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créés par CHARLES EHRESMANN en 1958

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# IN MEMORIAM ANDRÉE EHRESMANN



(Photo by J. Baez, 2014)

Andrée Ehresmann (née Bastiani)  
Nice, 7 September 1935 – Amiens, 21 August 2026  
Director of the *Cahiers de Topologie et Géométrie  
Différentielle Catégoriques* from 1979 to 2026





# CROSSED $\mathbb{N}$ -CUBES AND CROSSED $n$ -CUBE COMPLEXES

(VARIETIES OF SIMPLICIAL GROUPOIDS, II)

*Timothy Porter*

**Résumé.** Dans cet article, nous introduisons les notions de complexes de  $\mathbb{N}$ -cubes croisés et de complexes de  $n$ -cubes croisés, qui généralisent celles des complexes croisés et des  $n$ -cubes croisés d'Ellis-Steiner. Nous montrons qu'en faisant varier  $n$ , les différentes catégories de complexes de  $n$ -cubes croisés forment une tour réflexive de catégories au sein de celle des  $\mathbb{N}$ -cubes croisés.

**Abstract.** In this article, we introduce notions of crossed  $\mathbb{N}$ -cube and crossed  $n$ -cube complexes, which extend both those of crossed complexes and Ellis-Steiner crossed  $n$ -cubes. We show that varying  $n$ , the various categories of crossed  $n$ -cube complexes form a reflective tower of categories within that of crossed  $\mathbb{N}$ -cubes.

**Keywords.** Crossed complex, crossed  $n$ -cube

**Mathematics Subject Classification (2010).** 18D40 · 18G45.

## 1. Introduction

This paper is both a continuation of [23, 24] and an extension of [42].

In [42], a proof was given of a version of Loday's theorem from [36], Loday's paper showed that a certain simple algebraic structure, a  $\text{cat}^n$ -group, was able to model all connected homotopy  $(n + 1)$ -types, generalising the classical result of Mac Lane and Whitehead, [37]. Such objects were internal

$n$ -fold categories in the category of groups, but also equivalently have a description as crossed  $n$ -cubes of groups, a structure which generalises crossed modules to higher dimension.

In [23, 24], the problem that was addressed was to ask for an algebraic description of those simplicially enriched groupoids that corresponded to the crossed complexes defined by Brown and Higgins, for example in [11]. This was to be without initial reference to the homotopy theory that had been developed, as the aim was for a description as a *variety* within the category of simplicial groupoids, so specifying the ‘verbal subobject’ generating the kernel of the reflection into the variety. In other words, the aim of that work was to examine the algebraic nature of those models, and to initiate their study as objects of algebraic interest.

That study was in part motivated by J. H. C. Whitehead’s ICM address, [49], as quoted by Hans Baues in [2]:

“The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that ‘analytic’ is equivalent to ‘pure’ projective geometry.”

In this paper, we will to some extent expand that examination to higher dimensions. As some of the theory has only been developed for the connected case, and thus for simplicial groups rather than simplicially enriched groupoids, our treatment will, in the main, be handling that ‘reduced’ situation. The extension to the general case is not hard to do, but would take up quite a lot of space. This is an analogue of the simplicial analysis of homotopy types via coskeleta, but is much more algebraic. It is also related to the notion of towers of categories introduced by Baues in [2, 3], and even more closely to the use of a similar idea by Bullejos, Faro, and García-Muñoz, [15], which limits attention to crossed complexes.

There is an additional theme to this paper, however. Following on from Loday’s work on homotopy  $n$ -types, [36], crossed  $n$ -cubes, as introduced by Ellis and Steiner, [27], gave complete information on the homotopy type up to dimension  $n$ . Crossed complexes, on the other hand, give useful, but incomplete, information in *all* dimensions. In both situations, a van Kampen type theorem exists. For crossed complexes, this van Kampen theorem was due to Brown and Higgins, [9], and for crossed  $n$ -cubes, a corresponding

result was proved by Brown and Loday, [13]. Both generalised the crossed module version of [8]. This suggests that there might be, for each  $n$ , a class of models that we will call *crossed  $n$ -cube complexes*, or sometimes more briefly  *$n$ -crossed complexes*, which would give complete information on the  $n$ -type and yet give useful information in all dimensions. For  $n = 1$ , crossed complexes already achieve this aim.

More precisely, the hopes in looking for this tentative model are as follows:

(i) to find a fairly simple algebraic model for those homotopy types with vanishing Whitehead products above a given dimension,

(ii) to find a unified van Kampen theorem that includes both the Brown-Higgins and Brown-Loday forms as special cases,

(iii) to allow for a better treatment of homotopies of maps between of crossed  $n$ -cubes,  
and, if possible,

(iv) find a varietal, or at least more algebraic, description of those simplicially enriched groupoids that correspond to such models for each fixed  $n$ .

The results later on show that the model proposed looks promising in this regard, but these aims have not been explored adequately here<sup>1</sup>.

Although crossed  $n$ -cube complexes, once defined, should clearly have trivial Whitehead products in dimensions greater than  $n$ , the converse is almost certainly untrue, as it is already false for  $n = 1$ .

The second aim will not be examined here in detail and, to my knowledge, not much progress has been made on that problem. The problem seems to be that of turning the filtered spaces encountered in the Brown-Higgins theory into some sort of hypercube of fibrations as in the corresponding Brown-Loday theory.

About the third aim, it should perhaps be pointed out that crossed *modules* only have a truncated homotopy theory, since they have nothing above dimension one. Crossed complexes have, however, an extremely rich homotopy theory since they have tensor products and ‘internal homs’, and form

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<sup>1</sup>Of course, one such description of a somewhat different flavour was already available as a particular class of hyper-crossed complexes, as in [16] and [17], especially if combined with insights from Bullejos-Cegarra-Duskin, [14]. Another approach was initiated by Baues in [3], which explores the links with Whitehead’s programme in a different way.

a locally Kan simplicially enriched category, which is tensored and cotensored over simplicial sets, and hence have cylinder and cocylinder functors (*cf.*, Brown-Golanskiński, [7], Brown-Higgins, [11], and Kamps-Porter, [33]). The development of that theory depended to a large extent on the use of non-reduced crossed complexes, *i.e.*, crossed complexes over a groupoids (*cf.*, Brown-Higgins, [11]) and the many different categories equivalent to the category of these. It thus seems likely that a similar theory for crossed  $n$ -cube complexes will need crossed  $n$ -cube complexes over groupoids.

The notions of a crossed  $\mathbb{N}$ -cube and crossed  $n$ -cube complexes were introduced by the author in various sets of notes, starting back in the 1990s and 2000s, but the details were not published. The reason for rejuvenating these results now is the revival of interest in some closely related concepts, such as that of  $n$ -hypergroupoids, also known as (weak)  $n$ -groupoids, following on from work of Behrend and Getzler, [4], and Pridham, [44, 45], and derived from work of Duskin, [22], and Glenn, [30]. We note, in different contexts, the use of crossed complexes of Lie algebras by Kapranov, [34], and a conjectured use of crossed  $n$ -cubes in lattice gauge theory, [35], by Kapustin and Spodyneiko. We hope to consecrate a future article to how hypergroupoids fit into this overall picture, but, in the last section here, will briefly sketch out some of the ideas involved, and also mention some of the problems that will need to be surmounted if that conjectural theory is to thrive, emphasising that the theory developed here is just one of several possible ones involving algebraic or simplicial models for  $n$ -types with a ‘tail’ attached. At the moment, this feels like a puzzle in which some of the pieces have yet to be ‘turned over’, but in which we have a fairly clear view of what the overall picture looks like and of some of the problems that need to be resolved..

With those recent developments in mind, it should be said that although this paper is written in set-based way, the categorical and algebraic constructions used are available in many other contexts. For instance, for a semi-abelian category,  $\mathcal{A}$ , the category of  $n$ -fold internal groupoids in  $\mathcal{A}$  is also semi-abelian, by results of Bourn and Gran, [6], and most if not all of the theory developed here could be adapted, with care, to that setting.

Similarly, in categories of group objects in presheaf toposes, the availability of the Yoneda lemma and techniques of generalised elements, mean this theory goes across to that wider context, with some slight adjustment and in the last section we will recall that by using categories with covers, *e.g.*,

as in Wolfson, [50], or categories with a singleton Grothendieck pretopology, as in Zhu, [51], this theory should be able to be applied to several other more geometric settings.

Finally, it should be noted that a theory of models for higher dimensional algebra has been partially developed by various authors, in various flavours, using varied terminology. This can make it difficult to know what is known and what is unknown. I hope this paper will not contribute to the potential confusion, and will provide a place in which some of the varied sources can be linked together.

**Acknowledgements:** These results were presented, in the 1990s, in seminars in Bangor. They benefitted enormously from discussions with Ronnie Brown, Chris Wesley, Zekariya Arvasi and Ali Mutlu, who also assisted in the preparation of the initial draft version all those years ago. More recently, they and related areas were discussed in online sessions with Jim Stasheff, João Faria Martins, Francis Bischoff and others. I would like to thank all for their feedback on these ideas.

I dedicate this to the memory of Ronnie and the lively group discussions in Bangor on this and related areas, that we had many years ago.

## 2. Crossed gadgetry

For convenience, we start by recalling the definition of crossed modules, crossed squares, and then crossed  $n$ -cubes, as well as that of crossed complexes.

### 2.1 Crossed modules

We first review some definitions.

**Definition:** A *crossed module* is a group homomorphism,  $\partial : M \longrightarrow N$ , together with an action of  $N$  on  $M$  satisfying:

$$\text{CM 1) } \partial({}^n m) = n\partial(m)n^{-1}$$

and

CM 2)  $\partial^{(m)}m' = mm'm^{-1}$  for all  $m, m' \in M$ ,  $n \in N$ .

These seem first to have been defined by J. H. C. Whitehead in [48]. The second condition is known as the *Peiffer identity*.

A 2-dimensional version of a crossed module, called a crossed square, was defined by D. Guin-Waléry and J.-L. Loday, [31], in 1981. We refer the reader to that paper and also to [36], for the original definition of these. We will give an equivalent definition slightly later as a special case of a crossed  $n$ -cube. Crossed squares model all connected homotopy 3-types, but we note also that 2-crossed modules in the sense of Conduché, [19], do so as well. References to the connections between the two models are given in [20].

In [36], Loday introduced a notion of  $\text{cat}^n$ -group and showed that they model all connected homotopy  $n + 1$ -types, for which see slightly later.

## 2.2 Crossed complexes of groups

The next definition we need is that of a crossed complex of groups<sup>2</sup>. This was given by R. Brown and P. J. Higgins, [10], generalising earlier work of Whitehead (1949), [48]. The theory of crossed complexes has been extensively described in Brown-Higgins-Sivera, [12].

**Definition:** A *crossed complex of groups* is a sequence of groups,

$$C : \quad \cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

- (i)  $C_n$  is abelian for  $n \geq 2$ ,
  - (ii)  $C_0$  acts on  $C_n$  for  $n \geq 1$ , and the restriction to  $\partial_1 C_1$  acts trivially on  $C_n$  for  $n \geq 2$ ,
  - (iii) each  $\partial_n$  is a  $C_0$ -group homomorphism, with  $\partial_1 : C_1 \rightarrow C_0$  being a crossed module,
- and
- (iv) for  $i \geq 1$ ,  $\partial_{i+1}\partial_i = 1$ .

---

<sup>2</sup>The transition to crossed complexes of groupoids is relatively easy to do, but, so as to keep things relatively short and simple, we will not give it here. When more generally considering the groupoid case, a group crossed complex is often referred to as being a *reduced crossed complex*.

In other words, it is a non-negatively graded chain complex of groups in which the last map is a crossed module.

The category of crossed complexes will be denoted by  $Crs.Comp$ , for the moment, although this will be absorbed into a generic notation for crossed  $n$ -cube complexes later on, as we will be considering a family of generalisations of crossed complexes in which the bottom crossed module is replaced by more complex structures. Because of this we may sometimes refer to crossed complexes as *crossed 1-cube complexes*.

We note also the notion of  *$n$ -truncated crossed complexes*, although we will not use it that much. Such crossed complexes satisfy that  $C_m$  is trivial for  $m > n$ .

### 2.3 $Cat^n$ -groups and crossed $n$ -cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

**Definition:** A *cat<sup>n</sup>-group* is a group,  $G$ , together with  $2n$  endomorphisms  $s_i, t_i$ , ( $1 \leq i \leq n$ ), such that

$$s_i t_i = t_i, \text{ and } t_i s_i = s_i \text{ for all } i,$$

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \text{ for } i \neq j$$

and, for all  $i$ ,

$$[Ker s_i, Ker t_i] = 1.$$

A  $cat^n$ -group is thus a group with  $n$  independent  $cat^1$ -group structures on it. A  $cat^1$ -group can also be reformulated as an internal groupoid in the category of groups, so a  $cat^n$ -group, not surprisingly, leads to an internal  $n$ -fold groupoid in the same setting.

These were first described by Loday in [36] under a slightly different name. In that paper, his aim was to generalise the result of Mac Lane and Whitehead, [37], that showed that crossed modules modelled all homotopy 2-types<sup>3</sup>. Loday also proved that  $cat^2$ -groups were equivalent to the crossed squares that had been examined earlier in [31].

---

<sup>3</sup>Their 3-types are now called 2-types. We note that there is an error in the proof of the

The initial proof that  $\text{cat}^n$ -groups model  $(n + 1)$ -types is due to Loday, [36], but with amendments to his proof contributed by Gilbert, [29], and Steiner, [47]. Ellis and Steiner proved, in [27], the equivalence of  $\text{cat}^n$ -groups with crossed  $n$ -cubes (and much more). Porter, [42], and independently Bullejos-Cegarra-Duskin, [14], proved purely algebraic versions of Loday's result. Our treatment will, in the main, make use of [42].

The definition of crossed  $n$ -cube as an  $n$ -fold crossed module was initially suggested by Ellis in his thesis, [25]. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The elegant formulation that resulted from the joint work of Ellis and Steiner, [27], showed how that could be avoided by encoding the actions and the  $h$ -maps in the same structure.

We write  $\langle n \rangle$  for the set  $\{1, \dots, n\}$ .

**Definition:** A *crossed  $n$ -cube*,  $M$ , is a family of groups,  $\{M_A : A \subseteq \langle n \rangle\}$ , together with homomorphisms,  $\mu_i : M_A \rightarrow M_{A-\{i\}}$ , for  $i \in \langle n \rangle$ ,  $A \subseteq \langle n \rangle$ , and functions,  $h : M_A \times M_B \rightarrow M_{A \cup B}$ , for  $A, B \subseteq \langle n \rangle$ , such that if  ${}^a b$  denotes  $h(a, b)b$  for  $a \in M_A$  and  $b \in M_B$  with  $A \subseteq B$ , then for  $a, a' \in M_A$ ,  $b, b' \in M_B$ ,  $c \in M_C$  and  $i, j \in \langle n \rangle$ , the following axioms hold:

- (1)  $\mu_i a = a$  if  $i \notin A$
- (2)  $\mu_i \mu_j a = \mu_j \mu_i a$
- (3)  $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
- (4)  $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$  if  $i \in A \cap B$
- (5)  $h(a, a') = [a, a']$
- (6)  $h(a, b) = h(b, a)^{-1}$
- (7)  $h(a, b) = 1$  if  $a = 1$  or  $b = 1$
- (8)  $h(aa', b) = {}^a h(a', b)h(a, b)$
- (9)  $h(a, bb') = h(a, b){}^b h(a, b')$
- (10)  ${}^a h(h(a^{-1}, b), c){}^c h(h(c^{-1}, a), b){}^b h(h(b^{-1}, c), a) = 1$

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main result of Loday's paper. This was remedied in several other papers, *e.g.*, by Bullejos, Cegarra and Duskin, [14], or Porter, [42], as well as clarifications by other authors mentioned above.

$$(11) {}^a h(b, c) = h({}^a b, {}^a c) \text{ if } A \subseteq B \cap C.$$

A morphism of crossed  $n$ -cubes,

$$\{M_A\} \rightarrow \{M'_A\},$$

is a family of homomorphisms,  $\{f_A : M_A \rightarrow M'_A \mid A \subseteq \langle n \rangle\}$ , which commute with the maps,  $\mu_i$ , and the functions,  $h$ . This gives us a category,  $Crs^n$ , equivalent to that of  $\text{cat}^n$ -groups.

**Remark:** In the correspondence between  $\text{cat}^n$ -groups and crossed  $n$ -cubes (see Ellis and Steiner, [27]), the  $\text{cat}^n$ -group corresponding to a crossed  $n$ -cube,  $(M_A)$ , is constructed as a repeated semidirect product of the various  $M_A$ . Within the resulting “big group”, the  $h$ -functions interpret as being commutators. This partially explains the structure of the  $h$ -function axioms; see also the examples below.

We note the result of Ellis and Steiner.

**Theorem 2.1.** [27] *There are equivalences of categories,*

$$Crs^n \simeq Cat^n(Grp).$$

□

As the 11 axioms may look rather daunting when first seen, we will give some hopefully illuminating examples.

**Examples:** 1. Inclusion crossed  $n$ -cubes are possibly the simplest examples, whilst being almost completely general in a certain way.

A group,  $G$ , together with a list,  $(N_1, \dots, N_n)$ , of  $n$  (closed) normal subgroups, possibly with repeats, will be called a *normal  $(n+1)$ -ad of groups* and will be denoted  $(G; N_1, \dots, N_n)$ . Morphisms of such are morphisms of the big group compatible with the listed normal subgroups.

Let  $(G; N_1, \dots, N_n)$  be a normal  $(n+1)$ -ad of groups, then the following defines a crossed  $n$ -cube of groups:

- for  $A \subseteq \langle n \rangle$ , let  $M_A = \bigcap \{N_i \mid i \in A\}$ ;

- for  $i \in \langle n \rangle$ , let  $\mu_i : M_A \rightarrow M_{A \setminus \{i\}}$  be the obvious inclusion;  
and
- for  $A, B \subseteq \langle n \rangle$ , let  $h : M_A \times M_B \rightarrow M_{A \cup B}$  be given by the commutator,  $h(a, b) = [a, b]$ .

We call the resulting crossed  $n$ -cube the *inclusion crossed  $n$ -cube defined by the  $(n+1)$ -ad*,  $(G; N_1, \dots, N_n)$ . These do look rather special, but we note the following from [42]. It uses simplicial objects consisting of such  $(n+1)$ -ads of groups.

Let  $(G; N_1, \dots, N_n)$  be a simplicial normal  $(n+1)$ -ad of groups and define:

- for  $A \subseteq \langle n \rangle$ , let  $M_A = \pi_0(\bigcap \{N_i : i \in A\})$ ;
- for  $i \in \langle n \rangle$ , let  $\mu_i : M_A \rightarrow M_{A \setminus \{i\}}$  be induced by the obvious inclusion;

and

- for  $A, B \subseteq \langle n \rangle$ , let  $h : M_A \times M_B \rightarrow M_{A \cup B}$  be induced by the commutator,  $h(a, b) = [a, b]$ ,

then  $\mathcal{M} = \{M_A : A \subseteq \langle n \rangle, \{\mu_i\}, h\}$  is a crossed  $n$ -cube of groups.

This is not just a way of producing simple examples because, as is shown in [42], §5, that, for every crossed  $n$ -cube, there is a simply constructed simplicial normal  $(n+1)$ -ad of groups whose  $\pi_0$  is isomorphic to the given one.

2. For  $n = 1$ , the eleven axioms above reduce to the usual ones for crossed modules. Some conditions do not apply as the cases they describe cannot occur. Here is a tiny bit more detail. The 1-cube is  $M_{\langle 1 \rangle} \xrightarrow{\mu_1} M_\emptyset$ , with  $h : M_\emptyset \times M_{\langle 1 \rangle} \rightarrow M_{\langle 1 \rangle}$  being the main  $h$ -map. Axioms 8) - 11) just say that  ${}^a b$  is an action. (1) and (2) do not apply, (3) say  $\mu_1$  is equivariant for the action, and (4) is the Peiffer identity in disguise.

3. For  $n = 2$ , the definition gives a crossed square in the sense of [31]:

$$\begin{array}{ccc}
 M_{\langle 2 \rangle} & \xrightarrow{\mu_2} & M_{\{1\}} \\
 \mu_1 \downarrow & & \downarrow \mu_1 \\
 M_{\{2\}} & \xrightarrow{\mu_2} & M_\emptyset
 \end{array}$$

with the important  $h$ -map being  $h : M_{\{1\}} \times M_{\{2\}} \rightarrow M_{\langle 2 \rangle}$ . The other  $h$ -maps in the above definition correspond to the various actions as explained in the original definition in [31].

4. We will recall, slightly later, how to obtain a crossed  $n$ -cube from a simplicial group.

We next relate crossed  $n$ -cubes to crossed  $(n-1)$ -cubes. We will do this, initially, just in the case of crossed squares. It may help to think of a crossed  $n$ -cube as a morphism of crossed  $(n-1)$ -cubes, e.g., for  $n=2$ , if

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \chi' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

is a crossed square, then taken horizontally there is a morphism of crossed modules,

$$\begin{array}{ccc} L & & M \\ \chi' \downarrow & \longrightarrow & \downarrow \mu \\ N & & P, \end{array}$$

which is exactly the pair of morphisms,  $(\lambda, \mu')$ , in the crossed square. There is also a vertical morphism of crossed modules.

The following is Proposition 5 of [42], page 9.

**Proposition 2.2.** *Let  $\mathcal{M} = \{M_A : A \subseteq \langle n \rangle, \{\mu_i\}, h\}$  be a crossed  $n$ -cube of groups. For any  $i \in \langle n \rangle$ , let  $\mathcal{M}_1$  denote the family  $\{M_A : A \subseteq \langle n \rangle, i \in A\}$ , and  $\mathcal{M}_0$  the corresponding family with the condition  $i \notin A$ , then*

- a)  $\mathcal{M}_0$  and  $\mathcal{M}_1$  both have the structure of crossed  $(n-1)$ -cubes of groups;
- b)  $\mu_i : \mathcal{M}_1 \rightarrow \mathcal{M}_0$  is a morphism of crossed  $(n-1)$ -cubes such that for each  $B \subseteq \langle n-1 \rangle$ ,  $\text{Ker } \mu_{i,B}$  is central in  $\mathcal{M}_{1,B}$  and  $\text{Im } \mu_{i,B} \subseteq \mathcal{M}_{0,B}$  is a (closed) normal subgroup;
- c) denoting  $\mathcal{N}_B = \{\mathcal{M}_{0,B} / \text{Im } \mu_{i,B}, B \subseteq \langle n-1 \rangle\}$ , then  $\mathcal{N}$  has a natural structure of a crossed  $(n-1)$ -cube and the sequence

$$1 \rightarrow \text{Ker } \mu_i \rightarrow \mathcal{M}_1 \xrightarrow{\mu_i} \mathcal{M}_0 \rightarrow \mathcal{N} \rightarrow 1$$

is an exact sequence of crossed  $(n - 1)$ -cubes in the obvious sense.  $\square$

We will use this, in particular, when  $i = n$ .

Another useful construction is given as follows. It will be illustrated with crossed squares, but can be generalised. Within the category  $Crs^2$  of crossed squares, we will be considering ones of form

$$\begin{array}{ccc} M_{\langle 2 \rangle} & \xrightarrow{\mu_2} & M_{\{1\}} \\ \mu_1 \downarrow & & \downarrow \mu_1 \\ M_{\{2\}} & \xrightarrow{\mu_2} & M_{\emptyset} \end{array}$$

in which  $M_{\{2\}} = 1$ . This leaves a sequence of groups

$$M_{\langle 2 \rangle} \xrightarrow{\mu_2} M_{\langle 1 \rangle} \xrightarrow{\mu_1} M_{\emptyset},$$

which is, in fact, a 2-truncated crossed complex. (We will be using a generalisation of this observation later on.)

We have a functor,

$$Tr_2 Crs.Comp \rightarrow Crs^2,$$

that considers a 2-truncated crossed complex,  $D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0$ , as being a crossed square

$$D = \left( \begin{array}{ccc} D_2 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & D_0 \end{array} \right).$$

That inclusion functor has a left adjoint that we illustrate on an inclusion crossed square coming from a group,  $G$ , with  $M, N$  being two normal subgroups. This gives

$$C = \left( \begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & G \end{array} \right),$$

all the maps being inclusions and with  $h : N \times M \rightarrow M \cap N$  given by  $h(x, y) = [x, y]$ .

Suppose that we have a map from  $C$  to  $D$ , then clearly that map kills off  $N$ , within  $G$ , but also the action of  $N$  on  $M$ , and so on. We thus have that the map will factor through

$$\begin{array}{ccc} \frac{M \cap N}{[M, N]} & \longrightarrow & \frac{M}{[M, N]} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & G/N \end{array}$$

This will generalise not only to general crossed squares, but also to higher dimensions. However we have not yet introduced all the ideas and terminology needed to express that so will postpone further discussion of the general result until later

## 2.4 Squared complexes

It is natural curiosity to try to extend crossed squares and crossed complexes to a joint description. This should be seen as just the start of a hierarchy. A first step is given by squared complexes. Ellis, [26], gave the following definition.

**Definition:** A *squared complex* consists of a diagram of group homomorphisms,

$$\begin{array}{ccccccc} & & & & N & & \\ & & & \nearrow^{\lambda'} & & \searrow^{\mu} & \\ \dots & \longrightarrow & C_4 & \xrightarrow{\partial_4} & C_3 & \xrightarrow{\partial_3} & L \\ & & & \searrow_{\lambda} & & \nearrow_{\mu'} & \\ & & & & M & & P \end{array}$$

together with actions of  $P$  on  $L, N, M$  and  $C_i$  for  $i \geq 3$ , and a function  $h : M \times N \longrightarrow L$ . The following axioms need to be satisfied.

- (i) The square  $\begin{pmatrix} L & \xrightarrow{\lambda} & N \\ \lambda' \downarrow & & \downarrow \mu \\ M & \xrightarrow{\mu'} & P \end{pmatrix}$  is a crossed square;
- (ii) Each group,  $C_n$ , is abelian for  $n \geq 3$ ;

- (iii) The boundary homomorphisms satisfy  $\partial_n \partial_{n+1} = 1$  for  $n \geq 3$ , and  $\partial_3(C_3)$  lies in the intersection  $\text{Ker } \lambda \cap \text{Ker } \lambda'$ ;
- (iv) The action of  $P$  on  $C_n$  for  $n \geq 3$  is such that  $\mu M$  and  $\mu' N$  act trivially, thus each  $C_n$  is a  $\pi_0$ -module where  $\pi_0 = P/(\mu M \cdot \mu' N)$ .
- (v) The homomorphisms  $\partial_n$  are  $\pi_0$ -module homomorphisms for  $n \geq 3$ .

This last condition does make sense since the axioms for crossed squares from [31] imply that  $\text{Ker } \mu' \cap \text{Ker } \mu$  is a  $\pi_0$ -module.

This uses the original definition of crossed square given in [31], so is given here just for motivation for a later notion, not for detailed use in this paper.

A squared complex is thus a crossed square with a ‘tail’ attached. If we replace the crossed square by a 2-crossed module, in the sense of Conduché, [19], then we would have what is called a ‘2-crossed complex’; see, for instance, [18]. A variant of that notion is that of quadratic complex, *cf.*, Baues, [3].

We are now ready to give the generalisation to all dimensions.

