



CROSSED $\mathbb{N}$ -CUBES AND CROSSED n -CUBE COMPLEXES

(VARIETIES OF SIMPLICIAL GROUPOIDS, II)

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Résumé. Dans cet article, nous introduisons les notions de complexes de $\mathbb{N}$ -cubes croisés et de complexes de n -cubes croisés, qui généralisent celles des complexes croisés et des n -cubes croisés d'Ellis-Steiner. Nous montrons qu'en faisant varier n , les différentes catégories de complexes de n -cubes croisés forment une tour réflexive de catégories au sein de celle des $\mathbb{N}$ -cubes croisés.

Abstract. In this article, we introduce notions of crossed $\mathbb{N}$ -cube and crossed n -cube complexes, which extend both those of crossed complexes and Ellis-Steiner crossed n -cubes. We show that varying n , the various categories of crossed n -cube complexes form a reflective tower of categories within that of crossed $\mathbb{N}$ -cubes.

Keywords. Crossed complex, crossed n -cube

Mathematics Subject Classification (2010). 18D40 · 18G45.

1. Introduction

This paper is both a continuation of [23, 24] and an extension of [42].

In [42], a proof was given of a version of Loday's theorem from [36], Loday's paper showed that a certain simple algebraic structure, a cat^n -group, was able to model all connected homotopy $(n + 1)$ -types, generalising the classical result of Mac Lane and Whitehead, [37]. Such objects were internal

n -fold categories in the category of groups, but also equivalently have a description as crossed n -cubes of groups, a structure which generalises crossed modules to higher dimension.

In [23, 24], the problem that was addressed was to ask for an algebraic description of those simplicially enriched groupoids that corresponded to the crossed complexes defined by Brown and Higgins, for example in [11]. This was to be without initial reference to the homotopy theory that had been developed, as the aim was for a description as a *variety* within the category of simplicial groupoids, so specifying the ‘verbal subobject’ generating the kernel of the reflection into the variety. In other words, the aim of that work was to examine the algebraic nature of those models, and to initiate their study as objects of algebraic interest.

That study was in part motivated by J. H. C. Whitehead’s ICM address, [49], as quoted by Hans Baues in [2]:

“The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that ‘analytic’ is equivalent to ‘pure’ projective geometry.”

In this paper, we will to some extent expand that examination to higher dimensions. As some of the theory has only been developed for the connected case, and thus for simplicial groups rather than simplicially enriched groupoids, our treatment will, in the main, be handling that ‘reduced’ situation. The extension to the general case is not hard to do, but would take up quite a lot of space. This is an analogue of the simplicial analysis of homotopy types via coskeleta, but is much more algebraic. It is also related to the notion of towers of categories introduced by Baues in [2, 3], and even more closely to the use of a similar idea by Bullejos, Faro, and García-Muñoz, [15], which limits attention to crossed complexes.

There is an additional theme to this paper, however. Following on from Loday’s work on homotopy n -types, [36], crossed n -cubes, as introduced by Ellis and Steiner, [27], gave complete information on the homotopy type up to dimension n . Crossed complexes, on the other hand, give useful, but incomplete, information in *all* dimensions. In both situations, a van Kampen type theorem exists. For crossed complexes, this van Kampen theorem was due to Brown and Higgins, [9], and for crossed n -cubes, a corresponding

result was proved by Brown and Loday, [13]. Both generalised the crossed module version of [8]. This suggests that there might be, for each n , a class of models that we will call *crossed n -cube complexes*, or sometimes more briefly *n -crossed complexes*, which would give complete information on the n -type and yet give useful information in all dimensions. For $n = 1$, crossed complexes already achieve this aim.

More precisely, the hopes in looking for this tentative model are as follows:

- (i) to find a fairly simple algebraic model for those homotopy types with vanishing Whitehead products above a given dimension,
 - (ii) to find a unified van Kampen theorem that includes both the Brown-Higgins and Brown-Loday forms as special cases,
 - (iii) to allow for a better treatment of homotopies of maps between of crossed n -cubes,
- and, if possible,
- (iv) find a varietal, or at least more algebraic, description of those simplicially enriched groupoids that correspond to such models for each fixed n .

The results later on show that the model proposed looks promising in this regard, but these aims have not been explored adequately here¹.

Although crossed n -cube complexes, once defined, should clearly have trivial Whitehead products in dimensions greater than n , the converse is almost certainly untrue, as it is already false for $n = 1$.

The second aim will not be examined here in detail and, to my knowledge, not much progress has been made on that problem. The problem seems to be that of turning the filtered spaces encountered in the Brown-Higgins theory into some sort of hypercube of fibrations as in the corresponding Brown-Loday theory.

About the third aim, it should perhaps be pointed out that crossed *modules* only have a truncated homotopy theory, since they have nothing above dimension one. Crossed complexes have, however, an extremely rich homotopy theory since they have tensor products and ‘internal homs’, and form

¹Of course, one such description of a somewhat different flavour was already available as a particular class of hyper-crossed complexes, as in [16] and [17], especially if combined with insights from Bullejos-Cegarra-Duskin, [14]. Another approach was initiated by Baues in [3], which explores the links with Whitehead’s programme in a different way.

a locally Kan simplicially enriched category, which is tensored and cotensored over simplicial sets, and hence have cylinder and cocylinder functors (*cf.*, Brown-Golanskiński, [7], Brown-Higgins, [11], and Kamps-Porter, [33]). The development of that theory depended to a large extent on the use of non-reduced crossed complexes, *i.e.*, crossed complexes over a groupoids (*cf.*, Brown-Higgins, [11]) and the many different categories equivalent to the category of these. It thus seems likely that a similar theory for crossed n -cube complexes will need crossed n -cube complexes over groupoids.

The notions of a crossed $\mathbb{N}$ -cube and crossed n -cube complexes were introduced by the author in various sets of notes, starting back in the 1990s and 2000s, but the details were not published. The reason for rejuvenating these results now is the revival of interest in some closely related concepts, such as that of n -hypergroupoids, also known as (weak) n -groupoids, following on from work of Behrend and Getzler, [4], and Pridham, [44, 45], and derived from work of Duskin, [22], and Glenn, [30]. We note, in different contexts, the use of crossed complexes of Lie algebras by Kapranov, [34], and a conjectured use of crossed n -cubes in lattice gauge theory, [35], by Kapustin and Spodyneiko. We hope to consecrate a future article to how hypergroupoids fit into this overall picture, but, in the last section here, will briefly sketch out some of the ideas involved, and also mention some of the problems that will need to be surmounted if that conjectural theory is to thrive, emphasising that the theory developed here is just one of several possible ones involving algebraic or simplicial models for n -types with a ‘tail’ attached. At the moment, this feels like a puzzle in which some of the pieces have yet to be ‘turned over’, but in which we have a fairly clear view of what the overall picture looks like and of some of the problems that need to be resolved..

With those recent developments in mind, it should be said that although this paper is written in set-based way, the categorical and algebraic constructions used are available in many other contexts. For instance, for a semi-abelian category, $\mathcal{A}$, the category of n -fold internal groupoids in $\mathcal{A}$ is also semi-abelian, by results of Bourn and Gran, [6], and most if not all of the theory developed here could be adapted, with care, to that setting.

Similarly, in categories of group objects in presheaf toposes, the availability of the Yoneda lemma and techniques of generalised elements, mean this theory goes across to that wider context, with some slight adjustment and in the last section we will recall that by using categories with covers, *e.g.*,

as in Wolfson, [50], or categories with a singleton Grothendieck pretopology, as in Zhu, [51], this theory should be able to be applied to several other more geometric settings.

Finally, it should be noted that a theory of models for higher dimensional algebra has been partially developed by various authors, in various flavours, using varied terminology. This can make it difficult to know what is known and what is unknown. I hope this paper will not contribute to the potential confusion, and will provide a place in which some of the varied sources can be linked together.

Acknowledgements: These results were presented, in the 1990s, in seminars in Bangor. They benefitted enormously from discussions with Ronnie Brown, Chris Wesley, Zekariya Arvasi and Ali Mutlu, who also assisted in the preparation of the initial draft version all those years ago. More recently, they and related areas were discussed in online sessions with Jim Stasheff, João Faria Martins, Francis Bischoff and others. I would like to thank all for their feedback on these ideas.

I dedicate this to the memory of Ronnie and the lively group discussions in Bangor on this and related areas, that we had many years ago.

2. Crossed gadgetry

For convenience, we start by recalling the definition of crossed modules, crossed squares, and then crossed n -cubes, as well as that of crossed complexes.

2.1 Crossed modules

We first review some definitions.

Definition: A *crossed module* is a group homomorphism, $\partial : M \longrightarrow N$, together with an action of N on M satisfying:

$$\text{CM 1) } \partial({}^n m) = n\partial(m)n^{-1}$$

and

CM 2) $\partial^{(m)} m' = mm'm^{-1}$ for all $m, m' \in M$, $n \in N$.

These seem first to have been defined by J. H. C. Whitehead in [48]. The second condition is known as the *Peiffer identity*.

A 2-dimensional version of a crossed module, called a crossed square, was defined by D. Guin-Waléry and J.-L. Loday, [31], in 1981. We refer the reader to that paper and also to [36], for the original definition of these. We will give an equivalent definition slightly later as a special case of a crossed n -cube. Crossed squares model all connected homotopy 3-types, but we note also that 2-crossed modules in the sense of Conduché, [19], do so as well. References to the connections between the two models are given in [20].

