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MAREK ZAWADOWSKI:
A SCIENTIFIC BIOGRAPHY

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Résumé. Marek Zawadowski (1960–2024) était un mathématicien polonais travaillant dans les domaines de la théorie des catégories, de la logique et des fondements, et de la théorie de l'homotopie. Nous esquissons ses nombreuses contributions à ces domaines.

Abstract. Marek Zawadowski (1960–2024) was a Polish mathematician working in the areas of category theory, logic and foundations, and homotopy theory. We outline his manifold contributions to these areas.

Keywords. Marek Zawadowski, category theory, categorical logic, descent, duality, pretopos, model completion, Heyting algebra, sheaf representation, regular theory, analytic monad, operad, opetope, higher-dimensional category, ordered face structure.

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Contents

1. Introduction

— *Krzysztof Kapulkin and Karol Szumiło*

Marek Witold Zawadowski (1960–2024) was a Polish mathematician, specializing in category theory, logic, homotopy theory, and linguistics. Born in Montréal, Que. in 1960, Zawadowski grew up in Warsaw, Poland, where he attended Stanisław Staszic High School, a prestigious and highly selective school specializing in mathematics and computer science, where the mathematics courses were taught by faculty members from the nearby University of Warsaw. Upon graduation in 1978, he enrolled at the University of Warsaw to study mathematics. He graduated in 1983 with a master's degree, having already published his first paper, on the Löwenheim–Skolem theorem in toposes [58]. He joined the faculty of the University of Warsaw on October 1, 1983, and in 1986 returned to Canada for doctoral studies, receiving a Ph.D. in 1989 from the Université de Montréal for the dissertation *Un théorème de descente pour les prétopos* [60], written under the supervision of Gonzalo E. Reyes. Following a postdoctoral appointment in Montréal, he was a faculty member at the University of Warsaw from 1991 onward, where he received his habilitation in 2004 on the basis of the book, written with Silvio Ghilardi, *Sheaves, Games, and Model Completions: A Categorical Approach to Non-Classical Propositional Logics* [21].

Throughout his career, Zawadowski made deep contributions to several areas, both within and outside of mathematics, including logic and foundations, category theory, abstract homotopy theory, type theory, and linguistics. He was an active and valued member of many communities, a supervisor and instructor universally admired by his students, and, above all, a profoundly insightful researcher. This document focuses on the last of these, but let us briefly address the first two.

On December 13, 1981, Zawadowski's undergraduate years coincided with a national crisis when the Polish puppet government, installed by the Soviet Union, declared martial law. In response, the Independent Students' Association had organized a strike in 1981 that included occupying classrooms to demand academic freedom, the removal of mandatory Marxist and Russian-language courses, and the right to free association. This action helped pave the way for the broader democratic movement that eventually led to the fall of the Iron Curtain in Europe. Zawadowski was a central figure in the process, leading the strike activities in the Department of Mathematics. The University of Warsaw's main lecture hall, the Auditorium Maximum, became a scene of this strike, with students sleeping on its floor for four consecutive weeks.

The ancient wisdom of Marcus Aurelius and Epictetus teaches us that during times of external chaos, such as martial law, one should continue to fulfill one's duties. In Albert Camus's *The Plague*, Dr. Rieux insists on "living as normally as possible" as a catastrophe unfolds around him. This lesson stayed with many of Zawadowski's fellow countrymen, including his mentor, the Polish logician Helena Rasiowa, who lived through the Second World War and the difficult post-war years, rebuilding the institutions of the Polish school of mathematics in the 1950s. Zawadowski fits squarely into this tradition: as students occupied the Auditorium Maximum, he initiated numerous seminars and lecture series to ensure that students were still doing what students should be doing. Many of his then-fellow students still remember these lectures and reminisce about learning topos theory from him.

Zawadowski served as the supervisor of numerous master's and doctoral students. His doctoral student, Stanisław Szawiel, received his Ph.D. in 2015 with the dissertation *A Unified Approach to Opetopic Algebra* [53].

His recent M.Sc. students include Jacek Karwowski (M.Sc. thesis *Formal semantics of a reversible language: symmetric groups in Homotopy Type Theory*, and B.Sc. thesis *Products in positive opetopic sets*), Mateusz Zugaj (M.Sc. thesis *Algebraic presentation of sheaves on compact spaces*), Krzysztof Galias (M.Sc. thesis *Kleisli completion of the free adjunction*), and Łukasz Sienkiewicz (M.Sc. thesis *Formal Theory of Monoidal Objects*). The story of Karwowski is of particular interest. He first heard of category theory in his freshman linear algebra class, and found the subject esoteric, if not nonsensical — an opinion he then communicated to Zawadowski, while asking him to supervise his bachelor's thesis. To the surprise of many, Zawadowski agreed, and devised a project well suited to Karwowski's mathematical interests and programming skills. Today, Karwowski is a doctoral student at Oxford University, and he credits his acceptance into the program to Zawadowski's thoughtful supervision and guidance.

Scientifically, Zawadowski's career divides naturally into several periods. Early on, he worked on categorical logic and topos theory. He is best known for his thesis work on *descent and duality*, subsequently published in the *Annals of Pure and Applied Logic* [61]. It grew out of Makkai's duality, which asserts a duality between the category of models of a first-order theory and its syntactic category, a pretopos. Zawadowski extended this duality to the broader setting of pseudoelementary categories. A second striking result is the characterization of the effective descent morphisms of pretoposes and Barr-exact categories as precisely the conservative functors. This period of Zawadowski's career is surveyed in Section 2 by Jaap van Oosten, a fellow categorical logician and a long-time friend of Zawadowski.

Zawadowski's work on duality connects to his subsequent collaboration with Silvio Ghilardi, which began in the early 1990s and culminated in the book *Sheaves, Games, and Model Completions: A Categorical Approach to Non-Classical Propositional Logics* [21], published in 2002. As the title aptly indicates, the collaboration focused on non-classical propositional logics — chiefly modal and intuitionistic systems — by analyzing their categories of (finitely generated) models. It was an interplay between an algebraic question, the existence of model completions, and a proof-theoretic one, concerning the interpretability of second-order propositional logic in ordinary propositional logic. The techniques used to establish the results deserve mention as well: they include Zawadowski's "bread and butter," dualities and sheaf representations, as well as Ehrenfeucht–Fraïssé games and bounded bisimulations. Silvio Ghilardi surveys these developments in Section 3.

In the early 2000s, Zawadowski's interests shifted, following a shift in the interests of his mentor, Michael Makkai. Higher-dimensional category theory had emerged as a central theme in the field, and Zawadowski became an active contributor, focusing in particular on the opetopic approach pioneered by John Baez and James Dolan [3] and developed extensively by Claudio Hermida, Michael Makkai, and John Power [26, 27, 28]. Zawadowski's main contribution was an axiomatic, combinatorial description of opetopes through his framework of *ordered face structures*, contained primarily in [63, 64, 69]. He also pioneered the development of connections between the opetopic approaches to higher category theory and the standard methods of homotopical algebra. These seemingly disparate threads came together in another research program initiated by Zawadowski, which recast his early interests in logic — and specifically in algebraic theories — in operadic and higher-categorical language. Much of this work, though not all, was carried out in collaboration with his Ph.D. student Stanisław Szawiel [52, 54, 55]. It is surveyed by Martin Hyland in Section 4, while Pierre-Louis Curien's overview of Zawadowski's work on the opetopic approach to higher-dimensional category theory is found in Section 5.

In the latter part of the 2010s, Zawadowski's interests shifted once again, as he turned to linguistics. This final strand of his work lies outside the scope of the present article, which concentrates on his mathematical contributions; we mention it here only to convey the full breadth of his curiosity. Indeed, no single document could do justice to the entirety of Zawadowski's work, and we have not attempted to be exhaustive. The contributions that follow are deliberately brief, each offering a high-level overview of one facet of his mathematics, written by a colleague who knew that part of his work intimately. Together, we hope, they convey something of both the depth and the range of what Marek Zawadowski achieved.

2. Categorical logic

— *Jaap van Oosten*

In this section I would like to discuss to some extent two research themes in Categorical Logic that Marek Zawadowski pursued: the Löwenheim-Skolem theorems in toposes, and his work in the tradition of Makkai and Reyes.

2.1 Löwenheim-Skolem

Already at a very young age, Zawadowski had absorbed the basics of categorical logic, as laid out in Makkai and Reyes' monograph *First Order Categorical Logic* ([40]). His first papers on the subject were probably written in 1982 (they were sent to the journal in February 1983), when he was about 22 years old. The following is a much simplified rendering of his work in this direction.

Expectations were high, in the early 1980's, that the categorical point of view might offer a fruitful perspective for classical Model Theory. But the first thing at hand was to establish facts in arbitrary toposes (Grothendieck or not) which were analogous to the model-theoretic results for the category Set.

Zawadowski set out to develop the Löwenheim-Skolem (curiously, for him Skolem-Löwenheim) theorems in arbitrary Grothendieck toposes. These theorems contain cardinalities as an essential ingredient and since one of these appears as the cardinality of something "external" (the *language*), one thinks of a theory of external *powers* and this is what Zawadowski developed in the papers [58] and [59].

A class G of objects of a topos $\mathcal{E}$ is a *class of* Σ*-generators* if every object X of $\mathcal{E}$ can be written as the union of subobjects $A_i \to X$ with A_i from G. Clearly, every class of Σ-generators is a class of generators; conversely, a class of generators is a class of Σ-generators if it is closed under epimorphic images. A class of Σ-generators is a *class of power generators* if it is closed under subobjects. An important example is the class of subobjects of the terminal object 1; this is a class of generators (equivalently, Σ-generators; note that this class is closed under epimorphic images) if and only if the topos $\mathcal{E}$ is localic over Set ([30], 5.37).

Given a class G of power generators, we define for every object X the *power of X relative to G*, $\bar{X}^G$, as the least cardinality $|I|$ of a family $\{A_i \to X \mid i \in I\}$ such that $X = \bigcup_{i \in I} A_i$ and all A_i are from G. Clearly, taking for G the standard class of power generators in Set (the class containing just the terminal object) one gets $\bar{X}_G = |X|$, the usual cardinality of a set X.