### 3. Crossed $\mathbb{N}$ -cubes, and crossed $n$ -cube complexes.

We have seen how crossed modules generalised to crossed complexes, which have a better structured homotopical and homological algebra. We have seen, briefly in the previous section, how crossed squares generalise to give squared complexes in the sense of Ellis.

We end this sequence of ideas by looking at a simple theoretical treatment of a generalisation of both crossed complexes, and crossed  $n$ -cubes and one including squared complexes. These gadgets are related to the “Moore chain complexes of order  $(n + 1)$  of a simplicial group”, as briefly studied by Baues in [3], Chapter IV, Appendix B, but have some of the advantages of crossed squares over 2-crossed modules, namely they can be ‘non-symmetric’, and hence are easily specified by, say, an ‘inclusion crossed  $n$ -cube’ consisting of a simplicial group and  $n$  simplicial normal subgroups, which may not be distinct. This allows for extra freedom in constructions. Also the axioms are very much simpler!

The definition of a crossed  $n$ -cube involves the set  $\langle n \rangle = \{1, 2, \dots, n\}$ . One obvious way to extend this, eliminating dependence on  $n$ , is to try replacing  $\langle n \rangle$  by  $\mathbb{N} = \{1, 2, \dots\}$ , and specifying that the subsets  $A, B, C$ , in that definition must be finite, a condition previously automatic. This gives the notion of a crossed  $\mathbb{N}$ -cube.

If we write  $A \subset_{fin} \mathbb{N}$  as a shorthand for ‘ $A \subset \mathbb{N}$  with  $A$  finite’, we can replace  $A \subseteq \langle n \rangle$ , everywhere in the definition. We will not repeat the 11 axioms from earlier.

**Definitions:** A *crossed  $\mathbb{N}$ -cube*,  $\mathcal{M}$ , is a family of groups,  $\{M_A : A \subset_{fin} \mathbb{N}\}$ , together with homomorphisms,  $\mu_i : M_A \rightarrow M_{A-\{i\}}$ , ( $i \in \mathbb{N}$ ,  $A \subset_{fin} \mathbb{N}$ ), and functions,  $h : M_A \times M_B \rightarrow M_{A \cup B}$ , ( $A, B \subset_{fin} \mathbb{N}$ ), such that, if  ${}^ab$  denotes  $h(a, b)b$  for  $a \in M_A$  and  $b \in M_B$  with  $A \subseteq B$ , then for  $a, a' \in M_A$ ,  $b, b' \in M_B$ ,  $c \in M_C$  and  $i, j \in \mathbb{N}$ , the 11 axioms given above hold (with the evident adaptations).

Morphisms of crossed  $\mathbb{N}$ -cubes are defined in the obvious way, and give a category, which will be denoted  $Crs^{\mathbb{N}}$ . We will be looking at various reflective subcategories in  $Crs^{\mathbb{N}}$ , and the left adjoints to the inclusion functors in each case.

**Examples:** (i) As, for any  $n$ ,  $\langle n \rangle \subset \mathbb{N}$ , if  $M$  is a crossed  $n$ -cube, then we can extend it trivially to a crossed  $\mathbb{N}$ -cube by defining  $M_A = M_A$  if  $A \subseteq \langle n \rangle$ , and  $M_A = 1$  otherwise. The  $h$ -maps,  $M_A \times M_B \rightarrow M_{A \cup B}$ , are then clearly determined by those of the original crossed  $n$ -cube.

(ii) Suppose  $M = \{M_A, \mu_i, h\}$  is a crossed  $\mathbb{N}$ -cube, which is such that  $M_A$  is trivial unless  $A$  is of form  $\langle n \rangle$  for some  $n$ , (where we interpret  $\emptyset$  as being  $\langle 0 \rangle$ , so that  $M_\emptyset$  is not required to be trivial). We will write  $C_n = M_{\langle n \rangle}$  and  $\partial_n : C_n \rightarrow C_{n-1}$  for the morphism  $\mu_n : M_{\langle n \rangle} \rightarrow M_{\langle n-1 \rangle}$ .

We note that  $\partial_{n-1}\partial_n$  is trivial as it factorises via the group,  $M_A$ , where  $A = \langle n \rangle - \{n-1\}$ , which means that  $M_A = 1$ , as in

$$\begin{array}{ccc} M_{\langle n \rangle} & \longrightarrow & M_{\langle n-1 \rangle} \\ \downarrow & & \downarrow \\ M_A & \longrightarrow & M_{\langle n-2 \rangle}. \end{array}$$

We thus have that  $(C_n, \partial_n)_{n \in \mathbb{N}}$  is a chain complex of groups.

There are a pairings,

$$C_0 \times C_n \rightarrow C_n,$$

given by  $h : M_\emptyset \times M_{\langle n \rangle} \rightarrow M_{\langle n \rangle}$ , and thus an action

$${}^a b = h(a, b)b,$$

whilst  $\partial({}^a b) = {}^a \partial b$ , since  $\mu_n h(a, b) = h(\mu_n a, \mu_n b)$ , which is  $h(a, \mu_n b)$ , since, clearly,  $n \notin \emptyset$ .

The map  $\partial_1 : C_1 \rightarrow C_0$  is a crossed module by exactly the proof that a crossed 1-cube is a crossed module.

If  $a = \partial_1 b$ , then, for  $c \in C_n$ ,  $n \geq 2$ ,

$$\begin{aligned} {}^a c &= h(\partial_1 b, c)c \\ &= h(b, \mu_1 c)c, \end{aligned}$$

since  $1 \in \langle 1 \rangle \cap \langle n \rangle$ , but  $\mu_1 c \in M_{\langle n \rangle - \{1\}}$ , the trivial group, so

$${}^a c = c.$$

**Proposition 3.1.** *Given a crossed  $\mathbb{N}$ -cube such that  $M_A$  is trivial unless  $A$  is of form  $\langle n \rangle$  for some  $n$ , then the above construction gives the resulting chain complex of groups,  $(C_n, \partial_n)$ , has a natural crossed complex structure.*

*Conversely any crossed complex gives a crossed  $\mathbb{N}$ -cube in the evident way.*

**Sketch proof:** We will not systematically check *all* the axioms, here, but clearly from what has been said above,  $(C_n, \partial)$  is a crossed complex. (The detailed checking is left to the reader.) It is then fairly routine, given a crossed complex to construct the corresponding crossed  $\mathbb{N}$ -cube.  $\square$

These examples show that both crossed  $n$ -cubes, for any  $n$ , and crossed complexes are examples of crossed  $\mathbb{N}$ -cubes.

We will quietly abuse terminology, and may use the same symbol for a crossed complex as usually defined and for the corresponding crossed  $\mathbb{N}$ -cube. The inclusion of  $Crs$  into  $Crs^{\mathbb{N}}$  has a left adjoint as will become clear a bit later on.

An obvious question, given our previous discussion, is to try to put Ellis' squared complex in the same framework. There is an obvious method to try

out, and it works. One takes  $M_A = 1$  unless  $A = \langle n \rangle$  for some  $n \in \mathbb{N}$  or if  $A \subseteq \langle 2 \rangle$ . This does it, but it also indicates an effective way of encoding higher dimensional analogues of these squared complexes. We make the following definition.

**Definition:** For a given  $n$ , a *crossed  $n$ -cube complex* is a crossed  $\mathbb{N}$ -cube,  $\mathcal{M}$ , satisfying  $M_A = 1$  unless  $A = \langle m \rangle$  for some  $m \in \mathbb{N}$  or if  $A \subseteq \langle n \rangle$  for the given  $n$ .

We have a full subcategory,  $Crs^n.Comp$ , of the category of crossed  $\mathbb{N}$ -cubes specified by these crossed  $n$ -cube complexes. For instance,  $Crs^1.Comp$  becomes an alternative notation for the category of crossed complexes.

If  $\mathcal{M}$  is a crossed  $n$ -cube complex, then the groups and morphisms,  $M_{\langle m \rangle}$  and  $\mu_m$ , for  $m > n$ , form what will be loosely called the ‘tail’ of  $\mathcal{M}$ .

## 4. From simplicial groups to crossed $n$ -cube complexes

To show how these gadgets relate to the more usual models of homotopy types, we will show how to obtain a crossed  $n$ -cube complex from a simplicial group,  $G$ . To do this we will analyse the constructions giving crossed complexes and crossed square complexes.

### 4.1 From simplicial groups to crossed complexes

We give the formulae, which can, for instance, be found in [23, 24], or in relevant nLab pages, [40]. We first set up some notation.

In what follows,  $D_n$  denotes the subgroup of  $G_n$  generated by the degenerate elements in that dimension, whilst  $NG$  is the *Moore complex*,  $(NG, \partial)$ , of  $G$ , *i.e.*, the chain complex defined by

$$NG_n = \bigcap_{i=1}^n \text{Ker } d_i^n$$

with  $\partial_n : NG_n \rightarrow NG_{n-1}$  induced from  $d_0^n$  by restriction.

For crossed complexes, one used the relative homotopy groups of  $G$ . This gives a crossed complex,  $C(G)$ , with

$$\mathbb{C}(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

and with the base crossed module being

$$\frac{NG_1}{(NG_1 \cap D_1)d_0(NG_2 \cap D_2)} \rightarrow G_0,$$

but note that, in fact,  $NG_1 \cap D_1 = 1$ , since  $D_1$  is generated by the  $s_0(g)$  with  $g \in G_0$ .

For an arbitrary simplicial group,  $H$ , the usual associated crossed module, written  $M(H, 1)$ , is given by

$$\frac{NH_1}{d_0(NH_2)} \rightarrow H_0,$$

so, taking  $H = sk_1G$  to be the 1-skeleton of  $G$ , the base crossed module of  $\mathbb{C}(G)$  is  $M(sk_1G, 1)$ , as  $N(sk_1G)_2 = NG_2 \cap D_2$ .

As we will occasionally refer to the idea of an  $n$ -skeleton of a simplicial group,  $G$ , as here, but for various  $n$ , we should clarify precisely what this is, as there are two related, but slightly different meanings.

The one that we will use takes as the  $n$ -skeleton,  $sk_nG$ , of  $G$  to be the sub simplicial group of  $G$  generated by the group,  $G_n$ , of  $n$ -simplices of  $G$ . It thus includes all simplices of lower dimension than  $n$ , but in dimension  $n+1$ , there will be only elements that can be written as a product of elements, which are degenerate images of simplices of dimension  $\leq n$ . These form a subgroup,  $D_{n+1}$ , of  $G_{n+1}$ . We will rarely need to consider higher dimensions than that, so do not attempt to give a useful description of  $sk_n$  in those higher dimensions.

The other, more categorical form corresponds to an adjunction with a truncation functor and constructs the higher dimensions in as free a way as possible. There is a natural morphism onto the simpler algebraic one, but that does not seem to be a monomorphism in general.

## 4.2 ... and to crossed $n$ -cubes

For a crossed  $n$ -cube, the detailed formulae are given in [42], page 10, or in [41]. They lead to a crossed  $n$ -cube which, as there, will be denoted

$M(G, n)$ . We used this, in the last section, in a special case. The formulae are essentially as in Proposition 4.2, below, except that all the subsets  $A$  are subsets of  $\langle n \rangle$ , so several of the clauses do not apply.

Before we go ‘beyond’ finite  $n$ , we note a result (Proposition 9, page 12) from [42] for later use.

**Proposition 4.1.** *Given a simplicial group,  $G$ , and  $n \geq 1$  there is an exact sequence of crossed  $(n - 1)$ -cubes,*

$$1 \rightarrow K \rightarrow M(G, n)_1 \xrightarrow{\mu_1} M(G, n)_0 \rightarrow M(G, n - 1) \rightarrow 1,$$

where, for  $B \subseteq \langle n \rangle$ , but  $B \neq \langle n \rangle$ ,  $K_B = 1$ , the trivial group, whilst  $K_{\langle n-1 \rangle} \cong \pi_n(G)$ , the  $n^{\text{th}}$  homotopy group of  $G$ .  $\square$

### 4.3 ... and, finally, beyond

The analogous formula for crossed  $n$ -cube complexes is given as follows, which is both a proposition and a definition.

**Proposition 4.2.** *Let  $G$  be a simplicial group and  $n \in \mathbb{N}$ .*

*Define a family  $M_A$ ,  $A \subset \mathbb{N}$ , with  $A$  finite, by*

*(i) if  $A = \langle m \rangle$  and  $m > n$ , then*

$$M_A = \frac{NG_m}{(NG_m \cap D_m)d_0(NG_{m+1} \cap D_{m+1})};$$

*(ii) if  $A \subseteq \langle n \rangle$ ,*

$$\begin{aligned} M_A &= M(sk_n G, n)_A \\ &= \frac{\bigcap \{Ker d_j^n : j \in A\}}{d_0(Ker d_1^{n+1} \cap \bigcap \{Ker d_{j+1}^{n+1} : j \in A\} \cap D_{n+1})}; \end{aligned}$$

*(iii) if  $A$  is otherwise, then  $M_A$  is trivial.*

*Further define  $\mu_i : M_A \rightarrow M_{A-\{i\}}$  by*

*(iv) if  $i \notin A$ , then  $\mu_i$  is the identity morphism;*

*(v) if  $A = \langle m \rangle$ , with  $m > n$  and  $i = m$ , then  $\mu_m$  is induced by  $d_0$ , and is trivial if  $i \neq m$ ;*

(vi) if  $A \subseteq \langle n \rangle$ , then  $\mu_i$  is induced by the inclusions of intersections (i.e., as in  $M(sk_n G, n)$ );

(vii) otherwise  $\mu_i$  is trivial.

Finally define  $h : M_A \times M_B \rightarrow M_{A \cup B}$  by

(viii) if  $A = \emptyset$  and  $B = \langle m \rangle$  with  $m > n$  then, as  $M_\emptyset = G_{n-1}$  and  $M_B = C(G)_m$ , if  $a \in M_\emptyset$  and  $b \in M_B$ ,

$$h(a, b) = [s_0^{m-n+1}(a), b] \in M_B;$$

similarly if  $A = \langle m \rangle$  and  $B = \emptyset$ ;

(ix) if  $A, B \subseteq \langle n \rangle$ ,  $h$  is defined as in  $M(sk_n G, n)$ ;

(x) otherwise  $h$  is trivial.

This data defines a crossed  $\mathbb{N}$ -cube, which is, in fact, a crossed  $n$ -cube complex.

**Proof:** Much of this can be safely ‘left to the reader’. It uses results from [42]. Note that (viii) and (x) effectively say that it is only the  $s_0^{n-1}G_0$  part of  $G_{n-1}$  that acts on any  $M_{\langle m \rangle}$  for  $m > n$ , and even then the image of  $d_0 : NG_1 \rightarrow G_0$  acts trivially. To see this note that any  $a \in G_{n-1} = M_\emptyset$ , which is in some  $\text{Ker } d_i$ , is in the image of some  $\mu_i$ , hence  $a = \mu_i x$  say, but then

$$h(a, b) = h(\mu_i x, b) = h(x, \mu_i b) = 1,$$

by necessity if the structure is to be a crossed  $\mathbb{N}$ -cube, thus to check that the  $h$ -maps, and, in particular, those involved with part (viii) of the definition, satisfy the axioms, it suffices to use the methods for checking that  $C(G)$  was a crossed complex.  $\square$

We will denote this crossed  $n$ -cube complex by  $C(G, n)$ , as it combines both the technology of the  $M(G, n)$  and the construction of  $C(G)$ . These models have yet to be fully explored, but see [43] and below for some preliminary results.

#### 4.4 How does $C(G, n)$ compare with $C(G)$ and $M(G, n)$ ?

For  $n = 1$ , we have

$$C(G, 1)_{\langle 1 \rangle} \cong C(G)_1 = \frac{NG_1}{d_0(NG_1 \cap D_1)},$$

and, for  $C(G, 1)_{\langle m \rangle}$ , it is by definition  $C(G)_m$ . As the boundaries are all induced from  $d_0$ , we have

**Proposition 4.3.** *For any simplicial group,  $G$ ,  $C(G, 1)$  is the crossed 1-cube complex corresponding to  $C(G)$ .  $\square$*

More general comparison between  $C(G, n)$  and  $C(G)$  will be given in the next section. The importance of  $C(G)$  is that it has a quotient which is the simplicial group analogue of the complex of chains on the universal cover. This is a chain complex of modules over  $\pi_0(G)$ , the group of connected components of  $G$ . In the case that  $G = G(K)$ , the simplicial loop group on a reduced simplicial set,  $K$ , this is exactly the usual chain complex of  $\pi_1(K)$ -modules coming from the universal cover of  $K$ .

The crossed  $n$ -cube complex,  $C(G, n)$ , also includes the information on the  $n$ -type of  $G$ , since one can easily obtain  $M(G, n)$  from it, and that  $n$ -cube represents the homotopy  $n$ -type of  $G$  by Loday's theorem, [36], in the version given in [42].

**Proposition 4.4.** *Considering the crossed  $n$ -cube as a crossed  $n$ -cube complex, there is a natural epimorphism,  $C(G, n) \rightarrow M(G, n)$ , which universally trivialises the tail of  $C(G, n)$ .*

**Proof:** We look at the group  $C(G, n)_{\langle n \rangle}$ , which is the last group before the tail of  $C(G, n)$ , and also at  $C(G, n)_{\langle n+1 \rangle}$ , which is isomorphic to  $C(G)_{n+1}$ , the first group in that tail. We will also need the map,

$$\mu_{n+1} : C(G, n)_{\langle n+1 \rangle} \rightarrow C(G, n)_{\langle n \rangle}.$$

We have

$$C(G)_{n+1} = \frac{NG_{n+1}}{(NG_{n+1} \cap D_{n+1})d_0(NG_{n+2} \cap D_{n+2})},$$

whilst

$$\begin{aligned} C(G, n)_{\langle n \rangle} &= M(sk_n G, n)_{\langle n \rangle} \\ &= \frac{NG_n}{d_0(NG_{n+1} \cap D_{n+1})}. \end{aligned}$$

We look at the cokernel of  $\mu_{n+1}$  and compare it with  $M(G, n)_{\langle n \rangle}$ . They are the same, namely

$$M(G, n)_{\langle n \rangle} = \frac{NG_n}{d_0(NG_{n+1})}.$$

We get a map of crossed  $n$ -cube complexes,

$$C(G, n) \rightarrow M(G, n),$$

which kills off the tail. This is clearly an epimorphism and its universal property is easy to check.  $\square$

#### 4.5 From $n$ to $n - 1$ : collecting up some ideas and evidence

A useful feature of these crossed  $n$ -cube complexes is how simply they relate to crossed  $(n - 1)$ -cube complexes. Clearly going one way is simple as crossed  $(n - 1)$ -cube complexes are, by default, crossed  $n$ -complexes.

We noted, in Proposition 4.1, that given  $M(G, n)$ , the quotient crossed  $(n - 1)$ -cube was  $M(G, n - 1)$ . Is a similar result true here? Is there an epimorphism from  $C(G, n)$  to  $C(G, n - 1)$ ?

This is linked with another problem. We have a nested sequence of full categories of  $Crs^{\mathbb{N}}$ ,

$$Crs^1.Comp \subset Crs^2.Comp \subset \dots \subset Crs^n.Comp \subset \dots \subset Crs^{\mathbb{N}}.$$

Does the inclusion of  $Crs^{n-1}.Comp$  into  $Crs^n.Comp$  have a left adjoint? In other words, is  $Crs^{n-1}.Comp$  a reflective subcategory of  $Crs^n.Comp$ ?

This is very similar to the situation described in Bullejos, Faro and García-Muñoz, [15], page 240, where the objects of special interest were crossed complexes. In their case, the inclusions all have left adjoints. Replacing the inclusions by their adjoints would seem to give a tower of categories in the sense used by Baues in [2, 3]. It is therefore of some interest to see if here, also, those functors have left adjoints with nice properties.

We investigate this question first for  $n = 2$ , as this is, at the same time, easiest to see, is one of the most useful cases and also indicates how to proceed in general.

We have seen an example<sup>4</sup> of an inclusion crossed square with a 2-truncated crossed complex viewed as a crossed square. We can now do away with the restriction that the image be truncated.

Crossed square complexes can be neatly viewed as

$$C := \left( \begin{array}{ccccc} \dots & \longrightarrow & C_3 & \xrightarrow{\mu_3} & C_{\langle 2 \rangle} & \xrightarrow{\mu_2} & C_{\langle 1 \rangle} \\ & & & \mu_1 \downarrow & & & \downarrow \mu_1 \\ & & & C_{\{2\}} & \xrightarrow{\mu_2} & & C_\emptyset \end{array} \right),$$

whilst those corresponding to crossed complexes look like

$$D := \left( \begin{array}{ccccc} \dots & \longrightarrow & D_3 & \xrightarrow{\mu_3} & D_{\langle 2 \rangle} & \xrightarrow{\mu_2} & D_{\langle 1 \rangle} \\ & & & \mu_1 \downarrow & & & \downarrow \mu_1 \\ & & & 1 & \xrightarrow{\mu_2} & & D_\emptyset \end{array} \right).$$

A map,  $\varphi$ , in  $Crs^2.Comp$  from  $C$  to  $D$ , clearly, must kill off  $C_{\{2\}}$  and hence must also kill off  $\mu_2(C_{\{2\}})$ , which is normal in  $C_\emptyset$ . That is not all however. If  $a \in C_{\{2\}}$ ,  $b \in C_{\{1\}}$  or  $C_{\langle 2 \rangle}$ , then

$$\varphi(h(a, b)) = h(\varphi a, \varphi b) = 1,$$

as  $\varphi a = 1$ , thus  $\varphi$  must kill off the action of  $C_{\{2\}}$  on  $C_{\langle 2 \rangle}$ , and thus all elements of the form,  $h(a, b)$ , with  $a \in C_{\{2\}}$ ,  $b \in C_{\{1\}}$  or  $C_{\langle 2 \rangle}$ . As the image of  $C_2$  in  $C_\emptyset$  has been killed off, the action of that part of  $C_\emptyset$  on  $C_{\langle 1 \rangle}$  must be nullified as was the case with the inclusion crossed square that we considered earlier.

In general, we have:

**Proposition 4.5.** *The inclusion of  $Crs^1.Comp$  into  $Crs^2.Comp$  has a left adjoint,*

$$L : Crs^2.Comp \rightarrow Crs^1.Comp.$$

*The left adjoint is a reflection, fixing the objects of the subcategory.*

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<sup>4</sup>at the end of section 2.3,

**Proof:** If  $C$  is in  $Crs^2.Comp$ , define  $L(C)$  as follows:

- if  $A \not\subseteq \langle 2 \rangle$ ,  $L(C)_A = C_A$ ;
- if  $A = \langle 2 \rangle$ , define  $L(C)_{\langle 2 \rangle} = C_{\langle 2 \rangle}/H$ , where  $H$  is the normal subgroup generated by all  $h(a, c)$ ,  $a \in C_{\{2\}}$ ,  $b \in C_{\{1\}}$ , (normality is, in fact, automatic);
- if  $A = \{2\}$ ,  $L(C)_A = 1$ ;
- if  $A = \langle 1 \rangle$ ,  $L(C)_A = C_{\langle 1 \rangle}/K$ , where  $K$  is the normal subgroup generated by all  $h(\mu_2 a, b)$  with  $a \in C_{\{2\}}$ ,  $b \in C_{\langle 1 \rangle}$ .

The  $\mu_i$  and  $h$ -maps are induced from those of  $C$ .

It is clear from our previous discussion that a morphism,  $\varphi$ , from  $C$  to some  $D$  in  $Crs^1.Comp$  must factor through the various groups of the family,  $L(C)$ , as specified. What remains to be checked is that  $L(C)$  is a crossed  $\mathbb{N}$ -cube and hence is in  $Crs^1.Comp$ .

The  $\mu_i$  are induced from those of  $C$ .

If  $a \in C_{\{2\}}$ ,  $b \in C_{\langle 1 \rangle}$ , then  $\mu_2 h(a, b) = h(\mu_2 a, b)$ , and so is in  $K$ .

If  $a \in C_{\{2\}}$ ,  $b \in C_{\langle 2 \rangle}$ , then  $\mu_2 h(a, b) = h(\mu_2 a, \mu_2 b)$ , and so is in  $K$ .

We thus have that  $\mu_2 : L(C)_{\langle 2 \rangle} \rightarrow L(C)_{\langle 1 \rangle}$  is well defined.

Similarly, it is fairly routine to check that the induced  $h$ -maps here are well defined. The verification, then, of the axioms is straightforward, given the fact that the axioms are satisfied by the original structure in  $C$ .  $\square$

It is clear that  $L$  is a functor. It is the first of a series of reflections on the higher inclusions and, if we need to emphasise which one it is, we may write it as  $L_1^2$ .

#### 4.6 From $C(G, 2)$ to $C(G, 1)$

What happens if we apply  $L$  to  $C(G, 2)$ ? The answer is not that much of a surprise!

**Proposition 4.6.** *If  $G$  is a simplicial group, then there is a natural isomorphism,*

$$L(C(G, 2)) \cong C(G, 1).$$

(Of course, the ‘crossed 1-cube complex’  $C(G, 1)$  is just the crossed complex  $C(G)$  under another name.)

**Proof:** As, for  $A \not\subseteq \langle 2 \rangle$ ,  $L(C)_A = C_A$ , we may as well assume that  $G$  is 2-truncated, so that  $NG_3$  is trivial. The general case is obtained by using the 2-truncation functor or more simply by ‘observation’!

The crossed square complex,  $C(G, 2)$ , is now just  $M(G, 2)$  and up to isomorphism, this is:

$$\begin{array}{ccc} NG_2 & \xrightarrow{\mu_2} & Ker d_0 \\ \downarrow \mu_1 & & \downarrow \mu_1 \\ Ker d_1 & \xrightarrow{\mu_2} & G_1 \end{array}$$

where the  $\mu_i$  with domain  $NG_2$  are both induced by  $d_0$  and those with codomain  $G_1$  are the inclusions. The  $h$ -maps are defined using the isomorphisms coming from  $\mu_1$  and  $\mu_2$  of  $M(G, 2)$ . We have

$$\begin{aligned} Ker d_0 &\cong \frac{Ker d_1^2}{d_0(Ker d_1^3 \cap Ker d_2^3)}, \\ Ker d_1 &\cong \frac{Ker d_2^2}{d_0(Ker d_1^3 \cap Ker d_3^3)}, \\ G_1 &\cong \frac{G_2}{d_0(Ker d_1^3)}. \end{aligned}$$

Writing  $\bar{x}$  for the element of  $G_1$ , *etc.*, determined by  $x \in G_2$  via these isomorphisms, this gives  $h(\bar{x}, \bar{y}) = \overline{[x, y]}$ .

We now have to examine the subgroups  $H$  and  $K$  defined in the description of the functor  $L$ .

We note that if  $x \in Ker d_1^2$ , then either (i)  $x \in NG_2$ , (ii)  $x = s_1 a$  for some  $a \in NG_1$  or (iii)  $x$  is a product of elements of type (i) and (ii). Similarly,  $y \in Ker d_2^2$  is either in  $NG_2$ , or in  $s_0 NG_1$ , or is a product of elements from these subgroups. Because of this and the commutator identities, it suffices to examine those  $h(\bar{x}, \bar{y})$ , where  $x, y$  have each of the forms (i) or (ii) in turn, as the third form can be reduced to a product of the first two.

