightarrow G$ in $sPb((\mathbf{pOpeCard})^{op}, Set)$, we put

$$\widetilde{\alpha} = \{\widetilde{\alpha}_n : \widetilde{F}_n \longrightarrow \widetilde{G}_n\}_{n\in\omega}$$

such that

$$\widetilde{\alpha}_n = \coprod_S \alpha_S : \widetilde{F}_n \longrightarrow \widetilde{G}_n,$$

where the coproduct is taken over all (up to iso) positive opetopic cardinals S of dimension at most n. This ends the definition of the functor $\widetilde{(-)}$.

Proposition 13.3. *The functor*

$$\widetilde{(-)} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow \mathbf{pPoly}$$

is well defined.

Proof. The verification that $\widetilde{(-)}$ is a functor into ωCat is left for the reader. We shall verify that, for any special pullbacks preserving functor $F : \mathbf{pOpeCard}^{op} \longrightarrow Set$, $\widetilde{F}$ is a positive-to-one polygraph whose n-indeterminates are

$$|\widetilde{F}|_n = \coprod_{B\in\mathbf{pOpe}, dim(B)=n} F(B) \hookrightarrow \coprod_{S\in\mathbf{pOpeCard}, dim(S)\leq n} F(S) = \widetilde{F}_n.$$

We call R the object of $Set \downarrow D_{n-1}$ defined by $R = (|\widetilde{F}|_n, \widetilde{F}_{<n}, \mathbf{d}^{(n-1)}, \mathbf{c}^{(n-1)})$. We shall show that $\widetilde{F}_{\leq n}$ is in a bijective correspondence with $\overline{R}$ described in the previous section. We define a function

$$\varphi : \overline{R}_n \longrightarrow \widetilde{F}_n$$

as follows: for a cell $f : S^{\sharp,n} \longrightarrow R$ in $\overline{R}_n$, we put

$$\varphi(f) = \begin{cases} 1_{f_{n-1}(S)} & \text{if } dim(S) < n \\ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ principal},\ S_n = \{m_S\} \\ \varphi(f^{\uparrow a}) \circ_k \varphi(f^{\downarrow a}) & \text{if } dim(S) = n,\ a \in Sd(S)_k. \end{cases}$$

and the morphisms in $\varphi(f^{\downarrow a})$ and $\varphi(f^{\uparrow a})$ in $Set \downarrow D_{n-1}$ are obtained by compositions so that the diagram

$$\begin{tikzcd}
(S^{\downarrow a})^{\sharp,n} \arrow[dr] \arrow[drr, "f^{\downarrow a}"] & & \\
& S^{\sharp,n} \arrow[r, "f"] & R \\
(S^{\uparrow a})^{\sharp,n} \arrow[ur] \arrow[urr, "f^{\uparrow a}"'] & &
\end{tikzcd}$$

commutes. We need to verify, by induction on n, that φ is well defined, bijective and that it preserves compositions, domains, and codomains.

We shall only verify (partially) that φ is well defined, i.e., the definition φ for any positive opetopic cardinal S of dimension n that is not a positive opetope does not depend on the choice of the saddle point of S. Let $a, x \in Sd(S)$ so that $k = dim(x) < dim(a) = m$. Using Lemma 8.4 and the fact that $(-)^{\sharp,n}$ preserves special pushouts, we have

$$\begin{aligned}
&\varphi(f^{\uparrow a}) \circ_m \varphi(f^{\downarrow a}) = \\
&= (\varphi(f^{\uparrow a \uparrow x}) \circ_k \varphi(f^{\uparrow a \downarrow x})) \circ_m (\varphi(f^{\downarrow a \uparrow x}) \circ_k \varphi(f^{\downarrow a \downarrow x})) = \\
&= (\varphi(f^{\uparrow a \uparrow x}) \circ_m \varphi(f^{\downarrow a \uparrow x})) \circ_k (\varphi(f^{\uparrow a \downarrow x}) \circ_m \varphi(f^{\downarrow a \downarrow x})) = \\
&= (\varphi(f^{\uparrow x \uparrow a}) \circ_m \varphi(f^{\uparrow x \downarrow a})) \circ_k (\varphi(f^{\downarrow x \uparrow a}) \circ_m \varphi(f^{\downarrow x \downarrow a})) = \\
&= \varphi(f^{\uparrow x}) \circ_m \varphi(f^{\downarrow x}),
\end{aligned}$$

as required in this case. The reader can compare these calculations with those, in the same case, of Proposition 9.1 (F is replaced by φ and T is replaced by f). So there is no point to repeat the other calculations. □

For P in **pPoly**, we define a polygraph map

$$\eta_P : P \longrightarrow \widetilde{\widehat{P}}$$

so that, for $x \in P_n$, we put

$$\eta_{P,n}(x) = \tau_x : T^*_x \to P.$$

For F in $sPb((\mathbf{pOpeCard})^{op}, Set)$, we define a natural transformation

$$\varepsilon_F : \widehat{\widetilde{F}} \longrightarrow F$$

such that, for a positive opetopic cardinal S of dimension n,

$$(\varepsilon_F)_S : \widehat{\widetilde{F}}(S) \longrightarrow F(S)$$

and $g : S^* \to \widetilde{F} \in \widehat{\widetilde{F}}(S)$, we put

$$(\varepsilon_F)_S(g) = g_n(S).$$

Proposition 13.4. *The functors*

$$\begin{tikzcd} sPb((\mathbf{pOpeCard})^{op}, Set) \arrow[r, shift left, "\widetilde{(-)}"] & \mathbf{pPoly} \arrow[l, shift left, "\widehat{(-)} = \mathbf{pPoly}((\simeq)^*, -)"] \end{tikzcd}$$

together with the natural transformations η and ε defined above form an adjunction ($\widehat{(-)} \dashv \widetilde{(-)}$), which is an equivalence of categories.

Proof. The fact that both η and ε are bijective on each component follows immediately from Proposition 12.1.6. So we shall verify the triangular equalities only. Let P be a polygraph, and let F be a functor in $sPb((\mathbf{pOpeCard})^{op}, Set)$. We need to show that the triangles

$$\begin{tikzcd} & \widehat{\widetilde{\widehat{P}}} \arrow[dr, "\varepsilon_{\widehat{P}}"] & \\ \widehat{P} \arrow[ur, "\widehat{\eta_P}"] \arrow[rr, "1_{\widehat{P}}"'] & & \widehat{P} \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\widehat{\widetilde{F}}} \arrow[dr, "\widetilde{\varepsilon_F}"] & \\ \widetilde{F} \arrow[ur, "\eta_{\widetilde{F}}"] \arrow[rr, "1_{\widetilde{F}}"'] & & \widetilde{F} \end{tikzcd}$$

commute. So let $f : S^* \to P \in \widehat{P}(S)$, $dim(S) = n$. Then we have

$$\varepsilon_{\widehat{P}} \circ \widehat{\eta_P}(f) = \varepsilon_{\widehat{P}}(\eta_P \circ f) = (\eta_P \circ f)_n(S) =$$

$$= (\eta_P)_n(f_n(S)) = \tau_{f_n(S)} = f.$$

So let $x \in F(S) \longrightarrow \widetilde{F}_n$. Then we have

$$\widetilde{\varepsilon_F} \circ \eta_{\widetilde{F}}(x) = \widetilde{\varepsilon_F}(\tau_x) = (\tau_x)_n(1_{T_x}) = x.$$

So both triangles commutes, as required. □

From Propositions 13.1 and 13.4 we get immediately the following corollary.

Corollary 13.5. *The functor*

$$\widehat{(-)} : \mathbf{pPoly} \longrightarrow \widehat{\mathbf{pOpe}}$$

such that, for a positive-to-one polygraph X,

$$\widehat{X} = \mathbf{pPoly}((-)^*, X) : (\mathbf{pOpe})^{op} \longrightarrow Set$$

is an equivalence of categories.

14. The principal pushouts

Recall the positive opetopic cardinals α^n from section 9. A *total composition map* is an inner ω-functor whose domain is of form $(\alpha^n)^*$ (which we also write as $\alpha^{n,*}$), for some $n \in \omega$. If S is a positive opetopic cardinal of dimension n, then the total composition of S (in fact S^*) is denoted by

$$\boldsymbol{\mu}^{S^*} : \alpha^{n,*} \longrightarrow S^*.$$

It is uniquely determined by the condition $\boldsymbol{\mu}^{S^*}_n(\alpha^n) = S$. We have the following

Proposition 14.1. *Let* N *be a normal positive opetopic cardinal. With the notation as above, the square*

$$\begin{tikzcd}
N^* \arrow[r, "\mathbf{d}^*_{N^\bullet}"] & N^{\bullet,*} \\
\alpha^{n,*} \arrow[u, "\boldsymbol{\mu}^{N^*}"] \arrow[r, "\mathbf{d}^*_{\alpha^{n+1}}"'] & \alpha^{n+1,*} \arrow[u, "\boldsymbol{\mu}^{N^{\bullet,*}}"']
\end{tikzcd}$$

is a pushout in $\mathbf{pOpeCard}_\omega$.

Proof. This is an easy consequence of Proposition 7.3, particularly point 4. □

Pushouts described in the above Proposition are called *principal pushouts*.

From the above proposition we immediately get

Corollary 14.2. *If $n > 0$ and P is a positive opetope of dimension n, then the square*

$$\begin{tikzcd}
\mathbf{d}P^* \arrow[r, "\mathbf{d}^*_P"] & P^* \\
\alpha^{n-1,*} \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}P}"] \arrow[r, "\mathbf{d}_{\alpha^{n,*}}"'] & \alpha^{n,*} \arrow[u, "\boldsymbol{\mu}^{P}"']
\end{tikzcd}$$

is a (principal) pushout in $\mathbf{pOpeCard}_\omega$. $\Box$

Theorem 14.3 (V.Harnik). [25] *Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor. Then F preserves the principal pullbacks, as well.*

The proof of the above theorem will be divided into three Lemmas. Theorem 14.3 is a special case of Lemma 14.6, for $k = n - 1$.

Before we even formulate the next three Lemmas, we need to introduce some constructions on positive opetopic cardinals and define some ω-functors between positive opetopic cardinals. Along the way, we whall introduce some notations for them, and we shall make some comments on how they are going to be interpreted by special pullback preserving morphisms from $(\mathbf{pOpeCard}_\omega)^{op}$ to Set.

Notation for presheaves. To simplify the notation concerning presheaf $F : \mathbf{pOpeCard}_\omega{}^{op} \to Set$, for a morphism $g : P \to Q \in \mathbf{pOpeCard}_\omega$, and an element $a \in F(Q)$, we will write $a \cdot g$ for $F(g)(a)$, i.e., we treat F as a family of sets with a right action of the category $\mathbf{pOpeCard}_\omega$. This convention is explained, for example, in [13] page 121. Such notation suppresses the name of the presheaf F but it will be always clear from the context. In this section

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[25] The original statement of V. Harnik is saying that the nerve functor from ω-categories to (all) polygraphs is monadic. However, in the present context the argument given by V.Harnik, c.f. [9], is directly proving the present statement, i.e., that the principal pullbacks are preserved whenever the special ones are. This statement is used to show that the category of ω-categories is equivalent to the category of special pullback preserving functors from $(\mathbf{pOpeCard}_\omega)^{op}$ to Set, c.f. Corollary 15.2. From that statement, the monadicity of the nerve functor is an easy corollary, c.f. Theorem 16.5. In the remainder of this section, Harnik's argument, adopted to the present context, is presented.

we deal only with one presheaf named F that preserves special pullbacks. We also have $(a \cdot g) \cdot f = a \cdot (g \circ f)$ and $a \cdot id_Q = a$ whenever these formulas are well defined.