So, the power $\bar{X}^G$ depends on G, but in many cases there is a canonical choioe: the least class G of Σ-generators with the property that $\bar{1}^G = 1$. If the topos $\mathcal{E}$ is sheaves on a complete Heyting algebra then this is simply $\mathrm{Sub}(1)$, the class of subobjects of the terminal object; in the case of a presheaf category $\widehat{M} = \mathrm{Set}^{M^{\mathrm{op}}}$, one takes the closure of the set of representable presheaves under subobjects and epimorphic images; this yields the least class of power generators G. Moreover, $\bar{1}^G = 1$, if and only if the category M has a weak terminal object (an object m such that for every object n there is an arrow $n \to m$).

Now the Löwenheim-Skolem theorem for toposes is formulated by Zawadowski using the framework of *generalized quantifiers*, which I find a little too technical and notation-heavy to review here. Let me restrict myself to the following: for a (one-sorted) language L of predicate logic there is a standard notion of an *interpretation* in a topos: given an underlying universe A (object of $\mathcal{E}$) and interpretations of the non-logical symbols of L, one has an *L-structure* A and for every L-formula $\varphi(x_1, \ldots, x_n)$ a subobject $[\varphi]^A$ of the product A^n. A *substructure* B of A is a subobject B of A which interprets the non-logical symbols of L as the restrictions of those of A, to B; the substructure B is *elementary* if $[\varphi]^B = (\iota^n)^{-1}([\varphi]^A)$ for every formula $\varphi(x_1, \ldots, x_n)$, where $\iota : B \to A$ is the embedding. In other words, if

$$\begin{tikzcd}
{[\varphi]^B} \arrow[r] \arrow[d] & {[\varphi]^A} \arrow[d] \\
B^n \arrow[r] & A^n
\end{tikzcd}$$

is always a pullback.

We have then the following version of the Löwenheim-Skolem theorem:

Theorem 2.1 (Zawadowski). *Let $\mathcal{E}$ be a Grothendieck topos with canonical class G of power generators. Suppose $\omega \leq \kappa \leq \lambda$ are cardinal numbers, L a language of cardinality $\leq \kappa$, A an L-structure in $\mathcal{E}$ with $\bar{A}^G = \lambda$. If C is a* *subobject of A with $\bar{C}^G < \kappa$, then there is an elementary substructure B of A such that C is a subobject of B and $\bar{B}^G \leq \kappa$.*

2.2 Pretoposes

This concerns Marek Zawadowski's thesis work ([60, 61]). It touches on a number of research lines in categorical logic: Makkai and Reyes's *Conceptual Completeness Theorem* for pretoposes ([40]), Pitts' alternative proof using interpolation ([47]), Pitts' generalization of conceptual completeness to Heyting categories and the method of *descent* ([48]). A very clear outline of Conceptual Completeness is in Zawadowski's [62].

Fix a first-order multi-sorted language L. The (finitary) *coherent fragment* of first-order logic over L is the collection of L-formulas built from $\bot, \top, \wedge, \vee, \exists$; a *coherent L-theory* is a collection of *sequents* $\phi \Rightarrow_{\vec{x}} \psi$ where ϕ and ψ belong to the coherent fragment, and $\vec{x}$ is a sequence $x_1, \ldots, x_n$ of variables which contains all variables which occur free in ϕ or ψ.

A *coherent category* (Makkai and Reyes speak of 'logical category') is a regular category which has stable joins of subobjects. In such a category $\mathcal{R}$, an *L-structure* M consists of: for every sort S of the language L, an object $M(S)$; for every function symbol $f : S_1, \ldots, S_n \to T$ a morphism $M(f) : M(S_1) \times \cdots \times M(S_n) \to M(T)$; and for every relation symbol $R : S_1, \ldots, S_n$ a subobject $M(R)$ of the product $M(S_1) \times \cdots \times M(S_n)$. Given such a structure, for every coherent L-formula ϕ and every list of variables $\vec{x}$ containing all free variables of ϕ, a standard inductive construction gives $[\phi]^M_{\vec{x}}$ as subobject of $M(S_1) \times \cdots \times M(S_n)$ (where S_i is the sort of the variable x_i). We now say that the coherent sequent $\phi \Rightarrow_{\vec{x}} \psi$ is *true* in the structure M if $[\phi]^M_{\vec{x}} \leq [\psi]^M_{\vec{x}}$ as subobjects of $M(S_1) \times \cdots \times M(S_n)$. The structure M is a *model* of the L-theory T if every sequent of T is true in M.

Now if M and N are two L-structures in the coherent category $\mathcal{R}$, a *homomorphism* from M to N consists of a morphism $\alpha_S : M(S) \to N(S)$ for each sort S of L, which system has to satisfy the following two conditions:

for each function symbol $f : S_1, \ldots, S_n \to T$ of the language, the diagram

$$\begin{tikzcd}
M(S_1) \times \cdots \times M(S_n) \arrow[r, "M(f)"] \arrow[d, "\alpha_{S_1} \times \cdots \times \alpha_{S_n}"'] & M(T) \arrow[d, "\alpha_T"] \\
N(S_1) \times \cdots \times N(S_n) \arrow[r, "N(f)"'] & N(T)
\end{tikzcd}$$

should commute; and for each relation symbol $R : S_1, \ldots, S_n$ the composition $M(R) \to M(S_1) \times \cdots \times M(S)_n \overset{\alpha_{S_1} \times \cdots \times \alpha_{S_n}}{\longrightarrow} N(S_1) \times \cdots \times N(S_n)$ factors through the subobject $N(R)$ of $N(S_1) \times \cdots \times N(S_n)$.

We make the following remark on the definition of a homomorphism: it follows by a straighforward induction that for such a homomorphism α and a coherent L-formula ϕ with list of free variables $\vec{x}$, that the composition $[\phi]_{\vec{x}}^{M} \to M(S_1) \times \cdots \times M(S)_n \overset{\alpha_{S_1} \times \cdots \times \alpha_{S_n}}{\longrightarrow} N(S_1) \times \cdots \times N(S_n)$ factors through $[\phi]_{\vec{x}}^{N}$. So, a homomorphism looks like a natural transformation (and we shall shortly see that it actually is one).

Now we have two interesting constructions, from the point of view of categorical logic: the *canonical language* and the *syntactic category*. Let $\mathcal{R}$ be a coherent category. The canonical language $L_{\mathcal{R}}$ has a sort A for every object A of $\mathcal{R}$, and a function symbol $f : A \to B$ for every arrow $f : A \to B$ in $\mathcal{R}$. We have a straightforward $L_{\mathcal{R}}$-structure $M_{\mathcal{R}}$ in $\mathcal{R}$ which "is the identity", and we can discuss which $L_{\mathcal{R}}$-sequents are true in $M_{\mathcal{R}}$. For example:

a) A diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "h"'] & B \arrow[d, "f"] \\
 & C
\end{tikzcd}$$

commutes if and only if the sequent $\top \Rightarrow_x f(g(x)) = h(x)$ is true in $M_{\mathcal{R}}$.

b) The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

is a product diagram if and only if ther sequents

$$\begin{gathered}\top \Rightarrow_{y,z} \exists x(f(x) = y \wedge g(x) = z)\\ f(x) = f(x') \wedge g(x) = g(x') \Rightarrow_{x,x'} x = x'\end{gathered}$$

are both true in $M_{\mathcal{R}}$.

c) The diagram

$$\begin{tikzcd} A \arrow[r, "e"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

is an equalizer if and only if the sequents

$$\begin{gathered}g(y) = h(y) \Rightarrow_y \exists x(e(x) = y)\\ \top \Rightarrow_x g(e(x)) = h(e(x))\\ e(x) = e(x') \Rightarrow_{x,x'} x = x'\end{gathered}$$

are all true in $M_{\mathcal{R}}$.

d) Finally, the arrow $f : A \to B$ is a regular epi in $\mathcal{R}$ iff the sequent $\top \Rightarrow \exists x(f(x) = y$ is true in $M_{\mathcal{R}}$.

We see that the whole coherent structure of $\mathcal{R}$ is described by the coherent logic over $L_{\mathcal{R}}$. Makkai and Reyes call this the "first main fact".

The second construction, the *syntactic category* $\mathcal{C}_T$ or, in Zawadowski's words, the *Lindenbaum-Tarski category* of a coherent theory T, is as follows: objects are coherent formulas ϕ up to renaming of free variables. Morphisms from $\phi(\vec{x})$ to $\psi(\vec{y})$ (which by our convention we may assume to have disjoint sets of free variables) are coherent formulas $\theta(\vec{x}, \vec{y})$ which are, T-provably, functional relations: that is, the sequents

$$\begin{gathered}\theta(\vec{x}, \vec{y}) \Rightarrow_{\vec{x},\vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y})\\ \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{u}) \Rightarrow_{\vec{x},\vec{y},\vec{u}} \vec{y} = \vec{u}\\ \phi(\vec{x}) \Rightarrow_{\vec{x}} \exists \vec{y}\theta(\vec{x}, \vec{y})\end{gathered}$$

are true in every model of T.

The category $\mathcal{C}_T$ is coherent. We can now see that a model of T in a coherent category $\mathcal{R}$ is 'nothing but' a coherent funrtor $\mathcal{C}_T \to \mathcal{R}$ and that, modulo this equivalence, a homomorphism between models is 'nothing but' a natural transformation between the corresponding functors.