With our assumption that  $NG_3 = 1$ , we get, by a simple calculation, that if  $x \in Ker d_1$  and  $y \in NG_2$ , then

$$[x, y] = [s_0 d_0 x, s_1 d_0 y^{-1} \cdot s_0 d_0 y] \in NG_2 \cap D_2.$$

If, on the other hand,  $x \in NG_2$ , and  $y \in \text{Ker } d_2$ , then calculating  $[y, x]$  using that same sort of calculation, gives a similar result. Finally if both  $x$  and  $y$  are degenerate, then  $[x, y] \in NG_2 \cap D_2$ . The subgroup,  $H$ , generated by all  $h(\bar{x}, \bar{y})$  is, thus, contained in  $NG_2 \cap D_2$ .

We next use the Brown-Loday lemma (Lemma 5.7 of [13]). This can be rephrased to show that  $NG_2 \cap D_2$  is generated by the commutators of the form  $[s_0x, s_1y.s_0y^{-1}]$  with  $x, y \in NG_1$ , however then  $s_0x \in \text{Ker } d_2^2$  and  $s_1y.s_0y^{-1} \in \text{Ker } d_1^2$ , so this is  $h(s_0x, s_1y.s_0y^{-1})$  and is in  $H$ , thus  $H = NG_2 \cap D_2$ .

The identification of the subgroup,  $K$ , is simpler as, in the Brown-Loday lemma, used above,  $d_0(NG_2 \cap D_2) = [\text{Ker } d_0, \text{Ker } d_1] = K$ . Collecting things up, we find the result that  $L(C(G, 2))$  is exactly  $C(G, 1)$ .  $\square$

These are special cases of more general results that we will see shortly, but in the next section we explore crossed  $n$ -cube complexes in their own right.

## 5. Crossed $n$ -cube complexes

We now turn to the general case, trying to tease out some of the elementary properties of these new gadgets, the crossed  $n$ -cube complexes.

### 5.1 More on the tail

First a proposition:

**Proposition 5.1.** *Let  $\mathcal{M} = (M_A)$  be a crossed  $n$ -cube complex, then*

- (i) *if  $m > n$ ,  $M_{\langle m \rangle}$  is abelian and  $\mu_m \mu_{m+1} = 1$ ,*
- (ii) *the group  $M_\emptyset$  acts on all the  $M_{\langle m \rangle}$ ,  $m > n$ , in such a way that each of the normal subgroups,  $Im \mu_i$ , acts trivially on  $M_{\langle m \rangle}$ , if  $1 \leq i \leq n$ .*
- (iii) *All  $M_A$ , for which  $A \subseteq \langle n \rangle$ , and  $A \neq \emptyset$ , act trivially on  $M_{\langle m \rangle}$  for  $m > n$ .*

**Proof:** For (i): if  $a, b \in M_{\langle m \rangle}$ , then

$$[a, b] = h(a, b) = h(\mu_{m-1}a, b) = 1$$

since  $\{1, \dots, m-2, m\}$  is not a down segment, nor is it within  $\langle n \rangle$ .

To see (ii), it is clear that the  $h$ -map,

$$h : M_\emptyset \times M_{\langle m \rangle} \longrightarrow M_{\langle m \rangle},$$

defines an action:  ${}^a b = h(a, b)b$ , so

$${}^{aa'} b = h(aa', b)b = {}^a h(a', b) h(a, b)b,$$

whilst

$${}^a (a' b) = {}^a (h(a', b)b) = {}^a h(a', b) {}^a b,$$

which are clearly equal. If, on the other hand,  $a = \mu_i a'$ , we have

$${}^a b = h(\mu_i a', b)b = h(a', \mu_i b)b = b$$

since  $\mu_i : M_{\langle m \rangle} \rightarrow 1$ .

For (iii), similarly if  $a \in M_A$ ,  $b \in M_{\langle m \rangle}$ , and  $i \in A \subseteq \langle n \rangle$ , then again  $\mu_i b = 1$ , but  $h(a, b) = h(a, \mu_i b) = 1$ .  $\square$

**Remark:** In [42], a sort of ‘total quotient’ of a crossed  $n$ -cube was introduced. This is

$$Q(\mathcal{M}) = M_\emptyset / \prod_{i=1}^n \text{Im } \mu_i(M_{\{i\}}).$$

It is a substitute for  $\pi_0$ , but does not depend on the crossed  $n$ -cube being derived from a simplicial group. The above proposition shows that if  $\mathcal{M}$  is a crossed  $n$ -cube complex, with  $\mathcal{M}'$ , the crossed  $n$ -cube at its base, then

$$\cdots \rightarrow M_{\langle m \rangle} \rightarrow M_{\langle m-1 \rangle} \rightarrow \cdots \rightarrow M_{\langle n+1 \rangle}$$

is a chain complex of  $Q(\mathcal{M}')$ -modules.

This reinforces the intuition that a crossed  $n$ -cube complex consists of the crossed  $n$ -cube,  $\mathcal{M}'$ , with a chain complex of  $Q(\mathcal{M}')$ -modules attached to it.

## 5.2 The reflections

As was said earlier, the category,  $Crs^n.Comp$ , of crossed  $n$ -cube complexes is a full subcategory of  $Crs^{\mathbb{N}}$ , the category of crossed  $\mathbb{N}$ -cubes in such a way that if  $n < m$ , we have full inclusions,

$$Crs^n.Comp \subset Crs^m.Comp \subset Crs^{\mathbb{N}}.$$

Each subcategory is determined merely by specifying that certain positions of a  $\mathbb{N}$ -cube are trivial. By general intuitions on algebraic theories, this suggests, in particular, that the inclusions,

$$Crs^n.Comp \hookrightarrow Crs^{n+1}.Comp,$$

should each have a left adjoint,  $L_n^{n+1}$ . This left adjoint will be explicitly given shortly. The important point is to note that if, as claimed, crossed  $n$ -cube complexes form ever better algebraic approximations to integral homotopy types, with increasing  $n$ , the study of the passage from crossed  $(n+1)$ -cube complexes to crossed  $n$ -cube complexes via this adjunction could yield important information as to where exactly in the homotopy type the ‘lost information’ resides when we pass from one stage to the next. This will be especially so when studying the various functors from simplicial groups to these crossed models.

## 5.3 The functor, $L_n^{n+1} : Crs^{n+1}.Comp \rightarrow Crs^n.Comp$ .

We take  $\mathcal{M} = (M_A)$  to be a crossed  $(n+1)$ -cube complex and will write, for short,  $\mathcal{L} := L_n^{n+1}(\mathcal{M}) = (F_A)$  for the corresponding crossed  $n$ -cube complex. The form of the two structures makes it clear how to define  $F_A$  for most  $A$  :

- if  $A \not\subseteq \langle n \rangle$  and  $A$  is not a down segment,  $F_A = 1$ ,
- if  $A = \langle m \rangle$ ,  $n+1 < m$ ,  $F_A = M_A$ ,
- if  $A \subset \langle n \rangle$ , so  $A \neq \langle n \rangle$ , then  $F_A = M_A / \text{Im } \mu_{n+1}$ ,
- for  $A = \langle n \rangle$ , let  $N_{\langle n \rangle}$  be the normal subgroup of  $M_{\langle n \rangle}$  generated by all the elements  $\mu_{n+1}h(b, c)$ , where  $\{B, C\}$  forms a non-trivial partition of  $\langle n+1 \rangle$ , so  $C \neq \emptyset$ , with  $n+1 \in B$ ,  $b \in M_B$ , and  $c \in M_C$ , then  $F_{\langle n \rangle} = M_{\langle n \rangle} / N_{\langle n \rangle}$ .

- for  $A = \langle n+1 \rangle$ , let  $N_{\langle n+1 \rangle}$  be the normal subgroup of  $M_{\langle n+1 \rangle}$  generated by all elements,  $h(b, c)$ ,  $b \in M_B$ ,  $c \in M_C$ , where  $\{B, C\}$  is as before (above), then  $F_{\langle n+1 \rangle} = M_{\langle n+1 \rangle} / N_{\langle n+1 \rangle}$ .

The verification that the obvious definitions of  $\mu_i$  and  $h$ -maps give  $\mathcal{L}$  the structure of a crossed  $n$ -cube complex is now quite easy.

**Proposition 5.2.** *The structure,  $\mathcal{L}$ , is a crossed  $n$ -cube complex and the natural quotient,  $\eta : \mathcal{M} \rightarrow \mathcal{L}$ , a map of crossed  $(n+1)$ -cube complexes. These have the following universal property:*

*Given any map,  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ , of crossed  $(n+1)$ -cube complexes, where  $\mathcal{M}'$  is in  $\text{Crs}^n.\text{Comp}$ , there is a unique factorisation in the form*

$$\mathcal{M} \xrightarrow{\eta} \mathcal{L} \xrightarrow{\varphi'} \mathcal{M}'$$

where  $\eta$  is the canonical quotient map. As a consequence, the functor,  $\mathbb{L}_n^{n+1}$  is left adjoint to the inclusion of  $\text{Crs}^n.\text{Comp}$  into  $\text{Crs}^{n+1}.\text{Comp}$ .

**Remark:** We note that if we take the  $(n+1)$ -cube determined by  $\{F_A : A \subseteq \langle n+1 \rangle\}$ , the cokernel of  $\mu_{n+1}$  is exactly the same as that for  $\{M_A : A \subseteq \langle n+1 \rangle\}$ . In fact, the only case that is not immediate is that of the cokernel of  $\mu_{n+1} : F_{\langle n+1 \rangle} \rightarrow F_{\langle n \rangle}$ . This is induced by the  $\mu_{n+1}$  of  $\mathcal{M}$  on the quotients,

$$\frac{M_{\langle n+1 \rangle}}{N_{\langle n+1 \rangle}} \rightarrow \frac{M_{\langle n \rangle}}{N_{\langle n \rangle}},$$

but  $\mu_{n+1}N_{\langle n+1 \rangle} = N_{\langle n \rangle}$ , so the quotient is  $M_{\langle n \rangle} / \text{Im } \mu_{n+1}$  as claimed.

### Proof of Proposition 5.2:

The verification of the first part is routine, and so will be omitted.

Suppose given  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$  with  $\mathcal{M}'$  a crossed  $n$ -cube complex. As  $\mathcal{M}'_A = 1$  if  $A \not\subseteq \langle n \rangle$  and  $A$  is not a down segment, then  $\text{Ker } \varphi_A = M_A$  whatever  $\varphi$  is. Similar comments show that if  $A = \langle m \rangle$ , with  $m > n+1$ , then  $\varphi_A$  has no special properties and if  $A \subseteq \langle n \rangle$ ,  $A \neq \langle n \rangle$ , then  $\text{Ker } \varphi_A$  must contain  $\text{Im } \mu_{n+1}$ , since  $\varphi_A \mu_{n+1} a$  for  $a \in M_{A \cup \{n+1\}}$  must be the identity as  $\varphi(a) \in M'_{A \cup \{n+1\}}$ , which is itself zero.

There thus remain only the justification of the statements relative to  $\langle n+1 \rangle$  and  $\langle n \rangle$ . If  $A \cup B = \langle n+1 \rangle$  with  $A \cap B = \emptyset$ , but  $B \neq \emptyset$  and  $n+1 \in A$ , then

$M'_A = 1$ , so  $\varphi_{\langle n+1 \rangle} h(a, b) = 1$  for  $a \in M_A$ ,  $b \in M_B$ , thus  $\text{Ker } \varphi$  will always contain the subgroup denoted  $N_{\langle n+1 \rangle}$  in the statement of the proposition.

Similarly  $\varphi_{\langle n \rangle} \mu_{n+1} h(a, b) = 1$ , as it is  $\mu_{n+1} \varphi_{\langle n+1 \rangle} h(a, b)$ , so  $N_{\langle n \rangle} \subseteq \text{Ker } \varphi_{\langle n \rangle}$ .

We have thus that  $\text{Ker } \varphi$  is always at least as big as these subgroups. As these form the kernel of the map from  $\mathcal{M}$  to  $\mathcal{L}$ , which has been verified to be a map of  $(n+1)$ -crossed cubes, we have checked both the existence and the uniqueness of the claimed factorisation.  $\square$

The same basic construction will give the left adjoint to the inclusion of  $\text{Crs}^n.\text{Comp}$  into  $\text{Crs}^{\mathbb{N}}$ , and thus by restriction to that of  $\text{Crs}^n.\text{Comp}$  into  $\text{Crs}^m.\text{Comp}$ , for  $m > n$ . Of course, that will give a chain of functors that simplify a crossed  $n$ -cube complex killing off the groups at some corners, until one arrives at a crossed complex, and finally at a chain complex of modules over the total quotient,  $Q(\mathcal{M})$ , of the initial crossed  $n$ -cube complex.

## 6. Comments, queries and some ideas for further development

In this paper, the basic idea has been to take an algebraic model for homotopy  $(n+1)$ -types, to add a ‘tail’, and then to look at how the models for different  $n$  are related algebraically and categorically. There are several different well known algebraic or categorical models for homotopy  $(n+1)$ -types. In this paper, we have concentrated on crossed  $n$ -cubes as their algebra is fairly easy to explain as it, in fact, uses fairly elementary almost classical tool, but there are other useful models. Some of these have been developed in more general contexts than the one used here.

Parallel to the paper, [42], was the paper by Bullejos, Cegarra and Duskin, [14], which used more simplicial techniques, and, as their constructions worked inductively by working in a Barr-exact category, their results are directly applicable in a more general context. They replaced the category,  $\text{Grp}$ , of groups by that,  $\text{Grp}(E)$ , of group objects internal to such a Barr-exact category,  $E$ . As the categories of simplicial groups in  $E$  and of  $\text{cat}^1$ -groups in  $E$  are also Barr-exact, they could repeat their basic observations inductively.

Their end result was a  $\text{cat}^n$ -group rather than a crossed  $n$ -cube. These are equivalent models. As a consequence of their methods being within  $E$ , however, their methods can be applied directly to other contexts. In other words, given a simplicial group object in an exact category, one can obtain a  $\text{cat}^n$ -group in that category. The same goes for the results in this paper. We will return to this setting slightly later. We could have taken the  $\text{cat}^n$ -group model instead of the crossed  $n$ -cube one, and, adding a ‘tail’ to that. This should be quite straight forward and easy to do.

We note that a closely related idea was introduced by Baues in [3]. There was defined, in section II.2, the notion of a homotopy system of order  $(n+1)$ . Such an object consisted of an  $n$ -dimensional CW-complex,  $X^n$ , a chain complex,  $C$ , of free  $\pi_1(X^n)$ -modules and a homomorphism that links  $C_{n+1}$  with  $\pi_n(X^n)$ . Later, in Appendix B in chapter IV of [3], a method is given that produces a homotopy system of order 4 from a CW-complex via a construction defined on simplicial groups that is closely related to the two methods mentioned above here. That method involves the notions of *quadratic module* and *quadratic complex*. The first of these is like a Conduché 2-crossed module, as in [19], which corresponds to the Moore complex of a 2-truncated simplicial group, but with a certain nilpotency condition added, whilst the second one is the corresponding ‘model with a tail’.

We could also have taken as the basic model *n-hypercrossed complexes* in the sense of [17] and have then added a tail. This remains an interesting area to investigate, but requires more difficult algebraic manipulations than those used here.

That brings us to  $n$ -truncated simplicial group(oid)s, more exactly back to the  $n$ -hypergroupoids that were mentioned in the introduction and which combine a truncated simplicial structure with an algebraic condition in their top dimension. These provide another good model for homotopy  $n$ -types. Their initial development was in Duskin, [22], and Glenn, [30].

We will drop back, here, into using an exact category,  $E$ , as the base category instead of  $\text{Set}$ . We will need to revamp some of the ideas from the classical theory of simplicial sets, so as to adapt them to the context of simplicial objects in  $E$ . For instance, given  $(n, k)$  with  $0 \leq k \leq n$ , and an object,  $X$ , in  $\text{Simp}(E)$ , the category of simplicial objects in  $E$ , we can form an object,  $\Lambda[n, k](X)$ , of  $(n, k)$ -horns in  $X$ . There is a natural morphism,

$$\lambda_{n,k}^X : X_n \rightarrow \Lambda[n, k](X),$$

which sends  $x$  to  $(d_0x, \dots, d_{k-1}x, -, d_{k+1}x, \dots, d_n)$ . In the case where  $E = \mathbf{Set}$ ,  $X$  is a Kan complex if, for all  $(n, k)$ , every  $(n, k)$ -horn has a filler, and that implies that the above map is an epimorphism. That latter fact gives how to extend the notion to  $\mathbf{Simp}(E)$ . One says that  $X$  satisfies the Kan condition in dimension  $n$  if  $\lambda_{n,k}^X$  is an epimorphism, and  $X$  satisfies the *exact Kan condition in dimension  $n$*  if that same map is an *isomorphism*.

Finally an  $n$ -hypergroupoid is a simplicial object in  $E$  for which the morphisms,  $\lambda_{m,k}^X$ , are epimorphisms for all  $k, m$ , so  $X$  is a Kan complex object, but, in addition, these  $\lambda_{m,k}^X$  are isomorphism for all  $m > n$ .

As Kan complexes are often thought of as  $\infty$ -groupoid,  $n$ -hypergroupoids are sometimes called ‘(weak)  $n$ -groupoids’, *e.g.*, by Getzler, [28]. Because of the link with the exact Kan condition, and the related idea of an exact fibration in dimension  $n$ , Beke, [5], uses the term *exact  $n$ -type* for the same idea.

The requirements on  $E$  can be relaxed further. Instead of using epimorphisms in the definition above, one can require that a class of special morphisms be specified, which behave sufficiently like epimorphisms. These morphisms are typically called *covers*. This idea has been used, for instance, in Henriques, [32], Zhu, [51], Behrand and Getzler, [4], and others papers mentioned in the introduction. In this theory, there is a strong link with singleton Grothendieck topologies, see Henriques, cited above, Meyer and Zhu, [38], or Rogers and Zhu, [46]. In this context, a simplicial object,  $X$ , will be an  $n$ -hypergroupoid if each of the  $\lambda_{m,k}^X$  is a cover and is an isomorphism if  $m > n$ .

How does this relate to the notions from earlier in this paper? Hopefully we can extend, as suggested, the earlier discussion to considering an  $n$ -hypergroupoid ‘with a tail’, which will related to a suitable notion of crossed  $n$ -cube complex in the wider setting of categories with covers, and thus will include Lie, topological and other geometric analogues of the results here - and with possibly very little extra work.

But what is the evidence that this is possible, say, in low dimensions and with crossed complexes?

In [39], Nan Tie, working in an arbitrary category,  $\mathbb{C}$ , with finite limits, introduced crossed complexes in  $\mathbb{C}$ ,  $T$ -complexes in  $\mathbb{C}$ , and so on, then considered *filtered* systems of hypergroupoids and  $n$ -truncated crossed complexes in  $\mathbb{C}$  showing that they led to equivalent categories. This raises the

challenge of finding an analogous relationship between some form of filtered hypergroupoid and crossed cube complexes that works in the context of ‘categories with covers’, and thus in a wide variety of geometric settings.

A further indication as to a useful generalisation of these links occurs in Getzler’s [28], where there is discussion of (weak)  $\ell$ -groupoids, *i.e.*, Duskin-Glenn  $\ell$ -hypergroupoids, for which one has an explicit choice of fillers for horns. To axiomatise these (weak)  $\ell$ -groupoids, Getzler uses a weakened form of the axioms for simplicial  $T$ -complexes, as defined by Dakin, [21], and used by Ashley, [1], which are equivalent to crossed complexes - so here again there are some pieces of the puzzle to turn over.

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# VERY-WELL-BEHAVED EPIREFLECTIONS FOR CATEGORIES OF MODELS OF SKETCHES

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**Résumé.** Nous étudions les réflexions obtenues à partir d'autres adjonctions en factorisant l'unité de l'adjonction. L'objectif est de fournir des conditions garantissant la stabilité des unités de telles réflexions, et d'induire un système de factorisation *monotone-light*. En particulier, nous considérons les cas où l'adjonction est induite par une extension de Kan à gauche et à droite (respectivement). Dans la dernière section, le cas des extensions de Kan à droite à partir de catégories de préfaisceaux est restreint aux catégories de modèles de esquisses projectives, et une réflexion de catégories en préordres est donnée à titre d'exemple. Nous généralisons également aux 2- et n-catégories.

**Abstract.** We study reflections obtained from other adjunctions by factorizing the unit of the adjunction. The objective is to provide some conditions which ensure that such reflections have stable units, and induce a monotone-light factorization system. In particular, we consider the cases where the adjunction is induced by a left and right Kan extension (respectively). In last section, the case of right Kan extensions from categories of presheaves is restricted to categories of models of limit sketches, and a reflection of categories into preorders is given as an example. We also make a generalization to 2- and n-categories.

**Keywords.** Epireflection, stable units, prefactorization, monotone-light factorization, presheaves, simplicial set, Kan extensions, pseudo-filtered, cogen-

erating class, models of sketches,  $n$ -categories.

**Mathematics Subject Classification (2010).** 18A40, 18A32, 18E50, 18N50, 18F20, 18C30, 18N10.

## 1. Introduction

Let  $(\mathcal{F}, \theta)$  be a pointed endofunctor on a category  $\mathbb{C}$  with pullbacks, i.e.,  $\mathcal{F} : \mathbb{C} \rightarrow \mathbb{C}$  is an endofunctor and  $\theta : 1_{\mathbb{C}} \rightarrow \mathcal{F}$  a natural transformation from the identity functor into  $\mathcal{F}$  (see [6, §1.1]). If  $(\mathcal{E}, \mathcal{M})$  is an (orthogonal) factorization system on  $\mathbb{C}$  such that all the morphisms in  $\mathcal{E}$  are epimorphisms and  $\mathcal{E} = \mathcal{E}'$ , the largest subclass of  $\mathcal{E}$  stable under pullbacks, then there exists a *well-behaved* reflection  $I : \mathbb{C} \rightarrow \mathbb{M}$  whose unit  $\eta$  is given by the componentwise decomposition of  $\theta = \mu \circ \eta$  using the  $(\mathcal{E}, \mathcal{M})$ -factorization. Well-behaved means that the induced reflection  $I$  has stable units, i.e., every unit morphism  $\eta_C$  belongs to  $\mathcal{E}'_I$ , the largest subclass of  $\mathcal{E}_I$  which is stable under pullbacks, being  $(\mathcal{E}_I, \mathcal{M}_I)$  the reflective factorization system associated to the reflection  $I$  (see [1]). Having stable units is a stronger condition than semi-left-exactness (see [2]), also called admissibility in categorical Galois theory.

The process described in the previous paragraph allows that, from an adjunction from a category with pullbacks and a stable factorization system (whose left-class consists of epimorphisms), it is induced a well-behaved reflection and the corresponding Galois theories.

It may happen that the induced well-behaved reflection is moreover *very-well-behaved*, meaning that there is a monotone-light factorization system  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  associated to it (see [1]). In order to be so, it only needs that, for every  $C \in \mathbb{C}$ , there is an effective descent morphism (see [5]) into  $C$  such that its domain is in the reflective subcategory.

In section 2 below it is presented a detailed study of the process outlined in the paragraphs above, followed by some generic examples and some results. These results were applied afterward to left and right Kan extensions for presheaves in sections 3 and 4 respectively.

In section 3, a new class of monotone-light factorizations were presented, deriving from adjunctions given by left Kan extensions for presheaves. A non-trivial example of such was displayed in an astonishingly simple case (cf. 3.5 below).

Then, in the rest of the paper, we focused in the more interesting context of right Kan extensions for presheaves, where there is an elegant characterization of the functors  $K$  in  $Set^K \dashv Ran_K$  for which the existence of a very-well-behaved induced reflection is assured (see 4.5).

In the last section, Lemma 5.1 allows to move to the new level of well-behaved subreflections derived from the induced well-behaved reflections, from an adjunction of right Kan extensions for presheaves. The subreflection is obtained considering just the limit models of a sketch (cf. the beginning of section 5). The last part of the paper states that this process, ending in very-well-behaved subreflections of models of presheaves, encompasses the already known very-well-behaved reflections from  $n$ -categories into  $n$ -preorders (for every positive integer  $n$ ; cf. Examples 5.4, and subsections 5.1, 5.2).

It is noticeable that our main examples are about presheaves,  $\mathbb{S} = Set$ . In future work, we hope to enlarge this setting with new examples without this restriction, namely with  $\mathbb{S}$  a regular or a (Barr) exact category, and to explore the algebraic setting (it is well known that a variety of universal algebras can be identified with a category of models of a sketch for presheaves).

Anyway, we believe that in this work a very detailed and precise theoretical framework was established, which makes us optimistic about future developments applying it.

**Notice that all the basic concepts needed to read the following paper can be found in the bibliography, mostly in two items of it, namely [8] and [1].**

## 2. Revision of General Results with Improvements

### 2.1 Derivable well-behaved epireflections

Let us first show that, a *well-behaved* epireflection is derivable from a monad in a category with pullbacks, such that certain of them are preserved by the endofunctor, provided there is a stable factorization system whose left class contains only epimorphisms (the needed conditions are in fact much weaker, as the reader shall discover; see Theorem 2.6 below).

**Lemma 2.1.**

Let  $(\mathcal{F}, \theta)$  be a pointed endofunctor on  $\mathbb{C}$  (pictured by  $\mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C}$  with  $\theta$  and  $\mathcal{F}$ ).

If there is a prefactorization system  $(\mathcal{E}, \mathcal{M})$  on  $\mathbb{C}$  such that, for every  $C \in \mathbb{C}$ , each component factorizes as

$$\theta_C = \mu_C \circ \eta_C : C \rightarrow \mathcal{I}(C) \rightarrow \mathcal{F}(C),$$

with  $\eta_C \in \mathcal{E}$  and  $\mu_C \in \mathcal{M}$ , then there is a unique pointed endofunctor  $(\mathcal{I}, \eta)$  on  $\mathbb{C}$  such that  $\theta = \mu \cdot \eta$ , where  $\eta : 1_{\mathbb{C}} \rightarrow \mathcal{I}$  and  $\mu : \mathcal{I} \rightarrow \mathcal{F}$  are the obvious natural transformations.

*Proof.* The proof follows trivially from the definition of a prefactorization system (cf. [1, §2.1]). Confer the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & \mathcal{I}(C) & \xrightarrow{\mu_C} & \mathcal{F}(C) \\ f \downarrow & & \mathcal{I}f \downarrow & & \mathcal{F}f \downarrow \\ C' & \xrightarrow{\eta_{C'}} & \mathcal{I}(C') & \xrightarrow{\mu_{C'}} & \mathcal{F}(C'), \end{array}$$

where  $\mathcal{I}f$  is the unique morphism making it commutative. □

**Lemma 2.2.** Under the conditions of Lemma 2.1, if  $\eta_C : C \rightarrow \mathcal{I}(C)$  is an epimorphism in  $\mathbb{C}$  for every  $C \in \mathbb{C}$ , then the pointed endofunctor  $(\mathcal{I}, \eta)$  on  $\mathbb{C}$  is well-pointed, i.e.,  $\mathcal{I}\eta = \eta\mathcal{I}$ .

*Proof.*  $\eta$  is an epimorphism in the functor category  $\mathbb{C}^{\mathbb{C}}$  and

$$(\mathcal{I}\eta) \cdot \eta = \eta \circ \eta = (\eta\mathcal{I}) \cdot \eta$$

□

In the following Proposition 2.3, a statement in [6, §1.1] is proved in detail.

**Proposition 2.3.** An endofunctor  $(\mathcal{I}, \eta)$  on a category  $\mathbb{C}$  is idempotent (i.e., it is well-pointed and, in addition,  $\eta\mathcal{I}$  is an isomorphism) if and only if  $\text{Fix}(\mathcal{I}, \eta)$  is a reflective full replete subcategory of  $\mathbb{C}$  with reflection  $\eta$ , where  $\text{Fix}(\mathcal{I}, \eta)$  is determined by all the objects  $M \in \mathbb{C}$  for which  $\eta_M : M \rightarrow \mathcal{I}(M)$  is an isomorphism.