Fix $0 < k \leq n$, and a P positive opetope of dimension n. We say that P is *k-globular* iff $\mathbf{d}^{(l)}P$ is a positive opetope, for $k \leq l \leq n$, i.e., $\delta^{(l)}(\mathbf{p}_n)$ is a singleton, for $k \leq l \leq n$, where $P_n = \{\mathbf{p}_n\}$. The *k-globularization* $\boxed{k}P$ of P is the k-globular positive opetope of dimension n defined as follows. We put

$$\boxed{k}P_l = \begin{cases} \{\mathbf{p}_n\} & \text{for } l = n \\ \{\mathbf{q}_l, \mathbf{p}_l\} & \text{for } k \leq l < n \\ \delta^{(k-1)}(\mathbf{p}_n) \cup \{\mathbf{p}_{k-1}\} & \text{for } l = k-1 \\ P_l & \text{otherwise.} \end{cases}$$

For $x \in \boxed{k}P$,

$$\gamma^{\boxed{k}P}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l, \text{ for some } k \leq l < n \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

and

$$\delta^{\boxed{k}P}(x) = \begin{cases} \mathbf{q}_l & \text{if } x \in \boxed{k}P_{l+1}, \text{ for some } k < l < n \\ \delta^P(\mathbf{p}_k) & \text{if } x \in \boxed{k}P_k \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

Note that $\boxed{n}P$ is P itself and $\boxed{n}P$ is α^n. Thus the elements of the shape $\boxed{k}P^*$ are k-globularized versions of the elements of the shape P^*. As the following positive opetopic cardinals

$$\mathbf{c}^{(k)}P \;\cong\; \mathbf{c}^{(k)}\boxed{k}P \;\cong\; \mathbf{c}\mathbf{c}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{d}\mathbf{c}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{d}^{(k)}\boxed{k}P$$

are isomorphic, we can form the following special pushouts

$$\begin{tikzcd}
\mathbf{c}^{(k+1)}P^* \arrow[r, "\kappa_1"] & \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\
\mathbf{c}^{(k)}P^* \arrow[u, "\mathbf{c}^*_{\mathbf{c}^{(k+1)}P}"] \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k}P}"'] & \boxed{k}P^* \arrow[u, "\kappa_2"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{c}^{(k+1)}\boxed{k}P^* \arrow[r, "\kappa'_1"] & \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\
\mathbf{c}^{(k)}P^* \arrow[u, "\mathbf{c}^*_{\mathbf{c}^{(k+1)}\boxed{k}P}"] \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k}P}"'] & \boxed{k}P^* \arrow[u, "\kappa'_2"']
\end{tikzcd}$$

We describe in detail the positive opetopic cardinals we have just defined. Their faces are described in the following table:

dim	$\boxed{k+1}P$	$\boxed{k}P$	$P' = \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P}\boxed{k}P$	$P'' = \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P}\boxed{k}P$
n	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$
n–1	$\{\mathbf{q}_{n-1},\mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1},\mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1},\mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1},\mathbf{p}_{n-1}\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k+1	$\{\mathbf{q}_{k+1},\mathbf{p}_{k+1}\}$	$\{\mathbf{q}_{k+1},\mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1},\mathbf{q}_{k+1},\mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1},\mathbf{q}_{k+1},\mathbf{p}_{k+1}\}$
k	$\partial(\mathbf{p}_{k+1})$	$\{\mathbf{q}_k,\mathbf{p}_k\}$	$\delta(\mathbf{p}_{k+1})\cup\{\mathbf{q}_k,\mathbf{p}_k\}$	$\{\mathbf{r}_k,\mathbf{q}_k,\mathbf{p}_k\}$
k–1	P_{k-1}	$\partial(\mathbf{p}_k)$	P_{k-1}	$\partial(\mathbf{p}_k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l	P_l	P_l	P_l	P_l

$0 \leq l < k$. The functions γ and δ in $P' = \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P}\boxed{k}P$ are given by the following formulas

$$\gamma^{P'}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l \text{ and } k \leq l < n \\ \mathbf{q}_k & \text{if } x = \mathbf{r}_{k+1} \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

$$\delta^{P'}(x) = \begin{cases} \{\mathbf{q}_{l-1}\} & \text{if } x = \mathbf{q}_l \text{ and } k < l < n \\ & \text{or } x = \mathbf{p}_l \text{ and } k < l \leq n \\ \delta^P(\mathbf{p}_{k+1}) & \text{if } x = \mathbf{r}_{k+1} \\ \delta^P(\mathbf{p}_k) & \text{if } x = \mathbf{q}_k \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

The functions γ and δ in $P'' = \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P}\boxed{k}P$ are given by the following formulas

$$\gamma^{P''}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l \text{ and } k \leq l < n \\ \mathbf{q}_k & \text{or } x = \mathbf{r}_{k+1} \\ \mathbf{p}_{k-1} & \text{or } x = \mathbf{r}_k \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

$$\delta^{P''}(x) = \begin{cases} \{\mathbf{q}_{l-1}\} & \text{if } x = \mathbf{q}_l \text{ and } k < l < n \\ & \text{or } x = \mathbf{p}_l \text{ and } k < l \leq n \\ \{\mathbf{r}_k\} & \text{if } x = \mathbf{r}_{k+1} \\ \delta^P(\mathbf{p}_k) & \text{if } x \in \{\mathbf{r}_k, \mathbf{q}_k, \mathbf{p}_k\} \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

Now we shall define some ω-functors between some positive opetopic cardinals just defined. To describe their meaning, let us fix a special pullback preserving functors from $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$.

The ω-functors denoted by letter $\boldsymbol{\mu}$ are interpreted as operation that 'globularize' cells. We have two of them. The first one

$$\boldsymbol{\mu}^{S^*} : \alpha^{n,*} \longrightarrow S^*$$

was already introduced at the beginning of this section for any positive opetopic cardinal S. The second is the ω-functor

$$\boldsymbol{\mu}_{\boxed{k}} : \boxed{k}P^* \longrightarrow \boxed{k+1}P^*$$

such that

$$\boldsymbol{\mu}_{\boxed{k}}(X) = \begin{cases} (X - \{\mathbf{q}_k\}) \cup \delta(\mathbf{p}_{k+1}) & \text{if } \mathbf{q}_k \in X \\ X & \text{otherwise.} \end{cases}$$

for $X \in \boxed{k}P^*$.

The fact that these operations are interpreted as globularization of cells can be explained as follows. The function

$$F(\boldsymbol{\mu}_{\boxed{k}}) : F(\boxed{k+1}P^*) \longrightarrow F(\boxed{k}P^*),$$

takes a $(k+1)$-globular n-cell $a \in F(\boxed{k+1}P^*)$ and returns a k-globular n-cell $a \cdot \boldsymbol{\mu}_{\boxed{k}} = F(\boldsymbol{\mu}_{\boxed{k}})(a) \in F(\boxed{k}P^*)$. Intuitively, $F(\boldsymbol{\mu}_{\boxed{k}})$ is composing the k-domain of a leaving the rest 'untouched'. So it is a 'one-step globularization'. On the other hand, the function

$$F(\boldsymbol{\mu}^{S^*}) : F(S^*) \longrightarrow F(\alpha^{n,*})$$

is taking an n-cell $b \in F(S^*)$ of an arbitrary shape S^* of dimension n, and it is returning a globular n-cell $b \cdot \boldsymbol{\mu}^{S^*} \in F(\alpha^{n,*})$. This time $F(\boldsymbol{\mu}^{S^*})$ is composing all the domains and codomains in the cell b as much as possible, so that there is nothing left to be composed. This is the 'full globularization'.

We need a separate notation for the ω-functor $\boldsymbol{\mu}_k : \mathbf{c}^{(k+1)}{}_{\boxed{k}}P^* \longrightarrow \mathbf{c}^{(k+1)}P^*$ such that

$$\boldsymbol{\mu}_k(X) = \begin{cases} \mathbf{c}^{(k+1)}P & \text{if } X = \mathbf{c}^{(k+1)}{}_{\boxed{k}}P \\ \mathbf{d}^{(k)}P & \text{if } X = \mathbf{d}^{(k)}{}_{\boxed{k}}P \\ X & \text{otherwise,} \end{cases}$$

for $X \in \mathbf{c}^{(k+1)}{}_{\boxed{k}}P^*$. It is a version of $\boldsymbol{\mu}_{\boxed{k}}$. The ω-functor

$$\boldsymbol{\nu}_P : P^* \longrightarrow \mathbf{d}P^*$$

is given by

$$\boldsymbol{\nu}_P(X) = \begin{cases} \mathbf{d}P & \text{if } \mathbf{c}P \subseteq X \\ X & \text{otherwise,} \end{cases}$$

for $X \in P^*$. $\boldsymbol{\nu}_P$ is a kind of degeneracy map and it is interpreted as 'a kind of identity'. For a cell $t \in F(\mathbf{d}P^*)$, $t \cdot \boldsymbol{\nu}_P \in F(P^*)$ is 'identity on t' but with the codomain composed. The ω-functor

$$\boldsymbol{\beta}_k : \mathbf{c}^{(k)}P^* \longrightarrow \mathbf{d}^{(k)}P^*$$

such that

$$\boldsymbol{\beta}_k(X) = \begin{cases} \mathbf{d}^{(k)}P & \text{if } X = \mathbf{c}^{(k)}P \\ X & \text{otherwise.} \end{cases}$$

for $X \in P^*$, is the operation of 'composition of all the cells at the top' leaving the rest untouched. The map $\boldsymbol{\beta}_{n-1}$ is equal to the composition

$$\mathbf{c}P^* \xrightarrow{\mathbf{c}_P^*} P^* \xrightarrow{\boldsymbol{\nu}_P} \mathbf{d}P^*.$$

The following two ω-functors

$$[\mathbf{d}^{(k),*}_P \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P}, \boldsymbol{\mu}_P] : \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow P^*$$

and

$$[\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P}, 1_{\boxed{k}P}] : \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow \boxed{k}P^*$$

are defined as the unique ω-functors making the following diagrams

$$\begin{tikzcd}
 & \mathbf{d}^{(k)}P^* \arrow[dr, "\mathbf{d}^{(k),*}_P"] & \\
\mathbf{c}^{(k+1)}P^* \arrow[ur, "\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}"] \arrow[r, "\kappa_1"] & \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \arrow[r, "{[\mathbf{d}^{(k)} \circ \boldsymbol{\nu}, \boldsymbol{\mu}]}"] & \boxed{k+1}P^* \\
\mathbf{c}^{(k)}P^* \arrow[u, "\mathbf{c}^*_{\mathbf{c}^{(k+1)}P}"] \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k}P}"'] & \boxed{k}P^* \arrow[u, "\kappa_2"'] \arrow[ur, "\boldsymbol{\mu}_{\boxed{k}}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbf{d}^{(k)}\boxed{k}P^* \arrow[dr, "\mathbf{d}^{(k),*}_{\boxed{k}P}"] & \\
\mathbf{c}^{(k+1)}\boxed{k}P^* \arrow[ur, "\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}"] \arrow[r, "\kappa'_1"] & \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \arrow[r, "{[\mathbf{d}^{(k)} \circ \boldsymbol{\nu}, 1_{\boxed{k}P^*}]}"] & \boxed{k}P^* \\
\mathbf{c}^{(k)}P^* \arrow[u, "\mathbf{c}^*_{\mathbf{c}^{(k+1)}\boxed{k}P}"] \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k}P}"'] & \boxed{k}P^* \arrow[u, "\kappa'_2"'] \arrow[ur, "1_{\boxed{k}P^*}"'] &
\end{tikzcd}$$

commute in $\mathbf{pOpeCard}_\omega$.

Finally, we introduce two maps that are a kind of binary composition combined with wiskering. The first

$$\diamond_{\boxed{k+1}} : \boxed{k+1}P^* \longrightarrow \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$$

is given by

$$\diamond_{\boxed{k+1}}(X) = \begin{cases} X \cup \{\mathbf{r}_{k+1}, \mathbf{q}_k\} & \text{if } X \cap \{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\} \neq \emptyset \\ X & \text{otherwise,} \end{cases}$$

for $X \in \boxed{k+1}P^*$. The other composition map is

$$\diamond_k : \boxed{k}P^* \longrightarrow \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P,$$

given by the same defining formula as $\diamond_{\boxed{k+1}}$, for $X \in \boxed{k}P^*$.

In the following diagram all the morphisms that we introduced above are displayed. Most of the subscripts of the morphisms are suppressed for clarity of the picture.

$$\begin{tikzcd}
& \mathbf{c}^{(k+1)}P^* \arrow[rr, "\kappa_1"] \arrow[dl, shift left, "\boldsymbol{\nu}"] & & \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \arrow[dl, shift left, "{[d^*\circ \boldsymbol{\nu}, \boldsymbol{\mu}]}"] \\
\mathbf{d}^{(k)}P^* \arrow[ur, "\mathbf{d}^*"] \arrow[rr, "\mathbf{d}^{(k),*}"] & & \boxed{k+1}P^* \arrow[ur, "\diamond_{\boxed{k+1}}"] & \\
& \mathbf{c}^{(k+1)}\boxed{k}P^* \arrow[uu, "\boldsymbol{\mu}_k"] \arrow[rr, "\kappa'_1"] \arrow[dl, shift left, "\boldsymbol{\nu}"] & & \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \arrow[uu, "\boldsymbol{\mu}_k + 1"'] \arrow[dl, shift left, "{[d^* \circ \boldsymbol{\nu}, 1]}"] \\
\mathbf{c}^{(k)}P^* \arrow[uu, "\boldsymbol{\beta}_k"] \arrow[uuur, "\mathbf{c}^*"] \arrow[ur, shift left=2, "\mathbf{c}^*"] \arrow[ur, "\mathbf{d}^*"] \arrow[rr, "\mathbf{d}^{(k),*}"] & & \boxed{k}P^* \arrow[uu, "\boldsymbol{\mu}_{\boxed{k}}"] \arrow[uuur, "\kappa_2"] \arrow[ur, shift left=2, "\kappa'_2"] \arrow[ur, "\diamond_k"] &
\end{tikzcd}$$

The above cube contains two special pushouts mentioned above. The following lemma describes some other commutations.