Finally, we consider *interpretations* between coherent theories. If T_1 is an L_1-theory and T_2 an L_2-theory then an *interpretation* $I : T_1 \to T_2$ sends every sort S of L_1 to a sort $I(S)$ of L_2, every function symbol $f : S_1, \ldots, S_n \to T$ of L_1 to a function symbol $I(f) : I(S_1) \ldots I(S_n) \to I(T)$ and each relation symbol $R : S_1, \ldots, S_n$ to a coherent formula $I(R)$ with free variables of sorts $I(S_1), \ldots, I(S_n)$. This gives a translation I from coherent L_1-formulas to L_2-formulas; and we require that for every axiom of T_1, its I-image is true in every model of T_2. Note that I is also a coherent functor from $\mathcal{C}_{T_1}$ to $\mathcal{C}_{T_2}$. Let $\mathrm{Mod}(T_1)$ be the category of models of T_1, that is: the category of coherent functors $\mathcal{C}_{T_1} \to \mathrm{Set}$ and natural transformations/homomorphisms, and likewise for T_2. Composition with I gives a functor $I^* : \mathrm{Mod}(T_2) \to \mathrm{Mod}(T_1)$. Now suppose that the functor I^* is an equivalence of categories. What can we say about I? Here we have to take into account that there is categorical structure that is recognized by coherent logic, i.e. preserved by every coherent functor, yet not always present in every coherent category. This structure is: *finite coproducts and quotients of equivalence relations*. A coherent category with stable finite coproducts and quotients of equivalence relations is called a *pretopos*. The *Conceptual Completeness Theorem for Pretoposes* (Makkai and Reyes) reads: Let $I : T_1 \to T_2$ be an interpretation between coherent theories. If I^* is an equivalence and $\mathcal{C}_{T_1}$ is a pretopos, then I is an equivalence of categories.

In this theorem, the category Set plays a special role, as category of the models of theories. The completeness for Set-valued models is not quite constructive; it is equivalent to Gödel's Completeness Theorem for first-order logic. The constructivisation was formulated by A.M. Pitts in [47] and consists in taking models in larger families of pretoposes into account. In this version, the special role of Set disappears.

The paper [61] is a quite monumental piece of work. I have, above, dealt in some detail with one of the ingredients of the theory Zawadowski sets up: the Makkai-Reyes Conceptual Completeness Theorem. Two other ingredients are: Makkai's theory of *Ultracategories* and a notion of *descent*. Both are pieces of mathematics that cannot be explained in a few paragraphs. Let me restrict myself to formulate one application (Theorem 11.4). Every pretopos morphism $J : A \to B$ factors through a morphism $J_d : A \to \mathrm{Des}(J)$, the category of descent data. This category turns out to be equivalent to a category of *invariant* subobjects, yielding what the author claims a *global* *invariance theorem.*

The paper [4] claims to offer "a short argument for the descent theorems of Zawadowski ([61] (originally [60])) and Makkai ([41]), which were conjectured by Pitts after the descent theorem of Joyal and Tierney ([31]) for open geometric morphisms of (Grothendieck) toposes". It uses forcing. Due to (lack of) my university's permissions I was not able to consult the paper.

3. Model Completeness

— *Silvio Ghilardi*

In this section, we report the content of the cooperation between Marek Zawadowski and the second co-author: the cooperation started in Montreal 1992 and ended few years later in the publication of a book [21], collecting and extending all results from previous papers [17, 18, 19, 20].

The starting point of this work was a surprising theorem proved by A. Pitts in [49]. The theorem originated from the attempt of solving a well-known and still open question ("is any Heyting algebra the Heyting algebra of global sections of the subobject classifier Ω in an elementary topos?"), but it has an important independent interest. It can be stated as follows (below we use notations like $\theta(\underline{v})$ to say that the formula θ contains at most the variables $\underline{v}$):

Theorem [49, Pitts]. *For each propositional variable x and for each formula $\varphi(x, \underline{y})$ of intuitionistic propositional calculus (IpC), there exist formulas $[\exists^x \varphi]$ and $[\forall^x \varphi]$ (effectively computable from φ and containing at most the variables $\underline{y}$) such that for any formula $\psi(\underline{y}, \underline{z})$, we have*

$$\vdash_{IpC} [\exists^x \varphi] \to \psi \qquad \textit{iff} \qquad \vdash_{IpC} \varphi \to \psi$$

$$\vdash_{IpC} \psi \to [\forall^x \varphi] \qquad \textit{iff} \qquad \vdash_{IpC} \psi \to \varphi.$$

From the proof-theoretic point of view the theorem can be reformulated by saying that there exists a (non conservative) interpretation of second order intuitionistic calculus IpC^2 into IpC or, again, by saying that Craig's interpolation theorem holds in a very strong ('uniform') way for IpC. The starting problem in the above mentioned Marek Zawadowski's joint work were the following two interrelated problems:

(a) to reinterpret the above theorem in a different mathematical (categorical or model-theoretic) context;

(b) to supply a semantic proof of it.

Answering such questions does not only supply a better understanding of the theorem itself, but paves also the way of investigating extensions to other logical contexts.

The model theoretic interpretation of Pitts' Theorem relies on the following algebraic observations (taken from [21]). Below, we make a systematic identification of formulas of IpC and terms of the first-order theory of Heyting algebras. For such a formula/term $\varphi(x, \underline{y})$, and a matching tuple of elements $\underline{a}$ from a Heyting algebra H, we have that $H \models [\exists^x \varphi](\underline{a}) = 1$ iff H embeds into the Heyting algebra of polynomials $H[\mathbf{x}]$ divided by the congruence generated by the condition $\varphi(\underline{a}, \mathbf{x}) = 1$. Moreover $H \models [\forall^x \varphi](\underline{a}) = 1$ iff $H[\mathbf{x}] \models \varphi(\underline{a}, \mathbf{x}) = 1$. These observations shows that the first order theory of Heyting algebras *admits a model completion.* In fact, it turns out that a system of equations and inequations with parameters $\underline{a}$ from a Heyting algebra H

$$\exists x \begin{pmatrix} \varphi_1(\underline{a}, x) = 1 \;\&\; \cdots \;\&\; \varphi_n(\underline{a}, x) = 1 \;\& \\ \&\; \psi_1(\underline{a}, x) \neq 1 \;\&\; \cdots \;\&\; \psi_m(\underline{a}, x) \neq 1 \end{pmatrix}$$

is solvable in an extension of H iff the quantifier-free formula

$$[\exists^x \bigwedge_{i=1}^{n} \varphi_i](\underline{a}) = 1 \quad \& \;\; [\forall^x (\textstyle\bigwedge_{i=1}^{n} \varphi_i \to \psi_1)](\underline{a}) \neq 1 \;\& \;\cdots$$
$$\cdots \;\& \;[\forall^x (\textstyle\bigwedge_{i=1}^{n} \varphi_i \to \psi_m)](\underline{a}) \neq 1$$

is true in H; here the formulae/terms

$$[\exists^x \bigwedge_{i=1}^{n} \varphi_i],\; [\forall^x (\bigwedge_{i=1}^{n} \varphi_i \to \psi_1)], \ldots, [\forall^x (\bigwedge_{i=1}^{n} \varphi_i \to \psi_m)]$$

are computed according to Pitts' theorem. This proves that the class of existentially closed Heyting algebras is an elementary class and so *Pitts' theorem implies that the equational theory of Heyting algebras has a model completion.* The interesting fact is that a converse result also holds in an appropriate universal algebraic context.

In order to explain such converse result, Pitts' Theorem needs to be formulated in category-theoretic terms. In this equivalent formulation, the theorem says that *the opposite of the category of finitely presented Heyting algebras is a Heyting category*, where a Heyting category [40] is a coherent category where taking pullbacks on subobjects has not only a left but also a right adjoint. Using this formulation, it is shown in [21] that the first order theory of a variety $\mathbf{V}$ with equationally definable principal congruences and equationally definable principal meets *admits a model completion if and only if* $\mathbf{V}_{fp}^{op}$ (the opposite of the category of finitely presented $\mathbf{V}$-algebras) *is an r-Heyting category* (the notion of an r-Heyting category is a modification of the notion of a Heyting category taking care of the fact that epis might not be regular - a phenomenon that nevertheless does not occur in the logic-algebraic varieties investigated in the book [21], because there Beth definability property holds for the corresponding propositional logics).

Having reformulated Pitts's theorem both in model-theoretic and in category-theoretic terms, it remains the question on how to get a semantic proof of it. This problem was solved in [17] and independently also in [57]. Both [17] and [57] apply combinatorial arguments inspired from previous work on Gödel-Löb logic in [50], the difference lies in the fact that [17] recasts the problem in a sheaf-theoretic environment. Let $\mathbf{P}$ be the opposite of the category of finite Heyting algebras (this category can be equivalently described as the category of finite posets and open maps, i.e. order-preserving maps taking downsets to downsets). Let $\mathbf{J}$ be the canonical topology on $\mathbf{P}$. We have a functor

$$\mathbf{\Phi_H} : HA_{fp}^{op} \longrightarrow Sh(\mathbf{P}, \mathbf{J})$$

sending a finitely presented Heyting algebra H to a sheaf

$$\mathbf{\Phi_H}(H) \;=\; [\, H_0 \mapsto HA(H, H_0)\,]$$

(i.e. $\mathbf{\Phi_H}(H)$ associates to every finite Heyting algebra H_0 the set of all Heyting morphisms from H to H_0 - the functor operates in the obvious way, i.e. by composition, on morphisms of finite and of finitely presented Heyting algebras). $\mathbf{\Phi_H}$ preserves finite limits and is conservative. The strategy to prove Pitts' theorem is now to show that its essential image inherits images and dual images from the ambient sheaf category. To do this, such essential image is described via a duality theorem: in a nutshell, Ehrenfeucht-Fraissé games are introduced in order to be able to recover definable subobjects in the image of $\mathbf{\Phi_H}$. Armed with this extra feature, it is eventually possible to show that HA_{fp}^{op} is a Heyting category and that $\mathbf{\Phi_H}$ preserves such Heyting category structure.

This analysis is extended in [21] to modal logics, where a full classification of logics extending $S4$ (the modal logic arising from interior topology axioms) is obtained; negative examples include $S4$ itself, where Pitts' theorem is shown to fail [18].

The above techniques have further applications: for instance, the duality theorem is exploited in [21] to prove that free Heyting algebras have a dual co-Heyting algebra structure, to supply semantic proofs of Beth definability theorems in certain modal logics, to characterize finitely presented projective Heyting algebras. More recently, the same duality has been used in order to obtain a semantic proof of Ruitenburg's theorem [22] stating that certain endomorphisms of finitely generated free Heyting algebras are ultimately periodic (a remarkable consequence of Ruitenburg's theorem is the definability of least and greatest fixpoints of monotonic formulae in IpC).