In [36], Loday introduced a notion of cat^n -group and showed that they model all connected homotopy $n + 1$ -types, for which see slightly later.

2.2 Crossed complexes of groups

The next definition we need is that of a crossed complex of groups². This was given by R. Brown and P. J. Higgins, [10], generalising earlier work of Whitehead (1949), [48]. The theory of crossed complexes has been extensively described in Brown-Higgins-Sivera, [12].

Definition: A *crossed complex of groups* is a sequence of groups,

$$\mathcal{C} : \quad \cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

- (i) C_n is abelian for $n \geq 2$,
- (ii) C_0 acts on C_n for $n \geq 1$, and the restriction to $\partial_1 C_1$ acts trivially on C_n for $n \geq 2$,
- (iii) each ∂_n is a C_0 -group homomorphism, with $\partial_1 : C_1 \rightarrow C_0$ being a crossed module,

and

- (iv) for $i \geq 1$, $\partial_{i+1} \partial_i = 1$.

²The transition to crossed complexes of groupoids is relatively easy to do, but, so as to keep things relatively short and simple, we will not give it here. When more generally considering the groupoid case, a group crossed complex is often referred to as being a *reduced crossed complex*.

In other words, it is a non-negatively graded chain complex of groups in which the last map is a crossed module.

The category of crossed complexes will be denoted by $Crs.Comp$, for the moment, although this will be absorbed into a generic notation for crossed n -cube complexes later on, as we will be considering a family of generalisations of crossed complexes in which the bottom crossed module is replaced by more complex structures. Because of this we may sometimes refer to crossed complexes as *crossed 1-cube complexes*.

We note also the notion of *n -truncated crossed complexes*, although we will not use it that much. Such crossed complexes satisfy that C_m is trivial for $m > n$.

2.3 Cat^n -groups and crossed n -cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

Definition: A *cat^n -group* is a group, G , together with $2n$ endomorphisms $s_i, t_i, (1 \leq i \leq n)$, such that

$$s_i t_i = t_i, \text{ and } t_i s_i = s_i \text{ for all } i,$$

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \text{ for } i \neq j$$

and, for all i ,

$$[Ker s_i, Ker t_i] = 1.$$

A cat^n -group is thus a group with n independent cat^1 -group structures on it. A cat^1 -group can also be reformulated as an internal groupoid in the category of groups, so a cat^n -group, not surprisingly, leads to an internal n -fold groupoid in the same setting.

These were first described by Loday in [36] under a slightly different name. In that paper, his aim was to generalise the result of Mac Lane and Whitehead, [37], that showed that crossed modules modelled all homotopy 2-types³. Loday also proved that cat^2 -groups were equivalent to the crossed squares that had been examined earlier in [31].

³Their 3-types are now called 2-types. We note that there is an error in the proof of the

The initial proof that cat^n -groups model $(n + 1)$ -types is due to Loday, [36], but with amendments to his proof contributed by Gilbert, [29], and Steiner, [47]. Ellis and Steiner proved, in [27], the equivalence of cat^n -groups with crossed n -cubes (and much more). Porter, [42], and independently Bullejos-Cegarra-Duskin, [14], proved purely algebraic versions of Loday's result. Our treatment will, in the main, make use of [42].

The definition of crossed n -cube as an n -fold crossed module was initially suggested by Ellis in his thesis, [25]. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The elegant formulation that resulted from the joint work of Ellis and Steiner, [27], showed how that could be avoided by encoding the actions and the h -maps in the same structure.

We write $\langle n \rangle$ for the set $\{1, \dots, n\}$.

Definition: A *crossed n -cube*, M , is a family of groups, $\{M_A : A \subseteq \langle n \rangle\}$, together with homomorphisms, $\mu_i : M_A \rightarrow M_{A-\{i\}}$, for $i \in \langle n \rangle$, $A \subseteq \langle n \rangle$, and functions, $h : M_A \times M_B \rightarrow M_{A \cup B}$, for $A, B \subseteq \langle n \rangle$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n \rangle$, the following axioms hold:

- (1) $\mu_i a = a$ if $i \notin A$
- (2) $\mu_i \mu_j a = \mu_j \mu_i a$
- (3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
- (4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
- (5) $h(a, a') = [a, a']$
- (6) $h(a, b) = h(b, a)^{-1}$
- (7) $h(a, b) = 1$ if $a = 1$ or $b = 1$
- (8) $h(aa', b) = {}^a h(a', b)h(a, b)$
- (9) $h(a, bb') = h(a, b){}^b h(a, b')$
- (10) ${}^a h(h(a^{-1}, b), c){}^c h(h(c^{-1}, a), b){}^b h(h(b^{-1}, c), a) = 1$

main result of Loday's paper. This was remedied in several other papers, *e.g.*, by Bullejos, Cegarra and Duskin, [14], or Porter, [42], as well as clarifications by other authors mentioned above.

$$(11) {}^a h(b, c) = h({}^a b, {}^a c) \text{ if } A \subseteq B \cap C.$$

A morphism of crossed n -cubes,

$$\{M_A\} \rightarrow \{M'_A\},$$

is a family of homomorphisms, $\{f_A : M_A \rightarrow M'_A \mid A \subseteq \langle n \rangle\}$, which commute with the maps, μ_i , and the functions, h . This gives us a category, Crs^n , equivalent to that of cat^n -groups.

Remark: In the correspondence between cat^n -groups and crossed n -cubes (see Ellis and Steiner, [27]), the cat^n -group corresponding to a crossed n -cube, (M_A) , is constructed as a repeated semidirect product of the various M_A . Within the resulting “big group”, the h -functions interpret as being commutators. This partially explains the structure of the h -function axioms; see also the examples below.

We note the result of Ellis and Steiner.

Theorem 2.1. [27] *There are equivalences of categories,*

$$Crs^n \simeq Cat^n(Grp).$$

□

As the 11 axioms may look rather daunting when first seen, we will give some hopefully illuminating examples.

Examples: 1. Inclusion crossed n -cubes are possibly the simplest examples, whilst being almost completely general in a certain way.

A group, G , together with a list, $(N_1, \dots, N_n)$, of n (closed) normal subgroups, possibly with repeats, will be called a *normal $(n+1)$ -ad of groups* and will be denoted $(G; N_1, \dots, N_n)$. Morphisms of such are morphisms of the big group compatible with the listed normal subgroups.

Let $(G; N_1, \dots, N_n)$ be a normal $(n+1)$ -ad of groups, then the following defines a crossed n -cube of groups:

- for $A \subseteq \langle n \rangle$, let $M_A = \bigcap \{N_i \mid i \in A\}$;

- for $i \in \langle n \rangle$, let $\mu_i : M_A \rightarrow M_{A \setminus \{i\}}$ be the obvious inclusion;
- and
- for $A, B \subseteq \langle n \rangle$, let $h : M_A \times M_B \rightarrow M_{A \cup B}$ be given by the commutator, $h(a, b) = [a, b]$.

We call the resulting crossed n -cube the *inclusion crossed n -cube defined by the $(n+1)$ -ad*, $(G; N_1, \dots, N_n)$. These do look rather special, but we note the following from [42]. It uses simplicial objects consisting of such $(n+1)$ -ads of groups.

Let $(G; N_1, \dots, N_n)$ be a simplicial normal $(n+1)$ -ad of groups and define:

- for $A \subseteq \langle n \rangle$, let $M_A = \pi_0(\bigcap \{N_i : i \in A\})$;
- for $i \in \langle n \rangle$, let $\mu_i : M_A \rightarrow M_{A \setminus \{i\}}$ be induced by the obvious inclusion;

and

- for $A, B \subseteq \langle n \rangle$, let $h : M_A \times M_B \rightarrow M_{A \cup B}$ be induced by the commutator, $h(a, b) = [a, b]$,

then $\mathcal{M} = \{M_A : A \subseteq \langle n \rangle, \{\mu_i\}, h\}$ is a crossed n -cube of groups.

This is not just a way of producing simple examples because, as is shown in [42], §5, that, for every crossed n -cube, there is a simply constructed simplicial normal $(n+1)$ -ad of groups whose π_0 is isomorphic to the given one.

2. For $n = 1$, the eleven axioms above reduce to the usual ones for crossed modules. Some conditions do not apply as the cases they describe cannot occur. Here is a tiny bit more detail. The 1-cube is $M_{\langle 1 \rangle} \xrightarrow{\mu_1} M_\emptyset$, with $h : M_\emptyset \times M_{\langle 1 \rangle} \rightarrow M_{\langle 1 \rangle}$ being the main h -map. Axioms 8) - 11) just say that ${}^a b$ is an action. (1) and (2) do not apply, (3) say μ_1 is equivariant for the action, and (4) is the Peiffer identity in disguise.

3. For $n = 2$, the definition gives a crossed square in the sense of [31]:

$$\begin{array}{ccc} M_{\langle 2 \rangle} & \xrightarrow{\mu_2} & M_{\{1\}} \\ \mu_1 \downarrow & & \downarrow \mu_1 \\ M_{\{2\}} & \xrightarrow{\mu_2} & M_\emptyset \end{array}$$

with the important h -map being $h : M_{\{1\}} \times M_{\{2\}} \rightarrow M_{\langle 2 \rangle}$. The other h -maps in the above definition correspond to the various actions as explained in the original definition in [31].