Lemma 14.4. *With the notation as above we have, for $k \geq 1$,*

1. $\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = \boldsymbol{\beta}_k$,

2. $\kappa_1 \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = (\,\diamond_{\boxed{k+1}}) \circ \mathbf{d}^{(k),*}_{P}$,
3. $\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = 1_{\mathbf{d}^{(k)}P^*}$,
4. $(\,\diamond_{\boxed{k+1}}) \circ \boldsymbol{\mu}_{\boxed{k}} = (\boldsymbol{\mu}_k + 1_{\boxed{k}P}) \circ (\,\diamond_k)$,
5. $\boldsymbol{\beta}_k \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*} = \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \boldsymbol{\mu}_k$,
6. $[\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}] \circ (\,\diamond_k) = 1_{\boxed{k}P^*}$,
7. $[\mathbf{d}^{(k),*}_{P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\,\diamond_{\boxed{k+1}}) = 1_{\boxed{k+1}P^*}$.

Proof. Routine check. □

Lemma 14.5. *Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a positive opetope of dimension n. Then, for any $0 \leq k < n$, F preserves the pullback in $(\mathbf{pOpeCard}_\omega)^{op}$*

$$\begin{tikzcd}
\mathbf{d}^{(k)}P^* \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k+1}P}"] & \boxed{k+1}P^* \\
\mathbf{c}^{(k)}P^* \arrow[u, "\boldsymbol{\beta}_k"] \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k}P}"'] & \boxed{k}P^* \arrow[u, "\boldsymbol{\mu}_{\boxed{k}}"']
\end{tikzcd}$$

Proof. Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a positive opetope of dimension n. We need to show that the square

$$\begin{tikzcd}
F(\mathbf{d}^{(k)}P^*) \arrow[d, "F(\boldsymbol{\beta}_k)"'] & F(\boxed{k+1}P^*) \arrow[l, "F(\mathbf{d}^{(k),*}_{\boxed{k+1}P})"'] \arrow[d, "F(\boldsymbol{\mu}_{\boxed{k}})"] \\
F(\mathbf{c}^{(k)}P^*) & F(\boxed{k}P^*) \arrow[l, "F(\mathbf{d}^{(k),*}_{\boxed{k}P})"]
\end{tikzcd}$$

is a pullback in Set. Let us fix $t \in F(\mathbf{d}^{(k)}P^*)$ and $a \in F(\boxed{k}P^*)$ such that

$$t \cdot \boldsymbol{\beta}_k = a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}.$$

We will check that it is a pullback, by showing existence and uniqueness of an element $b \in F(\boxed{k+1}P^*)$ such that

$$b = t \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} \quad \text{and} \quad b = a \cdot \boldsymbol{\mu}_{\boxed{k}}\,.$$

Existence. Put $t' = t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}$. By Lemma 14.4.1, we have

$$t' \cdot \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \cdot \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = t \cdot \boldsymbol{\beta}_k = a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}.$$

Since F preserves special pullbacks and $\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ is a special pullback in $(\mathbf{pOpeCard}_\omega)^{op}$, we have an element

$$\langle t', a\rangle \in F(\mathbf{c}^{(k+1)}P) \times_{F(\mathbf{c}^{(k)}P)} F(\boxed{k}P) \cong F(\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P)$$

such that

$$\langle t', a\rangle \cdot \kappa_1 = t' \quad \text{and} \quad \langle t, a\rangle \cdot \kappa_2 = a.$$

We put $b = \langle t', a\rangle \cdot \diamond_{\boxed{k+1}} \in F(\boxed{k+1}P^*)$. We have

$$\begin{aligned}
b \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} &= (\text{def of } b)\\
= (\langle t', a\rangle \cdot \diamond_{\boxed{k+1}}) \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} &= (F \text{ presheaf})\\
= \langle t', a\rangle \cdot (\diamond_{\boxed{k+1}} \circ \mathbf{d}^{(k),*}_{\boxed{k+1}P}) &= (\text{Lemma } 14.4.2)\\
= \langle t', a\rangle \cdot (\kappa_1 \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) &= (F \text{ presheaf})\\
= (\langle t', a\rangle \cdot \kappa_1) \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} &= (F \text{ pres. special pb's})\\
= t' \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} &= (F \text{ pres. special pb's, def } t')\\
= (t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}) \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} &= (F \text{ presheaf})\\
= t \cdot (\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) &= (\text{Lemma } 14.4.3)\\
= t \cdot \; 1_{\mathbf{d}^{(k)}P^*} &= (F \text{ presheaf})\\
&= t
\end{aligned}$$

and

$$\begin{aligned}
b \cdot \boldsymbol{\mu}_{\boxed{k}} &= (\text{def of } b)\\
= (\langle t', a\rangle \cdot \diamond_{\boxed{k+1}}) \cdot \boldsymbol{\mu}_{\boxed{k}} &= (F \text{ presheaf})
\end{aligned}$$

$$
\begin{gathered}
= \langle t', a \rangle \cdot (\diamond_{\boxed{k+1}} \circ \boldsymbol{\mu}_{\boxed{k}}) = (\text{Lemma 14.4.4}) \\
= \langle t', a \rangle \cdot ((\boldsymbol{\mu}_k + 1) \circ \diamond_k) = (F \text{ presheaf}) \\
= (\langle t', a \rangle \cdot (\boldsymbol{\mu}_k + 1)) \cdot \diamond_k = (F \text{ pres. special pb}'\text{s}) \\
= \langle t' \cdot \boldsymbol{\mu}_k, a \rangle \cdot \diamond_k = (\text{def } t') \\
= \langle (t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}) \cdot \boldsymbol{\mu}_k, a \rangle \cdot \diamond_k = (F \text{ presheaf}) \\
= \langle (t \cdot (\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \boldsymbol{\mu}_k), a \rangle \cdot \diamond_k = (\text{Lemma 14.4.5}) \\
= \langle (t \cdot (\boldsymbol{\beta}_k \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}), a \rangle \cdot \diamond_k = (F \text{ presheaf}) \\
= \langle (t \cdot \boldsymbol{\beta}_k) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, a \rangle \cdot \diamond_k = (\text{assumption on } a \text{ and } t) \\
= \langle (a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, a \rangle \cdot \diamond_k = (F \text{ presheaf}) \\
= \langle a \cdot (\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}), a \rangle \cdot \diamond_k = (F \text{ pres. special pb}'\text{s}) \\
= (a \cdot [\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}]) \cdot \diamond_k = (F \text{ presheaf}) \\
= a \cdot ([\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}] \circ \diamond_k) = (\text{Lemma 14.4.6}) \\
= a \cdot 1_{\boxed{k}P^*} = (F \text{ presheaf}) \\
= a.
\end{gathered}
$$

Uniqueness. Now suppose that we have two elements $b, b' \in F(\boxed{k+1}P^*)$ such that $\mathbf{d}^{(k)}(b) = t = \mathbf{d}^{(k)}(b')$ and $\boldsymbol{\mu}(b) = a = \boldsymbol{\mu}(b')$. Then, using Lemma 14.4 7 and the assumption, we have (we won't mention that we use the fact that F is a sheaf anymore)

$$b = (\text{Lemma 14.4.7})$$

$$
\begin{gathered}
= b \cdot ([\mathbf{d}^{(k),*}_P \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\diamond_{\boxed{k+1}})) = \\
= \langle (b \cdot \mathbf{d}^{(k),*}_P) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, b \cdot \boldsymbol{\mu}_{\boxed{k}} \rangle \cdot (\diamond_{\boxed{k+1}}) = (\text{assumption on } b \text{ and } b') \\
= \langle (b' \cdot \mathbf{d}^{(k),*}_P) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, b' \cdot \boldsymbol{\mu}_{\boxed{k}} \rangle \cdot (\diamond_{\boxed{k+1}}) = \\
= b' \cdot ([\mathbf{d}^{(k),*}_P \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\diamond_{\boxed{k+1}})) = (\text{Lemma 14.4.7}) \\
= b'.
\end{gathered}
$$

So the element with these properties is unique. □

Lemma 14.6. *Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a positive opetope of dimension n. Then, for any $0 \leq k < n$, F preserves the following pullback in $(\mathbf{pOpeCard}_\omega)^{op}$*

$$\begin{tikzcd}
\mathbf{d}^{(k)}P^* \arrow[r, "\mathbf{d}^{(k),*}_{\boxed{k+1}P}"] & \boxed{k+1}P^* \\
\alpha^{k,*} \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}^{(k)}P^*}"] \arrow[r, "\mathbf{d}^{(k),*}_{\alpha^n}"'] & \alpha^{n,*} \arrow[u, "\boldsymbol{\mu}^{\boxed{k+1}P^*}"']
\end{tikzcd} \tag{8}$$

Proof. The proof is by double induction, on the dimension n of the positive opetopic P, and $k < n$. Note that if $k = 0$, then, for any $n > 0$, the vertical arrows in (8) are isomorphisms, so any functor from $(\mathbf{pOpeCard}_\omega)^{op}$ sends (8) to a pullback. This shows in particular that the Lemma holds for $n = 1$. As we already mentioned, if $k = n - 1$, the square (8) is an arbitrary special pushout.

Thus, we assume that $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ is a special pullback preserving functor, and that P is a positive opetope of dimension n, $0 \leq k < n$, F preserves the pullback (8). Moreover, for $m < n$ and the positive opetope Q of dimension m, F preserves the principal pullback in $(\mathbf{pOpeCard}_\omega)^{op}$:

$$\begin{tikzcd}
\mathbf{d}Q^* \arrow[r, "\mathbf{d}^*_Q"] & Q^* \\
\alpha^{m-1,*} \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}Q}"] \arrow[r, "\mathbf{d}^*_{\alpha^m}"'] & \alpha^{m,*} \arrow[u, "\boldsymbol{\mu}^{Q}"']
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{d}^{(m-1)}Q^* \arrow[r, "\mathbf{d}^{(m-1),*}_{\boxed{m}Q}"] & \boxed{m}Q^* \\
\alpha^{m-1,*} \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}^{(m-1)}Q^*}"] \arrow[r, "\mathbf{d}^{(m-1),*}_{\alpha^m}"'] & \alpha^{m,*} \arrow[u, "\boldsymbol{\mu}^{\boxed{m}Q^*}"']
\end{tikzcd}$$

We shall show that F preserves the pullback

$$\begin{tikzcd}
\mathbf{d}^{(k+1)}P^* \arrow[r, "\mathbf{d}^{(k+1),*}_{\boxed{k+2}P}"] & \boxed{k+2}P^* \\
\alpha^{k+1,*} \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}"] \arrow[r, "\mathbf{d}^{(k+1),*}_{\alpha^n}"'] & \alpha^{n,*} \arrow[u, "\boldsymbol{\mu}^{\boxed{k+2}P^*}"']
\end{tikzcd} \tag{9}$$

in $(\mathbf{pOpeCard}_\omega)^{op}$, as well. In the following diagram (most of the subscripts and some superscripts were suppressed for clarity):

$$\begin{tikzcd}
 & & \mathbf{d}^{(k+1)}P^* \arrow[rr, "\mathbf{d}^{(k+1),*}"] & & \boxed{k+2}P^* \\
 & & & I & \\
 & & \mathbf{c}^{(k+1)}P^* \arrow[uu, "\boldsymbol{\beta}_{k+1}"'] \arrow[rr, "\mathbf{d}^{(k+1),*}"] & & \boxed{k+1}P^* \arrow[uu, "\boldsymbol{\mu}"'] \\
 & & \mathbf{d}^{(k)}P^* \arrow[u, "\mathbf{d}^*"] \arrow[urr, "\mathbf{d}^{(k),*}"'] & & \\
 & III & & II & \\
\alpha^{k+1,*} \arrow[uuuuurr, "\boldsymbol{\mu}"] \arrow[uuurr, "\boldsymbol{\mu}"'] \arrow[rrrr, bend right, "\mathbf{d}^{(k+1),*}"'] & & \alpha^{k,*} \arrow[ll, "\mathbf{d}^*"'] \arrow[uu, "\boldsymbol{\mu}"] \arrow[rr, "\mathbf{d}^{(k),*}"] & & \alpha^{n,*} \arrow[uuu, "\boldsymbol{\mu}"']
\end{tikzcd}$$

all the squares and triangles commute. Moreover, F sends the squares I, II, III to pullbacks in Set: I by Lemma 14.5, II by inductive hypothesis for k, III by inductive hypothesis since $dim(\mathbf{c}^{(k+1)}P) < n$.