4. Algebraic theories

— Martin Hyland

As a byproduct of his research into Opetopes and Opetopic Sets, Marek Zawadowski developed foundations for notions of algebraic theory. High points are his treatment of two special cases: those of *rigid theories* and *regular theories*, where his results are definitive. Alongside these I describe the general setting which he used and indicate where he opened up fresh avenues for investigation. I hope that this will give a glimpse of his fertile mathematical imagination in which there was a productive tension between a preference for concrete presentations and an instinct to support them with abstract theory.

I start with some context. The notion of algebraic theory has three established manifestations. First there is the universal algebra notion of abstract clone, a direct abstraction from the syntax. In modern terms an abstract clone is a (single-sorted) cartesian multicategory. Secondly there is the closely related notion of a Lawvere theory, effectively a category with products generated by a single object. Thirdly there is the notion - abstracted from the semantics - of a finitary monad. A monad is finitary just when its functor preserves filtered colimits: this restriction captures the finiteness of standard syntax. In his work on restricted notions of algebraic theory, Zawadowski provided a general setting underlying the first approach and developed theory involving all three.

What happens when we look at restricted classes of algebraic theory? Typically we have an independent notion of theory presented as some kind of multicategory. From that we obtain a Lawvere theory and a monad. The obvious question is whether we characterise those that arise from our restricted notion amongst the collection of all theories or monads. Typically the Lawvere theory is freely generated from the multicategory and a characterisation expresses that fact. The monad comes from the semantics: for it things are less obvious. It turns out that characterisations can be given in terms of preservation properties of the functor and related properties of the unit and multiplication of the monad. But the results are subtle and we are still far from a full understanding.

4.1 Rigid theories

The original formulation of Opetopic Sets by Baez and Dolan [3] used some form of the theory of Opetopes but in detail it raised concerns as to how exactly to make the definitions precise. The leading suggestion was made by Hermida-Makkai-Power [26, 27, 28] in a series of three papers. They introduced the notion of *multicategories with non-standard amalgamation* and used it as the basis for an approach to higher categories via what they called multitopic sets. However in making things precise they created a new difficulty. The definition given in [27] is perplexing: it is not even obvious that algebras or models within the category of sets are multisorted algebras of some kind. Generally it was not clear how to relate the idea to other more familiar notions of algebraic theories.

Zawadowski was the first properly to understand the situation. One can make sense of multicategories with non-standard amalgamation from two more established and seemingly quite distinct perspectives. On the one hand they can be seen from the point of view of the theory of polynomials (see [16] for a detailed account). They can simply be identified with polynomial monads. On the other hand one can identify multicategories with non-standard amalgamation as special kinds of (many-sorted) operads, namely those for which the permutation actions are in an obvious sense free. Aficionados will call this Σ-free, avoiding confusion as they are not in any sense freely generated as operads. The characterisation as Σ-free operads led Zawadowski to call the corresponding theories *rigid*.

The two equivalences - with polynomial monads and with free or rigid operads - found by Zawadowski clarified what was a very murky state of affairs. The results are not prima facie obvious. I remember Zawadowski telling me some parts of the story in Peter Johnstone's rooms in St. John's College Cambridge at the PSSL meeting in April 2012. Some of the basic facts are already laid out in his [65] which dates from the end of 2009 and in 2012 he was presumably aware of the further aspects which appear in [52] written with his student Stanislas Szawiel.

4.2 Lax monoidal fibrations

The starting point for Zawadowski's treatment of algebraic theories is the paper [65] which he wrote for the 70th birthday of Michael (Mihaly) Makkai. In it, he introduced his notion of a *lax monoidal fibration* which he used consistently in his investigations of notions of algebraic theory. Zawadowski uses the language of fibrations but for purposes of exposition I shall combine it with the terminology of indexed categories. The essential features of the definition are as follows. In the first place we have a fibration $p : \mathbf{E} \to \mathbf{B}$. This is lax monoidal in the sense that for every object $b \in \mathbf{B}$, the fibre $\mathbf{E}_b = p^{-1}(b)$ has the structure of a monoidal category, while the reindexings come with the structure of a lax monoidal functor. The idea is that the fibration captures a flavour of algebraic signature. The monoidal structure corresponds to substitution and so monoids in the fibres correspond to algebraic theories. This picture is supported by a good deal of basic material presented in sections 3 and 4 of [65] and the rest of his paper is devoted to discussion of three main examples.

Burroni's T-categories The idea of a T-category for a monad T appears in an old paper [7] of Albert Burroni. The idea was neglected for many years but came to play a fundamental role in Leinster's work [39] on (non-symmetric) higher operads and categories. Leinster considered the situation which arises when the monad T is cartesian. It is a striking feature of the setting developed by Zawadowski that Burroni's idea falls within it without any restriction whatever. In section 5 of [65], he gives the details and he goes on to rework the definition of opetopes from [39] within his richer setting. All this should perhaps be seen as a proof of concept for lax monoidal fibrations. While this approach to opetopes did not play an explicit role in Zawadowski's work, it seems to me that understanding it fed into Zawadowski's study of algebraic theory.

Nonstandard amalgamation This is the crucial motivating example of [65]: Zawadowski shows how to use his notion to give a mathematically civilised presentation of the Hermida-Makkai-Power multicategories with nonstandard amalgamation [27]. Here I follow Zawadowski in suppressing the 2-dimensional features of the definition.

The basic issue is the treatment of substitution. The idea is that we have typed operations $f : u \to a$ with a string u of inputs and a single output a. If we have $\alpha : u \to a$ and $\beta : v \to b$ and if a appears at a particular point in the list v then we would like to substitute α for a in β. In the standard theory of non-symmetric operads (as developed in [39]) the input of the result is simply the result of inserting the list u for the entry a in the list v; and the output is b. That is standard amalgamation. Motivated by discussion of pasting diagrams in [26], the idea of non-standard amalgamation is systematically to allow the resulting input to be a different ordering of the standard list. This introduces a serious issue. Multiple substitutions for variables can be made in different orders and one wants to ensure that the result is independent of the order in which that is done. Conditions to ensure that are laid out in [27] but the resulting definition of multicategory is offputting.

In this difficult situation the first step taken by Zawadowski was simply to absorb the complications of the non-standard amalgamation into the total category of a monoidal fibration. The objects are signatures: they consist of a set O of types and a set A of operations equipped with a domain map $d : A \to O^*$ and a codomain map $c : A \to O$. (Here I am using the standard notation O^* for finite lists from O.) A map from a signature (O, A) to another (Q, B) consists in the first instance of functions $f : O \to Q$ and $F : A \to B$ where the function F is required to respect the lengths of the domains. This is augmented for each $\alpha \in A$ by a permutation σ_α of the length n say of $d(\alpha)$; and relative to these permutations the domains and codomains are required to match in the following sense. If $\beta : v \to b = F(\alpha : u \to a)$ is the image under F of $\alpha \in A$ then the domains match in the sense that $v_i = f(u_{\sigma_\alpha}(i))$ and the codomains agree, that is, $f(a) = b$. It is straightforward to see that this category is fibred over the category of sets. Moreover there is a clear and intuitive definition of a substitution tensor in each fibre. A multicategory with non-standard amalgamation in the sense of [27] is then simply a monoid in some fibre.

This very substantial reformulation is not merely a matter of clarification as it points to alternative characterisations. Most immediate is the connection with polynomials [16]. There is a lax monoidal fibration of finitary polynomial diagrams or equivalently of finitary polynomial endofunctors.

Theorem 4.1. *The (lax monoidal) fibration of signatures with non-standard amalgamation is equivalent to that of finitary polynomial endofunctors. Thus the fibration of multicategories with non-standard amalgamation is equivalent to that of polynomial monads.*

One can show this using symmetric operads but Zawadowski proves it via a concrete characterisation of the induced monads.

Proposition 4.2. *A monad* $(T, \eta.\mu)$ *on (slices of)* Sets *is polynomial just when the following holds.* T *is finitary, cartesian and preserving wide pullbacks; and* η *and* μ *are cartesian natural transformations.*

The same conditions characterise monads arising from multicategories with non-standard amalgamation and Theorem 4.1 follows. While this proof works, I do not think that Zawadowski was quite happy with the approach as he makes explicit mention of the fact that the characterisation is special to Sets. (Perhaps one should use instead the characterisation in [35]?)

Symmetric operads The third lax monoidal fibration of [65] is that of symmetric signatures or equivalently that of analytic functors. This is a natural many variable extension of Joyal's theory of species [32]. The algebraic theories which arise as monoids in the fibre over 1 are the familiar operads which already appear in May's work [44] on iterated loop spaces. Various characterisations of the resulting monads have been sketched. Zawadowski presents his own version.

Proposition 4.3. *A monad* $(T, \eta.\mu)$ *on (slices of)* Sets *is analytic just when the following holds.* T *is finitary and weakly preserves wide pullbacks; and* η *and* μ *are weakly cartesian natural transformations.*

Comparison with the characterisation of polynomial monads makes it clear that they are special analytic monads. That leads naturally enough to an identification of multicategories with non-standard amalgamation as special (multi-sorted) symmetric operads.

Comparison with Kleisli bicategories There is an alternative approach to algebraic theories in terms of Kleisli bicategories [14]. I give a brief sketch of how this is encompassed by Zawadowski's lax monoidal fibrations.

In the alternative approach one has a bicategory $\mathcal{B}$ equipped with an identity-on-objects, locally fully faithful homomorphism $\mathbf{B} \to \mathcal{B}$ of bicategories from (what is typically) a 2-category $\mathbf{B}$. The images $u_* : b \to c$ in $\mathcal{B}$ of 1-cells $u : b \to c$ from $\mathbf{B}$ are *maps* in the sense that they come with a right adjoint $u^* : c \to b$. (Essentially this is a proarrow equipment in the sense of Wood.) Then $\mathbf{B}$ - or if you prefer its underlying category - is the base for a lax monoidal fibration $\mathbf{E} \to \mathbf{B}$ obtained as follows. For $b \in \mathbf{B}$ we set the fibre $\mathbf{E}_b = \mathcal{B}(b, b)$. The monoidal structure is composition in $\mathcal{B}$. Reindexing uses the fact that 1-cells $u : b \to c$ in $\mathbf{B}$ give maps in $\mathcal{B}$ with $u_* : b \to c$ left adjoint to $u^* : c \to b$. So to reindex along $u : b \to c$ we can take $M \in \mathcal{B}(c, c)$ to the composite $u^*.M.u_* \in \mathcal{B}(b, b)$. This is a lax map of monoidal categories in an evident fashion.