*Proof.*  $\text{Fix}(\mathcal{I}, \eta)$  is a replete subcategory of  $\mathbb{C}$ , since if  $k : M \rightarrow N$  is an isomorphism from an object  $M \in \text{Fix}(\mathcal{I}, \eta)$ , then  $\eta_N = \mathcal{I}k \circ \eta_M \circ k^{-1}$  is also an isomorphism, assuring that  $N \in \text{Fix}(\mathcal{I}, \eta)$ .

( $\Rightarrow$ (only if))

If  $(\mathcal{I}, \eta)$  is idempotent, then any image  $\mathcal{I}(C)$  belongs to  $Fix(\mathcal{I}, \eta)$ ,  $C \in \mathbb{C}$  ( $\eta\mathcal{I}$  is an isomorphism by hypothesis). Name  $I = \mathcal{I}|_{Fix(\mathcal{I}, \eta)}$  the co-restriction of the functor  $\mathcal{I}$  to  $Fix(\mathcal{I}, \eta)$ , and  $H$  the full inclusion of  $Fix(\mathcal{I}, \eta)$  into  $\mathbb{C}$ . It is going to be shown that  $I$  is left adjoint to  $H$ , with unit  $\eta$ .

Let  $f : C \rightarrow M$  be any morphism whose codomain  $M$  belongs to  $Fix(\mathcal{I}, \eta)$ . Then,  $\eta_M \circ f = \mathcal{I}f \circ \eta_C \Leftrightarrow f = (\eta_M^{-1} \circ \mathcal{I}f) \circ \eta_C$ . It remains to check that  $f' = \eta_M^{-1} \circ \mathcal{I}f$  is the unique morphism such that  $f = f' \circ \eta_C$ :

$$\begin{aligned} f'' \circ \eta_C &= f' \circ \eta_C \Rightarrow \mathcal{I}(f'') \circ \mathcal{I}(\eta_C) = \mathcal{I}(f') \circ \mathcal{I}(\eta_C) \\ \Rightarrow \mathcal{I}(f'') \circ \eta_{\mathcal{I}(C)} &= \mathcal{I}(f') \circ \eta_{\mathcal{I}(C)} \quad (\mathcal{I}\eta = \eta\mathcal{I}, \text{ by hypothesis}) \\ \Rightarrow \eta_M \circ f'' &= \eta_M \circ f' \Rightarrow f'' = f' \quad (M \in Fix(\mathcal{I}, \eta)). \end{aligned}$$

( $\Leftarrow$ (if))

If  $\mathcal{I}(C)$  is the reflection of  $C$ , then  $\mathcal{I}(C) \in Fix(\mathcal{I}, \eta)$  and  $\eta_{\mathcal{I}(C)}$  is an isomorphism. Hence,  $\eta\mathcal{I}$  is an isomorphism. Then, as  $\eta\mathcal{I} \cdot \eta = \mathcal{I}\eta \cdot \eta$  and  $\eta$  is the unit of the reflection, it follows that  $\eta\mathcal{I} = \mathcal{I}\eta$ .  $\square$

**Theorem 2.4.** *Under the conditions of the previous Lemmas 2.1 and 2.2, if  $\mathcal{F}$  is a monad in which the unit is  $\theta : 1_{\mathbb{C}} \rightarrow \mathcal{F}$ , then  $(\mathcal{I}, \eta)$  is idempotent, i.e.,  $\eta\mathcal{I}$  is an isomorphism (and  $(\mathcal{I}, \eta)$  is well-pointed:  $\mathcal{I}\eta = \eta\mathcal{I}$ ).*

*More specifically, under the conditions of Lemma 2.2, if  $\mathcal{F} = GF$ , with  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  being an adjunction with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ , then there is a reflection  $H \vdash I : \mathbb{C} \rightarrow \mathbb{M}$ , where the right adjoint  $H$  is the full inclusion of  $\mathbb{M} = Fix(HI, \eta)$ . It also follows that  $F\eta$  is an isomorphism.*

*Notice that  $\mathbb{M} = Fix(\mathcal{I}, \eta)$  is equal to  $\mathcal{M}(F)$ , the full replete subcategory of  $\mathbb{C}$  determined by those objects  $M$  such that  $\theta_M : M \rightarrow \mathcal{F}(M)$  belongs to the right-class  $\mathcal{M}$  of the prefactorization system on  $\mathbb{C}$ .*

*Proof.* Consider the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & \mathcal{I}(C) & \xrightarrow{\mu_C} & GF(C) \\ & & & \searrow f & \downarrow G(f') \\ & & & & G(X) \end{array}$$

By the universality of the unit morphism  $\theta_C = \mu_C \circ \eta_C$ , there exists a unique morphism  $f' : F(C) \rightarrow X$  such that  $G(f') \circ \theta_C = f \circ \eta_C$ , for every

$f : \mathcal{I}(C) \rightarrow G(X)$ . Then  $G(f') \circ \mu_C = f$  because  $\eta_C$  is an epimorphism. Hence,  $\mu_C (\in \mathcal{M})$  is also a universal arrow from  $\mathcal{I}(C)$  to  $G$ , and so it must be isomorphic to  $\theta_{\mathcal{I}(C)} = \mu_{\mathcal{I}(C)} \circ \eta_{\mathcal{I}(C)}$  which implies that  $\eta_{\mathcal{I}(C)} (\in \mathcal{E})$  is an isomorphism, by the properties of a prefactorization system  $(\mathcal{E}, \mathcal{M})$  (cf. [1, Propositions 2.2(c) and 2.4]).

As  $F$  is a left-adjoint and every  $\eta_C : C \rightarrow \mathcal{I}(C)$  is an epimorphism in  $\mathbb{C}$ , then  $F\eta$  is an epimorphism. It is also split monic since  $F = \varepsilon F \cdot F\theta = (\varepsilon F \cdot F\mu) \cdot F\eta$ , with  $\varepsilon$  the counit of the adjunction  $G \vdash F$ . Therefore,  $F\eta$  is an isomorphism.  $\square$

**Remark 2.5.** Under the conditions of Theorem 2.4 and using the terminology in its statement, it is easy to conclude that  $\eta_{G(X)}$  is an isomorphism for all  $X \in \mathbb{X}$ , since  $G\varepsilon_X \circ \theta_{G(X)} = (G\varepsilon_X \circ \mu_{G(X)}) \circ \eta_{G(X)} = 1_{G(X)}$  and  $\eta_C$  is an epimorphism for all  $C \in \mathbb{C}$ , with  $\varepsilon$  the counit of the adjunction  $G \vdash F$ . Hence, the restriction  $G_{\mathbb{M}} : \mathbb{X} \rightarrow \mathbb{M}$  of the right adjoint  $G$  to the codomain  $\mathbb{M}$  is well defined. The left adjoint of  $G_{\mathbb{M}}$  is obviously the restriction  $F_{\mathbb{M}} : \mathbb{M} \rightarrow \mathbb{X}$  of  $F$  to  $\mathbb{M}$ . As  $G = H \circ G_{\mathbb{M}}$ , then  $F \cong F_{\mathbb{M}} \circ I$ . The unit of the adjunction  $F_{\mathbb{M}} \dashv G_{\mathbb{M}}$  is exactly the restriction of  $\mu$  to  $\mathbb{M}$ , provided one chooses  $\eta_M = 1_M$  in every factorization  $\theta_M = \mu_M \circ \eta_M$ , for every  $M \in \mathbb{M}$ , which is equivalent to the fact that the identity  $1_{\mathbb{M}}$  is the counit of the induced reflection ( $I(M) = M$ , for every  $M \in \mathbb{M}$ ).

**Theorem 2.6.** *Under the conditions of Theorem 2.4, for a category  $\mathbb{C}$  with pullbacks, the replete full reflection  $I \dashv H : \mathcal{M}(F) \rightarrow \mathbb{C}$  has stable units if the following two conditions hold:*

(a)  $\mathcal{F} = GF$  preserves the pullback squares of the form

$$\begin{array}{ccc} A \times_{\mathcal{I}(C)} C & \longrightarrow & C \\ g^*(\eta_C) \downarrow & & \downarrow \eta_C \\ A & \xrightarrow{g} & \mathcal{I}(C) ; \end{array}$$

(b)  $\eta_C \in \mathcal{E}'$  for every  $C \in \mathbb{C}$ , where  $\mathcal{E}'$  stands for the largest subclass of  $\mathcal{E}$  which is closed under pullbacks.

Notice also that  $I \dashv H : \mathcal{M}(F) \rightarrow \mathbb{C}$  does have stable units only if the condition (a) above holds, for a category  $\mathbb{C}$  with pullbacks and under the

conditions of Theorem 2.4.

*Proof.* The final remark in the statement is going to be proved first. If the reflection  $I \dashv H$  has stable units, then, by definition, the image by  $\mathcal{I}$  of the pullback diagram in the statement is also a pullback diagram, meaning that both  $\mathcal{I}\eta_C$  and  $\mathcal{I}g^*(\eta_C)$  are isomorphisms (the former is so since  $\mathcal{I}\eta = \eta\mathcal{I}$  is an isomorphism). Then, the image by  $\mathcal{F}\mathcal{I}$  of the pullback diagram in the statement is necessarily a pullback diagram. Finally,  $\mathcal{F}$  preserves the pullback diagram in the statement, since  $F\eta$  being an isomorphism implies  $\mathcal{F} \cong \mathcal{F}\mathcal{I}$ .

For the if part of the theorem, consider the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\eta_P} & \mathcal{I}(P) & \xrightarrow{\mu_P} & \mathcal{F}(P) \\ \downarrow g^*(\eta_C) & & \downarrow \mathcal{I}g^*(\eta_C) & & \downarrow \mathcal{F}g^*(\eta_C) \\ A & \xrightarrow{\eta_A} & \mathcal{I}(A) & \xrightarrow{\mu_A} & \mathcal{F}(A), \end{array}$$

where  $P = A \times_{\mathcal{I}(C)} C$ . We need to show that  $\mathcal{I}g^*(\eta_C)$  is an isomorphism, or equivalently that  $\mathcal{I}g^*(\eta_C) \in \mathcal{E} \cap \mathcal{M}$ , using the properties of prefactorization systems (cf. [1, Proposition 2.4]): as  $\eta_C \in \mathcal{E}'$ , then  $\eta_A \circ g^*(\eta_C) = \mathcal{I}g^*(\eta_C) \circ \eta_P \in \mathcal{E}$ , implying that  $\mathcal{I}g^*(\eta_C) \in \mathcal{E}$ ; as  $\mathcal{F}g^*(\eta_C)$  is an isomorphism (by hypothesis (a), since  $F\eta$  is an isomorphism) then  $\mu_A \circ \mathcal{I}g^*(\eta_C) \cong \mu_P \in \mathcal{M}$ , implying that  $\mathcal{I}g^*(\eta_C) \in \mathcal{M}$  ( $\mu_A \in \mathcal{M}$ ).  $\square$

**Remark 2.7.** We take the opportunity here to report a mistake in our paper [16]. Theorem 2.4 in [16] corresponds to Theorem 2.6 just above, but in the latter the two conditions (a) and (b) are now only sufficient for the derived reflection to have stable units. Fortunately, this mistake had no impact on the other results of [16], since only the sufficiency was used.

**Examples 2.8.** Consider any adjunction  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ , in which  $\mathbb{C}$  is a regular category (i.e., it has pullbacks, it has coequalizers of kernel pairs, and the class of its regular epimorphisms is closed under pullbacks) and the left adjoint  $F$  preserves pullbacks. Then, there is a reflection  $H \vdash I : \mathbb{C} \rightarrow \text{Mono}(F)$  with stable units, where  $\text{Mono}(F)$  is a full (replete) subcategory of  $\mathbb{C}$  determined by the objects  $M \in \mathbb{C}$  for which the unit morphism  $\theta_M : M \rightarrow GF(M)$  is a monomorphism. The

unit  $\eta : 1_{\mathbb{C}} \rightarrow HI$  consists componentwise of regular epimorphisms of  $\mathbb{C}$ ; in fact,  $(\mathcal{E}, \mathcal{M}) = (\text{RegularEpimorphisms}, \text{Monomorphisms})$  is an (orthogonal) factorization system on  $\mathbb{C}$ .

## 2.2 Derivable very-well-behaved epireflections

**Theorem 2.9.** *Under the conditions of Theorem 2.4, if the derived reflection  $H \vdash I : \mathbb{C} \rightarrow \mathcal{M}(F)$  has stable units, and for every object  $C (\in \mathbb{C})$  there is an effective descent morphism  $p : M \rightarrow C$  (in  $\mathbb{C}$ ) with domain in  $\mathcal{M}(F)$ , then there is a monotone-light factorization system  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  on  $\mathbb{C}$ , obtained by simultaneously stabilizing and localizing the reflective factorization system  $(\mathcal{E}_I, \mathcal{M}_I)$ .*

*Proof.* This small result (but crucial in this work) is included in Corollary 6.2 in [12], which follows from the main result of [1].  $\square$

**Examples 2.10.** Under the conditions of Examples 2.8, let  $\mathbb{C}$  be not only regular, but furthermore (Barr) exact, which is known to imply that in  $\mathbb{C}$  the effective descent morphisms are the regular epimorphisms (cf. [5]). Suppose also that  $\mathbb{C}$  is cocomplete, and that there is a full subcategory  $\mathbb{N}$  of  $\mathbb{C}$  which is dense in  $\mathbb{C}$  (i.e., every  $C \in \mathbb{C}$  is a colimit of objects of  $\mathbb{N}$ ) and closed in  $\text{Mono}(F)(= \mathcal{M}(F))$  for coproducts in  $\mathbb{C}$  (hence, the canonical presentation of every  $C \in \mathbb{C}$  as a colimit gives  $p : M \rightarrow C$ , with  $M = \coprod_{i \in I} N_i \in \text{Mono}(F)$  and  $p$  a regular epimorphism, that is, an effective descent morphism; cf. [8, §V.2]).

Then, by Theorems 2.6 and 2.9, it follows that the induced reflection, from the left adjoint functor  $F$  preserving pullbacks, produces a monotone-light factorization system on  $\mathbb{C}$ .

Remark that, by next Lemma 2.11(b), instead of demanding that the dense subcategory  $\mathbb{N}$  is closed in  $\text{Mono}(F)$  for coproducts, one could alternatively ask  $\mathbb{N}$  to be a dense subcategory of  $\mathbb{C}$  closed under coproducts and such that the restriction of the left-adjoint  $F$  to  $\mathbb{N}$  is faithful.

A class  $S$  of objects of a category  $\mathbb{C}$  is a generating class if, for every two distinct parallel morphisms  $f, f' \in \mathbb{C}$ , there exists an object  $N \in S$  and a morphism  $g$  with domain  $N$  such that  $f \circ g \neq f' \circ g$ .

Notice that the objects of a full subcategory  $\mathbb{N}$  which is dense in  $\mathbb{C}$  are obviously a generating class for  $\mathbb{C}$ .

The following Lemma 2.11 is needed in the proof of next Theorems 2.13 and 2.14, which give conditions under which an adjunction from a category of presheaves has *enough* effective descent morphisms, in the sense of Theorem 2.9.

**Lemma 2.11.** *Consider any adjunction  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ .*

*Suppose there is a full subcategory  $\mathbb{N}$  of  $\mathbb{C}$  whose class of objects  $ob(\mathbb{N})$  is a generating class for  $\mathbb{C}$ . Then:*

- (a) *for any object  $C \in \mathbb{C}$ , the unit morphism  $\theta_C : C \rightarrow GF(C)$  is a monomorphism if and only if, for every object  $N \in \mathbb{N}$ , the restrictions  $F_{N,C} : \mathbb{C}(N, C) \rightarrow \mathbb{X}(F(N), F(C))$  of the functor  $F$  are all injective;*
- (b) *[10, Lemma 4.2] each unit morphism  $\theta_N : N \rightarrow GF(N)$  is a monomorphism,  $N \in \mathbb{N}$ , if and only if the restriction of the left -adjoint  $F$  to  $\mathbb{N}$  is faithful.*

*Proof.* (a)

$$\begin{aligned} (\Rightarrow) \quad Ff &= Ff' : F(N) \rightarrow F(C) \Rightarrow GFf \circ \theta_N = GFf' \circ \theta_N \\ &\Rightarrow \theta_C \circ f = \theta_C \circ f' \quad (\theta \text{ is a natural transformation}) \\ &\Rightarrow f = f' \quad (\theta_C \text{ is a monomorphism by hypothesis}) \end{aligned}$$

(in fact, it was proved that  $F_{N,C}$  is injective for any  $N \in \mathbb{C}$ , provided  $\theta_C$  is a monomorphism, and not only for  $N \in \mathbb{N}$ ).

( $\Leftarrow$ ) Suppose, by *reductio ad absurdum*, that there are  $g \neq g' : C' \rightarrow C$  such that  $\theta_C \circ g = \theta_C \circ g'$ . Being  $ob(\mathbb{N})$  a generating class for  $\mathbb{C}$ , there exists  $\tau : N \rightarrow C'$  with  $N \in \mathbb{N}$ , such that  $g \circ \tau \neq g' \circ \tau$ . Then,  $\theta_C \circ g \circ \tau = \theta_C \circ g' \circ \tau$  implies that  $F(g \circ \tau) = F(g' \circ \tau)$ , because  $\theta_N$  is a unit morphism, and so  $g \circ \tau = g' \circ \tau$  by hypothesis, which is a contradiction.

(b) The proof of this item follows immediately from item (a) by just considering all  $C \in \mathbb{N}$ . □

The following general result of Lemma 2.12 is needed in the proof of the consecutive Theorem 2.13.

**Lemma 2.12.** *Consider an adjunction  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ . Suppose that, for the family  $(T_i)_{i \in I}$  of objects of  $\mathbb{C}$ , there exists its*

coproduct  $\coprod_{i \in I} T_i$  in  $\mathbb{C}$ . Then,

$$\theta_{\coprod_{i \in I} T_i} = \coprod_{i \in I} \theta_{T_i},$$

provided the right-adjoint  $G$  preserves the coproduct  $\coprod_{i \in I} F(T_i)$  in  $\mathbb{X}$ .

*Proof.* Let  $\iota_{T_i} : T_i \rightarrow \coprod_{i \in I} T_i$ ,  $\iota_{F(T_i)} = F\iota_{T_i} : F(T_i) \rightarrow \coprod_{i \in I} F(T_i) = F(\coprod_{i \in I} T_i)$  and  $\iota_{GF(T_i)} = G\iota_{F(T_i)} = GF\iota_{T_i} : GF(T_i) \rightarrow \coprod_{i \in I} GF(T_i) = G\coprod_{i \in I} F(T_i) = GF(\coprod_{i \in I} T_i)$  denote the three coproduct injections in question (recall that  $F$  preserves colimits, because it is a left adjoint).

Since  $\theta$  is a natural transformation,

$$\theta_{\coprod_{i \in I} T_i} \circ \iota_{T_i} = GF(\iota_{T_i}) \circ \theta_{T_i}.$$

By definition of  $\coprod_{i \in I} \theta_{T_i}$ ,

$$\coprod_{i \in I} \theta_{T_i} \circ \iota_{T_i} = \iota_{GF(T_i)} \circ \theta_{T_i}.$$

Hence, as  $GF(\iota_{T_i}) = \iota_{GF(T_i)}$ ,  $\theta_{\coprod_{i \in I} T_i} \circ \iota_{T_i} = \coprod_{i \in I} \theta_{T_i} \circ \iota_{T_i}$  for all  $i \in I$ , implying that  $\theta_{\coprod_{i \in I} T_i} = \coprod_{i \in I} \theta_{T_i}$  by the definition of a coproduct  $\coprod_{i \in I} T_i$ .  $\square$

It is convenient now to recall that, in a category of presheaves  $\text{Set}^{\mathbb{D}^{op}}$ , the representable functors  $\mathbb{D}(-, D)$  constitute a generating class,  $D \in \mathbb{D}$  (cf. [8, §III.7]).

**Theorem 2.13.** [10, See Remark 6.1] Consider any adjunction  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ , where  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$  is a category of presheaves: if

- $F \circ y$  is faithful  
(where  $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$  is the Yoneda embedding) and
- $G$  preserves coproducts  $\coprod_{i \in I} F(N_i)$ ,  
where each  $N_i = \mathbb{D}(-, D_i)$  belongs to the image of  $y$ ,

then

the unit morphism  $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$  is a monomorphism in  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ , for each family  $(D_i)_{i \in I}$  of objects of  $\mathbb{D}$ .

*Proof.* Since  $G$  preserves coproducts  $\coprod_{i \in I} F(N_i)$ , for each family  $(N_i)_{i \in I}$  of objects as in the statement,  $\theta_{\coprod_{i \in I} N_i} = \coprod_{i \in I} \theta_{N_i}$  (cf. Lemma 2.12). Hence,  $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$  is a monomorphism in  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ , for each family  $(D_i)_{i \in I}$  of objects of  $\mathbb{D}$ , by Lemma 2.11(b) and the fact that a monomorphism in  $\text{Set}^{\mathbb{D}^{op}}$  is a componentwise injection.  $\square$

The previous Theorem 2.13 will be helpful in section 3. The following Corollary 2.15 of Theorem 2.14 is needed in section 4.

**Theorem 2.14.** *Consider any adjunction  $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$  with unit  $\theta : 1_{\mathbb{C}} \rightarrow GF$ , where  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$  is a category of presheaves: if*

- *$F \circ y$  is faithful and injective on objects*  
*(where  $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$  is the Yoneda embedding),*
- *the coproduct injections in  $\mathbb{X}$  of the form  $F(\mathbb{D}(-, D_j)) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$*   
*are monomorphisms,*

- *the equality  $\iota_j \circ f = \iota_{j'} \circ f'$  implies  $\iota_j = \iota_{j'}$ ,*

*for every two coproduct injections*

*$\iota_j : F(\mathbb{D}(-, D_j)) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$  and  $\iota_{j'} : F(\mathbb{D}(-, D_{j'})) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$  in  $\mathbb{X}$ ,*

*and any pair of morphisms  $f, f'$  with common domain  $F(\mathbb{D}(-, D))$ , for some  $D \in \mathbb{D}$ ,*

*then*

*the unit morphism  $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$  is a monomorphism in  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ , for each family  $(D_i)_{i \in I}$  of objects of  $\mathbb{D}$ .*

*Proof.* The representable functors  $\mathbb{D}(-, D)$  in  $\text{Set}^{\mathbb{D}^{op}}$  are known to constitute a dense full subcategory of  $\text{Set}^{\mathbb{D}^{op}}$  (cf. [8, Theorem 1 in §III.7]), therefore a generating class for  $\text{Set}^{\mathbb{D}^{op}}$ . By Lemma 2.11(a), we have just to show that, for every family  $(D_i)_{i \in I}$  of objects in  $\mathbb{D}$ ,  $F_{\mathbb{D}(-, D), \coprod_{i \in I} \mathbb{D}(-, D_i)} :$

$\text{Set}^{\mathbb{D}^{op}}(\mathbb{D}(-, D), \coprod_{i \in I} \mathbb{D}(-, D_i)) \rightarrow \mathbb{X}(F(\mathbb{D}(-, D)), F(\coprod_{i \in I} \mathbb{D}(-, D_i)))$   
 is an injection, for every  $D \in \mathbb{D}$ .

It is easy to show, using the Yoneda Lemma, that for any  $\alpha : \mathbb{D}(-, D) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$  there exists  $\alpha_j : \mathbb{D}(-, D) \rightarrow \mathbb{D}(-, D_j)$ , with  $j \in I$ , such that  $\delta_j \circ \alpha_j = \alpha$ , where  $\delta_j : \mathbb{D}(-, D_j) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$  is a coproduct injection,  $j \in I$  ( $\alpha_j$  is determined by  $\alpha_D(1_D) : D \rightarrow D_j$ , i.e.,  $\alpha_{j,D}(1_D) = \alpha_D(1_D)$ ).

Let  $\alpha' : \mathbb{D}(-, D) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$  be another morphism in  $\text{Set}^{\mathbb{D}^{op}}$  such that  $F\alpha' = F\alpha$ ; then, there exists  $\alpha'_{j'} : \mathbb{D}(-, D) \rightarrow \mathbb{D}(-, D_{j'})$  such that  $\delta_{j'} \circ \alpha'_{j'} = \alpha'$ , where  $\delta_{j'} : \mathbb{D}(-, D_{j'}) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$  is a coproduct injection ( $j' \in I$ ); hence,  $F\delta_j \circ F\alpha_j = F\delta_{j'} \circ F\alpha'_{j'}$ . Notice that  $\iota_j = F\delta_j$  and  $\iota_{j'} = F\delta_{j'}$  are coproduct injections in  $\mathbb{X}$  (since  $F$  is a left-adjoint). By the hypotheses in the statement,  $\iota_j = \iota_{j'}$ , which implies  $\delta_j = \delta_{j'}$  ( $F\delta_j = F\delta_{j'} \Rightarrow F(\mathbb{D}(-, D_j)) = F(\mathbb{D}(-, D_{j'})) \Rightarrow \mathbb{D}(-, D_j) = \mathbb{D}(-, D_{j'})$ , because  $Fy$  is injective on objects) and  $F\alpha_j = F\alpha'_{j'}$  ( $\iota_j$  is a monomorphism by hypothesis). Then,  $\alpha_j = \alpha'_{j'}$  because  $Fy$  is faithful (it was concluded before that  $\alpha_j$  and  $\alpha'_{j'}$  are parallel morphisms in the dense subcategory of representable functors). Finally,  $\alpha = \delta_j \circ \alpha_j = \delta_{j'} \circ \alpha'_{j'} = \alpha'$ .  $\square$

**Corollary 2.15.** [10, See footnote in Remark 6.1] *If both  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$  and  $\mathbb{X} = \text{Set}^{\mathbb{B}}$  are categories of presheaves, and  $F \circ y$  is faithful and injective on objects (where  $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$  is the Yoneda embedding), then the unit morphism  $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$  is a monomorphism in  $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ , for each family  $(D_i)_{i \in I}$  of objects of  $\mathbb{D}$ .*

*Proof.* First, it is trivial that the coproduct injections are monomorphisms in  $\text{Set}^{\mathbb{B}}$ .

According to Theorem 2.14, if  $\iota_j \circ f = \iota_{j'} \circ f'$  implies that  $\iota_j = \iota_{j'}$  in  $\text{Set}^{\mathbb{B}}$  (where  $\iota_j : X_j \rightarrow \coprod_{i \in I} X_i$  and  $\iota_{j'} : X_{j'} \rightarrow \coprod_{i \in I} X_i$  are coproduct injections in  $\text{Set}^{\mathbb{B}}$ , for any family  $(X_i)_{i \in I}$  of objects in  $\text{Set}^{\mathbb{B}}$ ) then the statement would have been proved:

in fact, for every  $B \in \mathbb{B}$ ,  $\iota_{j,B} \circ f_B = \iota_{j',B} \circ f'_B$  implies that  $X_j(B) = X_{j'}(B)$  since  $(\coprod_{i \in I} X_i)(B) = \coprod_{i \in I} X_i(B)$  and  $\iota_{j,B}, \iota_{j',B}$  are inclusions in  $\text{Set}$ ; therefore,  $\iota_{j,B} = \iota_{j',B}$  for every  $B \in \mathbb{B}$ , that is,  $\iota_j = \iota_{j'}$ .  $\square$

### 2.3 Uplifting factorization systems to functor categories

The last result in this section 2 allows us to start with a factorization of the morphisms in a category  $\mathbb{S}$ , and then raise it to the factorization of the natural transformations in the functor category  $\mathbb{S}^{\mathbb{I}}$ .