4. We will recall, slightly later, how to obtain a crossed n -cube from a simplicial group.

We next relate crossed n -cubes to crossed $(n-1)$ -cubes. We will do this, initially, just in the case of crossed squares. It may help to think of a crossed n -cube as a morphism of crossed $(n-1)$ -cubes, *e.g.*, for $n=2$, if

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

is a crossed square, then taken horizontally there is a morphism of crossed modules,

$$\begin{array}{ccc} L & & M \\ \lambda' \downarrow & \longrightarrow & \downarrow \mu \\ N & & P, \end{array}$$

which is exactly the pair of morphisms, (λ, μ') , in the crossed square. There is also a vertical morphism of crossed modules.

The following is Proposition 5 of [42], page 9.

Proposition 2.2. *Let $\mathcal{M} = \{M_A : A \subseteq \langle n \rangle, \{\mu_i\}, h\}$ be a crossed n -cube of groups. For any $i \in \langle n \rangle$, let $\mathcal{M}_1$ denote the family $\{M_A : A \subseteq \langle n \rangle, i \in A\}$, and $\mathcal{M}_0$ the corresponding family with the condition $i \notin A$, then*

- a) $\mathcal{M}_0$ and $\mathcal{M}_1$ both have the structure of crossed $(n-1)$ -cubes of groups;*
- b) $\mu_i : \mathcal{M}_1 \rightarrow \mathcal{M}_0$ is a morphism of crossed $(n-1)$ -cubes such that for each $B \subseteq \langle n-1 \rangle$, $\text{Ker } \mu_{i,B}$ is central in $\mathcal{M}_{1,B}$ and $\text{Im } \mu_{i,B} \subseteq \mathcal{M}_{0,B}$ is a (closed) normal subgroup;*
- c) denoting $\mathcal{N}_B = \{\mathcal{M}_{0,B} / \text{Im } \mu_{i,B}, B \subseteq \langle n-1 \rangle\}$, then $\mathcal{N}$ has a natural structure of a crossed $(n-1)$ -cube and the sequence*

$$1 \rightarrow \text{Ker } \mu_i \rightarrow \mathcal{M}_1 \xrightarrow{\mu_i} \mathcal{M}_0 \rightarrow \mathcal{N} \rightarrow 1$$

is an exact sequence of crossed $(n - 1)$ -cubes in the obvious sense. $\square$

We will use this, in particular, when $i = n$.

Another useful construction is given as follows. It will be illustrated with crossed squares, but can be generalised. Within the category Crs^2 of crossed squares, we will be considering ones of form

$$\begin{array}{ccc} M_{\langle 2 \rangle} & \xrightarrow{\mu_2} & M_{\{1\}} \\ \mu_1 \downarrow & & \downarrow \mu_1 \\ M_{\{2\}} & \xrightarrow{\mu_2} & M_{\emptyset} \end{array}$$

in which $M_{\{2\}} = 1$. This leaves a sequence of groups

$$M_{\langle 2 \rangle} \xrightarrow{\mu_2} M_{\langle 1 \rangle} \xrightarrow{\mu_1} M_{\emptyset},$$

which is, in fact, a 2-truncated crossed complex. (We will be using a generalisation of this observation later on.)

We have a functor,

$$Tr_2 Crs.Comp \rightarrow Crs^2,$$

that considers a 2-truncated crossed complex, $D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0$, as being a crossed square

$$D = \left(\begin{array}{ccc} D_2 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & D_0 \end{array} \right).$$

That inclusion functor has a left adjoint that we illustrate on an inclusion crossed square coming from a group, G , with M, N being two normal subgroups. This gives

$$C = \left(\begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & G \end{array} \right),$$

all the maps being inclusions and with $h : N \times M \rightarrow M \cap N$ given by $h(x, y) = [x, y]$.

Suppose that we have a map from C to D , then clearly that map kills off N , within G , but also the action of N on M , and so on. We thus have that the map will factor through

$$\begin{array}{ccc} \frac{M \cap N}{[M, N]} & \longrightarrow & \frac{M}{[M, N]} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & G/N \end{array}$$

This will generalise not only to general crossed squares, but also to higher dimensions. However we have not yet introduced all the ideas and terminology needed to express that so will postpone further discussion of the general result until later

2.4 Squared complexes

It is natural curiosity to try to extend crossed squares and crossed complexes to a joint description. This should be seen as just the start of a hierarchy. A first step is given by squared complexes. Ellis, [26], gave the following definition.

Definition: A *squared complex* consists of a diagram of group homomorphisms,

$$\begin{array}{ccccccc} & & & & N & & \\ & & & \nearrow \lambda' & \searrow \mu & & \\ \dots & \longrightarrow & C_4 & \xrightarrow{\partial_4} & C_3 & \xrightarrow{\partial_3} & L \\ & & & \searrow \lambda & \nearrow \mu' & & \\ & & & & M & & P \end{array}$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

- (i) The square $\begin{pmatrix} L & \xrightarrow{\lambda} & N \\ \lambda' \downarrow & & \downarrow \mu \\ M & \xrightarrow{\mu'} & P \end{pmatrix}$ is a crossed square;
- (ii) Each group, C_n , is abelian for $n \geq 3$;

- (iii) The boundary homomorphisms satisfy $\partial_n \partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $\text{Ker } \lambda \cap \text{Ker } \lambda'$;
- (iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially, thus each C_n is a π_0 -module where $\pi_0 = P/(\mu M \cdot \mu' N)$.
- (v) The homomorphisms ∂_n are π_0 -module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares from [31] imply that $\text{Ker } \mu' \cap \text{Ker } \mu$ is a π_0 -module.

This uses the original definition of crossed square given in [31], so is given here just for motivation for a later notion, not for detailed use in this paper.

A squared complex is thus a crossed square with a ‘tail’ attached. If we replace the crossed square by a 2-crossed module, in the sense of Conduché, [19], then we would have what is called a ‘2-crossed complex’; see, for instance, [18]. A variant of that notion is that of quadratic complex, *cf.*, Baues, [3].

We are now ready to give the generalisation to all dimensions.

3. Crossed $\mathbb{N}$ -cubes, and crossed n -cube complexes.

We have seen how crossed modules generalised to crossed complexes, which have a better structured homotopical and homological algebra. We have seen, briefly in the previous section, how crossed squares generalise to give squared complexes in the sense of Ellis.

We end this sequence of ideas by looking at a simple theoretical treatment of a generalisation of both crossed complexes, and crossed n -cubes and one including squared complexes. These gadgets are related to the “Moore chain complexes of order $(n + 1)$ of a simplicial group”, as briefly studied by Baues in [3], Chapter IV, Appendix B, but have some of the advantages of crossed squares over 2-crossed modules, namely they can be ‘non-symmetric’, and hence are easily specified by, say, an ‘inclusion crossed n -cube’ consisting of a simplicial group and n simplicial normal subgroups, which may not be distinct. This allows for extra freedom in constructions. Also the axioms are very much simpler!

The definition of a crossed n -cube involves the set $\langle n \rangle = \{1, 2, \dots, n\}$. One obvious way to extend this, eliminating dependence on n , is to try replacing $\langle n \rangle$ by $\mathbb{N} = \{1, 2, \dots\}$, and specifying that the subsets A, B, C , in that definition must be finite, a condition previously automatic. This gives the notion of a crossed $\mathbb{N}$ -cube.

If we write $A \subset_{fin} \mathbb{N}$ as a shorthand for ‘ $A \subset \mathbb{N}$ with A finite’, we can replace $A \subseteq \langle n \rangle$, everywhere in the definition. We will not repeat the 11 axioms from earlier.

Definitions: A crossed $\mathbb{N}$ -cube, $\mathcal{M}$, is a family of groups, $\{M_A : A \subset_{fin} \mathbb{N}\}$, together with homomorphisms, $\mu_i : M_A \rightarrow M_{A-\{i\}}$, ($i \in \mathbb{N}$, $A \subset_{fin} \mathbb{N}$), and functions, $h : M_A \times M_B \rightarrow M_{A \cup B}$, ($A, B \subset_{fin} \mathbb{N}$), such that, if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \mathbb{N}$, the 11 axioms given above hold (with the evident adaptations).

Morphisms of crossed $\mathbb{N}$ -cubes are defined in the obvious way, and give a category, which will be denoted $Crs^{\mathbb{N}}$. We will be looking at various reflective subcategories in $Crs^{\mathbb{N}}$, and the left adjoints to the inclusion functors in each case.

Examples: (i) As, for any n , $\langle n \rangle \subset \mathbb{N}$, if $\mathcal{M}$ is a crossed n -cube, then we can extend it trivially to a crossed $\mathbb{N}$ -cube by defining $M_A = M_A$ if $A \subseteq \langle n \rangle$, and $M_A = 1$ otherwise. The h -maps, $M_A \times M_B \rightarrow M_{A \cup B}$, are then clearly determined by those of the original crossed n -cube.

(ii) Suppose $\mathcal{M} = \{M_A, \mu_i, h\}$ is a crossed $\mathbb{N}$ -cube, which is such that M_A is trivial unless A is of form $\langle n \rangle$ for some n , (where we interpret $\emptyset$ as being $\langle 0 \rangle$, so that $M_{\emptyset}$ is not required to be trivial). We will write $C_n = M_{\langle n \rangle}$ and $\partial_n : C_n \rightarrow C_{n-1}$ for the morphism $\mu_n : M_{\langle n \rangle} \rightarrow M_{\langle n-1 \rangle}$.