Zawadowski's approach gives added generality, in particular allowing for a general treatment of Burroni's T-multicategories. It also obeys Occam's Razor avoiding excess of structure in the bicategories. (In favour of the bicategorical approach - it involves relatively familiar category theory.)

4.3 Leading examples

The experience of [65] suggested a systematic approach to the study of varieties of algebraic theories. That is the focus of [52] written with Zawadowski's student Stanislaw Szawiel. For purposes of exposition the abstract setting of [65] is set aside in favour of concrete syntactic presentations. There are three leading examples.

- The general notion of an algebraic theory: in [52] these are called *equational theories*.
- Operadic theories: those axiomatised by equations between terms in which exactly the same variables occur exactly once; [52] say *regular-linear theories*. Usually we say *analytic* or *operadic*.)
- What [52] call *rigid theories*. These are special regular-linear theories $\mathbb{T}$. Let t be a term with distinct variables $\mathbf{x}$ and let t^σ be t with the variables permuted by some permutation σ. Then the only circumstances in which $\mathbb{T} \vdash t = t^\sigma$ is when σ is the identity.

Szawiel-Zawadowski treat these examples from the point of view of Lawvere Theories and of monads.

Lawvere Theories The basic perspective of Lawvere's thesis [36] is folklore. From an algebraic theory $\mathbb{T}$ one constructs a Lawvere theory $\mathsf{L}_\mathbb{T}$, essentially a category with products generated by a single object. The category of models of a Lawvere theory L is the category $\mathrm{Prod}(\mathsf{L}, \mathbf{Sets})$ of product preserving functors to sets.

For regular-linear and rigid theories $\mathbb{T}$ there is an analogous symmetric monoidal or operadic category $\mathsf{O}_\mathbb{T}$ generated by a single object. The category of models of an operadic category O is the category $\mathrm{Mon}(\mathsf{O}, \mathbf{Sets})$ of tensor preserving functors to $\mathbf{Sets}$. The natural question for the restricted classes of theories is to characterise the Lawvere theories $\mathsf{L}_\mathbb{T}$ which result from them. Evidently they are freely generated from the operad $\mathsf{O}_\mathbb{T}$. Szawiel-Zawadowski show how the resulting Lawvere theories can be characterised very elegantly in terms of a factorisation system together with the requirement that the category has *simple automorphisms*.

Monads The approach to algebraic theories via monads is again folk-lore: algebraic theories correspond to finitary monads.

For regular-linear theories [52] shows in detail that these correspond to analytic monads. So [65] characterised the monadic approach (Proposition

4.3 above). There are many alternative characterisations going back to Joyal [32]. In the same spirit, rigid theories correspond to polynomial monads. Again [65] gives a characterisation (Proposition 4.2 above). One obtains the polynomial condition from the analytic by dropping the 'weak'. The characterisation seems cleaner but that is misleading. There is a problem.

Proposition 4.4. *The problem whether an analytic theory presented by a finite set of regular-linear equations defines a rigid theory is undecidable.*

This is proved by Zawadowski et al in [6]. To my mind this undecidability reflects something a bit riddling about the world of polynomials.

4.4 Regular Theories

It is always good to consider a new and telling example and in [54] Szawiel and Zawadowski do just that. Already on [52] they called a theory $\mathbb{T}$ *regular* when it is axiomatised by equations between terms in which exactly the same variables occur on each side. (The terminology was established long ago: in another logic tradition one would say *relevant*.) The analysis in [54] is parallel to that of [52] but everything is new and Szawiel and Zawadowski adopt a concrete multicategory approach.

In this fresh perspective one gets algebraic theories by taking some (standard skeletal) category F of finite sets: algebraic theories are monoids in the corresponding functor category $[\mathsf{F}, \mathrm{Sets}]$ with substitution tensor product. Operads are obtained by replacing F by B the category of finite sets and bijections. Then regular theories are obtained by taking instead the category S of finite sets and surjections.

Szawiel and Zawadowski look at the corresponding Lawvere theories. They characterise those which arise from regular theories in the same style as their characterisation of operadic theories, that is, in terms of a factorisation system and the condition of simple automorphisms. They then turn to the corresponding monads which they call *semi-analytic*. In terms of the new perspective that means characterising those which arise from monoids in $[\mathsf{S}, \mathbf{Sets}]$ by left Kan extension. That is really not at all obvious and the result is an impressive technical achievement.

Proposition 4.5. *A monad* $(T, \eta.\mu)$ *on (slices of)* $\mathbf{Sets}$ *is semi-analytic just when the following holds.* T *is finitary and semi-analytic in the sense that it* *preserves pullbacks of monomorphisms; and η and μ are semi-cartesian in the sense that their naturality squares for monomorphisms are pullbacks.*

This is subtle. Note for example that it is not obvious from Propositions 4.3 and 4.5 that analytic monads are semi-analytic.

4.5 Other directions

Zawadowski liked working with detailed calculations in specific examples but he had an instinct for abstract theory. It seems worth mentioning some of his ideas which hint at deep structure.

Behind opetopes In their joint paper [51] Szawiel and Zawadowski look again at an idea which has been a puzzle since it first appeared in Baez-Dolan [3]. The idea is to have a theory of theories: theories should be algebras (or models) for some defining theory. This idea plays a role in approaches to the theory of opetopes but [51] is concerned to provide some deeper conceptual foundations. The setting is that of Zawadowski's lax monoidal fibrations and the main ingredient is a remarkable and surprising construction of what is called the *web monoid*. It involves the interplay between two monoidal structures: weirdly the free monoid for one on the unit for the other. This idea is explained in greater detail and with explanatory intuitions in Curien's contribution to this memorial for Zawadowski. I add the thought that Szawiel and Zawadowski found some of the ingredients for their construction in a neglected paper [5] by Baues, Jibladze and Tonks. But the almost completely formal way in which they work things out suggests that something very fundamental is at play. This line of work is surely worth further study.

An abstract setting In a long but unpublished paper [55], Szawiel and Zawadowski returned to the subject of their [52] but from a radically more abstract point of view. The paper starts from one surprising observation. There is an evident monoidal monad on the category of signatures whose Eilenberg-Moore category is essentially the category of finitary functors on Sets. The observation is that the Kleisli category is essentially that of polynomial functors (again with all natural transformations). That is unexpected but perhaps just an oddity. But it seems not as [55] weaves, out of that simple fact, a complicated story covering a range of examples. The paper is rich in detail but it is not easy to obtain a synoptic vision of the material. (For example as in [51] an important role is played by distributive laws but what is going on?) The paper seems not to have reached publication - a pity as there are many suggestive ideas.

Further Questions The papers just discussed are typical of the way in which Zawadowski's work opens up lines of enquiry. To conclude I mention two very general kinds of question suggested by the work.

The first concerns the scope of the general theoretical machinery. Algebraic theories correspond to monoids for the substitution tensor product on the category $[\mathsf{F}, \mathbf{Sets}]$ of set-valued functors on the category of finite sets. Using the categories B of bijections and S of surjections we get operadic and regular theories. Are there other interesting possibilities? Clearly there is I, the category of injections, and it is clear from [67] that Zawadowski contemplated looking at it and the interaction with S. Also there is N with just identities which gives rise to non-symmetric operads. For these there are subtleties laid out in section 6.2 of Leinster [39]. A side remark in [52] shows that Zawadowski was aware of the issue and thought to look more closely. Obviously we would like to know if there are more possibilities of this kind. What subcategories of F admit the substitution tensor product? Then orthogonal to that line of thought further is the special case of the rigid theories. That involves a global restriction. Is that a one-off or are there other similar cases? These kinds of question suggest looking yet more deeply at algebraic theories. One such direction appears in the recent paper Fiore and Ranchod [15].

The second kind of question which I think it worth pursuing concerns the characterisations of the induced monads alluded to above. The obvious puzzle is the contrast between weak properties or the weak preservation of properties and the more familiar preservation in an up to isomorphism sense. At the moment this all seems ad hoc. We have no story which makes sense of the different cases. When looking at the relation between Propositions 4.3 and 4.5, I am reminded of arguments from Gabriel-Ulmer. If C is a category with finite limits, a functor $F : \mathsf{C} \to \mathbf{Sets}$ preserves finite limits just when its category of elements is filtered. Is that part of the same story?

Last thought Zawadowski was a great pioneer in the world of algebraic theories. But that subject is very rich and his work surely shows that there remain many avenues to explore.

5. Opetopic sets

— Pierre-Louis Curien

This chapter reports on works around opetopic sets launched by an inspiring paper of Baez and Dolan published in 1998 [3]. Zawadowski became interested in this line of work in the first years of the present millenium, and since then, over the years and until his last moments, he made so many contributions to their study that he truly deserves the title of master of opetopes. As a milestone in his continuous effort of understanding these notoriously difficult mathematical objects, he organised a workshop on Opetopes, Opetopic Sets and Opetopic Categories in Warsaw in March 2013. When visiting my laboratory in 2022, he proposed the following undebatable definitions of opetopic sets and opetopes, which he called Sèvres definitions (in reference to the location of the Bureau International des Poids et Mesures `https://www.bipm.org/en`).

1. The category $\mathbf{OpeSet}$ of opetopic sets is the category of many-to-one polygraphs.
2. The category $\mathbf{Ope}$ of opetopes is the exponent category of $\mathbf{OpeSet}$, i.e., there is an equivalence of categories $\mathbf{OpeSet} \sim \mathbf{Set}^{\mathbf{Ope}^{op}}$.

Here is a brief dictionary. Polygraphs [2] (also known as computads) are strict infinity categories (or ω-categories) that are freely generated in all dimensions. Generating n-cells have an upper (or source) boundary and a lower (or target) boundary which are both (n-1)-cells. When all generators have a generating cell as target rather than some arbitrary codimension 1 cell, the polygraph is called many-to-one. If moreover the sources of generators are not allowed to be identities (or degeneracies) of codimension 1 cells, then the polygraph is called positive-to-one.