Let $f : X \longrightarrow F(\mathbf{d}^{(k+1)}P^*)$ and $g : X \longrightarrow F(\alpha^{n,*})$ be functions such that

$$F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}) \circ f = F(\mathbf{d}^{(k+1),*}_{\alpha^n}) \circ g.$$

Since F applied to II is a pullback in Set, and all squares and triangles in the above diagram commute, there is a unique function $h_1 : X \longrightarrow F(\boxed{k+1}P^*)$ such that

$$F(d^{(k),*}_{\boxed{k+1}P}) \circ h_1 = F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f \quad \text{and} \quad F(\boldsymbol{\mu}^{\boxed{k+1}P^*}) \circ h_1 = g. \tag{10}$$

To get a unique function $h_2 : X \longrightarrow F(\boxed{k+2}P^*)$ such that

$$F(\mathbf{d}^{k+1,*}_{\boxed{k+2}P}) \circ h_2 = f \quad \text{and} \quad F(\boldsymbol{\mu}_{\boxed{k+1}}) \circ h_2 = h_1, \tag{11}$$

we use the fact that F sends III to a pullback in Set. The application of F to the diagram above will give the following diagram in Set, where we added the additional functions f, g, h_1, and h_2:

$$\begin{tikzcd}
 & F(\mathbf{d}^{(k+1)}P^*) \arrow[d, "F(\boldsymbol{\beta}_{k+1})"'] \arrow[dddl, "F(\boldsymbol{\mu})"'] & F(\boxed{k+2}P^*) \arrow[l, "F(\mathbf{d}^{(k+1),*})"'] \arrow[d, "F(\boldsymbol{\mu})"] & X \arrow[l, "h_2"'] \arrow[dl, "h_1"] \arrow[ll, bend right, "f"'] \arrow[ddd, bend left, "g"] \\
 & F(\mathbf{c}^{(k+1)}P^*) \arrow[d, "F(\mathbf{d}^*)"'] \arrow[ddl, "F(\boldsymbol{\mu})"] & F(\boxed{k+1}P^*) \arrow[l, "F(\mathbf{d}^{(k+1),*})"'] \arrow[dl, "F(\mathbf{d}^{(k),*})"] \arrow[dd, "F(\boldsymbol{\mu})"] & \\
 & F(\mathbf{d}^{(k)}P^*) \arrow[d, "F(\boldsymbol{\mu})"'] & & \\
F(\alpha^{k+1,*}) \arrow[r, "F(\mathbf{d}^*)"] & F(\alpha^{k,*}) & F(\alpha^{n,*}) \arrow[l, "F(\mathbf{d}^{(k),*})"'] \arrow[ll, bend left, "F(\mathbf{d}^{(k+1),*})"] &
\end{tikzcd}$$

Thus in order to verify that

$$F(\boldsymbol{\beta}_{k+1}) \circ f = F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1$$

and to get h_2 satisfying (11), it is enough to verify that

$$F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1 \qquad (12)$$

and

$$F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1. \qquad (13)$$

For (12), we have

$$F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f =$$

$$= F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}) \circ f =$$

$$= F(\mathbf{d}^{(k+1),*}_{\alpha^n}) \circ g =$$

$$= F(\mathbf{d}_{\alpha^n}^{(k+1),*}) \circ F(\boldsymbol{\mu}^{\boxed{k+1}P^*}) \circ h_1 =$$

$$= F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\mathbf{d}_{\boxed{k+1}P}^{(k+1)^*}) \circ h_1,$$

and for (13), we have

$$F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f =$$

$$= F(d_{\boxed{k+1}P}^{(k),*}) \circ h_1 =$$

$$= F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\mathbf{d}_{\boxed{k+1}P}^{(k+1),*}) \circ h_1.$$

By uniqueness of both h_1 and h_2, h_2 is the unique function such that

$$\mathbf{d}_{\boxed{k+2}P}^{(k+1),*} \circ h_2 = f \quad \text{and} \quad \boldsymbol{\mu}^{\boxed{k+2}P^*} \circ h_2 = g,$$

i.e., F sends (9) to a pullback in Set, as required. □

15. A full nerve of ω-categories

Let $\mathcal{S}$ denote the category of simple ω-categories, described in [15]. It was proved there that any simple ω-category is isomorphic to one of form $(\alpha^{\vec{u}})^*$ for some ud-vector $\vec{u}$. In fact what we need here is that any simple ω-category can be obtained from those of form $(\alpha^n)^*$, with $n \in \omega$, via special pushouts. For more details the reader should consult [15].

As every simple ω-category is a positive opetopic cardinal (considered as an ω-category), we have a full inclusion functor

$$\mathbf{k} : \mathcal{S} \longrightarrow \mathbf{pOpeCard}_\omega$$

whose essential image is spanned by the opetopic cardinals all of whose faces are globular.

In [15] we have shown that $sPb(\mathcal{S}^{op}, Set)$, the category of special pullbacks preserving functors from the dual of $\mathcal{S}$ to Set, is equivalent to the category ω-categories. We have in fact an adjoint equivalence

$$\begin{tikzcd}
\omega Cat \arrow[r, shift right=1.5ex, "\widehat{(-)} = \omega Cat(\simeq, -)"'] & sPb(\mathcal{S}^{op}, Set) \arrow[l, shift right=1.5ex, "\widetilde{(-)}"']
\end{tikzcd}$$

where

$$\widehat{C} : \mathcal{S}^{op} \longrightarrow Set$$

is given by

$$\widehat{C}(A) = \omega Cat(A, C),$$

where A is a simple ω-category.

Proposition 15.1. *The adjunction*

$$\begin{tikzcd} \widehat{\mathcal{S}} \arrow[r, shift left, "Ran_{\mathbf{k}}"] & \widehat{\mathbf{pOpeCard}}_\omega \arrow[l, shift left, "\mathbf{k}^*"] \end{tikzcd}$$

restricts to an equivalence of categories.

$$\begin{tikzcd} sPb(\mathcal{S}^{op}, Set) \arrow[r, shift left, "Ran_{\mathbf{k}}"] & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \arrow[l, shift left, "\mathbf{k}^*"] \end{tikzcd}$$

where $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ is the category of the special pullbacks preserving functors.

Proof. First we shall describe the adjunction in details.

The counit. Let G be a functor in $sPb(\mathcal{S}^{op}, Set)$ and A be a simple ω-category. We have a functor

$$\begin{tikzcd} (\mathbf{k} \downarrow A)^{op} \arrow[r, "\pi^A"] & \mathcal{S}^{op} \arrow[r, "G"] & Set \end{tikzcd}$$

with the limit, say $(Lim(G \circ \pi^A), \sigma^A)$. Then the counit $(\varepsilon_G)_A$ is

$$(\varepsilon_G)_A : (Ran_{\mathbf{k}}(G) \circ \mathbf{k})(A) = Lim(G \circ \pi^A) \xrightarrow{\sigma^A_{1_A}.} G(A)$$

As $\mathbf{k}$ is full and faithful[26], for any G, ε_G is an iso. Thus ε is an iso.

The unit. Let F be a functor in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, T a positive opetopic cardinal. We have a functor

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[26] This condition translates to the fact that 1_A is the initial object in $(\mathbf{k} \downarrow A)^{op}$ and therefore that we have an iso $\sigma^A_{1_A} : Lim(G \circ \pi^A) \cong G \circ \pi^A(1_A) = G(A)$.

$$(\mathbf{k}\downarrow T^*)^{op} \xrightarrow{\pi^{T^*}} \mathcal{S}^{op} \xrightarrow{\mathbf{k}} \mathbf{pOpeCard}_\omega \xrightarrow{F} Set$$

with the limit, say $(Lim(F\circ\mathbf{k}\circ\pi^{T^*}), \sigma^{T^*})$. Then the unit $(\eta_F)_{T^*}$ is the unique morphism into the limit:

$$\begin{tikzcd}
F(T^*) \arrow[rr, "(\eta_F)_{T^*}"] \arrow[ddr, "F(a)"'] \arrow[ddrrr, "F(b)"] & & Ran_{\mathbf{k}}(F\circ\mathbf{k})(T^*) \arrow[d, equal] & \\
 & & Lim(F\circ\mathbf{k}\circ\pi^{T^*}) \arrow[dl, "\sigma^T_a"'] \arrow[dr, "\sigma^T_b"] & \\
 & F(A) \arrow[rr, "F(f)"'] & & F(B)
\end{tikzcd}$$

where the triangle in $\mathbf{pOpeCard}_\omega$

$$\begin{tikzcd}
 & T^* & \\
A \arrow[ur, "a"] & & B \arrow[ul, "b"'] \arrow[ll, "f"]
\end{tikzcd}$$

commutes.

Note that, as F preserves special pullbacks, and any simple ω-category can be obtained from those of form α^n with $n\in\omega$, we can restrict the limiting cone $(Lim(F\circ\mathbf{k}\circ\pi^{T^*}), \sigma^{T^*})$ to the objects of form α^n, with $n\in\omega$.

After this observation we shall prove, by induction on the size of a positive opetopic cardinal T, that $(\eta_F)_{T^*}$ is an iso.

If $dim(T)\leq 1$, then $(\eta_F)_{T^*}$ is obviously an iso.

Suppose T is not principal, i.e., we have $a\in Sd(T)$, for some $k\in\omega$. By inductive hypothesis the morphisms

$$(\eta_F)_{(T^{\downarrow a})^*},\quad (\eta_F)_{\mathbf{c}^{(k)}(T^{\downarrow a})^*},\quad (\eta_F)_{(T^{\uparrow a})^*}$$

are isos, and the square

$$\begin{tikzcd}
T^{\downarrow a} \arrow[r] & T \\
\mathbf{c}^{(k)}(T^{\downarrow a}) \arrow[u] \arrow[r] & T^{\uparrow a} \arrow[u]
\end{tikzcd}$$

is a special pushout (see Proposition 6.2) which is sent by F to a pullback. Hence the morphism

$$(\eta_F)_{T^*} = (\eta_F)_{(T^{\downarrow a})^*} \times (\eta_F)_{(T^{\uparrow a})^*}$$

is indeed an iso in this case.

If T is principal and $T = (\alpha^n)^*$, then the category $(\mathbf{k} \downarrow (\alpha^n)^*)^{op}$ has the initial object $1_{(\alpha^n)^*}$, so the morphism

$$(\eta_F)_{(\alpha^n)^*} : F((\alpha^n)^*) \longrightarrow Ran_{\mathbf{k}}(F \circ \mathbf{k})((\alpha^n)^*)$$

is an iso.

Finally, let us assume that T $(= P)$ is any positive opetope of dimension n. Thus, by Corollary 14.2, we have a principal pushout

$$\begin{tikzcd}
(\mathbf{d}P)^* \arrow[r, "\mathbf{d}_P^*"] & P^* \\
(\alpha^{n-1})^* \arrow[u, "\boldsymbol{\mu}^{\mathbf{d}P}"] \arrow[r, "\mathbf{d}^*_{\alpha^n}"'] & (\alpha^n)^* \arrow[u, "\boldsymbol{\mu}^P"']
\end{tikzcd}$$

which, by Theorem 14.3, is preserved by F. By induction hypothesis the morphisms

$$(\eta_F)_{(\mathbf{d}P)^*} \qquad (\eta_F)_{(\alpha^{n-1})^*} \qquad (\eta_F)_{(\alpha^n)^*}$$

are isos, so we have that the morphism

$$(\eta_F)_{P^*} = (\eta_F)_{(\mathbf{d}P)^*} \times (\eta_F)_{(\alpha^n)^*}$$

is an iso, as well. $\Box$

Corollary 15.2. *We have a commuting triangle of adjoint equivalences*

$$\begin{tikzcd}
\omega Cat \arrow[d, shift right=1.5ex, "\widehat{(-)}"'] \arrow[drr, shift left, "\widehat{(-)}"] & & \\
sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \arrow[u, shift right=1.5ex, "\widetilde{(-)}"'] \arrow[rr, shift left, "\mathbf{k}^*"] & & sPb(\mathcal{S}^{op}, Set) \arrow[ll, shift left, "Ran_{\mathbf{k}}"] \arrow[ull, shift left, "\widetilde{(-)}"]
\end{tikzcd}$$

In particular, the categories ωCat and $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ are equivalent.

Proof. It is enough to show that in the above diagram $\mathbf{k}^* \circ \widehat{(-)} = \widehat{(-)}$. But this is clear. $\Box$

16. A monadic adjunction

In this section we show that the inclusion functor $\mathbf{e} : \mathbf{pPoly} \longrightarrow \omega Cat$ has a right adjoint which is monadic.

First we will give an outline of the proof. Consider the following diagram of categories and functors

$$\begin{tikzcd}
\mathbf{pPoly} \arrow[rr, "\mathbf{e}"] \arrow[d, shift right=2, "\widehat{(-)}"'] & & \omega Cat \arrow[d, shift right=2, "\widehat{(-)}"'] \\
sPb((\mathbf{pOpeCard})^{op}, Set) \arrow[rr, shift left, "Lan_{\mathbf{j}}"] \arrow[u, shift right=2, "\widetilde{(-)}"'] & & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \arrow[ll, shift left, "\mathbf{j}^*"] \arrow[u, shift right=2, "\widetilde{(-)}"']
\end{tikzcd}$$

where $\mathbf{e}$ is just an inclusion of positive-to-one polygraphs into ω-categories and $\mathbf{j} = (-)^* : \mathbf{pOpeCard} \longrightarrow \mathbf{pOpeCard}_\omega$ is the essentially surjective inclusion functor. We have already shown (Proposition 13.4, Corollary 15.2) that the vertical functors constitute two adjoint equivalences. The proof that $\mathbf{e}$ has a right adjoint takes a few steps. We begin by presenting $Lan_{\mathbf{j}}$ as a familially representable functor (or local right adjoint). Then we check that $Lan_{\mathbf{j}}$ is well defined, i.e., we will check that the functor $Lan_{\mathbf{j}}(F)$, the left Kan extension of special pullbacks preserving functor F, preserves special pullbacks. Next we shall check that the above square commutes, i.e., $\widehat{(-)} \circ \mathbf{e} = Lan_{\mathbf{j}} \circ \widehat{(-)}$. This will reduce the problem of monadicity of ωCat over $\widehat{\mathbf{pOpe}}$ to verification whether $\mathbf{j}^*$, the left adjoint to $Lan_{\mathbf{j}}$, is monadic. The last statement is verified directly checking assumptions of Beck's monadicity theorem.