**Lemma 2.16.** *Let  $\mathcal{S}$  be a class of morphisms in the category  $\mathbb{S}$ . Define  $\mathcal{S}^{\mathbb{I}}$  as the class of all morphisms in the functor category  $\mathbb{S}^{\mathbb{I}}$  whose components are in  $\mathcal{S}$  ( $\alpha = (\alpha_i)_{i \in \mathbb{I}} \in \mathcal{S}^{\mathbb{I}}$  if and only if  $\alpha_i$  is in  $\mathcal{S}$  for all  $i \in \mathbb{I}$ ).*

(a) If  $(\mathcal{E}, \mathcal{M})$  is a prefactorization system on  $\mathbb{S}$ , then there is a prefactorization system  $(\mathcal{F}, \mathcal{N})$  on  $\mathbb{S}^{\mathbb{I}}$  such that  $\mathcal{E}^{\mathbb{I}} \subseteq \mathcal{F}$  and  $\mathcal{M}^{\mathbb{I}} \subseteq \mathcal{N}$ .

(b) If  $(\mathcal{E}, \mathcal{M})$  is a factorization system on  $\mathbb{S}$ , then  $(\mathcal{E}^{\mathbb{I}}, \mathcal{M}^{\mathbb{I}})$  is a factorization system on  $\mathbb{S}^{\mathbb{I}}$ .

*Proof.* (a) It has to be proved that, for any commutative square  $v \circ e = m \circ u$  in  $\mathbb{S}^{\mathbb{I}}$ , with  $e : A \rightarrow B \in \mathcal{E}^{\mathbb{I}}$  and  $m : C \rightarrow D \in \mathcal{M}^{\mathbb{I}}$ , there is one and only one morphism  $w : B \rightarrow C$  such that  $w \circ e = u$  and  $m \circ w = v$ .

If such a morphism  $w$  exists then it must be the unique  $w = (w_i)_{i \in \mathbb{I}}$  such that  $w_i \circ e_i = u_i$  and  $m_i \circ w_i = v_i$ , for every  $i \in \mathbb{I}$ . The question is if  $(w_i)_{i \in \mathbb{I}}$  is a natural transformation.

Let  $f : i \rightarrow j$  be a morphism in  $\mathbb{I}$ . It has to be shown that  $Cf \circ w_i = w_j \circ Bf$ , or equivalently (since  $e_i \downarrow m_j$ ),  $Cf \circ w_i \circ e_i = Cf \circ u_i = w_j \circ Bf \circ e_i$  and  $m_j \circ Cf \circ w_i = v_j \circ Bf = m_j \circ w_j \circ Bf$ :

$Cf \circ (w_i \circ e_i) = Cf \circ u_i$ ,  $w_j \circ (Bf \circ e_i) = (w_j \circ e_j) \circ Af = u_j \circ Af = Cf \circ u_i$ ;  
 $(m_j \circ Cf) \circ w_i = Df \circ (m_i \circ w_i) = Df \circ v_i = v_j \circ Bf$ ,  $(m_j \circ w_j) \circ Bf = v_j \circ Bf$ .

(b) Being  $(\mathcal{E}, \mathcal{M})$  a factorization system on  $\mathbb{S}$ , it is obvious that both  $\mathcal{E}^{\mathbb{I}}$  and  $\mathcal{M}^{\mathbb{I}}$  contain the identities and are closed under composition with isomorphisms. By (a), in order that  $(\mathcal{E}^{\mathbb{I}}, \mathcal{M}^{\mathbb{I}})$  is a factorization system on  $\mathbb{S}^{\mathbb{I}}$ , it remains to show that each morphism  $\alpha : A \rightarrow B$  in  $\mathbb{S}^{\mathbb{I}}$  factorises as  $\alpha = m \circ e$ , with  $m \in \mathcal{M}^{\mathbb{I}}$  and  $e \in \mathcal{E}^{\mathbb{I}}$  (cf. [1, §2.8]):

$$\begin{array}{ccccc} A(i) & \xrightarrow{e_i} & C(i) & \xrightarrow{m_i} & B(i) & m_i \circ e_i = \alpha_i \\ Af \downarrow & & Cf \downarrow & & Bf \downarrow & \\ A(j) & \xrightarrow{e_j} & C(j) & \xrightarrow{m_j} & B(j) & m_j \circ e_j = \alpha_j, \end{array}$$

where  $C : \mathbb{I} \rightarrow \mathbb{S}$  being a functor follows from  $(\mathcal{E}, \mathcal{M})$  being a (pre)factorization system.  $\square$

### 3. The case of left Kan extensions for presheaves

**Theorem 3.1.** *Consider a functor  $K : \mathbb{B} \rightarrow \mathbb{A}$ , such that  $\mathbb{B}$  is a small category and  $\mathbb{A}$  is a category with small hom-sets. Then, there is an adjunction  $\text{Set}^K \vdash \text{Lan}_K : \text{Set}^{\mathbb{B}} \rightarrow \text{Set}^{\mathbb{A}}$  given by left Kan extensions.*

*If the left-adjoint  $\text{Lan}_K$  preserves pullbacks and  $\text{Lan}_K \circ y$  is faithful (where  $y : \mathbb{B}^{\text{op}} \rightarrow \text{Set}^{\mathbb{B}}$  is the Yoneda embedding), then there is an induced*

reflection with stable units  $H \vdash I : \text{Set}^{\mathbb{B}} \rightarrow \text{Mono}(\text{Lan}_K)$  and a monotone-light factorization  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  on  $\text{Set}^{\mathbb{B}}$ .

*Proof.* The statement is a special case of Examples 2.10, with  $\mathbb{C} = \text{Set}^{\mathbb{B}}$ ,  $\mathbb{X} = \text{Set}^{\mathbb{A}}$ ,  $(\mathcal{E}, \mathcal{M}) = (\text{Epis}, \text{Monos})$  is the lifting given by Lemma 2.16(b) of the factorization system  $(\text{Surjections}, \text{Injections})$  on  $\text{Set}$ .

In fact, the representable functors  $\mathbb{B}(B, -)$  in  $\text{Set}^{\mathbb{B}}$  constitute a dense subcategory of  $\text{Set}^{\mathbb{B}}$ , whose unit morphisms

$$\theta_{\mathbb{B}(B, -)} : \mathbb{B}(B, -) \rightarrow \text{Lan}_K(\mathbb{B}(B, -)) \circ K$$

are by assumption monomorphisms ( $\Leftrightarrow \text{Lan}_K \circ y$  faithful) and their co-products are also in  $\text{Mono}(\text{Lan}_K)$  by Theorem 2.13 (the right adjoint  $\text{Set}^K$  preserves colimits).  $\square$

**Remark 3.2.** In Theorem 3.1 just above, instead of asking that  $\text{Lan}_K$  preserves all pullbacks, it could be only demanded the weaker condition that  $\text{Set}^K \circ \text{Lan}_K$  preserves pullbacks (in fact, only the ones given in Theorem 2.6(a)).

Notice that, if  $K = ! : \mathbb{J} \rightarrow \mathbf{1}$  is the functor whose codomain is the category  $\mathbf{1}$  (one object and no non-identity morphisms) and the domain  $\mathbb{J}$  is small, then the adjunction  $\text{Set}^K \vdash \text{Lan}_K : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$  is  $\Delta \vdash \text{Colim}$ , that is, the right adjoint is the diagonal functor and the left adjoint takes colimits.

Recall, as well, that a category  $\mathbb{J} = \coprod_{i \in I} \mathbb{J}_i$  is pseudo-filtered if its connected components  $\mathbb{J}_i$  are filtered (cf. Exercise 2 in [8, §IX.2]).

**Lemma 3.3.** *Consider the adjunction  $\Delta \vdash \text{Colim} : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$ , with unit  $\theta$ , from which derives the full reflective subcategory  $\text{Mono}(\text{Colim})$  as in Theorem 2.4, using the factorization system*

$$(\text{Epis}, \text{Monos}) = (\text{Surjections}^{\mathbb{J}}, \text{Injections}^{\mathbb{J}})$$

(cf. Lemma 2.16(b)). Let the category  $\mathbb{J}$  be pseudo-filtered (cf. last paragraph just before this Lemma 3.3). Then, the two following statements are valid.

- (a)  $M \in \text{Mono}(\text{Colim})$  if and only if  $Mu$  is injective, for every  $u$  in  $\mathbb{J}$ .
- (b) The functor  $\text{Colim} : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$  preserves pullbacks.

*Proof.* (a)

$(\Rightarrow) M \in \text{Mono}(\text{Colim}) \Leftrightarrow \forall_{j \in \mathbb{J}} \theta_{M,j} : M(j) \rightarrow \text{Colim}(M)$  is an injection, where  $\theta_{M,j}$  is the  $j$ -component of the unit morphism of  $M \in \text{Set}^{\mathbb{J}}$  (the colimit cocone of  $M : \mathbb{J} \rightarrow \text{Set}$ ), then

$$\forall_{u:j \rightarrow j'} \forall_{x,x' \in M(j)} Mu(x) = Mu(x') \Rightarrow x = x'.$$

In fact,  $\theta_{M,j}(x) = \theta_{M,j'} \circ Mu(x) = \theta_{M,j'} \circ Mu(x') = \theta_{M,j}(x')$ , and so  $x = x'$  because  $\theta_{M,j}$  is an injection:

$$\begin{array}{ccc} & \text{Colim}(M) & \\ \theta_{M,j} \nearrow & & \nwarrow \theta_{M,j'} \\ M(j) & \xrightarrow{Mu} & M(j') \end{array} .$$

$(\Leftarrow) \text{Colim}(M) = \coprod_{j \in \mathbb{J}} M(j)/E$ , where  $E$  is the equivalence relation which is about to be described. Let  $x \in M(j)$  and  $x' \in M(j')$ , then  $xE x'$  if and only if there exists  $u : j \rightarrow k$  and  $u' : j' \rightarrow k$  such that  $Mu(x) = Mu'(x')$  (cf. the proof of Theorem 1 in [8, §IX.2], where this result is stated for  $\mathbb{J}$  filtered; the generalization to  $\mathbb{J} = \coprod_{i \in I} \mathbb{J}_i$  pseudo-filtered is obvious, since the colimits are obtained as the disjoint union of the colimits of the filtered connected components  $\mathbb{J}_i$ ).

Let  $x, x' \in M(j)$  such that  $\theta_{M,j}(x) = \theta_{M,j}(x')$ , and therefore  $xE x'$ . Then, as  $j = j'$ , it follows that  $\exists j \xrightarrow[u']{u} k \xrightarrow{w} k'$  such that  $M(w \circ u)(x) = M(w \circ u')(x')$ , which implies by the hypothesis that  $x = x'$ , since  $w \circ u = w \circ u'$  (the existence of such  $w : k \rightarrow k'$  is assured by  $\mathbb{J}$  being pseudo-filtered).

(b) It is well known that finite limits commute with filtered colimits in *Set*. The statement is just a refinement of this fact (cf. Exercise 4 in [8, §IX.2]): pseudo-filtered colimits commute with pullbacks in *Set*; it follows that if  $\mathbb{J}$  is pseudo-filtered then *Colim* preserves pullbacks.  $\square$

**Theorem 3.4.** *Under the conditions of Lemma 3.3, if every morphism in  $\mathbb{J}$  is a monomorphism, then there is a monotone-light factorization system  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  on  $\text{Set}^{\mathbb{J}}$ .*

*Proof.* By Lemma 3.3(b), the left-adjoint  $\text{Lan}_! = \text{Colim}$  preserves pullbacks. According to Theorem 3.1, there exists the required monotone-light factorization, provided any representable functor  $\mathbb{J}(i, -)$  belongs to

$Mono(Colim)$ . This is the case if, for every  $u : j \rightarrow j'$  in  $\mathbb{J}$ , the image  $\mathbb{J}(i, -)u : \mathbb{J}(i, -)(j) \rightarrow \mathbb{J}(i, -)(j')$  is always an injection, by Lemma 3.3(a). It is obviously so, since the functions (compose with  $u$ )  $u_* : \mathbb{J}(i, j) \rightarrow \mathbb{J}(i, j')$  are injections if  $u$  is a monomorphism.  $\square$

**Examples 3.5.** Any preorder  $\mathbb{J}$ , such that each of its connected components is a filtered category (also called filtered set or directed set), gives rise to a very-well-behaved induced reflection  $Set^{\mathbb{J}} \rightarrow Mono(Colim)$ .

For instance, consider the very simple special case of Theorem 3.4 with  $\mathbb{J} = \mathbf{2}$ , the ordinal number with two objects 0, 1 and only one non-identity morphism  $u : 0 \rightarrow 1$ . As  $\mathbf{2}$  is a filtered preorder, the reflection  $H \vdash I : Set^{\mathbf{2}} \rightarrow Mono(Colim)$  is *very-well-behaved*, that is, has stable units and a monotone-light factorization  $(\mathcal{E}'_I, \mathcal{M}^*_I)$ , obtained by simultaneous stabilization and localization of the classes in the reflective factorization system  $(\mathcal{E}_I, \mathcal{M}_I)$ . It is going to be shown that this particular monotone-light factorization  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  is non-trivial, that is,  $\mathcal{E}'_I$  is strictly contained in  $\mathcal{E}_I (\Leftrightarrow \mathcal{M}^*_I$  strictly contains  $\mathcal{M}_I)$ .

The unit morphism  $\theta_M : M \rightarrow \Delta Colim(M) = \Delta M(1)$  consists in two functions  $\theta_{M,0} = Mu$  and  $\theta_{M,1} = id_{M(1)}$ , for all  $Mu = M(0) \rightarrow M(1)$ . Hence, from the  $(Surjections, Injections)$ -factorizations

$$\theta_{M,0} = \mu_{M,0} \circ \eta_{M,0}, \quad \theta_{M,1} = 1_{M(1)} = \mu_{M,1} \circ \eta_{M,1} = 1_{M(1)} \circ 1_{M(1)},$$

it is easy to conclude that the replete full subcategory  $Mono(Colim)$  is determined by all injections (cf. Lemma 3.3). Concluding then that a morphism  $\varphi = (\varphi_0, \varphi_1) : M \rightarrow N$  belongs to  $\mathcal{E}_I$  if and only if  $\varphi_1 : M(1) \rightarrow N(1)$  is a bijection and  $(Nu \circ \varphi_0)(M(0)) = Nu(N(0))$  (where  $Nu : N(0) \rightarrow N(1)$  and  $\varphi_0 : M(0) \rightarrow N(0)$ ):

indeed,  $\varphi = (\varphi_0, \varphi_1) \in \mathcal{E}_I$  if and only if  $I\varphi = I(\varphi_0, \varphi_1) = (I(\varphi)_0, I(\varphi)_1)$  is an isomorphism (cf. [1, (3.2) in 3.1]), that is, both components are bijections. Noticing that  $I(\varphi)_1 = \varphi_1$ , and that  $\varphi_1$  being a bijection implies that  $I(\varphi)_0$  is an injection, it remains to characterize the surjection of  $I(\varphi)_0$  in terms of the initial data:  $I(\varphi)_0$  is a surjection iff  $I(\varphi)_0 \circ \eta_{M,0}$  is a surjection iff  $\eta_{N,0} \circ \varphi_0$  is a surjection iff  $\eta_{N,0} \circ \varphi_0(M(0)) = I(N)(0)$  iff  $Nu \circ \varphi_0(M(0)) = Nu(N(0)) (\cong I(N)(0))$ .

Let  $M = N$ ,  $M(0) = \{x, y\}$ ,  $M(1) = \{x\}$  and  $\varphi_0(x) = x = \varphi_0(y)$ . This particular morphism  $\varphi = (\varphi_0, \varphi_1)$  belongs to  $\mathcal{E}_I$  but does not belong to

its largest subclass closed under pullbacks  $\mathcal{E}'_I$ . Indeed, take  $Qu : \{y\} \rightarrow \{x\}$ , and  $\psi = (\psi_0, \psi_1) : Q \rightarrow N$ , such that  $\psi_0$  and  $\psi_1$  are the inclusions, then the pullback  $\psi^*(\varphi)$  of  $\varphi$  along  $\psi$  does not belong to  $\mathcal{E}_I$ , since the empty set is the domain of  $(\psi^*(\varphi))_0 : \emptyset \rightarrow \{y\}$ .

## 4. The case of right Kan extensions for presheaves

The right Kan extensions adjunction  $Ran_K \vdash Set^K : Set^{\mathbb{A}} \rightarrow Set^{\mathbb{B}}$  is more interesting for the purpose of obtaining *very-well-behaved* reflections, than the left Kan extensions case of the precedent section 3.

As  $Set^K$  is left-exact “a priori” (in fact, it preserves both limits and colimits), Corollary 2.15 (of Theorem 2.14; cf. also Examples 2.10) states that  $H \vdash I : Set^{\mathbb{A}} \rightarrow Mono(Set^K)$  is *very-well-behaved* if  $Set^K \circ y$  is faithful ( $\Leftrightarrow \forall_{A \in \mathbb{A}} \mathbb{A}(A, -) \in Mono(Set^K)$ ) and injective on objects, where  $y : \mathbb{A}^{op} \rightarrow Set^{\mathbb{A}}$  is the Yoneda embedding. But, for the case  $Ran_K \vdash Set^K : Set^{\mathbb{A}} \rightarrow Set^{\mathbb{B}}$ , the injectivity on objects of  $Set^K \circ y$  is not needed, because coproduts preserve jointly monic morphisms in  $Set$  (see Definition 4.3 and Lemma 4.4).

Next Theorem 4.5 gives a useful characterization of such a case using the notion of *cogenerating class* (the dual of the notion of a generating class given two paragraphs before Lemma 2.11).

**Lemma 4.1.** *Consider an adjunction*

$$(\mathbb{S}^K, Ran_K, \theta, \varepsilon) : \mathbb{S}^{\mathbb{A}} \rightarrow \mathbb{S}^{\mathbb{B}},$$

*given by right Kan extensions calculated as pointwise limits. For every  $S \in \mathbb{S}^{\mathbb{A}}$ , the right Kan extension  $Ran_K(SK)$  is calculated for each  $A \in \mathbb{A}$  as*

$$Ran_K(SK)(A) = Lim((A \downarrow K) \xrightarrow{Q} \mathbb{B} \xrightarrow{SK} \mathbb{S}) = Lim_f SK(B),$$

*$f \in (A \downarrow K)$ , where  $Q$  is the projection of the comma category (cf. [8, §X.3]).*

*Then, the unit morphism  $\theta_S$  is the unique morphism in  $\mathbb{S}^{\mathbb{A}}$  such that*

$$\lambda_f^{SK, A} \circ \theta_{S, A} = Sf,$$

*for every  $f \in (A \downarrow K)$ , where  $\lambda_f^{SK, A} : Ran_K(SK)(A) \rightarrow SK(B)$  is the morphism in the limiting cone of  $Ran_K(SK)(A)$  associated with  $f : A \rightarrow K(B)$ .*

*Proof.* Theorem 1 in [8, §X.3] gives the counit and proves the universal property for the adjunction in the statement. Finding the unit is just a matter of applying the universal property in the special case of an identity natural transformation, as in every adjunction.  $\square$

**Lemma 4.2.** *Under the conditions in the previous Lemma 4.1, for each  $S \in \mathbb{S}^{\mathbb{A}}$  and  $A \in \mathbb{A}$ ,  $\theta_{S,A} : S(A) \rightarrow \text{Ran}_K(SK)(A)$  is a monomorphism in  $\mathbb{S}$  if and only if the morphisms in the set  $\{Sf : S(A) \rightarrow SK(B) \mid f : A \rightarrow K(B), B \in \mathbb{B}\}$  are jointly monic in  $\mathbb{S}$ .*

*Proof.* The easy proof is displayed in [16, Lemma 4.1]. The immediately above Lemma 4.1 completes that proof, by justifying the nature of the unit  $\theta$  assumed in [16, Lemma 4.1].  $\square$

**Definition 4.3.** *Consider, in a category  $\mathbb{S}$  with coproducts, a family  $(f_{ij} : s_i \rightarrow s_{ij})_{j \in J}$  of families of morphisms in  $\mathbb{S}$  with the same domain  $s_i$  which are jointly monic, for each  $i \in I$  (notice that  $J$  is fixed). It will be said that the coproducts preserve jointly monic morphisms in  $\mathbb{S}$  if, for any such family of families, the family  $(\coprod_{i \in I} f_{ij} : \coprod_{i \in I} s_i \rightarrow \coprod_{i \in I} s_{ij})_{j \in J}$  is also jointly monic, where  $\coprod_{i \in I} f_{ij}$  is the morphism uniquely determined by the coproduct diagram associated to  $(f_{ij})_{i \in I}$ .*

As an example, it is trivial that coproducts preserve jointly monic morphisms in the category *Set* of sets.

**Lemma 4.4.** *Under the conditions of Lemmas 4.1, suppose that  $\mathbb{S}$  has coproducts which preserve jointly monic morphisms (cf. previous Definition 4.3 just above).*

*If  $(S_i)_{i \in I}$  is a family of functors belonging to  $\mathbb{S}^{\mathbb{A}}$ , such that  $\theta_{S_i} : S_i \rightarrow \text{Ran}_K(S_i K)$  is monic, for all  $i \in I$ , then the unit  $\theta_{\coprod_{i \in I} S_i}$  of the coproduct is also monic.*

*Proof.* We need to show that  $\theta_{\coprod_{i \in I} S_i, A} : \coprod_{i \in I} S_i(A) \rightarrow \text{Ran}_K(\coprod_{i \in I} S_i K)(A)$  is a monomorphism for every  $A \in \mathbb{A}$ ; or, equivalently, that the morphisms in the sets

$$\left\{ \coprod_{i \in I} S_i f : \coprod_{i \in I} S_i(A) \rightarrow \coprod_{i \in I} S_i K(B) \mid f : A \rightarrow K(B), B \in \mathbb{B} \right\}$$

are jointly monic for every  $A \in \mathbb{A}$ , according to Lemma 4.2.

Since  $\theta_{S_i, A} : S_i(A) \rightarrow \text{Ran}_K(S_i K)(A)$  is a monomorphism by assumption, for all  $i \in I$  and  $A \in \mathbb{A}$ , then all the sets  $\{S_i f : S_i(A) \rightarrow S_i K(B) \mid f : A \rightarrow K(B), B \in \mathbb{B}\}$  are jointly monic, again by Lemma 4.2. Hence, as coproducts preserve jointly monic morphisms in  $\mathbb{S}$ , it follows that  $\theta_{\coprod_{i \in I} S_i}$  is monic.  $\square$

**Theorem 4.5.** *Let  $K : \mathbb{B} \rightarrow \mathbb{A}$  be a functor from a small category  $\mathbb{B}$  into a category  $\mathbb{A}$  with small hom-sets. There is an adjunction  $\text{Ran}_K \vdash \text{Set}^K : \text{Set}^{\mathbb{A}} \rightarrow \text{Set}^{\mathbb{B}}$  given by right Kan extensions calculated as pointwise limits. Then, factorizing its unit morphisms  $\theta_S : S \rightarrow \text{Ran}_K(SK)$  componentwise with the surjections and injections, gives rise to the full reflection with stable units  $H \vdash I : \text{Set}^{\mathbb{A}} \rightarrow \text{Mono}(\text{Set}^K)$ .*

*If  $K(\text{ob}\mathbb{B})$  is a cogenerating class for  $\mathbb{A}$ , then  $(\mathcal{E}'_I, \mathcal{M}^*_I)$  is a monotone-light factorization system on  $\text{Set}^{\mathbb{A}}$  (where  $K(\text{ob}\mathbb{B})$  is the image by  $K$  of every object in  $\mathbb{B}$ ).*

*Proof.* Consider two morphisms  $f, g : A' \rightarrow A$  in  $\mathbb{A}$ , and their image  $f^*, g^* : \mathbb{A}(A, K(-)) \rightarrow \mathbb{A}(A', K(-))$  by  $\text{Set}^K \circ y$ . These two morphisms  $f^*$  and  $g^*$  are distinct functors if and only if there exists  $B \in \mathbb{B}$  and a morphism  $h : A \rightarrow K(B)$  such that  $f^*_B(h) = h \circ f \neq h \circ g = g^*_B(h)$  (see [10], just before Theorem 6.1 there, where this part of the proof was given).

In the last paragraph, it has been shown that  $\text{Set}^K \circ y$  is faithful if and only if  $K(\text{ob}\mathbb{B})$  is a cogenerating class for  $\mathbb{A}$ . It then follows, by Lemma 2.11(b) that all representable functors  $\mathbb{A}(A, -)$  are in  $\text{Mono}(\text{Set}^K)$ ,  $A \in \mathbb{A}$ , if and only if  $K(\text{ob}\mathbb{B})$  is a cogenerating class for  $\mathbb{A}$ . Then, by the precedent Lemma 4.4, with  $\mathbb{S} = \text{Set}$ , all the coproducts  $\coprod_{i \in I} \mathbb{A}(A_i, -)$  are in  $\text{Mono}(\text{Set}^K)$ . Hence, the conditions of Theorem 2.9 hold (cf. Examples 2.10), and so there is a monotone-light factorization system derived from  $\text{Set}^K \dashv \text{Ran}_K$  if  $K(\text{ob}\mathbb{B})$  is a cogenerating class for  $\mathbb{A}$ .  $\square$

**Examples 4.6.** Consider the category  $\Delta$  of non-empty finite ordinal numbers  $[n]$  ( $n \geq 0$ ), where the singular set  $\{[0]\}$  is a generating class ( $\Delta = \Delta^+$  as in [8, VII.5]:  $[n]$  is the linear order whose objects are  $0, 1, \dots, n$ ; the morphisms of  $\Delta$  are the weakly monotone functions). Hence,  $\{[0]\}$  is a cogenerating class in  $\Delta^{op}$ . Let  $K : \mathbb{B} \rightarrow \Delta^{op}$  be a functor (from a small category  $\mathbb{B}$ ) such that  $[0]$  is in the image  $K(\text{ob}\mathbb{B})$  of the objects of  $\mathbb{B}$ . Then,  $H \vdash I : \text{Set}^{\Delta^{op}} \rightarrow \text{Mono}(\text{Set}^K)$  is a very-well-behaved reflection.

The same conclusion holds if one replaces  $\Delta$  by its full subcategory  $\Delta_n$ , determined by the  $n + 1$  objects  $[0], [1], \dots, [n]$ .

For instance, if  $\mathbb{B} = \Delta_0^{op} \cong \mathbf{1}$ , with  $K : \Delta_0^{op} \rightarrow \Delta^{op}$  the full inclusion, one gets the non-trivial (i.e.  $\mathcal{E}'_I$  strictly contained in  $\mathcal{E}_I$ ) monotone-light factorization for simplicial sets via ordered simplicial complexes (cf. [9]).

## 5. General Results for Sub-reflections from Models

In the present paper, a *sketch* consists of a category  $\mathbb{A}$ , together with a chosen set of distinguished cones of the form

$$\lambda : \Delta(a) \rightarrow L,$$

where  $\Delta : \mathbb{A} \rightarrow \mathbb{A}^{\mathbb{J}}$  is the diagonal functor,  $\mathbb{J}$  is a small category and  $L : \mathbb{J} \rightarrow \mathbb{A}$  is a functor ( $\mathbb{J}, L, a, \lambda$  vary from one distinguished cone to another).