Implicit to item (2) above is a theorem, namely that many-to-one polygraphs form a presheaf category. Alternatively, the category $\mathbf{Ope}$ could be defined first (like in [11]) and then (1) is a theorem to prove. Therefore, (1) and (2) are in fact criteria that have to be met for any of the numerous definitions that flourished since a quarter of century. Of course, like in computability theory, the Sèvres stamp is contagious: any definition equivalent to one that is Sèvres is also Sèvres.

This short and compelling definition is of course anachronic. Opetopic shapes were introduced by Baez and Dolan as a proposal for a theory of weak higher categories. There were some rough corners in their seminal paper though, that triggered works of Hermida, Makkai and Power, who proposed their own version of opetopic sets based on multicategories, which they therefore called *multitopic sets* [26, 27, 28], and works of Leinster [39] and Cheng [8]. Soon after arrived some key contributions in which Zawadowski took an active part. In joint work with Harnik and Makkai, he proved that multitopic sets are the same as many-to-one polygraphs [25] (see also [64]), hitting target (1) above. Target (2) had been already reached in [28]. This altogether made the multitopic setting the first Sèvres definition! Makkai and Zawadowski also proved that general polygraphs do not form a presheaf category [43]. In parallel, Zawadowski also pioneered combinatorial definitions of opetopic shapes, as we shall see.

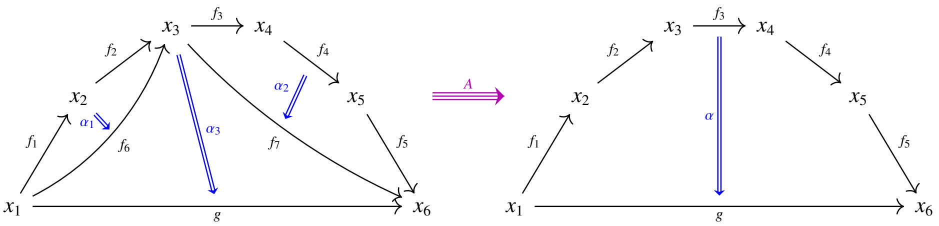
Then came a milestone contribution of Kock, Joyal, Batanin and Mascari [34], that offered both a categorical definition of opetopes based on polynomial diagrams, and a nice combinatorial description of opetopes via zoom complexes. Through its clarity and its appealing combinatorics, this paper made the whole field accessible to a wider audience (including the author of these notes!). In their setting, opetopes are defined through the so-called + construction, similar to the original slice construction of [3]. As we shall see, the mysteries and foundational value of these constructions were uncovered by Szawiel and Zawadowski [51] (see also the more extensive treatment in [53]). Also, in [68], Zawadowski unveiled not only a bijection but a categorical duality between zoom complexes and opetopes (in the positive-to-one case), prolonging a joint work with Makkai on the duality between simple ω-categories and Joyal's disks [42].

During all these years, Zawadowski kept an eye on the applications of opetopes to homotopical algebra and higher category theory. In [66], he showed that positive opetopes (equipped with suitable morphisms) form a test category. His last student Wojciech Dułinski is completing a PhD thesis investigating model structures on opetopic sets (again equipped with suitable morphisms). Let me also mention a recent application of opetopic structures in homotopy type theory: in [1], Allioux, Finster and Sozeau proved the celebrated result that all types form weak ∞-groupoids internally, in an "opetopic" extension of type theory. Yet another study of algebraic structures on opetopic sets is carried out by Ho Thanh and Subramaniam in [29].

The rest of this text proceeds as follows. In Section 5.1, we offer a brief introduction to opetopes. In Section 5.2, we aim at giving an overview of various combinatorial or type-theoretical approaches to opetopes. Finally, in Section 5.3, we zoom on polynomial diagrams and expose (our understanding of) the luminous foundational explanation of Szawiel and Zawadowski alluded to above.

5.1 Pictorial introduction to opetopes

There is a unique opetope ♦ of dimension 0, represented as a point. Starting from dimension 1, opetopes have (a finite set of) sources and one target (cf. "many-to-one" above). There is a unique opetope ■ of dimension 1. It has one source, and both its source and target are (copies of) ♦. It is represented as an arrow. We present 2-opetopes and 3-opetopes at once. The following picture features a 3-opetope A, which I shall explain in terms of what I would like to call geometry of unbiased associativity (just like associahedra express the geometry of biased associativity – we hope to be able to say more on this somewhen). Here, "biased" refers to binary composition as opposed to n-ary.



On the right of the picture is the target of A which is a 2-opetope α, having f_1, f_2, f_3, f_4, f_5 as sources, and g as target; all of them are (copies) of ■. I read α as the witness that g is an unbiased composite of the f_i's, i.e. α is a 2-cell or a homotopy from the formal sequence $f_1 f_2 f_3 f_4 f_5$ (or path, in diagrammatic order) to g. The sources of A are $\alpha_1, \alpha_2, \alpha_3$. Each of these

2-opetopes reads likewise as a composition (of f_1f_2, $f_3f_4f_5$ and f_6f_7, respectively). They are pasted together in such a way that the Poincaré dual of this arrangement of sources is a tree whose root node is α_3, with incoming edges f_6 and f_7 to which the nodes α_1 and α_2 are attached. A term notation for this is

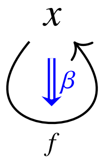
$$T = \alpha_3\langle f_6 \leftarrow \alpha_1, f_7 \leftarrow \alpha_2\rangle.$$

Implicitly, T has leaf edges f_1 and f_2 (coming into α_1) and f_3, f_4, f_5 (coming into α_2). The tree T can be read as a "more biased" recipe for composing the f_i's: first compose (in parallel) f_1f_2, yielding f_6 on one hand, and $f_3f_4f_5$, yielding f_7 on the other hand, and then compose f_6f_7, which is summarised in the parenthesised expression $((f_1f_2)(f_3f_4f_5))$. Under these glasses, A reads as a witness of associativity (or higher homotopy) between $((f_1f_2)(f_3f_4f_5))$ and $(f_1f_2f_3f_4f_5)$. One could further refine and build a 4-opetope, where in turn α_2 is refined as, say, $(f_3(f_4f_5))$. A picture of a 4-opetope can be found in [37]. The term notation can be used for opetopes in all dimensions ≥ 1. Here is the tree for α:

$$t = f_5\langle x_5 \leftarrow f_4\langle x_4 \leftarrow f_3\langle x_3 \leftarrow f_2\langle x_2 \leftarrow f_1\rangle\rangle\rangle\rangle.$$

Remarkably, the nodes of (the tree t of sources of) the target α of A are in one-to-one correspondence with the leaves of (the tree T of sources of) A. The expert reader will have recognised here a key feature of zoom complexes, on which we shall say a bit more in the following section.

Note that all opetopes in the picture are positive: there is always at least one source for each of A,α_i, and ■ is positive. Non-positivity starts at dimension 2. Here is the unique non-positive (or degenerate) opetope:



One can think of f as the identity on x, and β as a witness that it is such. As an example of a 3-opetope that is not positive, we can attach above β to α_2 along f_4 (renamed as f), forcing x_4 and x_5 to coincide, and then we have to modify the target α that now features the unbiased composition $(f_1f_2f_3f_5)$ (exit f_4!), and the modified 3-opetope reads as combined associativity-and-unit law: $((f_1f_2)(f_3(id)f_5)) = (f_1f_2f_3f_5)$.

As a final note, let us stress that 2-opetopes are in bijection with natural numbers (above, β is 0, α is 5, etc.). We refer to [9, 39, 11] for more extensive presentations of opetopes.

5.2 Combinatorial definitions

In [69, 63], Zawadowski gave axiomatic descriptions of opetopic shapes in terms of their face structures, consisting of a graded set $(S_n)_{n\in\omega}$ of faces, and two functions `s` and `t` assigning to an n-face (or cell) a its (finite) set $\delta(a)$ of sources and its target $\gamma(a)$, where $\delta(a) \subseteq S_{n-1}$ and $\gamma(a) \in S_{n-1}$, subject to a number of axioms. In the positive case, considered in [69], $\delta(a)$ is required to be always non-empty. In this way, Zawadowski defines what he calls *opetopic cardinals* (in the revision of his paper, published posthumously in this volume of the Cahiers). They form a strict ω-category which is the terminal positive-to-one polygraph. Opetopes are defined as the subset of *principal* opetopic cardinals. Jumping over time, in 2023, as part of her Master thesis project on comparing various combinatorial descriptions of opetopes, Louise Leclerc exhibited a simpler axiomatisation for positive opetopes, which she called positive *dendritic face complexes*, and showed that they were equivalent to principal opetopic cardinals [37]. One of Leclerc's axioms is called oriented thinness – a refinement informed by the source/target distinction of a familiar axiom in posets saying that when x, y, z are such that x covers y and y covers z, then there exists a unique other element y' diffferent from x, y, z in the interval $[x, z]$, which hence looks like a lozenge. Louise and I learned this axiom from Amar Hadzihasanovic, who has developped over the recent years a much more general theory of diagrammatic sets in which opetopes sit as special shapes [24, 23].

Zawadowski's axioms were more complicated for a reason: he wanted to capture not only the generators, but all cells of the terminal positive-to-one polygraph. This allowed him to provide a nice definition of their compositions in all dimensions, by means of pushouts. Zawadowski extended his face structures to cover all opetopes in [63].

Independently, Thorsten Palm had proposed a theory of *dendrotopic sets* [45, 46]. In his Bachelor's thesis [33], Chris Kapulkin proved that Palm's axioms were valid in Zawadowski's framework, while whether the converse holds is still uncertain today. Over the years, Marek has been always eager to learn about other approaches, striving to make sure that they were Sèvres.

It is remarkable that opetopes as decribed in the previous section and as face structures of various flavours contain redundancies:

1. The target of an opetope can be derived from its tree of sources, as will be clear from the material of the next section.
2. An n-opetope can be reconstructed from its sequence of targets together with some additional information.