We note that $\partial_{n-1}\partial_n$ is trivial as it factorises via the group, M_A , where $A = \langle n \rangle - \{n-1\}$, which means that $M_A = 1$, as in

$$\begin{array}{ccc} M_{\langle n \rangle} & \longrightarrow & M_{\langle n-1 \rangle} \\ \downarrow & & \downarrow \\ M_A & \longrightarrow & M_{\langle n-2 \rangle}. \end{array}$$

We thus have that $(C_n, \partial_n)_{n \in \mathbb{N}}$ is a chain complex of groups.

There are pairings,

$$C_0 \times C_n \rightarrow C_n,$$

given by $h : M_\emptyset \times M_{\langle n \rangle} \rightarrow M_{\langle n \rangle}$, and thus an action

$${}^a b = h(a, b)b,$$

whilst $\partial({}^a b) = {}^a \partial b$, since $\mu_n h(a, b) = h(\mu_n a, \mu_n b)$, which is $h(a, \mu_n b)$, since, clearly, $n \notin \emptyset$.

The map $\partial_1 : C_1 \rightarrow C_0$ is a crossed module by exactly the proof that a crossed 1-cube is a crossed module.

If $a = \partial_1 b$, then, for $c \in C_n$, $n \geq 2$,

$$\begin{aligned} {}^a c &= h(\partial_1 b, c)c \\ &= h(b, \mu_1 c)c, \end{aligned}$$

since $1 \in \langle 1 \rangle \cap \langle n \rangle$, but $\mu_1 c \in M_{\langle n \rangle - \{1\}}$, the trivial group, so

$${}^a c = c.$$

Proposition 3.1. *Given a crossed $\mathbb{N}$ -cube such that M_A is trivial unless A is of form $\langle n \rangle$ for some n , then the above construction gives the resulting chain complex of groups, (C_n, ∂_n) , has a natural crossed complex structure.*

Conversely any crossed complex gives a crossed $\mathbb{N}$ -cube in the evident way.

Sketch proof: We will not systematically check *all* the axioms, here, but clearly from what has been said above, (C_n, ∂) is a crossed complex. (The detailed checking is left to the reader.) It is then fairly routine, given a crossed complex to construct the corresponding crossed $\mathbb{N}$ -cube. $\square$

These examples show that both crossed n -cubes, for any n , and crossed complexes are examples of crossed $\mathbb{N}$ -cubes.

We will quietly abuse terminology, and may use the same symbol for a crossed complex as usually defined and for the corresponding crossed $\mathbb{N}$ -cube. The inclusion of Crs into $Crs^{\mathbb{N}}$ has a left adjoint as will become clear a bit later on.

An obvious question, given our previous discussion, is to try to put Ellis' squared complex in the same framework. There is an obvious method to try

out, and it works. One takes $M_A = 1$ unless $A = \langle n \rangle$ for some $n \in \mathbb{N}$ or if $A \subseteq \langle 2 \rangle$. This does it, but it also indicates an effective way of encoding higher dimensional analogues of these squared complexes. We make the following definition.

Definition: For a given n , a *crossed n -cube complex* is a crossed $\mathbb{N}$ -cube, $\mathcal{M}$, satisfying $M_A = 1$ unless $A = \langle m \rangle$ for some $m \in \mathbb{N}$ or if $A \subseteq \langle n \rangle$ for the given n .

We have a full subcategory, $Crs^n.Comp$, of the category of crossed $\mathbb{N}$ -cubes specified by these crossed n -cube complexes. For instance, $Crs^1.Comp$ becomes an alternative notation for the category of crossed complexes.

If $\mathcal{M}$ is a crossed n -cube complex, then the groups and morphisms, $M_{\langle m \rangle}$ and μ_m , for $m > n$, form what will be loosely called the ‘tail’ of $\mathcal{M}$.

4. From simplicial groups to crossed n -cube complexes

To show how these gadgets relate to the more usual models of homotopy types, we will show how to obtain a crossed n -cube complex from a simplicial group, G . To do this we will analyse the constructions giving crossed complexes and crossed square complexes.

4.1 From simplicial groups to crossed complexes

We give the formulae, which can, for instance, be found in [23, 24], or in relevant nLab pages, [40]. We first set up some notation.

In what follows, D_n denotes the subgroup of G_n generated by the degenerate elements in that dimension, whilst NG is the *Moore complex*, (NG, ∂) , of G , *i.e.*, the chain complex defined by

$$NG_n = \bigcap_{i=1}^n Ker d_i^n$$

with $\partial_n : NG_n \rightarrow NG_{n-1}$ induced from d_0^n by restriction.

For crossed complexes, one used the relative homotopy groups of G . This gives a crossed complex, $C(G)$, with

$$\mathcal{C}(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

and with the base crossed module being

$$\frac{NG_1}{(NG_1 \cap D_1)d_0(NG_2 \cap D_2)} \rightarrow G_0,$$

but note that, in fact, $NG_1 \cap D_1 = 1$, since D_1 is generated by the $s_0(g)$ with $g \in G_0$.

For an arbitrary simplicial group, H , the usual associated crossed module, written $M(H, 1)$, is given by

$$\frac{NH_1}{d_0(NH_2)} \rightarrow H_0,$$

so, taking $H = sk_1G$ to be the 1-skeleton of G , the base crossed module of $\mathcal{C}(G)$ is $M(sk_1G, 1)$, as $N(sk_1G)_2 = NG_2 \cap D_2$.

As we will occasionally refer to the idea of an n -skeleton of a simplicial group, G , as here, but for various n , we should clarify precisely what this is, as there are two related, but slightly different meanings.

The one that we will use takes as the n -skeleton, sk_nG , of G to be the sub simplicial group of G generated by the group, G_n , of n -simplices of G . It thus includes all simplices of lower dimension than n , but in dimension $n+1$, there will be only elements that can be written as a product of elements, which are degenerate images of simplices of dimension $\leq n$. These form a subgroup, D_{n+1} , of G_{n+1} . We will rarely need to consider higher dimensions than that, so do not attempt to give a useful description of sk_n in those higher dimensions.

The other, more categorical form corresponds to an adjunction with a truncation functor and constructs the higher dimensions in as free a way as possible. There is a natural morphism onto the simpler algebraic one, but that does not seem to be a monomorphism in general.

4.2 ... and to crossed n -cubes

For a crossed n -cube, the detailed formulae are given in [42], page 10, or in [41]. They lead to a crossed n -cube which, as there, will be denoted

$M(G, n)$. We used this, in the last section, in a special case. The formulae are essentially as in Proposition 4.2, below, except that all the subsets A are subsets of $\langle n \rangle$, so several of the clauses do not apply.

Before we go ‘beyond’ finite n , we note a result (Proposition 9, page 12) from [42] for later use.

Proposition 4.1. *Given a simplicial group, G , and $n \geq 1$ there is an exact sequence of crossed $(n - 1)$ -cubes,*

$$1 \rightarrow K \rightarrow M(G, n)_1 \xrightarrow{\mu_1} M(G, n)_0 \rightarrow M(G, n - 1) \rightarrow 1,$$

where, for $B \subseteq \langle n \rangle$, but $B \neq \langle n \rangle$, $K_B = 1$, the trivial group, whilst $K_{\langle n-1 \rangle} \cong \pi_n(G)$, the n^{th} homotopy group of G . $\square$

4.3 ... and, finally, beyond

The analogous formula for crossed n -cube complexes is given as follows, which is both a proposition and a definition.

Proposition 4.2. *Let G be a simplicial group and $n \in \mathbb{N}$.*

Define a family M_A , $A \subset \mathbb{N}$, with A finite, by

(i) if $A = \langle m \rangle$ and $m > n$, then

$$M_A = \frac{NG_m}{(NG_m \cap D_m)d_0(NG_{m+1} \cap D_{m+1})};$$

(ii) if $A \subseteq \langle n \rangle$,

$$\begin{aligned} M_A &= M(sk_n G, n)_A \\ &= \frac{\bigcap \{Ker d_j^n : j \in A\}}{d_0(Ker d_1^{n+1} \cap \bigcap \{Ker d_{j+1}^{n+1} : j \in A\} \cap D_{n+1})}; \end{aligned}$$

(iii) if A is otherwise, then M_A is trivial.

Further define $\mu_i : M_A \rightarrow M_{A-\{i\}}$ by

(iv) if $i \notin A$, then μ_i is the identity morphism;

(v) if $A = \langle m \rangle$, with $m > n$ and $i = m$, then μ_m is induced by d_0 , and is trivial if $i \neq m$;

(vi) if $A \subseteq \langle n \rangle$, then μ_i is induced by the inclusions of intersections (i.e., as in $M(sk_n G, n)$);

(vii) otherwise μ_i is trivial.

Finally define $h : M_A \times M_B \rightarrow M_{A \cup B}$ by

(viii) if $A = \emptyset$ and $B = \langle m \rangle$ with $m > n$ then, as $M_\emptyset = G_{n-1}$ and $M_B = C(G)_m$, if $a \in M_\emptyset$ and $b \in M_B$,

$$h(a, b) = [s_0^{m-n+1}(a), b] \in M_B;$$

similarly if $A = \langle m \rangle$ and $B = \emptyset$;

(ix) if $A, B \subseteq \langle n \rangle$, h is defined as in $M(sk_n G, n)$;

(x) otherwise h is trivial.

This data defines a crossed $\mathbb{N}$ -cube, which is, in fact, a crossed n -cube complex.