$\check{\mathbb{A}}(\mathbb{S})$  shall denote the full subcategory of the functor category  $\hat{\mathbb{A}}(\mathbb{S}) = \mathbb{S}^{\mathbb{A}}$  determined by the models of such a sketch, that is, whose objects are the functors  $M : \mathbb{A} \rightarrow \mathbb{S}$  for which  $M\lambda = M\Delta(a) \rightarrow ML$  is a limiting cone in  $\mathbb{S}$ , for each distinguished cone in the sketch.

If  $\mathbb{S} = \mathbf{Set}$  then we will simply write  $\hat{\mathbb{A}} (= \hat{\mathbb{A}}(\mathbf{Set}) = \mathbf{Set}^{\mathbb{A}})$  and  $\check{\mathbb{A}} (= \check{\mathbb{A}}(\mathbf{Set}))$ .

If there is a factorization system  $(\mathcal{E}, \mathcal{M})$  on  $\mathbb{S}$ , then  $(\hat{\mathcal{E}}, \hat{\mathcal{M}}) = (\mathcal{E}^{\mathbb{A}}, \mathcal{M}^{\mathbb{A}})$  will denote the factorization system which is the uplifting of  $(\mathcal{E}, \mathcal{M})$  to  $\hat{\mathbb{A}}(\mathbb{S})$  (cf. Lemma 2.16(b)).

**Lemma 5.1.** *Let  $(\mathcal{E}, \mathcal{M})$  be a factorization system on a category  $\mathbb{S}$ .*

*Consider any natural transformation  $\varphi : F \rightarrow G : \mathbb{J} \rightarrow \mathbb{S}$ , where both functors  $F, G$  are supposed to have limits in  $\mathbb{S}$ , with cones  $(\text{Lim}F, (\lambda_i^F)_{i \in \mathbb{J}})$  and  $(\text{Lim}G, (\lambda_i^G)_{i \in \mathbb{J}})$ ; and  $s$  is the object obtained by the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism  $\text{Lim}\varphi = \mu \circ \eta : \text{Lim}F \rightarrow s \rightarrow \text{Lim}G$ .*

*Consider also the functor  $D : \mathbb{J} \rightarrow \mathbb{S}$  obtained by factorizing  $\varphi = n \cdot f : F \rightarrow D \rightarrow G$  using the factorization system  $(\hat{\mathcal{E}}, \hat{\mathcal{M}})$ , which is the uplifting of the factorization system  $(\mathcal{E}, \mathcal{M})$  to  $\hat{\mathbb{S}}(\mathbb{J}) = \mathbb{S}^{\mathbb{J}}$  (i.e., using the  $(\mathcal{E}, \mathcal{M})$ -factorization for each component  $\varphi_i = n_i \circ f_i : F(i) \rightarrow D(i) \rightarrow G(i)$ ,  $i \in \mathbb{J}$ ).*

*If  $\text{Lim}D$  exists in  $\mathbb{S}$ , then it is isomorphic to the object  $s$ , provided  $\text{Lim}f : \text{Lim}F \rightarrow \text{Lim}D$  is in  $\mathcal{E}$ , with  $n \cdot f : F \rightarrow D \rightarrow G$  the  $(\hat{\mathcal{E}}, \hat{\mathcal{M}})$ -factorization of  $\varphi$ .*

*Proof.* It is known that  $\text{Lim } n : \text{Lim } D \rightarrow \text{Lim } G$  is in  $\mathcal{M}$ , since every  $n_i \in \mathcal{M}$ ,  $i \in \mathbb{J}$  (cf. [1, Proposition 2.2(d)]). Hence,  $\text{Lim } D \cong s$ , by definition of factorization system (cf. [1, §2.8]).  $\square$

The following Corollary 5.2 is intended for future applications, detailing sufficient conditions which are not for the most needed in the next applications to n-categories (in the present paper only the item (a) is relevant).

**Corollary 5.2.** *Under the conditions of the previous Lemma 5.1:*

- (a) *if  $\mathbb{J} = i \xrightarrow{u} k \xleftarrow{v} j$ , being  $\text{Lim } F$  and  $\text{Lim } G$  pullbacks, and if the pullback  $\text{Lim } D$  exists, then it is isomorphic to the object  $s$  obtained by the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism  $\varphi_i \times \varphi_j = \mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$ , provided  $f_i, f_j \in \mathcal{E}'$  (the largest subclass of  $\mathcal{E}$  stable under pullbacks) and  $n_k, \varphi_k \in \text{Mono}(\mathbb{S})(\Leftarrow \varphi_k \in \mathcal{M} \cap \text{Mono}(\mathbb{S}))$ , with  $\text{Mono}(\mathbb{S})$  the class of monomorphisms of  $\mathbb{S}$ ;*
- (b) *if  $\mathbb{J}$  is the discrete category generated by the finite set  $\{1, 2, \dots, n\}$ , ( $n \geq 0$ ), being  $\text{Lim } F = \prod_{i=1}^n F(i)$  and  $\text{Lim } G = \prod_{i=1}^n G(i)$   $n$ -products, then the  $n$ -product  $\text{Lim } D = \prod_{i=1}^n D(i)$  is isomorphic to the object  $s$  obtained by the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism  $\prod_{i=1}^n \varphi_i = \mu \circ \eta : \prod_{i=1}^n F(i) \rightarrow \prod_{i=1}^n G(i)$ , provided that  $\mathbb{S}$  has pullbacks and the class  $\mathcal{E}$  of morphisms of  $\mathbb{S}$  is stable under pullbacks ( $\mathcal{E} = \mathcal{E}'$ );*
- (c) *if  $\mathbb{J} = \Downarrow (= 0 \xrightarrow{u} 1)$ , being  $\text{Lim } F$  and  $\text{Lim } G$  equalizers, and if the equalizer  $\text{Lim } D$  exists, then it is isomorphic to the object  $s$  obtained by the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism  $\mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$ , provided the binary products  $F(0) \times F(1)$ ,  $D(0) \times D(1)$ ,  $G(0) \times G(1)$  exist,  $n_0 \times n_1 : D(0) \times D(1) \rightarrow G(0) \times G(1)$ ,  $\varphi_0 \times \varphi_1 : F(0) \times F(1) \rightarrow G(0) \times G(1)$  are monomorphisms in  $\mathbb{S}$  ( $\Leftarrow n_0, n_1, \varphi_0, \varphi_1 \in \text{Monos}(\mathbb{S})$ ), and  $f_0 \in \mathcal{E}'$ ;*
- (d) *if  $\mathbb{S}$  has finite limits, then  $\text{Lim } D$  is isomorphic to the object  $s$  obtained by the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism  $\text{Lim } \varphi = \mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$ , provided  $\mathbb{J}$  is finite,  $\varphi_i : F(i) \rightarrow G(i)$  and  $n_i : D(i) \rightarrow G(i)$  are monomorphisms for every  $i \in \mathbb{J}$ , and  $\mathcal{E} = \mathcal{E}'$ .*

*Proof.* (a) Consider the commutative diagram

$$\begin{array}{ccccc}
 \text{Lim}F & \xrightarrow{\lambda_j^F} & F(j) & & \\
 \downarrow \lambda_i^F & \searrow f_i \times f_j & \downarrow f_j & & \\
 & \text{Lim}D & \xrightarrow{\lambda_j^D} & D(j) & \\
 & \downarrow \lambda_i^D & \searrow n_i \times n_j & \downarrow n_j & \\
 & & \text{Lim}G & \xrightarrow{\lambda_j^G} & G(j) \\
 & & \downarrow \lambda_i^G & & \downarrow Gv \\
 F(i) & \xrightarrow{f_i} & D(i) & \xrightarrow{n_i} & G(i) & \xrightarrow{Gu} & G(k).
 \end{array}$$

The outside square and the middle square are pullback diagrams, since  $\varphi_k$  and  $n_k$  are monomorphisms by hypothesis:  $Gu \circ n_i \circ f_i = Gu \circ \varphi_i = \varphi_k \circ Fu$  and  $Gv \circ n_j \circ f_j = Gv \circ \varphi_j = \varphi_k \circ Fv$ ;  $Gu \circ n_i = n_k \circ Du$  and  $Gv \circ n_j = n_k \circ Dv$ .

According to the properties of a factorization system (cf. [1, §2.8] and last Lemma 5.1), one needs to show that  $n_i \times n_j \in \mathcal{M}$  and  $f_i \times f_j \in \mathcal{E}$  in order that  $s \cong \text{Lim}D$ . That is obviously so, since  $n_i \times n_j \in \mathcal{M}$  by the general property of prefactorization systems stated in [1, Proposition 2.2(d)], and  $f_i, f_j \in \mathcal{E}'$ .

(b) If  $n = 0$  and hence  $F = \emptyset = G$ , then the  $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism is  $t = t = t$ , where  $t = \text{Lim}\emptyset = \text{Lim}F = \text{Lim}G$  is a terminal object of  $\mathbb{S}$ .

If  $n = 1$  it is also obvious, since  $\text{Lim}F = F(1)$ ,  $\text{Lim}G = G(1)$  and  $\text{Lim}D = D(1)$  ( $\varphi_1 = n_1 \circ f_1 : F(1) \rightarrow D(1) \rightarrow G(1)$ ).

If  $n = 2$ , then  $\text{Lim}F = F(1) \times F(2)$ ,  $\text{Lim}G = G(1) \times G(2)$  are binary products. Consider the commutative diagram

$$\begin{array}{ccccc}
D(1) & \xleftarrow{\lambda_1^D} & D(1) \times D(2) & \xrightarrow{\lambda_2^D} & D(2) \\
\downarrow n_1 & \swarrow p_2^1 & \downarrow n_1 \times n_2 & \searrow p_2^2 & \downarrow n_2 \\
& P(1) & & P(2) & \\
& \swarrow p_1^1 & & \swarrow p_1^2 & \\
G(1) & \xleftarrow{\lambda_1^G} & G(1) \times G(2) & \xrightarrow{\lambda_2^G} & G(2),
\end{array}$$

where the three squares  $\lambda_1^G \circ p_1^1 = n_1 \circ p_2^1$ ,  $\lambda_2^G \circ p_1^2 = n_2 \circ p_2^2$  and  $p_1^1 \circ q_1 = p_1^2 \circ q_2$  are pullback diagrams,  $p_2^i \circ q_i = \lambda_i^D$  and  $n_1 \times n_2 = p_1^i \circ q_i$  ( $i = 1, 2$ ). It was just shown that  $\text{Lim} D = D(1) \times D(2)$  exists and can be obtained using pullbacks, which are supposed to exist in the category  $\mathbb{S}$ .

Consider now the next commutative diagram

$$\begin{array}{ccccc}
F(1) & \xleftarrow{\lambda_1^F} & F(1) \times F(2) & \xrightarrow{\lambda_2^F} & F(2) \\
f_1 \downarrow & & f_1 \times f_2 \downarrow & & \downarrow f_2 \\
D(1) & \xleftarrow{\lambda_1^D} & D(1) \times D(2) & \xrightarrow{\lambda_2^D} & D(2) \\
n_1 \downarrow & & n_1 \times n_2 \downarrow & & \downarrow n_2 \\
G(1) & \xleftarrow{\lambda_1^G} & G(1) \times G(2) & \xrightarrow{\lambda_2^G} & G(2),
\end{array}$$

where  $n_1 \times n_2 \in \mathcal{M}$  (cf. Proposition 2.2(d) in [1]). Then, one has just to show that  $f_1 \times f_2 \in \mathcal{E}$ , which is so since the binary product  $F(1) \times F(2)$  may be obtained via pullbacks (from the product diagram of  $D(1) \times D(2)$ , exactly in the same way as we showed that  $D(1) \times D(2)$  exists using the product diagram of  $G(1) \times G(2)$ ), and the class of morphisms  $\mathcal{E}$  of  $\mathbb{S}$  is stable under pullbacks ( $\mathcal{E} = \mathcal{E}'$ ).

For  $n \geq 3$ , the statement follows immediately by iteration of binary products.

(c) The proof of this item is just an adaptation of that of item (b), using the well known fact that any equalizer diagram is a pullback diagram if there is a binary product of the two objects involved other than the limit object.

Consider the commutative diagram

$$\begin{array}{ccccc}
\text{Lim}F & \xrightarrow{\lambda_0^F} & & & F(0) \\
& \searrow \text{Lim}f & & & \downarrow f_0 \\
& & \text{Lim}D & \xrightarrow{\lambda_0^D} & D(0) \\
& & \downarrow \lambda_0^D & \searrow \text{Lim}n & \downarrow n_0 \\
& & & & \text{Lim}G \xrightarrow{\lambda_0^G} G(0) \\
& & & & \downarrow \lambda_0^G \\
F(0) & \xrightarrow{f_0} & D(0) & \xrightarrow{n_0} & G(0) \xrightarrow{(1, Gu)} G(0) \times G(1). \\
& & & & \downarrow (1, Gv)
\end{array}$$

The outside square and the middle square are pullback diagrams, since  $\varphi_0 \times \varphi_1$  and  $n_0 \times n_1$  are monomorphisms by hypothesis:  $(1, Gu) \circ n_0 \circ f_0 = (1, Gu) \circ \varphi_0 = \varphi_0 \times \varphi_1 \circ (1, Fu)$  and  $(1, Gv) \circ n_0 \circ f_0 = (1, Gv) \circ \varphi_0 = \varphi_0 \times \varphi_1 \circ (1, Fv)$ ;  $(1, Gu) \circ n_0 = n_0 \times n_1 \circ (1, Du)$  and  $(1, Gv) \circ n_0 = n_0 \times n_1 \circ (1, Dv)$ .

According to the properties of a factorization system (cf. the references in the proof of (a)), one needs to show that  $\text{Lim}f = f_0 \times_{G(0) \times G(1)} f_0 \in \mathcal{E}$  in order that  $s \cong \text{Lim}D$ . That is so, since  $f_0 \in \mathcal{E}'$ .

(d) The proof follows from the proof of item (c), since any limit of a functor  $X : \mathbb{J} \rightarrow \mathbb{S}$  can be presented canonically using products and an equalizer of two specific morphisms  $\varphi^X, \gamma^X$  in  $\mathbb{S}$  (cf. [8, §V.2]). Just consider the following commutative diagram,

$$\begin{array}{ccccc}
\text{Lim} F & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} F(i)}^F} & \prod_{i \in \mathbb{J}} F(i) & & \\
\downarrow \lambda_{\prod_{i \in \mathbb{J}} F(i)}^F & \searrow \text{Lim} f & \downarrow \prod_{i \in \mathbb{J}} f_i & & \\
& & \text{Lim} D & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} D(i)}^D} & \prod_{i \in \mathbb{J}} D(i) \\
& & \downarrow \lambda_{\prod_{i \in \mathbb{J}} D(i)}^D & \searrow \text{Lim} n & \downarrow \prod_{i \in \mathbb{J}} n_i \\
& & & & \text{Lim} G & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} G(i)}^G} & \prod_{i \in \mathbb{J}} G(i) \\
& & & & \downarrow \lambda_{\prod_{i \in \mathbb{J}} G(i)}^G & & \downarrow (1, \gamma^G) \\
\prod_{i \in \mathbb{J}} F(i) & \xrightarrow{\prod_{i \in \mathbb{J}} f_i} & \prod_{i \in \mathbb{J}} D(i) & \xrightarrow{\prod_{i \in \mathbb{J}} n_i} & \prod_{i \in \mathbb{J}} G(i) & \xrightarrow{(1, \varphi^G)} & \prod_{i \in \mathbb{J}} G(i) \times \prod_u G(\text{cod } u),
\end{array}$$

which is the adaptation of the diagram in the proof of item (c), to the three equalizers arising from the given limits. Remark that, according to item (c), it is needed that  $\prod_{i \in \mathbb{J}} \varphi_i \times \prod_u \varphi_{\text{cod } u} \in \text{Mono}(\mathbb{S})$  ( $\Leftarrow \varphi_i \in \text{Mono}(\mathbb{S}), i \in \mathbb{J}$ ),  $\prod_{i \in \mathbb{J}} n_i \times \prod_u n_{\text{cod } u} \in \text{Mono}(\mathbb{S})$  ( $\Leftarrow n_i \in \text{Mono}(\mathbb{S}), i \in \mathbb{J}$ ), and  $\prod_{i \in \mathbb{J}} f_i \in \text{Mono}(\mathbb{S}) \in \mathcal{E}$  ( $\Leftarrow \mathcal{E} = \mathcal{E}'$  and the set of objects of  $\mathbb{J}$  is finite; cf. item (b) above).  $\square$

**Proposition 5.3.** *Consider an adjunction  $\text{Ran}_K \vdash \text{Set}^K : \hat{\mathbb{A}} = \text{Set}^{\mathbb{A}} \rightarrow \hat{\mathbb{B}} = \text{Set}^{\mathbb{B}}$ , as in Theorem 4.5.*

*Let  $\check{\mathbb{A}}$  be the full subcategory determined by the models of a (fixed) sketch (cf. the beginning of this section 5), such that  $\text{Ran}_K(M \circ K) \in \check{\mathbb{A}}$  and the canonical map  $\text{Lim}(M \circ L) \rightarrow \text{Lim}(D \circ L)$  is always a surjection, where  $D$  is the presheaf obtained by the (Surjections, Injections)-factorization of the unit morphism  $\theta_M : M \rightarrow \text{Ran}_K(M \circ K)$ , for every  $M \in \check{\mathbb{A}}$  and for every distinguished  $L : \mathbb{J} \rightarrow \mathbb{A}$ .*

*Then, there is a reflection with stable units  $\check{H} \vdash \check{I} : \check{\mathbb{A}} \rightarrow \check{\mathbb{M}}$  which is a subreflection of  $H \vdash I : \hat{\mathbb{A}} \rightarrow \mathbb{M} = \text{Mono}(\text{Set}^K)$ , and  $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}}) = (\mathcal{E}_I \cap \text{Mor}(\check{\mathbb{A}}), \mathcal{M}_I \cap \text{Mor}(\check{\mathbb{A}}))$ . Here,  $\check{\mathbb{M}}$  is the full subcategory of  $\mathbb{M}$  determined by the models of the given sketch on  $\hat{\mathbb{A}}$ ,  $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}})$  and  $(\mathcal{E}_I, \mathcal{M}_I)$  are the reflective factorization systems associated respectively to the reflections  $\check{H} \vdash \check{I}$  and  $H \vdash I$ , and  $\text{Mor}(\check{\mathbb{A}})$  is the class of all morphisms in  $\check{\mathbb{A}}$ .*

*Proof.* As, by hypothesis, for every  $M \in \check{\mathbb{A}}$ ,

$$\text{Ran}_K(MK) \in \check{\mathbb{A}}$$

( $\Leftrightarrow \text{Ran}_K(MK) \in \check{\mathbb{M}}$  by Remark 2.5), then  $I(M) \in \check{\mathbb{M}}$  according to Lemma 5.1 just above: consider, in the statement of Lemma 5.1,  $F = M \circ L, G = \text{Ran}_K(MK) \circ L : \mathbb{J} \rightarrow \mathbb{A} \rightarrow \text{Set} = \mathbb{S}, \varphi = \theta_M \circ L$ . Therefore,  $\check{\mathbb{M}}$  is indeed a full reflective subcategory of  $\check{\mathbb{A}}$ .

Let  $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}})$  be the reflective factorization system associated to  $\check{I} \dashv \check{H}$ . It is known that  $\gamma : M \rightarrow N$  is in  $\mathcal{E}_{\check{I}}$  if and only if  $\check{I}\gamma = I\gamma$  is an isomorphism (cf. [1, §3.1]), therefore  $\mathcal{E}_{\check{I}} = \mathcal{E}_I \cap \text{Mor}(\check{\mathbb{A}})$ . As  $\check{\mathbb{A}}$  is closed under limits in  $\hat{\mathbb{A}}$  (using the property of interchange of limits, cf. [8, §IX.2]), it follows that  $\mathcal{E}'_I \cap \text{Mor}(\check{\mathbb{A}}) \subseteq \mathcal{E}'_{\check{I}}$ . Hence,  $\check{I} \dashv \check{H}$  has stable units because  $I \dashv H$  has stable units (the units  $\check{\eta}_M = \eta_M \in \mathcal{E}'_I, M \in \check{\mathbb{A}}$ ). Being  $\check{H} \vdash \check{I}$  a reflection with stable units it is also simple, and then  $\mathcal{M}_{\check{I}} = \mathcal{M}_I \cap \text{Mor}(\check{\mathbb{A}})$ , because  $\check{\mathbb{A}}$  is closed under limits in  $\hat{\mathbb{A}}$  and by the characterization of the morphisms in  $\check{\mathbb{M}}$  in a simple reflection (cf. Theorem 4.1 in [2] or [1, §3.5]).  $\square$

**Examples 5.4.** Consider  $\mathbb{P} = \Delta_3^{op}$  the dual of the truncated simplicial category generated by the diagram (where  $0 \leq i \leq k+1$  for  $d_i^k$  and  $0 \leq j \leq k$  for  $s_j^k, k \in \{0, 1, 2\}$ )

$$[3] \xrightleftharpoons[s_j^2]{d_i^2} [2] \xrightleftharpoons[s_j^1]{d_i^1} [1] \xrightleftharpoons[s_0^0]{d_i^0} [0]$$

and the usual relations (cf. Examples 4.6 and [8, §VII.5]; notice that this example could as well be worked out using the dual of the entire simplicial category  $\mathbb{A} = \Delta^{op}$ )<sup>1</sup>.

Let  $K : \Delta_0^{op} \rightarrow \Delta_3^{op} = \mathbb{P}$  be the inclusion functor ( $\Delta_0^{op} \cong 1$ ). Since  $\{[0]\}$  is a cogenerating class for  $\Delta_3^{op}$  (because  $[0]$  is a generator in  $\Delta$ ), then, by Theorem 4.5, there is a reflection  $H \vdash I : \hat{\mathbb{P}} = \text{Set}^{\mathbb{P}} \rightarrow \mathbb{M} = \text{Monos}(\text{Set}^K)$  which is very-well-behaved (i.e., with stable units and a monotone-light factorization).  $\hat{\mathbb{P}}$  could be notated  $\text{Smp}_3$ , the category of truncated simplicial sets.

This reflection induces a subreflection with stable units  $\check{H} \vdash \check{I} : \check{\mathbb{P}} \rightarrow \check{\mathbb{M}}$ , for the sketch whose models  $M : \mathbb{P} \rightarrow \text{Set}$  are those functors such that  $Md_1^0 \circ Md_0^1 = Md_0^0 \circ Md_2^1$  and  $Md_2^1 \circ Md_0^2 = Md_0^1 \circ Md_3^2$  are pull-back squares, according to Proposition 5.3. Indeed,  $\text{Ran}_K \circ \text{Set}^K(S) \in \check{\mathbb{P}}$

<sup>1</sup>It follows from the Lemma in [8, §VII.5] that: truncating the dual of the simplicial category gives the same result as using the diagram above and the usual relations to generate  $\Delta_3^{op}$ .

for every  $S \in \hat{\mathbb{P}}$  (not only for every  $S \in \check{\mathbb{P}}$ ): it is easy to check that  $Ran_K(SK)([n]) = Lim(( [n] \downarrow K) \xrightarrow{Q} \Delta_0^{op} \xrightarrow{SK} Set) = S_0^{n+1}$ , the  $n+1$ -power of  $S_0 = S([0])$ ,  $0 \leq n \leq 3$ ; hence,  $Ran_K(SK)$  is an indiscrete category (a connected equivalence relation).  $\check{\mathbb{P}}$  can be identified with  $Cat$ , the category of all small categories and  $\mathbb{M}$  with the category  $Preord$  of all preordered sets.

Remark that Lemma 5.1 is needed so that  $Md_2^1 \circ Md_0^2 = Md_0^1 \circ Md_3^2$  is a pullback, but  $Md_1^0 \circ Md_0^1 = Md_0^0 \circ Md_2^1$  being a pullback is an immediate consequence of Corollary 5.2(a) (indeed, there  $\varphi_k = \theta_{M,[0]}$  is the identity).

The reflection  $Cat \rightarrow Preord$  was showed to be very-well-behaved in [13], where the monotone-light factorization was presented and showed to be non-trivial ( $\mathcal{E}'_I \neq \mathcal{E}_I$ ).

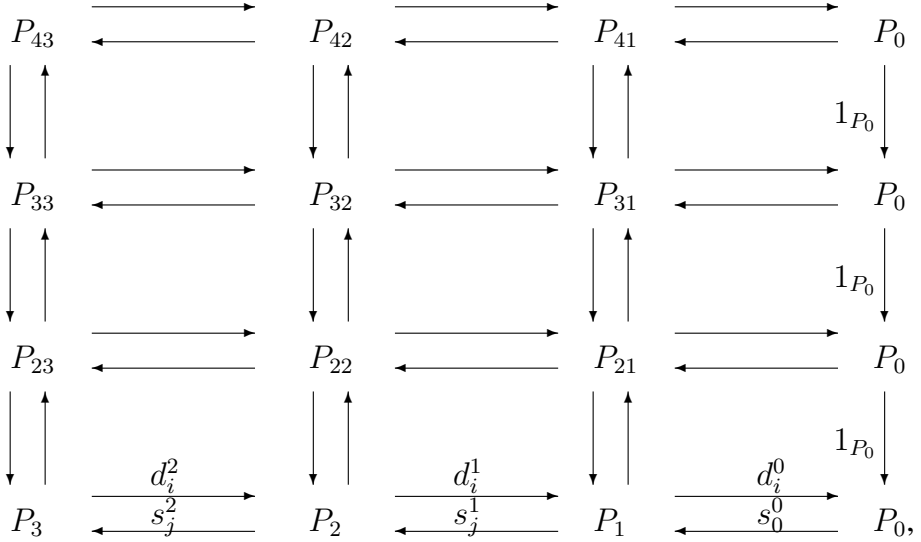
The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every category  $M \in \hat{\mathbb{P}}$ , there is an effective descent morphism  $p : E \rightarrow M$  in  $\hat{\mathbb{P}}$  which is also an effective descent morphism in  $\check{\mathbb{P}} = Cat$  (cf. the characterization of effective descent morphisms in  $Cat$  given in [5]);  $p$  is the canonical presentation of the presheaf  $M$ , where  $E$  is the coproduct of representable functors which are linear orders, and so  $E$  is obviously also a linear order (a special case of a preorder).

## 5.1 Discussion of the very-well-behaved reflection from 2-categories into 2-preorders

This and the following last subsection are presented in a lighter way than the rest of the paper. We expect to be more precise in a future paper.

The example discussed in this subsection was fully studied in [14], with a slightly different presentation from the one given here. In order to relate to that paper [14], so that the interested reader may have an easier task consulting it, we shall call *precategories* the objects of  $\hat{\mathbb{P}} = Set^{\mathbb{P}}$  in Examples 5.4 just above, and the diagram there which generates the category  $\mathbb{P}$  is now called a *precategory diagram*, with  $P_i = [i]$  ( $0 \leq i \leq 3$ ).