We describe the left Kan extension along $\mathbf{j}$ in a convenient way, c.f. [12], as a familially representable functor.

Proposition 16.1. *The functor of left Kan extension*

$$Lan_{\mathbf{j}} : \widehat{\mathbf{pOpeCard}} \longrightarrow \widehat{\mathbf{pOpeCard}}_\omega$$

along the functor

$$\mathbf{j} : \mathbf{pOpeCard} \to \mathbf{pOpeCard}_\omega$$

is defined, for $F \in \widehat{\mathbf{pOpeCard}}$, as follows. For a positive opetopic cardinal S, we have

$$\begin{tikzcd} Lan_{\mathbf{j}}(F)(S^*) = \coprod_{a:S^*\to T^* \text{ inner}} F(T) & F(T) \arrow[l, "\kappa_a^{S^*}"'] \end{tikzcd}$$

where the coproduct is taken over all up to iso inner maps in $\mathbf{pOpeCard}_\omega$ with the domain S^*, with the coprojections as shown.

If $h : S_1^* \longrightarrow S_2^*$ is an ω-functor and $a_2 : S_2^* \to T_2^*$ is inner, by Lemma 10.3, we can form a diagram

$$\begin{tikzcd} S_1^* \arrow[r, "h"] \arrow[d, "a_1"'] & S_2^* \arrow[d, "a_2"] \\ T_1^* \arrow[r, "(h')^*"'] & T_2^* \end{tikzcd}$$

with a_1 inner and h' a map of positive opetopic cardinals, i.e., the map $(h')^*$ is an outer map. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h, h', a_1, a_2 as above, the diagram

$$\begin{tikzcd} Lan_{\mathbf{j}}(F)(S_2^*) = \coprod_{a_2:S_2^*\to T_2^* \text{ inner}} F(T_2) \arrow[d, "Lan_{\mathbf{j}}(F)(h)"'] & F(T_2) \arrow[l, "\kappa_{a_2}^{S_2^*}"'] \arrow[d, "F(h')"] \\ Lan_{\mathbf{j}}(F)(S_1^*) = \coprod_{a_1:S_1^*\to T_1^* \text{ inner}} F(T_1) & F(T_1) \arrow[l, "\kappa_{a_1}^{S_1^*}"] \end{tikzcd}$$

commutes.

Proof. Fix F in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ for the whole proof. Let S be a positive opetopic cardinal. Then $Lan_{\mathbf{j}}(F)(S)$ is the colimit of the following functor

$$\mathbf{j}^{op}\downarrow S \xrightarrow{\pi^S} (\mathbf{pOpeCard})^{op} \xrightarrow{F} Set,$$

i.e., $Lan_{\mathbf{j}}(F)(S) = (F\circ\pi^S, \sigma^F)$. A map $f : a \longrightarrow b$ in $\mathbf{j}^{op}\downarrow S$, is a commuting triangle

$$\begin{tikzcd}
 & S^* \arrow[dl, "a"'] \arrow[dr, "b"] & \\
T_1^* \arrow[rr, "f^*"'] & & T_2^*
\end{tikzcd}$$

in $\mathbf{pOpeCard}_\omega$, and hence by Lemma 10.3 we can take the inner-outer factorizations, c.f. 10, of both $a = (a'')^*\circ a'$ and $b = (b'')^*\circ b'$, with a' and b' inner maps. Then, again by Lemma10.3, there is a morphism $f' : a' \longrightarrow b'$ in $\mathbf{pOpeCard}$ which must be an iso. In this way we get a commuting diagram

$$\begin{tikzcd}
 & & S^* \arrow[dll, "a'"'] \arrow[drr, "b'"] \arrow[ddll, "a"'] \arrow[ddrr, "b"] & & \\
T_3^* \arrow[rrrr, "(f')^*"'] \arrow[d, "(a'')^*"'] & & & & T_4^* \arrow[d, "(b'')^*"] \\
T_1^* \arrow[rrrr, "f^*"'] & & & & T_2^*
\end{tikzcd}$$

in $\mathbf{pOpeCard}_\omega$, which corresponds to the following part of the colimiting cocone:

$$\begin{tikzcd}
 & & Lan_{\mathbf{j}}(F)(S^*) & & \\
F(T_3) \arrow[urr, "\sigma^F_{a'}"] & & & & F(T_4) \arrow[ull, "\sigma^F_{b'}"'] \arrow[llll, "F(f')"'] \\
F(T_1) \arrow[u, "F(a'')"] \arrow[uurr, "\sigma^F_a"'] & & & & F(T_2) \arrow[u, "F(b'')"'] \arrow[uull, "\sigma^F_b"] \arrow[llll, "F(f)"]
\end{tikzcd}$$

Thus if there is a morphism $f : a \to b$ between two objects in $\mathbf{j}^{op}\downarrow S$, we have a commuting diagram

$$\begin{tikzcd}
 & a' \arrow[dl, "a''"'] \arrow[dr, "b''\circ f"] & \\
a \arrow[rr, "f"'] & & b
\end{tikzcd}$$

in $\mathbf{j}^{op} \downarrow S$ with a' being the inner part of both a and b. There are no other comparison maps between these objects. But this says that in fact

$$Lan_{\mathbf{j}}(F)(S^*) = \coprod_{a:S^*\to T^* \text{ inner}} F(T) \xleftarrow{\kappa_a^{S^*}} F(T),$$

where the coproduct is taken over all (up to iso) inner maps with the domain S^*, with the coprojections as shown.

To define $Lan_{\mathbf{j}}(F)$ on morphisms, fix an ω-functor $h : S_1^* \longrightarrow S_2^*$ in $\mathbf{pOpeCard}$ and an inner map $a_2 : S_2^* \to T_2^*$. By Lemma 10.3, we can form a diagram

$$\begin{tikzcd}
S_1^* \arrow[r, "h"] \arrow[d, "a_1"'] & S_2^* \arrow[d, "a_2"] \\
T_1^* \arrow[r, "(h')^*"'] & T_2^*
\end{tikzcd}$$

with a_1 inner and $(h')^*$ outer. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h', a_1, a_2 as above, the diagram

$$\begin{tikzcd}
Lan_{\mathbf{j}}(F)(S_2^*) = \coprod_{a_2:S_2^*\to T_2^* \text{ inner}} F(T_2) \arrow[d, "Lan_{\mathbf{j}}(F)(h)"'] & F(T_2) \arrow[l, "\kappa_{a_2}^{S_2^*}"'] \arrow[d, "F(h')"] \\
Lan_{\mathbf{j}}(F)(S_1^*) = \coprod_{a_1:S_1^*\to T_1^* \text{ inner}} F(T_1) & F(T_1) \arrow[l, "\kappa_{a_1}^{S_1^*}"]
\end{tikzcd}$$

commutes. This shows that the functor $Lan_{\mathbf{j}}$ is a familially representable functor. $\square$

Lemma 16.2. *The functor of the left Kan extension along* $\mathbf{j}$ *restricts to*

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set),$$

i.e., whenever $F : (\mathbf{pOpeCard})^{op} \longrightarrow Set$ *preserves special pullbacks, so does* $Lan_{\mathbf{j}}(F) : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$*. Moreover,* $Lan_{\mathbf{j}}$ *is the left adjoint to*

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set).$$

Proof. Note that once the first part of the statement will be proved, the part following 'moreover' will follow immediately.

Fix F in $sPb((\mathbf{pOpeCard})^{op}, Set)$ for the whole proof. We shall use the description of $Lan_{\mathbf{j}}(F)$ from Proposition 16.1 to show that $Lan_{\mathbf{j}}(F)$ preserves special pullbacks. So assume that S_1 and S_2 are positive opetopic cardinals such that

$$\mathbf{c}^{(k)}(S_1) = \mathbf{d}^{(k)}(S_2),$$

i.e., we have a pushout

$$\begin{tikzcd}
S_1 \arrow[r, "\kappa_1"] & S_1 \oplus_k S_2 \\
\mathbf{c}^{(k)}(S_1) \arrow[u, "\mathbf{c}^{(k)}_{S_1}"] \arrow[r, "\mathbf{d}^{(k)}_{S_2}"'] & S_2 \arrow[u, "\kappa_2"']
\end{tikzcd}$$

in $\mathbf{pOpeCard}$. We need to show that the square

$$\begin{tikzcd}
Lan_{\mathbf{j}}(F)(S_1) \arrow[d, "Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})"'] & Lan_{\mathbf{j}}(F)(S_1 \oplus_k S_2) \arrow[l, "Lan_{\mathbf{j}}(F)(\kappa_1)"'] \arrow[d, "Lan_{\mathbf{j}}(F)(\kappa_2)"] \\
Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}(S_1)) & Lan_{\mathbf{j}}(F)(S_2) \arrow[l, "Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})"]
\end{tikzcd}$$

is a pullback in Set, i.e., that the square

$$\begin{tikzcd}
\coprod_{a:S_1^* \to T^* \text{ inner}} F(T) \arrow[d, "Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})"'] & \coprod_{a:(S_1;_k S_2)^* \to T^* \text{ inner}} F(T) \arrow[l, "Lan_{\mathbf{j}}(F)(\kappa_1)"'] \arrow[d, "Lan_{\mathbf{j}}(F)(\kappa_2)"] \\
\coprod_{a:(\mathbf{c}^{(k)}(S_1))^* \to T^* \text{ inner}} F(T) & \coprod_{a:S_2^* \to T^* \text{ inner}} F(T) \arrow[l, "Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})"]
\end{tikzcd}$$

is a pullback in Set. So suppose we have

$$x_1 \in F(T_1) \xrightarrow{\kappa^{s^*}_{a_1}} \coprod_{a:S_1^* \to T^* \text{ inner}} F(T)$$

$$x_2 \in F(T_2) \xrightarrow{\kappa^{s^*}_{a_2}} \coprod_{a:S_2^* \to T^* \text{ inner}} F(T)$$

such that

$$Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})(x_1) = Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})(x_2),$$

i.e., we have a commuting diagram in **pOpeCard**

$$\begin{tikzcd}
S_1^* \arrow[d, "a_1"'] & (\mathbf{c}^{(k)}(S_1))^* = (\mathbf{d}^{(k)}(S_2))^* \arrow[l, "(\mathbf{c}^{(k)}_{S_1})^*"'] \arrow[r, "(\mathbf{d}^{(k)}_{S_2})^*"] \arrow[d, "a_0"'] & S_2^* \arrow[d, "a_2"] \\
T_1^* & T^* \arrow[l, "f_1^*"] \arrow[r, "f_2^*"'] & T_2^*
\end{tikzcd}$$

such that

$$F(f_1)(x_1) = F(f_2)(x_2).$$

By Proposition 10.2,

$$\mathbf{c}^{(k)}a_1 = a_0 = \mathbf{d}^{(k)}a_2 \qquad f_1^* = (\mathbf{c}^{(k)}_{T_1})^* \qquad f_2^* = (\mathbf{d}^{(k)}_{T_2})^*,$$

and the square

$$\begin{tikzcd}
T_1 \arrow[r, "\kappa_1'"] & T_1 \oplus_k T_2 \\
T \arrow[u, "f_1"] \arrow[r, "f_2"'] & T_2 \arrow[u, "\kappa_2'"']
\end{tikzcd}$$

is a special pushout. We have a commuting diagram

$$\begin{tikzcd}
 & S_1^* \arrow[rr] \arrow[dd, "a_1"] & & (S_1 \oplus_k S_2)^* \arrow[dd, "a_1 \oplus_k a_2"] \\
\mathbf{c}^{(k)}(S_1)^* \arrow[ur] \arrow[rr] \arrow[dd, "a_0"'] & & S_2^* \arrow[ur] \arrow[dd, "a_2"] & \\
 & T_1^* \arrow[rr] & & (T_1 \oplus_k T_2)^* \\
T^* \arrow[ur] \arrow[rr] & & T_2^* \arrow[ur] &
\end{tikzcd}$$

where the bottom square is the above square, and the top square is the one we formed earlier. All the horizontal morphisms are outer. Since a_1 and a_2 are inner, $a_1(S_1) = T_1$ and $a_2(S_2) = T_2$, we have

$$(a_1 \oplus_k a_2)(S_1 \oplus_k S_2) = a_1(S_1) \oplus_k a_2(S_2) = T_1 \oplus_k T_2,$$

i.e., $a_1 \oplus a_2 : (S_1 \oplus_k S_2)^* \longrightarrow (T_1 \oplus_k T_2)^*$ is inner, as well. So in fact all vertical morphisms in the above diagram are inner.