Proof: Much of this can be safely ‘left to the reader’. It uses results from [42]. Note that (viii) and (x) effectively say that it is only the $s_0^{n-1}G_0$ part of G_{n-1} that acts on any $M_{\langle m \rangle}$ for $m > n$, and even then the image of $d_0 : NG_1 \rightarrow G_0$ acts trivially. To see this note that any $a \in G_{n-1} = M_\emptyset$, which is in some $\text{Ker } d_i$, is in the image of some μ_i , hence $a = \mu_i x$ say, but then

$$h(a, b) = h(\mu_i x, b) = h(x, \mu_i b) = 1,$$

by necessity if the structure is to be a crossed $\mathbb{N}$ -cube, thus to check that the h -maps, and, in particular, those involved with part (viii) of the definition, satisfy the axioms, it suffices to use the methods for checking that $C(G)$ was a crossed complex. $\square$

We will denote this crossed n -cube complex by $C(G, n)$, as it combines both the technology of the $M(G, n)$ and the construction of $C(G)$. These models have yet to be fully explored, but see [43] and below for some preliminary results.

4.4 How does $C(G, n)$ compare with $C(G)$ and $M(G, n)$?

For $n = 1$, we have

$$C(G, 1)_{\langle 1 \rangle} \cong C(G)_1 = \frac{NG_1}{d_0(NG_1 \cap D_1)},$$

and, for $C(G, 1)_{\langle m \rangle}$, it is by definition $C(G)_m$. As the boundaries are all induced from d_0 , we have

Proposition 4.3. *For any simplicial group, G , $C(G, 1)$ is the crossed 1-cube complex corresponding to $C(G)$. $\square$*

More general comparison between $C(G, n)$ and $C(G)$ will be given in the next section. The importance of $C(G)$ is that it has a quotient which is the simplicial group analogue of the complex of chains on the universal cover. This is a chain complex of modules over $\pi_0(G)$, the group of connected components of G . In the case that $G = G(K)$, the simplicial loop group on a reduced simplicial set, K , this is exactly the usual chain complex of $\pi_1(K)$ -modules coming from the universal cover of K .

The crossed n -cube complex, $C(G, n)$, also includes the information on the n -type of G , since one can easily obtain $M(G, n)$ from it, and that n -cube represents the homotopy n -type of G by Loday's theorem, [36], in the version given in [42].

Proposition 4.4. *Considering the crossed n -cube as a crossed n -cube complex, there is a natural epimorphism, $C(G, n) \rightarrow M(G, n)$, which universally trivialises the tail of $C(G, n)$.*

Proof: We look at the group $C(G, n)_{\langle n \rangle}$, which is the last group before the tail of $C(G, n)$, and also at $C(G, n)_{\langle n+1 \rangle}$, which is isomorphic to $C(G)_{n+1}$, the first group in that tail. We will also need the map,

$$\mu_{n+1} : C(G, n)_{\langle n+1 \rangle} \rightarrow C(G, n)_{\langle n \rangle}.$$

We have

$$C(G)_{n+1} = \frac{NG_{n+1}}{(NG_{n+1} \cap D_{n+1})d_0(NG_{n+2} \cap D_{n+2})},$$

whilst

$$\begin{aligned} C(G, n)_{\langle n \rangle} &= M(sk_n G, n)_{\langle n \rangle} \\ &= \frac{NG_n}{d_0(NG_{n+1} \cap D_{n+1})}. \end{aligned}$$

We look at the cokernel of μ_{n+1} and compare it with $M(G, n)_{\langle n \rangle}$. They are the same, namely

$$M(G, n)_{\langle n \rangle} = \frac{NG_n}{d_0(NG_{n+1})}.$$

We get a map of crossed n -cube complexes,

$$C(G, n) \rightarrow M(G, n),$$

which kills off the tail. This is clearly an epimorphism and its universal property is easy to check. $\square$

4.5 From n to $n - 1$: collecting up some ideas and evidence

A useful feature of these crossed n -cube complexes is how simply they relate to crossed $(n - 1)$ -cube complexes. Clearly going one way is simple as crossed $(n - 1)$ -cube complexes are, by default, crossed n -complexes.

We noted, in Proposition 4.1, that given $M(G, n)$, the quotient crossed $(n - 1)$ -cube was $M(G, n - 1)$. Is a similar result true here? Is there an epimorphism from $C(G, n)$ to $C(G, n - 1)$?

This is linked with another problem. We have a nested sequence of full categories of $Crs^{\mathbb{N}}$,

$$Crs^1.Comp \subset Crs^2.Comp \subset \dots \subset Crs^n.Comp \subset \dots \subset Crs^{\mathbb{N}}.$$

Does the inclusion of $Crs^{n-1}.Comp$ into $Crs^n.Comp$ have a left adjoint? In other words, is $Crs^{n-1}.Comp$ a reflective subcategory of $Crs^n.Comp$?

This is very similar to the situation described in Bullejos, Faro and García-Muñoz, [15], page 240, where the objects of special interest were crossed complexes. In their case, the inclusions all have left adjoints. Replacing the inclusions by their adjoints would seem to give a tower of categories in the sense used by Baues in [2, 3]. It is therefore of some interest to see if here, also, those functors have left adjoints with nice properties.

We investigate this question first for $n = 2$, as this is, at the same time, easiest to see, is one of the most useful cases and also indicates how to proceed in general.

We have seen an example⁴ of an inclusion crossed square with a 2-truncated crossed complex viewed as a crossed square. We can now do away with the restriction that the image be truncated.

Crossed square complexes can be neatly viewed as

$$C := \left(\begin{array}{ccccc} \dots & \longrightarrow & C_3 & \xrightarrow{\mu_3} & C_{\langle 2 \rangle} & \xrightarrow{\mu_2} & C_{\langle 1 \rangle} \\ & & & \mu_1 \downarrow & & & \downarrow \mu_1 \\ & & & C_{\{2\}} & \xrightarrow{\mu_2} & & C_{\emptyset} \end{array} \right),$$

whilst those corresponding to crossed complexes look like

$$D := \left(\begin{array}{ccccc} \dots & \longrightarrow & D_3 & \xrightarrow{\mu_3} & D_{\langle 2 \rangle} & \xrightarrow{\mu_2} & D_{\langle 1 \rangle} \\ & & & \mu_1 \downarrow & & & \downarrow \mu_1 \\ & & & 1 & \xrightarrow{\mu_2} & & D_{\emptyset} \end{array} \right).$$

A map, φ , in $Crs^2.Comp$ from C to D , clearly, must kill off $C_{\{2\}}$ and hence must also kill off $\mu_2(C_{\{2\}})$, which is normal in $C_{\emptyset}$. That is not all however. If $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2 \rangle}$, then

$$\varphi(h(a, b)) = h(\varphi a, \varphi b) = 1,$$

as $\varphi a = 1$, thus φ must kill off the action of $C_{\{2\}}$ on $C_{\langle 2 \rangle}$, and thus all elements of the form, $h(a, b)$, with $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2 \rangle}$. As the image of C_2 in $C_{\emptyset}$ has been killed off, the action of that part of $C_{\emptyset}$ on $C_{\langle 1 \rangle}$ must be nullified as was the case with the inclusion crossed square that we considered earlier.

In general, we have:

Proposition 4.5. *The inclusion of $Crs^1.Comp$ into $Crs^2.Comp$ has a left adjoint,*

$$L : Crs^2.Comp \rightarrow Crs^1.Comp.$$

The left adjoint is a reflection, fixing the objects of the subcategory.

⁴at the end of section 2.3,

Proof: If C is in $Crs^2.Comp$, define $L(C)$ as follows:

- if $A \not\subseteq \langle 2 \rangle$, $L(C)_A = C_A$;
- if $A = \langle 2 \rangle$, define $L(C)_{\langle 2 \rangle} = C_{\langle 2 \rangle}/H$, where H is the normal subgroup generated by all $h(a, c)$, $a \in C_{\{2\}}$, $b \in C_{\langle 1 \rangle}$, (normality is, in fact, automatic);
- if $A = \{2\}$, $L(C)_A = 1$;
- if $A = \langle 1 \rangle$, $L(C)_A = C_{\langle 1 \rangle}/K$, where K is the normal subgroup generated by all $h(\mu_2 a, b)$ with $a \in C_{\{2\}}$, $b \in C_{\langle 1 \rangle}$.

The μ_i and h -maps are induced from those of C .

It is clear from our previous discussion that a morphism, φ , from C to some D in $Crs^1.Comp$ must factor through the various groups of the family, $L(C)$, as specified. What remains to be checked is that $L(C)$ is a crossed $\mathbb{N}$ -cube and hence is in $Crs^1.Comp$.

The μ_i are induced from those of C .

If $a \in C_{\{2\}}$, $b \in C_{\langle 1 \rangle}$, then $\mu_2 h(a, b) = h(\mu_2 a, b)$, and so is in K .

If $a \in C_{\{2\}}$, $b \in C_{\langle 2 \rangle}$, then $\mu_2 h(a, b) = h(\mu_2 a, \mu_2 b)$, and so is in K .

We thus have that $\mu_2 : L(C)_{\langle 2 \rangle} \rightarrow L(C)_{\langle 1 \rangle}$ is well defined.

Similarly, it is fairly routine to check that the induced h -maps here are well defined. The verification, then, of the axioms is straightforward, given the fact that the axioms are satisfied by the original structure in C . $\square$

It is clear that L is a functor. It is the first of a series of reflections on the higher inclusions and, if we need to emphasise which one it is, we may write it as L_1^2 .

4.6 From $C(G, 2)$ to $C(G, 1)$

What happens if we apply L to $C(G, 2)$? The answer is not that much of a surprise!