Consider the category  $2\mathbb{P}$  generated by the following 2-*precategory* diagram,



in which:

- each one of the four horizontal diagrams

$$\begin{array}{ccccccc}
 & \xrightarrow{d_i^{l2}} & & \xrightarrow{d_i^{l1}} & & \xrightarrow{d_i^{l0}} & \\
 P_{l3} & \xleftarrow{s_j^{l2}} & P_{l2} & \xleftarrow{s_j^{l1}} & P_{l1} & \xleftarrow{s_0^{l0}} & P_{l0}
 \end{array}$$

is a precategory diagram (remark that, in the 2-precategory diagram, for  $l = 1$ , the notation is simplified in order to be similar to the one in Examples 5.4), and each one of the four vertical diagrams

$$\begin{array}{ccccccc}
 & \xrightarrow{\check{d}_i^{l2}} & & \xrightarrow{\check{d}_i^{l1}} & & \xrightarrow{\check{d}_i^{l0}} & \\
 P_{4(l-1)} & \xleftarrow{\check{s}_j^{l2}} & P_{3(l-1)} & \xleftarrow{\check{s}_j^{l1}} & P_{2(l-1)} & \xleftarrow{\check{s}_0^{l0}} & P_{1(l-1)}
 \end{array}$$

is also a precategory diagram (the rightmost one is the trivial one with all morphisms identities, as portrayed in the 2-precategory diagram),  $l \in \{1, 2, 3, 4\}$ ;

- in the obvious sense given in [14], the  $\check{s}_j^{lk}$  provide precategory morphisms vertically, and the  $s_j^{lk}$  provide precategory morphisms horizontally,  $l \in \{1, 2, 3, 4\}$ ; similarly for the  $\check{d}_i^{lk}$  and  $d_j^{lk}$ .

Let  $K : \mathbb{P} \rightarrow 2\mathbb{P}$  be the inclusion functor of  $\mathbb{P}$  in  $2\mathbb{P}$

(recall, from Examples 5.4, that  $\mathbb{P}$  is generated by

$$P_3 \begin{array}{c} \xrightarrow{d_i^2} \\ \xleftarrow{s_j^2} \end{array} P_2 \begin{array}{c} \xrightarrow{d_i^1} \\ \xleftarrow{s_j^1} \end{array} P_1 \begin{array}{c} \xrightarrow{d_i^0} \\ \xleftarrow{s_0^0} \end{array} P_0 \text{ and the usual relations}).$$

$\{P_3 = P_{13}, P_2 = P_{12}, P_1 = P_{11}, P_0 = P_{10}\}$  is a cogenerating class for  $2\mathbb{P}$ , because it obviously is a generating class for  $2\mathbb{P}^{op}$ . Being  $K$  a full inclusion, it is well known that  $Ran_K \vdash Set^K$  is a full reflection (cf. Corollary 3 in [8, §X.3]), hence  $Ran_K$  can be seen as the inclusion of the category  $\hat{\mathbb{P}}$  in  $2\hat{\mathbb{P}}$ . Then, by Theorem 4.5, there is a very-well-behaved reflection

$$H \vdash I : 2\hat{\mathbb{P}} = Set^{2\hat{\mathbb{P}}} \rightarrow 2\mathbb{M} = Monos(Set^K).$$

This reflection induces a subreflection having stable units

$$\check{H} \vdash \check{I} : 2\check{\mathbb{P}} \rightarrow 2\check{\mathbb{M}},$$

for the sketch whose models are those functors  $M$  such that  $Md_1^{l0} \circ Md_0^{l1} = Md_0^{l0} \circ Md_2^{l1}$ ,  $Md_2^{l1} \circ Md_0^{l2} = Md_0^{l1} \circ Md_3^{l2}$  and  $M\check{d}_1^{l0} \circ M\check{d}_0^{l1} = M\check{d}_0^{l0} \circ M\check{d}_2^{l1}$ ,  $M\check{d}_2^{l1} \circ M\check{d}_0^{l2} = M\check{d}_0^{l1} \circ M\check{d}_3^{l2}$  are pullback squares for every horizontal and vertical precategory diagram respectively,  $l \in \{1, 2, 3, 4\}$ . Indeed, it is clear then that  $Ran_K \circ Set^K(M) \in 2\check{\mathbb{P}}$  for every  $M \in 2\check{\mathbb{P}}$ , so that the conditions of Proposition 5.3 hold.

$2\check{\mathbb{P}}$  can be identified with  $2Cat$ , the category of all 2-categories, and  $2\check{\mathbb{M}}$  with the category  $2Preord$  of all 2-preorders (cf. the beginning of §5 in [14]).

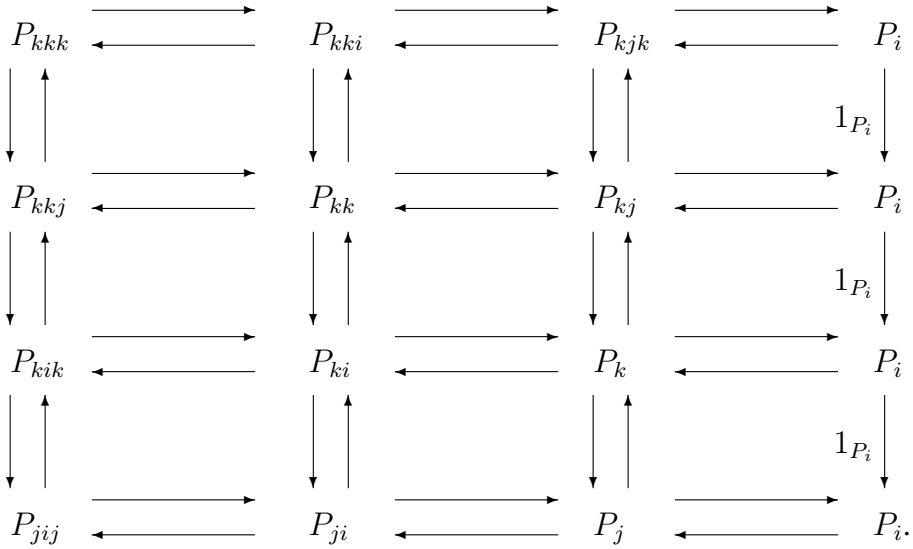
The reflection  $2Cat \rightarrow 2Preord$  was showed to be very-well-behaved in [14], where the monotone-light factorization was displayed and showed to be non-trivial ( $\mathcal{E}'_i \neq \mathcal{E}_i$ ). The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every 2-category  $M \in 2\check{\mathbb{P}}$ , there is an effective descent morphism  $p : E \rightarrow M$  in  $2\check{\mathbb{P}}$  which is also an effective descent morphism in  $2\check{\mathbb{P}} = 2Cat$ .

In [14], the proof that  $\check{H} \vdash \check{I} : 2Cat \rightarrow 2Preord$  has stable units is much more complicated than in here. In that paper, one used to the full extent the results in [11] to prove directly that the reflection  $\check{H} \vdash \check{I}$  has stable units, while in the present paper this reflection inherits this property (although, to do so, a small theory had to be developed).

## 5.2 Discussion of the very-well-behaved reflection from $n$ -categories into $n$ -preorders

This subsection generalizes the example of the previous subsection, whose results are subsumed here. Anyway, the somewhat repeating of subjects helps the exposition and discussion of these examples, which can be displayed using different theoretical paths.

Consider the category  $n\mathbb{P}$  generated by all the 2-precategory diagrams below (cf. [15, §10] and last subsection 5.1), for all integers  $i, j$  and  $k$  such that  $0 \leq i < j < k \leq n$ ,



The number  $\frac{(n-1)n(n+1)}{6}$  of these 2-precategory diagrams can easily be obtained by using the counting principle. Remark that these  $\frac{(n-1)n(n+1)}{6}$  diagrams are not independent since they share common objects and arrows (every arrow between two common objects), and that if  $n = 2$  there is only one 2-precategory diagram.

Let  $K : (n-1)\mathbb{P} \rightarrow n\mathbb{P}$  be the inclusion functor of  $(n-1)\mathbb{P}$  in  $n\mathbb{P}$ .

Being  $K$  a full inclusion, it is well known that  $Ran_K \vdash Set^K$  is a full

reflection (cf. Corollary 3 in [8, §X.3]), hence  $Ran_K$  can be seen as the inclusion of the category  $(n - \hat{1})\mathbb{P}$  in  $\hat{n}\mathbb{P}$ .

The objects of  $(n - 1)\mathbb{P}$  constitute a cogenerating class for  $n\mathbb{P}$ , because it obviously is a generating object for  $(n\mathbb{P})^{op}$ . Then, by Theorem 4.5, there is a very-well-behaved reflection

$$H \vdash I : \hat{n}\mathbb{P} = Set^{\hat{n}\mathbb{P}} \rightarrow n\mathbb{M} = Monos(Set^K).$$

This reflection induces a subreflection having stable units

$$\check{H} \vdash \check{I} : \check{n}\mathbb{P} \rightarrow \check{n}\mathbb{M},$$

for the sketch whose models are those functors  $M$  such that  $Md_1 \circ Md_0 = Md_0 \circ Md_2$  and  $Md_2 \circ Md_0 = Md_0 \circ Md_3$  are pullback squares for every horizontal and vertical precategory diagram (to understand this rather scanty notation, please cf. last subsection 5.1). Indeed, it is clear then that  $Ran_K \circ Set^K(M) \in \check{n}\mathbb{P}$  for every  $M \in \check{n}\mathbb{P}$ , so that the conditions of Proposition 5.3 hold.

$\check{n}\mathbb{P}$  can be identified with  $nCat$  (cf. [15, §10]), the category of all  $n$ -categories, and  $\check{n}\mathbb{M}$  with the category  $nPreord$  of all  $n$ -preorders (cf. the beginning of §12 in [15]).

The reflection  $nCat \rightarrow nPreord$  was showed to be very-well-behaved in [15], where the monotone-light factorization was displayed and showed to be non-trivial ( $\mathcal{E}'_I \neq \mathcal{E}_I$ ), for every  $n \geq 2$ . The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every  $n$ -category  $M \in \check{2}\mathbb{P}$ , there is an effective descent morphism  $p : E \rightarrow M$  in  $\hat{n}\mathbb{P}$  which is also an effective descent morphism in  $\check{n}\mathbb{P} = nCat$ .

In [15], the proof that  $\check{H} \vdash \check{I} : nCat \rightarrow nPreord$  has stable units was made using a quite different path from the one used here. In that paper, the category of  $n$ -categories was considered in the context of  $\mathcal{V}$ -categories; it is known that, considering  $\mathcal{V} = Set$  and then iterating,  $n$ -categories are obtained (see [7]). There, beginning with the well-behaved reflection  $Cat \rightarrow Preord$  and using the *change of enriching category* (see [3]), it was possible to show that the derived reflection from  $n$ -categories into  $n$ -preorders had stable units, while in the present paper this reflection inherits this property from larger reflections in which it is contained.

It is also relevant to state that the path in the context of  $\mathcal{V}$ -categories, in order to establish the stable units property, could be replaced by the process given in [11] which was used for 2-categories in [14].

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# CENTRAL EXTENSIONS OF NON-ASSOCIATIVE $K$ -ALGEBRAS

*Bernard STRAUSS*

**Résumé.** Soit  $K$  un anneau commutatif unitaire. Par  $K$ -algèbre, nous entendons tout  $K$ -module  $A$  muni d'un homomorphisme de  $K$ -modules  $A \otimes_K A \rightarrow A$ , noté  $a \otimes b \mapsto ab$ . Nous considérons la variété  $\text{Alg}_K$  de toutes les  $K$ -algèbres, ainsi que sa sous-variété  $\text{Alg}_K^{\text{AC}}$  des  $K$ -algèbres à la fois associatives et commutatives. Nous montrons qu'un épimorphisme régulier  $f : A \rightarrow B$  dans  $\text{Alg}_K$  est une extension centrale relativement à  $\text{Alg}_K^{\text{AC}}$  si et seulement si le noyau de  $f$  est contenu dans le centre de  $A$ , convenablement défini.

**Abstract.** Let  $K$  be a commutative ring with identity. By a  $K$ -algebra we mean any  $K$ -module  $A$  equipped with a  $K$ -module homomorphism  $A \otimes_K A \rightarrow A$  written as  $a \otimes b \mapsto ab$ . We consider the variety  $\text{Alg}_K$  of all  $K$ -algebras, and its subvariety  $\text{Alg}_K^{\text{AC}}$  of  $K$ -algebras that are both associative and commutative. We show that a regular epimorphism  $f : A \rightarrow B$  in  $\text{Alg}_K$  is a central extension with respect to  $\text{Alg}_K^{\text{AC}}$  if and only if the kernel of  $f$  is contained in the suitably defined centre of  $A$ .

**Keywords.** Central extension, non-associative algebra, associative algebra, commutative algebra, commutator, associator.

**Mathematics Subject Classification (2020).** 18E50, 17A01, 16B50, 16U70, 18E13.

## 1. Introduction

For a fixed commutative ring  $K$  with identity, we consider the reflection  $I : \text{Alg}_K \rightarrow \text{Alg}_K^{\text{AC}}$  from the variety  $\text{Alg}_K$  of  $K$ -algebras (by a  $K$ -algebra we mean any  $K$ -module  $A$  equipped with a  $K$ -module homomorphism  $A \otimes_K$

$A \rightarrow A$  written as  $a \otimes b \mapsto ab$ ) to the subvariety  $\text{Alg}_K^{\text{AC}}$  of associative and commutative  $K$ -algebras. For an algebra  $A$  in  $\text{Alg}_K$ , we denote by  $R(A)$  the kernel of the canonical epimorphism  $\eta_A : A \rightarrow I(A)$ , and by  $Z(A)$  the *centre* of  $A$ , which is defined as follows:

**Definition 1.1.** The *centre*  $Z(A)$  of a  $K$ -algebra  $A$  is the intersection of the sets

$$\{a \in A \mid \forall_{x \in A} [a, x] = 0\}$$

and

$$\{a \in A \mid \forall_{x \in A}, \forall_{y \in A} \langle a, x, y \rangle = \langle x, a, y \rangle = \langle x, y, a \rangle = 0\},$$

where  $[x, y] = xy - yx$  is the *commutator*, and  $\langle x, y, z \rangle = (xy)z - x(yz)$  the *associator*.

This definition of centre is the one used by non-associative algebraists, as shown e.g. in [5]. It is useful in the sense that it generalizes both the “commutative” centre used with associative algebras, as well as the “associative” centre used for example with Jordan algebras.

Using this, the following definition is then a non-associative generalization of Definition 1.1 in [3]:

**Definition 1.2.** A regular epimorphism (equivalently, a surjective homomorphism)  $f : A \rightarrow B$  of  $K$ -algebras is said to be:

1. *classically-central* if  $\text{Ker}(f) \subseteq Z(A)$ ;
2. *algebraically-central* if any two parallel morphisms  $u$  and  $v$ , both with codomain  $A$ , satisfy

$$fu = fv \Rightarrow R(u) = R(v);$$

3. *categorically-central* if the diagram

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\eta_{A \times_B A}} & I(A \times_B A) \\ \pi \downarrow & & \downarrow I(\pi) \\ A & \xrightarrow{\eta_A} & I(A) \end{array}$$

in which  $\pi$  is either of the two pullback projections  $A \times_B A \rightarrow A$ , is a pullback.

It was already shown in [4] (in particular, see Theorem 5.2) that, in the more general context of varieties of  $\Omega$ -groups,  $f$  is algebraically-central if and only if it is categorically-central. The purpose of this paper is then to show that  $f$  is classically-central if and only if it is algebraically-central (this is Theorem 2.5), thus tying this more general definition of classical centrality to Janelidze and Kelly's well-developed categorical theory (and its algebraic predecessor, mentioned in [4]; see also [2]). We do this "directly" (i.e. using algebraic identities), but one might also try several "less direct" approaches, such as using split extensions (as done by Janelidze in [3]) or using relative commutator theory (as introduced by Everaert in [2]).

**Remark 1.3.** Since  $\text{Alg}_K$  is a variety of  $\Omega$ -groups, it easily follows from Theorem 5.2 of [4] that a surjective homomorphism  $f : A \rightarrow B$  of  $K$ -algebras is algebraically-central if and only if

$$(x, y) \in R(A \times_B A) \Rightarrow x = y \quad (\dagger)$$

for all  $x, y \in A$ .

In fact, the same is true in the general context of [4], and even more generally for arbitrary semi-abelian varieties of universal algebras (where one can use Theorem 2.1 of [1] instead of Theorem 5.2 of [4]).

## 2. Central Extensions of $K$ -Algebras

For our proof of Theorem 2.5, it is also necessary that we define the reflection  $I$  explicitly: for an algebra  $A$ ,  $R(A)$  is given by the ideal generated by all elements of the forms  $[x, y]$  and  $\langle x, y, z \rangle$  where  $x, y, z \in A$ . This makes  $I(A) = A/R(A)$  associative and commutative, since  $I(A) = Z(I(A))$ .

The main result we will need is the fact that if  $\text{Ker}(f) \subseteq Z(A)$ , then  $R(A \times_B A)$  is a subalgebra of the diagonal  $1_A$  (where  $A \times_B A$  is the kernel-congruence of  $f$ ); this is shown in Lemmas 2.1 to 2.4. So, for the duration of these proofs, let  $\text{Ker}(f) \subseteq Z(A)$ , and note the following:

**Lemma 2.1.** For any  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times_B A$ ,

1.  $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [x_1, x_2])$ , and
2.  $\langle (x_1, y_1), (x_2, y_2), (x_3, y_3) \rangle = (\langle x_1, x_2, x_3 \rangle, \langle x_1, x_2, x_3 \rangle)$ .

*Proof.* We first rewrite each of the elements as  $(x_1, x_1 + k_1)$ ,  $(x_2, x_2 + k_2)$ , and  $(x_3, x_3 + k_3)$  for some  $k_1, k_2, k_3 \in \text{Ker}(f)$ . Applying the commutator yields

$$[(x_1, x_1 + k_1), (x_2, x_2 + k_2)] = ([x_1, x_2], [x_1 + k_1, x_2 + k_2]).$$

Then, a rudimentary calculation gives

$$\begin{aligned} [x_1 + k_1, x_2 + k_2] &= [x_1, x_2] + [x_1, k_2] + [k_1, x_2] + [k_1, k_2] \quad (\text{the commutator is multilinear}) \\ &= [x_1, x_2] \quad (\text{since } \text{Ker}(f) \subseteq Z(A)). \end{aligned}$$

On the other hand, applying the associator, we get

$$\begin{aligned} \langle (x_1, x_1 + k_1), (x_2, x_2 + k_2), (x_3, x_3 + k_3) \rangle \\ = \langle \langle x_1, x_2, x_3 \rangle, \langle x_1 + k_1, x_2 + k_2, x_3 + k_3 \rangle \rangle. \end{aligned}$$

Then, a similar simple calculation using the multilinearity of the associator (and the fact that  $\text{Ker}(f) \subseteq Z(A)$ ) yields

$$\langle x_1 + k_1, x_2 + k_2, x_3 + k_3 \rangle = \langle x_1, x_2, x_3 \rangle$$

which completes the proof. □

Next, consider the set

$$N(A, f) = \{x \in A \mid k \in \text{Ker}(f) \Rightarrow kx = 0 = xk\}$$

and note the following:

**Lemma 2.2.**  $N(A, f)$  is an ideal in  $A$ .

*Proof.* For  $x \in N(A, f)$  and  $a \in A$  we have

$$\begin{aligned} k(ax) &= (ka)x = (ak)x = a(kx) = a0 = 0, \\ (ax)k &= a(xk) = a0 = 0; \end{aligned}$$

similarly,  $k(xa) = 0 = (xa)k$ . Since  $N(A, f)$  is (clearly) closed under addition and scalar multiplication, this completes the proof. □

**Lemma 2.3.**  $R(A) \subseteq N(A, f)$ .

*Proof.* Since  $N(A, f)$  is an ideal in  $A$  (Lemma 2.2), it suffices to show that  $[x, y]$  and  $\langle x, y, z \rangle$  belong to  $N(A, f)$  for every  $x, y, z \in A$ . For  $k \in \text{Ker}(f)$ , we have

$$\begin{aligned} k[x, y] &= k(xy - yx) = (kx)y - y(kx) = 0, \text{ and} \\ k\langle x, y, z \rangle &= k((xy)z - x(yz)) = ((kx)y)z - (kx)(yz) = 0 \end{aligned}$$

since  $kx \in \text{Ker}(f) \subseteq Z(A)$ . Similarly,  $[x, y]k = 0$  and  $\langle x, y, z \rangle k = 0$ .  $\square$

**Lemma 2.4.**  $(\dagger)$  holds; that is,  $R(A \times_B A) \subseteq 1_A$ , where  $1_A$  is the diagonal (see Remark 1.3).

*Proof.* Let  $S = R(A \times_B A) \cap 1_A$ . By Lemma 2.1,  $S$  contains the canonical generators of  $R(A \times_B A)$ , and so it would suffice to prove that  $S$  is an ideal in  $A \times_B A$ . Since  $R(A \times_B A)$  is an ideal in  $A \times_B A$ , we only have to show that

$$\begin{aligned} &((x, x) \in R(A \times_B A) \ \& \ k \in \text{Ker}(f)) \\ \Rightarrow &(y, y + k)(x, x), (x, x)(y, y + k) \in 1_A \quad (*) \end{aligned}$$

for every  $y \in A$ . Since  $\eta$  is a natural transformation,  $(x, x)$  belongs to  $R(A \times_B A)$  if and only if  $x$  belongs to  $R(A)$ . Therefore, to prove  $(*)$  is equivalent to proving the implication

$$(x \in R(A) \ \& \ k \in \text{Ker}(f)) \Rightarrow kx = 0 = xk,$$

which follows immediately from Lemma 2.3.  $\square$

We are now in a position to prove the main result, using the results above and some results from [4]. In fact, the structure of the proof is also lifted from [4]: see there in particular the example after Theorem 5.2. We simply apply it to this new context, although with more results required (hence the above lemmas).

**Theorem 2.5.** Let  $f : A \rightarrow B$  be a regular epimorphism.  $f$  is classically-central if and only if it is algebraically-central.

*Proof.* ( $\Rightarrow$ ) Say  $f$  is classically-central, i.e.  $\text{Ker}(f) \subseteq Z(A)$ . Lemma 2.4 gives that  $(\dagger)$  is satisfied.

( $\Leftarrow$ ) Say  $f$  is algebraically-central. Take any  $k \in \text{Ker}(f)$ . Now, for any  $x \in A$  we calculate

$$[(x, x), (0, k)] = ([x, 0], [x, k]) = (0, [x, k]).$$

This being in  $R(A \times_B A)$ ,  $(\dagger)$  gives that  $0 = [x, k]$ . The same is of course true for any permutation of arguments, and so

$$0 = [x, k] = [k, x].$$

On the other hand, for any  $x, y \in A$ ,

$$\langle (x, x), (y, y), (0, k) \rangle = (\langle x, y, 0 \rangle, \langle x, y, k \rangle) = (0, \langle x, y, k \rangle).$$

Similar to the above,  $(\dagger)$  yields  $0 = \langle x, y, k \rangle$ , and the permutation of arguments yields

$$0 = \langle x, y, k \rangle = \langle x, k, y \rangle = \langle k, x, y \rangle.$$

Hence,  $k \in Z(A)$ , which finally gives  $\text{Ker}(f) \subseteq Z(A)$ .

□

As mentioned in the introduction, due to results from [4], this is enough to show the following:

**Corollary 2.6.** Let  $f : A \rightarrow B$  be a regular epimorphism of  $K$ -algebras.  $f$  is classically-central if and only if it is categorically-central.

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# ON THE GENERALIZED BRUNNIAN CONJECTURE

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**Résumé.** Nous démontrons la conjecture  $B(n, p, P)$  de [1, Conjecture 3.6 p. 62], que nous renommons ici “conjecture brunnienne”, dans les cas où  $p \geq 5$  et  $n$  n’est pas divisible par  $p-1$ . Nous introduisons aussi la “conjecture brunnienne généralisée”  $B(n, q, P)$  et nous la démontrons dans les cas où  $q \geq 4$  et  $n$  n’est pas divisible par  $q-1$ . En fin d’article, nous montrons aussi, analytiquement, que la stratégie intrinsèque que nous introduisons ici permet de conclure favorablement quant à la véracité de la conjecture dans une proportion significative des cas (voir Proposition 4.2 et Proposition 4.4)

**Abstract.** We prove the Conjecture  $B(n, p, P)$  of [1, Conjecture 3.6 p. 62], which is renamed here as the “Brunnian Conjecture”, in the case where  $p \geq 5$  and  $n$  is not a multiple of  $p-1$ . We also introduce, and prove the “Generalized Brunnian Conjecture”  $B(n, q, P)$  and prove it in the cases where  $q \geq 4$  and  $n$  is not divisible by  $q-1$ . At the end of the paper, we also show with through an analytical argument that the intrinsic strategy we introduce here leads to a proof of the conjecture in a significant proportion of cases (see Proposition 4.2 and Proposition 4.4).

**Keywords.** Brunnian Conjecture, finite fields, finite groups, linear groups, matrix groups, primitive polynomials.

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## 1. Introduction

The goal of this paper is to give a partial proof of the Conjecture  $B(n, p, P)$  ([1, Conjecture 3.6 p. 62]). We shall refer to it as the “Brunnian Conjecture” from now on. Our strategy consists in using Nick Gill’s proof of the Conjecture  $A(n, p, P)$  ([1, Conjecture 3.4 p. 62]) given in [3, Theorem 2 p. 230], corrected by the paper of Joel Brewster Lewis, see ([6, Theorem 7 p. 6 and Correction 3.3 p. 10]).

Our main result is:

**Theorem 1.1.** *Let  $p$  a prime number, and  $n \geq 3$  an integer, and*

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{F}_p[X]$$

*a primitive polynomial of degree  $n$ . Let  $C = C_f$  denote the companion matrix of  $f$ , namely:*

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_{n-2} \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix}.$$

*Let  $G$  denote the companion matrix of the polynomial  $X^n - 1$ , and let:*

$$\begin{aligned} G_1 &:= C, \\ G_{k+1} &:= GG_kG^{-1} \text{ for all } 1 \leq k \leq n-1. \end{aligned}$$

*Then, if  $p \geq 5$  and  $n$  is not divisible by  $p-1$ , then*

$$\langle G_1, G_2, \dots, G_n \rangle = \mathrm{GL}_n(\mathbb{F}_p).$$

## 2. Background and strategy

This story started with René Guitart’s works on Klein’s group (defined as the automorphisms group of the Klein’s quartic with 168 elements), described as a borromean objet ([4, section 3 p. 148]), which is nothing else that  $\mathrm{GL}_3(\mathbb{F}_2)$ . In [1], we extended René Guitart’s notion of borromean group

by introducing the notions of  $n$ -cyclable groups ([1, Definition 1.1 p. 58]) and  $n$ -brunnian groups ([1, Definition 1.2 and Definition 1.3 p. 58]). The so-called “Brunnian conjecture” refers to the following statement: if  $p$  is a prime number, and  $n \geq 2$ , then the group  $\mathrm{GL}_n(\mathbb{F}_p)$  can be generated by the companion matrix of a primitive polynomial  $P(X) \in \mathbb{F}_p[X]$  of degree  $n$ , and the conjugates by  $G^k$  for  $1 \leq k \leq n-1$  of the companion matrix of polynomial  $X^n - 1$ . This conjecture is correlated to this other one: the group  $\mathrm{GL}_n(\mathbb{F}_p)$  can be generated by the companion matrix of a primitive polynomial  $P(X) \in \mathbb{F}_p[X]$  of degree  $n$ , denoted by  $C$ , and the companion matrix of the polynomial  $X^n - 1$ , denoted by  $G$ . Thus:

$$\mathrm{GL}_n(\mathbb{F}_p) = \langle C, G \rangle.$$

Then, Nick Gill, using the classification of finite simple groups, gave a proof of the latter statement, as a consequence of the following slightly more general result ([3, Theorem 2 p. 230]): if  $\mathbb{F}$  is a finite field of order  $q$  and  $f, g$  are distinct polynomials of degree  $n$  such that  $f$  is primitive, and if the constant term of  $g$  is non-zero, then:

$$\langle C_f, C_g \rangle = \mathrm{GL}_n(\mathbb{F}_p).$$

Consequently, we can infer from Nick Gill’s theroem a quite simple way to prove the Brunnian conjecture: it sufficient to build a companion matrix  $K$  (of a polynomial  $g$  of degree  $n$ ) with the following properties:

- $K \in \langle G_1, G_2, \dots, G_n \rangle$ ;
- $g(0) \neq 0$ ;
- $K \neq G_1$ .