Suppose we have another inner map u and outer maps κ_1'', κ_2'' so that the squares

$$\begin{tikzcd}
S_1^* \arrow[r, "\kappa_1^*"] \arrow[d, "a_1"'] & (S_1 \oplus_k S_2)^* \arrow[d, "u"] & S_2^* \arrow[l, "\kappa_2^*"'] \arrow[d, "a_2"] \\
T_1^* \arrow[r, "\kappa_1''^*"'] & U^* & T_2^* \arrow[l, "\kappa_2''^*"]
\end{tikzcd}$$

commute. A diagram chasing shows that

$$\kappa_1''^* \circ f_1^* \circ a_1 = \kappa_2''^* \circ f_2^* \circ a_1.$$

As inner-outer factorization is essentially unique, it follows that

$$\kappa_1''^* \circ f_1^* = \kappa_2''^* \circ f_2^*.$$

By the universal property of the pushout $(T_1 \oplus_k T_2)^*$, we have an ω-functor

$$v : (T_1 \oplus_k T_2)^* \longrightarrow U^*$$

such that

$$\kappa''_1 = u \circ \kappa'_1 \qquad \kappa''_2 = u \circ \kappa'_2.$$

Then again, by a diagram chasing, we get

$$u \circ \kappa_i = v \circ (a_1 \oplus_k a_2) \circ \kappa_i,$$

for $i = 1, 2$. Hence, by universal property of the pushout $(S_1 \oplus_k S_2)^*$, we have that $u = v \circ (a_1 \oplus_k a_2)$. But both u and $(a_1 \oplus_k a_2)$ are inner so by uniqueness of factorization, see Lemma 10.3, v must be an iso, as well. This means that we need to find an

$$x \in F(T_1 \oplus_k T_2) \xrightarrow{\kappa^{(S_1 \oplus_k S_2)^*}_{(a_1 \oplus_k a_2)}} \coprod_{a:(S_1 \oplus_k S_2)^* \to T^* \text{ inner}} F(T)$$

such that

$$Lan_{\mathbf{j}}(F)(\kappa_1)(x) = x_1, \quad Lan_{\mathbf{j}}(F)(\kappa_2)(x) = x_2.$$

But F sends special pushouts in $\mathbf{pOpeCard}$ to pullbacks in Set so the square

$$\begin{tikzcd}
F(T_1) \arrow[d, "F(f_1)"'] & F(T_1 \oplus_k T_2) \arrow[l, "F(\kappa'_1)"'] \arrow[d, "F(\kappa'_2)"] \\
F(T) & F(T_2) \arrow[l, "F(f_2)"]
\end{tikzcd}$$

is a pullback in Set. Thus indeed there is a unique $x \in F(T_1 \oplus_k T_2)$ such that $F(\kappa'_i)(x) = x_i$ for $i = 1, 2$. This shows that $Lan_{\mathbf{j}}(F)$ preserves special pullbacks. $\Box$

Lemma 16.3. *The following square*

$$\begin{tikzcd}
\mathbf{pPoly} \arrow[rr, "\mathbf{e}"] \arrow[d, "\widehat{(-)}"'] & & \omega Cat \arrow[d, "\widehat{(-)}"] \\
sPb((\mathbf{pOpeCard})^{op}, Set) \arrow[rr, "Lan_{\mathbf{j}}"'] & & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)
\end{tikzcd}$$

commutes, up to an isomorphism.

Proof. We shall define two natural transformations φ and ψ which are mutually inverse, i.e., for a positive-to-one polygraph Q we define

$$\begin{tikzcd} Lan_{\mathbf{j}}\left(\mathbf{pPoly}((-)^*, Q)\right) \arrow[r, shift left, "\varphi_Q"] & \omega Cat((-)^*, Q). \arrow[l, shift left, "\psi_Q"] \end{tikzcd}$$

Let $a : S^* \longrightarrow T^*$ be an inner map and $g : T^* \to Q$ be a polygraph map, i.e., g is in the following coproduct/

$$g \in \mathbf{pPoly}(T^*, Q) \xrightarrow{\kappa^{S^*}_a} \textstyle\coprod_{S^* \to R^* \text{ inner}} \mathbf{pPoly}(R^*, Q).$$

Then we put

$$\varphi_Q(g) = g \circ a.$$

On the other hand, for an ω-functor $f : S^* \to Q \in \omega Cat(S^*, Q)$, by Proposition 12.2.4, we have a factorization

$$\begin{tikzcd} S^* \arrow[rr, "f"] \arrow[dr, "f^{in}"'] & & Q \\ & T^*_{f(S)} \arrow[ur, "\tau_{f(S)}"'] & \end{tikzcd}$$

Then we put

$$\psi_Q(f) = \tau_{f(S)} \in \mathbf{pPoly}(T^*_{f(S)}, Q) \xrightarrow{\kappa^{S^*}_{f^{in}}} \textstyle\coprod_{S^* \to R^* \text{ inner}} \mathbf{pPoly}(R^*, Q).$$

The fact that these transformations are mutually inverse follows from the fact that the above factorization is essentially unique.

The verifications that these transformations are natural is left for the reader. $\square$

Theorem 16.4. *The functor*

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

is monadic.

Proof. We are going to verify Beck's conditions for monadicity. As $\mathbf{j}$ is essentially surjective, $\mathbf{j}^*$ is conservative. By Lemma 16.2, the adjunction $Lan_{\mathbf{j}} \dashv j^*$ restricts to the above categories. So $\mathbf{j}^*$ has a left adjoint. It remains to show that $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ has coequalizers of $\mathbf{j}^*$-contractible coequalizer pairs and that $\mathbf{j}^*$ preserves them. To this aim, let us assume that we have a parallel pair

$$\begin{tikzcd} A \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & B \end{tikzcd}$$

of morphisms in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ such that

$$\begin{tikzcd} A((-)^*) \arrow[r, shift left=2, "F_{(-)^*}"] \arrow[r, shift right=2, "G_{(-)^*}"'] & B((-)^*) \arrow[l, "t"'] \arrow[r, shift left, "q"] & Q \arrow[l, shift left, "s"] \end{tikzcd}$$

is a split coequalizer in $sPb((\mathbf{pOpeCard})^{op}, Set)$, i.e., the following equations

$$q \circ s = 1_Q \quad q \circ G_{(-)^*} = q \circ F_{(-)^*} \quad F_{(-)^*} \circ t = 1_{B((-)^*)} \quad G_{(-)^*} \circ t = s \circ q$$

hold. We are going to construct a special pullbacks preserving functor

$$C : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$$

and a natural transformation

$$H : B \longrightarrow C$$

so that the diagram in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$

$$\begin{tikzcd} A \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & B \arrow[r, "H"] & C \end{tikzcd}$$

is a coequalizer, and $H_{(-)^*} = q$.

The functor C on a morphism $f : T_1^* \longrightarrow T_2^*$ is defined as in the diagram

$$\begin{tikzcd}
C(T_1^*) \arrow[r, "C(f)"] \arrow[d, equal] & C(T_2^*) \arrow[d, equal] \\
Q(T_1) \arrow[d, "s_{T_1}"'] & Q(T_2) \\
B(T_1^*) \arrow[r, "B(f)"'] & B(T_2^*) \arrow[u, "q_{T_2}"']
\end{tikzcd}$$

i.e., $C(T_i) = Q(T_i)$, for $i = 1, 2$ and $C(f) = q_{T_2} \circ B(f) \circ s_{T_1}$.

The natural transformation H is given by

$$H_{T^*} = q_T,$$

for $T \in \mathbf{pOpeCard}$. It remains to verify that

1. C is a functor;
2. H is a natural transformation;
3. $C((-)^*) = Q$;
4. $H_{(-)^*} = q$;
5. C preserves the special pullbacks;
6. H is a coequalizer.

Ad 1. Let

$$\begin{tikzcd}
T_1^* & T_2^* \arrow[l, "f"'] & T_3^* \arrow[l, "g"']
\end{tikzcd}$$

be a pair of morphisms in $\mathbf{pOpeCard}_\omega$. We calculate

$$C(g) \circ C(f) = q_{T_3} \circ B(g) \circ s_{T_2} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= q_{T_3} \circ B(g) \circ G_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ G_{T_3^*} \circ A(g) \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ F_{T_3^*} \circ A(g) \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(g) \circ F_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= q_{T_3} \circ B(g) \circ 1_{B(T_2)^*} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(g) \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(f \circ g) \circ s_{T_1} = C(f \circ g),$$

i.e., C preserves compositions. If T is a positive opetopic cardinal, we also have

$$C(1_{T^*}) = q_T \circ B(1_{T^*}) \circ s_T = q_T \circ s_T = 1_{Q(T)} = 1_{C(T^*)},$$

i.e., C preserves identities, as well.

Ad 2. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. We have

$$H_{T_2^*} \circ B(f) = q_{T_2} \circ B(f) =$$
$$= q_{T_2} \circ B(f) \circ F(T_1^*) \circ t_{T_1} =$$
$$= q_{T_2} \circ F(T_2^*) \circ A(f) \circ t_{T_1} =$$
$$= q_{T_2} \circ G(T_2^*) \circ A(f) \circ t_{T_1} =$$
$$= q_{T_2} \circ B(f) \circ G(T_1^*) \circ t_{T_1} =$$
$$= q_{T_2} \circ B(f) \circ s_{T_1} \circ q_{T_1} =$$
$$= C(f) \circ q_{T_1} = C(f) \circ H_{T_1^*},$$

i.e., H is a natural transformation.

Ad 3. Let $f : T_2 \longrightarrow T_1$ be a morphism in $\mathbf{pOpeCard}$. Thus q is natural with respect to f. So we have

$$C(f^*) = q_{T_2} \circ B(f^*) \circ s_{T_1} = Q(f) \circ q_{T_1} \circ s_{T_1} = Q(f) \circ 1_{T_1} = Q(f),$$

i.e., $C_{(-)^*} = Q$.

Ad 4. $H_{(-)^*} = q$ holds by definition.

Ad 5. Since special pullbacks involve only the outer morphisms (i.e., those that come from $\mathbf{pOpeCard}$), and Q preserves special pullbacks, so does C.

Ad 6. Finally, we shall show that H is a coequalizer. Let $p : B \longrightarrow Z$ be a natural transformation in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ such that $pF = pG$. We put $k = s; p : C \longrightarrow Z$, so that we have a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & B \arrow[r, "H"] \arrow[rd, "p"'] & C \arrow[d, "k = s;p"] \\
 & & Z
\end{tikzcd}$$

We need to verify that k is a natural transformation in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, such that $p = H;k$. Then the uniqueness of k will follow from the fact that q is a split epi. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. Then

$$k_{T_2^*} \circ C(f) = k_{T_2^*} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= p_{T_2^*} \circ s_{T_2} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= p_{T_2^*} \circ G_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= p_{T_2^*} \circ F_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= p_{T_2^*} \circ B(f) \circ s_{T_1} =$$
$$= D(f) \circ p_{T_1^*} \circ s_{T_1} = D(f) \circ k_{T_1^*},$$

i.e., k is a natural transformation and hence H is indeed a coequalizer of F and G in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, as required. □

Theorem 16.5. *The nerve functor*

$$\widehat{(-)} : \omega Cat \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

sending the ω-category C to the presheaf

$$\omega Cat((-)^*, C) : (\mathbf{pOpeCard})^{op} \longrightarrow Set$$

is monadic.

Proof. This is obtained by combining the previous theorem with Corollaries 13.5 and 15.2. □

Proposition 16.6. *The functor*

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_{\omega})^{op}, Set)$$

preserves connected limits.

Proof. This is a consequence of Lemma 16.1, where $Lan_{\mathbf{j}} : \widehat{\mathbf{pOpeCard}} \longrightarrow \widehat{\mathbf{pOpeCard}}_{\omega}$ is described as a familially representable functor. In particular, it preserves connected limits. The above functor is a restriction of a familially representable functor to the category of functors preserving special pullbacks. Since limits commute with limits, the functor

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_{\omega})^{op}, Set)$$

preserves the connected limits, as well. □

Theorem 16.7. *The embedding functor*

$$\mathbf{e} : \mathbf{pPoly} \longrightarrow \omega Cat$$

preserves connected limits. □

Proof. This follows immediately from Propositions 13.4, 16.6, Lemma 16.3 and Corollary 15.2. □

17. More on monadic adjunctions and distributive laws

We have shown that ωCat is monadic over $\mathbf{pPoly}$ with the free functor being the embedding $\mathbf{pPoly} \to \omega Cat$. We also know that the category of positive-to-one polygraphs is equivalent to the category of presheaves on $\mathbf{pOpe}$ and to the subcategory of special pullback preserving functors $sPb((\mathbf{pOpeCard})^{op}, Set)$ of the presheaf category $\widehat{\mathbf{pOpeCard}}$. Because of the last equivalence we shall freely use the notation $X(Q)$ when X is a presheaf on $\mathbf{pPoly}$ and Q ranges over all positive opetopic cardinals. In this section we shall describe explicitly the whole strongly cartesian monad $(T_{\omega}, \eta_{\omega}, \mu_{\omega})$ on $\widehat{\mathbf{pOpe}}$ whose category of algebras is equivalent to ωCat. We also show that this monad decomposes into two other strongly cartesian monads of 'pure composition' (T_c, η_c, μ_c) and of 'adding identities' $(T_{\iota}, \eta_{\iota}, \mu_{\iota})$, in analogy with the decomposition of the strongly cartesian free monoid monad T_{mon} into free semigroup monad and pointed set monad, c.f. [3, p. 258]. In particular, the nerve theorem, c.f. [17, 6], applies.