The authors of [34] substantiated (2) by defining *zoom complexes* as sequences of trees featuring the successive targets, with some additional information which in the positive-to-one case consists in matching the leaves of the (n-1)-th target with the nodes of the n-th target (cf. T and t in Section 5.1). Additional book-keeping is needed in the many-to-one case and involves the notion of a tree with subdivisions, allowing some nodes to be only “traversed” by an edge. More precisely, to maintain the tight connection between the (n-1)-th target and the n-th target, the n-th target needs to be complemented by these subdivisions. The nodes of the targets are called black nodes, and the special traversed nodes are called white nodes. For example, the 3-opetope representing the identity law $(f_1(id)f_2) = (f_1f_2)$ has a tree of sources of the form $\gamma\langle f \leftarrow 0\rangle$ – where γ has sources f_1, f, f_2, and 0 is the degenerate 2-opetope –, and its target has $t = f_1\langle x \leftarrow f_2\rangle$ as tree of sources. This information does not suffice to reconstruct where the identity is introduced – here between f_1 and f_2. To record this, the zoom complex comprises a “fattened” version of t, in which a white node representing f is inserted on the edge x of t, If instead we want to describe the opetope for $((id)f_1f_2) = (f_1f_2)$, then we have to insert the white node on the leaf edge of t. We refer to [34] for nice pictures of zoom complexes. A variant of zoom complexes has been implemented by Eric Finster (see `https://ncatlab.org/nlab/show/opetopic+type+theory`). In [38], Leclerc extended her setting to dendritic face complexes (removing the positive restriction), showing them to be equivalent to zoom complexes.

In joint works, I have aimed at substantiating (1), in a type-theoretical style and in a combinatorial style, respectively. With Cédric Ho Thanh and Samuel Mimram [11], we developed a type-theoretical framework for opetopes coded as expressions in the style of the term notation used here, with a typing system guaranteeing that opetopes are exactly the correctly typed expressions. The issues with loops discussed above for zoom complexes were treated by maintaining tables of identifications forced by their presence (cf. $x_4 = x_5$ in the last example of Section 5.1).

With Leclerc [12], we defined *positive epiphytes* and showed them to be equivalent to positive dendritic face complexes. Among all combinatorial definitions of opetopes, epiphytes are arguably the closest to the polynomial definition (see next section): n-epiphytes are trees whose nodes are decorated with (n-1)-epiphytes and whose edges are decorated with (n-2)-epiphytes, in some compatible way. We refrained to release this paper, as we are still working on extending the picture to all opetopes. For this, the "technology" of trees with subdivision will be needed.

The last combinatorial definition in date, to my knowledge, is due to Taichi Uemura. When reading his work, I could recognise in his axioms ideas present in Palm's dendrotopic setting, and other ideas present in the dendritic setting of Leclerc, both of which he ignored at the time of writing his preprint [56], which makes his work all the more intriguing.

5.3 Opetopes via polynomial diagrams

Polynomial diagrams [16] are triples of maps $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} Q$ in $\mathbf{Set}$. We shall be mostly interested in the case where $O = Q$ (endodiagrams). One should think of O as a set of colours (operadic language), objects (multicategorical language) or sorts (logical language), and of B as a set of operations with an output sort given by $\mathtt{t}$ and input sorts given by $\mathtt{s}$. More precisely, the fiber $\mathtt{E}_b = \mathtt{p}^{-1}(b)$ (usually required to be finite) is the arity (or set of sources) of b and the restriction of $\mathtt{s}$ to $\mathtt{E}_b$, together with the target $\mathtt{t}(b)$, gives the typing of b. A morphism of polynomial (endo)diagrams is given by maps as below:

$$\begin{tikzcd}
O' \arrow[d, "\varphi"] & \mathtt{E}' \arrow[l, "\mathtt{s}'"'] \arrow[r, "\mathtt{p}'"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B' \arrow[r, "\mathtt{t}"] \arrow[d, "f"] & O' \arrow[d, "\varphi"] \\
O & \mathtt{E} \arrow[l, "\mathtt{s}"] \arrow[r, "\mathtt{p}"'] & B \arrow[r, "\mathtt{t}'"'] & O
\end{tikzcd} \tag{1}$$

The pullback ensures that an operation b is mapped to an operation with equipotent arity.

The category $\mathbf{Poly}$ of polynomial diagrams, together with the projection to $\mathbf{Set}$ mapping a diagram as above to its set of sorts O, forms a fibration. It is in fact a lax monoidal fibration in the sense of [65]. In particular, each fiber $\mathbf{Poly}(O)$ is equipped with a tensor product $\otimes_O$ defined on objects as follows. If $\mathbb{P}$ and $\mathbb{P}'$ are polynomial diagrams as above, then $\mathbb{P} \otimes_O \mathbb{P}' = \mathbb{P}''$ is defined as follows:

- $B'' = \{(b; \{b'_i \mid i \in \mathtt{E}_b\}) \mid b \in B, b'_i \in B'$ and $\mathtt{t}'(b'_i) = \mathtt{s}(i)\}$,
- with target given by the target of b and arity and sources given by the coproduct of the arities $\mathtt{E}_{b'_i}$, and the corresponding restrriction of $\mathtt{s}'$.

Operations of $\mathbb{P}''$ are thus trees of height 2 formed by an operation of $\mathbb{P}$ on all sources of which operations of $\mathbb{P}'$ are grafted. The unit $I_\otimes$ is given by the diagram $O \xleftarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O$.

In [65], it is proved that $\mathbf{Poly}$ is isomorphic to the category $\mathbf{Sig}_a$ of signatures with non standard amalgamation (in the terminology of Hermida, Makkai and Power). When O consists of one sort only, monoids for this tensor product are exactly the Σ-free operads (i.e. symmetric operads in which the action of the symmetric groups is free). Amalgamation here refers to the bijection that relates the arity of $(b; \{b'_i\})$ (which is the set of leaves of the tree described above) to the arity of the composite operation $b \circ \{b'_i\}$ given by the monoid structure. In the case of non-symmetric operads, all sources are (standardly) ordered, and the bijection is the unique order-isomorphism between two equipotent total orders. In the case of symmetric operads, so to say, all bijections are allowed, but equivariance laws governing the commutation of permutations and composition are required. As a matter of fact, these three flavours have led historically to three distinct ways of defining opetopic sets: non-symmetric in Leinster's work [39], Σ-free for the multitopic sets of [28], as well as in most Zawadowski's works and in the present exposition, symmetric in the original setting of [3]. Eugenia Cheng made an extensive comparison work between these three approaches (see e.g. [10] and references therein).

$\mathbf{Poly}(O)$ admits a free monoid construction, left adjoint to the forgetful functor from the category *Mon*$(\mathbf{Poly}(O))$ of $\otimes_O$-monoids to $\mathbf{Poly}(O)$. In fact, in [51], general continuity conditions are given on a monoidal category with coproducts that suffice to prove the existence of free monoid objects;

they are explicitly described as expected as a colimit mimicking the construction of the free monoid X^* in $\mathit{Mon} = \mathit{Mon}(\mathbf{Set})$. Instantiating this to $\mathbf{Poly}(O)$, we get that the free monoid $\mathbb{P}^*$ over some polynomial diagram $\mathbb{P}$ with data $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} O$ has B-trees as operations. These are rooted trees whose nodes are decorated with operations of B and whose edges – comprising root edge and leaf edges – are decorated with sorts of O, in a source/target respecting way. The target function returns the decoration of the root edge, the arity is the set of *leaves* and the source function returns the decorations of the leaf edges. We write $\mathit{Tree}(B)$ for the set of B-trees. The unit of $\mathbb{P}^*$ is specified by assigning to each sort o the node-less tree reduced to its root edge which is also the unique leaf edge, decorated with o. The composition operation $t \circ \{t_l\}$ consists in *grafting* a tree t_l on each leaf of t.

The illuminating point in [34] is the following observation: if $\mathbb{P} = \mathbb{M}$ is moreover equipped with a monoid structure, then the same set of operations $\mathit{Tree}(B)$ is also part of a different polynomial diagram, not in $\mathbf{Poly}(O)$, but in $\mathbf{Poly}(B)$. The target function is given by the evaluation, i.e. the effective composition of the tree, viewed as a specification of operations eligible to be composed. The arity of a tree is now its set of *nodes*, and the source function returns the operation decorating each node. It turns out that the resulting polynomial diagram $\mathbb{M}^+$ is also a monoid, so that this construction, known as the + *construction*, can be iterated, leading to the following "four-line" definition of opetopes: for $n \geq 1$, n-opetopes are the operations of the diagram obtained by iterating the + construction (n-1) times on the monoid $\{\blacklozenge\} \xleftarrow{\mathtt{s}} \{*\} \xrightarrow{\mathtt{p}} \{\blacksquare\} \xrightarrow{\mathtt{t}} \{\blacklozenge\}$, where $\mathtt{s}$, $\mathtt{p}$, $\mathtt{t}$ are the obvious bijections, and where $\blacklozenge$ is the unique 0-opetope. One sees in particular that there is a unique 1-opetope $\blacksquare$ with a unique source.

A composite $t \circ \{t_n\}$ in $\mathbb{M}^+$ consists in the *replacement* of each node of t by the tree t_n. The definitions ensure that the leaves of t_n are in bijection with the arity of the operation decorating n in t, ensuring precise rewiring instructions after replacement. We place $\mathbb{M}^*$ and $\mathbb{M}^+$ in perspective to stress the ascension: operations of $\mathbb{M}$ have become sorts of $\mathbb{M}^+$!

$$\begin{array}{ll} \mathbb{M}^* & O \xleftarrow{\mathtt{s}} \mathtt{Leaves} \longrightarrow \mathit{Tree}(B) \xrightarrow{\mathtt{t}} O \\ \mathbb{M}^+ & B \xleftarrow{\mathtt{label}} \mathtt{Nodes} \longrightarrow \mathit{Tree}(B) \xrightarrow{\mathtt{eval}} B. \end{array}$$

Although quite natural, the + construction as explained above "by hand" lacks a foundational blessing. We have seen that, when B is the set of operations of a monoid in $\mathbf{Poly}(O)$, *Tree*(B) presents the remarkable feature of having two monoid structures, given by grafting and replacement, respectively, but they do not (seem to) live in the same category: the relevant monoidal structures are in $\mathbf{Poly}(O)$ and $\mathbf{Poly}(B)$, respectively. Two natural questions arise:

1. Can we build a (single) category equipped with two monoidal structures (compatible in some way), relatively to which replacement and grafting appear as (compatible) monoid structures?
2. Can we exhibit the + construction as a free construction?