Thus, as  $\langle G_1, K \rangle \subset \langle G_1, G_2, \dots, G_n \rangle \subset \mathrm{GL}_n(\mathbb{F}_p)$ , whenever  $\langle G_1, K \rangle = \mathrm{GL}_n(\mathbb{F}_p)$ , we are done!

In this paper, our main contribution is to build such a matrix  $K$  in a generic way... Our machinery works whenever  $-a_0 \neq 1$  and  $(-a_0)^n \neq 1$ , which is guaranteed by the assumptions  $p \geq 5$  and  $n$  not divisible par  $p-1$ . Moreover, we must assume that  $n \geq 3$ , as Joel Brewster Lewis pointed it out.

### 3. Statements and proofs

**Definition 3.1.** *With the notations of the main theorem, assuming  $n \geq 3$ , we define the matrix  $K$  by:*

$$K := G_n G_{n-1} \cdots G_2 G_1 .$$

**Proposition 3.2.** *We have  $K = C(G^{-1}C)^n$ .*

*Proof.* We start by showing, thanks to a recurrence-based reasoning, the following result:

$$G_k G_{k-1} \cdots G_1 = G^{k-1} C (G^{-1} C)^{k-1} \text{ for all } 2 \leq k \leq n .$$

Indeed, if  $k = 2$ , this equality is equivalent to:

$$G_2 G_1 = G^{2-1} C (G^{-1} C)^{2-1} ;$$

therefore, it is true because the right hand side equals  $G C G^{-1} C = G_2 G_1$ . We assume the recurrence hypothesis at rank  $k$ , and we show it at rank  $k+1$ . We have:

$$\begin{aligned} G_{k+1} G_k G_{k-1} \cdots G_1 &= G_{k+1} \cdot G^{k-1} C (G^{-1} C)^{k-1} \\ &= G^k C G^{-k} \cdot G^{k-1} C (G^{-1} C)^{k-1} \\ &= G^k \cdot C \cdot G^{-k} \cdot G^{k-1} C \cdot (G^{-1} C)^{k-1} \\ &= G^k \cdot C \cdot G^{-1} C \cdot (G^{-1} C)^{k-1} \\ &= G^k \cdot C \cdot (G^{-1} C)^k . \end{aligned}$$

We then get:  $G_n G_{n-1} \cdots G_1 = G^{n-1} C (G^{-1})^{n-1} C$ . We conclude that:

$$\begin{aligned} G_1 G_n G_{n-1} \cdots G_1 &= G_1 G^{n-1} C (G^{-1})^{n-1} , \\ &= C G^{-1} C (G^{-1})^{n-1} , \\ &= C (G^{-1} C)^n , \end{aligned}$$

QED. □

**Proposition 3.3.** *Let  $E_k$  denote the  $k^{\text{th}}$  vector of the canonical basis, for  $1 \leq k \leq n$ . Then we get:*

1.  $G^{-1}C.E_k = E_k$  for  $1 \leq k \leq n-1$ ;
2.  $G^{-1}C.E_n = -a_0E_n + \sum_{j=0}^{n-1} -a_jE_j$ ,

id est :

$$G^{-1}C = \begin{bmatrix} 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \\ 0 & 0 & \dots & 0 & -a_0 \end{bmatrix}.$$

*Proof.* The first statement is a consequence of the fact that the product of the matrix  $G$  (respectively  $C$ ) by the column vector  $E_k$  is equal to the column vector  $E_{k+1}$  for  $1 \leq k \leq n-1$ . Thus,  $G^{-1}C.E_k = G^{-1}.CE_k = G^{-1}.E_{k+1} = E_k$ .

To show the second statement, we proceed as follows:

$$\begin{aligned} G^{-1}C.E_n &= G^{-1} \cdot \left( \sum_{i=1}^n -a_{i-1}E_i \right) \\ &= \sum_{i=1}^n -a_{i-1}G^{-1}.E_i \\ &= \sum_{i=1}^n -a_{i-1} \begin{cases} E_{i-1} & \text{si } 2 \leq i \leq n \\ E_n & \text{si } i = 1 \end{cases} \\ &= -a_0G^{-1}E_1 + \sum_{i=2}^n -a_{i-1}G^{-1}E_i \\ &= -a_0E_n + \sum_{i=2}^n -a_{i-1}E_{i-1} \\ &= -a_0E_n + \sum_{j=1}^n -a_jE_j \end{aligned}$$

thanks to the change of variables  $j = i - 1$ . □

**Proposition 3.4.** *We assume that  $-a_0 \neq 1$ . Then be the matrix  $G^{-1}C$  can be written as  $Q^{-1}G^{-1}CQ = D$ , where  $Q \in \text{GL}_n(\mathbb{F}_p)$  and  $D$ , diagonal*

matrices, express like:

$$Q = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{a_1}{1+a_0} \\ 0 & 1 & \dots & 0 & \frac{a_2}{1+a_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{a_{n-1}}{1+a_0} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -a_0 \end{bmatrix},$$

We deduce that:  $K = G^{n-1}(G^{-1}C)^n = C(QDQ^{-1})^n = CQD^nQ^{-1}$ .

*Proof.* The characteristic polynomial of the matrix  $G^{-1}C$  is  $\chi_{G^{-1}C}(X) = (X - 1)^{n-1}(X + a_0)$ . Therefore, the eigenvalues are 1, of order  $n - 1$  and  $-a_0$ , or order 1. From Proposition 3.3, (1), page 4, we can infer that  $Q$  will satisfy  $Q.E_k = E_k$  for  $1 \leq k \leq n - 1$ . Consequently, we must find a column vector  $V$  such that  $G^{-1}CV = -a_0V$ , as eigenvector associated to the eigenvalue  $-a_0$ .

Therefore, we write :

$$V := \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \text{ hence } G^{-1}CV = \begin{pmatrix} v_1 - a_1v_n \\ v_2 - a_2v_n \\ \vdots \\ v_{n-1} - a_{n-1}v_n \\ -a_0v_n \end{pmatrix}, \text{ and}$$

$$-a_0V = \begin{pmatrix} -a_0v_1 \\ -a_0v_2 \\ \vdots \\ -a_0v_n \end{pmatrix}$$

Solving this Cramer linear system in  $v_1, v_2, \dots, v_n$ , we get:

1.  $v_n \in \mathbb{F}_p$ ;
2.  $v_i = \frac{a_i v_n}{1+a_0}$  for  $1 \leq i \leq n - 1$ .

We now just have to choose  $v_n = 1$  to get the previously announced matrix  $Q$ . □

**Proposition 3.5.** *If  $-a_0 \neq 1$  and  $(-a_0)^n \neq 1$ , the matrix*

$$K := G_n G_{n-1} \cdots G_2 G_1$$

*is a companion matrix, different from  $C$ .*

*Proof.* Under the action of  $Q$  and  $D$ , the column vectors  $E_k$  for  $1 \leq k \leq n-1$  remain unchanged. We thus have:

$$\begin{aligned} K.E_k &= CQD^nQ^{-1}.E_k \\ &= C.E_k \\ &= E_{k+1}. \end{aligned}$$

Therefore, we must evaluate  $K.E_n = CQD^nQ^{-1}.E_n$

Let  $U := Q^{-1}E_n$ . Then  $E_n = QU$ . To find the value of  $U$ , it is sufficient to solve the equation. However, if

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 & q_1 \\ 0 & 1 & \cdots & 0 & q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & q_{n-1} \\ 0 & 0 & \cdots & 0 & q_n \end{bmatrix} \quad \text{and } U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix},$$

then

$$QU = \begin{bmatrix} u_1 + q_1 u_n \\ u_2 + q_2 u_n \\ \vdots \\ u_{n-1} + q_{n-1} u_n \\ q_n u_n \end{bmatrix} \quad \text{equalized to } E_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

enables to get :  $u_n = \frac{1}{q_n} = 1$  (because  $q_n$  is a notation for the coefficient in row  $n$  and column  $n$  of  $Q$ , which equals 1), and:

$$u_k = -q_k u_n = -q_k = \frac{-a_k}{1 + a_0}.$$

Thus, we have:

$$U = \begin{bmatrix} \frac{-a_1}{1+a_0} \\ \frac{-a_2}{1+a_0} \\ \vdots \\ \frac{-a_{n-1}}{1+a_0} \\ 1 \end{bmatrix}.$$

Now, let  $W := D^n U$ . Then:

$$W = \begin{bmatrix} \frac{-a_1}{1+a_0} \\ \frac{-a_2}{1+a_0} \\ \vdots \\ \frac{-a_{n-1}}{1+a_0} \\ (-a_0)^n \end{bmatrix},$$

and  $CQD^n U = CQW$ . Now, the following holds:

$$QW = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-a_0)^n \end{bmatrix},$$

To summarize, we get:

$$\begin{aligned} K.E_n &= CQD^n Q^{-1}.E_n \\ &= CQD^n U \\ &= CQWW \\ &= C(-a_0)^n E_n \\ &= (-a_0)^n C E_n \\ &= \sum_{i=1}^n -a_{i-1} (-a_0)^n E_i. \end{aligned}$$

Therefore, the matrix  $K$  is a companion matrix, different from  $C$ , because we assumed that  $(-a_0)^n \neq 1$ .  $\square$

**Theorem 3.6** (Main result). *Let  $p \geq 5$  a prime number. Then  $-a_0 \neq 1$ . Moreover, if  $n$  is not divisible by  $p-1$  then  $(-a_0)^n \neq 1$ . In this case, the Brunnian conjecture  $B(n, p, P)$  is true.*

*Proof.* As the polynomial  $f$  is primitive,  $(-1)^n a_0$  generates the multiplicative group  $\mathbb{F}_p^\times$ , so if  $p \geq 5$ , the equality  $-a_0 = 1$  can not occur. Indeed, if  $-a_0 = 1$  held, we would have  $a_0 = -1$ , then  $(-1)^n a_0 = (-1)^{n+1}$  would be equal to  $-1$  or  $1$ , which is never a generator of  $\mathbb{F}_p^\times$  when  $p \geq 5$ .

If  $n$  is not divisible by  $p-1$ , let us assume that  $(-a_0)^n = 1$  in  $\mathbb{F}_p^\times$ . We can write  $n = (p-1)q + r$ , where  $1 \leq r < p-1$  is an integer. Then  $(-a_0)^r = 1$ , because according to our assumption,  $1 = (-a_0)^n = (-a_0)^{(p-1)q}(-a_0)^r$  and  $(-a_0)^{p-1} = 1$  (because  $-a_0 \in \mathbb{F}_p^\times$ ). Now, it turns out that  $(-1)^n a_0 = (-1)^{r-1}(-a_0)$  has an order which divides  $r < p-1$ . This is a contradiction with the fact that  $(-1)^n a_0$  is a generator of  $\mathbb{F}_p^\times$ , the order of which is  $p-1$ . Consequently, if  $n$  is not divisible by  $p-1$ , then  $(-a_0)^n \neq 1$ .  $\square$

## 4. Further remarks

In [1], we gave some “experimental checkings”, and we were unable to conclude whether the conjecture was true for certain couples  $(n, p)$ , due to a lack of power of computations and/or high complexity of calculus. Thanks to our new method, some answers can be given. We sum up them in the following tables. If “(False)” is written in last column, it means that the conjecture can be true, but it is not a consequence of the main result of this paper.

Primitive polynomials over  $\mathbb{F}_5$

| $n$ | $p$ | $P(x)$                                                                          | $B(n, p, P)$ |
|-----|-----|---------------------------------------------------------------------------------|--------------|
| 7   | 5   | $x^7 + x^6 + 2$                                                                 | True         |
| 8   | 5   | $x^8 + x^5 + x^3 + 3$                                                           | (False)      |
| 9   | 5   | $x^9 + x^7 + x^6 + 3$                                                           | True         |
| 10  | 5   | $x^{10} + x^9 + x^7 + 3$                                                        | True         |
| 11  | 5   | $x^{11} + x^{10} + 2$                                                           | True         |
| 12  | 5   | $x^{12} + x^7 + x^4 + 3$                                                        | (False)      |
| 13  | 5   | $x^{13} + 4x^2 + 3x + 3$                                                        | True         |
| 14  | 5   | $x^{14} + x^7 + 4x^5 + 4x^4 + 2x^3 + 3x^2 + x + 2$                              | True         |
| 15  | 5   | $x^{15} + 2x^5 + 4x^5 + 3x^3 + 3x^2 + 4x + 3$                                   | True         |
| 16  | 5   | $x^{16} + x^7 + 4x^6 + 4x^5 + 4x^4 + 4x^3 + 4x^2 + x + 2$                       | (False)      |
| 17  | 5   | $x^{17} + 3x^2 + 2x + 3$                                                        | True         |
| 18  | 5   | $x^{18} + x^{12} + x^{11} + x^{10} + x^9 + 2x^8 + 2x^6 + x^5 + 2x^3 + 2x^2 + 2$ | True         |

Primitive polynomials over  $\mathbb{F}_7$

| $n$ | $p$ | $P(x)$                   | $B(n, p, P)$ |
|-----|-----|--------------------------|--------------|
| 6   | 7   | $x^6 + x^5 + x^4 + 3$    | (False)      |
| 7   | 7   | $x^7 + x^5 + 4$          | True         |
| 8   | 7   | $x^8 + x^7 + 3$          | True         |
| 9   | 7   | $x^9 + x^8 + x^3 + 2$    | True         |
| 10  | 7   | $x^{10} + x^9 + x^8 + 3$ | True         |

**Primitive polynomials over  $\mathbb{F}_{11}$** 

| $n$ | $p$ | $P(x)$                | $B(n, p, P)$ |
|-----|-----|-----------------------|--------------|
| 5   | 11  | $x^5 + x^4 + x^3 + 3$ | True         |
| 6   | 11  | $x^6 + x^5 + x + 7$   | True         |
| 7   | 11  | $x^7 + x^6 + 4$       | True         |
| 8   | 11  | $x^8 + x^7 + x^6 + 7$ | True         |

**Primitive polynomials over  $\mathbb{F}_{13}$** 

| $n$ | $p$ | $P(x)$                 | $B(n, p, P)$ |
|-----|-----|------------------------|--------------|
| 5   | 13  | $x^5 + x^4 + x^3 + 6$  | True         |
| 6   | 13  | $x^6 + x^5 + x^3 + 6$  | True         |
| 7   | 13  | $x^7 + x^4 + 2$        | True         |
| 8   | 13  | $x^8 + x^7 + x^6 + 11$ | True         |

**Primitive polynomials over  $\mathbb{F}_{17}$** 

| $n$ | $p$ | $P(x)$          | $B(n, p, P)$ |
|-----|-----|-----------------|--------------|
| 4   | 17  | $x^4 + x^3 + 5$ | True         |
| 5   | 17  | $x^5 + x^4 + 5$ | True         |
| 6   | 17  | $x^6 + x^5 + 3$ | True         |
| 7   | 17  | $x^7 + x^6 + 7$ | True         |

**Primitive polynomials over  $\mathbb{F}_{19}$** 

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 4   | 19  | $x^4 + x^3 + 2$  | True         |
| 5   | 19  | $x^5 + x^4 + 5$  | True         |
| 6   | 19  | $x^6 + x^5 + 15$ | True         |
| 7   | 19  | $x^7 + x^6 + 5$  | True         |

**Primitive polynomials over  $\mathbb{F}_{23}$** 

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 4   | 23  | $x^4 + x^3 + 20$ | True         |
| 5   | 23  | $x^5 + x^4 + 6$  | True         |
| 6   | 23  | $x^6 + x^5 + 7$  | True         |

Primitive polynomials over  $\mathbb{F}_{29}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 4   | 29  | $x^4 + x^3 + 2$  | True         |
| 5   | 29  | $x^5 + x^4 + 2$  | True         |
| 6   | 29  | $x^6 + x^5 + 11$ | True         |

Primitive polynomials over  $\mathbb{F}_{31}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 4   | 31  | $x^4 + x^3 + 13$ | True         |
| 5   | 31  | $x^5 + x^4 + 10$ | True         |
| 6   | 31  | $x^6 + x^5 + 12$ | True         |

Primitive polynomials over  $\mathbb{F}_{37}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 37  | $x^3 + x^2 + 17$ | True         |
| 4   | 37  | $x^4 + x^3 + 22$ | True         |
| 5   | 37  | $x^5 + x^4 + 2$  | True         |

Primitive polynomials over  $\mathbb{F}_{41}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 41  | $x^3 + x^2 + 11$ | True         |
| 4   | 41  | $x^4 + x^3 + 26$ | True         |
| 5   | 41  | $x^5 + x^4 + 11$ | True         |

Primitive polynomials over  $\mathbb{F}_{43}$

| $n$ | $p$ | $P(x)$          | $B(n, p, P)$ |
|-----|-----|-----------------|--------------|
| 3   | 43  | $x^3 + x^2 + 9$ | True         |
| 4   | 43  | $x^4 + x + 20$  | True         |
| 5   | 43  | $x^5 + x^4 + 9$ | True         |

Primitive polynomials over  $\mathbb{F}_{47}$

| $n$ | $p$ | $P(x)$          | $B(n, p, P)$ |
|-----|-----|-----------------|--------------|
| 3   | 47  | $x^3 + x^2 + 2$ | True         |
| 4   | 47  | $x^4 + x^3 + 5$ | True         |
| 5   | 47  | $x^5 + x^4 + 6$ | True         |

Primitive polynomials over  $\mathbb{F}_{53}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 53  | $x^3 + x^2 + 2$  | True         |
| 4   | 53  | $x^4 + x^3 + 2$  | True         |
| 5   | 53  | $x^5 + x^4 + 12$ | True         |

Primitive polynomials over  $\mathbb{F}_{59}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 59  | $x^3 + x^2 + 9$  | True         |
| 4   | 59  | $x^4 + x^3 + 18$ | True         |
| 5   | 59  | $x^5 + x^4 + 4$  | True         |

Primitive polynomials over  $\mathbb{F}_{61}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 61  | $x^3 + x^2 + 6$  | True         |
| 4   | 61  | $x^4 + x^3 + 17$ | True         |
| 5   | 61  | $x^5 + x^4 + 55$ | True         |

Primitive polynomials over  $\mathbb{F}_{67}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 67  | $x^3 + x^2 + 6$  | True         |
| 4   | 67  | $x^4 + x^3 + 12$ | True         |

Primitive polynomials over  $\mathbb{F}_{71}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 71  | $x^3 + x^2 + 8$  | True         |
| 4   | 71  | $x^4 + x^3 + 13$ | True         |

Primitive polynomials over  $\mathbb{F}_{73}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 73  | $x^3 + x^2 + 5$  | True         |
| 4   | 73  | $x^4 + x^3 + 33$ | True         |

Primitive polynomials over  $\mathbb{F}_{79}$

| $n$ | $p$ | $P(x)$          | $B(n, p, P)$ |
|-----|-----|-----------------|--------------|
| 3   | 79  | $x^3 + x^2 + 2$ | True         |
| 4   | 79  | $x^4 + x^3 + 7$ | True         |

Primitive polynomials over  $\mathbb{F}_{83}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 83  | $x^3 + x^2 + 11$ | True         |
| 4   | 83  | $x^4 + x^3 + 24$ | True         |

Primitive polynomials over  $\mathbb{F}_{89}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 89  | $x^3 + x^2 + 6$  | True         |
| 4   | 89  | $x^4 + x^3 + 14$ | True         |

Primitive polynomials over  $\mathbb{F}_{97}$

| $n$ | $p$ | $P(x)$           | $B(n, p, P)$ |
|-----|-----|------------------|--------------|
| 3   | 97  | $x^3 + x^2 + 5$  | True         |
| 4   | 97  | $x^4 + x^3 + 15$ | True         |

We shall conclude this paper by giving to results which enable to see that our results cover a large part of the cases.

**Definition 4.1.** For an integer  $N \geq 5$ :

1. we denote by  $\mathcal{P}(N)$  the set of prime numbers  $p$  for which the following inequalities holds:  $5 \leq p \leq N$ ;
2. we denote by  $\mathcal{Q}(N)$  the set of primary numbers  $q$  such that there exists a prime number  $5 \leq p \leq N$  and an integer  $r \geq 1$  such that  $q = p^r$ . Then  $q \geq 4$ .

If  $n \geq 3$ ,  $N \geq 5$ , and  $p \in \mathcal{P}(N)$ , we aim to evaluate the probability for  $n$  to is not divisible by  $p - 1$ , and we hope that this probability is sufficiently large. The result is given by the following proposition.

**Proposition 4.2.** For  $N \geq 5$ , let:

$$\mathcal{E}_N = \{(p, n) \mid p \in \mathcal{P}(N), 3 \leq n \leq N\};$$

$$\mathcal{D}_N = \{(p, n) \mid p \in \mathcal{P}, 3 \leq n \leq N, n \in (p - 1)\mathbb{Z}\}.$$

Then:

1.  $\#\mathcal{E}_N = (\pi(N) - 2)(N - 2)$  and  $\#\mathcal{D}_N = \sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor$ ;
2.  $\frac{\#\mathcal{D}_N}{\#\mathcal{E}_N} \leq \frac{\sqrt{N}}{N-2}$  and  $\frac{\#\overline{\mathcal{D}}_N}{\#\mathcal{E}_N} \geq 1 - \frac{\sqrt{N}}{N-2}$ .

*Proof.* Indeed, the number of elements in  $\mathcal{P}(N)$  is the number of prime numbers less than  $N$ , except 2 and 3. Therefore:

$$\#\mathcal{E}_N = (\pi(N) - 2)(N - 2).$$

Also:

$$\begin{aligned} \#\mathcal{D}_N &= \left\{ (p, k(p-1)) \mid p \in \mathcal{P}(N), \frac{3}{p-1} \leq k \leq \frac{N}{p-1} \right\}, \\ &= \sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor. \end{aligned}$$

Then the first point is proved. Now, by recurrence on  $N \geq 5$ , we can prove (see the Lemma 4.3 below) that:

$$\#\mathcal{D}(N) \leq \sqrt{N}(\pi(N) - 2)$$

.

Dividing by  $\#\mathcal{E}(N)$ , we get the two inequalities of the second point.  $\square$

**Lemma 4.3.** (i) Let  $p \geq 5$  a prime number with  $p \leq N$  for an integer  $N \geq 5$ . Then:

$$\sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor \leq \sum_{p \in \mathcal{P}(N)} \sqrt{N}.$$

*Proof.* Initialisation step: if  $N = 5$ , then  $\mathcal{P}(N) = \{5\}$ . Consequently :

$$\sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor = \left\lfloor \frac{5}{4} \right\rfloor = 1, 25 \leq \sqrt{5} = \sum_{p \in \mathcal{P}(N)} \sqrt{N}.$$

Recurrence step: we must prove the if the property is true for an integer  $N \geq 5$ , it is also true for the integer  $N + 1$ . We have to justify that:

$$\sum_{p \in \mathcal{P}(N+1)} \left\lfloor \frac{N+1}{p-1} \right\rfloor \leq \sum_{p \in \mathcal{P}(N+1)} \sqrt{N+1}.$$

Either  $N + 1$  is not prime, or it is.

Let us start by the case where  $N + 1$  is not prime. Then  $\mathcal{P}(N + 1) = \mathcal{P}(N)$ , and we get:

$$\begin{aligned}
 \sum_{p \in \mathcal{P}(N+1)} \left\lfloor \frac{N+1}{p-1} \right\rfloor &= \sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N+1}{p-1} \right\rfloor \\
 &= \sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor \\
 &\leq \sum_{p \in \mathcal{P}(N)} \sqrt{N} \text{ (recurrence)} \\
 &= \sum_{p \in \mathcal{P}(N+1)} \sqrt{N} \\
 &< \sum_{p \in \mathcal{P}(N+1)} \sqrt{N+1}.
 \end{aligned}$$

QED.

Let us now manage with the case where  $N + 1$  is prime. Then  $\mathcal{P}(N + 1) = \mathcal{P}(N) \cup \{N + 1\}$ , and we get:

$$\sum_{p \in \mathcal{P}(N+1)} \left\lfloor \frac{k+1}{p-1} \right\rfloor = \sum_{p \in \mathcal{P}(N+1)} \left\lfloor \frac{k+1}{p-1} \right\rfloor + \left\lfloor \frac{k+1}{k} \right\rfloor.$$

Thanks to a property of integer part:

$$\left\lfloor \frac{k+1}{k} \right\rfloor = 1 \text{ for all non-zero integer } k,$$

we can infer that:

$$\sum_{p \in \mathcal{P}(N+1)} \left\lfloor \frac{N+1}{p-1} \right\rfloor \leq \sum_{p \in \mathcal{P}(N)} \left\lfloor \frac{N}{p-1} \right\rfloor + 1 < \sum_{p \in \mathcal{P}(N)} \sqrt{N} + 1$$

and:

$$\sum_{p \in \mathcal{P}(N)} \sqrt{N} + 1 \leq \sum_{p \in \mathcal{P}(N+1)} \sqrt{N+1},$$

QED. □

**Proposition 4.4.** *For  $N \geq 5$ , let:*

$$\mathcal{E}^*_N = \{(q, n) \mid q \in \mathcal{Q}(N), 3 \leq n \leq N\};$$

$$\mathcal{D}^*_N = \{(q, n) \mid q \in \mathcal{Q}(N), 3 \leq n \leq N, n \in (q-1)\mathbb{Z}\}.$$

*Then:*

1.  $\#\mathcal{E}^*_N = (\pi(N) - 2)(N - 2)$  and  $\#\mathcal{D}^*_N = \sum_{q \in \mathcal{Q}(N)} \left\lfloor \frac{N}{q-1} \right\rfloor$ ;
2.  $\frac{\#\mathcal{D}^*_N}{\#\mathcal{E}^*_N} \leq \frac{\sqrt{N}}{N-2}$  and  $\frac{\overline{\#\mathcal{D}^*_N}}{\overline{\#\mathcal{E}^*_N}} \geq 1 - \frac{\sqrt{N}}{N-2}$ .

*Proof.* The proof is almost the same. □

Thus:

$$\lim_{N \rightarrow +\infty} \frac{\overline{\mathcal{D}}_N}{\overline{\mathcal{E}}_N} = 1 \text{ and } \lim_{N \rightarrow +\infty} \frac{\overline{\mathcal{D}^*_N}}{\overline{\mathcal{E}^*_N}} = 1.$$

If  $n \geq 3$  and  $p \geq 5$ , the Brunnian Conjecture  $B(n, p, P)$  (respectively the Generalized Brunnian Conjecture  $B(n, q, P)$ ) is “almost always true” when  $p \geq 5$  and  $n$  is not divisible by  $p-1$  (respectively when  $q = p^r \geq 4$  and  $n$  is not divisible by  $q-1$ ).

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