The monad T_ω

We write $P \overset{q}{\ominus\!\!\!\!\rightarrow} Q$ to indicate that the map q is inner, i.e., it is an ω-functor between ω-categories P^* and Q^* so that $q(P) = Q$. Let $u : X \to Y$ be a map of presheaves on $\mathbf{pOpe}$, and S be a positive opetope. Then $T_\omega(X)(S)$ is given by the coproduct [27] $T_\omega(X)(S) =$

$$\coprod_{S \overset{q}{\ominus\!\!\!\!\rightarrow} Q} X(Q) = \{\langle x, q\rangle | S \overset{q}{\ominus\!\!\!\!\rightarrow} Q \in \mathbf{pOpeCard},\ \ x : Q \to X \in X(Q)\},$$

with coprojections

$$\kappa^1_q : X(Q) \longrightarrow \coprod_{S \overset{q}{\ominus\!\!\!\!\rightarrow} Q} X(Q) = T_\omega(X)(S)$$

$$X(Q) \ni x \mapsto \langle x, q\rangle.$$

Moreover we set

$$T_\omega(X)_f : T_\omega(X)(S) \longrightarrow T_\omega(X)(S')$$

$$\langle x, q\rangle \mapsto \langle x \circ f', q'\rangle,$$

where (q', f') is the inner-outer factorization of $q \circ f$

$$\begin{tikzcd}
S' \arrow[r, "f"] \arrow[d, "q'"'] & S \arrow[d, "q"] \\
Q' \arrow[r, "f'"'] & Q
\end{tikzcd}$$

and

$$T_\omega(u)_S : T_\omega(X)(S) \longrightarrow T_\omega(Y)(S)$$

$$\langle x, q\rangle \mapsto \langle u \circ x, q\rangle.$$

---

[27] (PLC) For example, Q can be of the form $S_1 \oplus_n S_2$, with S_1, S_2 opetopes of the same dimension n as S, with q mapping the generator S to the free composite of the generators S_1 and S_2, so that an element of $X(Q)$ is a pair of an element in $X(S_1)$ and an element in $X(S_2)$. Then the (Q, q)-component of a T_ω-algebra α will provide an actual composition of these elements.

The iteration $T^2_\omega(X)(S)$ is given by the coproduct

$$T^2_\omega(X)(S) = \coprod_{S \overset{q'}{\circ\!\!\!\rightarrow} R} \ \coprod_{R \overset{q}{\circ\!\!\!\rightarrow} Q} X(Q) =$$

$$= \{\langle x, q, q'\rangle | S \overset{q'}{\circ\!\!\!\rightarrow} R, R \overset{q}{\circ\!\!\!\rightarrow} Q \in \mathbf{pOpeCard}, \ \ x \in X(Q)\},$$

with coprojections

$$\kappa^2_{q,q'} : X(R) \longrightarrow \coprod_{S \overset{q'}{\circ\!\!\!\rightarrow} Q} \ \coprod_{Q \overset{q}{\circ\!\!\!\rightarrow} R} X(R) = T^2_\omega(X)(S)$$

$$X(R) \ni x \mapsto \langle x, q, q'\rangle.$$

The unit is given by

$$((\eta_\omega)_X)_S = \kappa^1_{id_S} : X(S) \longrightarrow T_\omega(X)(S) = \coprod_{S \overset{q}{\circ\!\!\!\longrightarrow} Q} X(Q)$$

$$X(S) \ni x \mapsto \langle x, id_S\rangle$$

and the multiplication is the unique map commuting with the following coprojections

$$\begin{tikzcd}
T^2_\omega(X)(S) = \coprod_{S \overset{q'}{\circ\!\!\!\rightarrow} Q} \ \coprod_{Q \overset{q}{\circ\!\!\!\rightarrow} R} X(R) \arrow[rr, "((\mu_\omega)_X)_S"] & & \coprod_{S \overset{k}{\circ\!\!\!\longrightarrow} R} X(R) = T_\omega(X)(S) \\
& X(R) \arrow[ul, "\kappa^2_{q,q'}"] \arrow[ur, "\kappa^1_{q \circ q'}"'] &
\end{tikzcd}$$

i.e.,

$$\langle x, q, q'\rangle \mapsto \langle x, q \circ q'\rangle.$$

The factorization of a morphism

$$f : P \longrightarrow T_\omega(1)$$

through an inner map q_f is

$$\begin{tikzcd}
P \arrow[r, "q_f"'] & T_\omega(Q_f) \arrow[r] & T_\omega(1),
\end{tikzcd}$$

where $q_f = f_P(id_P)$.

The comparison functor K

Now we will describe the comparison functors between the category of ω-categories ωCat and the category of T_ω-algebras $\widehat{\mathbf{pOpe}}^{T_\omega}$, i.e., we shall define the functor K in the diagram

$$\begin{tikzcd}
\omega Cat \arrow[rr, shift right, "K"'] \arrow[dr, shift left, "N"] & & \widehat{\mathbf{pOpe}}^{T_\omega} \arrow[ll, shift right, "L"'] \arrow[dl, shift left, "U^{T_\omega}"] \\
& \widehat{\mathbf{pOpe}} \arrow[ul, shift left, "F_\omega"] \arrow[ur, shift left, "F^{T_\omega}"] &
\end{tikzcd}$$

and its left adjoint L. [28]

If $H : \mathcal{C} \to \mathcal{C}'$ is an ω-functor, and $f : S \to S'$ a morphism in $\mathbf{pOpe}$, then

$$K(\mathcal{C})(S) = \omega Cat(S^*, \mathcal{C}),$$

and

$$K(\mathcal{C})(f) : K(\mathcal{C})(S') \to K(\mathcal{C})(S)$$

$$h : S'^* \to \mathcal{C} \mapsto h \circ f.$$

Moreover, the T_ω-algebra map

$$\xi_\mathcal{C} : T_\omega(K(\mathcal{C})) \longrightarrow K(\mathcal{C}),$$

for $S \in \mathbf{pOpe}$ is given by

$$(\xi_\mathcal{C})_S : T_\omega(K(\mathcal{C}))(S) \longrightarrow K(\mathcal{C})(S)$$

$$\langle S \overset{q}{\circ\!\!\!\!\!\!\rightarrow} S', S'^* \xrightarrow{h} \mathcal{C} \rangle \mapsto h \circ q : S^* \to \mathcal{C}.$$

In particular, for $P, S \in \mathbf{pOpe}$,

---

[28] (PLC) Here, in reference to Section 1, we can set $F_\omega = \mathbf{e} \circ \widetilde{(-)} \circ Ran_{\mathbf{i}}$, i.e., F_ω is $\mathbf{e}$ up to the characterization of $\mathbf{pPoly}$ as $\widehat{\mathbf{pOpe}}$. One can chase a candidate for N similarly. One then could embark on proving the commutation of the triangles, which could provide an alternative direct proof of the monadicity result of Section 16.

$$\begin{array}{rcl} K(P^*)(S) & = & \coprod_{S \overset{h}{\mathrel{-\!\!\!\!\ominus}} S'} \mathbf{pOpe}(S', P) \\ & \cong & \{\langle h, k\rangle | h : S \mathrel{-\!\!\!\!\ominus} S',\ k : S' \to P \in \mathbf{pOpe}\} \\ & \cong & T_\omega(P)(S). \end{array}$$

For an ω-functor $f : P^* \to Q^*$ in $\mathbf{pOpe}_\omega$, the map

$$K(f) : T_\omega(P)(S) \longrightarrow T_\omega(Q)(S)$$

is given by

$$\langle q : S \mathrel{-\!\!\!\!\ominus} S', x : S' \to P\rangle \mapsto \langle q' : S \mathrel{-\!\!\!\!\ominus} S'', x' : S'' \to Q\rangle,$$

where (q', x') is the inner-outer factorization of the map in $\mathbf{pOpe}_\omega$

$$\begin{tikzcd} S^* \arrow[r, "q"] & S'^* \arrow[r, "x^*"] & P^* \arrow[r, "f"] & Q^*. \end{tikzcd}$$

The comparison functor L

Below we describe explicitly the left adjoint, the essential inverse to the functor K.

For a T_ω-algebra $(X, \xi : T_\omega(X) \to X)$, the ω-category $L((X, \xi)$ is defined as follows. The set of n-cells is

$$L(X, \xi)_n = X(\alpha^n),$$

and the map

$$m_{n,k,n} : X(\alpha^n) \times_{X(\alpha^k)} X(\alpha^n) \cong X(\alpha^{n,k,n}) \longrightarrow X(\alpha^n),$$

composing n-cells compatible over dimension k is the composition of the maps

$$\begin{tikzcd} X(\alpha^{n,k,n}) \arrow[r, "\kappa_{m_{n,k,n}}"] & T_\omega(X)(\alpha^n) \arrow[r, "\xi_{\alpha^n}"] & T_\omega(\alpha^n). \end{tikzcd}$$

Moreover, as there is a unique map from α^n to any opetopic cardinal Q of dimension less or equal n $\alpha^n \mathrel{-\!\!\!\!\ominus} Q$, we have

$$L(T_\omega(P))_n = T_\omega(P)(\alpha^n) = \coprod_{din(S') \leq n} \mathbf{pOpe}(S', P).$$

Generic maps and the generic closure.

Recall the notion of a T_ω-generic map from [17, 6].

Let $g : P \to T_\omega(D)$ be a map in $\widehat{\mathbf{pOpe}}$. Then we have a unique extension $\overline{g}$ as in the diagram

$$\begin{tikzcd}
P \arrow[rr, "\eta"] \arrow[dr, "g"'] & & T_\omega(P) \arrow[dl, "\overline{g}"] \\
& T_\omega(D) &
\end{tikzcd}$$

The morphism $g : P \to T_\omega(D)$ is T_ω-generic iff z is an isomorphism, where

$$\overline{g}_P(id_P) = \langle P \overset{q}{\to\!\!\!\!\!\circ\,} P', P' \overset{z}{\to} D\rangle.$$

To see this, consider maps $w \to T_\omega(X)$, $v : Q \to Y$ and $u : X \to Y$ in $\widehat{\mathbf{pOpe}}$, so that the diagram

$$\begin{tikzcd}
P \arrow[r, "w"] \arrow[d, "g"'] & T_\omega(X) \arrow[d, "T_\omega(u)"] \\
T_\omega(Q) \arrow[r, "T_\omega(v)"'] & T_\omega(Y)
\end{tikzcd}$$

commutes. Then if $w(id_P) = \langle q', x'\rangle$ with $x = x' \circ q' : P \to X$, and $g(id_P) = \langle q, id_Q\rangle$, then the above commutation is equivalent to the commutation of the square

$$\begin{tikzcd}
P \arrow[r, "x"] \arrow[d, "q"'] & X \arrow[d, "u"] \\
Q \arrow[r, "v"'] & Y
\end{tikzcd}$$

Since (q, v) is an inner-outer factorization and u is outer, there is an outer map $d : Q \to X$ making the square commute. Hence $T_\omega(d)$ is the lift in the previous square showing that g is indeed generic. Moreover, if z as above is not an isomorphism then we do not have a lifting d in general, and hence g is not a generic morphism.

Thus, since any positive opetopic cardinal is a codomain of a T_ω-generic map whose domain is a positive opetope, in fact α^n, for some $n \in \omega$, see page 242, the T_ω-generic closure of $\mathbf{pOpe}$ consists of all opetopic cardinals, i.e., $\mathbf{pOpeCard}$ is the T_ω-closure of $\mathbf{pOpe}$.

T_ω as a local right adjoint

The functor T_ω is a local right adjoint as it is familialy representable. Below we describe explicitly the left adjoint $L_{\omega,1}$ to the functor

$$T_{\omega,1} : \widehat{\mathbf{pOpe}} \longrightarrow \widehat{\mathbf{pOpe}}/T_\omega(1).$$

Let $\pi : X \to T_\omega(1)$ be an object of $\widehat{\mathbf{pOpe}}/T_\omega(1)$. We have a functor

$$D_X : \int_{\mathbf{pOpe}} X \longrightarrow \widehat{\mathbf{pOpe}}$$

such that the image of a morphism

$$\begin{tikzcd}
P \arrow[rr, "f"] \arrow[dr, "h"'] & & P' \arrow[dl, "h'"] \\
& X &
\end{tikzcd}$$

in $\int_{\mathbf{pOpe}} X$ is the map $D_X(f) : D_X(P,h) \to D_X(P',h')$ defined from the diagram

$$\begin{tikzcd}
P \arrow[rr, "f"] \arrow[dr, "h"'] \arrow[dd, "q"'] & & P' \arrow[dl, "h'"] \arrow[dd, "q'"] \\
& X \arrow[dd, "\pi"] & \\
D_X(P,h) \arrow[rr, "D_X(f)"'] \arrow[dr, "\bar{h}"'] & & D_X(P',h') \arrow[dl, "\bar{h}'"] \\
& T_\omega(1) &
\end{tikzcd}$$

where $(q, \bar{h})$ is an inner-outer factorization of $\pi \circ h$, $(q', \bar{h}')$ is an inner-outer factorization of $\pi \circ h'$, and $D_X(f)$ is the unique map making the whole diagram commute. Then $L_{\omega,1}(X, \pi)$ is the colimit of the functor D_X.