Szawiel and Zawadowski give positive answers to these two questions, which we try to recount below. But before that, I'd like to present an answer to Question (2) obtained by making the link between the + construction and the (idea of) Baez and Dolan's slice construction explicit. Let me start by recalling that any polynomial diagram $\mathbb{P}$ gives rise to a *polynomial functor* $\underline{\mathbb{P}} : \mathbf{Set}/O \to \mathbf{Set}/Q$, mapping a function $f : X \to O$ to $\underline{\mathbb{P}}(f) = \sum_{q\in Q}\sum_{b\in B,\mathtt{t}(b)=q}\prod_{i\in \mathtt{E}_b} f^{-1}(\mathtt{s}(i))$, with the obvious projection to Q. Pictorially, an element of the fiber of $\underline{\mathbb{P}}(f)$ over q is an operation whose target is of type q and whose sources are additionally decorated with elements of X in a compatible way, i.e., if x decorates i, then $f(x) = \mathtt{s}(i)$. The above sum of monomials explains the name polynomial! It is also known that $\underline{\mathbb{P}}$ determines $\mathbb{P}$.

We shall give evidence of an alternative description of $\underline{\mathbb{M}^+}$. For this, I also need to recall the fact that if $\mathbb{C}$ is a monoidal category and M a monoid object in $\mathbb{C}$, then the slice category $\mathbb{C}/M$ comes equipped with a monoidal structure defined by taking I_M to be the unit of the monoid and by setting $f \otimes_M g = \mathtt{m} \circ (f \otimes g)$, where $\mathtt{m}$ is the multiplication of the monoid. We shall apply this to $\mathbf{Poly}(O)$ and $\mathbb{M}$. But before doing that, we need one more observation. It is easily seen that the polynomial diagram underlying $\mathbb{M}$ induces an isomorphism between $\mathbf{Poly}(O)/\mathbb{M}$ and $\mathbf{Set}/B$ (observe that we are in a fiber of $\mathbf{Poly}$, i.e., in the diagram (1) above, we have $\varphi = id$, and all the rest of the data is inferred from the bottom line and from f). Through this isomorphism, the induced tensor product of $f : X \to B$ and $g : Y \to B$

is $h : Z \to B$, where

$$Z = \{((x, b); \{(y_i, b_i) \mid i \in \mathtt{E}_b\}) \mid b, b_i \in B, f(x) = b, g(y_i) = b_i, \mathtt{t}'(b_i) = \mathtt{s}(i)\}$$

and where h is given by the composition $b \circ \{b_i\}$. In plain words, an operation in Z is an operation of $\mathbb{M} \otimes_O \mathbb{M}$ in which each node receives an additional decoration from X or Y, in a compatible way. Then iterating this tensor and applying the general free monoid construction sketched above, we get the following description of the free monoid over $f : X \to O$: its operations are B-trees whose nodes are further decorated with elements of X, respecting the following invariant: if a node is decorated with (x, b), then $f(x) = b$. But this is exactly $\underline{\mathbb{M}^+}(f)$. Therefore $\mathbb{M}^+$ is the polynomial diagram whose underlying polynomial functor is the free monoid functor for the monoidal structure sketched above.

So we have hit the goal of exhibiting the + construction as a free construction (a fact known to experts). But this was not made internally, in the sense that we had to go back and forth between diagrams and actual functors. Moreover, Question (1) is not yet answered. We repair this now, by extracting and commenting a key construction that we found in [51], expressed there in the language of signatures and translated here in the language of polynomial diagrams. This construction is a sort of more informed version of the slice construction above.

We start again with a monoid $\mathbb{M}$ given by $O \xleftarrow{\mathtt{s}} \mathtt{E} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} O$ in *Mon*$(\mathbf{Poly}(O))$. We consider the category $\mathbf{Poly}[\mathbb{M}] = \mathbf{Poly}(B)$, which we name differently to stress the fact that we shall take profit of the monoid structure of $\mathbb{M}$ to define a second monoidal product $\odot$ on $\mathbf{Poly}[\mathbb{M}]$, in addition to $\otimes = \otimes_B$ – the "resident" monoidal product in $\mathbf{Poly}(B)$. Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be two objects of $\mathbf{Poly}(B)$, whose data are $B \xleftarrow{\mathtt{s}^1} \mathtt{E}^1 \xrightarrow{\mathtt{p}^1} X \xrightarrow{\mathtt{t}^1} B$ and $B \xleftarrow{\mathtt{s}^2} \mathtt{E}^2 \xrightarrow{\mathtt{p}^2} Y \xrightarrow{\mathtt{t}^2} B$, respectively. We define $\mathbb{P}^1 \odot \mathbb{P}^2 = \mathbb{P}$ as $B \xleftarrow{\mathtt{s}'} \mathtt{E}' \xrightarrow{\mathtt{p}'} Z \xrightarrow{\mathtt{t}'} B$,

- where $\mathtt{t}'$ is the tensor product of $\mathtt{t}^1$ and $\mathtt{t}^2$ in $\mathbf{Set}/B$ as defined above, i.e., Z consists of decorated trees of hight 2 denoted as $((x, b); \{(y_i, b_i)\})$, where $b, b_i \in B$, i ranges over $\mathtt{E}_b$, $x \in X$, $\mathtt{t}^1(x) = b$, $y_i \in Y$, $\mathtt{t}^2(y_i) = b_i$ and $\mathtt{t}(b_i) = \mathtt{s}(i)$, and $\mathtt{t}'$ is the evaluation map returning $b \circ \{b_i\}$;
- the arity is given by taking the coproduct of the arities $(\mathtt{E}^1)_x$ and $(\mathtt{E}^2)_{y_i}$ and finally $\mathtt{s}'$ is given by copairing the relevant restrictions of $\mathtt{s}^1$ and $\mathtt{s}^2$.

Beyond the technicalities, note that the arity is ”node-wise”, as the above coproduct is indexed over all the nodes of $((x, b); \{(y_i, b_i)\})$. The unit $I_\odot$ for this second monoidal structure is given by the diagram $B \longleftarrow \emptyset \longrightarrow O \xrightarrow{\eta} B$, where η is given by the unit of the monoid $\mathbb{M}$.

With this, the goal of endowing $\mathbf{Poly}[\mathbb{M}]$ with two monoidal structures is reached. The two structures interact in a nice way: most notably, there is a distribution of $\otimes$ over $\odot$, and $\mathbf{Poly}[\mathbb{M}]$ satisfies the conditions of Szawiel and Zawadowski's Three tensors theorem – the main theorem in [51] –, which asserts that $\mathcal{F}_\odot(I_\otimes)$, i.e., the free $\odot$-monoid over the $\otimes$ unit, is also a $\otimes$-monoid. To unravel this, applying once more the general recipe for the explicit construction of free monoids, we have to examine $I_\otimes \odot \cdots \odot I_\otimes$. For $n = 2$, replacing above $\mathbb{P}^1$ and $\mathbb{P}^2$ both with $I_\otimes = O \xleftarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O \xrightarrow{\mathtt{id}} O$, we see that x, y_i are determined by b and hence each summand of the coproduct above is a singleton, so that the arity of an operation in $I_\otimes \odot I_\otimes$ is just its set of nodes. Taking the colimit, we recover *Tree*(B) with nodes as sources! In other words, we obtain our by now old friend $\mathbb{M}^+$, and as a bonus, by the Three tensors theorem, we get *for free* that $\mathbb{M}^+ = \mathcal{F}_\odot(I_\otimes)$ is a monoid in $\mathbf{Poly}(B)$, which gives us replacement as $\otimes$-multiplication.

Let us give evidence that the $\odot$-multiplication is the grafting operation: an operation in $\mathbb{M}^+ \odot \mathbb{M}^+$ is of the form $((t, b); \{(t_i, b_i)\})$, where t is B-tree that evaluates to b, which entails that the leaves of t are the sources of b, so that $(t; \{t_i\})$ is an operation of $\mathbb{M}^*$. By the freeness of $\mathbb{M}^+$ as a $\odot$-monoid and the general construction of free monoids, we have that the composite is the “concatenation” of t and the t_i, which here means the tree obtained by grafting the t_i's on the leaves of t. Thus the $\odot$-monoid structure of $\mathbb{M}^+$ in $\mathbf{Poly}[\mathbb{M}]$ litterally transcribes the $\otimes_O$-monoid structure of $\mathbb{M}^*$ in $\mathbf{Poly}(O)$. This all beautifully answers Questions (1) and (2) at once. We summarise the discussion in the following table.

grafting:	$\mathbb{M}^*$ as free $\otimes_O$-monoid in $\mathbf{Poly}(O)$ $\rightsquigarrow$ $\mathbb{M}^+$ as free $\odot$-monoid in $\mathbf{Poly}[\mathbb{M}]$
replacement:	$\mathbb{M}^+$ as $\otimes_B$-monoid in $\mathbf{Poly}(B)$ $\rightsquigarrow$ $\mathbb{M}^+$ as $\otimes$-monoid in $\mathbf{Poly}[\mathbb{M}]$

I would like to end this promenade in the opetopic world of Zawadowski with a mention of the nice connections with the line of work on mathematical accounts of syntax carried out by Marcelo Fiore and coworkers. The Three tensors theorem is an instance of a more general theorem stated by Fiore and Saville in [13]. In yet unpublished work presented in some talks starting in 2017[1], Fiore has offered an insightful syntactic reading of the two monoid structures on trees : the leaves stand for variables and grafting for first-order substitution, while the nodes stand for function symbols and replacement for second-order substitution (cf. the term notation used in Section 5.1). He has also transposed and generalised the opetopic scene in the world of generalised species. Marek's legacy is very much alive!

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