Proposition 4.6. *If G is a simplicial group, then there is a natural isomorphism,*

$$L(C(G, 2)) \cong C(G, 1).$$

(Of course, the ‘crossed 1-cube complex’ $C(G, 1)$ is just the crossed complex $C(G)$ under another name.)

Proof: As, for $A \not\subseteq \langle 2 \rangle$, $L(C)_A = C_A$, we may as well assume that G is 2-truncated, so that NG_3 is trivial. The general case is obtained by using the 2-truncation functor or more simply by ‘observation’!

The crossed square complex, $C(G, 2)$, is now just $M(G, 2)$ and up to isomorphism, this is:

$$\begin{array}{ccc} NG_2 & \xrightarrow{\mu_2} & Ker d_0 \\ \downarrow \mu_1 & & \downarrow \mu_1 \\ Ker d_1 & \xrightarrow{\mu_2} & G_1 \end{array}$$

where the μ_i with domain NG_2 are both induced by d_0 and those with codomain G_1 are the inclusions. The h -maps are defined using the isomorphisms coming from μ_1 and μ_2 of $M(G, 2)$. We have

$$\begin{aligned} Ker d_0 &\cong \frac{Ker d_1^2}{d_0(Ker d_1^3 \cap Ker d_2^3)}, \\ Ker d_1 &\cong \frac{Ker d_2^2}{d_0(Ker d_1^3 \cap Ker d_3^3)}, \\ G_1 &\cong \frac{G_2}{d_0(Ker d_1^3)}. \end{aligned}$$

Writing $\bar{x}$ for the element of G_1 , *etc.*, determined by $x \in G_2$ via these isomorphisms, this gives $h(\bar{x}, \bar{y}) = \overline{[x, y]}$.

We now have to examine the subgroups H and K defined in the description of the functor L .

We note that if $x \in Ker d_1^2$, then either (i) $x \in NG_2$, (ii) $x = s_1 a$ for some $a \in NG_1$ or (iii) x is a product of elements of type (i) and (ii). Similarly, $y \in Ker d_2^2$ is either in NG_2 , or in $s_0 NG_1$, or is a product of elements from these subgroups. Because of this and the commutator identities, it suffices to examine those $h(\bar{x}, \bar{y})$, where x, y have each of the forms (i) or (ii) in turn, as the third form can be reduced to a product of the first two.

With our assumption that $NG_3 = 1$, we get, by a simple calculation, that if $x \in Ker d_1$ and $y \in NG_2$, then

$$[x, y] = [s_0 d_0 x, s_1 d_0 y^{-1} \cdot s_0 d_0 y] \in NG_2 \cap D_2.$$

If, on the other hand, $x \in NG_2$, and $y \in \text{Ker } d_2$, then calculating $[y, x]$ using that same sort of calculation, gives a similar result. Finally if both x and y are degenerate, then $[x, y] \in NG_2 \cap D_2$. The subgroup, H , generated by all $h(\bar{x}, \bar{y})$ is, thus, contained in $NG_2 \cap D_2$.

We next use the Brown-Loday lemma (Lemma 5.7 of [13]). This can be rephrased to show that $NG_2 \cap D_2$ is generated by the commutators of the form $[s_0x, s_1y.s_0y^{-1}]$ with $x, y \in NG_1$, however then $s_0x \in \text{Ker } d_2^2$ and $s_1y.s_0y^{-1} \in \text{Ker } d_1^2$, so this is $h(s_0x, s_1y.s_0y^{-1})$ and is in H , thus $H = NG_2 \cap D_2$.

The identification of the subgroup, K , is simpler as, in the Brown-Loday lemma, used above, $d_0(NG_2 \cap D_2) = [\text{Ker } d_0, \text{Ker } d_1] = K$. Collecting things up, we find the result that $L(C(G, 2))$ is exactly $C(G, 1)$. $\square$

These are special cases of more general results that we will see shortly, but in the next section we explore crossed n -cube complexes in their own right.

5. Crossed n -cube complexes

We now turn to the general case, trying to tease out some of the elementary properties of these new gadgets, the crossed n -cube complexes.

5.1 More on the tail

First a proposition:

Proposition 5.1. *Let $\mathcal{M} = (M_A)$ be a crossed n -cube complex, then*

- (i) *if $m > n$, $M_{\langle m \rangle}$ is abelian and $\mu_m \mu_{m+1} = 1$,*
- (ii) *the group $M_\emptyset$ acts on all the $M_{\langle m \rangle}$, $m > n$, in such a way that each of the normal subgroups, $Im \mu_i$, acts trivially on $M_{\langle m \rangle}$, if $1 \leq i \leq n$.*
- (iii) *All M_A , for which $A \subseteq \langle n \rangle$, and $A \neq \emptyset$, act trivially on $M_{\langle m \rangle}$ for $m > n$.*

Proof: For (i): if $a, b \in M_{\langle m \rangle}$, then

$$[a, b] = h(a, b) = h(\mu_{m-1}a, b) = 1$$

since $\{1, \dots, m-2, m\}$ is not a down segment, nor is it within $\langle n \rangle$.

To see (ii), it is clear that the h -map,

$$h : M_{\emptyset} \times M_{\langle m \rangle} \longrightarrow M_{\langle m \rangle},$$

defines an action: ${}^a b = h(a, b)b$, so

$${}^{aa'} b = h(aa', b)b = {}^a h(a', b) h(a, b)b,$$

whilst

$${}^a ({}^{a'} b) = {}^a (h(a', b)b) = {}^a h(a', b) {}^a b,$$

which are clearly equal. If, on the other hand, $a = \mu_i a'$, we have

$${}^a b = h(\mu_i a', b)b = h(a', \mu_i b)b = b$$

since $\mu_i : M_{\langle m \rangle} \rightarrow 1$.

For (iii), similarly if $a \in M_A$, $b \in M_{\langle m \rangle}$, and $i \in A \subseteq \langle n \rangle$, then again $\mu_i b = 1$, but $h(a, b) = h(a, \mu_i b) = 1$. $\square$

Remark: In [42], a sort of ‘total quotient’ of a crossed n -cube was introduced. This is

$$Q(\mathcal{M}) = M_{\emptyset} / \prod_{i=1}^n \text{Im } \mu_i(M_{\{i\}}).$$

It is a substitute for π_0 , but does not depend on the crossed n -cube being derived from a simplicial group. The above proposition shows that if $\mathcal{M}$ is a crossed n -cube complex, with $\mathcal{M}'$, the crossed n -cube at its base, then

$$\cdots \rightarrow M_{\langle m \rangle} \rightarrow M_{\langle m-1 \rangle} \rightarrow \cdots \rightarrow M_{\langle n+1 \rangle}$$

is a chain complex of $Q(\mathcal{M}')$ -modules.

This reinforces the intuition that a crossed n -cube complex consists of the crossed n -cube, $\mathcal{M}'$, with a chain complex of $Q(\mathcal{M}')$ -modules attached to it.

5.2 The reflections

As was said earlier, the category, $Crs^n.Comp$, of crossed n -cube complexes is a full subcategory of $Crs^{\mathbb{N}}$, the category of crossed $\mathbb{N}$ -cubes in such a way that if $n < m$, we have full inclusions,

$$Crs^n.Comp \subset Crs^m.Comp \subset Crs^{\mathbb{N}}.$$

Each subcategory is determined merely by specifying that certain positions of a $\mathbb{N}$ -cube are trivial. By general intuitions on algebraic theories, this suggests, in particular, that the inclusions,

$$Crs^n.Comp \hookrightarrow Crs^{n+1}.Comp,$$

should each have a left adjoint, L_n^{n+1} . This left adjoint will be explicitly given shortly. The important point is to note that if, as claimed, crossed n -cube complexes form ever better algebraic approximations to integral homotopy types, with increasing n , the study of the passage from crossed $(n+1)$ -cube complexes to crossed n -cube complexes via this adjunction could yield important information as to where exactly in the homotopy type the ‘lost information’ resides when we pass from one stage to the next. This will be especially so when studying the various functors from simplicial groups to these crossed models.

5.3 The functor, $L_n^{n+1} : Crs^{n+1}.Comp \rightarrow Crs^n.Comp$.

We take $\mathcal{M} = (M_A)$ to be a crossed $(n+1)$ -cube complex and will write, for short, $\mathcal{L} := L_n^{n+1}(\mathcal{M}) = (F_A)$ for the corresponding crossed n -cube complex. The form of the two structures makes it clear how to define F_A for most A :

- if $A \not\subseteq \langle n \rangle$ and A is not a down segment, $F_A = 1$,
- if $A = \langle m \rangle$, $n+1 < m$, $F_A = M_A$,
- if $A \subset \langle n \rangle$, so $A \neq \langle n \rangle$, then $F_A = M_A / \text{Im } \mu_{n+1}$,
- for $A = \langle n \rangle$, let $N_{\langle n \rangle}$ be the normal subgroup of $M_{\langle n \rangle}$ generated by all the elements $\mu_{n+1}h(b, c)$, where $\{B, C\}$ forms a non-trivial partition of $\langle n+1 \rangle$, so $C \neq \emptyset$, with $n+1 \in B$, $b \in M_B$, and $c \in M_C$, then $F_{\langle n \rangle} = M_{\langle n \rangle} / N_{\langle n \rangle}$.

- for $A = \langle n+1 \rangle$, let $N_{\langle n+1 \rangle}$ be the normal subgroup of $M_{\langle n+1 \rangle}$ generated by all elements, $h(b, c)$, $b \in M_B$, $c \in M_C$, where $\{B, C\}$ is as before (above), then $F_{\langle n+1 \rangle} = M_{\langle n+1 \rangle} / N_{\langle n+1 \rangle}$.