The distributive law.

If we replace in the above formulas the inner maps by inner epis (resp. inner monos), we still get strongly cartesian monads $(T_{\omega,\iota}, \eta_{\omega,\iota}, \mu_{\omega,\iota})$ $((T_{\omega,c}, \eta_{\omega,c}, \mu_{\omega,c}))$ on $\widehat{\mathbf{pOpe}}$. These monads do compose to the monad $(T_\omega, \eta_\omega, \mu_\omega)$. This is because there is a (cartesian) distributive law

$$\lambda_\omega : T_{\omega,c} \circ T_{\omega,\iota} \longrightarrow T_{\omega,\iota} \circ T_{\omega,c}.$$

For X in $\widehat{\mathbf{pOpe}}$ and S in $\mathbf{pOpe}$, both $((T_{\omega,c} \circ T_{\omega,\iota})_X)_S$ and $((T_{\omega,\iota} \circ T_{\omega,c})_X)_S$ are given by double coproducts with coprojections as displayed

$$\sigma^{c,\iota}_{m,e} : X(R) \longrightarrow \coprod_{S \overset{m}{\circ\!\!\!\!\!\rightarrow} Q \; mono} \quad \coprod_{Q \overset{e}{\circ\!\!\!\!\!\rightarrow} R \; epi} X(R) = ((T_{\omega,c} \circ T_{\omega,\iota})_X)_S$$

$$\sigma^{\iota,c}_{e',m'} : X(R) \longrightarrow \coprod_{S \overset{e'}{\circ\!\!\!\!\!\rightarrow} Q' \; epi} \quad \coprod_{Q' \overset{m'}{\circ\!\!\!\!\!\rightarrow} R \; mono} X(R) = ((T_{\omega,\iota} \circ T_{\omega,c})_X)_S.$$

The component

$$((\lambda_\omega)_X)_S : ((T_{\omega,c} \circ T_{\omega,\iota})_X)_S \longrightarrow ((T_{\omega,\iota} \circ T_{\omega,c})_X)_S$$

of the distributive law λ_ω is the unique map making all the following triangles

$$\begin{tikzcd}
\coprod_{S \overset{m}{\circ\!\!\!\!\!\rightarrow} Q \; mono} \coprod_{Q \overset{e}{\circ\!\!\!\!\!\rightarrow} R \; epi} X(R) \arrow[rr, "((\lambda_\omega)_X)_S"] & & \coprod_{S \overset{e'}{\circ\!\!\!\!\!\rightarrow} Q' \; epi} \coprod_{Q' \overset{m'}{\circ\!\!\!\!\!\rightarrow} R \; mono} X(R) \\
& X(R) \arrow[ul, "\sigma^{c,\iota}_{m,e}"] \arrow[ur, "\sigma^{\iota,c}_{e',m'}"'] &
\end{tikzcd}$$

commute, where e', m' is the epi-mono factorization of the inner map $e \circ m$, i.e., we have the following square of inner epi's and mono's in $\mathbf{pOpe}_\omega$

$$\begin{tikzcd}
S \arrow[r, tail, "m"] \arrow[d, two heads, "e'"'] & Q \arrow[d, two heads, "e"] \\
Q' \arrow[r, tail, "m'"'] & R
\end{tikzcd}$$

that commutes.

All the above constructions and considerations can be to truncated to the level n, for $n \in \omega$. Thus $n\mathbf{Cat}$ is monadic over $\widehat{pOpe_n}$ and the corresponding monad (T_n, η_n, μ_n), the n-truncation of the monad $(T_\omega, \eta_\omega, \mu_\omega)$ is strongly cartesian decomposing into two strongly cartesian monads $(T_{n,\iota}, \eta_{n,\iota}, \mu_{n,\iota})$ and $(T_{n,c}, \eta_{n,c}, \mu_{n,c})$, related by a distributive law

$$\lambda_n : T_{n,c} \circ T_{n,\iota} \longrightarrow T_{n,\iota} \circ T_{n,c},$$

so that we have

$$T_n = T_{n,\iota} \circ T_{n,c}$$

as monads.

18. Appendix: a definition of positive-to-one polygraphs

The category of positive-to-one polygraphs $\mathbf{pPoly}$, a replete subcategory of the category of ω-categories ωCat, is a limit of a tower of categories of positive-to-one polygraphs of dimension n $\mathbf{pPoly}_n$, replete subcategories of the category of n-categories $n\mathbf{Cat}$. Thus we shall describe a diagram

$$\begin{tikzcd}
\mathbf{pPoly} \arrow[r, "\varphi_\omega"] \arrow[d] & \omega Cat \arrow[d] \\
\vdots & \vdots \\
\mathbf{pPoly}_{n+1} \arrow[r, "\varphi_{n+1}"] \arrow[d, "tr_n"'] & (n+1)\mathbf{Cat} \arrow[d, "tr_n"] \\
\mathbf{pPoly}_n \arrow[r, "\varphi_n"'] & n\mathbf{Cat} \\
\vdots & \vdots \\
\mathbf{pPoly}_1 \arrow[r, "\varphi_1"] \arrow[d, "tr_0"'] & \mathbf{Cat} \arrow[d, "tr_0"] \\
\mathbf{pPoly}_0 \arrow[r, "\varphi_0"'] & Set
\end{tikzcd}$$

Having defined the subcategory $\mathbf{pPoly}_n$ of $n\mathbf{Cat}$, we define the category $\mathbf{CatPoly}_{n+1}$ to be a non-full subcategory of the category of $(n+1)$-categories $(n+1)\mathbf{Cat}$ such that an $(n+1)$-category C is an object of $\mathbf{CatPoly}_{n+1}$ iff its truncation to $n\mathbf{Cat}$ is in $\mathbf{pPoly}_n$, and an $(n+1)$-functor $f : C \to D$ is $\mathbf{CatPoly}_{n+1}$ iff its truncation to $n\mathbf{Cat}$ is in $\mathbf{pPoly}_n$.

The first two stages of the above tower are built as follows.

$$\begin{tikzcd}
Set & & \mathbf{pPoly}_1 \arrow[ll, "D_1"'] \arrow[r, "\varphi_1"] \arrow[d] & \mathbf{CatPoly}_1 \arrow[r, "\psi_1"] \arrow[d, "tr_0"] \arrow[dll, "U_1"] & \mathbf{Cat} \arrow[d, "tr_0"] \\
& Set \downarrow D_0 \arrow[ur, "\overline{(-)}^1"] \arrow[dl, "|-|_1"'] \arrow[dr, "t_0"'] \arrow[urr, "F_1"] & \mathbf{pPoly}_0 & & \\
Set & & \mathbf{pPoly}_0 \arrow[ll, "D_0"] \arrow[r, "\varphi_0"'] & \mathbf{CatPoly}_0 \arrow[r, "\psi_0"'] & Set
\end{tikzcd}$$

The categories $\mathbf{pPoly}_0$ and $\mathbf{CatPoly}_0$ are the category of sets, the functors φ_0 and ψ_0 are the identity on Set, and the functor D_0 sends set X to its product $X \times X$. $\mathbf{CatPoly}_1$ is $\mathbf{Cat}$ and ψ_1 is an identity on $\mathbf{Cat}$. Having this data, we form a comma category $Set \downarrow D_0$ with projections

$$|-|_1 : Set \downarrow D_0 \to Set \quad \text{and} \quad t_0 : Set \downarrow D_0 \to \mathbf{pPoly}_0.$$

There is a forgetful functor from $U_1 : \mathbf{CatPoly}_1 = \mathbf{Cat} \to Set \downarrow D_0$ sending a category $(C, d, c, \circ, i)$ to (C, d, c). The functor U_1 has a left adjoint F_1 building a free category on a graph. Then $\mathbf{pPoly}_1$ is the replete subcategory of $\mathbf{Cat}$ such that

$$\begin{tikzcd}
Set \downarrow D_0 \arrow[r, "\overline{(-)}^1"] & \mathbf{pPoly}_1 \arrow[r, "\varphi_1"] & \mathbf{Cat}
\end{tikzcd}$$

is a full and faithful/essentially surjective factorization. Thus $\mathbf{pPoly}_1$ is a category of free categories over graphs with functors that send generating morphisms to generating morphisms. Finally, we define the functor

$$D_1 : \mathbf{pPoly}_1 \longrightarrow Set$$

$X \mapsto \{\langle x, y\rangle \in X_1 \mid x \| y, y \text{ is a generator and } x \text{ is a non-identity morphism}\}$,

i.e., sending a free category X to the set of parallel pairs of morphisms $\langle x, y\rangle$, i.e., $d(x) = d(y)$ and $c(x) = c(y)$, noted $x\|y$, such that y is a generating morphism and x is a non-identity morphism.

The inductive stage in the construction of the above tower is similar to the second one.

$$\begin{tikzcd}
Set & & \mathbf{pPoly}_{n+1} \arrow[ll, "D_{n+1}"'] \arrow[r, "\varphi_{n+1}"] \arrow[dd] & \mathbf{CatPoly}_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[dd, "tr_n"] \arrow[dl, shift left, "U_{n+1}"] & (n+1)\mathbf{Cat} \arrow[dd, "tr_n"] \\
& Set \downarrow D_n \arrow[ur, "\overline{(-)}^{n+1}"] \arrow[urr, shift left, "F_{n+1}"] \arrow[dl, "|-|_{n+1}"'] \arrow[dr, "t_n"] & & & \\
Set & & \mathbf{pPoly}_n \arrow[ll, "D_n"] \arrow[r, "\varphi_n"'] & \mathbf{CatPoly}_n \arrow[r, "\psi_n"'] & n\mathbf{Cat}
\end{tikzcd}$$

Having defined the subcategory $\mathbf{pPoly}_n$ of $n\mathbf{Cat}$ and the functor D_n : $\mathbf{pPoly}_n \to Set$, we built the other parts of the above diagram. We form a comma category $Set \downarrow D_n$ with projections

$$|-|_{n+1} : Set \downarrow D_n \to Set \quad \text{and} \quad t_n : Set \downarrow D_n \to \mathbf{pPoly}_n.$$

There is a forgetful functor from $U_{n+1} : \mathbf{CatPoly}_{n+1} \to Set \downarrow U_n$ sending an $(n+1)$-category C in $\mathbf{CatPoly}_{n+1}$ to its n-truncation together with the set of $(n+1)$-cells and functions assigning their domains and codomains. The functor U_{n+1} has a left adjoint F_{n+1} – building free $(n+1))$-categories.Then $\mathbf{pPoly}_{n+1}$ is the replete subcategory of $\mathbf{CatPoly}_{n+1}$ such that

$$\begin{tikzcd} Set \downarrow D_n \arrow[r, "\overline{(-)}^{n+1}"] & \mathbf{pPoly}_{n+1} \arrow[r, "\varphi_{n+1}"] & \mathbf{CatPoly}_{n+1} \end{tikzcd}$$

is a full and faithful/essentially surjective factorization. Thus $\mathbf{pPoly}_{n+1}$ is a category of free $(n+1)$-categories whose generators/indeterminates are cells of positive-to-one shapes, and of $(n+1)$-functors that send indeterminates to indeterminates. [29] Finally, we define the functor

$$D_{n+1} : \mathbf{pPoly}_{n+1} \longrightarrow Set$$

$$X \mapsto \{\langle x, y\rangle \in X_{n+1} |\ x \| y,\ \ y \text{ indeterminate },\ x \text{ non-identity cell}\},$$

i.e., sending a free category X to the set of parallel pairs of cells $\langle x, y\rangle$, i.e., $d(x) = d(y)$ and $c(x) = c(y)$, noted $x\|y$, such that y is an indeterminate and x is a non-identity map.[30]

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[29] (PLC) More precisely, F_{n+1} builds an $(n+1)$-category whose n-truncation is a positive-to-one polygraph of dimension n and whose $(n+1)$-morphisms are freely obtained from a set of generators. By construction, every positive-to-one polygraph of dimension $n+1$ comes with such a set of $(n+1)$-generators, which is left implicit here, but is explicit in the definition given in [7, 1].

[30] (PLC) The definition of the functors D_n guarantees that the shapes of generators in all dimensions are positive-to-one. Note also that by construction the truncation tr_n : $(n+1)\mathbf{Cat} \to n\mathbf{Cat}$ restricts and corestricts to $tr_n = \mathbf{CatPoly}_{n+1} \to \mathbf{CatPoly}_n$ and $tr_n : \mathbf{pPoly}_{n+1} \to \mathbf{pPoly}_n$.

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