The verification that the obvious definitions of μ_i and h -maps give $\mathcal{L}$ the structure of a crossed n -cube complex is now quite easy.

Proposition 5.2. *The structure, $\mathcal{L}$, is a crossed n -cube complex and the natural quotient, $\eta : \mathcal{M} \rightarrow \mathcal{L}$, a map of crossed $(n+1)$ -cube complexes. These have the following universal property:*

Given any map, $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$, of crossed $(n+1)$ -cube complexes, where $\mathcal{M}'$ is in $\text{Crs}^n.\text{Comp}$, there is a unique factorisation in the form

$$\mathcal{M} \xrightarrow{\eta} \mathcal{L} \xrightarrow{\varphi'} \mathcal{M}'$$

where η is the canonical quotient map. As a consequence, the functor, $\mathbb{L}_n^{n+1}$ is left adjoint to the inclusion of $\text{Crs}^n.\text{Comp}$ into $\text{Crs}^{n+1}.\text{Comp}$.

Remark: We note that if we take the $(n+1)$ -cube determined by $\{F_A : A \subseteq \langle n+1 \rangle\}$, the cokernel of μ_{n+1} is exactly the same as that for $\{M_A : A \subseteq \langle n+1 \rangle\}$. In fact, the only case that is not immediate is that of the cokernel of $\mu_{n+1} : F_{\langle n+1 \rangle} \rightarrow F_{\langle n \rangle}$. This is induced by the μ_{n+1} of $\mathcal{M}$ on the quotients,

$$\frac{M_{\langle n+1 \rangle}}{N_{\langle n+1 \rangle}} \rightarrow \frac{M_{\langle n \rangle}}{N_{\langle n \rangle}},$$

but $\mu_{n+1}N_{\langle n+1 \rangle} = N_{\langle n \rangle}$, so the quotient is $M_{\langle n \rangle} / \text{Im } \mu_{n+1}$ as claimed.

Proof of Proposition 5.2:

The verification of the first part is routine, and so will be omitted.

Suppose given $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ with $\mathcal{M}'$ a crossed n -cube complex. As $\mathcal{M}'_A = 1$ if $A \not\subseteq \langle n \rangle$ and A is not a down segment, then $\text{Ker } \varphi_A = M_A$ whatever φ is. Similar comments show that if $A = \langle m \rangle$, with $m > n+1$, then φ_A has no special properties and if $A \subseteq \langle n \rangle$, $A \neq \langle n \rangle$, then $\text{Ker } \varphi_A$ must contain $\text{Im } \mu_{n+1}$, since $\varphi_A \mu_{n+1} a$ for $a \in M_{A \cup \{n+1\}}$ must be the identity as $\varphi(a) \in M'_{A \cup \{n+1\}}$, which is itself zero.

There thus remain only the justification of the statements relative to $\langle n+1 \rangle$ and $\langle n \rangle$. If $A \cup B = \langle n+1 \rangle$ with $A \cap B = \emptyset$, but $B \neq \emptyset$ and $n+1 \in A$, then

$M'_A = 1$, so $\varphi_{\langle n+1 \rangle} h(a, b) = 1$ for $a \in M_A$, $b \in M_B$, thus $\text{Ker } \varphi$ will always contain the subgroup denoted $N_{\langle n+1 \rangle}$ in the statement of the proposition.

Similarly $\varphi_{\langle n \rangle} \mu_{n+1} h(a, b) = 1$, as it is $\mu_{n+1} \varphi_{\langle n+1 \rangle} h(a, b)$, so $N_{\langle n \rangle} \subseteq \text{Ker } \varphi_{\langle n \rangle}$.

We have thus that $\text{Ker } \varphi$ is always at least as big as these subgroups. As these form the kernel of the map from $\mathcal{M}$ to $\mathcal{L}$, which has been verified to be a map of $(n+1)$ -crossed cubes, we have checked both the existence and the uniqueness of the claimed factorisation. $\square$

The same basic construction will give the left adjoint to the inclusion of $Crs^n.Comp$ into $Crs^{\mathbb{N}}$, and thus by restriction to that of $Crs^n.Comp$ into $Crs^m.Comp$, for $m > n$. Of course, that will give a chain of functors that simplify a crossed n -cube complex killing off the groups at some corners, until one arrives at a crossed complex, and finally at a chain complex of modules over the total quotient, $Q(\mathcal{M})$, of the initial crossed n -cube complex.

6. Comments, queries and some ideas for further development

In this paper, the basic idea has been to take an algebraic model for homotopy $(n+1)$ -types, to add a ‘tail’, and then to look at how the models for different n are related algebraically and categorically. There are several different well known algebraic or categorical models for homotopy $(n+1)$ -types. In this paper, we have concentrated on crossed n -cubes as their algebra is fairly easy to explain as it, in fact, uses fairly elementary almost classical tool, but there are other useful models. Some of these have been developed in more general contexts than the one used here.

Parallel to the paper, [42], was the paper by Bullejos, Cegarra and Duskin, [14], which used more simplicial techniques, and, as their constructions worked inductively by working in a Barr-exact category, their results are directly applicable in a more general context. They replaced the category, Grp , of groups by that, $Grp(E)$, of group objects internal to such a Barr-exact category, E . As the categories of simplicial groups in E and of cat^1 -groups in E are also Barr-exact, they could repeat their basic observations inductively.

Their end result was a cat^n -group rather than a crossed n -cube. These are equivalent models. As a consequence of their methods being within E , however, their methods can be applied directly to other contexts. In other words, given a simplicial group object in an exact category, one can obtain a cat^n -group in that category. The same goes for the results in this paper. We will return to this setting slightly later. We could have taken the cat^n -group model instead of the crossed n -cube one, and, adding a ‘tail’ to that. This should be quite straight forward and easy to do.

We note that a closely related idea was introduced by Baues in [3]. There was defined, in section II.2, the notion of a homotopy system of order $(n+1)$. Such an object consisted of an n -dimensional CW-complex, X^n , a chain complex, C , of free $\pi_1(X^n)$ -modules and a homomorphism that links C_{n+1} with $\pi_n(X^n)$. Later, in Appendix B in chapter IV of [3], a method is given that produces a homotopy system of order 4 from a CW-complex via a construction defined on simplicial groups that is closely related to the two methods mentioned above here. That method involves the notions of *quadratic module* and *quadratic complex*. The first of these is like a Conduché 2-crossed module, as in [19], which corresponds to the Moore complex of a 2-truncated simplicial group, but with a certain nilpotency condition added, whilst the second one is the corresponding ‘model with a tail’.

We could also have taken as the basic model *n-hypercrossed complexes* in the sense of [17] and have then added a tail. This remains an interesting area to investigate, but requires more difficult algebraic manipulations than those used here.

That brings us to n -truncated simplicial group(oid)s, more exactly back to the n -hypergroupoids that were mentioned in the introduction and which combine a truncated simplicial structure with an algebraic condition in their top dimension. These provide another good model for homotopy n -types. Their initial development was in Duskin, [22], and Glenn, [30].

We will drop back, here, into using an exact category, E , as the base category instead of Set . We will need to revamp some of the ideas from the classical theory of simplicial sets, so as to adapt them to the context of simplicial objects in E . For instance, given (n, k) with $0 \leq k \leq n$, and an object, X , in $\text{Simp}(E)$, the category of simplicial objects in E , we can form an object, $\Lambda[n, k](X)$, of (n, k) -horns in X . There is a natural morphism,

$$\lambda_{n,k}^X : X_n \rightarrow \Lambda[n, k](X),$$

which sends x to $(d_0x, \dots, d_{k-1}x, -, d_{k+1}x, \dots, d_n)$. In the case where $E = \mathbf{Set}$, X is a Kan complex if, for all (n, k) , every (n, k) -horn has a filler, and that implies that the above map is an epimorphism. That latter fact gives how to extend the notion to $\mathbf{Simp}(E)$. One says that X satisfies the Kan condition in dimension n if $\lambda_{n,k}^X$ is an epimorphism, and X satisfies the *exact Kan condition in dimension n* if that same map is an *isomorphism*.

Finally an n -hypergroupoid is a simplicial object in E for which the morphisms, $\lambda_{m,k}^X$, are epimorphisms for all k, m , so X is a Kan complex object, but, in addition, these $\lambda_{m,k}^X$ are isomorphism for all $m > n$.

As Kan complexes are often thought of as ∞ -groupoid, n -hypergroupoids are sometimes called ‘(weak) n -groupoids’, *e.g.*, by Getzler, [28]. Because of the link with the exact Kan condition, and the related idea of an exact fibration in dimension n , Beke, [5], uses the term *exact n -type* for the same idea.

The requirements on E can be relaxed further. Instead of using epimorphisms in the definition above, one can require that a class of special morphisms be specified, which behave sufficiently like epimorphisms. These morphisms are typically called *covers*. This idea has been used, for instance, in Henriques, [32], Zhu, [51], Behrand and Getzler, [4], and others papers mentioned in the introduction. In this theory, there is a strong link with singleton Grothendieck topologies, see Henriques, cited above, Meyer and Zhu, [38], or Rogers and Zhu, [46]. In this context, a simplicial object, X , will be an n -hypergroupoid if each of the $\lambda_{m,k}^X$ is a cover and is an isomorphism if $m > n$.

How does this relate to the notions from earlier in this paper? Hopefully we can extend, as suggested, the earlier discussion to considering an n -hypergroupoid ‘with a tail’, which will related to a suitable notion of crossed n -cube complex in the wider setting of categories with covers, and thus will include Lie, topological and other geometric analogues of the results here - and with possibly very little extra work.

But what is the evidence that this is possible, say, in low dimensions and with crossed complexes?

In [39], Nan Tie, working in an arbitrary category, $\mathbb{C}$, with finite limits, introduced crossed complexes in $\mathbb{C}$, T -complexes in $\mathbb{C}$, and so on, then considered *filtered* systems of hypergroupoids and n -truncated crossed complexes in $\mathbb{C}$ showing that they led to equivalent categories. This raises the

challenge of finding an analogous relationship between some form of filtered hypergroupoid and crossed cube complexes that works in the context of ‘categories with covers’, and thus in a wide variety of geometric settings.

A further indication as to a useful generalisation of these links occurs in Getzler’s [28], where there is discussion of (weak) ℓ -groupoids, *i.e.*, Duskin-Glenn ℓ -hypergroupoids, for which one has an explicit choice of fillers for horns. To axiomatise these (weak) ℓ -groupoids, Getzler uses a weakened form of the axioms for simplicial T -complexes, as defined by Dakin, [21], and used by Ashley, [1], which are equivalent to crossed complexes - so here again there are some pieces of the puzzle to turn over.

References

- [1] N. Ashley, *Simplicial T-Complexes: a non abelian version of a theorem of Dold and Kan*, *Dissertations Math.*, 165, (1989), 11 – 58.
- [2] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press.
- [3] H. J. Baues, 1991, *Combinatorial homotopy and 4-dimensional complexes*, *De Gruyter Expositions in Mathematics*, Walter de Gruyter.
- [4] K. Behrend and E. Getzler, 2017, *Geometric higher groupoids and categories*, in *Geometry, Analysis and Probability: In Honor of Jean-Michel Bismut*, volume 310 of *Progress in Mathematics*, 1–45, Springer.
- [5] T. Beke, *Higher Čech Theory*, *K Theory*, 32, (2004), 293 – 322.
- [6] D. Bourn and M. Gran, *Central extensions in semi-abelian categories*, *Journal of Pure and Applied Algebra*, 175, (2002), 31–44.
- [7] R. Brown and M. Golasiński, *A model structure for the homotopy theory of crossed complexes*, *Cah. Top. Géom. Diff. cat.*, 30, (1989), 61 – 82.
- [8] R. Brown and P. J. Higgins, *On the connection between the second relative homotopy groups of some related spaces*, *Proc. London Math. Soc.*, (3) 36, (1978), 193 – 212.

- [9] R. Brown and P. J. Higgins, *Colimit theorems for relative homotopy groups*, J. Pure Appl. Alg., 22, (1981), 11 – 41.
- [10] R. Brown and P. J. Higgins, 1982, *Crossed complexes and non-abelian extensions*, in *Int. Conf. Cat. Theory, Gumpersbach*, volume 962 of *Lecture Notes in Maths*, Springer-Verlag.
- [11] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33 – 57.
- [12] R. Brown, P. J. Higgins and R. Sivera, 2010, *Nonabelian Algebraic Topology. Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids, with contributions by Christopher D. Wensley and Sergei V. Soloviev*, volume 15 of *EMS Tracts in Mathematics*, European Mathematical Society.
- [13] R. Brown and J.-L. Loday, *Van Kampen Theorems for diagrams of spaces*, Topology, 26, (1987), 311 – 337.
- [14] M. Bullejos, A. M. Cegarra. and J. Duskin, *On cat^n -groups and homotopy types*, J. Pure Appl. Alg., 86, (1993), 135 – 154.
- [15] M. Bullejos, E. Faro and M. A. García-Muñoz, *Postnikov invariants of crossed complexes*, Journal of Algebra, 285, (2005), 238–291.
- [16] P. Carrasco, 1987, *Complejos Hiper cruzados, Cohomologia y Extensiones*, Ph.D. thesis, Universidad de Granada.
- [17] P. Carrasco and A. M. Cegarra, *Group-theoretic Algebraic Models for Homotopy Types*, J. Pure Appl. Alg., 75, (1991), 195 – 235.
- [18] P. Carrasco and T. Porter, *Coproduct of 2-crossed modules: applications to a definition of a tensor product for 2-crossed complexes*, Collectanea mathematica, 67, (2016), 485–517.
- [19] D. Conduché, *Modules croisés généralisés de longueur 2*, J. Pure Appl. Alg., 34, (1984), 155 – 178.
- [20] D. Conduché, *Simplicial Crossed Modules and Mapping Cones*, Georgian Math. J., 10, (2003), 623 – 636.

- [21] M. K. Dakin, 1977, *Kan complexes and multiple groupoid structures*, Ph.D. thesis, University of Wales, Bangor.
- [22] J. Duskin, 1979, *Higher-dimensional torsors and the cohomology of topoi: the abelian theory*, in *Applications of Sheaves*, number 753 in *Lecture Notes in Maths.*, Springer-Verlag.
- [23] P. J. Ehlers and T. Porter, *Varieties of simplicial groupoids I: Crossed complexes*, *J. Pure Appl. Alg.*, 120, (1997), 221 – 233.
- [24] P. J. Ehlers and T. Porter, *Erratum to “Varieties of simplicial groupoids I: Crossed complexes”*, *J. Pure Appl. Alg.*, 134, (1999), 207 – 209.
- [25] G. J. Ellis, 1984, *Crossed modules and their higher dimensional analogues*, Ph.D. thesis, University of Wales Bangor.
- [26] G. J. Ellis, *Crossed Squares and Combinatorial Homotopy*, *Math. Z.*, 214, (1993), 93 – 110.
- [27] G. J. Ellis and R. Steiner, *Higher dimensional crossed modules and the homotopy groups of $(n + 1)$ -ads*, *J. Pure Appl. Alg.*, 46, (1987), 117–136.
- [28] E. Getzler, *Lie theory for nilpotent L_∞ -algebras*, *Annals of Mathematics*, 170, (2009), 271–301.
- [29] N. D. Gilbert, 1988, *The fundamental cat^n -group of an n -cube of spaces*, in J. Aguadé and R. Kane, eds., *Proc. Conf. Algebraic Topology, Barcelona, 1986*, volume 1298, of *Lecture Notes in Math.*, 124 – 139, Springer.
- [30] P. G. Glenn, *Realization of cohomology classes in arbitrary exact categories*, *J. Pure. Appl. Alg.*, 25, (1982), 33 – 105.
- [31] D. Guin-Walery and J.-L. Loday, 1981, *Obstructions à l’excision en K -théorie algébrique*, in *Evanston Conference on Algebraic K -theory, 1980*, volume 854 of *Lecture Notes in Maths.*, 179 – 216, Springer.
- [32] A. Henriques, *Integrating L_∞ -algebras*, *Compositio Mathematica*, 144, (2008), 1017–1045.

- [33] K. H. Kamps and T. Porter, 1997, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific Publishing Co. Inc., River Edge, N. J.
- [34] M. Kapranov, 2015, *Membranes and higher groupoids*, URL <https://arxiv.org/abs/1502.06166>.
- [35] A. Kapustin and L. Spodyneiko, 2025, *Higher symmetries, anomalies, and crossed squares in lattice gauge theory*, URL <https://arxiv.org/abs/2507.16966>.
- [36] J.-L. Loday, *Spaces with finitely many homotopy groups*, J. Pure Appl. Alg., 24, (1982), 179 – 202.
- [37] S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41 – 48.
- [38] R. Meyer and C. Zhu, *Groupoids in categories with pretopology*, Theory and Applications of Categories, 30, (2015), 1906–1998, URL <http://www.tac.mta.ca/tac/volumes/30/55/30-55.pdf>.
- [39] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Alg., 56, (1989.), 177 – 194.
- [40] nLab authors, 2026, *crossed complex*, <https://ncatlab.org/nlab/show/crossed+complex>, Revision 45.
- [41] nLab authors, 2026, *crossed n-cube*, <https://ncatlab.org/nlab/show/crossed+n-cube>, Revision 20.
- [42] T. Porter, *n-types of simplicial groups and crossed n-cubes*, Topology, 32, (1993), 5 – 24.
- [43] T. Porter, >2026, *Profinite Algebraic Homotopy*, (unpublished draft, an old version is available from the n-Lab), URL ncatlab.org/timporter/files/ProfAlgHomotopy.pdf.
- [44] J. P. Pridham, 2013, *Notes characterising higher and derived stacks concretely*, URL <https://arxiv.org/abs/1105.4853>.

- [45] J. P. Pridham, *Presenting higher stacks as simplicial schemes*, Advances in Mathematics, 238, (2013), 184–245.
- [46] C. Rogers and C. Zhu, *On the homotopy theory for Lie ∞ -groupoids, with an application to integrating L_∞ -algebras*, Algebraic & Geometric Topology, 20, (2020), 1127–1219.
- [47] R. Steiner, *Resolutions of spaces by n -cubes of fibrations*, J. London Math. Soc., 34, (1986), 169 – 176.
- [48] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453 – 496.
- [49] J. H. C. Whitehead, 1950, *Algebraic Homotopy Theory*, in *Proc. Int. Cong. of Mathematics, Harvard*, volume 2, 354 – 357.
- [50] J. Wolfson, *Descent for n -bundles*, Advances in Mathematics, 288, (2016), 527–575.
- [51] C. Zhu, *n -Groupoids and Stacky Groupoids*, International Mathematics Research Notices, 2009:4087–4141.

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